Self-interested agents (Game theory) :

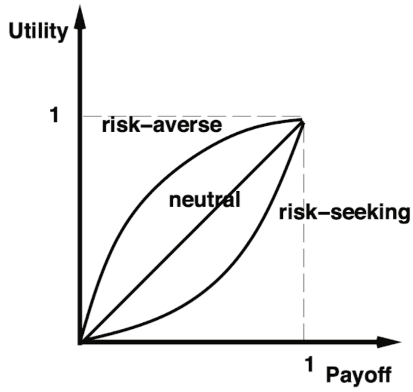
- Dominant strategy equilibrium : For every actions of the player, one action gives always better payoff. Can be strictly (all strict inequality), weakly strictly (at least one is strict inequality), very weakly (all no strict inequality) dominant strategy. There is an equilibrium when all the players play their dominant strategy as a pure strategy.

- Pure strategy equilibrium : Don't always exist. Each player play only one action and have no interest in changing that action. The pure strategy equilibrium don't guarantee better social outcome for general-sum games (see prisoner's dilemma).

- Mixed equilibrium minimax strategies : Always exist. Assume that the agent wants to maximize its gain supposing the opponent wants to minimize its loss. It consists of a probability distributions of actions for each player. All the players don't want to deviate from their probability distribution to maximize their gain or minimize their loss. The value of the game is the average gain of player A. The support of actions are the actions played by the player at equilibrium. Suppose player A will play with (p, 1-p), he wants to make player B indifferent of his actions (same expected payoff). Resolve the linear equation. This works only if there is a pure Nash equilibrium for each actions.

- Fictitious play : Each round the first player assume all other players will have fixed strategy and optimize the best strategy. Then, it is the turn of the second player, … Each player update his strategy by playing his new strategy with probability 1/N and his old strategy with probability (N-1)/N. In zero-sum games, it will always converge to a Nash equilibrium. For general-sum games, the convergence is not guaranteed.

- Utility : Give a subjective value for a resource and a subject. We can measure the attitudes toward risk in terms of utility. We can calculate the preference between the payoff of a lottery versus the sure return utility.



Attitude	u(10%)	u(20%)	...	u(90%)	u(0.1)	u(L)
risk-averse	40%	60%	...	99%	0.4	0.1
risk-seeking	1%	3%	...	60%	0.01	0.1

Find $u(r)$ so that $E[u(r)] = U$ is equivalent to sure return U.

The utility function is calculated using the preference relation between ressources. The preference relation should satisfy multiple conditions :

- Completeness : Exist for every pair of outcomes.
- Transitivity : if $a \succ b$ and $b \succ c$, then $a \succ c$
- Subsidability : If a is equally preferred at b, then each lottery with outcome a is equally preferred to lottery with outcome b.
- Monotonicity : if $o_1 \succ o_2$ and $p > q$, then $[p : o_1; (1-p) : o_2] \succ [q : o_1; (1-q) : o_2]$
- Continuity : if $o_1 \succ o_2 \succ o_3$, there exists p such that $o_2 \succ [p : o_1; (1-p) : o_3]$

From the continuity axiom, we can elicit a probability and calculate the system of equations $p_j(u(o_j) - u(o_{j+2})) = u(o_{j+1}) - u(o_{j+2})$ with $u(o_1) = 1, u(o_k) = 0$.

- <u>Nash equilibrium</u> : Dominant strategy and mixed strategy equilibrium. There is at least one set of mixed Nash equilibrium strategies for general-sum games. There is at least a Nash equilibrium in zero-sum games but not necessarily minimax. The Nash equilibrium is a linear complementarity problem usually solved by the Lemke's method.

- <u>Dominated actions elimination</u> : Rational players will never chose dominated actions (i.e. at least another action have better payoff no matter the opponent strategy). Eliminating weakly dominated actions may eliminate some Nash equilibria but not all Nash equilibria. An algorithm may be for all support for player A take the support of actions of player B that are not conditionally dominated given sup(A) (Actions_B). If there is no action in sup(A) that are conditionally dominated given actions_B, then for all sup(B) that are in Actions_B verify that no actions in sup(A) are conditionally dominated given sup(B). If all conditions have been verified, then it may be a Nash equilibrium.

- Stackelberg games : Decisions in that games are made sequentially where there is a leader (first decision) and a follower (second decision). Stackelberg equilibrium can be very different from Nash equilibrium. The leader can play such as it will have higher payoff than when playing under Nash equilibrium.

Real-world games :

- Bayes-Nash equilibrium : Have some belief about opponent. After each round, the belief about the opponent is updated using the Bayes rule. Compute a Nash equilibrium given the guessed type of the opponent. Also called the ex ante equilibrium. Players are assumed to know the actions and payoff of their opponents after each rounds.

- Ex-post Nash equilibrium : Don't always exist. Strategies that gives the highest utilities no matter the opponent's type (dominant strategy). For example, bidding equal to your true value for the Vickrey auction.

- Mediator : Improvement so that the Nash equilibrium is shifted toward a cooperation and the sum of reward is better. A player can plays the mediator and the mediator give another action depending on if the other player plays the mediator too.

			B	
		M	C	D
	M	(9,9)	(10,0)	(5,5)
A	C	(0,10)	(9,9)	(0,10)
	D	(5,5)	(10,0)	(5,5)

- Correlated equilibrium : Assume having a coordinator that propose an action present in a set of strategies with uniform probability. The best response for the agent is to choose the action suggested by the coordinator. The expected payoff is fair and give the best sum of payoffs. The set of strategies for the correlated equilibrium is not necessarily symmetric. Can create a bigger payoff matrix to visualize better the coordinator. The payoff of the game can be better if the two players coordinate their two actions.

- Coarse correlated equilibrium : If all player play a no-regret strategy, the game is certain to converge to a coarse correlated equilibrium. The difference about the correlated equilibrium is that the opponent has more freedom, he can play stupid action and the best strategies is to coordinate its action to the opponent.

$$\sum_s p(s)u_i(s) \geq max_{x \in S_i} \sum_s p(s)u_i(x, s_{-i})$$

$$DSE \subset PSNE \subset MSNE \subset CE \subset CCE$$

- Price of anarchy and stability : We can take the equilibrium that are coordinated and the one that are not. With that, we can calculate the price of anarchy :

$$PoA = \frac{max_{\underline{s}\in S} R(\underline{s})}{min_{\underline{s}\in E} R(\underline{s})}$$

and the price of stability :

$$PoA = \frac{max_{\underline{s}\in S} R(\underline{s})}{max_{\underline{s}\in E} R(\underline{s})}$$

. A (lambda,mu) smooth game is such that

$$\sum_{i\in\mathcal{A}} r_i(s_i^*, \underline{s}_{-i}) \geq \lambda R(\underline{s}^*) - \mu R(\underline{s})$$

and we can prove that for this type of game the price of anarchy is bounded to

$$\lambda/(1+\mu)$$

.

- Strategic negotiation : Each round an agent makes an offer and the other ones reject or not. the rounds continue until there is no agreement. If the negotiation fails, the agents accept the conflict payoff. The player having the last offer can force the negotiation. Can make a negotiation where the reward decrease with time to have agreement on coordination. However, the first player is the one that finish with the most reward usually. Agents want to maximize their utility

$$u_i(D_j) = \left[\sum_{g\in G(D_j)} w_i(g)\right] - c_i(D_j)$$

where Dj is the joint plan deal, wi(g) the worth of g and ci(Dj) the cost of the joint deal. Dc is the conflict deal.

- Nash Bargaining solution : A best joint deal Dj should satisfy 6 conditions :
    - Feasibility
    - Pareto-optimal : There is no deals where every agent get better utilities.
    - Individually rationality : For every agents the deal is better than the conflict deal.
    - Independence of sub-optimal alternatives : If the deal is optimal for the set of strategies, then it is optimal for every subsets of strategies.
    - Independence of linear transformations : If the utility is linearly transformed, then the new solution too.
    - Symmetry : If the game is symmetric for all players, then the agents get the same expected payoff.

  The single solution to the Nash Bargaining problem should maximize

$$(u_1(\overline{D}), ..., u_n(\overline{D})) = sup_D \Pi_{i=1}^{n} \left(u_i(\overline{D}) - u_i(D_c)\right)$$

  but need a mediator to collect and compute. An alternative without mediator can be that each agent propose a deal and chose the deal that maximize the product of utility differences. Multiple ways to obtain it :

- Monotonic concession protocol (Zeuthen) : Alternating offers where after each rounds the agent that have the most to lose by negotiation failure (conflict deals) makes a new concession. If an agent rejects offer Dj and makes offer Di, by individual rationality, we can calculate the risk that the agent takes $\frac{u_i(D_i) - u_i(D_j)}{u_i(D_i) - u_i(D_c)}$ which represent the limit probability that the negotiation will fail with that new proposition.

- Monotonic concession protocol (Rosenschein) : The agent with the smallest risk tolerance make the concession. If the utility of Di is equal to Dc, then set risk of the agent i to 1. We can show that it converge to the Nash Bargaining solution : limit the deal that give large p

  $$(u_i(D_j) - u_i(D_c))(u_j(D_j) - u_j(D_c)) \quad > \quad (u_i(D_i) - u_i(D_c))(u_j(D_i) - u_j(D_c))$$

  The propositions are made in parallel

Auctions and Mechanisms design :

- Dutch auction : Open-cry and first-price. The seller decrease his bid until someone agree to the bid. Speculate to bid a little bit lower.

- English auction : Open-cry. The buyer higher their bids until no one want to get higher. The winner will pay a little more than the second-highest bid.

- Discriminatory auction : Sealed-bid and first-price. The buyer can speculate and give bids a little less than their actual value. The strategy is as follow

$$b(t) = t - \frac{\int_A^t F(x)^{n-1} dx}{F(t)^{n-1}}$$

where t is the true value, A the lowest valuation and F the density function where the valuations are taken from the other players.

- Vickrey auction : Sealed-bid and second-price. The optimal strategy is to be truthful (incentive compatible). For multi units auctions, each agent pay the bid equal to the first bid that are not in the winning bids.

- (M+1)st auction : A double auction is when N buyers have different buy bids and M sellers have different sell bids. Setting an uniform price to a maximum of transactions. One way is to take the median. The M-th price is incentive-compatible for seller and the (M+1)-th price is incentive-compatible for buyers. This give rise to the impossibility result because the buyer pay less and somebody has to pay the difference.

- McAfee auction : Take the average price between the last pair that can be still matched up. It is incentive-compatible.

- Mechanism : Creation of a game which aim to create a social choice function (either via a dominant strategy or a Bayes-Nash equilibrium). No mechanism satisfy impossibility result, incentive-compatibility, efficiency and budget-balance (need to add money for the mechanism to work).

- Revelation principle : For any mechanism, there is a truthful mechanism with the same outcome and payments.

- <u>The Vickrey-Clarke-Graves tax mechanism</u> : The only social choice function for quasilinear utilities where an incentive-compatible scheme exists. Every agent pay a tax equal to the damage it makes to other agents

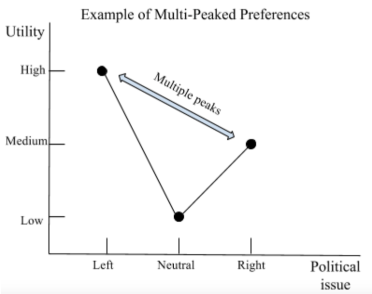
$$tax(A_i) = \sum_{A_j \in A, A_j \neq A_i} (v_j(o_{-i}) - v_j(o_{all}))$$

We can prove that it is a generalization of the Vickrey auction. The task is :
    - non-negative
    - individual rational : For every agent paying the tax and participating has a better utility than doing nothing. The goal of the agent is to optimize the sum of all valuations.
    - incentive-compatible : The declared valuation made by the agent won't change the price of the tax for him.
    - No pareto-optimal : The utility is lost due to the tax.
    - Sensitive to collusion : If multiple agents agree on the same valuation, they can exaggerate it and don't pay the tax.

  The VCG tax is an affine maximizer, it maximize a weighted sum of values.

$$f = argmax_{o \in O' \subset O}(c_o + \sum_i w_i v_i(o))$$

  . All incentive-compatible social choice function are weakly monotone.

- <u>The median voting mechanism</u> : Suppose that we can sort all the possible outcomes (e.g. from right to left in a political voting) and that each agent has a single-peaked preference 

  . Then, the incentive compatible mechanism is taking the median outcome, half of the agents prefer lower outcomes and half of the agents prefer higher outcomes.

- <u>Truthful information extraction</u> : Use a peer subject report. If the peer subject gives the same information than the agent, then a reward is given. For the subjective data, more tricky. We can find a mechanism that is Bayes-Nash (ex-ante)

$$pay(r,s) = \begin{cases} 1/p(r) - 1 & \text{if } r = s \\ -1 & \text{otherwise} \end{cases}$$

  incentive-compatible

Coalitions and group decisions :

- The core : Payoff distribution for which the grand coalition is stable. A coalition is stable if no subset of the coalition give better utility for every agent in that subset than in the original coalition. The core may be an empty set. A core is non empty if it satisfy the Bondereva-Shapley condition : $v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S)$ where N is the grand coalition, lambda a balanced function and v the total payoff of the coalition.
  Convex games have non empty cores $\forall S, T \subset N, v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$
  Convex games are part of the superadditive games
  $$\forall S, T \subset N, if S \cap T = \phi, v(S \cup T) \geq v(S) + v(T)$$

- Shapley values : Stable payoff distribution for each agent in the carrier (i.e. the minimal coalition of agents that determine the result of a game). The Shapley values are in the core for convex games. A permutation of agents give the same permutation of Shapley values. Agents playing two games in parallel, the Shapley values is the sum of the Shapley values for the two games. The Shapley value is the average of the value over all ordering.

- Weighted graph games : A graph representing reward of agents and pair of agents. In this case
  $$\begin{aligned} Shapleyvalue(a_i) &= w((a_i, a_i)) \\ &+0.5 \sum_{\{e_i \mid e_i = (a_i, a_j), j \neq i\}} w(e_i) \end{aligned}$$

- Marginal contribution networks : Generalization of the weighted graph game where we can represent all the possible coalitions. However, it is more difficult to calculate the Shapley value.

- Condorcet winner : Decision that beat all other decisions in a pairwise majority-voting. Don't always exist. Condorcet winner satisfies :
  - Pareto-optimality : If every agent prefer di over dj, di can't be preferred over dj.
  - Monotonicity : If one agent decide to increase its preference to the winner, the winner will remain.
  - Independence of losing alternatives : If another decision is added, it will not change that di is preferred over dj.

  This can be represented as a majority graph where an edge going from node ai to node aj means that the majority prefer ai over aj. The Condorcet winner is the node with only outgoing edges. If there is a cycle, the order of votes is determining for choosing the winner. Organizer can manipulate the vote by making the winner, the last introduced decision.

- Plurality voting : The agents only choose their preferred decision. No guarantee that it will pick the Condorcet winner. An alternative is using plurality with elimination. At each round, the decision with the less vote will be removed.

- Borda count : Each agent give a value to each decision. The least prefered decision get a zero value while the favourite decision get a (size-1) value. The problem with Borda count is that adding or removing a decision can have huge effect on the voting result. It is not independent for losing alternatives.

- Slater ranking : Make a majority graph with the last discrepancy between the different agents' orders (how many edges differ from the majority graph).

- Kemeny score : For a decision ordering, for each di and di+1 calculate the number of people that have a different order between the two. The sum of all counts is the Kemeny score. The order with the lowest Kemeny score is chosen.

- The Gibbard-Satterthwaite theorem : Every deterministic protocol has one of these 3 properties :
    - Dictatorial : one agent always decide of the final decision.
    - Some candidates cannot win under any circumstances.
    - There exist some situations where one agent can manipulate the vote by being non-truthful.