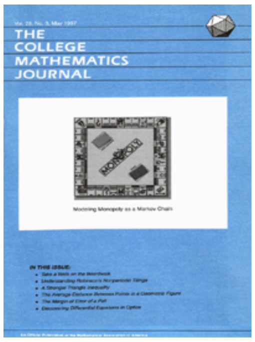

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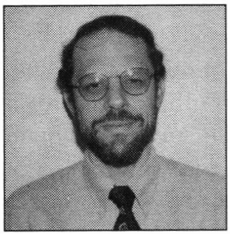

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Consider the following experiment: Find the distance between two points chosen at random (i.e., independently, each with uniform distribution) from a convex region, say a disk of diameter d. Repeat the experiment several times, and you can find an average distance between the points. The average distance depends in some way on the diameter d; the larger the disk, the farther apart on average we expect the points to be. Here is my question: *What do you think is the functional relationship between the average distance between the points and the diameter?* That is, as the diameter goes up, does the average distance increase as the square root of the diameter, the diameter, the square of the diameter, or in some more complicated relationship?

I have posed this question to mathematicians and scientists, and while many get it right, a sizable portion are misled by their intuition about distances and areas. In some places in the biological literature the wrong answer is even in print!

The aim of this article is to pose this problem in three specific geometric situations, deriving the functional relationship from dimensional analysis, by simulation, and finally by direct calculation. I will give a biological motivation for asking this mathematical question, then provide an answer based on dimensional analysis. Many scientists use dimensional analysis informally, but how many know that dimensional analysis has a rigorous foundation? A dimensional analysis shows the proportionality of the average distance to the diameter, but it does not give the constant of proportionality. We can do some simulations using simple statistics to get a confidence interval for the proportionality constant, but to rigorously evaluate the constant, we need to do some mathematics.

Finding the constant of proportionality for the square requires a fourfold iterated integral, but by simple symmetry arguments and changes of variable we can carry out the calculations without too much difficulty. The analogous calculations for the disk and the ball do not work, for lack of an obvious symmetry to exploit. Instead, applying a clever idea from geometrical probability known as Crofton's formula, we can replace the evaluation of a multiple integral with the easy task of solving a first-order linear differential equation. The biological problem leads us to apply many elementary techniques—from dimensional analysis, simulations, and statistical sampling to basic probability theory, changes of variable in multiple integrals, and finally an unexpected appearance of differential equations—the range of which highlights the unity and the utility of the undergraduate mathematics curriculum.

Incidentally, the answer is that the average distance between points in a disk is $64d/(45\pi) \approx 0.45271d$. The average distance between points in a square is

$\left(\{2+[5\ln(\sqrt{2}+1)+2]\sqrt{2}\}/30\right) d \approx 0.36869d$, where $d = s\sqrt{2}$ is the diameter of a square of side length s. That is, the average distance between points in a region is directly proportional to the diameter. The constant of proportionality is somewhat less than $1/2$, as you might guess, but points can be farther apart, on average, in a disk than in a square of the same diameter because of the greater area.

The Biological Motivation

Some species of birds or fish maintain an exclusive territory for hunting or feeding. Some bird and mammal species have a two-dimensional area on the ground or a volume of space in the forest canopy for their territory. Some fish species maintain a volume of space in a lake or ocean as their territory. The territorial size is not arbitrary; the territory must be large enough to provide sufficient food or resources such as nesting material, yet not so large that defending it against intruders expends too much energy.

To create a mathematical model for this situation we must make some assumptions. Let's consider the two-dimensional case and assume for simplicity that the territory is a bounded convex planar region R of diameter d. (Recall that the diameter of a convex region is the maximum distance between any two points in the region, $d = \max_{x,y\in R}\|x - y\|$. The diameter of a disk is the diameter, and the diameter of a square is the length of the diagonal.) Then it is reasonable to assume that the resources available, say seeds on the ground or dried grass for nests, will increase as the area of the region. If we represent the benefit or gain from the territory as G, then the simplest assumption is that $G = \alpha d^2$. More realistically, we might assume that the functional relationship between gain and area is concave and approaches a limit as the area becomes large, a statement of diminishing returns [**9**], [**7**].

What about the costs of maintaining the territory? Many costs can be plausibly associated with a territory, but here is one possibility. There is an energy cost associated with moving to an intruder to threaten the intruder. Since the defending animal and the intruder could be anywhere in the territory, we will hypothesize that the cost of defending a territory is proportional to the average distance $A(d)$ between two arbitrary or random points in the region, [**10**]. Other costs F might involve routine patrolling or marking the territory. Assume that these costs F are either fixed or at most proportional to the diameter. So the costs C of maintaining a territory become

$$C = \beta A(d) + F.$$

Now the problem is clear. We must find values of d for which

$$G = \alpha d^2 > \beta A(d) + F = C$$

If $A(d)$ grows more slowly than the square of the diameter, this inequality has a solution. This gives a lower bound on the economically satisfactory territory size—a prediction that biologists can test with experiments and observation. If the benefits or gain function is sigmoidal in the diameter, our model may also predict an upper bound on the possible territory size.

Mathematically, our biological modeling reduces to the following "geometric probability" question: *How does the average distance between two randomly chosen points in a convex figure grow as the figure diameter grows?*

Dimensional Analysis

Dimensional analysis formalizes the old adage that "you can't add apples and oranges." It is a predictive version of the physics or engineering student's practice of "checking the answer by checking the units to see that they agree." Dimensional analysis predicts possible relations among variables by requiring their dimensions to have mathematically consistent relationships [**6**], [**11**]. It is particularly useful in fluid dynamics [**1**], [**2**].

The Buckingham Pi theorem, which rigorously summarizes dimensional analysis, states: A relation in a mathematical model that is unchanged for any system of measurement units can be rewritten as a relation among dimensionless combinations of the original quantities. The number of independent dimensionless combinations involved is equal to the difference between the number of original quantities and the number of fundamental dimensional units involved.

In our cost–benefit analysis of territories, we have two dimensional quantities: $A(d)$ and the diameter d of the convex set, both measured in units of length. We assume that they have a functional relationship, $A = f(d)$.

A special case of the Buckingham Pi theorem [**1**, p. 90] asserts that the relation may instead be expressed as $A = kd$ for some constant k. It is easy to understand this theorem when f is a power function, $A = kd^r$. Under a linear magnification by 2, both A and d (indeed all linear dimensions of the territory) will double, which implies that the exponent r must be 1. The average distance between points in a convex region is thus proportional to the diameter. The units check, so we have an answer in general terms to our original biological question. But it leaves an interesting mathematical question: *What is the constant of proportionality and how is it related to the shape of the convex region?*

Simulations

An easy way to discover the approximate proportionality constant is by simulation. As before, let's choose a pair of points at random (meaning independent uniform distributions) and calculate the distance between them; we do this several times and calculate the average. Since the average distance is proportional to the diameter, it suffices to use sets of diameter 1. This gives us a quick estimate of the average distance. But can we trust it, or should we be more precise about its accuracy?

Since the points are random, the distance between them is a random variable. Unfortunately, we do not know the distribution of this random variable—which is why we are doing the simulation! But we do know that since the convex set has diameter d, the random variable is bounded by d, so it has finite mean and finite variance. Thus the average of several of these random variables will be approximately normal, by the central limit theorem. Of course, the mean and variance of the approximately normal distribution of the average distance from our several simulations is still unknown. What if we repeat the experiment several times and treat the group average data values as being a sample from a population with an approximately normal distribution with unknown mean and unknown variance? This is precisely what we need for a Student-t confidence interval. We can now do the experiment and derive confidence intervals for some simple sets.

An excellent exercise is to write the simulation on a programmable calculator or using some statistical simulation program. In the classroom, groups of students can find several pairs of random points and the corresponding distances. Each group's mean distance will be approximately normally distributed, and the class mean and sample standard deviation of the group averages will yield a Student-t confidence interval.

Table 1 summarizes the results of 20 simulations. Averaging 50 distances gives an approximately normally distributed estimate.

Table 1. Some Simulations of the Average Distance

Shape of set of diameter 1	*Point estimate of constant*	95% *confidence interval for constant*
Square	0.366	$(0.355, 0.377)$
Disk	0.461	$(0.443, 0.478)$
Ball	0.513	$(0.501, 0.525)$

A warm-up exercise on the interval. Before tackling the problem of finding the average distance between points in planar figures, let's do a warm-up exercise on the interval. This will illustrate the mathematical technique of finding the average with multiple integrals [4].

On an interval $[0, s]$, let two points x_1 and x_2 be chosen independently with uniform distribution. Then the joint distribution of x_1 and x_2 is uniform on the square, and the the average distance between the points is

$$\begin{aligned} A &= \frac{1}{s^2}\int_0^s\int_0^s |x_1 - x_2| dx_2\, dx_1 \\ &= \frac{1}{s^2}\left[\int_0^s\int_0^{x_1} x_1 - x_2 dx_2\, dx_1 + \int_0^s\int_{x_1}^s x_2 - x_1 dx_2\, dx_1\right] \\ &= \frac{1}{s^2}\left[\frac{s^3}{6} + \frac{s^3}{6}\right] = \frac{s}{3}. \end{aligned}$$

The interval can be considered a one-dimensional "square" or "disk," so this easy calculation points to our first generalization.

Average Distance in a Square

Assume that in a square of side length s, two points with coordinates (x_1, y_1) and (x_2, y_2) are chosen independently with a uniform probability distribution. We wish to determine the average distance between the points. That average distance can be computed from

$$A = \frac{1}{s^4}\int_0^s\int_0^s\int_0^s\int_0^s \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\, dx_2\, dx_1\, dy_2\, dy_1.$$

This integral seems to be impossible to integrate directly, but with some simple and natural changes of variables we can evaluate it. First, use some symmetry in the pairs of variables x_1, x_2 and y_1, y_2 to reduce the integration over the squares to integration over a pair of triangular regions; see Figure 1. We obtain

$$A = \frac{4}{s^4}\int_0^s\int_0^{y_1}\int_0^s\int_0^{x_1} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\, dx_2\, dx_1\, dy_2\, dy_1.$$

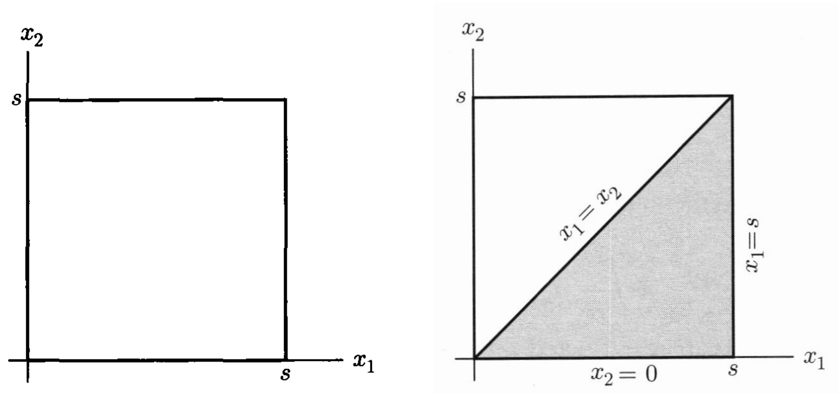

Figure 1. Symmetry reduction of integration over the square to a triangle.

Next, on the innermost double integral make the change of variables $z_1 = x_1 - x_2$ and $z_2 = x_1 + x_2$ (and equivalently, $x_1 = (z_1 + z_2)/2$ and $x_2 = (z_2 - z_1)/2$) with Jacobian determinant $J = \frac{\partial(x_1,x_2)}{\partial(z_1,z_2)} = \frac{1}{2}$. Figure 2 shows the corresponding regions of integration. In the new coordinates, the integration becomes

$$\begin{aligned} A &= \frac{2}{s^4}\int_0^s\int_0^{y_1}\int_0^s\int_{z_1}^{2s-z_1}\sqrt{z_1^2+(y_1-y_2)^2}\,dz_2\,dz_1\,dy_2\,dy_1 \\ &= \frac{2}{s^4}\int_0^s\int_0^{y_1}\int_0^s(2s-2z_1)\sqrt{z_1^2+(y_1-y_2)^2}\,dz_1\,dy_2\,dy_1. \end{aligned}$$

Now make the similar change of variables $w_1 = y_1 - y_2$ and $w_2 = y_1 + y_2$ so that the outermost pair of integrals reduces in the same way to

$$A = \frac{4}{s^4}\int_0^s\int_0^s(s-z_1)(s-w_1)\sqrt{z_1^2+w_1^2}\,dz_1\,dw_1.$$

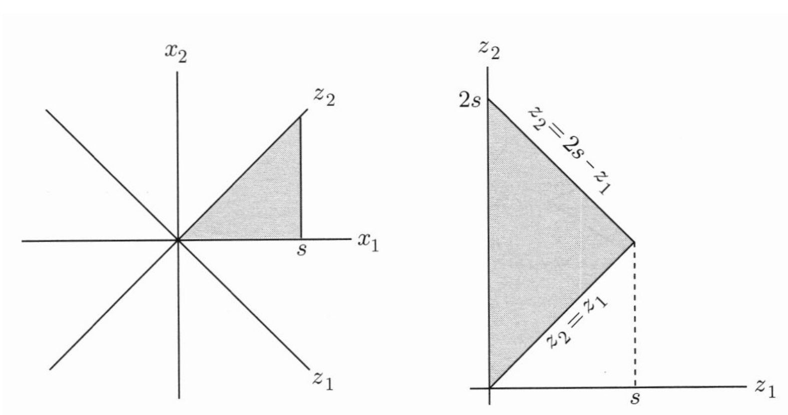

Figure 2. Change of variables from x_1x_2 to z_1z_2.

Again use the symmetry of the integrand in the variables z_1 and w_1 (see Figure 1 again) to reduce the integration to

$$A = \frac{8}{s^4} \int_0^s \int_0^{z_1} (s - z_1)(s - w_1)\sqrt{z_1^2 + w_1^2}\, dz_1\, dw_1.$$

Finally, change the integral to polar coordinates with $z_1 = r\cos\theta$ and $w_1 = r\sin\theta$:

$$A = \frac{8}{s^4} \int_0^{\pi/4} \int_0^{s/\cos\theta} (s - r\cos\theta)(s - r\sin\theta)\, r^2\, dr\, d\theta.$$

This integral can be evaluated with standard methods, by tables, or with a symbolic calculus program such as *Maple* or *Mathematica.* The result is

$$A = \frac{8}{s^4}\, \frac{2 + 5\sqrt{2}\ln(\sqrt{2}+1) + 2\sqrt{2}}{120\sqrt{2}}\, s^5.$$

Using the diameter $d = s\sqrt{2}$ and simplifying, we obtain

$$A = \frac{2 + 5\sqrt{2}\ln(\sqrt{2}+1) + 2\sqrt{2}}{30}\, d \approx 0.36869d.$$

Using Crofton's Theorem for the Disk

Consider now the apparently similar problem of finding the average distance between two random points in a disk of radius R. We choose two points with polar coordinates (r_1, θ_1) and (r_2, θ_2) independently with a uniform probability distribution. By the law of cosines, the distance between the points is

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\phi(\theta_1, \theta_2)}\,,$$

where $0 \le \phi(\theta_1, \theta_2) \le \pi$ is the central angle between the points; see Figure 3. The average distance is

$$A = \frac{1}{\pi^2 R^4} \int_0^R \int_0^R \int_0^{2\pi} \int_0^{2\pi} \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\phi(\theta_1, \theta_2)}\, r_1 r_2\, d\theta_1\, d\theta_2\, dr_1\, dr_2.$$

And you thought the integral for the square was tough to evaluate directly!

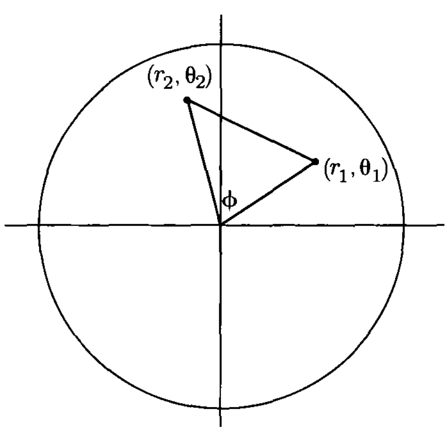

Figure 3. The distance between two random points in a disk.

Solomon's *Geometric Probability* [**8,** p. 97] is worth quoting here:

> It is to M. W. Crofton, an English mathematician whose works appeared in the latter half of the 19th century, that we are indebted for approaches by which results in geometrical probability that would normally would require difficult and sometimes intractable integration are achieved by ingenious artifices.

Crofton's mean value theorem is a generalization of Leibniz's theorem on differentiation of an integral. The essential idea of his proof is easily grasped. Following Solomon, let n points $\xi_1, \xi_2, \ldots, \xi_n$ be independently and randomly distributed on the domain S and let X be some random variable that depends on the positions of the n points. Let S' denote a domain slightly smaller than S, with $S' \subset S$, as in Figure 4. Let $\Delta S = S \setminus S'$ be the part of S that is not in S'. If we take ΔS small enough that all the higher order terms depending on ΔS are much smaller than ΔS itself, then by conditional expectations

$$\begin{aligned} E(X) \approx{} & E(X|\xi_1, \xi_2, \ldots, \xi_n \in S')\, P[\xi_1, \xi_2, \ldots, \xi_n \in S'] \\ & + \sum_{j=1}^{n} E(X|\xi_j \in \Delta S, \xi_i \in S' \text{ for all } i \neq j)\, P[\xi_i \in S' \text{ for all } i \neq j]. \end{aligned}$$

Recall that the probability that a point is in a set is proportional to the area or, more generally, the measure of the set. Therefore, if $s, s', \Delta s$ denote the measures of $S, S', \Delta S$ respectively, then

$$E(X) \approx E(X|\xi_1, \xi_2, \ldots, \xi_n \in S') \frac{s'^n}{s^n} + nE(X|\xi_1 \in \Delta S, \xi_2, \ldots, \xi_n \in S') \frac{s'^{n-1} \Delta s}{s^n}.$$

Letting $\Delta S \to 0$ in an appropriate way, $s'^n/s^n = (s - \Delta s)^n/s^n \approx 1 - n\Delta s/s$. Therefore,

$$\begin{aligned} E(X) \approx{} & E(X|\xi_1, \xi_2, \ldots, \xi_n \in S') \frac{1 - n\Delta s}{s} \\ & + nE(X|\xi_1 \in \Delta S, \xi_2, \ldots, \xi_n \in S') \frac{s'^{n-1} \Delta s}{s^n}. \end{aligned}$$

We subtract $E(X(S'))$ from both sides of the conditional expectation to obtain

$$\begin{aligned} E(X(S)) - E(X(S')) \approx{} & nE(X|\xi_1 \in \Delta S, \xi_2, \ldots, \xi_n \in S') \frac{s'^{n-1} \Delta s}{s^n} \\ & - nE(X|\xi_1, \xi_2, \ldots, \xi_n \in S') \frac{\Delta s}{s}. \end{aligned}$$

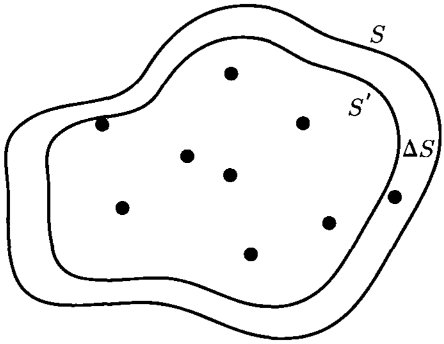

Figure 4. Diagram for the proof of Crofton's theorem.

Dividing by Δs and passing to the limit yields a differential equation:

$$\frac{dE(X(s))}{ds} = n\frac{E(X|\xi_1 \in \partial S) - E(X)}{s}.$$

Now, how do we apply Crofton's theorem? We have a pair of points randomly distributed in a disk of radius R, and the distance X between the points is a random function that depends on their positions. This is ideally set up for Crofton's theorem, and applying the formula we obtain the differential equation for the average distance A between two random points in a disk of radius R,

$$\frac{dA}{d(\pi R^2)} = 2\frac{E(X|\xi_1 \in \partial S) - A}{\pi R^2}.$$

We need to find the average distance from a randomly chosen point inside the disk to a randomly chosen point on the boundary circle, namely the term $E(X|\xi_1 \in \partial S)$. Fortunately, this is an easy integral, since the symmetry of the circle allows us to fix the point on the boundary, say at the point $(R, 0)$. (A recent article on geometric probability **[5]** also uses this rotational invariance of the probability distribution.) We use the law of cosines again for the distance from $(R, 0)$ to (r, θ); see Figure 5. Already symmetry lets us reduce the integral to angles $0 \leq \theta \leq \pi$. We now have

$$E(X|\xi_1 \in \partial S) = \frac{2}{\pi R^2}\int_0^\pi \int_0^R \sqrt{R^2 + r^2 - 2Rr\cos(\theta)}\, r\, dr d\theta.$$

In such terms, it makes more sense to evaluate the double integral in polar coordinates centered at $(R, 0)$. Then the double integral becomes (see Figure 5 again)

$$\int_0^{\pi/2} \int_0^{2R\cos\phi} \rho^2\, d\rho\, d\phi = \frac{16R^3}{9}.$$

Using this in the differential equation and rearranging, we obtain

$$\frac{dA}{dR} + \frac{4}{R}A = \frac{128}{9\pi}.$$

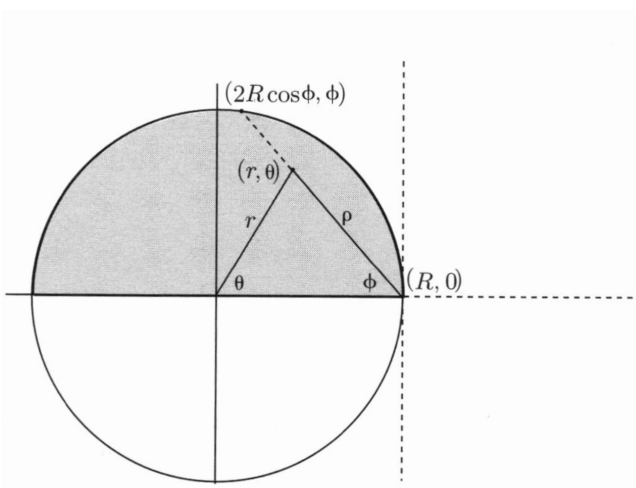

Figure 5. The domains of integration for the conditional expectation.

The average distance between points in a disk of radius 0 is 0, so our initial condition is $A(0) = 0$. Then the solution of this linear differential equation is easily found to be

$$A = \frac{128}{45\pi} R$$

In terms of the diameter of the disk, the solution is

$$A = \frac{64}{45\pi} d \approx 0.45271d$$

This result is derived in a more sophisticated way in Solomon [**8,** p. 36, eq. 2.58]. Using his results, it is true in general that for any convex figure of area S, the average distance between points in the figure satisfies the inequality

$$A \geq \frac{128}{45\pi} \sqrt{\frac{S}{\pi}}.$$

The comparison of the case of the square and the disk gives a specific example.

Using Crofton's Theorem for the Ball

Let's use Crofton's theorem to obtain the average distance between two points chosen independently from the uniform distribution on a ball S of radius R. Let $A = A(R)$ be the average distance between the points as a function of R. Let $s = 4\pi R^3/3$ be the measure or volume of the ball and let X be the distance between the randomly chosen points ξ_1 and ξ_2. Then Crofton's theorem says

$$\frac{dA}{ds} = 2[E(X|\xi_1 \in \partial S) - A]s^{-1}.$$

Now we need to evaluate the conditional expectation, which is the average distance of a randomly chosen point in the ball from a point on the boundary. By the symmetry of the ball, we can choose a convenient point on the boundary, say the point at the south pole. Then

$$\begin{aligned} E(X|\xi_1 \in \partial S) &= \frac{1}{(4/3)\pi R^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2R\cos\phi} r \cdot r^2 \sin\phi \, dr \, d\theta \, d\phi \\ &= 6R \int_0^{\pi/2} \cos^4\phi \sin\phi \, d\phi = \frac{6R}{5}. \end{aligned}$$

Therefore the differential equation for the average from Crofton's theorem becomes (after the change of variable $s = 4\pi R^3/3$)

$$\frac{dA}{dR} = 2(\frac{6R}{5} - A) \cdot \frac{3}{4\pi R^3} \cdot 4\pi R^2$$

or, more simply,

$$R\frac{dA}{dR} + 6A = \frac{36R}{5},$$

with the initial condition $A(0) = 0$. The solution is $A = 36R/35$. In terms of the diameter of the ball, the average distance between points in the ball is $A = 18d/35 = 0.5\overline{142857}d$.

The Constant for Balls in Higher Dimensions

Did you notice that the constant of proportionality for the ball is larger than the constant for the disk? Furthermore, it is a simpler expression, in fact a rational number! A natural question to ask is: *Does the pattern continue on into higher dimensions?* If the constant continues to increase, it would indicate that higher dimensional space is "roomier" on average than our familiar three-dimensional space. It would also be interesting to know if the sequence of constants approaches a limit as the dimension increases. Toward that end, we can use Crofton's theorem and some analytic geometry of n-space to evaluate the constants.

For the moment, let $\beta_n R = E(X|\xi_1 \in \partial S)$ be the average distance between a random point in the ball of radius R and a point on the boundary of the ball. The average distance is found by solving a linear differential equation as above:

$$A(R) = \frac{2n}{2n+1}\beta_n R.$$

Notice that as n increases, the average distance between points in the ball approaches the average distance between a point in the ball and a point on the boundary. This confirms our geometric intuition that most of the average is due to points near the edge of the ball.

After a lengthy integration in n-space in spherical coordinates (see [3]) and considerable simplification, we reach the final expression of β_n. It is easiest to express the results in terms of odd and even cases of n:

$$\beta_n = \begin{cases} \frac{2^{6m+1}(m!)^2(2m)!}{(2m+1)(4m)!\pi}, & \text{if } n = 2m \\ \frac{2^{2m+2}[(2m+1)!]^3}{(2m+2)(m!)^2(4m+2)!}, & \text{if } n = 2m+1. \end{cases}$$

From this we can now see that the average distance between two points chosen at random from a ball in even-dimensional space will be an irrational multiple of the radius, while in odd-dimensional space it will be a rational multiple of the radius.

With some additional work using Stirling's formula, we can show that $\beta_n \to \sqrt{2}$ from below as $n \to \infty$. This shows that in high-dimensional spaces, the average distance between two randomly chosen points in a unit ball is almost as large as the distance between the ends of two orthogonal unit vectors. Most of the average comes from the boundary of the ball.

Conclusion

Our original biological question has led us from a simple investigation using dimensional analysis, through some statistical investigations, to analysis in higher dimensions using special functions. The range of techniques used is interesting too, from changes of variables in multiple integrals to an application of differential equations in an unexpected context to avoid direct integration. Geometric probability theory offers much pleasure and yields some useful biological results.

Acknowledgment. David Stephens first posed the question to me in the biological context—and got the answer right the first time. I thank him for many pleasant conversations about this and other questions of mathematical biology. I also thank Richard Bernatz for comments on an early version of this work.

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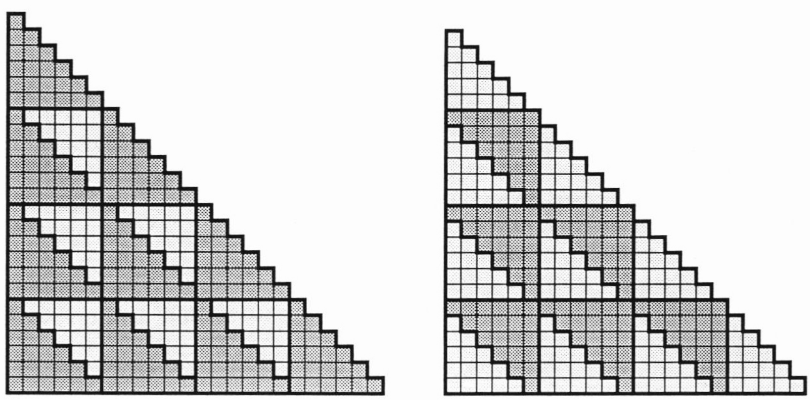
Two Identities for Triangular Numbers

$$t_n = 1 + 2 + 3 + \cdots + n \Rightarrow$$



$t_n t_k + t_{n-1} t_{k-1} = t_{nk}$ $\qquad$ $t_{n-1} t_k + t_n t_{k-1} = t_{nk-1}$

—*Roger B. Nelsen*
Lewis & Clark College