# Theory of stream rates

Syntax, semantics, Kleene-like algebra

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November 17, 2025

## 1 Introduction

I will attempt to begin formalizing a theory of stream rates, beginning with a proposed syntax, continuing with a set-based semantics, and finally proposing a Kleene-like algebra for equational reasoning, which I will prove correct with respect to the semantics.

## 2 Syntax

**Definition 2.1.** A stream rate expression is defined by the following grammar:

$$\theta ::= @n/t \mid \theta \cdot \theta \mid \theta \parallel \theta \mid \theta + \theta \mid \theta^* \mid \theta \wedge \theta \mid \top \mid \bot \mid \epsilon \mid \#n \mid \overline{@}n/t$$

I will quickly give some notes and some hints at the semantics, which will be expanded upon in the next section.

- $n$ is a non-negative integer. $t$ is a positive real.
- For both $@n/t$ and $\overline{@}n/t$, we require that these denote stream timelines of length $\geq t$. The definition of stream timelines and length will be given below, when I discuss semantics.
- $\#n$ has no constraint on length.

## 3 Set Semantics

I will now give a semantics to the syntax. First, we define a notion of *stream timeline*.

**Definition 3.1.** A stream timeline (which we can refer to using one of $S, T, U, V$) is a *finite multiset* of *non-negative real numbers*.

- The **size** of a stream timeline $S$ is the *number of elements* in $S$. We denote the size of a stream timeline $S$ by `size(S)`.
- The **length** of a stream timeline $S$ is the *maximum element* of $S$ if the size of $S$ is $> 0$; if the $S$ is empty (the size of $S$ is 0), then length is 0. We denote the length of a stream timeline $S$ by `length(S)`.

**Definition 3.2.** A *discrete* stream timeline is a stream timeline where all elements are *non-negative integers*.

**Definition 3.3.** $\Theta$ is the set of all possible stream timelines.

Equipped with these definitions, I will now give a recursive definition to each of the syntactic constructs. Something to note right off the bat: these semantics are all denotationally defined as *possibly infinite sets of stream timelines*.

## 3.1 $[\![@n/t]\!]$

I start by defining a function `count` that operates on stream timelines.

**Definition 3.4** (`count`)**.** Let `count(S, start, end)`, where `start`, `end` are non-negative real numbers, `end` $\geq$ `start`, and `S` is a stream timeline. `count(start, end, S)` is then the **size** of the stream timeline defined by:

$$\{e \mid e \in \texttt{S} \wedge \texttt{start} \leq e < \texttt{end}\}$$

$[\![@n/t]\!]$ is then defined as the set of all stream timelines $S$ for which the following predicate is true (note the encoding of the requirement that the stream length is $\geq t$):

$$\forall e \in S, \texttt{count(S, e, e + t)} \leq n \wedge \texttt{length(S)} \geq t$$

Note the slight abuse of font style: $S$ and `S` both refer to the same stream timeline and the difference in font style is irrelevant.

## 3.2 $[\![\theta_1 \cdot \theta_2]\!]$

I again start by defining a function, this time called `shift`, that operates on stream timelines.

**Definition 3.5** (`shift`)**.** Let `shift(S, offset)`, where `offset` is a non-negative real number and `S` is a stream timeline, evaluate to the stream timeline:

$$\{e + \texttt{offset} \mid e \in S\}$$

$[\![\theta_1 \cdot \theta_2]\!]$ is then defined as the following set of stream timelines:

$$\{S \cup \texttt{shift(T, length(S))} \mid S \in [\![\theta_1]\!], \texttt{T} \in [\![\theta_2]\!]\}$$

Note again the slight abuse of typography. Also, note that this definition is essentially $\cup$ (set union) mapped over the *Cartesian product* between $[\![\theta_1]\!]$ (which is, as a reminder, a set of stream timelines) and the set resulting from mapping `shift` over $[\![\theta_2]\!]$.

## 3.3 $[\![\theta_1 \parallel \theta_2]\!]$

$[\![\theta_1 \parallel \theta_2]\!]$ is defined as the following set of stream timelines:

$$\{S \cup T \mid S \in [\![\theta_1]\!], T \in [\![\theta_2]\!]\}$$

## 3.4 $[\![\theta_1 + \theta_2]\!]$

$[\![\theta_1 + \theta_2]\!]$ is defined as the following set of stream timelines:

$$[\![\theta_1]\!] \cup [\![\theta_2]\!]$$

## 3.5 $[\![\theta_1 \wedge \theta_2]\!]$

$[\![\theta_1 \wedge \theta_2]\!]$ is defined as:

$$[\![\theta_1]\!] \cap [\![\theta_2]\!]$$

## 3.6 $[\![\theta^*]\!]$

Intuitively, $[\![\theta^*]\!]$ should be the set of all finite repetitions of members of $[\![\theta]\!]$ (including 0 repetitions). To make this a bit more precise, notice that this is just unbounded concatenation, including $\epsilon$. Thus, we can recursively define $[\![\theta^*]\!]$ as:

$$\epsilon \cup [\![\theta]\!] \cup [\![\theta \cdot \theta]\!] \cup [\![(\theta \cdot \theta) \cdot \theta]\!] \ldots$$

Formally, given the following definitions:

$$[\![\theta^0]\!] = \{\emptyset\}$$
$$[\![\theta^1]\!] = [\![\theta]\!]$$
$$[\![\theta^{i+1}]\!] = [\![\theta^i]\!] \cdot [\![\theta]\!], \forall i \in \mathbb{N}$$

we define $[\![\theta^*]\!]$ as $\bigcup_{i \geq 0} [\![\theta^i]\!]$.

### 3.7 $[\![\top]\!]$

$[\![\top]\!]$ is the set of all possible stream timelines. In other words, the set of all multisets of non-negative reals.

### 3.8 $[\![\bot]\!]$

$[\![\bot]\!]$ is the empty set $\emptyset$.

### 3.9 $[\![\epsilon]\!]$

$[\![\epsilon]\!]$ is the set consisting of just the empty stream timeline (in other words, the set containing just the empty multiset): $\{\emptyset\}$

### 3.10 $[\![\#n]\!]$

$[\![\#n]\!]$ is the set of all stream timelines with size $n$. Perhaps more formally, it is the set of stream timelines defined by:

$$\{S \mid \texttt{size(S)} = n\}$$

### 3.11 $[\![\overline{@}n/t]\!]$

$[\![\overline{@}n/t]\!]$ is defined as the set of stream timelines $S$ for which the following predicate is true (again, note the encoding of the requirement that the length of $S$ is $\geq t$):

$$\forall \texttt{i} \in \mathbb{N} \cup \{0\}, \texttt{count(S, i*t, (i+1)*t)} \leq n \wedge \texttt{length(S)} \geq t$$

## 4 Equational Reasoning