Fokker-Plank equation

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Abstract

This text aims to summarise the content studied during the period in the Department of Mathematics at Imperial College London with professor Dante Kalise. The object of study is the Fokker-Planck equation and Optimal Control.

Contents

Notation

This list is part of the notation we use throughout the text:

- $\langle X \rangle = \mathbb{E}[X]$ is the expected value of the random variable X.

- For $f, g \in H^1(\Omega)$, we define $\langle f, g \rangle = \int_\Omega f(x)g(x)\,dx$

1 Introduction

Consider the Stochastic Differential Equation

$$dX_t = -\nabla V(x,t)\, dt + \sqrt{2\nu}dB_t.$$

The probability density function ρ of the process $\{X_t\}_{t\in\mathbb{R}_+}$ is the solution of

$$\rho_t(x,t) = \nabla \cdot J(x,t), \ \text{ where } J(x,t) = \nu\nabla\rho(x,t) + \rho(x,t)\nabla V(x,t), (x,t) \in \Omega \times (0,+\infty),$$

subject to $J(x,t) \cdot \vec{n} = 0$ on $\partial\Omega \times (0,+\infty)$ and $\rho(x,0) = \rho_0(x)$ in Ω. Consider the potential

$$V(x,t) = G(x) + u(t)\alpha(x),$$

such that u is a control function and α is a control shape function that satisfies $\nabla\alpha\cdot\vec{n} = 0$ on $\partial\Omega$. With that, we get that the boundary condition reduces to

$$0 = (\nu\nabla\rho + \rho\nabla V) \cdot \vec{n} = \nu(\nabla\rho) \cdot \vec{n} + \rho(\nabla G) \cdot \vec{n} + \cancel{u\,\rho(\nabla\alpha)\cdot\vec{n}},$$

witch implies that

$$(\nu\nabla\rho + \rho\nabla G) \cdot \vec{n} = 0 \text{ on } \partial\Omega.$$

Let ρ_∞ be the steady state for the uncontrolled system, i.e., the function that solves

$$\nabla \cdot (\nu\nabla\rho_\infty + \rho_\infty\nabla G) = 0 \text{ in } \Omega \times (0,\infty),$$

subject to $(\nu\nabla\rho_\infty + \rho_\infty\nabla G) \cdot \vec{n} = 0$ on $\partial\Omega$. Define $y = \rho - \rho_\infty$. The problem turns to

$$
\begin{aligned}
y_t &= \nabla \cdot [\nu\nabla(y + \rho_\infty) + (y + \rho_\infty)\nabla V] \\
&= \nabla \cdot (\nu\nabla y + y\nabla V + u\nabla\alpha\rho_\infty) + \cancel{\nabla \cdot (\nu\nabla\rho_\infty + \rho_\infty\nabla G)},
\end{aligned}
$$

in $\Omega \times (0,\infty)$ subject to

$$0 = (\nu\nabla(y + \rho_\infty) + (y + \rho_\infty)\nabla G) \cdot \vec{n} = (\nu\nabla y + y\nabla G) \cdot \vec{n} \text{ on } \partial\Omega$$

and $y(x,0) = \rho_0(x) - \rho_\infty(x)$ in Ω. We can write the above problem as a bilinear control system

$$\dot{y} = \mathcal{A}y + u\mathcal{N}y + \mathcal{B}u, \quad y(0) = y_0,$$

where $\mathcal{B} = \mathcal{N}\rho_\infty$.

The first feedback strategy is based on the cost functional

$$\mathcal{J}(y, u) = \frac{1}{2} \int_0^\infty \langle y, \mathcal{M}y \rangle + |u|^2 \, dt,$$

and it is obtained solving the Riccatti Equation

$$\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\mathcal{B}^*\Pi + \mathcal{M} = 0$$

and $u = -\mathcal{B}^*\Pi y$.

The second strategy is based on the solution Γ to a Lyapunov equation

$$\mathcal{A}^*\Upsilon + \Upsilon\mathcal{A} + 2\mu I = 0,$$

for a properly chosen parameter $\mu > 0$.

Some additional references:

- [1]: covers the existence and the uniqueness of a class of Fokker-Planck equations with coefficients in Sobolov spaces $W^{1,p}$.

- [3]: existence of steady state of Fokker-Planck equation in $\Omega \subseteq \mathbb{R}^n$.

2 Well-posedness

We search for a solution $\rho \in W(0, T) = L^2(0, T; H^1(\Omega)) \cap H^1(0, t; (H^1(\Omega))^*)$ and

$$\langle \rho_t, \phi \rangle + \langle \nu \nabla \rho(t) + \rho(t) \nabla G, \nabla \phi \rangle + u(t) \langle \rho(t) \nabla \alpha, \nabla \phi \rangle = 0, \forall \phi \in H^1(\Omega),$$

where integration by parts was applied. It is assumed that

$$G, \alpha \in W^{1,\infty}(\Omega) \cap W^{2,\max(2,n)}(\Omega).$$

With that in mind, is possible to prove that for every $u \in L^2(0, T)$ and $\rho_0 \in L^2(\Omega)$, there exists an unique solution to Fokker-Planck equation.

Moreover the solution satisfies

(i) For every $t \in [0, T]$, we have that $\langle \rho(t) - \rho_0, 1_\Omega \rangle = 0$.

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(ii) If $\rho_0 \geq 0$ almost everywhere on Ω, then $\rho(x,t) \geq 0$ for all $t > 0$ and almost all $x \in \Omega$.

3 Operator form

Define the operators

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to L^2(\Omega),$$
$$\mathcal{D}(\mathcal{A}) = \{\rho \in H^2(\Omega) : (\nu\nabla\rho + \rho\nabla G) \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$$
$$\mathcal{A}\rho = \nu\Delta\rho + \nabla \cdot (\rho\nabla G),$$

and

$$\mathcal{N} : H^1(\Omega) \to L^2(\Omega), \mathcal{N}\rho = \nabla \cdot (\rho\nabla\alpha).$$

The adjoint operators are well-defined and given by

$$\mathcal{A}^*\varphi = \nu\Delta\varphi - \nabla G \cdot \nabla\varphi, (\nu\nabla\varphi) \cdot \vec{n} = 0 \text{ on } \partial\Omega$$

and

$$\mathcal{N}^*\varphi = -\nabla\varphi \cdot \nabla\alpha.$$

Considering the uncontrolled system $\dot\rho = \mathcal{A}\rho$, introduce the function $\Phi(x) = \log\nu + \frac{G(x)}{\nu}$. Further, define the operator $\mathcal{A}_s = e^{\Phi/2}\mathcal{A}e^{-\Phi/2}$. Then,

$$\mathcal{A}\left(e^{-\Phi/2}\rho\right) = \nu e^{-\Phi/2}\left(\Delta\rho + \frac{1}{2}\rho\Delta\Phi - \frac{1}{4}\rho\nabla\Phi \cdot \nabla\Phi\right).$$

The following results can be obtained:

- $\mathcal{A}_s$ is self-adjoint.

- The spectrum $\sigma(\mathcal{A}_s)$ consists of non-positive pure points with $0 \in \sigma(\mathcal{A}_s)$.

- The eigenfunctions $\{\psi_i\}_{i=0}^{\infty}$ form a complete orthogonal set.

- $\sigma(\mathcal{A}_s) = \sigma(\mathcal{A})$ and ψ_i is eigenfunction of $\mathcal{A}$ iff $e^{\Psi/2}\psi_i$ is eigenfunction of $\mathcal{A}_s$.

- ψ_i is eigenfunction of $\mathcal{A}$ iff $e^{\Psi}\psi_i$ is eigenfunction of $\mathcal{A}^*$.

- $\rho_\infty = ce^{-\Phi}$ is eigenfunction of $\mathcal{A}$ associated with the eigenvalue 0.

3.1 Decoupling the Fokker-Planck equation

Consider the projection transform $\mathcal{P}$ onto $1^{\perp}$ along ρ_{∞}, written as

$$\mathcal{P}y = y - \int_{\Omega} y \, dx \rho_{\infty},$$

and its complementary $\mathcal{Q}y = (I - \mathcal{P})y$. If $y \in \mathcal{Y}$, we can decompose $\mathcal{Y} = im(\mathcal{P}) \oplus im(\mathcal{Q})$, which implies $y = \mathcal{P}y + \mathcal{Q}y = y_{\mathcal{P}} + y_{\mathcal{Q}}$. Therefore, Fokker-Planck equation turns to

$$\dot{y}_{\mathcal{P}} + \dot{y}_{\mathcal{Q}} = \mathcal{A}(y_{\mathcal{P}} + y_{\mathcal{Q}}) + u\mathcal{N}(y_{\mathcal{P}} + y_{\mathcal{Q}}) + \mathcal{B}u.$$

Applying $\mathcal{P}$ and $\mathcal{Q}$ to this equation and noticing that $\mathcal{P}(\mathcal{P}y) = \mathcal{P}y$, $\mathcal{Q}(\mathcal{Q}y) = \mathcal{Q}y$, $\mathcal{P}(\mathcal{Q}y) = \mathcal{Q}(\mathcal{P}y) = 0$, we get the system

$$\begin{bmatrix} \dot{y}_{\mathcal{P}} \\ \dot{y}_{\mathcal{Q}} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{bmatrix} + u \begin{bmatrix} \mathcal{P}\mathcal{N} & \mathcal{P}\mathcal{N} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{bmatrix} + u \begin{bmatrix} \mathcal{P}\mathcal{B} \\ 0 \end{bmatrix},$$

where other identities were also used.

4   Ricatti-based feedback control

It is considered the linearised version of the system, that is, uNy term is dropped. In this section, they derive the Ricatti equation.

5   Methods for solving a PDE

(a) Finite differences: [2] build a scheme where positivity and conservation of mass are maintained.

(b) Finite elements

(c) Collocation methods, which is a spectral method in the strong sense

(d) Spectral-Legendre method

   The equation we are trying to solve is

$$y_t = \nabla \cdot (\nu \nabla y + y \nabla G + uy \nabla \alpha + u \rho_{\infty} \nabla \alpha),$$

subject to $(\nu \nabla y + y \nabla G) \cdot \vec{n} = 0$ in the boundary and $y = \rho_0 - \rho_\infty$ as initial condition. In the weak formulation, for every $\phi \in H^1(\Omega)$, we have

$$\langle y_t, \phi \rangle = -\langle \nu \nabla y + y \nabla G, \nabla \phi \rangle - u(t) \langle \nabla \alpha y, \nabla \phi \rangle - u(t) \langle \nabla \alpha \rho_\infty, \nabla \phi \rangle.$$

5.1   Spectral-Legendre method

Let us consider the one-dimensional case for now: $\Omega = [a, b]$. To simplify future calculations, we consider the variable

$$\tilde{y}(x, t) = y \left(\left(\frac{b - a}{2} \right) x + \left(\frac{a + b}{2} \right), t \right), \forall x \in [-1, 1], t > 0.$$

We do the same for G, α and ρ_∞. For sake of conciseness, we drop $\sim$ for now. The formulation turns to

$$\langle y_t, \phi \rangle = -\left(\frac{2}{b - a} \right)^2 \langle \nu y_x + y \dot{G} + u y \dot{\alpha} + u \rho_\infty \dot{\alpha}, \dot{\phi} \rangle.$$

Consider the space

$$X_n = \{ \phi \in P_n : \nu \dot{\phi}(\pm 1) + \phi(\pm 1) \dot{G}(\pm 1) = 0 \},$$

where P_n is the space of polynomials with degree up to n. Notice that $\dim(X_n) = n - 1$. Let $\{\phi_i\}_{i=0}^{n-2}$ be a basis for X_n and write

$$y(x, t) \approx \sum_{j=0}^{n-2} y_j(t) \phi_j(x).$$

Considering the set of test equal to the trial functions, we get in the following formulation

$$\sum_{j=0}^{n-2} \dot{y}_j(t) \langle \phi_j, \phi_i \rangle = -\left(\frac{2}{b-a} \right)^2 \sum_{j=0}^{n-2} y_j(t) \left(\nu \langle \dot{\phi}_j, \dot{\phi}_i \rangle + \langle \dot{G} \phi_j, \dot{\phi}_i \rangle + u(t) \langle \dot{\alpha} \phi_j, \dot{\phi}_i \rangle \right)$$

$$+ u(t) \langle \dot{\alpha} \rho_\infty, \dot{\phi}_i \rangle,$$

which can be rewritten as a system of ODEs:

$$\Phi \dot{y}(t) = -\left(\frac{2}{b-a}\right)^2 (\Lambda + \Theta^1 + u(t)\Theta^2) y(t) - \left(\frac{2}{b-a}\right)^2 u(t) v.$$

Following the suggestion from [4, p.7], we consider

$$\phi_k(x) = L_k(x) + \alpha_k L_{k+1}(x) + \beta_k L_{k+2}(x),$$

where L_k is the Legendre polynomial of degree k and the coefficients are chosen to satisfy the boundary conditions. Let us calculate these quantities:

$$\nu(\dot{L}_k(\pm 1) + \alpha_k \dot{L}_{k+1}(\pm 1) + \beta_k \dot{L}_{k+2}(\pm 1)) + \dot{G}(\pm 1)(L_k(\pm 1) + \alpha_k L_{k+1}(\pm 1) + \beta_k L_{k+2}(\pm 1)) = 0.$$

which can be written in matrix formulation

$$\begin{bmatrix} \nu(k+1)(k+2) - 2g_- & -\nu(k+2)(k+3) + 2g_- \\ \nu(k+1)(k+2) + 2g_+ & \nu(k+2)(k+3) + 2g_+ \end{bmatrix} \begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} \nu k(k+1) - 2g_- \\ -\nu k(k+1) - 2g_+ \end{bmatrix},$$

where $g_\pm = \dot{G}(\pm 1)$ and the system is numerically solved.

Let's now pre-calculate the matrices in terms of the legendre polynomials.

$$\begin{aligned}
\Phi_{ij} = \langle \phi_i, \phi_j \rangle &= \langle L_i, L_j \rangle + \alpha_i \langle L_{i+1}, L_j \rangle + \beta_i \langle L_{i+2}, L_j \rangle \\
&\quad + \alpha_j \langle L_i, L_{j+1} \rangle + \alpha_i \alpha_j \langle L_{i+1}, L_{j+1} \rangle + \beta_i \alpha_j \langle L_{i+2}, L_{j+1} \rangle \\
&\quad + \beta_j \langle L_i, L_{j+2} \rangle + \alpha_i \beta_j \langle L_{i+1}, L_{j+2} \rangle + \beta_i \beta_j \langle L_{i+2}, L_{j+2} \rangle \\
&= 2 \frac{\delta_{ij}}{2i+1} + 2\alpha_i \frac{\delta_{i+1,j}}{2(i+1)+1} + \beta_i \frac{\delta_{i+2,j}}{2(i+2)+1} \\
&= 2\alpha_j \frac{\delta_{i,j+1}}{2i+1} + 2\alpha_i \alpha_j \frac{\delta_{ij}}{2(i+1)+1} + 2\beta_i \alpha_j \frac{\delta_{i+1,j}}{2(i+2)+1} \\
&= 2\beta_j \frac{\delta_{i,j+2}}{2i+1} + 2\alpha_i \beta_j \frac{\delta_{i,j+1}}{2(i+1)+1} + 2\beta_i \beta_j \frac{\delta_{ij}}{2(i+2)+1}
\end{aligned}$$

$$\begin{aligned}
\Lambda_{ij} = \nu \langle \dot{\phi}_i, \dot{\phi}_j \rangle &= \nu[\langle \dot{L}_i, \dot{L}_j \rangle + \alpha_i \langle \dot{L}_{i+1}, \dot{L}_j \rangle + \beta_i \langle \dot{L}_{i+2}, \dot{L}_j \rangle \\
&\quad + \alpha_j \langle \dot{L}_i, \dot{L}_{j+1} \rangle + \alpha_i \alpha_j \langle \dot{L}_{i+1}, \dot{L}_{j+1} \rangle + \beta_i \alpha_j \langle \dot{L}_{i+2}, \dot{L}_{j+1} \rangle \\
&\quad + \beta_j \langle \dot{L}_i, \dot{L}_{j+2} \rangle + \alpha_i \beta_j \langle \dot{L}_{i+1}, \dot{L}_{j+2} \rangle + \beta_i \beta_j \langle \dot{L}_{i+2}, \dot{L}_{j+2} \rangle],
\end{aligned}$$

where $\langle \dot{L}_i, \dot{L}_j \rangle = L_{\max(i,j)}(1) \dot{L}_{\min(i,j)}(1) - L_{\max(i,j)}(-1) \dot{L}_{\min(i,j)}(-1)$, integrating by parts and observing that Legendre polynomials are orthogonal to other polynomials with

smaller degree. Therefore,

$$\langle \dot{L}_i, \dot{L}_j \rangle = \frac{1}{2} \min(i,j)(\min(i,j) + 1) \left(1 + (-1)^{i+j}\right).$$

Finally, we calculate

$$\Theta_{ij}^1 = \langle \dot{G}\phi_j, \dot{\phi}_i \rangle, \Theta_{ij}^2 = \langle \dot{\alpha}\phi_j, \dot{\phi}_i \rangle \text{ and } v_i = \langle \dot{\alpha}\rho_\infty, \dot{\phi}_i \rangle$$

This is the method of solving the PDE. Before solving it, we need to compute the optimal control. For that, we have to discretise the operators $\mathcal{A}, \mathcal{B}$ and $\mathcal{M}$:

The discretised versions are

$$A_{ij} = \langle \mathcal{A}\phi_j, \phi_i \rangle = -\left(\frac{2}{b-a}\right)^2 \langle \nu\dot{\phi}_j + \dot{G}\phi_j, \dot{\phi}_i \rangle \implies A = -\left(\frac{2}{b-a}\right)^2 (\Lambda + \Theta^1)$$

and

$$B_i = \langle \mathcal{B} \cdot 1, \phi_i \rangle = -\left(\frac{2}{b-a}\right)^2 \langle \rho_\infty \dot{\alpha}, \dot{\phi}_i \rangle \implies B = -\left(\frac{2}{b-a}\right)^2 v.$$

With that in mind, we have to solve the discrete Ricatti equation

$$A^T \Pi + \Pi A + \Pi B B^T \Pi + M = 0.$$

with $u(t) = -B^T \Pi y(t)$. With this feedback, we solve

$$\begin{aligned}
\Phi \dot{y} &= (A + B^T \Pi y(t) \Theta^2) y(t) - B B^T \Pi y(t) \\
&= (A - B B^T \Pi + B^T \Pi y(t) \Theta^2) y(t).
\end{aligned}$$

After solving this system, we have to come back to the original coordinate system.

Remark 5.1.1. *Notice that* $M_{ij} = \langle \mathcal{M}\phi_j, \phi_i \rangle$. *For instance, if* $\mathcal{M}$ *is the identity operator, we have* $M = \Lambda$.

6 Finite elements

The second method we consider is the finite elements method. In this case, we also consider the weak formulation and look for a basis $\{\phi_i\}_{i=0}^n$. In the interval $[a, b]$, set the *node points* $a = x_0 < \cdots < x_n = b$. The *elements* are the sub-intervals $[x_i, x_{i+1}]$ with $x_{i+1} - x_i = h$, for $i = 0, \ldots, n-1$. We set the values $\phi_k(x_i) = \delta_{ik}$ and for each element

a linear function. Therefore, define the function

$$\phi(x) = \begin{cases} x/h, & \text{if } x \in [0, h] \\ 2 - x/h, & \text{if } x \in [h, 2h] \\ 0, & \text{otherwise} \end{cases}$$

and $\phi_k(x) = \phi(x - x_{k-1})$, for $k = 0, \dots, n$ and $x \in [a, b]$, setting $x_{-1} = a - h$. With that in mind, it remain to calculate the same matrices as the previous method. Here, the rescale for $[-1, 1]$ is not necessary. Then, if $i \geq j$,

$$\begin{aligned} \Phi_{ij} = \langle \phi_i, \phi_j \rangle &= \int_a^b \phi(x - x_{i-1})\phi(x - x_{j-1})\,dx \\ &= \int_0^{2h} \phi(y)\phi(y + x_{i-1} - x_{j-1})\,dy \\ &= \int_0^h \frac{y}{h}\phi(y + (i-j)h)\,dy + \int_h^{2h} \left(2 - \frac{y}{h}\right)\phi(y + (i-j)h)\,dy \end{aligned}$$

Notice that for $i - j \geq 2$, both integrals are 0, since $\phi(y + 2h) = 0$ for $y \geq 0$. If $i = j + 1$, the second integral is zero, while the first is

$$\int_0^h \frac{y}{h}\left(2 - \frac{y+h}{h}\right)\,dy = h \int_0^1 z(1 - z)\,dz = \frac{h}{6}.$$

If $n > i = j > 0$, the integrals are

$$\int_0^h \frac{y^2}{h^2}\,dy = \frac{h}{3}, \quad \int_h^{2h} \left(2 - \frac{y}{h}\right)^2\,dy = h \int_0^1 z^2\,dz = \frac{h}{3},$$

which implies $\Phi_{ii} = 2h/3$ and, finally, $\Phi_{00} = \Phi_{nn} = h/3$. For $i < j$, $\Phi_{ij} = \Phi_{ji}$. For calculating Λ, for $i \geq j$,

$$\begin{aligned} \Lambda_{ij} = \nu \langle \dot{\phi}_i, \dot{\phi}_j \rangle &= \int_a^b \dot{\phi}(x - x_{i-1})\dot{\phi}(x - x_{j-1})\,dx \\ &= \int_0^{2h} \dot{\phi}(y)\dot{\phi}(y + x_{i-1} - x_{j-1})\,dx \\ &= \frac{1}{h}\int_0^h \dot{\phi}(y + (i-j)h)\,dy - \frac{1}{h}\int_h^{2h} \dot{\phi}(y + (i-j)h)\,dy. \end{aligned}$$

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If $i - j \geq 2$, both integrals are 0. If $i = j + 1$, the second integral vanishes, while the first is

$$\int_0^h -\frac{1}{h}\, dx = -1 \implies \Lambda_{ij} = -\frac{\nu}{h}.$$

If $i = j$, the first integral is 1, while the second is -1, implying that $\Lambda_{ii} = 2\nu/h$. Finally, if $i < j$, then $\Lambda_{ij} = \Lambda_{ji}$.

The other three matrices have some simplification, but need numerical integration.

$$
\begin{aligned}
\Theta_{i,i+1}^1 = \langle \dot{G}\phi_{i+1}, \dot{\phi}_i \rangle &= \int_h^{2h} \dot{G}(y + x_{i-1})\phi(y - h)\dot{\phi}(y)\, dy \\
&= -\frac{1}{h} \int_h^{2h} \dot{G}(y + (a + (i-1)h)) \left(\frac{y - h}{h} \right) dy \\
&= -\frac{1}{h} \left[\int_h^{2h} \dot{G}(y + a + ih - h)\frac{y}{h}\, dy - G(a + ih + h) + G(a + ih) \right] \\
&= -\frac{1}{h} \left[G(a + (i+1)h) - \frac{1}{h} \int_h^{2h} G(y + a + ih - h)\, dy \right] \\
&= -\frac{1}{h} \left[G(a + (i+1)h) - \frac{1}{h} \int_{a+ih}^{a+(i+1)h} G(y)\, dy \right],
\end{aligned}
$$

for $i = 0, \ldots, n - 1$.

$$
\begin{aligned}
\Theta_{i,i-1}^1 = \langle \dot{G}\phi_{i-1}, \dot{\phi}_i \rangle &= \int_0^h \dot{G}(y + x_{i-1})\phi(y + h)\dot{\phi}(y)\, dy \\
&= \frac{1}{h} \int_0^h \dot{G}(y + (a + (i-1)h)) \left(2 - \frac{y + h}{h} \right) dy \\
&= -\frac{1}{h} \left[\int_0^h \dot{G}(y + a + ih - h)\frac{y}{h}\, dy - G(a + ih) + G(a + (i-1)h) \right] \\
&= -\frac{1}{h} \left[G(a + (i-1)h) - \frac{1}{h} \int_0^h G(y + a + ih - h)\, dy \right] \\
&= -\frac{1}{h} \left[G(a + (i-1)h) - \frac{1}{h} \int_{a+(i-1)h}^{a+ih} G(y)\, dy \right],
\end{aligned}
$$

for $i = 1, \ldots, n$

$$\Theta_{i,i}^1 = \langle \dot{G}\phi_i, \dot{\phi}_i \rangle = \int_0^{2h} \dot{G}(y + x_{i-1})\phi(y)\dot{\phi}(y)\,dy$$

$$= \frac{1}{h} \left[\int_0^h \dot{G}(y + x_{i-1}) \left(\frac{y}{h} \right) dy - \int_h^{2h} \dot{G}(y + x_{i-1}) \left(2 - \frac{y}{h} \right) dy \right]$$

$$= \frac{1}{h} \left[\int_0^{2h} \dot{G}(y + x_{i-1}) \left(\frac{y}{h} \right) dy - 2(G(x_{i+1}) - G(x_i)) \right]$$

$$= \frac{1}{h} \left[2G(a + ih) - \frac{1}{h} \int_{a+(i-1)h}^{a+(i+1)h} G(y)\,dy \right],$$

for $i = 1, \ldots, n-1$. For $i = 0$, we only have the decreasing part.

$$\Theta_{00}^1 = -\frac{1}{h} \int_h^{2h} \dot{G}(y + (a - h)) \left(2 - \frac{y}{h} \right) dy = \frac{1}{h} \left[G(a) - \frac{1}{h} \int_a^{a+h} G(y)\,dy \right].$$

On the other hand, for $i = n$, we only have the increasing part.

$$\Theta_{nn}^1 = \frac{1}{h} \int_0^h \dot{G}(y + (b - h)) \left(\frac{y}{h} \right) dy = \frac{1}{h} \left[G(b) - \frac{1}{h} \int_{b-h}^b G(y)\,dy \right].$$

On the other hand, for $i = n$, we only have the increasing part. If $i - j \geq 2$, we have $\Theta_{ij}^1 = 0$. Notice that Θ^2 has similar calculations substituting G by α.

Finally, we calculate the vector v. For $i = 1, \ldots, n-1$,

$$v_i = \langle \dot{\alpha}\rho_\infty, \dot{\phi}_i \rangle = \frac{1}{h} \left[\int_{a+(i-1)h}^{a+ih} \dot{\alpha}(y)\rho_\infty(y)\,dy - \int_{a+ih}^{a+(i+1)h} \dot{\alpha}(y)\rho_\infty(y)\,dy \right]$$

and

$$v_0 = -\frac{1}{h} \int_a^{a+h} \dot{\alpha}(y)\rho_\infty(y)\,dy, \quad v_n = \frac{1}{h} \int_{b-h}^b \dot{\alpha}(y)\rho_\infty(y)\,dy.$$

The discrete matrices A, B can be obtained from the same manner. Besides that, we need to consider the boundary condition: $\nu y_x + y G_x = 0$ in the boundary points. Therefore,

$$\sum_{j=0}^n y_j(t) \left[\nu \dot{\phi}_j(x) + \dot{G}(x)\phi_j(x) \right] = 0, \, x = a, b, t \geq 0,$$

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which defines to linear conditions,

$$y_0(t)\left[-\nu/h + \dot{G}(a)\right] + y_1(t)\left[\nu/h\right] = 0, y_{n-1}(t)\left[-\nu/h\right] + y_n(t)\left[\nu/h + \dot{G}(b)\right] = 0.$$

So we can write $y_0(t) = \nu y_1(t)/(\nu - h\dot{G}(a))$ and $y_n(t) = \nu y_{n-1}(t)/(\nu + h\dot{G}(b))$. This tells us that we can decrease the system of EDOs in two variables.

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