Counting Problems

Q1 UNUSUAL — 3 repeated letters

assign subsets of 5 letters – find a combination
– there are 5 distinct letters

{U, N, S, A, L} – this is the only subset of 5 letters that exists for the word UNUSUAL because U is repeated 3 times, so we only have 5 unique objects to pick from ($\binom{5}{5} = 1$), so only **one** subset can be formed

# of strings made from 5 of those 7 letters – find a permutation

$\frac{5!}{3!} = 5 \cdot 4 =$ **20 unique permutations** of the characters exist, so 20 strings of length 5 exist

Q2 # of ways to form 5 card hand w/ 2 pairs.

$\binom{13}{2}$ – choose 2 number values for the pairs of the 13 available

$\times \binom{4}{2}$ – choose 2 suits of the 4

$\times \binom{4}{2}$ – choose 2 other suits for the second pair

$\times \binom{11}{1}$ – choose the last card's value from cards

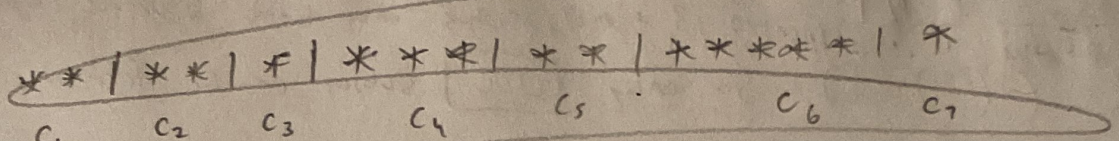
$\times \binom{4}{1}$ – pick the last card's suit

$$\frac{13!}{2!(13-2)!} \cdot \frac{4!}{2!(4-2)!} \cdot \frac{11!}{2!(11-2)!} \cdot \frac{4!}{2!(4-2)!} \cdot \frac{4!}{1!(4-1)!}$$

$= 78 \cdot 6 \cdot 6 \cdot 11 \cdot 4 =$ **123,552**

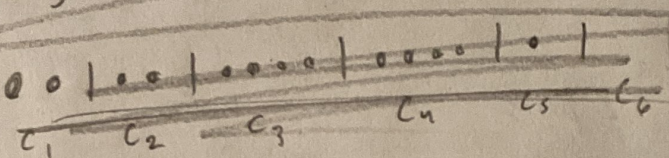
Q3 16 songs to be played by 7 couples
1 couple only allows 1 song

~~16 indistinguishable balls, 6 indistinguishable boxes~~, 1 distinguishable box
- stars/bars

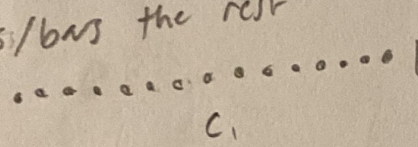


$\binom{22}{6} = 74613$ ways if each couple gets between 0 and 16 songs

now subtract the number of arrangements where 1 box has more than 1 song - do stars/bars again with 15 songs this time



assume the fighting couple has 1 song
stars/bars the rest



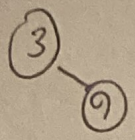
$\binom{20}{5} = 15,504$ combinations when the fighting couple gets 1 song

stars/bars w/ 16 songs to get $\binom{21}{5} = 20,349$ combinations

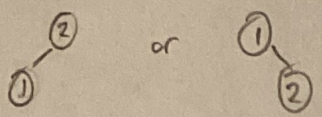
add the two combinations to get the total # of ways the violinist can play: $15,504 + 20,349 = \boxed{35,853}$

there are no possible duplicates b/c he plays a different number of songs to these people in both scenarios

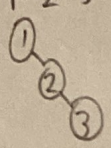
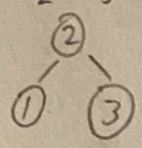
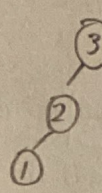
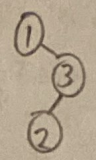
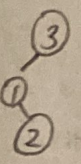
Q4 BST w/ 12 nodes



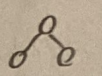
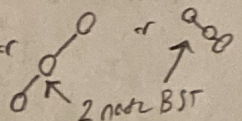
w/ 2 nodes
- 2 BSTs



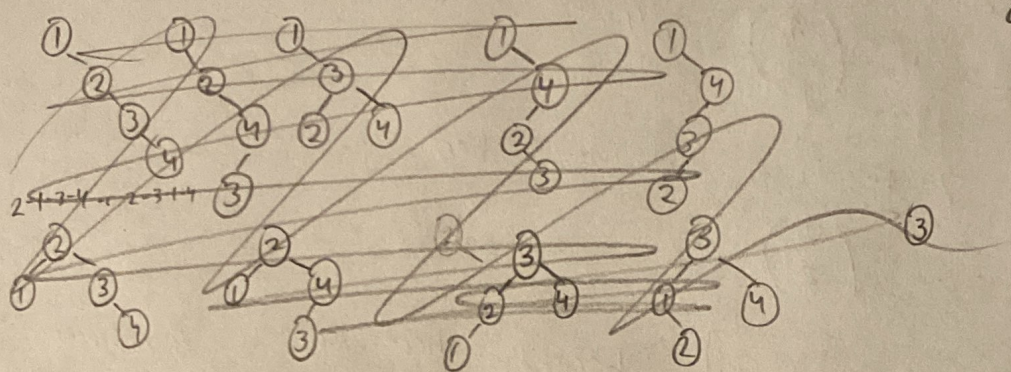
w/ 3 nodes
- 5 BSTs

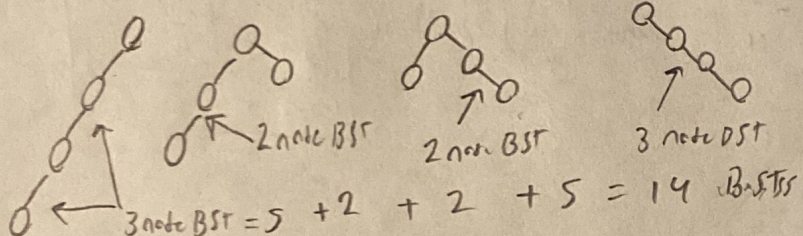
1-2-3  or 2-1-3 or 2-3-1  or 3-2-1  or 1-3-2  or 3-1-2 

if inserted after a median, small/large order doesn't matter

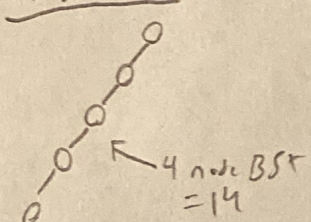
aka  or 

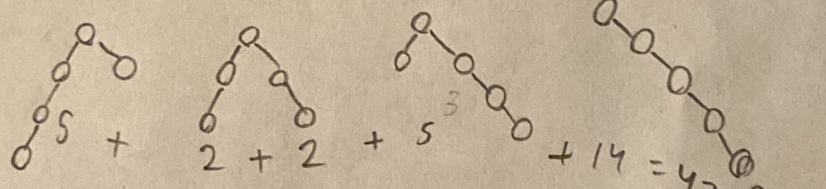
w/ 4

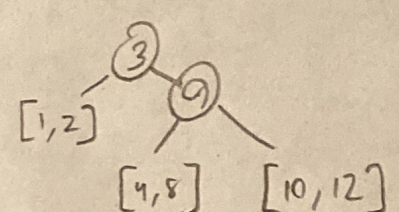




w/ 5 nodes = 42 BSTs







- [1,2] = 2 (2 nodes)
- [4,8] = 42 (5 nodes)
- [10,12] = 5 (3 nodes)

so total BSTs = 2 · 42 · 5

= 420 BSTs

Q5 10 friends, 4 nurses administering shots

w/ 4 nurses: ••|••|•••|••

$\binom{13}{3}$ each

w/ 3 nurses: ••••|•••|•••

$\binom{12}{2}$

since each nurse always gets 1 patient, in each example, we can only bother calculating the friends who aren't automatically assigned to a nurse

so w/ 4 nurses: 6 friends to distribute, 3 bars

$\binom{9}{3} = 84$

w/ 3 nurses: 7 friends to distribute, 2 bars

$\binom{9}{2} = 36$

we can add $84 + 36 = \boxed{120}$ because there can't be any repeats of combinations if each nurse is guaranteed 1 patient & we have a different number of nurses