Assignment 0: O Brother, How Far Art Thou?

Computational Statistics
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General guidance

- State and prove all non-trivial mathematical results necessary to substantiate your arguments;
- Do not forget to add appropriate scholarly references *at the end* of the document;
- Mathematical expressions also receive punctuation;
- Please hand in a single PDF file as your final main document.

  Code appendices are welcome, *in addition* to the main PDF document.

Background

A large portion of the content of this course is concerned with computing high-dimensional integrals *via* simulation. Today you will be introduced to a simple-looking problem with a complicated closed-form solution and one we can approach using simulation.

Suppose you have a disc C_R of radius R. Take $p = (p_x, p_y)$ and $q = (q_x, q_y) \in C_R$ two points in the disc. Consider the Euclidean distance between p and q, $||p - q|| = \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2} = |p - q|$.

Problem A: What is the *average* distance between pairs of points in C_R if they are picked uniformly at random?

Questions

1. To start building intuition, let's solve a related but much simpler problem. Consider an interval $[0, s]$, with $s > 0$ and take $x_1, x_2 \in [0, s]$ *uniformly at random*. Show that the average distance between x_1 and x_2 is $s/3$.

2. Show that Problem A is equivalent to computing

$$I = \frac{1}{\pi^2 R^4} \int_0^R \int_0^R \int_0^{2\pi} \int_0^{2\pi} \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\phi(\theta_1, \theta_2)} r_1 r_2 \, d\theta_1 \, d\theta_2 \, dr_1 \, dr_2,$$

   where $\phi(\theta_1, \theta_2)$ is the central angle between r_1 and r_2.

   Hint: Draw a picture.

3. Compute I in closed-form.

   Hint: Look up *Crofton's mean value theorem* or *Crofton's formula.*

4. Propose a simulation algorithm to approximate I. Provide point and interval estimates and give theoretical guarantees about them (consistency, coverage, etc).