Chapter 8

Dark Matter

Cosmologists, over the years, have dedicated a large amount of time and effort to determining the matter density of the universe. There are many reasons for this obsession. First, the density parameter in matter, $\Omega_{m,0}$, is important in determining the spatial curvature and expansion rate of the universe. Even if the cosmological constant is non-zero, the matter content of the universe is not negligible today, and was the dominant component in the fairly recent past. Another reason for wanting to know the matter density of the universe is to find out what the universe is made of. What fraction of the density is made of stars, and other familiar types of baryonic matter? What fraction of the density is made of dark matter? What constitutes the dark matter – cold stellar remnants, black holes, exotic elementary particles, or some other substance too dim for us to see? These questions, and others, have driven astronomers to make a census of the universe, and find out what types of matter it contains, and in what quantities.

We have already seen, in the previous chapter, one method of putting limits on $\Omega_{m,0}$. The apparent magnitude (or flux) of type Ia supernovae as a function of redshift is consistent with a flat universe having $\Omega_{m,0} \approx 0.3$ and $\Omega_{\Lambda,0} \approx 0.7$. However, neither $\Omega_{m,0}$ nor $\Omega_{\Lambda,0}$ is individually well-constrained by the supernova observations. The supernova data are consistent with $\Omega_{m,0} = 0$ if $\Omega_{\Lambda,0} \approx 0.4$; they are also consistent with $\Omega_{m,0} = 1$ if $\Omega_{\Lambda,0} \approx 1.7$. In order to determine $\Omega_{m,0}$ more accurately, we will have to adopt alternate methods of estimating the matter content of the universe.

8.1 Visible matter

Some types of matter, such as stars, help astronomers to detect them by broadcasting photons in all directions. Stars primarily emit light in the infrared, visible, and ultraviolet range of the electromagnetic spectrum. Suppose, for instance, you install a B-band filter on your telescope. Such a filter allows only photons in the wavelength range $4.0 \times 10^{-7}\,\mathrm{m} < \lambda < 4.9 \times 10^{-7}\,\mathrm{m}$ to pass through.[1] The "B" in B-band stands for "blue"; however, in addition to admitting blue light, a B-band filter also lets through violet light. The Sun's luminosity in the B band is $\mathrm{L}_{\odot,B} = 4.7 \times 10^{25}\,\mathrm{watts}$. [2]

In the B band, the total luminosity density of stars within a few hundred megaparsecs of our Galaxy is

$$j_{\star,B} = 1.2 \times 10^{8}\,\mathrm{L}_{\odot,B}\,\mathrm{Mpc}^{-3}\ . \tag{8.1}$$

To convert a luminosity density $j_{\star,B}$ into a mass density $\rho_\star$, we need to know the *mass-to-light ratio* for the stars. That is, we need to know how many kilograms of star, on average, it takes to produce one watt of starlight in the B band. If all stars were identical to the Sun, we could simply say that there is one solar mass of stars for each solar luminosity of output power, or $\langle M/L_B\rangle = 1\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B}$. However, stars are not uniform in their properties. They have a wide range of masses and a wider range of B-band luminosities. For main sequence stars, powered by hydrogen fusion in their centers, the mass-to-light ratio ranges from $M/L_B \sim 10^{-3}\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B}$ for the brightest, most massive stars (the O stars in the classic OBAFGKM spectral sequence) to $M/L_B \sim 10^{3}\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B}$ for the dimmest, least massive stars (the M stars).

Thus, the mass-to-light ratio of the stars in a galaxy will depend on the mix of stars which it contains. As a first guess, let's suppose that the mix of stars in the solar neighborhood is not abnormal. Within 1 kiloparsec of the Sun, the mass-to-light ratio of the stars works out to be

$$\langle M/L_B\rangle \approx 4\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B} \approx 170{,}000\,\mathrm{kg\,watt}^{-1}\ . \tag{8.2}$$

Although a mass-to-light ratio of 170 tons per watt doesn't seem, at first glance, like a very high efficiency, you must remember that the mass of a star includes all the fuel which it will require during its entire lifetime.

[1] For comparison, your eyes detect photons in the wavelength range $4 \times 10^{-7}\,\mathrm{m} < \lambda < 7 \times 10^{-7}\,\mathrm{m}$.

[2] This is only 12% of the Sun's total luminosity. About 6% of the luminosity is emitted at ultraviolet wavelengths, and the remaining 82% is emitted at wavelengths too long to pass through the B filter.

If the mass-to-light ratio of the stars within a kiloparsec of us is not unusually high or low, then the mass density of stars in the universe is

$$\rho_{\star,0} = \langle M/L_B \rangle j_{\star,B} \approx 5 \times 10^8 \, \mathrm{M}_\odot \, \mathrm{Mpc}^{-3} \; . \tag{8.3}$$

Since the current critical density of the universe is equivalent to a mass density of $\rho_{c,0} = \varepsilon_{c,0}/c^2 = 1.4 \times 10^{11} \, \mathrm{M}_\odot \, \mathrm{Mpc}^{-3}$, the current density parameter of stars is

$$\Omega_{\star,0} = \frac{\rho_{\star,0}}{\rho_{c,0}} \approx \frac{5 \times 10^8 \, \mathrm{M}_\odot \, \mathrm{Mpc}^{-3}}{1.4 \times 10^{11} \, \mathrm{M}_\odot \, \mathrm{Mpc}^{-3}} \approx 0.004 \; . \tag{8.4}$$

Stars make up less than 1/2% of the density necessary to flatten the universe. In truth, the number $\Omega_{\star,0} \approx 0.004$ is not a precisely determined one, largely because of the uncertainty in the number of low-mass, low-luminosity stars in galaxies. In our Galaxy, for instance, $\sim$ 95% of the stellar luminosity comes from stars more luminous than the Sun, but $\sim$ 80% of the stellar mass comes from stars *less* luminous than the Sun. The density parameter in stars will be further increased if you include in the category of "stars" stellar remnants (such as white dwarfs, neutron stars, and black holes) and brown dwarfs. A brown dwarf is a self-gravitating ball of gas which is too low in mass to sustain nuclear fusion in its interior. Because brown dwarfs and isolated cool stellar remnants are difficult to detect, their number density is not well determined.

Galaxies also contain baryonic matter which is not in the form of stars, stellar remnants, or brown dwarfs. The interstellar medium contains significant amounts of gas. In our Galaxy and in M31, for instance, the mass of interstellar gas is roughly equal to 10% of the mass of stars. In irregular galaxies such as the Magellanic Clouds, the ratio of gas to stars is even higher. In addition, there is a significant amount of gas between galaxies. Consider a rich cluster of galaxies such as the Coma cluster, located 100 Mpc from our Galaxy, in the direction of the constellation Coma Berenices. At visible wavelengths, as shown in Figure 8.1, most of the light comes from the stars in the cluster's galaxies. The Coma cluster contains thousands of galaxies; their summed luminosity in the B band comes to $L_{\mathrm{Coma},B} = 8 \times 10^{12} \, \mathrm{L}_{\odot,B}$. If the mass-to-light ratio of the stars in the Coma cluster is $\langle M/L_B \rangle \approx 4 \, \mathrm{M}_\odot / \, \mathrm{L}_{\odot,B}$, then the total mass of stars in the Coma cluster is $M_{\mathrm{Coma},\star} \approx 3 \times 10^{13} \, \mathrm{M}_\odot$. Although 30 trillion solar masses represents a lot of stars, the stellar mass in the Coma cluster is small compared to the mass of the hot, intracluster gas between the galaxies in the cluster. X-ray images, such as the one shown in

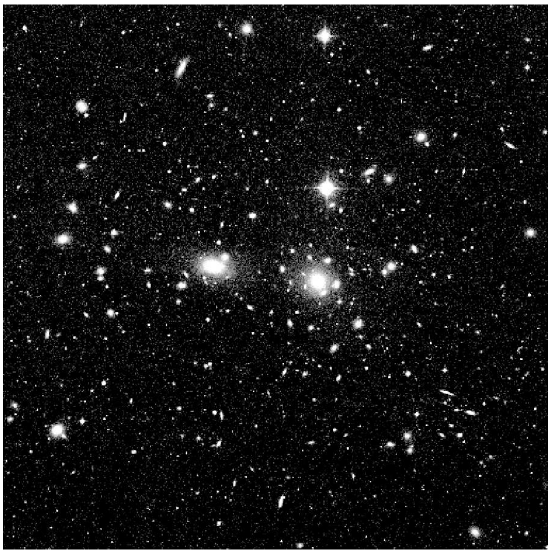

Figure 8.1: The Coma cluster as seen in visible light. The image shown is 35 arcminutes across, equivalent to $\sim 1\,\mathrm{Mpc}$ at the distance of the Coma cluster. [From the Digitized Sky Survey, produced at the Space Telescope Science Institute]

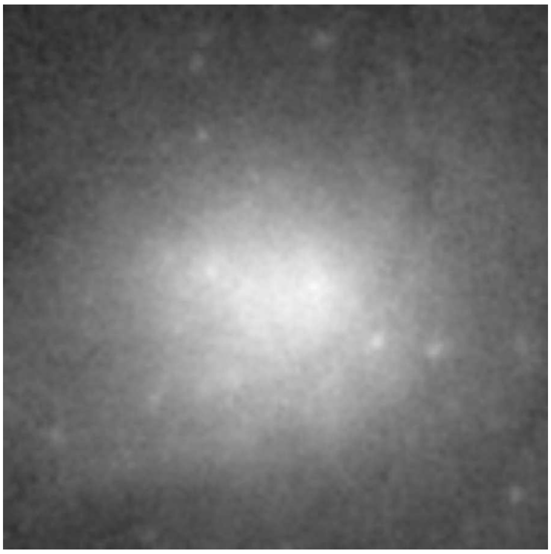

Figure 8.2: The Coma cluster as seen in x-ray light. The scale is the same as that of the previous image. [From the ROSAT x-ray observatory; courtesy Max-Planck-Institut für extraterrestriche Physik. This figure and the previous figure were produced by Raymond White, using NASA's SkyView facility.]

Figure 8.2, reveal that hot, low-density gas, with a typical temperature of $T \approx 1 \times 10^8\,\mathrm{K}$, fills the space between clusters, emitting x-rays with a typical energy of $E \sim kT_{\mathrm{gas}} \sim 9\,\mathrm{keV}$. The total amount of x-ray emitting gas in the Coma cluster is estimated to be $M_{\mathrm{Coma,gas}} \approx 2 \times 10^{14}\,\mathrm{M}_\odot$, roughly six or seven times the mass in stars.

As it turns out, the best current limits on the baryon density of the universe come from the predictions of primordial nucleosynthesis. As we will see in Chapter 10, the efficiency with which fusion takes place in the early universe, converting hydrogen into deuterium, helium, lithium, and other elements, depends on the density of protons and neutrons present. Detailed studies of the amounts of deuterium and other elements present in primordial gas clouds indicate that the density parameter of baryonic matter must be

$$\Omega_{\mathrm{bary},0} = 0.04 \pm 0.01 \ , \tag{8.5}$$

an order of magnitude larger than the density parameter for stars. When you stare up at the night sky and marvel at the glory of the stars, you are actually marveling at a minority of the baryonic matter in the universe. Most of the baryons are too cold to be readily visible (the infrared emitting brown dwarfs and cold stellar remnants) or too diffuse to be readily visible (the low density x-ray gas in clusters).

8.2 Dark matter in galaxies

The situation, in fact, is even more extreme than stated in the previous section. Not only is most of the baryonic matter undetectable by our eyes, most of the matter is not even baryonic. The majority of the matter in the universe is *nonbaryonic dark matter*, which doesn't absorb, emit, or scatter light of any wavelength. One way of detecting dark matter is to look for its gravitational influence on visible matter. A classic method of detecting dark matter involves looking at the orbital speeds of stars in spiral galaxies such as our own Galaxy and M31. Spiral galaxies contain flattened disks of stars; within the disk, stars are on nearly circular orbits around the center of the galaxy. The Sun, for instance, is on such an orbit – it is $R = 8.5\,\mathrm{kpc}$ from the Galactic center, and has an orbital speed of $v = 220\,\mathrm{km\,s^{-1}}$.

Suppose that a star is on a circular orbit around the center of its galaxy. If the radius of the orbit is R and the orbital speed is v, then the star experiences an acceleration

$$a = \frac{v^2}{R} \ , \tag{8.6}$$

directed toward the center of the galaxy. If the acceleration is provided by the gravitational attraction of the galaxy, then

$$a = \frac{GM(R)}{R^2} \ , \tag{8.7}$$

where $M(R)$ is the mass contained within a sphere of radius R centered on the galactic center.[3] The relation between v and M is found by setting equation (8.6) equal to equation (8.7):

$$\frac{v^2}{R} = \frac{GM(R)}{R^2} \ , \tag{8.8}$$

or

$$v = \sqrt{\frac{GM(R)}{R}} \ . \tag{8.9}$$

The surface brightness I of the disk of a spiral galaxy typically falls off exponentially with distance from the center:

$$I(R) = I(0)\exp(-R/R_s) \ , \tag{8.10}$$

with the scale length R_s typically being a few kiloparsecs. For our Galaxy, $R_s \approx 4\,\mathrm{kpc}$; for M31, a somewhat larger disk galaxy, $R_s \approx 6\,\mathrm{kpc}$. Once you are a few scale lengths from the center of the spiral galaxy, the mass of stars inside R becomes essentially constant. Thus, if stars contributed all, or most, of the mass in a galaxy, the velocity would fall as $v \propto 1/\sqrt{R}$ at large radii. This relation between orbital speed and orbital radius, $v \propto 1/\sqrt{R}$, is referred to as "Keplerian rotation", since it's what Kepler found for orbits in the Solar System, where the mass is strongly concentrated toward the center.[4]

The orbital speed v of stars within a spiral galaxy can be determined from observations. Consider a galaxy which has the shape of a thin circular disk. In general, we won't be seeing the disk perfectly face-on or edge-on;

[3] Equation (8.7) assumes that the mass distribution of the galaxy is spherically symmetric. This is not, strictly speaking, true (the stars in the disk obviously have a flattened distribution), but the flattening of the galaxy provides only a small correction to the equation for the gravitational acceleration.

[4] 99.8% of the Solar System's mass is contained within the Sun.

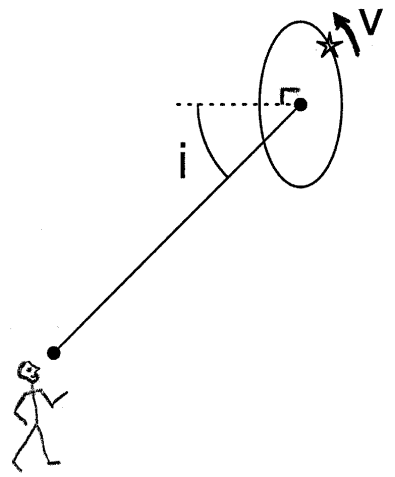

Figure 8.3: An observer sees a disk at an inclination angle i.

we'll see it at an inclination i, where i is the angle between our line of sight to the disk and a line perpendicular to the disk (see Figure 8.3). The disk we see in projection will be elliptical, not circular, with an axis ratio

$$b/a = \cos i \ . \tag{8.11}$$

For example, the galaxy M31 looks extremely elongated as seen from Earth, with an observed axis ratio $b/a = 0.22$. This indicates that we are seeing M31 fairly close to edge-on, with an inclination $i = \cos^{-1}(0.22) = 77°$. By measuring the redshift of the absorption, or emission, lines in light from the disk, we can find the radial velocity $v_r(R) = cz(R)$ along the apparent long axis of the galaxy. Since the redshift contains only the component of the stars' orbital velocity which lies along the line of sight, the radial velocity which we measure will be

$$v_r(R) = v_{\rm gal} + v(R)\sin i \ , \tag{8.12}$$

where $v_{\rm gal}$ is the radial velocity of the galaxy as a whole, resulting from the expansion of the universe, and $v(R)$ is the orbital speed at a distance R from the center of the disk. We can thus compute the orbital speed $v(R)$ in terms of observable properties as

$$v(R) = \frac{v_r(R) - v_{\rm gal}}{\sin i} = \frac{v_r(R) - v_{\rm gal}}{\sqrt{1 - b^2/a^2}} \ . \tag{8.13}$$

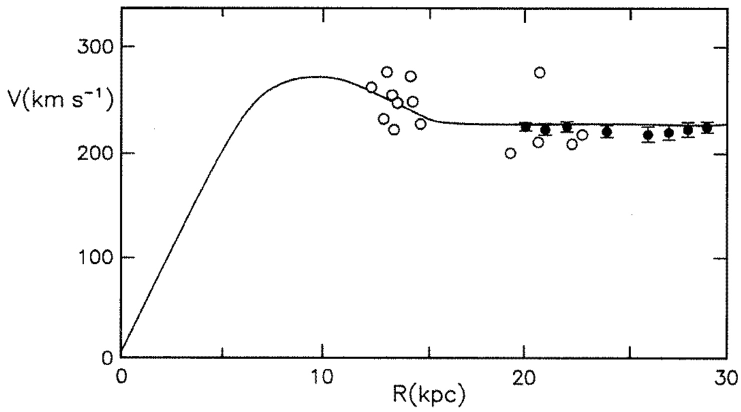

Figure 8.4: The orbital speed v as a function of radius in M31. The open circles show the results of Rubin and Ford (1970, ApJ, 159, 379) at visible wavelengths; the solid dots with error bars show the results of Roberts and Whitehurst (1975, ApJ, 201, 327) at radio wavelengths (figure from van den Bergh, 2000).

The first astronomer to detect the rotation of M31 was Vesto Slipher, in 1914. However, given the difficulty of measuring the spectra at low surface brightness, the orbital speed v at $R > 3R_s = 18\,\text{kpc}$ was not accurately measured until more than half a century later. In 1970, Vera Rubin and Kent Ford looked at emission lines from regions of hot ionized gas in M31, and were able to find the orbital speed $v(R)$ out to a radius $R = 24\,\text{kpc} = 4R_s$. Their results, shown as the open circles in Figure 8.4, give no sign of a Keplerian decrease in the orbital speed. Beyond $R = 4R_s$, the visible light from M31 was too faint for Rubin and Ford to measure the redshift; as they wrote in their original paper, "extrapolation beyond that distance is a matter of taste." At $R > 4R_s$, there is still a small amount of atomic hydrogen in the disk of M31, which can be detected by means of its emission line at $\lambda = 21\,\text{cm}$. By measuring the redshift of this emission line, M. Roberts and R. Whitehurst found that the orbital speed stayed at a nearly constant value of $v(R) \approx 230\,\text{km}\,\text{s}^{-1}$ out to $R \approx 30\,\text{kpc} \approx 5R_s$, as shown by the solid dots in Figure 8.4. Since the orbital speed of the stars and gas at large radii ($R > 3R_s$) is greater than it would be if stars and gas were the only matter present, we deduce the presence of a *dark halo* within which the visible stellar disk is embedded. The mass of the dark halo provides the necessary gravitational "anchor" to keep the high-speed stars and gas from being flung out into intergalactic space.

M31 is not a freak; most, if not all, spiral galaxies have comparable dark halos. For instance, our own Galaxy has an orbital speed which actually seems to be rising slightly at $R > 15\,\mathrm{kpc}$, instead of decreasing in a Keplerian fashion. Thousands of spiral galaxies have had their orbital velocities $v(R)$ measured; typically, v is roughly constant at $R > R_s$. If we approximate the orbital speed v as being constant with radius, the mass of a spiral galaxy, including both the luminous disk and the dark halo, can be found from equation (8.9):

$$M(R) = \frac{v^2 R}{G} = 9.6 \times 10^{10}\,\mathrm{M}_\odot \left(\frac{v}{220\,\mathrm{km\,s^{-1}}}\right)^2 \left(\frac{R}{8.5\,\mathrm{kpc}}\right) . \tag{8.14}$$

The values of v and R in the above equation are scaled to the Sun's location in our Galaxy. Since our Galaxy's luminosity in the B band is estimated to be $L_{\mathrm{Gal},B} = 2.3 \times 10^{10}\,\mathrm{L}_{\odot,B}$, this means that the mass-to-light ratio of our Galaxy, taken as a whole, is

$$\langle M/L_B \rangle_{\mathrm{Gal}} \approx 50\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B} \left(\frac{R_{\mathrm{halo}}}{100\,\mathrm{kpc}}\right) , \tag{8.15}$$

using $v = 220\,\mathrm{km\,s^{-1}}$ in equation (8.14). The quantity R_{halo} is the radius of the dark halo surrounding the luminous disk of our galaxy. The exact value of R_{halo} is poorly known. At $R \approx 20\,\mathrm{kpc}$, where the last detectable gas exists in the disk of our Galaxy, the orbital speed shows no sign of a Keplerian decrease; thus, $R_{\mathrm{halo}} > 20\,\mathrm{kpc}$. A rough estimate of the halo size can be made by looking at the velocities of the globular clusters and satellite galaxies (such as the Magellanic Clouds) which orbit our Galaxy. For these hangers-on to remain gravitationally bound to our Galaxy, the halo must extend as far as $R_{\mathrm{halo}} \approx 75\,\mathrm{kpc}$, implying a total mass for our Galaxy of $M_{\mathrm{Gal}} \approx 8 \times 10^{11}\,\mathrm{M}_\odot$, and a total mass-to-light ratio $\langle M/L_B \rangle_{\mathrm{Gal}} \approx 40\,\mathrm{M}/\,\mathrm{L}_{\odot,B}$. This mass-to-light ratio is ten times greater than that of the stars in our Galaxy, implying that the dark halo is an order of magnitude more massive than the stellar disk. Some astronomers have speculated that the dark halo is actually four times larger in radius, with $R_{\mathrm{halo}} \approx 300\,\mathrm{kpc}$; this would mean that our halo stretches nearly halfway to M31. With $R_{\mathrm{halo}} \approx 300\,\mathrm{kpc}$, the mass of our Galaxy would be $M_{\mathrm{Gal}} \approx 3 \times 10^{12}\,\mathrm{M}_\odot$, and the total mass-to-light ratio would be $\langle M/L_B \rangle_{\mathrm{Gal}} \approx 150\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B}$.

If our Galaxy is typical in having a dark halo 10 to 40 times more massive than its stellar component, then the density parameter of galaxies (including their dark halos) must be

$$\Omega_{\rm gal,0} = (10 \rightarrow 40)\Omega_{\star,0} \approx 0.04 \rightarrow 0.16 \ . \tag{8.16}$$

Although the total density of galaxies is poorly known, given the uncertainty in the extent of their dark halos, it is likely to be larger than the density of baryons, $\Omega_{\rm bary,0} = 0.04 \pm 0.01$. Thus, some part of the dark halos of galaxies is likely to be comprised of *nonbaryonic* dark matter.

8.3 Dark matter in clusters

The first astronomer to make a compelling case for the existence of large quantities of dark matter was Fritz Zwicky, in the 1930's. In studying the Coma cluster of galaxies (shown in Figure 8.1), he noted that the dispersion in the radial velocity of the cluster's galaxies was very large – around $1000\,{\rm km\,s^{-1}}$. The stars and gas visible within the galaxies simply did not provide enough gravitational attraction to hold the cluster together. In order to keep the galaxies in the Coma cluster from flying off into the surrounding voids, Zwicky concluded, the cluster must contain a large amount of "dunkle Materie", or (translated into English) "dark matter".[5]

To follow, at a more mathematical level, Zwicky's reasoning, let us suppose that a cluster of galaxies is comprised of N galaxies, each of which can be approximated as a point mass, with a mass m_i ($i = 1, 2, \ldots, N$), a position $\vec{x}_i$, and a velocity $\dot{\vec{x}}_i$. Clusters of galaxies are gravitationally bound objects, not expanding with the Hubble flow. They are small compared to the horizon size; the radius of the Coma cluster is $R_{\rm Coma} \approx 3\,{\rm Mpc} \approx 0.0002 d_{\rm hor}$. The galaxies within a cluster are moving at non-relativistic speeds; the velocity dispersion within the Coma cluster is $\sigma_{\rm Coma} \approx 1000\,{\rm km\,s^{-1}} \approx 0.003c$. Because of these considerations, we can treat the dynamics of the Coma cluster, and other clusters of galaxies, in a Newtonian manner. The acceleration of the

[5]Although Zwicky's work popularized the phrase "dark matter", he was not the first to use it in an astronomical context. For instance, in 1908, Henri Poincaré discussed the possible existence within our Galaxy of "matière obscure" (translated as "dark matter" in the standard edition of Poincaré's works).

i^{th} galaxy in the cluster, then, is given by the Newtonian formula

$$\ddot{\vec{x}}_i = G \sum_{j \neq i} m_j \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3} \ . \tag{8.17}$$

Note that equation (8.17) assumes that the cluster is an isolated system, with the gravitational acceleration due to matter outside the cluster being negligibly small.

The gravitational *potential energy* of the system of N galaxies is

$$W = -\frac{G}{2} \sum_{\substack{i,j \\ j \neq i}} \frac{m_i m_j}{|\vec{x}_j - \vec{x}_i|} \ . \tag{8.18}$$

This is the energy that would be required to pull the N galaxies away from each other so that they would all be at infinite distance from each other. (The factor of $1/2$ in front of the double summation ensures that each pair of galaxies is only counted once in computing the potential energy.) The potential energy of the cluster can also be written in the form

$$W = -\alpha \frac{GM^2}{r_h} \ , \tag{8.19}$$

where $M = \sum m_i$ is the total mass of all the galaxies in the cluster, α is a numerical factor of order unity which depends on the density profile of the cluster. and r_h is the *half-mass* radius of the cluster – that is, the radius of a sphere centered on the cluster's center of mass and containing a mass $M/2$. For observed clusters of galaxies, it is found that $\alpha \approx 0.4$ gives a good fit to the potential energy.

The *kinetic energy* associated with the relative motion of the galaxies in the cluster is

$$K = \frac{1}{2} \sum_i m_i |\dot{\vec{x}}_i|^2 \ . \tag{8.20}$$

The kinetic energy K can also be written in the form

$$K = \frac{1}{2} M \langle v^2 \rangle \ , \tag{8.21}$$

where

$$\langle v^2 \rangle \equiv \frac{1}{M} \sum_i m_i |\dot{\vec{x}}_i|^2 \tag{8.22}$$

is the mean square velocity (weighted by galaxy mass) of all the galaxies in the cluster.

It is also useful to define the *moment of inertia* of the cluster as

$$I \equiv \sum_i m_i |\vec{x}_i|^2 \ . \tag{8.23}$$

The moment of inertia I can be linked to the kinetic energy and the potential energy if we start by taking the second time derivative of I:

$$\ddot{I} = 2\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i + \dot{\vec{x}}_i \cdot \dot{\vec{x}}_i) \ . \tag{8.24}$$

Using equation (8.20), we can rewrite this as

$$\ddot{I} = 2\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) + 4K \ . \tag{8.25}$$

To introduce the potential energy W into the above relation, we can use equation (8.17) to write

$$\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) = G \sum_{\substack{i,j \\ j\neq i}} m_i m_j \frac{\vec{x}_i \cdot (\vec{x}_j - \vec{x}_i)}{|\vec{x}_j - \vec{x}_i|^3} \ . \tag{8.26}$$

However, we could equally well switch around the i and j subscripts to find the equally valid equation

$$\sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j) = G \sum_{\substack{j,i \\ i\neq j}} m_j m_i \frac{\vec{x}_j \cdot (\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \ . \tag{8.27}$$

Since

$$\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) = \sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j) \tag{8.28}$$

(it doesn't matter whether we call the variable over which we're summing i or j or k or "Fred"), we can combine equations (8.26) and (8.27) to find

$$\begin{aligned} \sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) &= \frac{1}{2}\left[\sum_i m_i(\vec{x}_i \cdot \ddot{\vec{x}}_i) + \sum_j m_j(\vec{x}_j \cdot \ddot{\vec{x}}_j)\right] \\ &= -\frac{G}{2}\sum_{\substack{i,j \\ j\neq i}} \frac{m_i m_j}{|\vec{x}_j - \vec{x}_i|} = W \ . \end{aligned} \tag{8.29}$$

Thus, the first term on the right hand side of equation (8.25) is simply $2W$, and we may now write down the simple relation

$$\ddot{I} = 2W + 4K \ . \tag{8.30}$$

This relation is known as the *virial theorem*. It was actually first derived in the nineteenth century in the context of the kinetic theory of gases, but as we have seen, it applies perfectly well to a self-gravitating system of point masses.

The virial theorem is particularly useful when it is applied to a system in steady state, with a constant moment of inertia. (This implies, among other things, that the system is neither expanding nor contracting, and that you are using a coordinate system in which the center of mass of the cluster is at rest.) If $I =$ constant, then the *steady-state virial theorem* is

$$0 = W + 2K \ , \tag{8.31}$$

or

$$K = -W/2 \ . \tag{8.32}$$

That is, for a self-gravitating system in steady state, the kinetic energy K is equal to $-1/2$ times the potential energy W. Using equation (8.19) and (8.21) in equation (8.32), we find

$$\frac{1}{2}M\langle v^2\rangle = \frac{\alpha}{2}\frac{GM^2}{r_h} \ . \tag{8.33}$$

This means we can use the virial theorem to estimate the mass of a cluster of galaxies, or any other self-gravitating steady-state system:

$$M = \frac{\langle v^2\rangle r_h}{\alpha G} \ . \tag{8.34}$$

Note the similarity between equation (8.14), used to estimate the mass of a rotating spiral galaxy, and equation (8.34), used to estimate the mass of a cluster of galaxies. In either case, we estimate the mass of a self-gravitating system by multiplying the square of a characteristic velocity by a characteristic radius, then dividing by the gravitational constant G.

Applying the virial theorem to a real cluster of galaxies, such as the Coma cluster, is complicated by the fact that we have only partial information about the cluster, and thus do not know $\langle v^2\rangle$ and r_h exactly. For instance, we can find the line-of-sight velocity of each galaxy from its redshift, but the velocity perpendicular to the line of sight is unknown. From measurements of the redshifts of hundreds of galaxies in the Coma cluster, the mean redshift of the cluster is found to be

$$\langle z \rangle = 0.0232 \ , \tag{8.35}$$

which can be translated into a radial velocity

$$\langle v_r \rangle = c\langle z \rangle = 6960\,\mathrm{km\,s^{-1}} \tag{8.36}$$

and a distance

$$d_{\mathrm{Coma}} = (c/H_0)\langle z \rangle = 99\,\mathrm{Mpc} \ . \tag{8.37}$$

The velocity dispersion of the cluster along the line of sight is found to be

$$\sigma_r = \langle (v_r - \langle v_r \rangle)^2 \rangle^{1/2} = 880\,\mathrm{km\,s^{-1}} \ . \tag{8.38}$$

If we assume that the velocity dispersion is isotropic, then the three-dimensional mean square velocity $\langle v^2 \rangle$ will be equal to three times the one-dimensional mean square velocity σ_r^2, yielding

$$\langle v^2 \rangle = 3(880\,\mathrm{km\,s^{-1}})^2 = 2.32 \times 10^{12}\,\mathrm{m^2\,s^{-2}} \ . \tag{8.39}$$

Estimating the half-mass radius r_h of the Coma cluster is even more peril-ridden than estimating the mean square velocity $\langle v^2 \rangle$. After all, we don't know the distribution of dark matter in the cluster *a priori*; in fact, the total amount of dark matter is what we're trying to find out. However, if we assume that the mass-to-light ratio is constant with radius, then the sphere containing half the mass of the cluster will be the same as the sphere containing half the luminosity of the cluster. If we further assume that the cluster is intrinsically spherical, then the observed distribution of galaxies within the Coma cluster indicates a half-mass radius

$$r_h \approx 1.5\,\mathrm{Mpc} \approx 4.6 \times 10^{22}\,\mathrm{m} \ . \tag{8.40}$$

After all these assumptions and approximations, we may estimate the mass of the Coma cluster to be

$$M_{\mathrm{Coma}} = \frac{\langle v^2 \rangle r_h}{\alpha G} \approx \frac{(2.32 \times 10^{12}\,\mathrm{m^2\,s^{-2}})(4.6 \times 10^{22}\,\mathrm{m})}{(0.4)(6.7 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}})} \tag{8.41}$$

$$\approx 4 \times 10^{45}\,\mathrm{kg} \approx 2 \times 10^{15}\,\mathrm{M_\odot} \ . \tag{8.42}$$

Thus, less than two percent of the mass of the Coma cluster consists of stars ($M_{\rm Coma,\star} \approx 3 \times 10^{13}\,{\rm M}_\odot$), and only ten percent consists of hot intracluster gas ($M_{\rm Coma,gas} \approx 2 \times 10^{14}\,{\rm M}_\odot$). Combined with the luminosity of the Coma cluster, $L_{{\rm Coma},B} = 8 \times 10^{12}\,{\rm L}_{\odot,B}$, the total mass of the Coma cluster implies a mass-to-light ratio of

$$\langle M/L_B\rangle_{\rm Coma} \approx 250\,{\rm M}_\odot/\,{\rm L}_{\odot,B} \ , \tag{8.43}$$

greater than the mass-to-light ratio of our Galaxy.

The presence of a vast reservoir of dark matter in the Coma cluster is confirmed by the fact that the hot, x-ray emitting intracluster gas, shown in Figure 8.2, is still in place; if there were no dark matter to anchor the gas gravitationally, the hot gas would have expanded beyond the cluster on time scales much shorter than the Hubble time. The temperature and density of the hot gas in the Coma cluster can be used to make yet another estimate of the cluster's mass. If the hot intracluster gas is supported by its own pressure against gravitational infall, it must obey the equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} \ , \tag{8.44}$$

where P is the pressure of the gas, ρ is the density of the gas, and M is the *total* mass inside a sphere of radius r, including gas, stars, dark matter, lost socks, and anything else.[6] Of course, the gas in Coma isn't perfectly spherical in shape, as equation (8.44) assumes, but it's close enough to spherical to give a reasonable approximation to the mass.

The pressure of the gas is given by the perfect gas law,

$$P = \frac{\rho kT}{\mu m_p} \ , \tag{8.45}$$

where T is the temperature of the gas, and μ is its mass in units of the proton mass (m_p). The mass of the cluster, as a function of radius, is found by combining equations (8.44) and (8.45):

$$M(r) = \frac{kT(r)r}{G\mu m_p}\left[-\frac{d\ln\rho}{d\ln r} - \frac{d\ln T}{d\ln r}\right] \ . \tag{8.46}$$

[6]Equation (8.44) is the same equation which determines the internal structure of a star, where the inward force due to gravity is also exactly balanced by an outward force due to a pressure gradient.

The above equation assumes that μ is constant with radius, as you'd expect if the chemical composition and ionization state of the gas is uniform throughout the cluster.

The x-rays emitted from the hot intracluster gas are a combination of bremsstrahlung emission (caused by the acceleration of free electrons by protons and helium nuclei) and line emission from highly ionized iron and other heavy elements. Starting from an x-ray spectrum, it is possible to fit models to the emission and thus compute the temperature, density, and chemical composition of the gas. In the Coma cluster, for instance, temperature maps reveal relatively cool regions (at $kT \approx 5\,\mathrm{keV}$) as well as hotter regions (at $kT \approx 12\,\mathrm{keV}$), averaging to $kT \approx 9\,\mathrm{keV}$ over the entire cluster. The mass of the Coma cluster, assuming hydrostatic equilibrium, is computed to be $(3 \to 4) \times 10^{14}\,\mathrm{M}_\odot$ within 0.7 Mpc of the cluster center and $(1 \to 2) \times 10^{15}\,\mathrm{M}_\odot$ within 3.6 Mpc of the center, consistent with the mass estimate of the virial theorem.

Other clusters of galaxies besides the Coma cluster have had their masses estimated, using the virial theorem applied to their galaxies or the equation of hydrostatic equilibrium applied to their gas. Typical mass-to-light ratios for clusters lie in the range $\langle M/L_B \rangle = 200 \to 300\,\mathrm{M}_\odot/\,\mathrm{L}_{\odot,B}$, so the Coma cluster is not unusual in the amount of dark matter which it contains. If the masses of all the clusters of galaxies are added together, it is found that their density parameter is

$$\Omega_{\mathrm{clus},0} \approx 0.2 \ . \tag{8.47}$$

This provides a *lower limit* to the matter density of the universe, since any matter which is smoothly distributed in the intercluster voids will not be included in this number.

8.4 Gravitational lensing

So far, I have outlined the classical methods for detecting dark matter via its gravitational effects on luminous matter.[7] We can detect dark matter around spiral galaxies because it affects the motions of stars and interstellar gas. We can detect dark matter in clusters of galaxies because it affects the motions of

[7] The roots of these methods can be traced back as far as the year 1846, when Leverrier and Adams deduced the existence of the dim planet Neptune by its effect on the orbit of Uranus.

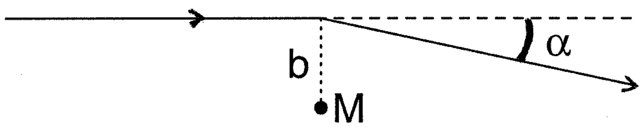

Figure 8.5: Deflection of light by a massive compact object.

galaxies and intracluster gas. However, as Einstein realized, dark matter will affect not only the trajectory of matter, but also the trajectory of photons. Thus, dark matter can bend and focus light, acting as a *gravitational lens*. The effects of dark matter on photons have been used to search for dark matter within the halo of our own Galaxy, as well as in distant clusters of galaxies.

To see how gravitational lensing can be used to detect dark matter, start by considering the dark halo surrounding our Galaxy. Some of the dark matter in the halo might consist of massive compact objects such as brown dwarfs, white dwarfs, neutron stars, and black holes. These objects have been collectively called MACHOs, a slightly strained acronym for MAssive Compact Halo Objects. If a photon passes such a compact massive object at an impact parameter b, as shown in Figure 8.5, the local curvature of space-time will cause the photon to be deflected by an angle

$$\alpha = \frac{4GM}{c^2 b} \,, \tag{8.48}$$

where M is the mass of the compact object. For instance, light from a distant star which just grazes the Sun's surface should be deflected through an angle

$$\alpha = \frac{4G\,\mathrm{M}_\odot}{c^2\,\mathrm{R}_\odot} = 1.7\,\mathrm{arcsec}\ . \tag{8.49}$$

In 1919, after Einstein predicted a deflection of this magnitude, an eclipse expedition photographed stars in the vicinity of the Sun. Comparison of the eclipse photographs with photographs of the same star field taken six months earlier revealed that the apparent positions of the stars were deflected by the amount which Einstein had predicted. This result brought fame to Einstein and experimental support to the theory of general relativity.

Since a star, or a brown dwarf, or a stellar remnant, can deflect light, it can act as a lens. Suppose a MACHO in the halo of our Galaxy passes

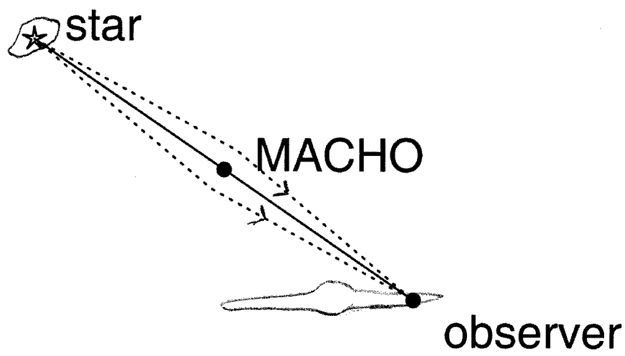

Figure 8.6: Light from star in the Large Magellanic Cloud is deflected by a MACHO on its way to an observer in the disk of our Galaxy (seen edge-on in this figure).

directly between an observer in our Galaxy and a star in the Large Magellanic Cloud. Figure 8.6 shows such a situation, with a MACHO which happens to be halfway between the observer and the star. As the MACHO deflects the light from the distant star, it produces an image of the star which is both distorted and amplified. If the MACHO is *exactly* along the line of sight between the observer and the lensed star, the image produced is a perfect ring, with angular radius

$$\theta_E = \left(\frac{4GM}{c^2 d} \frac{1-x}{x} \right)^{1/2} , \qquad (8.50)$$

where M is the mass of the lensing MACHO, d is the distance from the observer to the lensed star, and xd (where $0 < x < 1$) is the distance from the observer to the lensing MACHO. The angle θ_E is known as the *Einstein radius.* If $x \approx 0.5$ (that is, if the MACHO is roughly halfway between the observer and the lensed star), then

$$\theta_E \approx 4 \times 10^{-4} \text{arcsec} \left(\frac{M}{1\,\mathrm{M}_\odot} \right)^{1/2} \left(\frac{d}{50\,\mathrm{kpc}} \right)^{-1/2} . \qquad (8.51)$$

If the MACHO does not lie perfectly along the line of sight to the star, then the image of the star will be distorted into two or more arcs instead of a single unbroken ring. Although the Einstein radius for an LMC star being lensed by a MACHO is too small to be resolved, it is possible, in some cases, to detect the amplification of the flux from the star. For the amplification to be significant, the angular distance between the MACHO and the lensed star, as seen from Earth, must be comparable to, or smaller than, the Einstein radius. Given the small size of the Einstein radius, the probability of any particular star in the LMC being lensed at any moment is tiny. It has been calculated that if the dark halo of our galaxy were entirely composed of MACHOs, then the probability of any given star in the LMC being lensed at any given time would still only be $P \sim 5 \times 10^{-7}$.

To detect lensing by MACHOs, various research groups took up the daunting task of monitoring millions of stars in the Large Magellanic Cloud to watch for changes in their flux. Since the MACHOs in our dark halo and the stars in the LMC are in constant relative motion, the typical signature of a "lensing event" is a star which becomes brighter as the angular distance between star and MACHO decreases, then becomes dimmer as the angular distance increases again. The typical time scale for a lensing event is the time it takes a MACHO to travel through an angular distance equal to θ_E as seen from Earth; for a MACHO halfway between here and the LMC, this is

$$\Delta t = \frac{d\,\theta_E}{2v} \approx 90\,\mathrm{days}\left(\frac{M}{1\,\mathrm{M}_\odot}\right)^{1/2}\left(\frac{v}{200\,\mathrm{km\,s^{-1}}}\right)^{-1} , \tag{8.52}$$

where v is the relative transverse velocity of the MACHO and the lensed star as seen by the observer on Earth. Generally speaking, more massive MACHOs produce larger Einstein rings and thus will amplify the lensed star for a longer time.

The research groups which searched for MACHOs found a scarcity of short duration lensing events, suggesting that there is not a significant population of brown dwarfs (with $M < 0.08\,\mathrm{M}_\odot$) in the dark halo of our Galaxy. The total number of lensing events which they detected suggest that as much as 20% of the halo mass could be in the form of MACHOs. The long time scales of the observed lensing events, which have $\Delta t > 35\,\mathrm{days}$, suggest typical MACHO masses of $M > 0.15\,\mathrm{M}_\odot$. (Perhaps the MACHOs are old, cold white dwarfs, which would have the correct mass.) Alternatively, the observed lensing events could be due, at least in part, to lensing objects within the LMC itself. In any case, the search for MACHOs suggests that most of the matter in the dark halo of our galaxy is due to a smoothly distributed component, instead of being congealed into MACHOs of roughly stellar mass.

Gravitational lensing occurs at all mass scales. Suppose, for instance, that a cluster of galaxies, with $M \sim 10^{14}\,\mathrm{M}_\odot$, at a distance $\sim 500\,\mathrm{Mpc}$ from

Figure 8.7: A Hubble Space Telescope picture of the rich cluster Abell 2218, displaying gravitationally lensed arcs. The region shown is roughly 2.4 arcmin by 1.2 arcmin, equivalent to 0.54 Mpc by 0.27 Mpc at the distance of Abell 2218 (courtesy of W. Couch [University of New South Wales] and NASA).

our Galaxy, lenses a background galaxy at $d \sim 1000\,\mathrm{Mpc}$. The Einstein radius for this configuration will be

$$\theta_E \approx 0.5\,\mathrm{arcmin}\left(\frac{M}{10^{14}\,\mathrm{M}_\odot}\right)^{1/2}\left(\frac{d}{1000\,\mathrm{Mpc}}\right)^{-1/2} . \tag{8.53}$$

The arc-shaped images into which the background galaxy is distorted by the lensing cluster can thus be resolved. For instance, Figure 8.7 shows an image of the cluster Abell 2218, which has a redshift $z = 0.18$, and hence is at a proper distance $d = 770\,\mathrm{Mpc}$. The elongated arcs seen in Figure 8.7 are not oddly shaped galaxies within the cluster; instead, they are background galaxies, at redshifts $z > 0.18$, which are gravitationally lensed by the cluster mass. The mass of clusters can be estimated by the degree to which they lens background galaxies. The masses calculated in this way are in general agreement with the masses found by applying the virial theorem to the motions of galaxies in the cluster or by applying the equation of hydrostatic equilibrium to the hot intracluster gas.

8.5 What's the matter?

I have described how to detect dark matter by its gravitational effects, but I've been dodging the essential question: "*What is it?*" Adding together the masses of clusters of galaxies gives a lower limit on the matter density of the universe, telling us that $\Omega_{m,0} \geq 0.2$. However, the density parameter of baryonic matter is only $\Omega_{\rm bary,0} \approx 0.04$. Thus, the density of nonbaryonic matter is at least four times the density of the familiar baryonic matter of which people and planets and stars are made.

As you might expect, conjecture about the nature of the nonbaryonic dark matter has run rampant (some might even say it has run amok). A component of the universe which is totally invisible is an open invitation to speculation. To give a taste of the variety of speculation, some scientists have proposed that the dark matter might be made of axions, a type of elementary particle with a rest energy of $m_{\rm ax}c^2 \sim 10^{-5}\,{\rm eV}$, equivalent to $m_{\rm ax} \sim 2 \times 10^{-41}\,{\rm kg}$. This is a rather low mass – it would take some 50 billion axions (if they indeed exist) to equal the mass of one electron. On the other hand, some scientists have conjectured that the dark matter might be made of primordial black holes, with masses up to $m_{\rm BH} \sim 10^5\,{\rm M}_\odot$, equivalent to $m_{\rm BH} \sim 2 \times 10^{35}\,{\rm kg}$.[8] This is a rather high mass – it would take some 30 billion Earths to equal the mass of one primordial black hole (if they indeed exist). It is a sign of the vast ignorance concerning nonbaryonic dark matter that these two candidates for the role of dark matter differ in mass by 76 orders of magnitude.

One nonbaryonic particle which we know exists, and which seems to have a non-zero mass, is the neutrino. As stated in section 5.1, there should exist today a cosmic background of neutrinos. Just as the Cosmic Microwave Background is a relic of the time when the universe was opaque to photons, the Cosmic Neutrino Background is a relic of the time when the universe was hot and dense enough to be opaque to neutrinos. The number density of each of the three flavors of neutrinos (ν_e, ν_μ, and ν_τ) has been calculated to be 3/11 times the number density of CMB photons, yielding a total number density of neutrinos

$$n_\nu = 3(3/11)n_\gamma = (9/11)(4.11 \times 10^8\,{\rm m}^{-3}) = 3.36 \times 10^8\,{\rm m}^{-3}\ . \tag{8.54}$$

[8]A *primordial* black hole is one which forms very early in the history of the universe, rather than by the collapse of a massive star later on.

This means that at any moment, about twenty million cosmic neutrinos are zipping through your body, "like photons through a pane of glass". In order to provide *all* the nonbaryonic mass in the universe, the average neutrino mass would have to be

$$m_\nu c^2 = \frac{\Omega_{\mathrm{dm},0}\varepsilon_{c,0}}{n_\nu} \ . \tag{8.55}$$

Given a density parameter in nonbaryonic dark matter of $\Omega_{\mathrm{dm},0} \approx 0.26$, this implies that a mean neutrino mass of

$$m_\nu c^2 \approx \frac{0.26(5200\,\mathrm{MeV\,m^{-3}})}{3.36 \times 10^8\,\mathrm{m^{-3}}} \approx 4\,\mathrm{eV} \tag{8.56}$$

would be necessary to provide all the nonbaryonic dark matter in the universe.

Evidence indicates that neutrinos do have some mass. But how much? Enough to contribute significantly to the energy density of the universe? The observations of neutrinos from the Sun, as mentioned in section 2.4, indicate that electron neutrinos oscillate into some other flavor of neutrino, with the difference in the squares of the masses of the two neutrinos being $\Delta(m_\nu^2 c^4) \approx 3 \times 10^{-5}\,\mathrm{eV}^2$. Observations of muon neutrinos created in the Earth's atmosphere indicate that muon neutrinos oscillate into tau neutrinos, with $\Delta(m_\nu^2 c^4) \approx 3 \times 10^{-3}\,\mathrm{eV}^2$. The minimum neutrino masses consistent with these results would have one flavor with $m_\nu c^2 \sim 0.05\,\mathrm{eV}$, another with $m_\nu c^2 \sim 0.005\,\mathrm{eV}$, and the third with $m_\nu c^2 \ll 0.005\,\mathrm{eV}$. If the neutrino masses are this small, then the density parameter in neutrinos is only $\Omega_\nu \sim 10^{-3}$, and neutrinos make up less than 0.5% of the nonbaryonic dark matter. If, on the other hand, neutrinos make up all the nonbaryonic dark matter, the masses of the three species would have to be very nearly identical; for instance, one neutrino flavor with $m_\nu c^2 = 4.0\,\mathrm{eV}$, another with $m_\nu c^2 = 4.0004\,\mathrm{eV}$, and the third with $m_\nu c^2 = 4.000004\,\mathrm{eV}$ would be in agreement with the deduced values of $\Delta(m_\nu^2 c^4)$.

If the masses of all three neutrinos turn out to be significantly less than $m_\nu c^2 \sim 4\,\mathrm{eV}$, then the bulk of the nonbaryonic dark matter in the universe must be made of some particle other than neutrinos. Particle physicists have provided several possible candidates for the role of dark matter. For instance, consider the extension of the Standard Model of particle physics known as supersymmetry. Various supersymmetric models predict the existence of massive nonbaryonic particles such as photinos, gravitinos, axinos, sneutrinos, gluinos, and so forth. None of these "inos" have yet been detected in laboratories. The fact that supersymmetric particles such as photinos have not yet been seen in particle accelerator experiments means that they must be massive (if they exist), with $mc^2 > 10\,\mathrm{GeV}$.

Like neutrinos, the hypothetical supersymmetric particles interact with other particles only through gravity and through the weak nuclear force, which makes them intrinsically difficult to detect. Particles which interact via the weak nuclear force, but which are much more massive than the upper limit on the neutrino mass, are known generically as Weakly Interacting Massive Particles, or WIMPs.[9] Since WIMPs, like neutrinos, do interact with atomic nuclei on occasion, experimenters have set up WIMP detectors to discover cosmic WIMPs. So far, no convincing detections have been made – but the search goes on.

Suggested reading

[Full references are given in the "Annotated Bibliography" on page 286.]

Liddle (1999), ch. 8: A brief sketch of methods for detecting dark matter.

Peacock (1999), ch. 12: Dark matter in the universe, both baryonic and nonbaryonic. Also, chapter 4 gives a good review of gravitational lensing.

Rich (2001), ch. 2.4: A discussion of the dark matter candidates.

Problems

(8.1) Suppose it were suggested that black holes of mass $10^{-8}\,\mathrm{M}_\odot$ made up all the dark matter in the halo of our Galaxy. How far away would you expect the nearest such black hole to be? How frequently would you expect such a black hole to pass within 1 AU of the Sun? (An order-of-magnitude estimate is sufficient.)

Suppose it were suggested that MACHOs of mass $10^{-3}\,\mathrm{M}_\odot$ (about the mass of Jupiter) made up all the dark matter in the halo of our Galaxy. How far away would you expect the nearest MACHO to be? How

[9] The acronym "MACHO", encountered in the previous section, was first coined as a humorous riposte to the acronym "WIMP".

frequently would such a MACHO pass within 1 AU of the Sun? (Again, an order-of-magnitude estimate will suffice.)

(8.2) The Draco galaxy is a dwarf galaxy within the Local Group. Its luminosity is $L = (1.8 \pm 0.8) \times 10^5\,\mathrm{L}_\odot$ and half its total luminosity is contained within a sphere of radius $r_h = 120 \pm 12\,\mathrm{pc}$. The red giant stars in the Draco galaxy are bright enough to have their line-of-sight velocities measured. The measured velocity dispersion of the red giant stars in the Draco galaxy is $\sigma_r = 10.5 \pm 2.2\,\mathrm{km\,s^{-1}}$. What is the mass of the Draco galaxy? What is its mass-to-light ratio? Describe the possible sources of error in your mass estimate of this galaxy.

(8.3) A light ray just grazes the surface of the Earth ($M = 6.0 \times 10^{24}\,\mathrm{kg}$, $R = 6.4 \times 10^6\,\mathrm{m}$). Through what angle α is the light ray bent by gravitational lensing? (Ignore the refractive effects of the Earth's atmosphere.) Repeat your calculation for a white dwarf ($M = 2.0 \times 10^{30}\,\mathrm{kg}$, $R = 1.5 \times 10^7\,\mathrm{m}$) and for a neutron star ($M = 3.0 \times 10^{30}\,\mathrm{kg}$, $R = 1.2 \times 10^4\,\mathrm{m}$).

(8.4) If the halo of our Galaxy is spherically symmetric, what is the mass density $\rho(r)$ within the halo? If the universe contains a cosmological constant with density parameter $\Omega_{\Lambda,0} = 0.7$, would you expect it to significantly affect the dynamics of our Galaxy's halo? Explain why or why not.

(8.5) In the previous chapter, I noted that galaxies in rich clusters are poor standard candles, because they tend to grow brighter with time as they merge with other galaxies. Let's estimate the galaxy merger rate in the Coma cluster to see whether it's truly significant. The Coma cluster contains $N \approx 1000$ galaxies within its half-mass radius of $r_h \approx 1.5\,\mathrm{Mpc}$. What is the mean number density of galaxies within the half-mass radius? Suppose that the typical cross-section of a galaxy is $\Sigma \approx 10^{-3}\,\mathrm{Mpc}^2$. How far will a galaxy in the Coma cluster travel, on average, before it collides with another galaxy? The velocity dispersion of the Coma cluster is $\sigma \approx 880\,\mathrm{km\,s^{-3}}$. What is the average time between collisions for a galaxy in the Coma cluster? Is this time greater than or less than the Hubble time?