

Assignment I: Reducing the variance of Monte Carlo Estimators.

Computational Statistics
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Background

Suppose we are interested in an integrable function $\varphi(\cdot)$ and its expectation under a probability distribution with law π , $I = \int_{\mathbb{X}} \varphi(x)\pi(x)dx$. Monte Carlo is a large class of sampling algorithms to approximate integrals. It depends on constructing an estimator

$$\hat{I}_n = \frac{1}{n} \sum_{i=1}^n \varphi(X_i), \quad (1)$$

where $X_i \sim \pi$. The estimator in (1) can be improved in a number of ways. In particular, the procedure known as *Rao-Blackwellisation* allows one to obtain estimators that have lower variance with the same bias, i.e., are more efficient.

Questions

Hint: I suggest you employ $\varphi(x) = x^2$.

1. Show that, if $\pi(x) = \int_{\mathbb{Y}} g(x | y)h(y) dy$, then both

$$\begin{aligned} \hat{I}_n^{\text{MC}} &= \frac{1}{n} \sum_{i=1}^n \varphi(X_i), \quad X_i \sim \pi, \text{ and} \\ \hat{I}_n^{\text{RB}} &= \frac{1}{n} \sum_{i=1}^n E[\varphi(X) | Y_i], \quad Y_i \sim h, \end{aligned}$$

converge to $E_{\pi}[\varphi(X)]$.

2. For each of the following cases, generate random variables X_i and Y_i and compute the average and variance of the “vanilla” Monte Carlo and Rao-Blackwellised estimators:
 - (a) $X | Y \sim \text{Poisson}(Y)$, $Y \sim \text{Gamma}(a, b) \implies X \sim \text{Negative-Binomial}$;
 - (b) $X | Y \sim \text{Normal}(0, Y)$, $Y \sim \text{Gamma}(a, b) \implies X \sim \text{Generalised-t}$;

(c) $X | Y \sim \text{Binomial}(n, Y), Y \sim \text{Beta}(a, b) \implies X \sim \text{Beta-Binomial}$;

3. Propose a Rejection Control (see Question 5 in exercise sheet 1) algorithm to sample from each of the targets in the previous question;
4. Compute (empirically) the n_{eff} for the results in the previous question. How does the algorithm perform in each case? Do you have to fiddle with the value of $c > 0$ in order to obtain better performance? Discuss.
5. Compare the variance of the Rejection Control estimate with the variance of the “vanilla” Monte Carlo estimator and the Rao-Blackwellised estimator;
6. What other improvements (in terms of variance and bias) would you propose? Can you come up with an algorithm that dominates all algorithms studied here for the targets under consideration?