Chapter 12

The Formation of Structure

The universe can be approximated as being homogeneous and isotropic only if we smooth it with a filter $\sim 100\,\text{Mpc}$ across. On smaller scales, the universe contains density fluctuations ranging from subatomic quantum fluctuations up to the large superclusters and voids, $\sim 50\,\text{Mpc}$ across, which characterize the distribution of galaxies in space. If we relax the strict assumption of homogeneity and isotropy which underlies the Robertson-Walker metric and the Friedmann equation, we can ask (and, to some extent, answer) the question, "How do density fluctuations in the universe evolve with time?"

The formation of relatively small objects, such as planets, stars, or even galaxies, involves some fairly complicated physics. Consider a galaxy, for instance. As mentioned in Chapter 8, the luminous portions of galaxies are typically much smaller than the dark halos in which they are embedded. In the usual scenario for galaxy formation, this is because the baryonic component of a galaxy radiates away energy, in the form of photons, and slides to the bottom of the potential well defined by the dark matter. The baryonic gas then fragments to form stars, in a nonlinear magnetohydrodynamical process.

In this chapter, however, I will be focusing on the formation of structures larger than galaxies – clusters, superclusters, and voids. Cosmologists use the term "large scale structure of the universe" to refer to all structures bigger than individual galaxies. A map of the large scale structure of the universe, as traced by the positions of galaxies, can be made by measuring the redshifts of a sample of galaxies and using the Hubble relation, $d = (c/H_0)z$, to compute their distances from our own Galaxy. For instance, Figure 12.1 shows a redshift map from the 2dF Galaxy Redshift Survey. By plotting redshift as

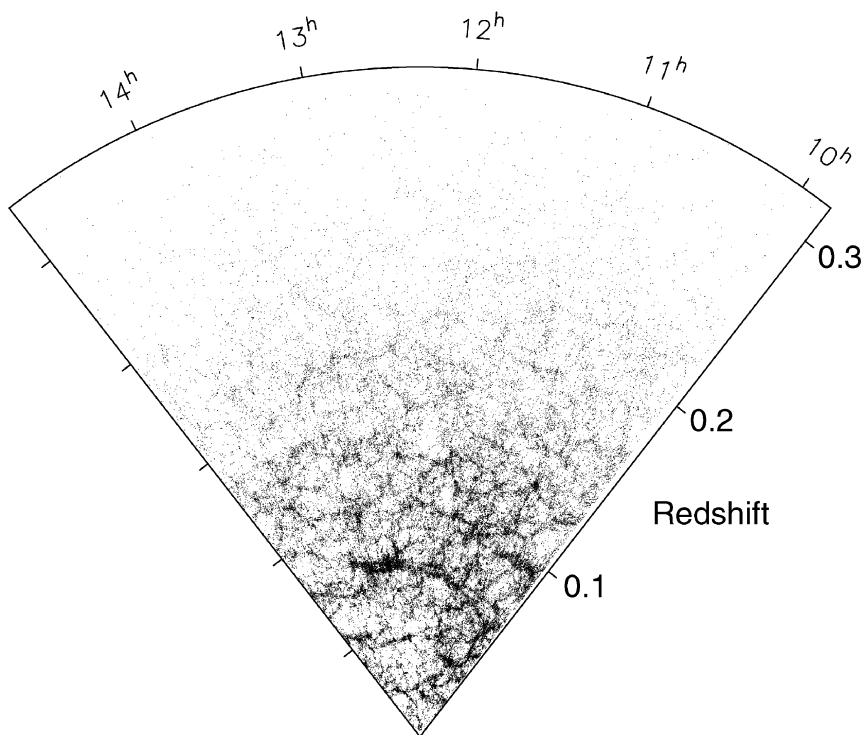

Figure 12.1: A redshift map of $\sim 10^5$ galaxies, in a strip $\sim 75°$ long, from right ascension $\alpha \approx 10\,\mathrm{h}$ to $\alpha \approx 15\,\mathrm{h}$, and $\sim 8°$ wide, from declination $\delta \approx -5°$ to $\delta \approx 3°$. (Image courtesy of the 2dF Galaxy Redshift Survey team.)

Figure 12.2: The northeastern United States and southeastern Canada at night, as seen by a satellite from the Defense Meteorological Satellite Program (DMSP).

a function of angular position for galaxies in a long, narrow strip of the sky, a "slice of the universe" can be mapped. In a slice such as that of Figure 12.1, which reaches to $z \approx 0.3$, or $d_p(t_0) \approx 1300\,\mathrm{Mpc}$, the galaxies obviously do not have a random Poisson distribution. The most prominent structures in Figure 12.1 are superclusters and voids. Superclusters are objects which are just in the process of collapsing under their own self-gravity. Superclusters are typically flattened (roughly planar) or elongated (roughly linear) structures. A supercluster will contain one or more clusters embedded within it; a cluster is a fully collapsed object which has come to equilibrium (more or less), and hence obeys the virial theorem, as discussed in section 8.3. In comparison to the flattened or elongated superclusters, the underdense voids are roughly spherical in shape. When gazing at the large scale structure of the universe, as traced by the distribution of galaxies, cosmologists are likely to call it "bubbly" or "spongy" or "frothy" or "foamy".

Being able to describe the distribution of galaxies in space doesn't automatically lead to an understanding of the origin of large scale structure. Consider, as an analogy, the distribution of luminous objects shown in Figure 12.2. The distribution of illuminated cities on the Earth's surface is obviously not random. There are "superclusters" of cities, such as the Boswash supercluster stretching from Boston to Washington. There are "voids" such as the Appalachian void. However, the influences which determine the exact location of cities are often far removed from fundamental physics.[1]

Fortunately, the distribution of galaxies in space is more closely tied to fundamental physics than is the distribution of cities on the Earth. The basic mechanism for growing large structures, such as voids and superclusters, is *gravitational instability*. Suppose that at some time in the past, the density of the universe had slight inhomogeneities. We know, for instance, that such density fluctuations occurred at the time of last scattering, since they left their stamp on the Cosmic Microwave Background. When the universe is matter-dominated, the overdense regions expand less rapidly than the universe as a whole; if their density is sufficiently great, they will collapse and become gravitationally bound objects such as clusters. The dense clusters will, in addition, draw matter to themselves from the surrounding underdense regions. The effect of gravity on density fluctuations is sometimes referred to as the Matthew Effect: "For whosoever hath, to him shall be given, and he shall have more abundance; but whosoever hath not, from him shall be taken away even that he hath" (Matthew 13:12). In less biblical language, the rich get richer and the poor get poorer.

12.1 Gravitational instability

To put our study of gravitational instability on a more quantitative basis, consider some component of the universe whose energy density $\varepsilon(\vec{r}, t)$ is a function of position as well as time. At a given time t, the spatially averaged energy density is

$$\bar{\varepsilon}(t) = \frac{1}{V} \int_V \varepsilon(\vec{r}, t) d^3 r \ . \tag{12.1}$$

To ensure that we have found the true average, the volume V over which we are averaging must be large compared to the size of the biggest structure in the universe. It is useful to define a dimensionless density fluctuation

$$\delta(\vec{r}, t) \equiv \frac{\varepsilon(\vec{r}, t) - \bar{\varepsilon}(t)}{\bar{\varepsilon}(t)} \ . \tag{12.2}$$

The value of δ is thus negative in underdense regions and positive in overdense regions. The minimum possible value of δ is $\delta = -1$, corresponding to $\varepsilon = 0$.

[1] Consider, for instance, the complicated politics that went into determining the location of Washington, DC.

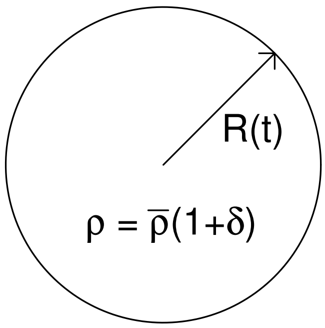

Figure 12.3: A sphere of radius $R(t)$ expanding or contracting under the influence of the density fluctuation $\delta(t)$.

In principle, there is no upper limit on δ. You, for instance, represent a region of space where the baryon density has $\delta \approx 2 \times 10^{30}$.

The study of how large scale structure evolves with time requires knowing how a small fluctuation in density, with $|\delta| \ll 1$, grows in amplitude under the influence of gravity. This problem is most tractable when $|\delta|$ remains very much smaller than one. In the limit that the amplitude of the fluctuations remains small, we can successfully use linear perturbation theory.

To get a feel for how small density contrasts grow with time, consider a particularly simple case. Start with a static, homogeneous, matter-only universe with uniform mass density $\bar{\rho}$. (At this point, we stumble over the inconvenient fact that there's no such thing as a static, homogeneous, matter-only universe. This is the awkward fact that inspired Einstein to introduce the cosmological constant. However, there are conditions under which we can consider some region of the universe to be approximately static and homogeneous. For instance, the air in a closed room is approximately static and homogeneous; it is stabilized by a pressure gradient with a scale length which is much greater than the height of the ceiling.) In a region of the universe which is *approximately* static and homogeneous, we add a small amount of mass within a sphere of radius R, as seen in Figure 12.3, so that the density within the sphere is $\bar{\rho}(1+\delta)$, with $\delta \ll 1$. If the density excess δ is uniform within the sphere, then the gravitational acceleration at the sphere's surface, due to the excess mass, will be

$$\ddot{R} = -\frac{G(\Delta M)}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar{\rho}\delta\right) , \tag{12.3}$$

or

$$\frac{\ddot{R}}{R} = -\frac{4\pi G\bar{\rho}}{3}\delta(t) \ . \tag{12.4}$$

Thus, a mass excess ($\delta > 0$) will cause the sphere to collapse inward ($\ddot{R} < 0$).

Equation (12.4) contains two unknowns, $R(t)$ and $\delta(t)$. If we want to find an explicit solution for $\delta(t)$, we need a second equation involving $R(t)$ and $\delta(t)$. Conservation of mass tells us that the mass of the sphere,

$$M = \frac{4\pi}{3}\bar{\rho}[1+\delta(t)]R(t)^3 \ , \tag{12.5}$$

remains constant during the collapse. Thus, we can write another relation between $R(t)$ and $\delta(t)$ which must hold true during the collapse:

$$R(t) = R_0[1+\delta(t)]^{-1/3} \ , \tag{12.6}$$

where

$$R_0 \equiv \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3} = \text{constant} \ . \tag{12.7}$$

When $\delta \ll 1$, we may make the approximation

$$R(t) \approx R_0[1 - \frac{1}{3}\delta(t)] \ . \tag{12.8}$$

Taking the second time derivative yields

$$\ddot{R} \approx -\frac{1}{3}R_0\ddot{\delta} \approx -\frac{1}{3}R\ddot{\delta} \ . \tag{12.9}$$

Thus, mass conservation tells us that

$$\frac{\ddot{R}}{R} \approx -\frac{1}{3}\ddot{\delta} \tag{12.10}$$

in the limit that $\delta \ll 1$. Combining equations (12.4) and (12.10), we find a tidy equation which tells us how the small overdensity δ evolves as the sphere collapses:

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta \ . \tag{12.11}$$

The most general solution of equation (12.11) has the form

$$\delta(t) = A_1 e^{t/t_{\rm dyn}} + A_2 e^{-t/t_{\rm dyn}} \ , \tag{12.12}$$

where the dynamical time for collapse is

$$t_{\rm dyn} = \frac{1}{(4\pi G\bar{\rho})^{1/2}} = \left(\frac{c^2}{4\pi G\bar{\varepsilon}}\right)^{1/2} . \tag{12.13}$$

Note that the dynamical time depends only on $\bar{\rho}$, and not on R. The constants A_1 and A_2 in equation (12.12) depend on the initial conditions of the sphere. For instance, if the overdense sphere starts at rest, with $\dot{\delta} = 0$ at $t = 0$, then $A_1 = A_2 = \delta(0)/2$. After a few dynamical times, however, only the exponentially growing term of equation (12.12) is significant. Thus, gravity tends to make small density fluctuations in a static, pressureless medium grow exponentially with time.

12.2 The Jeans length

The exponential growth of density perturbations is slightly alarming, at first glance. For instance, the density of the air around you is $\bar{\rho} \approx 1\,{\rm kg\,m^{-3}}$, yielding a dynamical time for collapse of $t_{\rm dyn} \approx 9\,{\rm hours}$.[2] What keeps small density perturbations in the air from undergoing a runaway collapse over the course of a few days? The answer, of course, is pressure. A non-relativistic gas, as shown in section 4.3, has an equation-of-state parameter

$$w \approx \frac{kT}{\mu c^2} , \tag{12.14}$$

where T is the temperature of the gas and μ is the mean mass per gas particle. Thus, the pressure of a ideal gas will never totally vanish, but will only approach zero in the limit that the temperature approaches absolute zero.

When a sphere of gas is compressed by its own gravity, a pressure gradient will build up which tends to counter the effects of gravity.[3] However, hydrostatic equilibrium, the state in which gravity is exactly balanced by a pressure gradient, cannot always be attained. Consider an overdense sphere

[2]Slightly longer if you are using this book for recreational reading as you climb Mount Everest.

[3]A star is the prime example of a dense sphere of gas in which the inward force of gravity is balanced by the outward force provided by a pressure gradient.

with initial radius R. If pressure were not present, it would collapse on a timescale

$$t_{\rm dyn} \sim \frac{1}{(G\bar{\rho})^{1/2}} \sim \left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2} . \tag{12.15}$$

If the pressure is nonzero, the attempted collapse will be countered by a steepening of the pressure gradient within the perturbation. The steepening of the pressure gradient, however, doesn't occur instantaneously. Any change in pressure travels at the sound speed.[4] Thus, the time it takes for the pressure gradient to build up in a region of radius R will be

$$t_{\rm pre} \sim \frac{R}{c_s} , \tag{12.16}$$

where c_s is the local sound speed. In a medium with equation-of-state parameter $w > 0$, the sound speed is

$$c_s = c\left(\frac{dP}{d\varepsilon}\right)^{1/2} = \sqrt{w}c . \tag{12.17}$$

For hydrostatic equilibrium to be attained, the pressure gradient must build up before the overdense region collapses, implying

$$t_{\rm pre} < t_{\rm dyn} . \tag{12.18}$$

Comparing equation (12.15) with equation (12.16), we find that for a density perturbation to be stabilized by pressure against collapse, it must be smaller than some reference size λ_J, given by the relation

$$\lambda_J \sim c_s t_{\rm dyn} \sim c_s \left(\frac{c^2}{G\bar{\varepsilon}}\right)^{1/2} . \tag{12.19}$$

The length scale λ_J is known as the *Jeans length*, after the astrophysicist James Jeans, who was among the first to study gravitational instability in a cosmological context. Overdense regions larger than the Jeans length collapse under their own gravity. Overdense regions smaller than the Jeans length merely oscillate in density; they constitute stable sound waves.

[4] What is sound, after all, but a traveling change in pressure?

A more precise derivation of the Jeans length, including all the appropriate factors of π, yields the result

$$\lambda_J = c_s \left(\frac{\pi c^2}{G\bar{\varepsilon}} \right)^{1/2} = 2\pi c_s t_{\rm dyn} \ . \tag{12.20}$$

The Jeans length of the Earth's atmosphere, for instance, where the sound speed is a third of a kilometer per second and the dynamical time is nine hours, is $\lambda_J \sim 10^5\,\text{km}$, far longer than the scale height of the Earth's atmosphere. You don't have to worry about density fluctuations in the air undergoing a catastrophic collapse.

To consider the behavior of density fluctuations on cosmological scales, consider a spatially flat universe in which the mean density is $\bar{\varepsilon}$, but which contains density fluctuations with amplitude $|\delta| \ll 1$. The characteristic time for expansion of such a universe is the Hubble time,

$$H^{-1} = \left(\frac{3c^2}{8\pi G\bar{\varepsilon}} \right)^{1/2} \ . \tag{12.21}$$

Comparison of equation (12.13) with equation (12.21) reveals that the Hubble time is comparable in magnitude to the dynamical time $t_{\rm dyn}$ for the collapse of an overdense region:

$$H^{-1} = \left(\frac{3}{2} \right)^{1/2} t_{\rm dyn} \approx 1.22 t_{\rm dyn} \ . \tag{12.22}$$

The Jeans length in an expanding flat universe will then be

$$\lambda_J = 2\pi c_s t_{\rm dyn} = 2\pi \left(\frac{3}{2} \right)^{1/2} \frac{c_s}{H} \ . \tag{12.23}$$

If we focus on one particular component of the universe, with equation-of-state parameter w and sound speed $c_s = \sqrt{w}c$, the Jeans length for that component will be

$$\lambda_J = 2\pi \left(\frac{2}{3} \right)^{1/2} \sqrt{w} \frac{c}{H} \ . \tag{12.24}$$

Consider, for instance, the "radiation" component of the universe. With $w = 1/3$, the sound speed in a gas of photons or other relativistic particles is

$$c_s = c/\sqrt{3} \approx 0.58c \ . \tag{12.25}$$

The Jeans length for radiation in an expanding universe is then

$$\lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H} \approx 3.0\frac{c}{H} \ . \tag{12.26}$$

Density fluctuations in the radiative component will be pressure-supported if they are smaller than three times the Hubble distance. Although a universe containing nothing but radiation can have density perturbations smaller than $\lambda_J \sim 3c/H$, they will be stable sound waves, and will not collapse under their own gravity.

In order for a universe to have gravitationally collapsed structures much smaller than the Hubble distance, it must have a non-relativistic component, with $\sqrt{w} \ll 1$. The gravitational collapse of the *baryonic* component of the universe is complicated by the fact that it was coupled to photons until a redshift $z_{\rm dec} \approx z_{\rm ls} \approx 1100$. In section 9.5, the Hubble distance at the time of last scattering (effectively equal to the time of decoupling) was shown to be $c/H(z_{\rm dec}) \approx 0.2\,{\rm Mpc}$. The energy density of baryons at decoupling was $\varepsilon_{\rm bary} \approx 2.8 \times 10^{11}\,{\rm MeV\,m^{-3}}$, corresponding to a mass density $\rho_{\rm bary} \approx 5.0 \times 10^{-19}\,{\rm kg\,m^{-3}}$, and the energy density of photons was $\varepsilon_\gamma \approx 3.8 \times 10^{11}\,{\rm MeV\,m^{-3}} \approx 1.4\varepsilon_{\rm bary}$.

Prior to decoupling, the photons, electrons, and baryons were all coupled together to form a single photon-baryon fluid. Since the photons were still dominant over the baryons at the time of decoupling, with $\varepsilon_\gamma > \varepsilon_{\rm bary}$, we can regard the baryons (with only mild exaggeration) as being a dynamically insignificant contaminant in the photon-baryon fluid. Just *before* decoupling, if we regard the baryons as a minor contaminant, the Jeans length of the photon-baryon fluid was roughly the same as the Jeans length of a pure photon gas:

$$\lambda_J({\rm before}) \approx 3c/H(z_{\rm dec}) \approx 0.6\,{\rm Mpc} \approx 1.9 \times 10^{22}\,{\rm m} \ . \tag{12.27}$$

The *baryonic Jeans mass*, M_J, is defined as the mass of baryons contained within a sphere of radius λ_J;

$$M_J \equiv \rho_{\rm bary}\left(\frac{4\pi}{3}\lambda_J^3\right) \ . \tag{12.28}$$

Immediately before decoupling, the baryonic Jeans mass was

$$\begin{aligned} M_J({\rm before}) &\approx 5.0 \times 10^{-19}\,{\rm kg\,m^{-3}}\left(\frac{4\pi}{3}\right)(1.9 \times 10^{22}\,{\rm m})^3 \\ &\approx 1.3 \times 10^{49}\,{\rm kg} \approx 7 \times 10^{18}\,{\rm M_\odot} \ . \end{aligned} \tag{12.29}$$

This is approximately 3×10^4 times greater than the estimated baryonic mass of the Coma cluster, and represents a mass greater than the baryonic mass of even the largest supercluster seen today.

Now consider what happens to the baryonic Jeans mass immediately after decoupling. Once the photons are decoupled, the photons and baryons form two separate gases, instead of a single photon-baryon fluid. The sound speed in the photon gas is

$$c_s(\text{photon}) = c/\sqrt{3} \approx 0.58c \; . \tag{12.30}$$

The sound speed in the baryonic gas, by contrast, is

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2}\right)^{1/2} c \; . \tag{12.31}$$

At the time of decoupling, the thermal energy per particle was $kT_{\rm dec} \approx 0.26\,\text{eV}$, and the mean rest energy of the atoms in the baryonic gas was $mc^2 = 1.22 m_p c^2 \approx 1140\,\text{MeV}$, taking into account the helium mass fraction of $Y_p = 0.24$. Thus, the sound speed of the baryonic gas immediately after decoupling was

$$c_s(\text{baryon}) \approx \left(\frac{0.26\,\text{eV}}{1140\times10^6\,\text{eV}}\right)^{1/2} c \approx 1.5\times10^{-5}c \; , \tag{12.32}$$

only 5 kilometers per second. Thus, once the baryons were decoupled from the photons, their associated Jeans length decreased by a factor

$$F = \frac{c_s(\text{baryon})}{c_s(\text{photon})} \approx \frac{1.5\times10^{-5}}{0.58} \approx 2.6\times10^{-5} \; . \tag{12.33}$$

Decoupling causes the baryonic Jeans mass to decrease by a factor $F^3 \approx 1.8\times10^{-14}$, plummeting from $M_J(\text{before}) \approx 7\times10^{18}\,\text{M}_\odot$ to

$$M_J(\text{after}) = F^3 M_J(\text{before}) \approx 1\times10^{5}\,\text{M}_\odot \; . \tag{12.34}$$

This is comparable to the baryonic mass of the smallest dwarf galaxies known, and is very much smaller than the baryonic mass of our own Galaxy, which is $\sim 10^{11}\,\text{M}_\odot$.

The abrupt decrease of the baryonic Jeans mass at the time of decoupling marks an important epoch in the history of structure formation. Perturbations in the baryon density, from supercluster scales down the the size of the smallest dwarf galaxies, couldn't grow in amplitude until the time of photon decoupling, when the universe had reached the ripe old age of $t_{\rm dec} \approx 0.35\,{\rm Myr}$. After decoupling, the growth of density perturbations in the baryonic component was off and running. The baryonic Jeans mass, already small by cosmological standards at the time of decoupling, dropped still further with time as the universe expanded and the baryonic component cooled.

12.3 Instability in an expanding universe

Density perturbations smaller than the Hubble distance can grow in amplitude only when they are no longer pressure-supported. For the baryonic matter, this loss of pressure support happens abruptly at the time of decoupling, when the Jeans length for baryons drops suddenly by a factor $F \sim 3 \times 10^{-5}$. For the *dark* matter, the loss of pressure support occurs more gradually, as the thermal energy of the dark matter particles drops below their rest energy. When considering the Cosmic Neutrino Background, for instance, which has a temperature comparable to the Cosmic Microwave Background, we found (see equation 5.18) that neutrinos of mass m_ν became non-relativistic at a redshift

$$1 + z = \frac{1}{a} \approx \frac{m_\nu c^2}{5 \times 10^{-4}\,{\rm eV}} \ . \qquad (12.35)$$

Thus, if the universe contains a Cosmic WIMP Background comparable in temperature to the Cosmic Neutrino Background, the WIMPs, if they have a mass $m_{\rm W} c^2 \gg 2\,{\rm eV}$, would have become non-relativistic long before the time of radiation-matter equality at $z_{\rm rm} \approx 3570$.

Once the pressure (and hence the Jeans length) of some component becomes negligibly small, does this imply that the amplitude of density fluctuations is free to grow exponentially with time? Not necessarily. The analysis of section 12.1, which yielded $\delta \propto \exp(t/t_{\rm dyn})$, assumed that the universe was *static* as well as pressureless. In an expanding Friedmann universe, the timescale for the growth of a density perturbation by self-gravity, $t_{\rm dyn} \sim (c^2/G\bar{\varepsilon})^{1/2}$, is comparable to the timescale for expansion, $H^{-1} \sim (c^2/G\bar{\varepsilon})^{1/2}$. Self-gravity, in the absence of global expansion, causes overdense regions to become *more dense* with time. The global expansion of the universe, in the absence of self-gravity, causes overdense regions to become *less dense* with time. Because the timescales for these two competing processes are similar, they must both be taken into account when computing the time evolution of a density perturbation.

To get a feel how small density perturbations in an expanding universe evolve with time, let's do a Newtonian analysis of the problem, similar in spirit to the Newtonian derivation of the Friedmann equation given in Chapter 4. Suppose you are in a universe filled with pressureless matter which has mass density $\bar{\rho}(t)$. As the universe expands, the density decreases at the rate $\bar{\rho}(t) \propto a(t)^{-3}$. Within a spherical region of radius R, a small amount of matter is added, or removed, so that the density within the sphere is

$$\rho(t) = \bar{\rho}(t)[1 + \delta(t)] \ , \tag{12.36}$$

with $|\delta| \ll 1$. (In performing a Newtonian analysis of this problem, we are implicitly assuming that the radius R is small compared to the Hubble distance and large compared to the Jeans length.) The total gravitational acceleration at the surface of the sphere will be

$$\ddot{R} = -\frac{GM}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}\rho R^3\right) = -\frac{4\pi}{3}G\bar{\rho}R - \frac{4\pi}{3}G(\bar{\rho}\delta)R \ . \tag{12.37}$$

The equation of motion for a point at the surface of the sphere can then be written in the form

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta \ . \tag{12.38}$$

Mass conservation tells us that the mass inside the sphere,

$$M = \frac{4\pi}{3}\bar{\rho}(t)[1 + \delta(t)]R(t)^3 \ , \tag{12.39}$$

remains constant as the sphere expands. Thus,

$$R(t) \propto \bar{\rho}(t)^{-1/3}[1 + \delta(t)]^{-1/3} \ , \tag{12.40}$$

or, since $\bar{\rho} \propto a^{-3}$,

$$R(t) \propto a(t)[1 + \delta(t)]^{-1/3} \ . \tag{12.41}$$

That is, if the sphere is slightly overdense, its radius will grow slightly less rapidly than the scale factor $a(t)$. If the sphere is slightly underdense, it will grow slightly more rapidly than the scale factor.

Taking two time derivatives of equation (12.41) yields

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} \ , \tag{12.42}$$

when $|\delta| \ll 1$. Combining equations (12.38) and (12.42), we find

$$\frac{\ddot{a}}{a} - \frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho} - \frac{4\pi}{3}G\bar{\rho}\delta \ . \tag{12.43}$$

When $\delta = 0$, equation (12.43) reduces to

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}G\bar{\rho} \ , \tag{12.44}$$

which is the correct acceleration equation for a homogeneous, isotropic universe containing nothing but pressureless matter (compare to equation 4.44). By subtracting equation (12.44) from equation (12.43) to leave only the terms linear in the perturbation δ, we find the equation which governs the growth of small perturbations:

$$-\frac{1}{3}\ddot{\delta} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} = -\frac{4\pi}{3}G\bar{\rho}\delta \ , \tag{12.45}$$

or

$$\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta \ , \tag{12.46}$$

remembering that $H \equiv \dot{a}/a$. In a static universe, with $H = 0$, equation (12.46) reduces to the result which we have already found in equation (12.11):

$$\ddot{\delta} = 4\pi G\bar{\rho}\delta \ . \tag{12.47}$$

The additional term, $\propto H\dot{\delta}$, found in an expanding universe, is sometimes called the "Hubble friction" term; it acts to slow the growth of density perturbations in an expanding universe.

A fully relativistic calculation for the growth of density perturbations yields the more general result

$$\ddot{\delta} + 2H\dot{\delta} = \frac{4\pi G}{c^2}\bar{\varepsilon}_m\delta \ . \tag{12.48}$$

This form of the equation can be applied to a universe which contains components with non-negligible pressure, such as radiation ($w = 1/3$) or a cosmological constant ($w = -1$). In multiple-component universes, however, it should be remembered that δ represents the fluctuation in the density of *matter* alone. That is,

$$\delta = \frac{\varepsilon_m - \bar{\varepsilon}_m}{\bar{\varepsilon}_m} \ , \tag{12.49}$$

where $\bar{\varepsilon}_m(t)$, the average matter density, might be only a small fraction of $\bar{\varepsilon}(t)$, the average total density. Rewritten in terms of the density parameter for matter,

$$\Omega_m = \frac{\bar{\varepsilon}_m}{\varepsilon_c} = \frac{8\pi G\bar{\varepsilon}_m}{3c^2H^2} , \tag{12.50}$$

equation (12.48) takes the form

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2\delta = 0 . \tag{12.51}$$

During epochs when the universe is not dominated by matter, density perturbations in the matter do not grow rapidly in amplitude. Take, for instance, the early radiation-dominated phase of the universe. During this epoch, $\Omega_m \ll 1$ and $H = 1/(2t)$, meaning that equation (12.51) takes the form

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} \approx 0 , \tag{12.52}$$

which has a solution of the form

$$\delta(t) \approx B_1 + B_2 \ln t . \tag{12.53}$$

During the radiation-dominated epoch, density fluctuations in the dark matter grew only at a logarithmic rate. In the far future, if the universe is indeed dominated by a cosmological constant, the density parameter for matter will again be negligibly small, the Hubble parameter will have the constant value $H = H_\Lambda$, and equation (12.51) will take the form

$$\ddot{\delta} + 2H_\Lambda\dot{\delta} \approx 0 , \tag{12.54}$$

which has a solution of the form

$$\delta(t) \approx C_1 + C_2 e^{-2H_\Lambda t} . \tag{12.55}$$

In a lambda-dominated phase, therefore, fluctuations in the matter density reach a constant fractional amplitude, while the average matter density plummets at the rate $\bar{\varepsilon}_m \propto e^{-3H_\Lambda t}$.

It is only when matter dominates the energy density that fluctuations in the matter density can grow at a significant rate. In a flat, matter-dominated universe, $\Omega_m = 1$, $H = 2/(3t)$, and equation (12.51) takes the form

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0 . \tag{12.56}$$

If we guess that the solution to the above equation has the power-law form Dt^n, plugging this guess into the equation yields

$$n(n-1)Dt^{n-2} + \frac{4}{3t}nDt^{n-1} - \frac{2}{3t^2}Dt^n = 0 \ , \tag{12.57}$$

or

$$n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0 \ . \tag{12.58}$$

The two possible solutions for this quadratic equation are $n = -1$ and $n = 2/3$. Thus, the general solution for the time evolution of density perturbations in a spatially flat, matter-only universe is

$$\delta(t) \approx D_1 t^{2/3} + D_2 t^{-1} \ . \tag{12.59}$$

The values of D_1 and D_2 are determined by the initial conditions for $\delta(t)$. The decaying mode, $\propto t^{-1}$, eventually becomes negligibly small compared to the growing mode, $\propto t^{2/3}$. When the growing mode is the only survivor, the density perturbations in a flat, matter-only universe grow at the rate

$$\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z} \tag{12.60}$$

as long as $|\delta| \ll 1$.

When an overdense region attains an overdensity $\delta \sim 1$, its evolution can no longer be treated with a simple linear perturbation approach. Studies of the nonlinear evolution of structure are commonly made using numerical computer simulations, in which the matter filling the universe is modeled as a distribution of point masses interacting via Newtonian gravity. In these simulations, as in the real universe, when a region reaches an overdensity $\delta \sim 1$, it breaks away from the Hubble flow and collapses. After one or two oscillations in radius, the overdense region attains virial equilibrium as a gravitationally bound structure. If the baryonic matter within the structure is able to cool efficiently (by bremsstrahlung or some other process) it will radiate away energy and fall to the center. The centrally concentrated baryons can then proceed to form stars, becoming the visible portions of galaxies that we see today. The less concentrated nonbaryonic matter forms the dark halo within which the stellar component of the galaxy is embedded.

If baryonic matter were the only type of non-relativistic matter in the universe, then density perturbations could have started to grow at $z_{\rm dec} \approx 1100$,

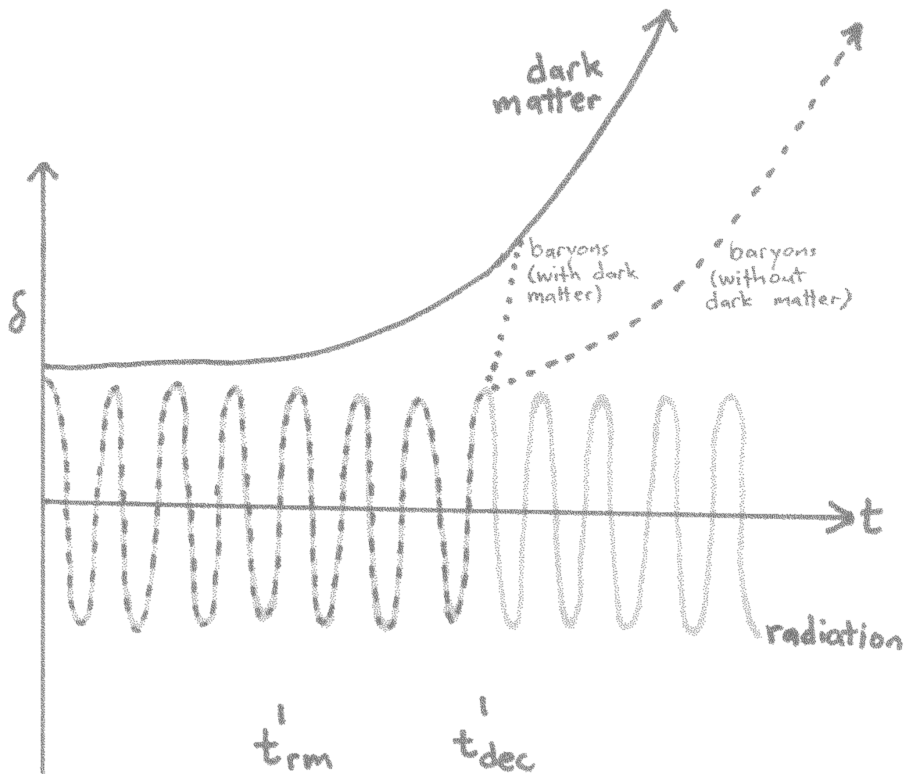

Figure 12.4: A highly schematic drawing of how density fluctuations in different components of the universe evolve with time.

and they could have grown in amplitude only by a factor ~ 1100 by the present day. However, the mominant form of non-relativistic matter is dark matter. The density perturbations in the dark matter started to grow effectively at $z_{\rm rd} \approx 3570$. At the time of decoupling, the baryons fell into the preexisting gravitational wells of the dark matter. The situation is schematically illustrated in Figure 12.4. Having nonbaryonic dark matter allows the universe to get a "head start" on structure formation; perturbations in the matter density can start growing at $z_{\rm rd} \approx 3570$ rather than $z_{\rm dec} \approx 1100$, as they would in a universe without dark matter.

12.4 The power spectrum

When deriving equation (12.46), which determines the growth rate of density perturbations in a Newtonian universe, I assumed that the perturbation was spherically symmetric. In fact, equation (12.46) and its relativistically correct brother, equation (12.48), both apply to low-amplitude perturbations of any shape. This is fortunate, since the density perturbations in the real universe are not spherically symmetric. The bubbly structure shown in redshift maps of galaxies, such as Figure 12.1, has grown from the density perturbations which were present when the universe became matter dominated. Great oaks from tiny acorns grow – but then, great pine trees from tiny pinenuts grow. By looking at the current large scale structure (the "tree"), we can deduce the properties of the early, low-amplitude, density fluctuations (the "nut").[5]

Let us consider the properties of the early density fluctuations at some time t_i when they were still very low in amplitude ($|\delta| \ll 1$). As long as the density fluctuations are small in amplitude, the expansion of the universe is still nearly isotropic, and the geometry of the universe is still well described by the Robertson-Walker metric (equation 3.25):

$$ds^2 = -c^2dt^2 + a(t)^2[dr^2 + S_\kappa(r)^2 d\Omega^2] \ . \tag{12.61}$$

Under these circumstances, it is useful to set up a comoving coordinate system. Choose some point as the origin. In a universe described by the Robertson-Walker metric, as shown in section 3.4, the proper distance of any point from the origin can be written in the form

$$d_p(t_i) = a(t_i)r \ , \tag{12.62}$$

where the comoving distance r is what the proper distance would be at the present day ($a = 1$) *if* the expansion continued to be perfectly isotropic. If we label each bit of matter in the universe with its comoving coordinate position $\vec{r} = (r, \theta, \phi)$, then $\vec{r}$ will remain very nearly constant as long as $|\delta| \ll 1$. Thus, when considering the regime where density fluctuations are small, cosmologists typically consider $\delta(\vec{r})$, the density fluctuation at a comoving location $\vec{r}$, at some time t_i. (The exact value of t_i doesn't matter, as long as it's a time after the density perturbations are in place, but before

[5] At the risk of carrying the arboreal analogy too far, I should mention that the temperature fluctuations of the Cosmic Microwave Background, as shown in Figures 9.2 and 9.5, offer us a look at the "sapling".

they reach an amplitude $|\delta| \sim 1$. Switching to a different value of t_i, under these restrictions, simply changes the amplitude of $\delta(\vec{r})$, and not its shape.)

When discussing the temperature fluctuations of the Cosmic Microwave Background, back in Chapter 9, I pointed out that cosmologists weren't interested in the exact pattern of hot and cold spots on the last scattering surface, but rather in the statistical properties of the field $\delta T/T(\theta,\phi)$. Similarly, cosmologists are not interested in the exact locations of the density maxima and minima in the early universe, but rather in the statistical properties of the field $\delta(\vec{r})$. When studying the temperature fluctuations of the CMB, it is useful to expand $\delta T/T(\phi,\theta)$ in spherical harmonics. A similar decomposition of $\delta(\vec{r})$ is also useful. Since δ is defined in three-dimensional space (rather than on the surface of a sphere), a useful expansion of δ is in terms of Fourier components.

Within a large comoving box, of comoving volume V, the density fluctuation field $\delta(\vec{r})$ can be expressed as

$$\delta(\vec{r}) = \frac{V}{(2\pi)^3}\int \delta_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}} d^3k \ , \tag{12.63}$$

where the individual Fourier components $\delta_{\vec{k}}$ are found by performing the integral

$$\delta_{\vec{k}} = \frac{1}{V}\int \delta(\vec{r}) e^{i\vec{k}\cdot\vec{r}} d^3r \ . \tag{12.64}$$

In performing the Fourier transform, you are breaking up the function $\delta(\vec{r})$ into an infinite number of sine waves, each with comoving wavenumber $\vec{k}$ and comoving wavelength $\lambda = 2\pi/k$. If you have complete, uncensored knowledge of $\delta(\vec{r})$, you can compute all the Fourier components $\delta_{\vec{k}}$; conversely, if you know all the Fourier components, you can reconstruct the density field $\delta(\vec{r})$.

Each Fourier component is a complex number, which can be written in the form

$$\delta_{\vec{k}} = |\delta_{\vec{k}}| e^{i\phi_{\vec{k}}} \ . \tag{12.65}$$

When $|\delta_{\vec{k}}| \ll 1$, then each Fourier component obeys equation (12.51),

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - \frac{3}{2}\Omega_m H^2 \delta_{\vec{k}} = 0 \ , \tag{12.66}$$

as long as the proper wavelength, $a(t)2\pi/k$, is large compared to the Jeans length and small compared to the Hubble distance c/H.[6] The phase $\phi_{\vec{k}}$ remains constant as long as the amplitude $|\delta_{\vec{k}}|$ remains small. Even after fluctuations with a short proper wavelength have reached $|\delta_{\vec{k}}| \sim 1$ and collapsed, the growth of the longer wavelength perturbations is still described by equation (12.66). This means, helpfully enough, that we can use linear perturbation theory to study the growth of very large scale structure even after smaller structures, such as galaxies and clusters of galaxies, have already collapsed.

The mean square amplitude of the Fourier components defines the *power spectrum*:

$$P(k) = \langle |\delta_{\vec{k}}|^2 \rangle \ , \tag{12.67}$$

where the average is taken over all possible orientations of the wavenumber $\vec{k}$. (If $\delta(\vec{r})$ is isotropic, then no information is lost, statistically speaking, if we average the power spectrum over all angles.) When the phases $\phi_{\vec{k}}$ of the different Fourier components are uncorrelated with each other, then $\delta(\vec{r})$ is called a *Gaussian field*. If a Gaussian field is homogeneous and isotropic, then all its statistical properties are summed up in the power spectrum $P(k)$. If $\delta(\vec{r})$ is a Gaussian field, then the value of δ at a randomly selected point is drawn from the Gaussian probability distribution

$$p(\delta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) \ , \tag{12.68}$$

where the standard deviation σ can be computed from the power spectrum:

$$\sigma = \frac{V}{(2\pi)^3} \int P(k) d^3k = \frac{V}{2\pi^2} \int_0^\infty P(k) k^2 dk \ . \tag{12.69}$$

The study of Gaussian density fields is of particular interest to cosmologists because most inflationary scenarios predict that the density fluctuations created by inflation (see section 11.5) will be an isotropic, homogeneous Gaussian field. In addition, the expected power spectrum for the inflationary fluctuations has a scale-invariant, power-law form:

$$P(k) \propto k^n \ , \tag{12.70}$$

[6]When a sine wave perturbation has a wavelength large compared to the Hubble distance, its crests are not causally connected to its troughs. As long as the crests remain out of touch with the troughs (that is, as long as $a(t)2\pi/k > c/H(t)$), the amplitude of a perturbation grows at the rate $\delta(t) \propto a(t)$.

with the favored value of the power-law index being $n = 1$. The preferred power spectrum, $P(k) \propto k$, is often referred to as a Harrison-Zel'dovich spectrum.

What would a universe with $P(k) \propto k^n$ look like? Imagine going through such a universe and marking out randomly located spheres of comoving radius L. The mean mass of each sphere (considering only the non-relativistic matter which it contains) will be

$$\langle M \rangle = \frac{4\pi}{3} L^3 \frac{\varepsilon_{m,0}}{c^2} . \tag{12.71}$$

However, the actual mass of each sphere will vary; some spheres will be slightly underdense, and others will be slightly overdense. The mean square density fluctuation of the mass inside each sphere is a function of the power spectrum and of the size of the sphere:

$$\left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle} \right)^2 \right\rangle \propto k^3 P(k) , \tag{12.72}$$

where the comoving wavenumber associated with the sphere is $k = 2\pi/L$. Thus, if the power spectrum has the form $P(k) \propto k^n$, the root mean square mass fluctuation within spheres of comoving radius L will be

$$\frac{\delta M}{M} \equiv \left\langle \left(\frac{M - \langle M \rangle}{\langle M \rangle} \right)^2 \right\rangle^{1/2} \propto L^{-(3+n)/2} . \tag{12.73}$$

This can also be expressed in the form $\delta M/M \propto M^{-(3+n)/6}$. For $n < -3$, the mass fluctuations diverge on large scales, which would be Bad News for our assumption of homogeneity on large scales. (Note that if you scattered point masses randomly throughout the universe, so that they formed a Poisson distribution, you would expect mass fluctuations of amplitude $\delta M/M \propto N^{-1/2}$, where N is the expected number of point masses within the sphere. Since the average mass within a sphere is $M \propto N$, a Poisson point distribution has $\delta M/M \propto M^{-1/2}$, or $n = 0$. The Harrison-Zel'dovich spectrum, with $n = 1$, thus will produce more power on small length scales than a Poisson distribution of points.) Note that the potential fluctuations associated with mass fluctuations on a length scale L will have an amplitude $\delta\Phi \sim G\delta M/L \propto \delta M/M^{1/3} \propto M^{(1-n)/6}$. Thus, the Harrison-Zel'dovich spectrum, with $n = 1$, is the only power law which prevents the divergence of the potential fluctuations on both large and small scales.

12.5 Hot versus cold

Immediately after inflation, the expected power spectrum for density perturbations has the form $P(k) \propto k^n$, with an index $n = 1$ being predicted by most inflationary models. However, the shape of the power spectrum will be modified between the end of inflation at t_f and the time of radiation-matter equality at $t_{\rm rm} \approx 4.7 \times 10^4$yr. The shape of the power spectrum at $t_{\rm rm}$, when density perturbations start to grow significantly in amplitude, depends on the properties of the dark matter. More specifically, it depends on whether the dark matter is predominantly *cold dark matter* or *hot dark matter.*

Cold dark matter consists of particles which were non-relativistic at the time they decoupled from the other components of the universe. For instance, WIMPs would have had thermal velocities much smaller than c at the time they decoupled, and hence qualify as cold dark matter. If any primordial black holes had formed in the early universe, their peculiar velocities would have been much smaller than c at the time they formed; thus primordial black holes would also act as cold dark matter. Axions are a type of elementary particle first proposed by particle physicists for non-cosmological purposes. If they exist, however, they would have formed out of equilibrium in the early universe, with very low thermal velocities. Thus, axions would act as cold dark matter, as well.

Hot dark matter, by contrast, consists of particles which were *relativistic* at the time they decoupled from the other components of the universe, and which remained relativistic until the mass contained within a Hubble volume (a sphere of proper radius c/H) was large compared to the mass of a galaxy. In the Benchmark Model, the Hubble distance at the time of radiation-matter equality was

$$\frac{c}{H(t_{\rm rm})} = \frac{c}{\sqrt{2}H_0}\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2} \approx 1.8ct_{\rm rm} \approx 0.026\,{\rm Mpc}\ , \tag{12.74}$$

so the mass within a Hubble volume at that time was

$$\frac{4\pi}{3}\frac{c^3}{H(t_{\rm rm})^3}\frac{\Omega_{m,0}\rho_{c,0}}{a_{\rm rm}^3} = \frac{\sqrt{2}\pi}{3}\frac{c^3}{H_0^3}\frac{\Omega_{r,0}^{3/2}}{\Omega_{m,0}^2}\rho_{c,0} \approx 1.4\times10^{17}\,{\rm M}_\odot\ , \tag{12.75}$$

much larger than the mass of even a fairly large galaxy such as our own ($M_{\rm gal} \approx 10^{12}\,{\rm M}_\odot$). Thus, a weakly interacting particle which remains relativistic until the universe becomes matter-dominated will act as hot dark matter. For instance, neutrinos decoupled at $t \sim 1\,\mathrm{s}$, when the universe had a temperature $kT \sim 1\,\mathrm{MeV}$. Thus, a neutrino with mass $m_\nu c^2 \ll 1\,\mathrm{MeV}$ was hot enough to be relativistic at the time it decoupled. Moreover, as discussed in section 5.1, a neutrino with mass $m_\nu c^2 < 2\,\mathrm{eV}$ doesn't become non-relativistic until after radiation-matter equality, and hence qualifies as hot dark matter.[7]

To see how the existence of hot dark matter modifies the spectrum of density perturbations, consider what would happen in a universe filled with weakly interacting particles which are relativistic at the time they decouple. The initially relativistic particles cool as the universe expands, until their thermal velocities drop well below c when $3kT \sim m_h c^2$. This happens at a temperature

$$T_h \sim \frac{m_h c^2}{3k} \sim 8000\,\mathrm{K}\left(\frac{m_h c^2}{2\,\mathrm{eV}}\right) . \tag{12.76}$$

In the radiation-dominated universe, this corresponds to a cosmic time (equation 10.2)

$$t_h \sim 2 \times 10^{12}\,\mathrm{s}\left(\frac{m_h c^2}{2\,\mathrm{eV}}\right)^{-2} . \tag{12.77}$$

Prior to the time t_h, the hot dark matter particles move freely in random directions with a speed close to that of light. This motion, called *free streaming*, acts to wipe out any density fluctuations present in the hot dark matter. Thus, the net effect of free streaming in the hot dark matter is to wipe out any density fluctuations whose wavelength is smaller than $\sim ct_h$. When the hot dark matter particles become non-relativistic, there will be no density fluctuations on scales smaller than the physical scale

$$\lambda_{\min} \sim ct_h \sim 20\,\mathrm{kpc}\left(\frac{m_h c^2}{2\,\mathrm{eV}}\right)^{-2} , \tag{12.78}$$

corresponding to a comoving length scale

$$L_{\min} = \frac{\lambda_{\min}}{a(t_h)} \sim \frac{T_h}{2.725\,\mathrm{K}}\lambda_{\min} \sim 60\,\mathrm{Mpc}\left(\frac{m_h c^2}{2\,\mathrm{eV}}\right)^{-1} . \tag{12.79}$$

[7]It may seem odd to refer to neutrinos as "hot" dark matter, when the temperature of the Cosmic Neutrino Background is only two degrees above absolute zero. The label "hot", in this case, simply means that the neutrinos were hot enough to be relativistic back in the radiation-dominated era.

The total amount of matter within a sphere of comoving radius $L_{\rm min}$ is

$$M_{\rm min} = \frac{4\pi}{3} L_{\rm min}^3 \Omega_{m,0} \rho_{c,0} \sim 5 \times 10^{16}\,{\rm M}_\odot \left(\frac{m_h c^2}{2\,{\rm eV}} \right)^{-3} , \tag{12.80}$$

assuming $\Omega_{m,0} = 0.3$. If the dark matter is contributed by neutrinos with rest energy of a few electron volts, then the free streaming will wipe out all density fluctuations smaller than superclusters.

The upper panel of Figure 12.5 shows the power spectrum of density fluctuations in hot dark matter, once the hot dark matter has cooled enough to become non-relativistic. Note that for wavenumbers $k \ll 2\pi/L_{\rm min}$, the power spectrum of hot dark matter (shown as the dotted line) is indistinguishable from the original $P \propto k$ spectrum (shown as the dashed line). However, the free streaming of the hot dark matter results in a severe loss of power for wavenumbers $k \gg 2\pi/L_{\rm min}$. The lower panel of Figure 12.5 shows that the root mean square mass fluctuations in hot dark matter, $\delta M/M \propto (k^3 P)^{1/2}$, have a maximum amplitude at a mass scale $M \sim 10^{16}\,{\rm M}_\odot$. This implies that in a universe filled with hot dark matter, the first structures to collapse are the size of superclusters. Smaller structures, such as clusters and galaxies then form by fragmentation of the superclusters. (This scenario, in which the largest observable structures form first, is called the *top-down* scenario.)

If most of the dark matter in the universe were hot dark matter, such as neutrinos, then we would expect the oldest structures in the universe to be superclusters, and that galaxies would be relatively young. In fact, the opposite seems to be true in our universe. Superclusters are just collapsing today, while galaxies have been around since at least $z \sim 6$, when the universe was less than a gigayear old. Thus, most of the dark matter in the universe must be *cold* dark matter, for which free streaming has been negligible.

The evolution of the power spectrum of cold dark matter, given the absence of free streaming, is quite different from the evolution of the power spectrum for hot dark matter. Remember, when the universe is radiation-dominated, density fluctuations $\delta_{\vec{k}}$ in the dark matter do not grow appreciably in amplitude, as long as their proper wavelength $a(t)2\pi/k$ is small compared to the Hubble distance $c/H(t)$. However, when the proper wavelength of a density perturbation is large compared to the Hubble distance, its amplitude will be able to increase, regardless of whether the universe is radiation-dominated or matter-dominated. If the cold dark matter consists of WIMPs, they decouple from the radiation at a time $t_d \sim 1\,{\rm s}$, when the

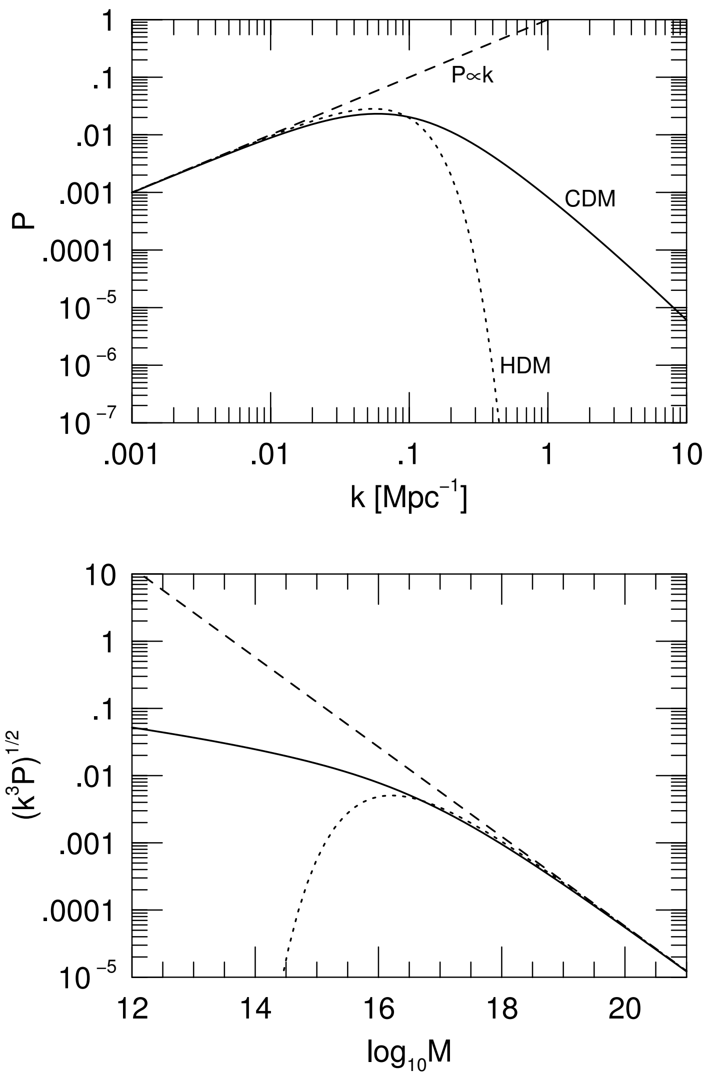

Figure 12.5: Upper panel – The power spectrum at the time of radiation-matter equality for cold dark matter (solid line) and for hot dark matter (dotted line). The initial power spectrum produced by inflation (dashed line) is assumed to have the form $P(k) \propto k$. The normalization of the power spectrum is arbitrary. Lower panel – The root mean square mass fluctuations, $\delta M/M \propto (k^3P)^{1/2}$, are shown as a function of $M \propto k^{-3}$ (masses are in units of $M_\odot$). The line types are the same as in the upper panel.

scale factor is $a_d \sim 3 \times 10^{-10}$. At the time of WIMP decoupling, the Hubble distance is $c/H \sim 2ct_d \sim 6 \times 10^8$ m, corresponding to a comoving wavenumber

$$k_d \sim \frac{2\pi a_d}{2ct_d} \sim 10^5 \,\mathrm{Mpc}^{-1} \ . \tag{12.81}$$

Thus, density fluctuations with a wavenumber $k < k_d$ will have a wavelength greater than the Hubble distance at the time of WIMP decoupling, and will be able to grow freely in amplitude, as long as their wavelength remains longer than the Hubble distance. Density fluctuations with $k > k_d$ will remain frozen in amplitude until matter starts to dominate the universe at $t_{\rm rm} \approx 4.7 \times 10^4$ yr, when the scale factor has grown to $a_{\rm rm} \approx 2.8 \times 10^{-4}$. At the time of radiation-matter equality, the Hubble distance, as given in equation (12.74), is $c/H \approx 1.8ct_{\rm rm} \approx 0.026$ Mpc, corresponding to a comoving wavenumber

$$k_{\rm rm} \approx \frac{2\pi a_{\rm rm}}{1.8ct_{\rm rm}} \approx 0.07 \,\mathrm{Mpc}^{-1} \ . \tag{12.82}$$

Thus, density fluctuations with a wavenumber $k < k_{\rm rm} \approx 0.07\,\mathrm{Mpc}^{-1}$ will grow steadily in amplitude during the entire radiation-dominated era, and for wavenumbers $k < k_{\rm rm} \approx 0.07\,\mathrm{Mpc}^{-1}$, the power spectrum for cold dark matter retains the original $P(k) \propto k$ form which it had immediately after inflation (see the upper panel of Figure 12.5).

By contrast, cold dark matter density perturbations with a wavenumber $k_d > k > k_{\rm rm}$ will be able to grow in amplitude only until their physical wavelength $a(t)/(2\pi k) \propto t^{1/2}$ is smaller than the Hubble distance $c/H(t) \propto t$. At that time, their amplitude will be frozen until the time $t_{\rm rm}$, when matter dominates, and density perturbations smaller than the Hubble distance are free to grow again. Thus, for wavenumbers $k > k_{\rm rm}$, the power spectrum for cold dark matter is suppressed in amplitude, with the suppression being greatest for the largest wavenumbers (corresponding to shorter wavelengths, which come within the horizon at an earlier time). The top panel of Figure 12.5 shows, as the solid line, the power spectrum for cold dark matter at the time of radiation-matter equality. Note the broad maximum in the power spectrum at $k \sim k_{\rm rm} \approx 0.07\,\mathrm{Mpc}^{-1}$. The root mean square mass fluctuations in the cold dark matter, shown in the bottom panel of Figure 12.5 are largest in amplitude for the smallest mass scales. This implies that in a universe filled with cold dark matter, the first objects to form are the *smallest*, with galaxies forming first, then clusters, then superclusters. This scenario, called the *bottom-up* scenario, is consistent with the observed ages of galaxies and superclusters.

Assuming that the dark matter consists of nothing but hot dark matter gives a poor fit to the observed large scale structure of the universe. Assuming that the dark matter is purely cold dark matter gives a much better fit. However, there is strong evidence that neutrinos do have some mass, and thus that the universe contains at least *some* hot dark matter. Cosmologists studying the large scale structure of the universe can adjust the assumed power spectrum of the dark matter, by mixing together hot and cold matter. (It's a bit like adjusting the temperature of your bath by tweaking the hot and cold water knobs.) Comparison of the assumed power spectrum to the observed large scale structure (as seen, for instance, in figure 12.1) reveals that $\sim 13\%$ or less of the matter in the universe consists of hot dark matter. For $\Omega_{m,0} = 0.3$, this implies $\Omega_{\mathrm{HDM},0} \leq 0.04$. If there were more hot dark matter than this amount, free streaming of the hot dark matter particles would make the universe too smooth on small scales. Some like it hot, but most like it cold – the majority of the dark matter in the universe must be *cold* dark matter.

Suggested reading

[Full references are given in the "Annotated Bibliography" on page 286.]

Liddle & Lyth (2000): The origin of density perturbations during the inflationary era, and their growth thereafter.

Longair (1998): For those who want to know more about galaxy formation, and how it ties into cosmology.

Rich (2001), ch. 7: The origin and evolution of density fluctuations.

Problems

(12.1) Consider a spatially flat, matter-dominated universe ($\Omega = \Omega_m = 1$) which is *contracting* with time. What is the functional form of $\delta(t)$ in such a universe?

(12.2) Consider an empty, negatively curved, expanding universe, as described in section 5.2. If a dynamically insignificant amount of matter ($\Omega_m \ll 1$) is present in such a universe, how do density fluctuations in the matter evolve with time? That is, what is the functional form of $\delta(t)$?

(12.3) A volume containing a photon-baryon fluid is adiabatically expanded or compressed. The energy density of the fluid is $\varepsilon = \varepsilon_\gamma + \varepsilon_{\rm bary}$, and the pressure is $P = P_\gamma = \varepsilon_\gamma/3$. What is $dP/d\varepsilon$ for the photon-baryon fluid? What is the sound speed, c_s? In equation (12.27), how large of an error did I make in my estimate of λ_J(before) by ignoring the effect of the baryons on the sound speed of the photon-baryon fluid?

(12.4) Suppose that the stars in a disk galaxy have a constant orbital speed v out to the edge of its spherical dark halo, at a distance $R_{\rm halo}$ from the galaxy's center. What is the average density $\bar{\rho}$ of the matter in the galaxy, including its dark halo? (Hint: go back to section 8.2.) What is the value of $\bar{\rho}$ for our Galaxy, assuming $v = 220\,{\rm km\,s^{-1}}$ and $R_{\rm halo} = 100\,{\rm kpc}$? If a bound structure, such as a galaxy, forms by gravitational collapse of an initially small density perturbation, the minimum time for collapse is $t_{\rm min} \approx t_{\rm dyn} \approx 1/\sqrt{G\bar{\rho}}$. Show that $t_{\rm min} \approx R_{\rm halo}/v$ for a disk galaxy. What is $t_{\rm min}$ for our own Galaxy? What is the maximum possible redshift at which you would expect to see galaxies comparable in v and $R_{\rm halo}$ to our own Galaxy? (Assume the Benchmark Model is correct.)

(12.5) Within the Coma cluster, as discussed in section 8.3, galaxies have a root mean square velocity of $\langle v^2\rangle^{1/2} \approx 1520\,{\rm km\,s^{-1}}$ relative to the center of mass of the cluster; the half-mass radius of the Coma cluster is $r_h \approx 1.5\,{\rm Mpc}$. Using arguments similar to those of the previous problem, compute the minimum time $t_{\rm min}$ required for the Coma cluster to form by gravitational collapse.

(12.6) Derive equation (12.74), giving the Hubble distance at the time of radiation-matter equality. What was the Hubble distance at the time of matter-lambda equality, in the Benchmark Model? How much matter was contained within a Hubble volume at the time of matter-lambda equality?

(12.7) Warm dark matter is defined as matter which became non-relativistic when the amount of matter within a Hubble volume had a mass comparable to that of a galaxy. In the Benchmark Model, at what time $t_{\rm WDM}$ was the mass contained within a Hubble volume equal to $M_{\rm gal} = 10^{12}\,{\rm M}_\odot$? If the warm dark matter particles have a temperature equal to that of the cosmic neutrino background, what mass must they have in order to have become non-relativistic at $t \sim t_{\rm WDM}$?