Série2_SP

1. The Virial Theorem shows that for a gravitationally bound system in equilibrium:

$$\langle U \rangle + 2\,\langle K \rangle = 0 \tag{1}$$

From this we see that if twice of the internal kinetic energy of the system is less than its gravitational potencial energy then a gravitational instability arises causing the system to collapse. James Jeans noted that for an interstellar cloud of gas this collapse starts when a critical mass is exceeded. This mass was named *Jeans mass* and can be deduced by solving Equation 1. The kinetic energy of a ideal gas particle is related to its temperature:

$$\langle K \rangle = \frac{3}{2}\,k\,T$$

Considering a spherical cloud of gas with radius $\boldsymbol{R}$, mass $\boldsymbol{M}$ and constant density $\boldsymbol{\rho}$, we know that its potencial gravitational energy is:

$$\langle U \rangle = -G\,\frac{M^2}{R}$$

Since the cloud is homogeneous $M \propto \rho\,R^3$ therefore we can describe R in terms of total mass and density, obtaining that $U \propto \rho^{\frac{1}{3}}\,M^{\frac{5}{3}}$. To finaly calculate the mass of Jeans, $\boldsymbol{M_J}$, we solve for the critical mass of Equation 1:

$$C\,\rho^{\frac{1}{3}}\,M_J^{\frac{5}{3}} = 3\,N\,k\,T \;\Leftrightarrow\; C\,\rho^{\frac{1}{3}}\,M_J^{\frac{5}{3}} = 3\,\frac{M_J}{m}\,k\,T$$

where C is a constant of proportionality and we multiply the kinetic energy of a single particle by the number of particles of the system, N, that is the total mass divided by the mean particle mass, m. Continuing this equation we get:

$$M_J^{\frac{2}{3}} = C'\,\rho^{-\frac{1}{3}}\,T \;\Leftrightarrow\; M_J \propto \rho^{-\frac{1}{2}}\,T^{\frac{3}{2}} \tag{2}$$

2. The moment of inertia $\boldsymbol{I}$ of any object is given by:

$$I = \int dI = \int r^2\,dm$$

where dm is the mass element. Solving this integral for each of the cartesian axis we obtain:

$$\begin{cases} I_x = \int (y^2 + z^2)\,\rho(r)\,dm \\ I_y = \int (z^2 + x^2)\,\rho(r)\,dm \\ I_z = \int (x^2 + y^2)\,\rho(r)\,dm \end{cases}$$

considering that the object is a planetary type spherical object with density $\boldsymbol{\rho} \equiv \rho(r)$ and $dm = \rho(r)\,dV$. Making the assumption that it has spherical symmetry, we know that its momentum of inertia is the same with respect to the each axis so $I = I_x = I_y = I_z$. In this case:

$$3I = I_x + I_y + I_z = \int 2(x^2 + y^2 + z^2)\,\rho(r)\,dV$$

Changing to spherical coordinates we have:

$$3I = 4\pi \int_0^R 2r^2\,\rho(r)\,r^2\,dr \Rightarrow\; I = \frac{8\pi}{3}\int_0^R \rho(r)\,r^4\,dr$$

3. The relation between a planet radius, $\boldsymbol{R}$, and its mass, $\boldsymbol{M}$, is given by:

$$R(M) = 2\gamma_k\,M^{\frac{1}{3}}\left(\frac{Z}{A}\right)^{\frac{5}{3}}\left(\gamma_e\frac{Z^2}{A^{\frac{4}{3}}} + \gamma_G M^{\frac{2}{3}}\right)^{-1} \tag{3}$$

where Z and A are the atomic number and mass number, respectively, of the main element that composes the planet, and the gamma parameters will be set as $\gamma_e = 10^8$, $\gamma_k = 7 \times 10^6$ and $\gamma_G = 6.67408 \times 10^{-11}$.

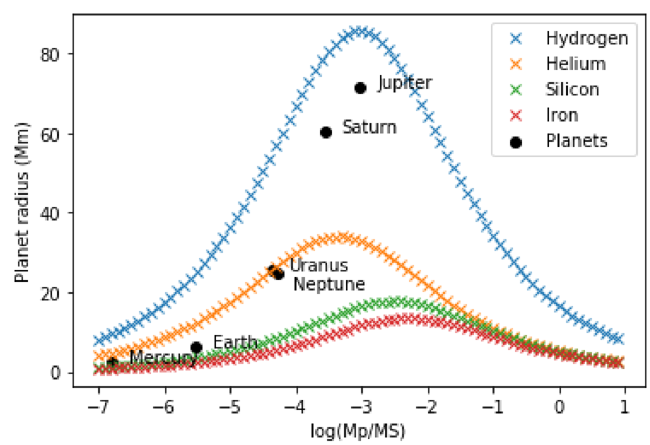

Figure 1. Graph of the relation of a planet radius, in millions of meters, and the logarithm of its mass expressed in solar masses, for different planetary ccompositions. The values of different solar system planets were overlaped in the graph.

The graph of the planet radius in function of the logarithm of its mass expressed in Solar Masses was plotted using *python's matplotlib* library with a list of 80 mass points, uniformely distibuted on the log scale, from 10^{-7} to 10 solar masses and considering 4 main types of planetary composition - hydrogen ($A = 1, Z = 1$), helium ($A = 4, Z = 2$), silicon ($A = 28.1, Z = 14$) and iron ($A = 55.8, Z = 26$). Then the mass and radius values of different planets of the solar system were scatered on the plot as seen in Figure 1.

4.
a. (discussed with Rafael Silva) We start by calculating

Jupiters critical mass, $\boldsymbol{M_c}$. For this we consider a planet made of hydrogen, the main component of Jupiter, closest planetary composition type as in Figure 1 and the element that the aliens are adding, i.e. increasing its relative abudance. From Equation 3:

$$\frac{1}{R} = \frac{\gamma_e}{2\gamma_k}\,(ZA)^{\frac{1}{3}}\,M^{-\frac{1}{3}} + \frac{\gamma_G}{2\gamma_k}\left(\frac{A}{Z}\right)^{\frac{5}{3}} M^{\frac{1}{3}} \;\Rightarrow$$

$$-\frac{1}{R^2}\frac{\mathrm{d}R}{\mathrm{d}M} = -\frac{\gamma_e}{6\gamma_k}\,(ZA)^{\frac{1}{3}}\,M^{-\frac{4}{3}} + \frac{\gamma_G}{6\gamma_k}\left(\frac{A}{Z}\right)^{\frac{5}{3}} M^{-\frac{2}{3}}$$

But M_c is reached when $\frac{dR}{dM} = 0$:

$$\frac{\gamma_e}{\gamma_G}\,(ZA)^{\frac{1}{3}}\,M_c^{-\frac{4}{3}} = \left(\frac{A}{Z}\right)^{\frac{5}{3}} M_c^{-\frac{2}{3}} \;\Rightarrow\; M_c = \left(\frac{\gamma_e}{\gamma_G}\right)^{\frac{3}{2}} \frac{Z^3}{A^2}$$

So we see that $M_c \sim 1.83\times10^{27}$ kg, very close to Jupiter's $1,89 \times 10^{27}$ kg. This means that Jupiter is in a critical regime where adding mass could trigger thermonuclear processes, creating a star. We can conclude that there *could* be a Jupiter radius star, since its radius would stay relatively the same when mass is added, however the processes and mechanisms that arise from considering a star, and not a planet, should be further analysed for a more in-depth analysis.

b. Stellar formation theory says that the minimum mass needed for thermonuclear reactions at the core of a stellar body is around 2.3×10^{27} kg. Jupiter's mass is about 1.8×10^{27} kg, so the aliens would have to add 5×10^{26} kg of hydrogen for it to ignite and fuse in the core, so about 5 Neptunes worth!

5. To estimate the cooling time by conduction of a planetary object with thermal diffusivity, $\boldsymbol{K}$, as a function of its radius, $\boldsymbol{R}$, we have to consider the the equation of heat diffusion given by:

$$\frac{\partial T}{\partial t} = K\nabla^2 T$$

where $T \equiv T(t,x,y,z)$ represents the temperature at any given point inside the planet and time since its formation, and no internal heat sources were taken into account.

Since we want only to estimate the order of magnitude of cooling time, we can approximate the derivative of T over time to the change of temperature during the cooling time. Considering the planet a body with spherical symmetry, the laplacian will only be applied along the radius so again we can approximate this operator to the change T over the square of the radius, since it´s a second order derivative and the planet cools completely.

Applying this approximations we derive that:

$$\frac{\partial T}{\partial t} = K\nabla^2 T \;\simeq\; \frac{\Delta T}{t_c} = K\frac{\Delta T}{R^2}$$

so the time of cooling of a planet is $t_c \approx \frac{R^2}{K}$.

Considering the objects formed in the primitive solar system, i.e. $\sim 4\times10^9$ years, the upper limit for the object radius knowing that it already lost all its internal heat and that $K \simeq 10^{-6}m^2s^{-1}$ is:

$$R \approx \sqrt{K\,t_c} \approx 355\,\mathrm{km} \tag{4}$$

This equates that only objects with radius in the order of the hundreds of kilometers or less have completely cooled since the formation of the solar system. This indicates that for example Vesta has already completely cooled off, considering no significant heat input since formation or internal heat sources.

6.
a. The observational success of Dione is maximized at the points where Earth and Saturn are closest. This distance is minimal when Earth is at aphelion and Saturn is at perihelion and the planets are aligned with the Sun (Sun-Earth-Saturn).

b. Since our objective is to study the leading and trailing hemispheres of Dione, the best interval of time for successive observation will be half its orbital period, i.e $66/2 = 33$ hours.

c. Saturn's axis is tilted by 26.73º with respect to its orbit around the Sun. Since its northen hemisphere is in the summer solistice and Dione in opposition in relation to Saturn, we need to confirm if our observations will not be compromised by Saturn's body or rings.

For this we compare the anglular distance, $\boldsymbol{\theta}$, between each body - Saturn, rings and Dione - and the orbital plane from Earth's prespective. We consider that Saturn is in the orbital plane, and the conditions in a. are met so the Earth-Saturn distance, $\boldsymbol{D_S}$, is the difference between Earth's aphelion and Saturn's perihelion:

$$\tan\theta_S = \frac{R_S}{D_S} = \frac{58\,232}{(1\,352.555 - 152.099)\times10^6}$$

that is $\theta_S \approx 0.00278^{\circ}$. For the rings, we'll consider that they are in the plane of Saturn's equator, i.e. $\phi = 26.73^{\circ}$, and that it extends until ring A, at around $D_R = 136\,775$ km.

$$\tan\theta_R = \frac{h_R}{D_S + d_R} = \frac{D_R\sin\phi}{D_S + D_R\cos\phi}$$

Doing the calculations we obtain that $\theta_R \approx 0.00294^{\circ}$, so we could barely see the rings popping out at the back of Saturn. The calculation of Dione's angular distance from Earth is similar to Saturn's rings but considering its orbital distance, $D_D = 377\,346$ km, and relative inclination to Saturn equatorial plane. This yields that $\theta_D \approx 0.00809^{\circ}$, which means that Dione is not eclipsed by Saturn and this would not pose a problem to our observations.