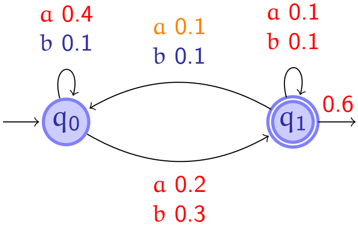
Weighted Finite Automata (WFA)

Example with 2 states and alphabet $\Sigma = \{a, b\}$

Operator Representation



$$\boldsymbol{\alpha}_0 = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}$$

$$\boldsymbol{\alpha}_\infty = \begin{bmatrix} 0.0 \\ 0.6 \end{bmatrix}$$

$$\mathbf{A}_a = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$$

$$\mathbf{A}_b = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}$$

$$f(ab) = \boldsymbol{\alpha}_0^\top \mathbf{A}_a \mathbf{A}_b \boldsymbol{\alpha}_\infty$$

Weighted Finite Automata (WFA)

Notation:

- Σ: alphabet – finite set
- n: number of states – positive integer
- $\boldsymbol{\alpha}_0$: initial weights – vector in $\mathbb{R}^n$ (features of empty prefix)
- $\boldsymbol{\alpha}_\infty$: final weights – vector in $\mathbb{R}^n$ (features of empty suffix)
- $\mathbf{A}_\sigma$: transition weights – matrix in $\mathbb{R}^{n \times n}$ $(\forall \sigma \in \Sigma)$

Definition: WFA with n states over Σ

$$A = \langle \boldsymbol{\alpha}_0, \boldsymbol{\alpha}_\infty, \{\mathbf{A}_\sigma\} \rangle$$

Compositional Function: Every WFA A defines a function $f_A : \Sigma^\star \to \mathbb{R}$

$$f_A(x) = f_A(x_1 \dots x_T) = \boldsymbol{\alpha}_0^\top \mathbf{A}_{x_1} \cdots \mathbf{A}_{x_T} \boldsymbol{\alpha}_\infty = \boldsymbol{\alpha}_0^\top \mathbf{A}_x \boldsymbol{\alpha}_\infty$$

Example – Hidden Markov Model

- Assigns probabilities to strings $f(x) = \mathbb{P}[x]$
- Emission and transition are conditionally independent given state

$$\boldsymbol{\alpha}_0^\top = [0.3 \;\; 0.3 \;\; 0.4]$$

$$\boldsymbol{\alpha}_\infty^\top = [1 \;\; 1 \;\; 1]$$

$$\mathbf{A}_a = \mathbf{O}_a \cdot \mathbf{T}$$

$$\mathbf{T} = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0 & 0.75 & 0.25 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

$$\mathbf{O}_a = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$