The Simplex Method

Most textbooks in mathematical optimization, especially linear programming, deal with the simplex method. In this note we study the simplex method. It requires basically elementary linear algebra, but many calculations are in block matrix form. We do not treat the degenerate case as we said before, but we will make some remarks in this case.

1. Algebraic and geometric treatments

Consider an LP problem in the standard form

$$\begin{aligned} \min \quad & c^t x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Solutions to the system of linear equations

$$Ax = b \tag{1}$$

correspond to an hypersurface in $\mathbb{R}^n$ and the positivity requirement $x \geq 0$. If the matrix A is an $n \times m$-matrix of full rank, then the hyperplane $\{x | Ax = b\}$ has dimension $n - m$. We shall assume the full rank in the sequel.

If you are in a feasible point, you perhaps would like to go in the negative gradient direction, that is, $-c$, for minimizing the objective function. However, this direction normally lies outside the hypersurface $\{x | Ax = b\}$. Naturally, there are two ways to deal with the problem. (Think why it is natural.)

- To project the direction $-c$ onto the hyperplane $\{x | Ax = b\}$ and go along this projected direction
- To solve the m variables by (1) and express them in terms of the rest of $n - m$ variables.

The simplex method uses the second alternative.

Without loss of generality, we assume that the first m columns of the matrix A are linearly independent. Then (1) can be written as

$$A_B x_B + A_N x_N = b \tag{2}$$

where x_B is a vector containing the first m components of x, and x_N the rest of $n - m$. The matrix A_B is formed by the first m columns of A and A_N the rest of columns. The sub-indexes B, N stand for basic and nonbasic respectively. They can also be interpreted as multiindex sets. Then B consists of those indexes where the variables are included in x_B in the order they appear in the problem. For instance, $B = (1, 4, 3)$, then $x_B = (x_1, x_4, x_3)$, and $A_B = (a_1, a_4, a_3)$.

Notice that A_B is an $m \times m$ matrix. By our full rank assumption, it has full rank, and hence, inevitable. Consequently x_B can be solved uniquely. More precisely, by pre-multiplying A_B^{-1} we obtain

$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b,$$

and then

$$x_B = A_B^{-1} b - A_B^{-1} A_N x_N.$$

We know by elementary linear algebra that x_N can be chosen freely. If we set $x_N = 0$, we have a solution to (1). This is by definition a basic solution. The matrix A_B is called the corresponding base matrix. The components in x_B are basic variables and x_N are nonbasic variables. Clearly, the basic solution $x_B = A_B^{-1}b, x_N = 0$ is feasible if $x \geq 0$, that is, $x_B = A_B^{-1}b \geq 0$. As we showed in the previous section, a basic feasible solution lies at a corner of the feasible set, i.e. an extreme point of the convex feasible set.

Now we can compute the objective function at this basic solution. We partition the vector c as $c^t = (c_B^t, c_N^t)$

$$\begin{aligned} z =& c_B^t x_B + c_N^t x_N = c_B^t A_B^{-1} b - c_B^t A_B^{-1} A_N x_N + c_N^t x_N \\ =& (c_N^t - c_B^t A_B^{-1} A_N) x_N + c_B^t A_B^{-1} b =: \bar{c}_N^t x_N + \bar{z} \end{aligned}$$

At the basic solution $x_N = 0$ and $z = \bar{z}$.

The quantity $\bar{c}^t = c_N^t - c_B^t A_B^{-1} A_N$ is called *reduced cost*. It tells us how much the objective function varies as x_N varies. Observe that c_N has two parts: One is c_N^t which represents a direct contribution to the objective function from x_N, and the other, $-c_B^t A_B^{-1} A_N$, an indirect contribution from $\bar{c}_N$ through x_B, that varies as x_N varies.

Now denote $\bar{A}_N = A_B^{-1} A_N$ and $\bar{b} = A_B^{-1} b$ we have the following transformed problem

$$\begin{aligned} \min \quad & z = \bar{c}^t x_N + \bar{z} = \bar{z} + \sum_{j \in N} \bar{c}_j x_j \\ \text{s.t.} \quad & x_B = \bar{b} - \bar{A}_N x_N \\ & x_B, x_N \geq 0. \end{aligned} \tag{3}$$

The corresponding basic solution is $(x_B^t, x_N^t) = (\bar{b}^t, 0)$. Assume that this basic solution is feasible, that is $\bar{b} \geq 0$. If some components in $\bar{c}_N$, e.g. $\bar{c}_j$, is negative, we see that it is meaningful to increase x_j from 0. If only x_j increases, then x_B varies according to

$$x_B = \bar{b} - \bar{a}_j x_j \tag{4}$$

where $\bar{a}_j$ is the column corresponding to x_j in the matrix $\bar{A}_N$. This solution is feasible as long as $x_B \geq 0$. Equation (4) shows that

$$(x_B)_i = \bar{b}_i - \bar{a}_{ij} x_j \quad j \in N.$$

Note the following facts:

- If $\bar{a}_{ij} \leq 0$, the corresponding variable $(x_B)_i$ increases or stays constant as x_j increase. The condition $(x_B)_i \geq 0$ has no effect in this case, because $(x_B)_i$ will never become 0.
- If, on the other hand, $\bar{a}_{ij} > 0$, the corresponding variable $(x_B)_i$ decreases and it becomes 0 when $x_j = \frac{\bar{b}_i}{\bar{a}_{ij}}$. This shows that the solution is feasible only if $x_j \leq \min_i \{ \frac{\bar{b}_i}{\bar{a}_{ij}} | \bar{a}_{ij} > 0 \}$.

Note that, if $x_j = \min_i \{ \frac{\bar{b}_i}{\bar{a}_{ij}} | \bar{a}_{ij} > 0 \}$ (we call this the *smallest replacement quantity*), then some of basic variable become 0, that is, that/those for which the minimum with respect to i is attained and we reach another extreme point.

Now it is the time to change coordinate system in the hypersurface $\{x | Ax = b\}$. The variable x_j now is greater than 0 and it becomes basic variable. The variable which became 0 is a nonbasic variable. This is the *essence* of the simplex method. In short, the simplex method is based on the fact that a given feasible basic solution can be improved if some component in $\bar{c}_N$ is negative. It is natural to ask: What happens if $\bar{c}_N \geq 0$? Is the solution optimal then? The answers are affirmative.

Theorem 1.1. *Assume that $\bar{b} \geq 0$ and $\bar{c}_N \geq 0$ in the transformed LP problem* (3). *Then the corresponding basic solution is optimal.*

Proof. We show this by relaxation. Since $\bar{b} \geq 0$, the basic solution $(x_B^t, x_N^t) = (\bar{b}^t, 0)$ is feasible. If we relax the condition $x_B = \bar{b} - \bar{A}_N x_N$ from the transformed LP problem (3), we get a relaxed problem

$$\begin{aligned} \min \quad & z = \bar{c}_N^t x_N + \bar{z} = \sum_{j \in N} \bar{c}_j x_j + \bar{z} \\ \text{s.t.} \quad & x_B, x_N \geq 0 \end{aligned}$$

Because $\bar{c}_j \geq 0$ and we can minimize for each variable x_j separately, it is obvious that $(x_B^t, x_N^t) = (\bar{b}^t, 0)$ is optimal to the relaxed problem above, and hence to the transformed problem (3), since the solution is feasible. It gives automatically same value of objective function, for we have the same objective function. □

We summarize the basic structure of the simplex to close this section:

(i) START at a feasible basic solution.
(ii) Transform the problem to the form of (3) by determining the basic variables.
(iii) If all $\bar{c}_j \geq 0$, STOP. We are at optimum.
(iv) If some $\bar{c}_j < 0$, increase the corresponding x_j until $(x_B)_i = 0$. We are now at a new basic solution. GO TO Step 2.

2. Tableau form

For hand calculation of the simplex method, we usually put all data in a tableau. Now we do an example of the simplex method step by step through the example given in the previous section. We repeat it here.

$$\begin{aligned} \max \quad & 80x_1 + 60x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 100 \\ & 2x_1 + x_2 \leq 150 \\ & 5x_1 + 10x_2 \leq 800 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The feasible region is depicted in the previous section. Number the corners from the origin anticlockwise we have corner 1: $(0, 0)$, corner 2: $(75, 0)$, corner 3: $(50, 50)$, corner 4: $(40, 60)$, corner 5: $(0, 80)$.

First transform the problem to minimizing $-80x_1 - 60x_2$. Then, introduce slack variables $x_3, x_4, x_5 \geq 0$. We put the equations and the objective function in the following matrix form:

x_1	x_2	x_3	x_4	x_5	b
1	1	1	0	0	100
2	1	0	1	0	150
5	10	0	0	1	800
−80	−60	0	0	0	0

The tableau is of the form $\begin{bmatrix} A & b \\ c^t & 0 \end{bmatrix}$. This is called the *simplex tableau.*

This tableau is already in a feasible basic solution form. The feasible basic solution is $x_1 = 0, x_2 = 0, x_3 = 100, x_4 = 150, x_5 = 800$. It corresponds to corner 1.

Now we have to decide which variable enters the basic variable, and which variable departs from the basic variable. In our notation, $N = \{1, 2\}$ and $B = \{3, 4, 5\}$. So $\bar{c}_N^t = (-80, -60)$. The last row in the tableau shows that there are negative costs $\bar{c}_1$ and $\bar{c}_2$. It seems natural to choose the first component x_1 because it corresponds the most negative one in the reduced cost. To decide the departing variable we compute $100/1, 150/2, 800/5$. Then the second quantity is the minimum of these three. Hence the variable x_4 departs B. Now $B = \{3, 1, 5\}$, and $N = \{4, 2\}$. The new tableau is

x_1	x_2	x_3	x_4	x_5	b
0	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	25
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	75
0	$\frac{15}{2}$	0	$-\frac{5}{2}$	1	425
0	-20	0	40	0	6000

The new basic solution is $x_1 = 75, x_2 = 0, x_3 = 25, x_4 = 0, x_5 = 425$. It corresponds to corner 2. There is one negative component in $\bar{c}_N$. That is $\bar{c}_2$. So the variable x_2 enters the index set B. From $\min\{25/\frac{1}{2}, 75/\frac{1}{2}, 425/\frac{15}{2}\} = 50$, the variable x_3 departs from the basic variables. So the new index sets are $B = \{2, 1, 5\}$ and $N = \{4, 3\}$. The next tableau is

x_1	x_2	x_3	x_4	x_5	b
0	1	2	-1	0	50
1	0	-1	1	0	50
0	0	-15	5	1	50
40	0	40	20	40	7000

The new basic solution is $x_1 = x_2 = x_5 = 50, x_3 = x_4 = 0$, that is, corner 3. Remember that we obtained the same optimal solution geometrically.

In this problem the solution is unique and bounded. In general, boundedness and uniqueness may be violated. Moreover if there are two optimal feasible solutions, then there are infinitely many optimal solutions. (Prove it as an exercise.) This happens if all basic variables are positive and in the optimal tableau a zero occurs in the objective row in a column corresponding to a nonbasic variable and including a positive entry.

Example 2.1. Example of infinitely many solutions.

$$\begin{aligned} \min \quad & z = -40x_1 - 100x_2 \\ \text{s.t.} \quad & 10x_1 + 5x_2 \leq 2500 \\ & 4x_1 + 10x_2 \leq 2000 \\ & 2x_1 + 3x_2 \leq 900 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Now we turn to discussing when a solution is unbounded. Remember that we stopped increasing a nonbasic variable x_j with negative reduced cost $\bar{c}_j$ when some basic variable became zero. If all $\bar{a}_{ij} \leq 0$ then no basic variable will decrease as x_j increases. No matter how much x_j increases, we have feasible solutions. Since the objective function decreases as x_j increases, the objective function will decrease unboundedly. We say that the problem has an unbounded solution.

Example 2.2. Example of unbounded solution.

$$\begin{aligned} \min \quad & z = -3x_1 - 2x_2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 1 \\ & 3x_1 - 2x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Compute these two examples as exercise!

In conclusion, we have the following algorithm: the simplex method.

Step 1. Obtain the *initial basic solution* by setting each slack variable equal to the corresponding resource and setting the other variables equal to zero.

Step 2. Choose the entering variable from among those having negative reduced cost.

Step 3. Choose the departing variable to be the basic variable in row producing the smallest replacement quantity.

Step 4. There is *no lower bound* on the objective function when one column has entirely nonpositive coefficients above a negative component of the reduced cost.

Step 5. The solution is *optimal* when all entries in the objective row are nonnegative.

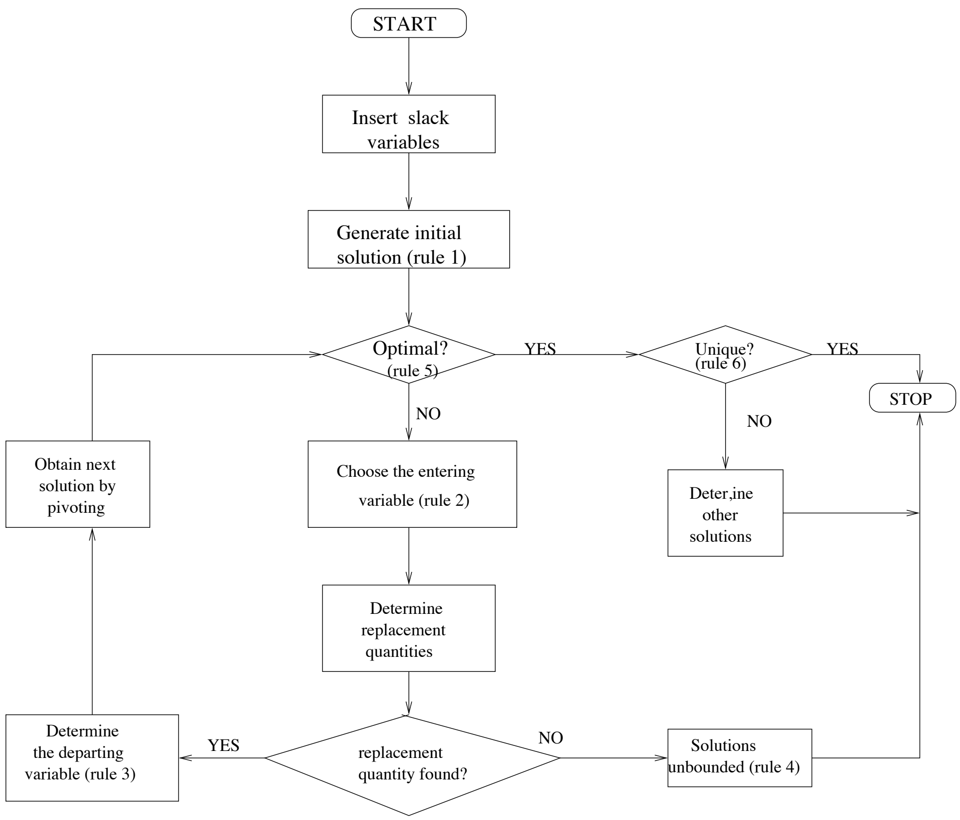
Step 6. If all basic variables are positive, an optimal solution is *not unique* if in the optimal tableau a zero occurs in the objective row in a column corresponding to a nonbasic variable and including a positive entry.

Finally we make an important remark. We have seen that the simplex method stops in two cases: (i) $\bar{c} \geq 0$. In this case, we have an optimal solution. (ii) For some $\bar{c}_j < 0$ all $\bar{a}_{ij} \leq 0$. In this case we have unbounded solution.

The question is whether or not we must have these two cases. We take a close look now. In every iteration of the simplex method, the objective function decreases or stays constant: If x_j is the entering variable then x_j increases from 0 to $\Delta x_j = \min\{\frac{\bar{b}_i}{\bar{a}_{ij}} | a_{ij} > 0\} \geq 0$. The change in the objective function is therefore $\Delta z = \bar{c}_j \Delta x_j = 0$ only if $\Delta x_j = \min\{\frac{\bar{b}_i}{\bar{a}_{ij}} | a_{ij} > 0\} = 0$, which means that a basic variable is zero in a basic solution. In an earlier iteration there are two basic variables at the same time as the entering variable increases. However, only one of these variables has departed from the basis. This situation is called *degenerate case* and means that more than $n - m$ hyperplanes of type $(x_B)_i = 0$ meet at the same point in the hyersurface $\{x | Ax = b\}$. One can argue that it is a rare situation. Nevertheless it does happen to all LP problems. If the degeneration never happens the objective function strictly decreases in each iteration. Then we can never come back to an extreme point we have already visited, because it will give the same value of the objective function. Moreover, we know that it has only finitely many basic solutions. So if degeneration never occurs, the simplex method must stop after finitely many steps, in either case (i) or case (ii). At degeneration the simplex method will go on as a circle. It is called *cyclic*. In practice, the cyclic situation seldom will be a case. We will do not discuss this in details. See for example our textbook for further discussion.

Next, even though the simplex method is finite, it is of no interest in practice, because a practically small problem such as 100 constraints and 200 variables can have up to 10^{30} feasible basic solutions. If the simplex method were to pass all these basic solutions in such a problem and uses only 10^{-9} seconds per basic solution, then it would take 10^{21} seconds which is about $3.17 * 10^{13}$ years, that is, much more than the length of the earth's life. However, the simplex method in practice takes much less time. Nowadays the problem of the abovementioned size takes just several seconds on a PC. One usually counts as follows: The number of iterations increases approximately linearly by the number of constraints, while work on each iteration increases as square of the number of constraints. This can be used as a practical complexity to the simplex method.

For implementation the simplex method the following flow chart can be used.



3. The two phase simplex method

In the preceding section we assumed implicitly that the initial simplex tableau is ready for further computing, i.e. we already have a basic feasible solution. It is clear that one needs to find it. This is the task of this section. It turns out that the problem of finding such a feasible basic solution can be solved by the simplex method!

Now consider the system of equations

$$Ax = b, \quad x \geq 0. \tag{5}$$

We assume that $b \geq 0$. (Note that this is always possible.)

Introduce now the so-called *artificial variables* $y \geq 0$. In other words, we can think of this as measuring the error between the left and right hand sides in the equations. Then we have another system of linear equations with all variables nonnegative:

$$Ax + y = b, \quad x \geq 0, y \geq 0 \tag{6}$$

In this second equation, it is easy to find a feasible basic solution, that is $y = b, x = 0$. If we want to have a feasible solution to the first equation we have to drive y to zero. Then we can solve the following *Phase I problem.*

$$\begin{aligned} \min \quad & \sum_{i=1}^{m} y_i \\ \text{s.t.} \quad & Ax + y = b \\ & x, y \geq 0 \end{aligned} \tag{7}$$

It is still an LP problem, and therefore it can be solved by the simplex method.

Theorem 3.1. *The LP problem* (7) *always have a finite positive optimal value. This optimal value is given by the simplex method. If the optimal value is greater than* 0*, then the system* (5) *has no feasible solution. If the optimal value is zero then the system* (5) *has a feasible solution, and the simplex method gives the desired result.*

Proof. Assume that $\bar{x}$ is a feasible solution to (5). Then, $(\bar{x}^t, 0)$ is a feasible basic solution to (6). Thus the optimal value of (7) is equal to 0. If the optimal value of (7) is greater than 0, then (5) cannot have a feasible solution.

Conversely, if the optimal value of (7) is 0, then the simplex method will give the optimal basic solution with value 0. Since $\sum y_i = 0$ and $y_i \geq 0$, it must hold that all $y_i = 0$ and we obtain a feasible solution of (5). $\square$

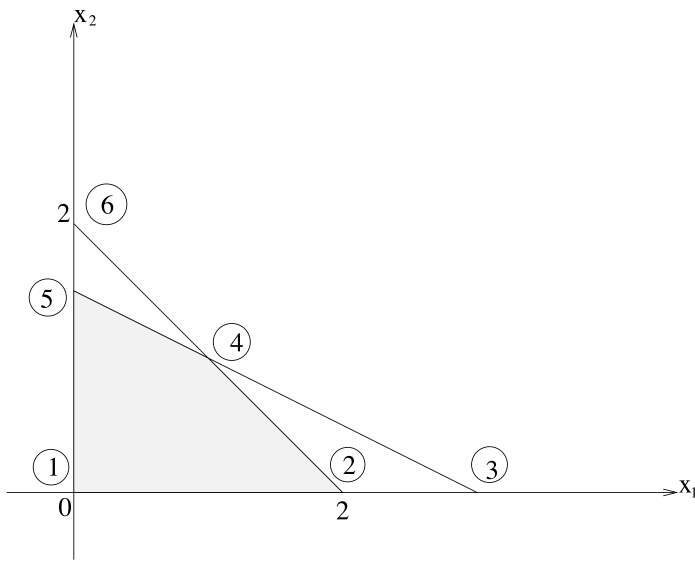
Example 3.1.

$$\begin{aligned} \min \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & x_1 + 2x_2 \geq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

From the figure we see that the optimum is attained at point 6. First we introduce surplus variables x_3 and x_4 to obtain equality constraints. So we get the following tableau.

x_1	x_2	x_3	x_4	b
1	1	−1	0	2
1	2	0	−1	3
3	2	0	0	0

From this we do not see any natural initial basis. Thus we introduce artificial variables y_1 and y_2. In Phase I we shall minimize $y_1 + y_2$. This gives the next tableau.



x_1	x_2	x_3	x_4	y_1	y_2	b
1	1	-1	0	1	0	2
1	2	0	-1	0	1	3
3	2	0	0	0	0	0
0	0	0	0	1	1	0

The last row in this tableau is the objective function of Phase I while the third row is the actual objective row. As said before, it is suitable to choose y_1, y_2 as basic variables. Eliminate the 1's in the corresponding columns yields the initial simplex tableau.

x_1	x_2	x_3	x_4	y_1	y_2	b
1	1	-1	0	1	0	2
1	2	0	-1	0	1	3
3	2	0	0	0	0	0
-2	-3	1	1	0	0	-5

Thus $y_1 = 2, y_2 = 3$, $x_1 = x_2 = x_3 = x_4 = 0$ form a basic solution, corner 1. In this tableau $\bar{c}_1$ and $\bar{c}_2$ are negative. We let x_1 be entering variable. Next $\min\{2/1, 3/1\} = 2$, which meets at first row so y_1 is a departing variable. Now we pivot component $(1, 1)$ and get

x_1	x_2	x_3	x_4	y_1	y_2	b
1	1	-1	0	1	0	2
0	1	1	-1	-1	1	1
0	-1	3	0	-3	0	-6
0	-1	-1	1	2	0	-1

The corresponding basic solution is $x_1 = 2, y_2 = 1$ the rest rest variables are 0, which is corner 2. Now $\bar{c}_2, \bar{c}_3$ are negative. Choose x_2 to enter the basis. According to $\min\{2/1, 1/1\} = 1$ which meets at row 2. So y_2 departs. Pivoting $(2, 2)$-component gives a new tableau

x_1	x_2	x_3	x_4	y_1	y_2	b
1	0	-2	1	2	-1	1
0	1	1	-1	-1	1	1
0	0	4	-1	-4	1	-5
0	0	0	0	1	1	0

The basic solution now is $x_1 = 1, x_2 = 1$, and the rest is 0. So corner 4 is reached. Now $y_1 = y_2 = 0$ and the artificial objective function is zero. Thus, Phase I is finished and we have found a basic feasible solution to solve the original LP by simplex. We can drop the last row to continue Phase I. In the third row, we see that $\bar{c}_4 < 0$ and x_4 enters the basis. Now there is only one choice of the departing variable, because $(2, 4)$-component is negative. Pivoting $(1, 4)$-component gives

x_1	x_2	x_3	x_4	y_1	y_2	b
1	0	-2	1	2	-1	1
1	1	-1	0	1	0	2
1	0	2	0	-2	0	-4

The corresponding basic solution is $x_2 = 2, x_4 = 1$ and the rest is 0, that is corner 6. Now all reduced costs are nonnegative, so we are at optimum.

The whole structure of the simplex method is

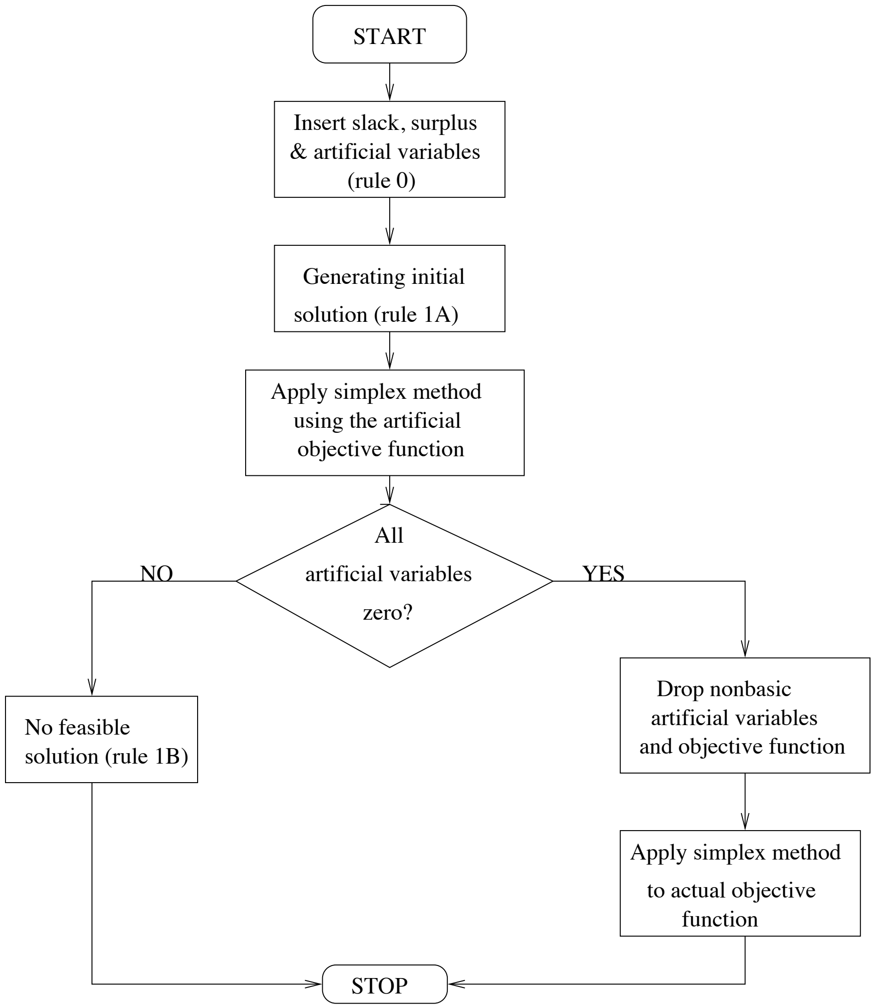
where

> **Step 1A.** From the *artificial objective function* as the sum of the artificial variables, obtain the initial feasible basic solution by setting each slack variable and each artificial variable equal to the corresponding constant on the right hand side of the constraint.
> **Step 1B.** There is no feasible solution if the sum of the artificial variables is greater than zero.

4. Duality theory

Consider the LP problem in standard form. The dual problem is $\max b^t u$ s.t $A^t u \leq c$.

Note that there is no sign restriction on the variables u. We know that the variables u is Lagrange multipliers. So it is natural to see if the simplex multipliers have something to do with the dual problem. Assume the primal problem is solved until the optimum is attained. Then x_B is basic variable. The optimality condition is $\bar{c}_N \geq 0$, that is $c_N^t - \pi^t N \geq 0$ or $N^t \pi \leq c_N$. However, the constraints on the dual variable u is $c \geq A^t u = [B, N]^t u = \begin{bmatrix} B^t \\ N^t \end{bmatrix} u, \Leftrightarrow c_B \geq B^t u,\ c_N \geq N^t u$. Therefore, the simplex multipliers satisfy "half" of the dual constraints, if the optimality condition is satisfied. The other half is $c_B \geq B^t)u$ is satisfied automatically with equality, since $\pi = (c_B^t B^{-1})^t = (B^t)^{-1} c_B$ and $B^t \pi = c_B \Rightarrow \pi$ is a feasible dual solution.
The above argument means that π is optimal if it gives the same objective function value as the primal solution does (according to weak duality). The primal objective value is $\bar{z} = c_B^t A_B^{-1} b$ while π yields the dual objective value $b^t \pi = b^t (A_B^t)^{-1} c_B = c_B^t A_B^{-1} b = \bar{z}$. Therefore, we have proved:
Theorem. Ass. $x_B = A_B^{-1} b, x_N = 0$ is simplex-optimal (i.e. $\bar{c} \geq 0$) basic solution to an LP in standard form. Then the corresponding simplex multiplier, $c_B^t A_B^{-1}$, is optimal dual solution. Moreover, the primal and dual optimal values are equal.
How primal and dual relation looks like in other cases. If an LP problem is solved



by the simplex method, according to above discussion, the following cases occur:
• Phase I gives optimal value greater than 0, that is there is no feasible solution. ($\emptyset$)
• Phase I gives optimal value 0 and Phase II gives a bounded optimum. (B)
• Phase I gives optimal value 0 and Phase II gives unbounded solution. (∞)
If P_∞ occurs then the dual cannot have a feasible solution according to weak duality. So the dual has no solution, $D_\emptyset$. Similarly, D_∞ implies $P_\emptyset$. By the theorem P_B implies D_B and vise verse (due to the symmetry). The only case we have not considered is the relation between $P_\emptyset$ and $D_\emptyset$.
Ex. $\min -x_2$, s.t. $x_1 - x_2 \geq 1$ $-x_1 + x_2 \geq 0$, $x_1, x_2 \geq 0$. Its dual: $\min u_1$ s.t. $u_1 - u_2 \leq 0$, $-u_1 + u_2 \geq -1$ $u_1, u_2 \geq 0$. None of these problems has solutions.
We conclude by the following tableau. (The cross stands for impossible situation.)

	$D_\emptyset$	D_B	D_∞
$P_\emptyset$		X	
P_B	X		X
P_∞		X	X

Complementary slackness: Let $(\bar{x}, \bar{y})$ be a pair of feasible solutions to the primal P resp. D. Then $\bar{x}$ and $\bar{y}$ are resp. optimal to P and D iff $(c - A^t\bar{y})_j\bar{x}_j = 0$ for $j = 1, ..., n$.

Exerceises.

(1) Solve the following LP problem:

$$\min \quad -7x_1 - x_2 + 3x_3$$

$$\text{s.t.} \quad \begin{cases} 2x_1 - x_3 \le -1 \\ 3x_1 + 2x_2 - x_3 \le 1 \\ -2x_1 + x_2 + x_3 \le 2 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

(2) Consider the following LP problem:

$$\max \quad 2x_1 + 2x_2 + x_3$$

$$\text{s.t.} \quad \begin{cases} x_1 + x_2 + x_3 \le 5 \\ x_1 + x_2 - x_3 \le 4 \\ x_1 - x_2 \le 1 \\ x_1, x_2, x_3 \ge 0 \end{cases}$$

Find the optimal value and an optimal point. Are there other optimal points? Give all optimal points if yes.

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