Computational statistics 2021.2
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Problem sheet 1

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Exercício 1. *(Inversion and Rejection)*

1. *Let $Y \sim \text{Exp}(\lambda)$ and let $a > 0$. We consider the variable after restricting its support to be $[a, +\infty)$. That is, let $X = Y_{|Y \geq a}$, i.e. X has the law of Y conditionally on being in $[a, +\infty)$. Calculate $F_X(x)$, the cumulative distribution function of X, and $F_X^{-1}(u)$, the quantile function of X. Describe an algorithm to simulate X from $U \sim \text{Unif}[0, 1]$.*

   If $x \geq a$, we have that

$$
\begin{aligned}
F_X(x) &= \mathbb{P}(Y \leq x \mid Y \geq a) \\
&= \frac{\mathbb{P}(Y \in [a, x])}{\mathbb{P}(Y \geq a)} \\
&= \frac{1 - e^{-\lambda x} - (1 - e^{-\lambda a})}{e^{-\lambda a}} \\
&= 1 - e^{-\lambda(x-a)},
\end{aligned}
$$

   otherwise, $F_X(x) = 0$. Let $u = 1 - e^{-\lambda(x-a)}$. Inverting this function, we get that

$$
F_X^{-1}(u) = a - \frac{\log(1-u)}{\lambda}.
$$

   A simple algorithm is the following

   (i) Let $U \sim \text{Unif}[0, 1]$.
   (ii) Define $X = F_X^{-1}(U)$. Then X has the desired distribution by the inversion method.

2. *Let a and b be given, with $a < b$. Show that we can simulate $X = Y_{|a \leq Y \leq b}$ from $U \sim \text{Unif}[0, 1]$ using*

$$
X = F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U),
$$

   i.e. show that if X is given by the formula above, then $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x \mid a \leq Y \leq b)$. Apply the formula to simulate an exponential random variable conditioned to be greater than a, as in the previous question.

   Using the properties of the (generalized) inverse and some affine transformations, note that

$$
\begin{aligned}
\mathbb{P}(X \leq x) &= \mathbb{P}(F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U) \leq x) \\
&= \mathbb{P}(F_Y(a)(1-U) + F_Y(b)U \leq F_Y(x)) \\
&= \mathbb{P}(U(F_Y(b) - F_Y(a)) \leq F_Y(x) - F_Y(a)) \\
&= \mathbb{P}\left(U \leq \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}\right) = \frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)}.
\end{aligned}
$$

However,

$$\frac{F_Y(x) - F_Y(a)}{F_Y(b) - F_Y(a)} = \frac{\mathbb{P}(Y \le x) - \mathbb{P}(Y \le a)}{\mathbb{P}(Y \le b) - \mathbb{P}(Y \le a)} = \mathbb{P}(Y \le x \mid Y \in [a, b]),$$

what concludes that X has the same distribution of $F_Y^{-1}(F_Y(a)(1-U) + F_Y(b)U)$. Taking $b = +\infty$, we can simulate $U \sim \text{Unif}[0,1]$ and use

$$X = F_Y^{-1}(F_Y(a)(1-U) + U).$$

3. *Here is a simple algorithm to simulate $X = Y_{|Y>a}$ for $Y \sim \text{Exp}(\lambda)$:*

   (a) *Let $Y \sim \text{Exp}(\lambda)$. Simulate $Y = y$.*

   (b) *If $Y > a$ then stop and return $X = y$, and otherwise, start again at step (a).*

   Show that this is just a rejection algorithm, by writing the proposal and target densities π and q, as well as the bound $M = \max_x \pi(x)/q(x)$. Calculate the expected number of trials to the first acceptance. Why is inversion to be preferred for $a \gg 1/\lambda$?

   The target density $\pi(x) = \frac{d}{dx}F_X(x) = \lambda e^{-\lambda(x-a)}1_{\{x \ge a\}}$ is the density of X, while the proposal density is the exponential $q(x) = \lambda e^{-\lambda x}1_{\{x \ge 0\}}$. Therefore, the bound is

   $$M = \sup_{x \ge 0} \frac{\pi(x)}{q(x)} = \sup_{\{x \ge a\}} e^{\lambda a} = e^{\lambda a}.$$

   The probability of accepting $X = y$ is

   $$\alpha(y) = \frac{\pi(y)}{Mq(y)} = \begin{cases} 0, & \text{if } y \le a \\ 1, & \text{if } y > a. \end{cases}.$$

   This is only the rejection sampling algorithm. Let N be the number os trials to the first acceptance. We already know that N is geometrically distributed with parameter $M^{-1} = e^{-\lambda a}$. In our case, this is easy to see, because,

   $$\mathbb{P}(N > n) = \mathbb{P}(Y \le a)^n = (1 - e^{-\lambda a})^n.$$

   We conclude that $\mathbb{E}[N] = e^{\lambda a}$. When $a \gg 1/\lambda$, we have that $\mathbb{E}[N] \gg e$ and several trials are rejected until a desired sample come. In that case, is much simpler to use the inversion method.