Policy Gradient Methods: Estimation

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CS/Stat 184(0): Introduction to Reinforcement Learning

Fall 2024

Today

- Feedback from last lecture
- Recap
- Estimation: REINFORCE
- Variance Reduction
    - Other Gradient Expressions
    - Baselines and Advantages
- Examples

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The Learning Setting:

We don't know the MDP, but we can obtain trajectories.

The Finite Horizon, Learning Setting. We can obtain trajectories as follows:

- We start at $s_0 \sim \mu$.
- We can act for H steps and observe the trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$

Note that with a simulator, we can sample trajectories as specified in the above.

Optimization Objective

- Consider a parameterized class of policies:

$$\{\pi_\theta(a \mid s) \mid \theta \in \mathbb{R}^d\}$$

(why do we make it stochastic?)

- Objective $\max_\theta J(\theta)$, where

$$J(\theta) := \mathbb{E}_{s_0 \sim \mu}\left[V^{\pi_\theta}(s_0)\right] = \mathbb{E}_{\tau \sim \rho_{\pi_\theta}}\left[\sum_{h=0}^{H-1} r(s_h, a_h)\right]$$

- Policy Gradient Descent:

$$\theta^{k+1} = \theta^k + \eta \nabla J(\theta^k)$$

Example Policy Parameterizations

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy

Feature vector $\phi(s, a, h) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s, h) = \frac{\exp(\theta^\top \phi(s, a, h))}{\sum_{a'} \exp(\theta^\top \phi(s, a', h))}$$

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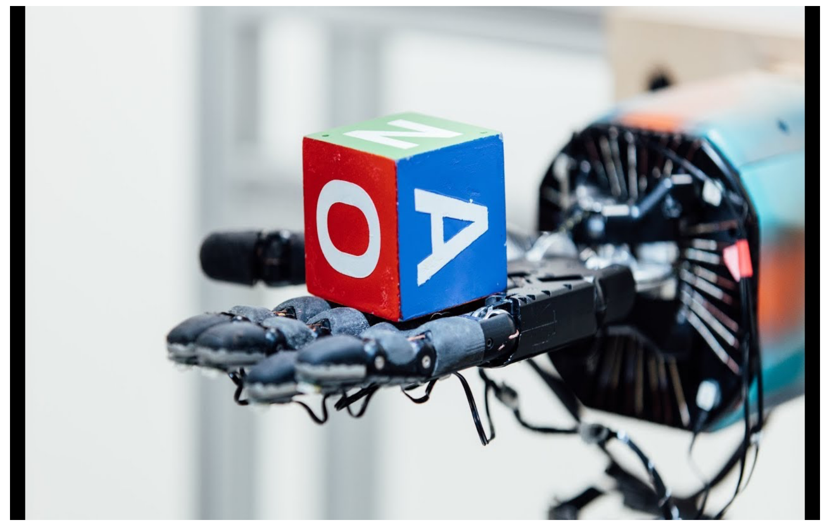
2. Neural Policy:

Neural network
$f_\theta : S \times A \times [H] \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s, h) = \frac{\exp(f_\theta(s, a, h))}{\sum_{a'} \exp(f_\theta(s, a', h))}$$

Example Policy Parameterization for “Controls”



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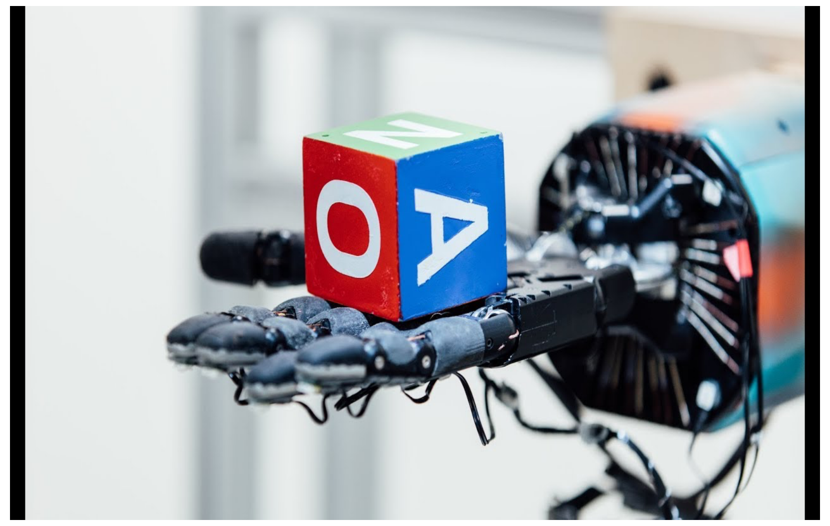

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3. Gaussian + Linear Model

- Feature vector: $\phi(s, h) \in \mathbb{R}^d$,
- Parameters: $\theta \in \mathbb{R}^{k \times d}$,
  (and maybe $\sigma \in R^+$)

- Policy: sample action from a (multivariate) Normal with mean $\theta \cdot \phi(s, h)$ and variance $\sigma^2 I$, i.e.
  $$\pi_{\theta,\sigma}(\cdot \mid s, h) = \mathcal{N}\left(\theta \cdot \phi(s, h), \sigma^2 I\right)$$
- Sampling:
  $$a = \theta \cdot \phi(s, h) + \eta, \text{ where } \eta \sim \mathcal{N}(0, \sigma^2 I)$$

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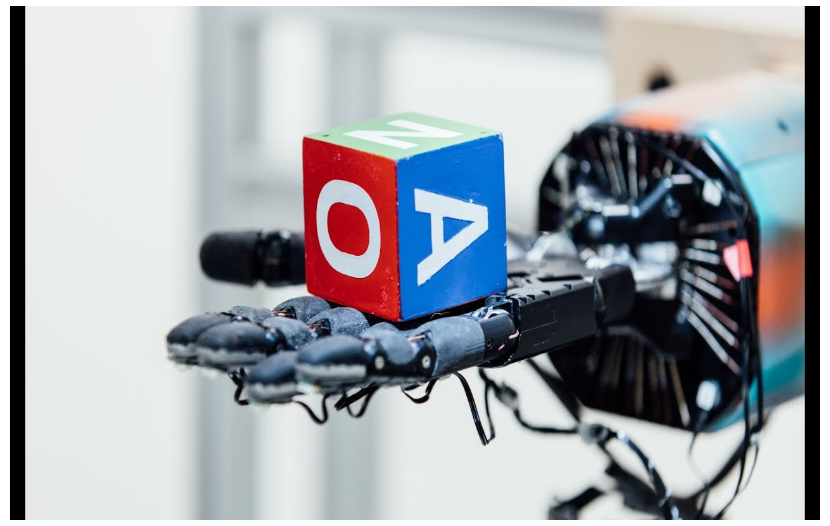

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- Neural network $g_\theta : S \times [H] \mapsto \mathbb{R}^k$
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$$\begin{aligned}\nabla_\theta J(\theta) &= \sum_x \nabla_\theta P_\theta(x) f(x)\\ &= \sum_x P_\theta(x)\frac{\nabla_\theta P_\theta(x)}{P_\theta(x)} f(x)\end{aligned}$$

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- We can lower variance by drawing N i.i.d. samples from P_θ and averaging:

$$\widehat{\nabla}_\theta J(\theta) = \frac{1}{N} \sum_{i=1}^{N} \nabla_\theta \log P_\theta(x_i)\, f(x_i)$$

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- Let $\rho_\theta(\tau)$ be the probability of a trajectory $\tau = \{s_0, a_0, s_1, a_1, \ldots, s_{H-1}, a_{H-1}\}$, i.e.

$$\rho_\theta(\tau) = \mu(s_0)\pi_\theta(a_0 \,|\, s_0)P(s_1 \,|\, s_0, a_0)\ldots P(s_{H-1} \,|\, s_{H-2}, a_{H-2})\pi_\theta(a_{H-1} \,|\, s_{H-1})$$

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- But $\rho_\theta(\tau)$ involves the dynamics P, which we assumed we don't know!

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We have: $\mathbb{E}[g(\theta, \tau)] = \nabla_\theta J(\theta)$

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We still have that at the kth step, g is unbiased for $\nabla_\theta J(\theta)$ evaluated at θ^k

Today

- Feedback from last lecture
- Recap
- Estimation: REINFORCE
- Variance Reduction
  - Other Gradient Expressions
  - Baselines and Advantages
- Examples

Other PG formulas
(that are lower variance for sampling)

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta}\left[\left(\sum_{h=0}^{H-1} \nabla_\theta \ln \pi_\theta(a_h \mid s_h)\right) R(\tau)\right] \qquad \text{(REINFORCE)}$$

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Intuition: Changing the action distribution at h only affects rewards later on…

HW: You will show these simplified version are also valid PG expressions

An improved policy gradient procedure:

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On a trajectory τ, define:

$$R_h(\tau) = \sum_{t=h}^{H-1} r_t$$

And define:

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Comments:

- We still have unbiased gradient estimates.
- Easy to use a mini-batch algorithm to reduce variance.
- Easy to compute the gradient in “one pass” over the data.

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This is (basically) the method of control variates.

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- To see this, first note:
  $$\mathbb{E}_{x \sim P_\theta}\left[\nabla_\theta \log P_\theta(x)\ c\right] \overset{\text{LRM}}{=} \nabla_\theta \mathbb{E}_{x\sim P_\theta}[c] = \nabla_\theta c = 0$$

Proof:

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- Thus for any constant c,
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- Returning to RL, we have:
  $$\mathbb{E}_{\tau\sim\rho_\theta}\left[\sum_{h=0}^{H-1}\nabla_\theta \ln \pi_\theta(a_h|s_h)\big(R_h(\tau)-b_h(s_h)\big)\right] = \sum_{h=0}^{H-1}\mathbb{E}_{s_h\sim\rho_\theta}\left[\mathbb{E}_{a_h\sim\pi_\theta(\cdot|s_h)}\left[\nabla_\theta \ln \pi_\theta(a_h|s_h)\big(R_h(\tau)-b_h(s_h)\big)\right]\right]$$
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(where $s_h \sim \rho_\theta$ is a sample from the marginal state distribution at time h)

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The Advantage Function (finite horizon)

$$V_h^\pi(s) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| s_h = s\right] \qquad Q_h^\pi(s, a) = \mathbb{E}\left[\sum_{t=h}^{H-1} r(s_t, a_t) \middle| (s_h, a_h) = (s, a)\right]$$

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- The second step follows by choosing $b_h(s) = V_h^\pi(s)$.
- In practice, the most common approach is to use $b_h(s)$ that's an estimate of $V_h^\pi(s)$.

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Note that regardless of our choice of $\widetilde{b}$, we still get unbiased gradient estimates.

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Policy Parameterizations

Recall that we consider parameterized policy $\pi_\theta(\cdot \mid s) \in \Delta(A), \forall s$

1. Softmax linear Policy

Feature vector $\phi(s,a) \in \mathbb{R}^d$, and parameter $\theta \in \mathbb{R}^d$

$$\pi_\theta(a \mid s) = \frac{\exp(\theta^\top \phi(s,a))}{\sum_{a'} \exp(\theta^\top \phi(s,a'))}$$

2. Neural Policy:

Neural network $f_\theta : S \times A \mapsto \mathbb{R}$

$$\pi_\theta(a \mid s) = \frac{\exp(f_\theta(s,a))}{\sum_{a'} \exp(f_\theta(s,a'))}$$

Softmax Policy Properties

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Two properties (see HW):

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- We have:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1} Q_h^{\pi_\theta}(s_h, a_h)\left(\phi(s_h, a_h) - \mathbb{E}_{a' \sim \pi_\theta(\cdot \mid s_h)}[\phi(s_h, a')]\right)\right]$$

$$= \mathbb{E}_{\tau \sim \rho_\theta}\left[\sum_{h=0}^{H-1} A_h^{\pi_\theta}(s_h, a_h)\phi(s_h, a_h)\right]$$

Summary:

1. REINFORCE (a direct application of the likelihood ratio method)
2. Variance Reduction: with baselines

Attendance:

bit.ly/3RcTC9T

Feedback:

bit.ly/3RHtlxy