Our notation of fake surfaces relies on ordering the vertices and edges within a 1-skeleton and also ordering the 1-skeletons of a given complexity. We do this in the following way.

Viewing the 1-skeletons as 4-regular multigraphs, consider the map $\mathsf{D} : M_n(\mathbb{Z}) \to \mathbb{Z}$ that gives a decimal representation of a matrix by simply concatenating the rows from top to bottom into an integer with n^2 digits. An example makes this clear:

$$\mathsf{D}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1234.$$

Explicitly, if $A = (a_{i,j})$,

$$\mathsf{D}(A) = \sum_{i,j=1}^{n} 10^{n(n-i)+n-j} a_{i,j}$$

as an integer.

For a given 1-skeleton of complexity n, we choose the adjacency matrix $A = (a_{i,j})$ and therefore an ordering of the vertices that maximizes $\mathsf{D}(A)$. Note this also provides the canonical ordering of the 1-skeleta for a given complexity: from largest value of $\mathsf{D}(A)$ to smallest.

We then label the edges in the order they appear in the upper-right triangle of the chosen adjacency matrix, read left-to-right and top-to-bottom. To be precise, using our ordering of the vertices, let (i_1, j_1) and (i_2, j_2) be edges satisfying $i_1 \leq j_1$ and $i_2 \leq j_2$. Then $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$ or $i_1 = i_2$ and $j_1 < j_2$. We then label edges from 1 to $2n$ in accordance with this ordering. The order of multiple edges between two vertices can be chosen arbitrarily.

We provide an example to make this clear. Consider the graph Γ given by the adjacency matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{bmatrix}.$$

Observe that this ordering of the vertices is the one that maximizes $\mathsf{D}(A)$. Then the self-loop is first edge, the two other edges at that vertex are the next two edges, and the three edges connecting the other two vertices are the fourth, fifth, and sixth edges. Thus we label the edges by any of the 12 valid labelings. One such labeling is shown in Fig. 1:

Figure 1: Valid Edge Labeling for Γ