Série4_SP

1. a. Considering an impact with relative speed and potential energy negligible, the speed of the impactor during impact with the surface will be the escape velocity. This is the minimum velocity needed for a free, non-propelled object to escape from the gravitational influence of a primary body, thus reaching an infinite distance from it.

$$\frac{1}{2}mv_{esc}^2 = G\frac{Mm}{R} \Rightarrow v_{esc} = \sqrt{2\frac{GM}{R}} \tag{1}$$

where **G** is the gravitational constant, **M** and **R** the mass and radius of the primary body, **m** is the mass of the object. Considering Mars as the primary body and an impacting object of density $\boldsymbol{\rho}$ and radius **R**, the kinetic energy with which the impactor reaches the planet's surface, **K**, is given by:

$$K = \frac{1}{2}mv_{esc}^2 \overset{(1)}{=} G\frac{M_{Mars}}{R_{Mars}} \times \frac{4}{3}\rho\pi R^3$$

considering a spherical impactor.

b. We know that $K_{proj} = K/2$, $\mathbf{K_{proj}}$ being the kinetic enery of the projected material. So the energy per mass unit of this material is:

$$\frac{K_{proj}}{m_{proj}} = \frac{\frac{2}{3}\rho\pi R^3 G\frac{M_{Mars}}{R_{Mars}}}{\frac{4}{3}\rho_{proj}\pi R_{proj}^3} = \frac{G}{2}\frac{\rho}{\rho_{proj}}\frac{M_{Mars}}{R_{Mars}}\left(\frac{R}{R_{proj}}\right)^3$$

considering that the mass of material projected is equivalent to a sphere of radius $\mathbf{R_{proj}}$ and density $\boldsymbol{\rho_{proj}}$.

c.

$$\frac{K_{proj}}{m_{proj}} = \frac{v_{proj}^2}{2} = \frac{1}{2}\frac{\rho}{\rho_{proj}}\left(\frac{R}{R_{proj}}\right)^3\frac{GM_{Mars}}{R_{Mars}}$$

From equation 1 we can obtain the speed of the projected material as a function of the escape velocity. Defining now $v_{esc} \equiv v_{esc}(Mars)$:

$$\frac{v_{proj}^2}{2} = \frac{1}{2}\frac{\rho}{\rho_{proj}}\left(\frac{R}{R_{proj}}\right)^3\frac{v_{esc}^2}{2} \Leftrightarrow$$

$$v_{proj} = \frac{1}{\sqrt{2}}\left(\frac{\rho}{\rho_{proj}}\right)^{\frac{1}{2}}\left(\frac{R}{R_{proj}}\right)^{\frac{3}{2}}v_{esc} \tag{2}$$

d. If $R_{proj} = R$ and $\rho_{proj} = \rho$ we obtain, from equation 2, that the velocity of the projected material is $v_{proj} = v_{esc}/\sqrt{2} < v_{esc}$, so this material could not escape the planet's gravitational influence.

e. If however the projected material originates from the crust, i.e. $\rho_{proj} = \rho_{crust}$, we can obtain an escape condition for that material.

$$v_{proj} \geq v_{esc} \overset{(2)}{\Leftrightarrow} \frac{1}{\sqrt{2}}\left(\frac{\rho}{\rho_{proj}}\right)^{\frac{1}{2}}\left(\frac{R}{R_{proj}}\right)^{\frac{3}{2}}v_{esc} \geq v_{esc}$$

making the escape condition as follows:

$$\frac{\rho}{\rho_{proj}}\left(\frac{R}{R_{proj}}\right)^3 \geq 2$$

2. We seek to calculate the $\tan\theta/2$ for a hyperbolic orbit, where $\boldsymbol{\theta}$ is the supplementary angle of the angle, $\boldsymbol{\alpha}$, between the asymptotes of the hyperbola ($\theta = \pi - \alpha$). For hiperbolic trajectories:

$$r = \frac{a(e^2-1)}{1+e\cos(\phi-\phi_0)}$$

We define α as the diference of the angles when $r \to \pm\infty$ so when $1+e\cos(\phi-\phi_0) \to 0 \Leftrightarrow \cos(\phi-\phi_0) \to -1/e$. Since ϕ_0 is the angle at the perielium $\cos\alpha/2 = -1/e$. From trigonometry we know the half angle formulas of tangent and cosine, respectively:

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}\,,\ \cos(2x) = 2\cos^2 x - 1 \tag{3}$$

As we seek $\tan\theta/2$ we have to know $\cos\theta$ or $\cos(\pi-\alpha) = -\cos\alpha$. But, from equation 3 we know that:

$$\cos\alpha = 2\cos^2\alpha/2 - 1 = \frac{2}{e^2} - 1 \Rightarrow \cos\theta = 1 - \frac{2}{e^2}$$

so,

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\left(1-2/e^2\right)}{1+1-2/e^2}} = \sqrt{\frac{1}{e^2-1}} = (e^2-1)^{-1/2}$$

3. We consider that the limit case where the elliptic orbit of the object that falls into the Sun as it leaves Mars, has a null minor axis and a major axis of $\boldsymbol{a} = a_{Mars}/2$. We further admit that Mars follows a circular orbit of radius $a_{Mars} = 1.5237$ AU and disregard initial velocity and the Sun's radius ($R_\odot/a_{Mars} = 0.3\%$).

By Kepler's third law (deduction in exercise 4) we know that every body in Solar system follows the $T^2 = a^3$ rule, with T in years and a in AU. So $T = a^{3/2} = (1.5237/2)^{3/2} \approx 0.66$ years. However, this is the time needed to complete one orbit, but just a half orbit is made until the object falls into the Sun. So the time it takes for an object to fall from Mars to the Sun is ~ 0.33 years, or 121 days.

4. Kepler deduced its third law of motion by the centripetal and gravitational forces of a planetary body.

$$ma\left(\frac{2\pi}{T}\right)^2 = G\frac{mM}{a^2} \rightarrow T^2 = C\,a^3$$

where $C = 4\pi^2/GM$. This is constant for any body in a given stellar system, considering that its central body has all the mass of the system. In the case of Earth $T = 1$ year and $a = 1$ AU, such as $C = 4\pi^2/GM_\odot = 1$ and defining the Kepler's third law as $T^2(year) = a^3(AU)$ for the solar system. However if the mass of the Sun, $M_\odot$, were to change, C would change accordingly, making this direct relationship no longer valid.

5. From the orbit equation we have that:

$$r_i = \frac{a\,(1-e^2)}{1 + e\cos\left(u_i - \omega\right)} \tag{4}$$

so,

$$r_2 - r_1 = \frac{a\,(1-e^2)}{1 + e\cos\left(u_2 - \omega\right)} - \frac{a\,(1-e^2)}{1 + e\cos\left(u_1 - \omega\right)} =$$

$$a(1-e^2)\Big(1 - 1 + \frac{1}{1 + e\cos\left(u_2 - \omega\right)} - \frac{1}{1 + e\cos\left(u_1 - \omega\right)}\Big)$$

$$= a\,(1-e^2)\Big(\frac{e\cos\left(u_1 - \omega\right)}{1 + e\cos\left(u_1 - \omega\right)} - \frac{e\cos\left(u_2 - \omega\right)}{1 + e\cos\left(u_2 - \omega\right)}\Big)$$

substituting equation 4 we have,

$$r_1\, e\cos\left(u_1 - \omega\right) - r_2\, e\cos\left(u_2 - \omega\right) =$$

$$r_1 e(\cos u_1 \cos\omega + \sin u_1 \sin\omega) - r_2 e(\cos u_2 \cos\omega + \sin u_2 \sin\omega)$$

We note that $r_i \cos u_i = r_i \cdot \mathrm{e}_1$ and $r_i \sin u_i = r_i \cdot \mathrm{e}_2$, e_1 being the unit vector over the node line's direction and e_2 being the unit vector perpendicular to the node line and contained in the plane (r_1, r_2, r_3). This way we resolve that,

$$r_2 - r_1 = e\cos\omega(r_1 - r_2)\cdot \mathrm{e}_1 - e\sin\omega(r_1 - r_2)\cdot \mathrm{e}_2$$

For r_3 instead of r_2, the calculations are analogous.

6. a. The mean anomaly, **M**, gives the angular distance from the pericenter at arbitrary time **t**. It is defined as $M = 2\pi(t-\tau)/T$, where **T** is the orbital period and $\boldsymbol{\tau}$ the time at which the body is at the pericenter. We'll consider $\tau = 0$ at May 20th at 0h U.T. and t the time elapsed until June 7, at 6:00 AM so $t = 18 * 24 + 6 = 438$ h. We know that $T_{Mer} = 87.97 * 24 = 2111.28$ h and the eccentricity is $e = 0.21$. Therefore,

$$M = \frac{2\pi}{2111.28}(438) = 1.3$$

The mean anomaly is defined as $M = \epsilon - e\sin\epsilon$ by Kepler's equation, where ϵ is the eccentric anomaly. Knowing ϵ we can calculate the distance of Mercury to the Sun at June 7, 6:00 AM by the equation $r = a(1 - e\cos\epsilon)$. However we cannot solve explicity Kepler's equation, so we'll solve it iteractively. This yields $\epsilon = 1.51$ radians therefore $r = 57\,170\,167$ km, or 0.38 AU, considering $a_{Mer} = 57\,909\,050$ km.

b. The relation between mean and true anomalies is:

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\epsilon}{2}$$

where f is the true anomaly. Doing the calculations:

$$\tan\frac{f}{2} = 1.165 \;\Rightarrow\; f = 1.72\,\text{radians}$$

(All the exercises were discussed among all the students of the course)