Computational statistics 2021.2
School of Applied Mathematics, Fundação Getulio Vargas
Professor Luiz Max de Carvalho

Problem sheet 2

Lucas Machado Moschen

Exercício 1. *(Monte Carlo for Gaussian)*
Let us consider the normal multivariate density on $\mathbb{R}^d$ with identity covariance, that is

$$\pi(x) = \frac{1}{(2\pi)^{d/2}} \exp\left\{-\frac{1}{2}x^T x\right\}.$$

1. *(Cameron-Martin formula). Show that for any $\theta \in \mathbb{R}^d$ and function $\phi : \mathbb{R}^d \to \mathbb{R}$*

$$\mathbb{E}[\phi(X)] = \mathbb{E}\left[\phi(X+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T X\right)\right].$$

   Let ϕ be any measurable function and $\theta \in \mathbb{R}^d$. Denote I_2 the quantity in the right of the equation. Then,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^d} \phi(x+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T x\right)\pi(x)\,dx \\
&= \int_{\mathbb{R}^d} \phi(x+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T x\right)\pi(x)\,dx \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}}\phi(x+\theta)\exp\left(-\frac{1}{2}(x+\theta)^T(x+\theta)\right)\,dx \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}}\phi(y)\exp\left(-\frac{1}{2}y^T y\right)\,dx \\
&= \mathbb{E}[\phi(X)].
\end{aligned}$$

2. *It follows directly from the Cameron-Martin formula and the strong law of large numbers that, for independent $X_1, \dots, X_n \sim \pi$, the estimator*

$$\hat{I}_n(\theta) = \frac{1}{n}\sum_{i=1}^n \phi(X_n+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T X_i\right)$$

   of $\mathbb{E}[\phi(X)]$ is strongly consistent for any $\theta \in \mathbb{R}^d$ such that

$$\mathbb{E}\left[|\phi(X+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T X\right)|\right] < +\infty.$$

   The case $\theta = 0$ corresponds to the usual Monte Carlo estimate. The variance of $\hat{I}_n(\theta)$ is given by $\sigma^2(\theta)/n$ where

$$\sigma^2(\theta) = \mathrm{Var}\left(\phi(X+\theta)\exp\left(-\frac{1}{2}\theta^T\theta - \theta^T X\right)\right).$$

We assume in the sequel that $\sigma^2(\theta) < \infty$ for any θ. Show that

$$\sigma^2(\theta) = \mathbb{E}\left[\phi^2(X)\exp\left(-\frac{1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] - (\mathbb{E}[\phi(X)])^2$$

Let $\sigma^2(\theta) = \mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$ to simplify the writing. We already know that $\mathbb{E}[Y] = \mathbb{E}[\phi(X)]$ by the last exercise. Therefore, it remains to prove that

$$\mathbb{E}[Y^2] = \mathbb{E}\left[\phi^2(X)\exp\left(-\frac{1}{2}X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right].$$

For that,

$$\begin{aligned}
\mathbb{E}[Y^2] &= \int_{\mathbb{R}^d} \phi^2(x+\theta)\exp\left(-\theta^T\theta - 2\theta^Tx\right)\pi(x)\,dx \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}}\phi^2(x+\theta)\exp\left(-\theta^T\theta - 2\theta^Tx - \frac{1}{2}x^Tx\right)\,dx \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}}\phi^2(x+\theta)\exp\left(-(x+\theta)^T(x+\theta) + \frac{1}{2}x^Tx\right)\,dx \\
&= \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}}\phi^2(y)\exp\left(-y^Ty + \frac{1}{2}(y-\theta)^T(y-\theta)\right)\,dx \\
&= \mathbb{E}\left[\phi^2(X)\exp\left(-X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right],
\end{aligned}$$

as we wanted to prove.

3. *A twice differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ is strictly convex if $\nabla^2 f(\theta)$ (called the Hessian of f) is a positive definite matrix for any $\theta \in \mathbb{R}^d$. Deduce from the expression of $\sigma^2(\theta)$ given in (2) that the function $\theta \to \sigma^2(\theta)$ is strictly convex.*

   For that, we will use the derived expression in the last exercise and we differentiate under the expected value using the Leibniz Rule. Then,

$$\nabla_\theta\sigma^2(\theta) = \mathbb{E}\left[\phi^2(X)(\theta - X)\exp\left(-X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right]$$

   and

$$\nabla_\theta^2\sigma^2(\theta) = \mathbb{E}\left[\phi^2(X)\exp\left(-X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\left((\theta-X)^T(\theta-X)+1\right)\right],$$

   which is clearly positive definite since $(\theta - X)^T(\theta - X)$ is semi definite positive.

4. Show that the minimum of $\theta \to \sigma^2(\theta)$ is reached at θ^* such that

$$\mathbb{E}[\phi^2(X)(\theta^* - X)\exp(-\theta^{*T}X)] = 0.$$

   Since $\sigma^2(\theta)$ is differentiable, its critical points are the solution of $\nabla_\theta\sigma^2(\theta) = 0$,

$$\begin{aligned}
&\mathbb{E}\left[\phi^2(X)(\theta - X)\exp\left(-X^TX + \frac{1}{2}(X-\theta)^T(X-\theta)\right)\right] = 0 \\
\implies &\mathbb{E}\left[\phi^2(X)(\theta - X)\exp\left(-\frac{1}{2}X^TX - \theta^TX\right)\right] = 0,
\end{aligned}$$

since $e^{\theta^T \theta/2}$ is a positive constant. Since the function is strictly convex, we already know that there is only one minimal and it occurs when the above expression is zero.