Cryptography I - Notes

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These notes are based upon the free Coursera Course called "Cryptography I", teached by Dan Boneh from Standford, as well as information gathered from Wikipedia.
First, it's a personal project from an amateur in the domain : it is not exempt from errors and imprecisions (as well as grammar/typos since I am not an native English speaker).
Second, the way the document is structured does not make it easy for a complete beginner to understand cryptography : I assume readers have to be somewhat proficient in mathematics and computer science to understand big-O notations, P/NP problems, and basic algebra (sets, group theory, etc.).

This document was written using ShareLatex, a powerful web-based editor, and shared using Github (lucasg/crypto-report1).

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Chapter 1

Introduction

Cryptography is the study of ways of secure communications in presence of adversaries. In computer science, it revolves around the construction of protocols which are robust against confidentiality and tampering.
One thing to consider is there is no such thing as perfect security in the real world : cryptography cannot prevent against all existent and futures attack vectors. Cryptography only provide ways to mitigate against such attacks : it's a trade-off between the simplicity of your transmission system (more secure means more expensive to operate) and the opportunity cost for the attacker who wants to break your system.
A cipher consist of an encoder and a decoder. The algorithm behind the encoder and the decoder has to be public in order to ensure the integrity and the robustness of the cipher. The ciphers are typically open source projects, reviewed by security experts and some are voted into standards by institutions such as NIST[1].

1.1 Definitions

$$\begin{aligned}
\textit{Encoder} : & \; E : K \times M \mapsto C \\
\textit{Decoder} : & \; D : K \times C \mapsto M \\
\textit{Equality} : & \; D(k, E(k, m)) = m
\end{aligned}$$

The encoder E maps a message m into the ciphertext space using a parameter k (which is the secret key) and the decipher D does the reverse, using the same key k to reveal the original message. E is often randomised (see **??**) whereas D is always deterministic. A cipher consist of the encoder and decoder : cipher $= (E, D)$.

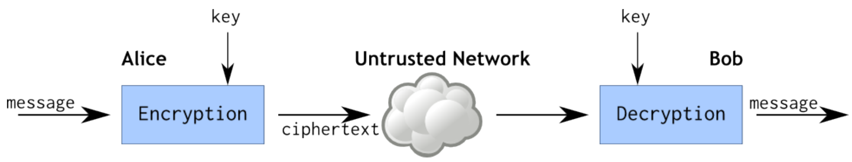


Figure 1.1: Example of a cipher

A common mistake is to think that an ad-hoc crypto algorithm with closed source is safer than open source standards : the big problem with closed source algorithms is that, when they are breached (and they usually are at least once), the user does not know.

[1] National Institute of Standards and Technology

Uses of cryptography

- Secure communication : private conversations without eavesdropping
- Digital signatures : secure identification (no tampering)
- Anonymous communication : secure and private communication without any of the participants know the identity of the others
- Anonymous computation : outsourcing computation without giving the purpose of the calculus to the contractor (e.g. Amazon w3s)

1.1.1 Trusted authority

One way to ensure confidentiality is to outsource the task to a third party which has credibility and trust, like when we give our last will to a exterior person which does not have any involvement with the family(typically a *notary*).
However, the trusted authority solution - like any centralized mechanism - creates a single point of failure, so the trusted authority might not be always a good solution.

Theorem 1
Any computation done by trusted authority can be done without it.

1.1.2 Zero proof of knowledge

Definition 1 (Aim of cryptography) *Prove that, under a certain threat vector, forge the signature comes to solve a NP-problem.*

The definition of a NP-problem is that it is computationally "hard" to solve (non-deterministic polynomial time), but easy to verify a solution (called signature).
A zero-knowledge proof is a signature the user present to the server, which will answer true or false based on the signature's validity. Under zero-knowledge hypothesis, the user does not learn anything more than the validity of the signature.

1.1.3 History of ciphers

Cryptology is an ancient matter : all sorts of encoding scheme has been invented throughout History.

Definition 2 *$E(k, m)$ is the encryption of message m using key k.(the key always first)*

Definition 3 *$D(k, c)$ is the decryption of ciphertext c using key k.(the key always first)*

Substitution cipher

The substitution cipher (also called Caesar cipher in its weak form) is a simple encoder, yet relatively effective : the key K is a bijective map between two alphabets. For example, A becomes C, B becomes O, etc.

input	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
output	c	o	a	h	l	z	m	v	n	e	b	q	s	j	u	k	w	r	p	g	t	i	d	x	f	y

Table 1.1: Exemple of a substitution table

plaintext	:	s	e	v	e	r	a	l	•	f	l	a	m	e	t	h	r	o	w	i	n	g	•	s	h	a	r	k	s
ciphertext	:	p	l	i	l	r	c	q	•	z	q	c	s	l	g	v	r	u	d	n	j	m	•	p	v	c	r	b	p

Table 1.2: Exemple of a encryption using the substitution procedure

Example of a encryption :

The encryption is simply the substitution of every letter in the message m by its counterpart in the map K.
The substitution cipher is however breakable just by looking at ciphertexts. The study of the letters' frequencies in the ciphertexts can give huge informations on the map K. In English, the letter e is the most used so, by looking which letter is the most used in the ciphertexts, we can give a pretty good guess of the substitution of e.

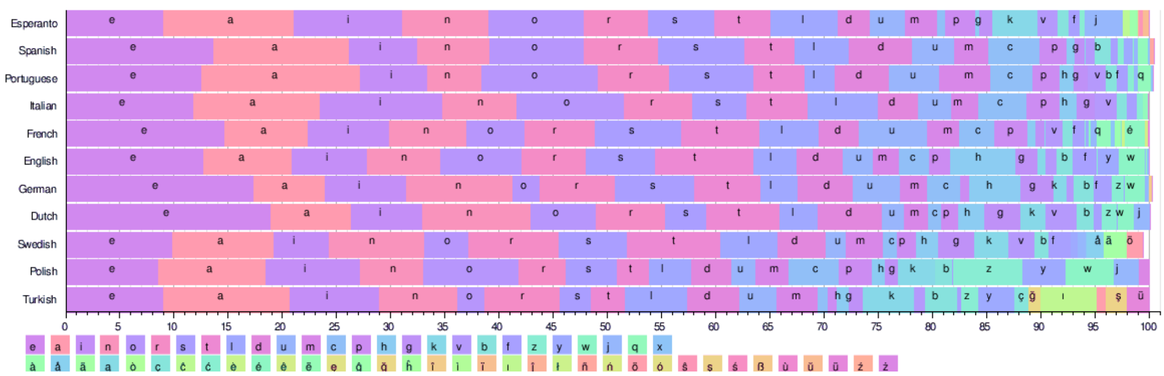


Figure 1.2: Letter frequencies in several languages. It is fairly easy to guess the message's language just by looking at letter's repartition.
source : Wikipedia

The Caesar cipher is a simpler (and weaker) version of the substitution cipher : every character in the alphabet is offsetted by a constant value (modulo the alphabet's length). For example A become C, B become D, and so one... In this form, the key is not a map anymore, but only a constant integer,which is far smaller.

Vigenere cipher

The Vigenere cipher is built from the substitution code. The cipher use a work as key : every character will be used as a Caesar cipher encryption scheme.

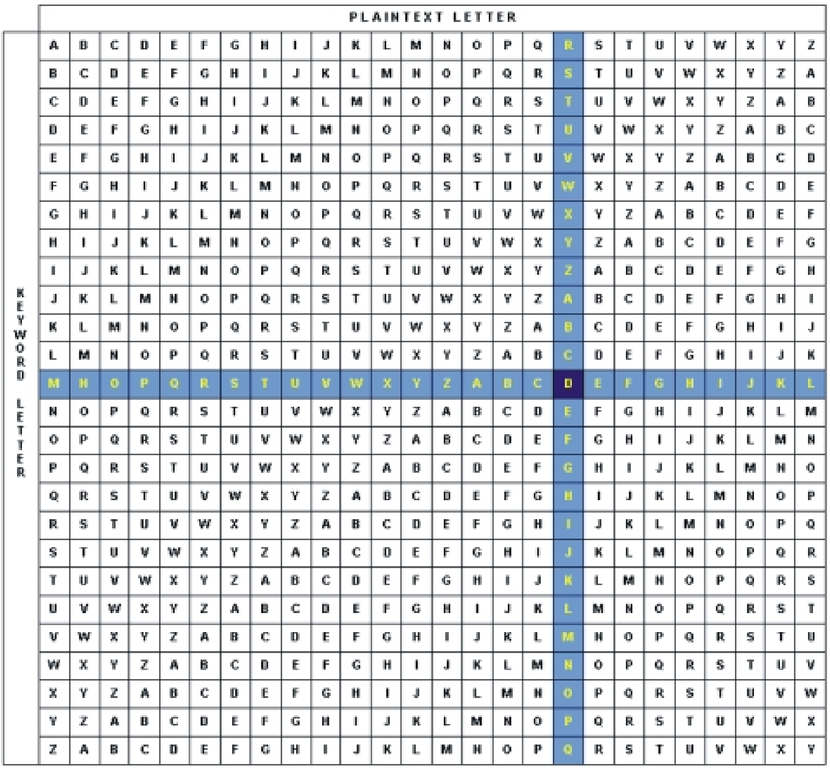
plaintext	:	f	l	a	m	e	•	t	h	r	o	w	e	r
		+	+	+	+	+	•	+	+	+	+	+	+	+
key	:	v	i	g	e	n	•	e	r	e	c	o	d	e
		=	=	=	=	=	•	=	=	=	=	=	=	=
ciphertext	:	z	t	g	q	r	•	a	y	v	q	k	h	v

Table 1.3: Exemple of a encryption using the vigenere code

If the encryption is based on a sum of letter position in the alphabet modulo the alphabet's length, the decryption is as simple since it's a difference:

ciphertext	:	z	t	g	q	r	•	a	y	v	q	k	h	v
		-	-	-	-	-	•	-	-	-	-	-	-	-
key	:	v	i	g	e	n	•	e	r	e	c	o	d	e
		=	=	=	=	=	•	=	=	=	=	=	=	=
plaintext	:	f	l	a	m	e	•	t	h	r	o	w	e	r

Table 1.4: Exemple of a decryption using the vigenere code


PLAINTEXT LETTER

KEYWORD LETTER

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A
C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C
E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E
G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F
H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I
K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J
L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K
M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M
O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N
P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X
Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y

Figure 1.3: Vigenere *tabula recta* for handmade encryption
source : illuminations.nctm.org

The Vigenere code is a fairly simple encryption scheme yet it were quite powerful (before the invention of automatic computation) since the encryption/decryption is a time-linear operation (table lookups) whereas the brute-force attack involve time-polynomial operations (multiple tables creation and lookups). It is still fairly easily breakable if the key's length is known to the attacker because it revert the cipher to a substitution one. Moreover, there are several methods which estimate the key's length based on frequency analysis (the vigenere cipher does mask all letter frequency patterns).

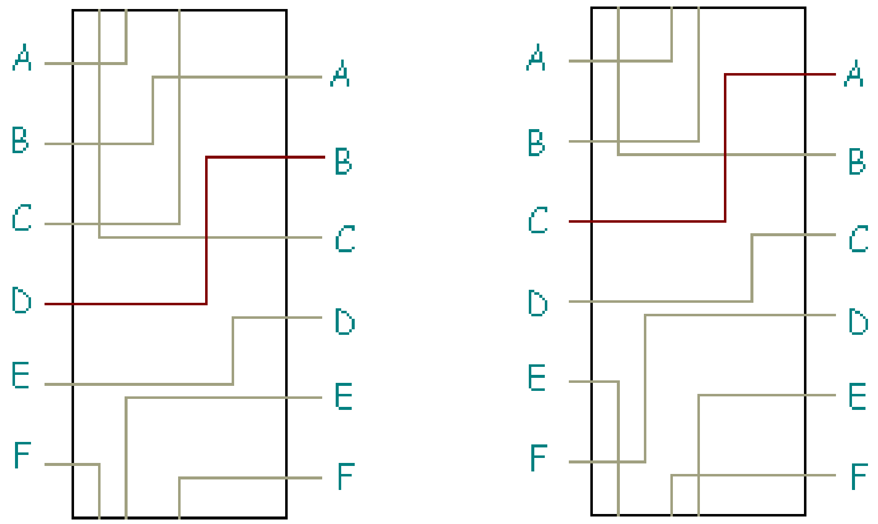
On a side note, if the key is smaller than the encoding message, they key is repeated and padded to match the message's length:

Enigma

The enigma machine is a device built by a German company around 1920 and consist of rotating substitution ciphers :

ciphertext	:	s	e	v	e	r	a	l	f	l	a	m	e	t	h	r	o	w	i	n	g	s	h	a	r	k	s
		+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+
key	:	s	m	a	l	l	s	m	a	l	l	s	m	a	l	l	s	m	a	l	l	s	m	a	l	l	s
		=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=	=
plaintext	:	f	l	a	m	e	•	t	h	r	o	w	e	r												

Table 1.5: Exemple of an encryption using a padded key.



(a) A is mapped to B (b) A is now mapped to C

Figure 1.4: Enigma rotor : the rotor rotate by a increment for every new input, which lead to a different substitution table.
Source : www.bibmath.net

In order to strengthen the protocol, a typical Enigma machine use 3 rotors and a reflector[2]. The reflector is a fixed rotor which acts as a mirror (it's a simple substitution cipher) and each rotor is incremented by a special transition from the rotor on the left (apart from the most left one, which is incremented at each new input). In this configuration, the rotors' configurations act as public keys while the rotors' initial position as well as the rotors' order is the private one. The rotors can be sold or stolen without revealing anything about the encryption (apart from degenerate rotor configurations). The machine being symmetrical, the cipher is symmetric too : using the same private key, typing the ciphertext into the machine will print the plaintext message[3].
The enigma machines were largely used during World War II by the German Army as a crypto-system. The breaking of Enigma did occupy a lot of mathematicians and logicians at that time, and bring numerous breakthroughs in cryptology. Finally, Alan Turing [4] did crack the cipher using "crib" techniques : by analysing ciphertexts and some partial plaintexts (such as headers, known contents,...), it was possible to deduce the inner structure of the machine, and creating an automatic decipher called *Bombe*.

[2]There is also a plugboard, which swap at most two pairs of letters.
[3]Mathematical proof of the Enigma's symmetrical property can be found here : ??? .
[4]also known as the most badass computer scientist of the 20th century

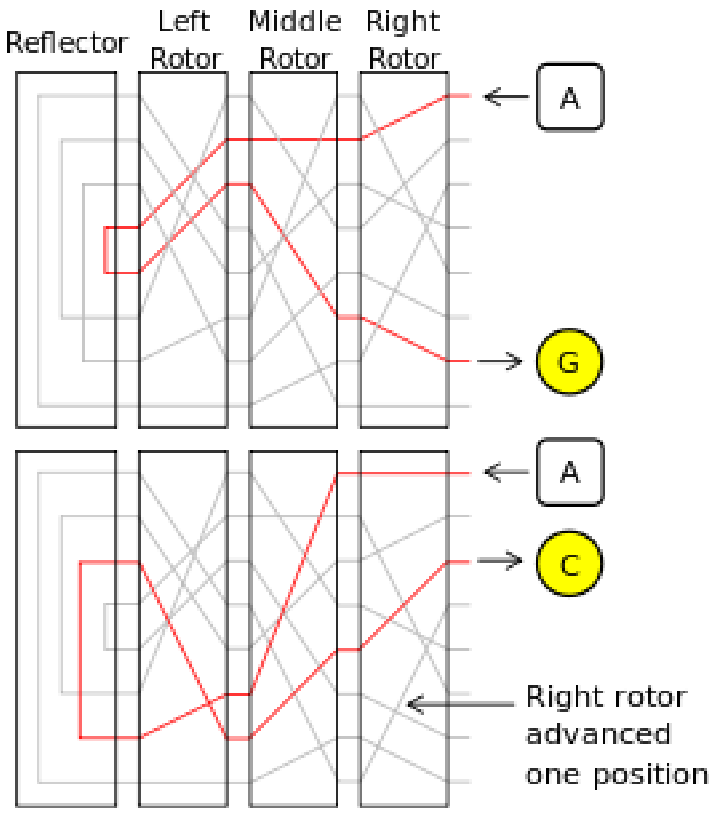


Figure 1.5: Enigma machine with 3 rotors and a reflector.
source : Wikipedia

Chapter 2

Theory

2.1 Definitions

2.1.1 Symmetric/Asymmetric Ciphers

Definition 4 *A cipher is symmetric if the encryption E and the decryption D use the same key k.*

Definition 5 *A cipher is asymmetric if the encryption E and the decryption D do not the same key. The encryption key is often called "public key" and the decryption one "private key".*

The symmetric-key encryption is the oldest class of cryptography process. The major flaw of symmetric ciphers is the obligation for both parties (sender and receiver) to share the secret (the key). Nowadays, it is recommended to use non-symmetric encryption (also known as public-key encryption).

2.2 Pseudo-Random Generation

2.2.1 Statistical test for Randomness

Pseudorandomness is the property of a function to appear random, while being completely deterministic. In cryptography, the pseudorandomness property for a bit generator is an important one : the security is often built upon it since the attacker cannot predict the output. That's why a lot of different statistical test were conceived to separate true pseudorandom generators from broken ones (bit-frequency, chi-square Test, arithmetic mean, ...).

Definition 6 *A statistical test is an algorithm which, given a generator, output 0 or 1 based on the stream of numbers generated, 1 being random and 0 deterministic.*

Advantage

Let F an oracle[1] which we want to study and G a perfect one implementing true randomness. The advantage Adv over F, using the statistical test A, is:

Definition 7 $Adv_F[A, G] = |Pr[A(G(k)) == 1 - Pr[A(G(r)) == 1]| \in [0, 1]$

[1] an oracle is a computational black box

An advantage "close to one"[2] is considered to break the pseudorandom function, because the test A can distinguish pseudo-randomness from true randomness.

2.2.2 Pseudo-Random Functions (PRF)

A pseudo-random function is a fairly easily computable function (polynomial) which simulate randomness while being completely deterministic.

Definition

Let F a function from KxD to R which maps a two set (the domain M and the range R) using a key parameter from K : the function F is a PRF if no efficient adversary with significant advantage can distinguish F from a random oracle. Pseudo-random functions are vital tools in the construction of cryptographic primitives.

Secure PRF

A pseudorandom function is secure if an attacker cannot solve the following game with a significant advantage :

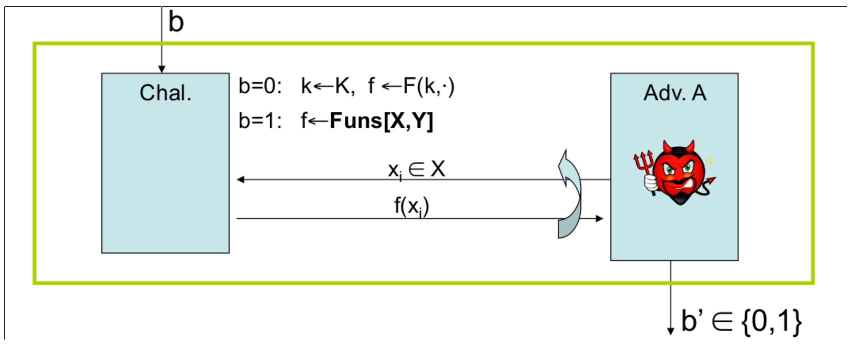


Figure 2.1: indistinguishability game for PRF

Let b a binary value and $F : X \times K \to Y$ a PRF from X to Y. If b is null, the oracle will chose a random key $k \in K$, otherwise it will choose a complete random function $f : X- > Y$. Then the adversary A submit one or several value(s) $x_i \in X$ and get the result $y_i \in Y$, from either the pseudo-random function or the randomly chosen one (according to the value of b). The attacker must find which function the oracle used, and if it find it with an advantage(see 2.2.1), the PRF is considered insecure.

2.2.3 Pseudo-Random Permutation (PRP)

Definition

A PRP differs from a PRF in the way that the domain D and the range R are the same. Since the function map the two same sets, we can deduce that $E(k, .)$ (k is fixed) is bijective : it is a permutation then. A major propriety of the PRP (comparatively to the PRF) is that the PRP can be inverted, thus improving the decoding speed.[3]

[2]it really depends on the security margin

[3]Another useful property is that permutation can be chained or cascaded since they all have the same working domain.

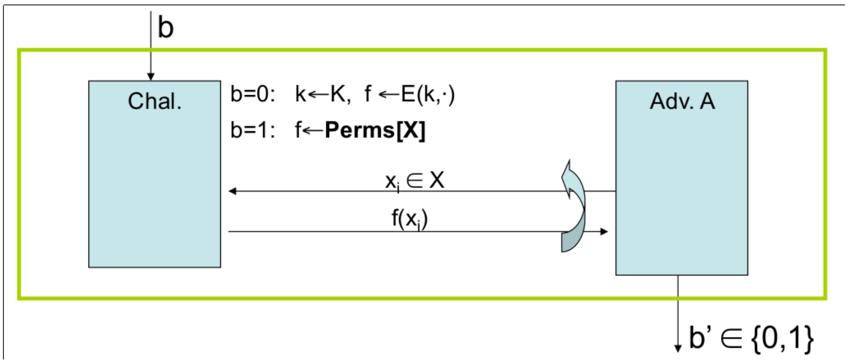


Figure 2.2: indistinguishability game for PRP

Secure PRP

The challenge for PRP is not much different from the one on PRF : see 2.2.2.

2.2.4 Pseudo Random Generator (PRG)

Definition

A pseudo random generator (PRG) is a deterministic bit generator with the property of unpredictability :

Definition 8 *$G : [\![0,1]\!]^s -> [\![0,1]\!]^n$ with $n >> s$.*
($S \in [\![0,1]\!]^s$ is often called "the seed" and is randomly chosen.)

Stream ciphers are easily built from PRG : $c = m \oplus G(k)$ (one time pad with pseudo-random generator)

Secure PRG

A PRG is secure if, for all "efficient" statistical test A, $Adv(A, G)$ is negligible.
On a side note, we don't know all the existing efficient statistical tests so we can't prove that PRG is secure, we can only prove we didn't found an efficient test which break it.

2.3 Confidentiality

2.3.1 Perfect Secrecy

Theorem 2 (Shannon perfect secrecy)
$\forall m_1, m_2$ such as $len(m_1) = len(m_2)$, $Pr[E(k, m_1) = c] = Pr[E(k, m_2) = c]$

(k uniform in K)

In layman terms, perfect secrecy means that, given two messages and the ciphertext of one of the two plaintext messages, the attacker cannot know from which message the ciphertext has been created (equal probability). A corollary is, under perfect secrecy conditions, the key space K cardinality must be equal or larger than the ciphertext space C cardinality, which has to be equal or larger than the message space M cardinality :

Theorem 3 (Shannon perfect secrecy corollary)
$|M| \leq |C| \leq |K|$.

2.3.2 Semantic Security

The semantic security is a weaker form of Shannon's perfect secrecy (which is not usable in reality) : the distributions $P_1 = Pr[E(k, m_1) = c, k \in K]$ and P_2 does not have to be equal, just computationally equivalent.

Challenge The semantic security is often defined using a "challenge" : an experiment of though where and adversary has to find an important information given a certain protocol and certain rights. The challenge is the following one :

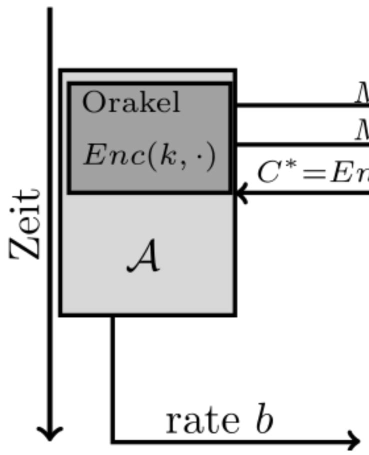


Figure 2.3: Semantic security challenge

The challenger (the "defender") choose a fixed-key and a variable b randomly from two values (0 and 1 to be simple). The attacker then will send the challenger $2 * n$ messages (plain or encrypted) and the challenger only process half of them (given the value b) and send the result back to the attacker.
The attacker has to estimate the value b chosen : if the attacker can guess the value of b with a significant advantage, the encryption cipher is considered insecure. Otherwise, it has semantic security against the corresponding attack channel.

Semantic Security against Chosen Plaintext Attack (CPA)
In this configuration, the attacker can choose which plaintext message(s) to send to the challenger, and has the result of the encryption of half of them.

Theorem 4 (Semantic Security under CPA)
E is semantically secure under chosen plaintext attack if, for all adversary A,

$$Adv_{CPA}[A, E] = Pr() \textit{ is negligible.}$$

Semantic Security against Chosen Ciphertext (Adaptive) Attack ($CCA{-}1/CCA{-}2$)
In this configuration, the attacker can choose which ciphertext message(s) to send to the challenger, and has the result of the decryption of half of them. Under $CCA - 2$ he can also makes incremental changes to the ciphertext sent given the output the previous decryption, enabling linear and differential attacks.
The CCA assumption allow the attacker a wide range of access to information : the

protocols which are secure against CCA are extremely useful since they can withstand a great variety of attacks [4].

2.4 Integrity

2.4.1 Secure MAC

The challenge used to define MAC security is a bit different from the one for semantic security : the challenger choose a key randomly. The attacker can submit n messages m_i and receive their corresponding tags t_i. Then he can submit a new forged message-tag pair (m, t) which will be verified by the challenger. If the challenger certified the pair to be true, the MAC protocol is considered insecure, otherwise it is secure.

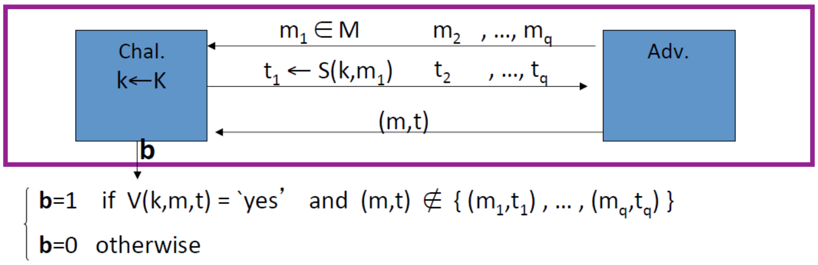


Figure 2.4: MAC integrity challenge

Definition 9 *I= (S, V) is considered secure if* $\forall$ *efficient statistical test A,*

$$Adv_{MAC}[A, I] = Pr[b == 1] \text{ is negligible.}$$

[4] of course, there still are side-channel attacks.

Chapter 3

Attack Vectors

3.1 Exhaustive Search Attack

The Exhaustive search attack, or brute-force attack, is to try every key possible until we retrieve the plaintext message (i.e. finding the secret key). Although it's a very crude and somewhat inefficient vector attack, some ciphers are unfortunately weak against it: the DES is one of them.

Definition : Given (m_i, c_i) in $[\![1, N]\!]$ (with N a small integer), find key k.

DES Challenge (by RSA)
The aim of the DES Challenge is to find the secret key given three messages and their encryption. It was organized by the RSA company, to demonstrate the vulnerability of DES (the good ol' naming and shaming technique).

1997 : 3 months using Internet-distributed computation ("cloud" computing)

1998 : 3 days using EFF[1]'s deep crack machine.

1999 : 22 hours

2006 : 7 days using 10k$'s FPGAs.

Prevention You only way to prevent a cipher from ESA is to augment the key size until no existing computation power can break the cipher in a reasonable time (like centuries).

3.2 Cryptographic attacks

The cryptographic attacks are the generic attack types used when talking about theoretical security (like semantic security).

Known Plaintext
The attacker has in his possession a plaintext message and its corresponding encryption.

[1]Electronic Frontier Foundation : a NGO focusing on net freedom

Cyphertext Only
The attacker does not know the plaintext message and has to find out (see Cesar cipher's breaking technique).

Chosen Plaintext/Cyphertext
The attacker can respectively encrypt/decrypt any plaintext/cyphertext message and study the result. It is the most common paradigm when studying public key cryptography (the attacker know the public key and can encrypt any message of his will).

Adaptative Plaintext/Cyphertext
The attacker can iterate the encryption/decryption based on the previous result (used for linear and differential attacks).

3.3 Side-channel attacks

Side-channel attacks are "real-world" attacks, i.e. they don't rely on theoretical cryptographic knowledge but rather the implementation of ciphers (buffer overflows, hardware failure, timing attacks, etc.). [2]

Attack on the implementation
Some perfectly secure ciphers can be compromised if they are implemented in the wrong way : buffer overflows on user input, stack overflows, information leaking, insufficient entropy, ...

Hardware Attacks
Another way to look at implementation is to consider the hardware : the analysis of the time and power required to encode/decode messages (using oscilloscopes and multimeter) can reveal important informations about the cipher used.

Fault Attacks
This attack is a follow up of the previous ones : by using laser to provoke run-time segfault (by twiddling the RAM bits), the attacker's aim is to create computing errors which can reveal informations about the key.

3.4 Meet in the middle attacks

Definition
The MITM attacks is a generic cryptoanalysis used originally to break n-rounds block ciphers. Given the ability to encrypt and decrypt any data, the attacker will try to find the encryption of a certain plaintext which match the decryption of another certain cyphertext. In others words, the encryption and decryption algorithm does each one half of the work and "meet".

Definition 10 (Meet in the middle Attack :)
Given (E,D) encryption/decryption system, find (m,c) such that : $E(m) == D(c)$.

[2] They are considered "inelegant", but the result is what matters, is it not ?

A famous example which invalidate the theory of "more rounds equal better security" is the double-DES : once demonstrated that DES is insecure, they tried to update the standard without changing everything by simply cascading a DES into another one, using two separate keys. However, 2-DES is badly vulnerable from MITM attacks.

Definition 11 (Challenge :)
Given (m,c), find $k = (k_1, k_2)$ *such that* $E(k_1, E(k_2, m)) == c$

Since k_1 and k_2 are independent keys, we have : $E(k_2, m) = x = D(k_1, m)$. Breaking a $2 - DES$ cipher is close to break DES, just twice longer. That's why $2 - DES$ was never used and standard chosen is $3 - DES$.

3.5 Linear and Differential Attacks

3.5.1 Linear Attacks

3.5.2 Differential Attacks

3.6 Quantum attacks

Quantum attacks are based on quantum computing. Unlike classical computers, quantum computers take advantage of quantum state superposition and entanglement to speed up solving time of equations, especially when combinatory calculus are present.

Generic Search Problem

Definition 12 *Let* $f : X \to 0, 1$ *a generic oracle,*
Generic Search Problem : find $x \in X$ *such as* $f(x) == true$.

On classical computer, the best generic solver is linear ($O(|X|)$) whereas on a quantum computer, it's root-squared ($O(|X|^{\frac{1}{2}})$).

Consequences The practical consequence of quantum computing over cryptography is to halve the key space : a way to prevent ciphers from quantum attacks is to double the key space (a 1024-key RSA cipher is as strong against quantum computers as a 512-key RSA cipher against classical computers). Anyway, quantum attacks are already studied and prevented (see `http://pqcrypto.org/`).

3.7 Collision attacks

When talking about data integrity and resistance to tampering, it is important to speak about the Birthday Paradox(which stipulates that there are way more random collisions than you would guess).

3.7.1 Generic Birthday attacks

The Birthday Paradox rest upon the following question : given a random group of people, what's the odds of having at least two persons with the same birthday ? The question is equivalent of estimating the probability of a collision from the output of a bounded random number generator.

Intuitively, we would think that the odds grow linearly with the group's size whereas, in reality, the odds grow way faster : when the group has 23 person, there is a 50% chance of a collision ! The paradox lies here : you only need of tenth of sample from the random pool to have a fairly high chance of collision, which is not intuitive (we would rather think of needing half of the pool - more or less 180 persons - to have a 50% chance).

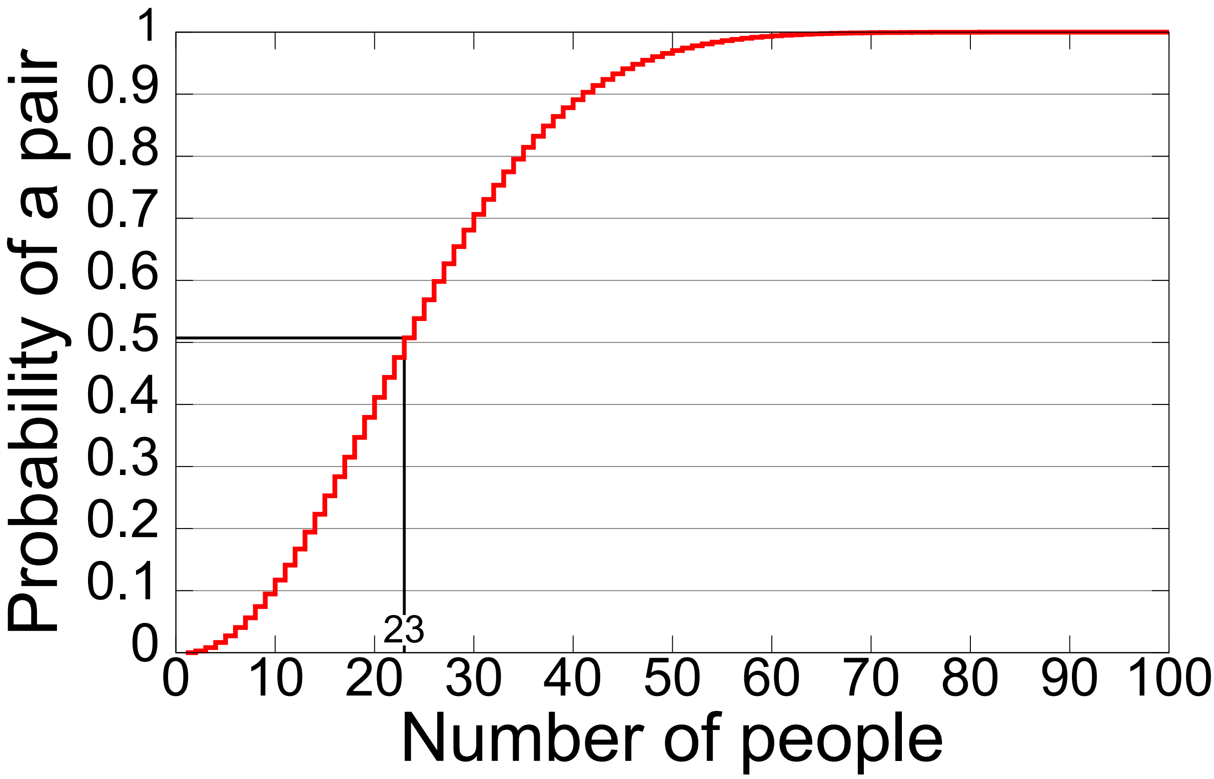


Figure 3.1: Probability of a match, according to the group's size
source : Wikipedia

Theorem 5 (Birthday Paradox)
Let $r_1, r_2, ...r_n in 1, ...B$ *n independent integers chosen uniformly,*

$$\textit{Then, when } n = 1.2 \times B^{\frac{1}{2}},\ Pr[collision] \geq \frac{1}{2}$$

The mathematical proof of this theorem is fairly simple : let compute the probability $Q(n, B)$ of not having a collision given n persons and B possible birthdays. Q can be seen as multiples draws without duplicates :

$$
\begin{aligned}
Q(n,B) = &\quad 1 \times \frac{B-1}{B} \times ... \times \frac{B-n}{B} && (3.1)\\
= &\quad \frac{B!}{(B-n)! * B^n} && (3.2)
\end{aligned}
$$

$P(n,B)$, the probability of having a birthday collision given n persons and B possible birth dates, is $Q(n,b)$ complement.
Using Taylor series expansion we can rewrite $\frac{B-k}{B}$ as $e^{\frac{-k}{B}}$.

$$
\begin{aligned}
Q(n,B) = &\quad \prod_{k=0}^{n} e^{\frac{-k}{B}} && (3.3)\\
= &\quad e^{\sum_{k=0}^{n} \frac{k}{B}} && (3.4)\\
= &\quad e^{\frac{-(\frac{*n(n-1)}{2})}{B}} && (3.5)\\
P(n,B) \simeq &\quad 1 - e^{\frac{-n^2}{2*B}} && (3.6)
\end{aligned}
$$

Using the previous relation, we can easily compute $P(n,B) \geq 0.5$ and find the result presented in the theorem.

The attackers take advantage of the fact that a fairly low number of random guess have a reasonable chance for one of them of being correct. For example, there was a DNS spoofing method using it :

The attacker's aim is to trick the DNS cache server into thinking that the IP of www.mybank.com is the spoofed server's one, not the real one. This attack is used for phishing : as soon as the DNS cache server is compromised, users connecting to www.mybank.com will send their credentials to the attackers.
In order to do so, the attacker will send multiple queries : each DNS resolution is accompanied with a transaction ID which identify the query. The attacker's goal is to send resolution answers from this server with the correct transaction ID. The spoofed server send multiple resolution answers with random transaction ID. Since a DNS transaction ID is coded on 2 bytes, the pool of ID is no larger than 65535 : the birthday theorem stipulate that, with $n = 1.2 * \frac{\sqrt{65535}}{2} = 153$ [3]queries, the attack has a 50% chance of success[4]. To further improve the probability of success, the attacker can also cripple the other server (the real one) with DDOS or bad routed packets in order to prevent him from sending a correct DNS resolution answer.

3.8 Social Attacks

Attacks target the weakest link of a system : sometimes it is the human nature the weakest link. This finding led to the famous "social engineering" attacks and the less known (but way more dangerous) rubberhose attacks.

[3] n resolution queries+n answers $= 2 * n$ total queries where looking for collision
[4]with 400 queries, the probability is over 99%

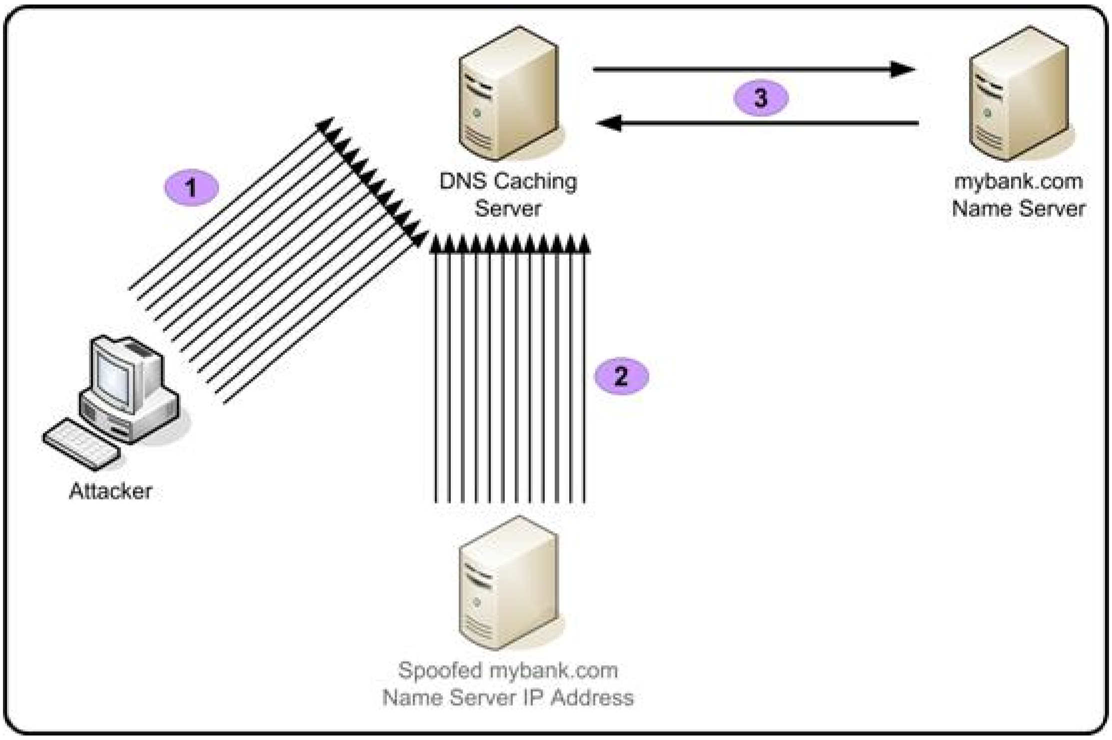


Figure 3.2: DNS Spoofing using multiple queries and Birthday Paradox
source : http://www.technicalinfo.net

Social Engineering Social Engineering describe a motley crew of methods designed to "punching a hole" into a system from the inside, without informing the insider of what he has done.
The most famous example is to leave some booby-trapped pendrives (and even more creative methods [5]) on the company site and hope for one employee to plug it in his workstation . Once it is plugged, the rigged pendrive which usually contains a Trojan, execute its script in order to obtain a privilege escalation from within the system and connect to the attacker then.

A new type of Social Engineering has arisen with the emergence of social networks (Facebook, LinkedIn, ...) : in this type of attacks, the goal is to obtain a remote account of a target using infos about disseminated around the Web (email addresses, universities he has been, birthday, birth place, ...) and use it to gain access to his email account (using password reset and secret questions). Once the email account has been compromised, it can be used to launch phishing attacks on the real target and retrieve important secrets.

Rubberhose Cryptoanalysis The rubberhose attack describe the use of force (legal, hierarchical or physical) and/or torture on a physical person in order to extract information about the system (keys, ciphers used, ...). It is difficult to prevent those attacks since they are outside the scope of cryptanalysis (most of them belong to sociology/politics). The only way to mitigate against those attacks would be agent partial-blindness or plausible deniability.

[5] A mouse device with modified firmware was sent as a gift to tech-based company employees for a security test : `http://www.theregister.co.uk/2011/06/27/mission_impossible_mouse_attack/`.

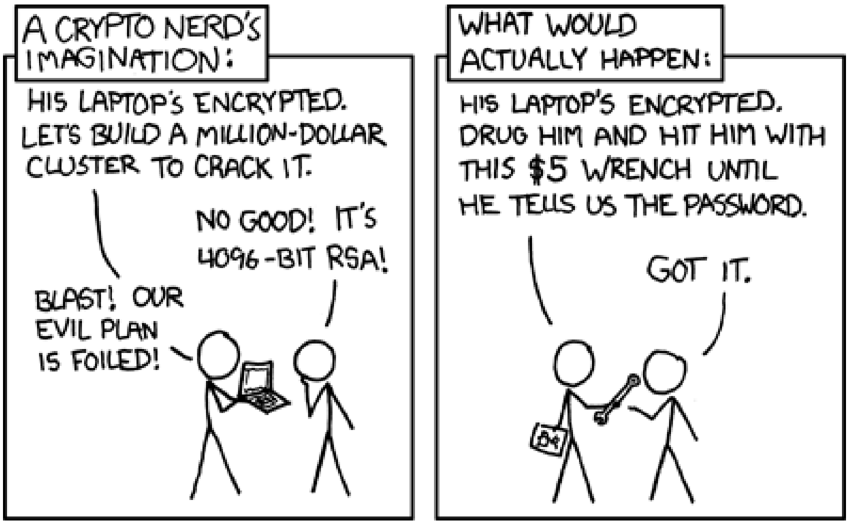


Figure 3.3: Real-world 0-day exploit.
source : XKCD

Chapter 4

Integrity

This chapter will present the other aspect of cryptography : tampering prevention. An adversary can be able to do more than just eavesdropping : he can actually modify ciphertext messages on-the-go (using fro example a Man in the Middle attack). Therefore we need methods to ensure the non modification of the message during transmission.
In Dan Boneh's lecture, the integrity enforcement mechanisms are a minor part of the course, so this chapter will be quite succinct (especially since some generic constructions were already explained in the confidentiality chapter).

4.1 Message Authentication Code (MAC)

The most common way to enforce message integrity is to add a tag to the message, which will be verified upon reception. Moreover, the tag needs to be created using a secret key in order to prevent an attacker from fooling the verification algorithm.

Definition 13 $MAC = (S, V)$

Tag Generator	*Verification Algorithm*
$S : (k, m) - > t$	$V : (k, m, t) - > 0, 1$

The tag and tag generation are more often called respectively "hash" and "hashing" : a hash function is an algorithm which takes variable-length data as input and outputs a fixed-length image of the input data. While being created in order to lessen the memory footprint of databases and speed-up the lookup of elements (hash tables, caches), hashing functions are also vastly used in cryptography.

4.1.1 Secure Mac

The experiment needed to describe the security of a MAC mechanism is the same as for the cipher's semantic security : the attacker can submit 2 messages q times, and receive the tag of n ones. If it can't forge a new valid pair (m,t) to submit to V with a significant advantage, the MAC algorithm is considered secure.

4.1.2 Collision Resistance

The strength of a MAC against forged tags are closely related to the algorithm's resistance against collision attacks. A collision is a pair of messages (m_0, m_1) such that $H(m_0) ==$

$H(m_1)$. We can clearly see that m_0 and m_1, if the MAC is using the hash functions H, have great chances to have the same tag. The verification algorithm will take one for another, which is a breach of security.
Therefore the algorithm for the tag generator has to be built upon collision resistant hashing functions.

4.1.3 MAC Padding

As for block ciphers, a hash algorithm works usually on a fixed length of plaintext information. However, contrary to the former, it is not possible to just pad the input text with 0's because it is insecure : the attacker can then send parts of the same message to retrieve important parts of information (block length, last digit digest).
The standard (ISO) currently used is to pad with one 1 and the rest with 0's.

4.2 Constructions

4.2.1 Construction from PRF

A secure MAC can be easily constructed from a PRF family. The following theorem is important since a lot of real-world MAC use it, in various environment (Internet, Banks, Defence, ..).

Theorem 6
If $F : K \times X \leftarrow Y$ is a secure PRF and $card(Y)$ is large, then $I_F = (F, V_F)$ is a secure MAC and :

$$Adv_{MAC} \leq Adv_{PRF} + \frac{1}{|Y|}$$

The main interest of this theorem is we can produce fairly large hash function from concise pseudorandom function, as long as the output domain is large. The following paragraph will present well-known constructions to treat large input files.

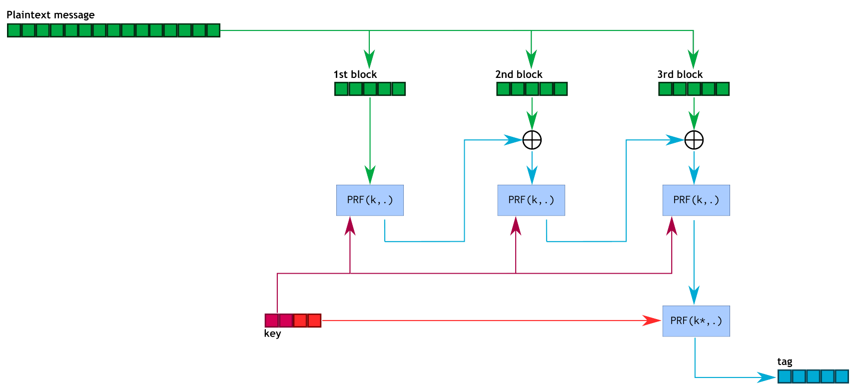


Figure 4.1: CBC-MAC construction

CBC-MAC The CBC-MAC is constructed from a single PRP by cascading the output of the previous block into the current block's input. The last tag is pad with ??? and then hashed a final time with a different key. Finally, the last block is hashed by the same function but with a different key to produce the tag.

Theorem 7
For every q-query adversary A attacking the CBC-MAC, there exist an adversary B attacking the PRP function F such that :

$$Adv_{PRP}[A, F_{ECBC}] = Adv_{PRP}[B, F] + 2 \times \frac{q^2}{2.|K|}$$

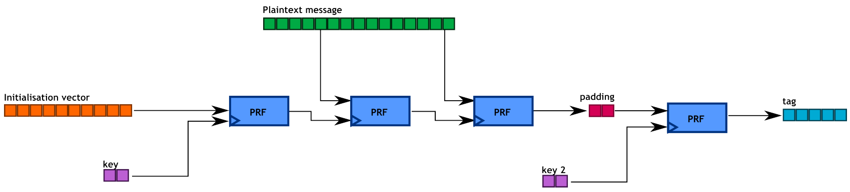


Figure 4.2: NMAC construction

NMAC In the NMAC construction, the output of a a block is used as the key for the next block. Like CBC-MAC, the process is fundamentally sequential.

Theorem 8
For every q-query adversary A attacking the NMAC, there exist an adversary B attacking the PRF function F such that :

$$Adv_{PRF}[A, F_{NMAC}] = Adv_{PRF}[B, F] + 2 \times \frac{q^2}{|X|}$$

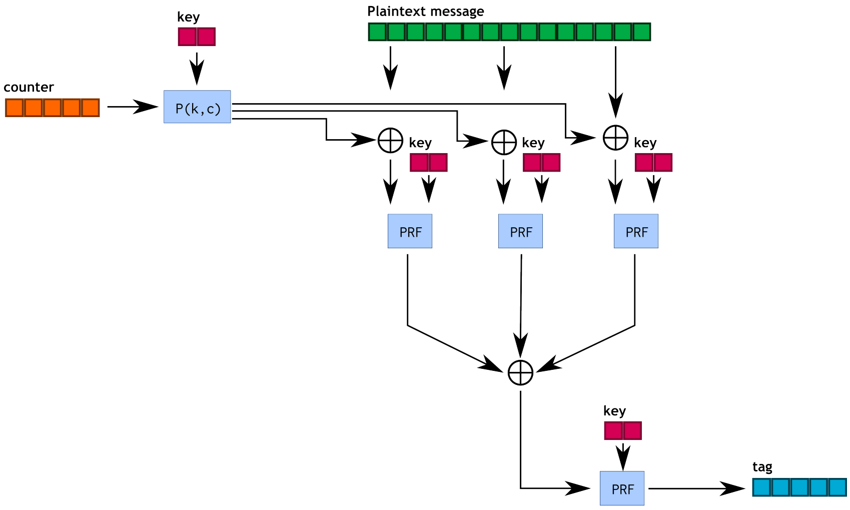


Figure 4.3: PMAC construction

PMAC The PMAC, or Parallel MAC, is obviously a parallel MAC construction : P is a simple expansion function which will be used to fuzz the input data for each hash block. Like the previous constructions, the overall result is padded and hash one last time a second time to prevent oracle attack.

Theorem 9
For every q-query adversary A attacking the PMAC, there exist an adversary B attacking the PRF function F such that :

$$Adv_{PRF}[A, F_{PMAC}] = Adv_{PRF}[B, F] + 2 \times \frac{q^2 \times L^2}{|X|}$$

4.2.2 Construction from hash functions

There is another method to construct secure MAC, from collision-resistant hash functions. A lot of famous MAC use this construction : SHA-1, SHA-2, etc.

Compression functions Compression functions are function that map two domains (let call them M and T) with the following property : the output domain M is several orders of size smaller than the input domain T ($|M| \ll |T|$). Compression functions often take a fixed-size message and a key and output a fixed-sized message, which halves the overall message length.
Collision resistance is a fundamental property for compression functions used in secure MAC mechanisms. Others useful properties : easily computable, pre-image and second pre-image resistance.
Compression functions are often constructed from ciphers : for example the Davies-Meyer construction use a encryption cipher and some fuzzying steps to produce a collision-based hash function. This construction has the advantage of saving space, since the cipher can be used to encrypt and hash data (more on 4.3).

Merkle-Damgard Paradigm The Merkle-Damgard architecture is a general layout to construct secure MAC mechanisms from collision-resistant hashing functions. It relies heavily on compression functions :

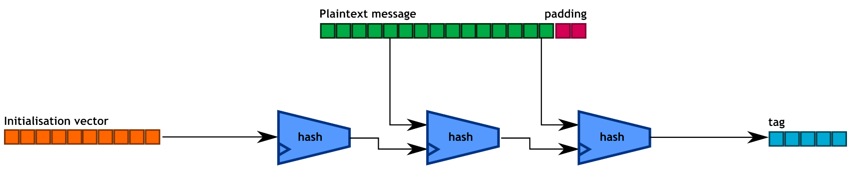


Figure 4.4: Merkle-Damgard construction

The construction is really similar to block ciphers : there is an initialization vector IV to prevent replay attacks, several chained blocks of the same compression function, and a final padding.

HMAC HMAC has two padding system : the inner pad *ipad* and the outer pad *opad*.

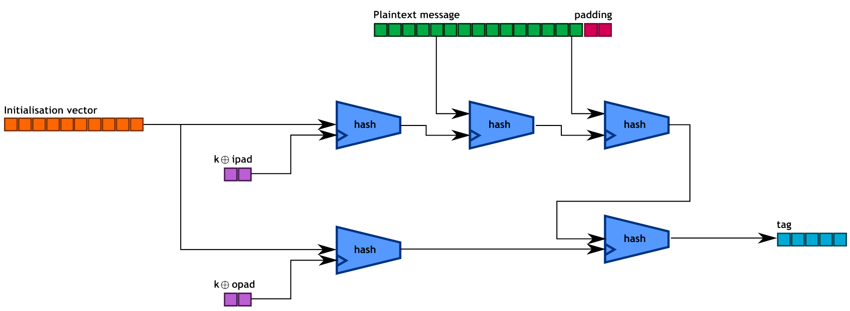


Figure 4.5: HMAC construction

4.3 Authenticated Encryption

Authenticated encryption is the following step in cryptography : ensuring integrity as well as confidentiality. In order to do that, it has to mix a MAC mechanism and a encrypting cipher. There is three approaches to the mixing :

Encrypt-then-MAC : This is the standard method, which yield the best results in terms of security.

Encrypt-and-MAC : Used for SSH.

MAC-then-Encrypt : Used for SSL/TLS.

4.3.1 Definition

An authenticated encryption system consist of a cipher (E, D) where :

- $E : KxM- > C$
- $D : KxC- > M \cup \perp$ ($\perp$ is the rejection symbol).

The system decrypt a ciphertext only if the MAC associated with has been validated. Otherwise, it will just output $\perp$. This symbol is important since it can prevent against oracle padding attacks : whether it's a invalid MAC or a pad error, the system has to output the same symbol in both cases.

Chapter 5

Key Exchange Protocols

This chapter will focus on the initialisation aspect of cryptography, and more exactly on the part which contains the key-exchange protocols. As we seen in the last two chapters, there are robust systems to ensure confidentiality and integrity of data communication against eavesdropping and active attackers (given those systems were correctly implemented). However, these mechanisms does not describe how the two actors (generally called Alice and Bob) create a shared secret (i.e. key) used to encrypt data.
Key-exchange protocol is the last part of Coursera Cryptography I class and it's more detailed in the second class.

5.1 Trusted 3rd party

The most straightforward method for key exchange between two parties (or more) is to use a third one as proxy/notary. Each party gives their secret key to the TTP[1] and, in return gives a shared one. While having many implementations (notary for wills and contract, Paypal for money transactions, certifications authorities (CA) ...) this procedure is not completely secure since it create a single point of failure (e.g. the TTP being pwn'ed) and the "trusted" part is sometimes false (as seen with the NSA leak) : that's why it is preferable to look at zero-knowledge protocols for key exchange.

5.2 Merkle Puzzles

Merkle puzzles has been developed as a way to exchange keys using a generic symmetric cipher between two persons without a third-party. It's constructed upon a "puzzle", which means a computational difficult problem.[2]
As usual , we are in a situation of security against eavesdropping (Merkle puzzles are insecure against active attacks such as AP poisoning). Eve wants to retrieve the shared key that will be use for data transmission, and can only listen the communications between Alice and Bob.

Steps for key-exchange using Merkle puzzles :

[1]Trusted Third Party

[2]I rather not try to define what a "difficult" computational problem means, let's say it's more or less NP-hard problems.

1. Alice prepares n problems, and send it to Bob. Each problem has a message encrypted containing an identifier and a secret key
2. Bob solves one, and send the identifier in plaintext back to Alice
3. Alice fetch the secret key corresponding to the identifier sent by Bob
4. the two parties can communicate securely using their shared key.

Complexity The strength of the Merkle puzzle scheme lies in the asymmetry of the problem. Let a Merkle Puzzle be of complexity linear $O(m)$. Alice send n puzzles to Bob which solves one so Bob need $O(n+m)$ time computation. Bob sending back an identifier, Alice only need to solve one puzzle, thus $O(n+m)$ too. However, Eve needs to solve "all" (until the identifier is found) puzzles, so $O(n*m)$: Alice and Bob needs linear time computation, while Eve needs quadratic time.

While being quite useful, the best gap in complexity provided by Merkle Puzzles is quadratic at best : for many real-world cases it is not enough[3]. That's why exponential time gap scheme has been created (by Diffie and Hellman or RSA).

5.3 Public Key Encryption

The public key encryption class regroup a list of ciphers useful for key-exchange. They all rely on the same idea as Merkle Puzzle's, meaning finding a problem which is exponentially harder to solve than checking if a solution is a valid one.

5.3.1 Diffie-Hellman protocol

The Diffie-Hellman protocol is a key exchange scheme created in 1976, but still fairly used nowadays. The strength of this protocol is based on the difficulty of the discrete logarithm problem.
Steps :

- Alice and Bob choose a finite group (generally $\frac{\mathbb{Z}}{p\mathbb{Z}}$) and g a generator from this group.
- Alice choose randomly a number a, and send to Bob g^a
- Bob does the same with b
- $g^{a.b}$ is the shared secret used to encrypt communications. Any eavesdropper has access to g^a and g^b, but cannot easily compute the shared secret from these two numbers (since he has to compute either a or b using discrete log).

Discrete exponentiation is fairly easy (linear time) whereas discrete log is hard (best known algo in $exp(O(\sqrt[3]{x}))$: this protocol use the asymmetry of the operation $\times$ to protect the key exchange, thus gaining the name of asymmetric-cryptography. However, if it were to be found an effective to the discrete log problem, the Diffie-Hellman would

[3]Merkle Puzzles are also vulnerable to quantum computation

become insecure (as many others systems like El-Gamal).

This exchange is secure against eavesdropping, but not against Man-in-the-Middle attacks. The mitigation against it is to send, as well as g^x, a publicly certified signature from a trusted third party[4].

Multi-party communication As seen previously the Diffie-Hellman protocol gives a secure way for two parties to exchange a secret. What about more than two people ? Unfortunately, this a currently an open problem [5] : there is no efficient way to create a multi-party shared secret.

5.3.2 RSA encryption

Trapdoor Functions RSA encryption relies heavily on trapdoor functions for security. Trapdoor functions - or “one-way functions” - are easily computable in one direction, yet difficult in the other one [6].

In cryptography, a trapdoor function is a triplet (G, F, F_{-1}) where :

- G: *seed* $\rightarrow (pk, sk)$ a randomized algorithm which produce a public key and a secret one.
- $F : (pk, X) \rightarrow Y$ used to encrypt message X using public key pk.
- $F_{-1} : (sk, Y) \rightarrow X$ used to decrypt ciphertext Y using secret key sk.

Theorem 10
$\forall (pk, sk)$ *generated by* G,
$\forall m \in M,\ F_{-1}(sk, F(pk, m)) = m$

RSA Trapdoor The RSA trapdoor function relies on the prime factorization problem and the modular e-roots' problem.
First, the prime factorization problem : given two prime numbers, it's easy to compute their product. However, given the product of two primes numbers, it is fairly difficult to find the factorization. That's mean we can communicate the primes' product publicly without lessening the security of the protocol - the two primes still has to be big enough -. In the case of RSA, the primes' product number will be used as a generator for (Z/nZ).
Secondly, the modular e-roots' problem : given a number and an exponent, it is easy to compute its exponentiation in Z/nZ, but it is difficult to compute the e-root in the same group (in the math sense).

ECC Trapdoor Another famous trapdoor is the Elliptic Curve (Cryptography) trapdoor, which is more secure than RSA's trapdoor for the same key-length.

[4] More on digital signature in the second part of the Coursera course

[5] there is nonetheless a simple solution for 3 parties.

[6] “easy” and “hard” does not have any formal meaning, we just look at difficulty from a empirical point-of-view

Definition 14 (Elliptic Curve) *An elliptic curve has the following equation in the plane : $P[x] = Q[y]$ where P and Q are polynoms of degree 3.*

The Elliptic Curve trapdoor is a bit more complicated than modular e-roots and need some solid geometric bases. You can find a primer here : `http://blog.cloudflare.com/a-relatively-easy-to-understand-primer-on-elliptic-curve-cryptography`

5.3.3 El Gamal

El Gamal use Diffie-Hellman protocol to create a secure public-key protocol. It's currently used by GPG, some version of PGP and the algorithm isn't patented like RSA's.

Definition

ElGamal is built from a cyclic finite group G ($\mathbb{Z}_p^*$ for example, with p a "big" prime number) : let g be a generator[7] of G. Like any other authenticated encryption system, it has a symmetric cipher (E, D) for encryption and a hash function H for integrity.

Key Generation : Alice and Bob choose respectively a and b from the set of powers used to generate G, and send to the other party resp. g^a and g^b which are used in the Diffie-Hellman protocol.

- Alice's secret key : a
- Alice's public key : $pk_A = (g, g^a)$
- Bob's secret key : b
- Bob's public key : $pk_B = (g, g^b)$

Encryption : Alice wants to send a message m to Bob, using Bob's public key. First, Alice create the encryption key k using the hash function H : $k = H(g^b, g^{a.b})$. Then, using k, Alice encrypt the message : $c = E(k, m)$. Alice output (g^a, c)

Decryption : The decryption is really straightforward : Bob, using his secret key b can compute $g^{a.b}$ from g^a sent by Alice along with the message. $g^{a.b}$ is then used by the hash function to compute the secret key k which will be used by the decipher D.

Security :

The ElGamal security theorem is built upon Diffie-Hellman security, and especially Diffie-Hellman Computational/Decisional Assumption[8].

Theorem 11 (DH Computational Assumption)
Let G a cyclic finite group, and g a generator of G. Then, $\forall$ efficient algorithms A,

$$Pr[A(g, g^a, g^b) == g^{a.b}] \text{ is negligible.}$$

In layman terms, given the generator and the partial public keys (that's what the attacker has access to), it is almost impossible to find an algorithm which can compute $g^{a.b}$.

[7] a generator g can build the set G only from it's geometric sequence : $g^i, i \in N == G$.

[8] The Decisional version is a generalisation of the Computational version. For more infos : `http://en.wikipedia.org/wiki/Decisional_Diffie%E2%80%93Hellman_assumption`.)