Topological Data Analysis - Exercises

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1 General topology

1.1 Important definitions

Definition 1. A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. for every infinite collection $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, we have $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. for every finite collection $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$, we have $\bigcap_{1 \leq i \leq n} O_i \in \mathcal{T}$.

Definition 2. Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r, denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, ||x - y|| < r\}$.

Definition 3. Let $A \subset \mathbb{R}$ and $x \in A$. We say that A is open around x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is open if for every $x \in A$, A is open around x.

Definition 4. Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. We define the subspace topology on Y as the following set:

$$T_{|Y} = \{O \cap Y, O \in \mathcal{T}\}$$

Definition 5. Let $f : X \to Y$ be a map. We say that f is continuous if for every $O \in U$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.

1.2 Exercises

Exercise 1. Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?

As we know every Topology contains $\emptyset$ and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\}$ - $\mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\}$
  $\{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\}$
  $\{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\}$
  $\{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

EXERCISE 2. Let $\mathbb{Z}$ be the set of integers. Consider the *cofinite topology* $\mathcal{T}$ on $\mathbb{Z}$, defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or ${}^cO$ is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in $\mathbb{Z}$

1. Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

   Let's verify the three axioms:

   (a) $\emptyset$ is an open set by definition and $\mathbb{Z}$ is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.

   (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcup_{\alpha \in A} O_\alpha\right) = \bigcap_{\alpha \in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall \alpha \in A$. If $\forall \alpha, O_\alpha = \emptyset$, then ${}^cO = {}^c\emptyset \implies O = \emptyset$ and O is open. On the other hand, if there exists $\alpha \in A$ such that $O_\alpha \neq \emptyset$ we have ${}^cO_\alpha$ being finite, so is ${}^cO$, given the inclusion. We conclude O is open set.

   (c) Let $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcap_{1 \leq i \leq n} O_i\right) = \bigcup_{1 \leq i \leq n} {}^cO_i$. If $O_i = \emptyset$ for some $1 \leq i \leq n$, $O = \emptyset$ because of the intersection. Alternatively, if $\forall i, O_i \neq \emptyset$ we have that ${}^cO_i$ is finite and a finite union of finites is finite. We conclude that O is open set.

   By (a), (b) and (c), $\mathcal{T}$ is a topology on $\mathbb{Z}$.

2. Exhibit an sequence of open sets $\{O_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n \in \mathbb{N}} O_n$ is not an open set.

   Let $O_n = {}^c\{1, ..., n\}$. Thus ${}^cO_n = \{1, ..., n\}$ is finite and

$$ {}^c\left(\bigcap_{n \in \mathbb{N}} O_n\right) = \bigcup_{n \in \mathbb{N}} {}^cO_n = \bigcup_{n \in \mathbb{N}} \{1, ..., n\} = \mathbb{N}, $$

   that is not finite. Therefore, this intersection is not an open set.

EXERCISE 3. Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that

$$ \mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r) $$

Let $z \in \mathcal{B}(y, r - ||x - y||)$, so $||z - y|| < r - ||x - y|| \implies ||z - y|| + ||x - y|| < r$. We can conclude that, by the triangular inequality,

$$ ||x - z|| \leq ||x - y|| + ||z - y|| < r. $$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$.

Remark. In the notes, the exercise is to prove $\mathcal{B}(y, ||x - y||) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take y next the border of $\mathcal{B}(x, r)$, $||x - y|| \approx r$ and $B(y, r - \epsilon) \not\subset B(x, r)$.

EXERCISE 4. Let $x, y \in \mathbb{R}^n$, and $r = ||x - y||$. Show that

$$ \mathcal{B}\left(\frac{x + y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r) $$

Define $m = \dfrac{x+y}{2}$. Take $z \in \mathcal{B}\left(m, \dfrac{r}{2}\right)$. Thus, using the triangular inequality,

$$||x-z|| \leq ||x-m|| + ||m-z|| = \frac{1}{2}||x-y|| + ||m-z|| < r/2 + r/2 = r$$

$$||y-z|| \leq ||y-m|| + ||m-z|| = \frac{1}{2}||y-x|| + ||m-z|| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x,r)$, $z \in \mathcal{B}(y,r)$ and $z \in \mathcal{B}(x,r) \cap \mathcal{B}(y,r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x,r) \cap \mathcal{B}(y,r)$.

Exercise 5. Show that the open balls $\mathcal{B}(x,r)$ of R^n are open sets (with respect to the Euclidean topology).

We have to prove that for every $y \in \mathcal{B}(x,r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y,\epsilon) \subset \mathcal{B}(x,r)$. Put $\epsilon = r - ||x-y||$. As we have proved in exercise 3, $\mathcal{B}(y,\epsilon) \subset \mathcal{B}(x,r)$. So $\mathcal{B}(x,r)$ is open set.

Exercise 6. Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?

1. $[0,1]$. It's not open set because for every $\epsilon > 0, \mathcal{B}(0,\epsilon) = (-\epsilon, \epsilon) \not\subset [0,1]$. It's closed because $[0,1]^c = (-\infty, 0) \cup (1, \infty)$ is an union of two open sets, as we prove in item 3.
2. $[0,1)$. It's not open for the same reason as before. It's not closed because $B(1,\epsilon) = (1-\epsilon, 1+\epsilon) \not\subset (-\infty, 0) \cup [1, \infty]$.
3. $(-\infty, 1)$. It's open because: take $x < 1$. Put $r = 1 - x$ and take $z \in \mathcal{B}(x,r)$. If $z > x$, $|x-z| < 1-x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty, 1)$ is open. It's not closed cause $\forall \epsilon > 0, \mathcal{B}(1,\epsilon) \not\subset [1, \infty)$.
4. the singletons. It's not open cause $\forall \epsilon > 0, x + \epsilon/2 \in \mathcal{B}(x,\epsilon)$. It's close cause $(-\infty, x) \cup (x, \infty)$ is union of open sets.
5. $\mathbb{Q}$. It's not open because for every open ball around a rational, there is irrationals, that is, for $x \in \mathbb{Q}$ and $\forall \epsilon > 0$, exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x,\epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

Exercise 7. A map is continuous if and only if the preimage of closed sets are closed sets.

First we shall prove that $f^{-1}(\,^cA) = \,^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}(\,^cA)$. So there exists $y \in \,^cA$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in \,^c(f^{-1}(A))$.

Now take $x \in \,^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in \,^cA \implies x \in f^{-1}(\,^cA)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose f is a continuos map and take a closed set F. We shall prove that $f^{-1}(F)$ is closed. Well, $^c(f^{-1}(F)) = f^{-1}(\,^cF)$ is open, because ^cF is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set F, we have $f^{-1}(F)$ being closed. We will use that A is open if ^cA is closed. This is true because $^c(^c(A)) = A$. Take an open set A. $^c(f^{-1}(A)) = f^{-1}(\,^cA)$ is closed, because ^cA is. Thus $f^{-1}(A)$ is open and we have proved the continuity of f.

2 Homeomorphisms

2.1 Important definitions

2.2 Exercises