

# TOPOLOGICAL DATA ANALYSIS - EXERCISES

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## 1 General topology

### 1.1 Important definitions

DEFINITION 1.1.1. A topological space is a pair  $(X, \mathcal{T})$  where  $X$  is a set and  $\mathcal{T}$  is a collection of subsets of  $X$  such that:

1.  $\emptyset \in \mathcal{T}, X \in \mathcal{T}$ .
2. for every infinite collection  $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ , we have  $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$ .
3. for every finite collection  $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$ , we have  $\bigcap_{1 \leq i \leq n} O_i \in \mathcal{T}$ .

DEFINITION 1.1.2. Let  $x \in \mathbb{R}^n$  and  $r > 0$ . The open ball of center  $x$  and radius  $r$ , denoted  $\mathcal{B}(x, r)$ , is defined as:  $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$ .

DEFINITION 1.1.3. Let  $A \subset \mathbb{R}$  and  $x \in A$ . We say that  $A$  is open around  $x$  if there exists  $r > 0$  such that  $\mathcal{B}(x, r) \subset A$ . We say that  $A$  is open if for every  $x \in A$ ,  $A$  is open around  $x$ .

DEFINITION 1.1.4. Let  $(X, \mathcal{T})$  be a topological space, and  $Y \subset X$ . We define the subspace topology on  $Y$  as the following set:

$$T|_Y = \{O \cap Y, O \in \mathcal{T}\}$$

DEFINITION 1.1.5. Let  $f : X \rightarrow Y$  be a map. We say that  $f$  is continuous if for every  $O \in \mathcal{U}$ , the preimage  $f^{-1}(O) = \{x \in X, f(x) \in O\}$  is in  $\mathcal{T}$ .

### 1.2 Exercises

EXERCISE 1. Let  $X = \{0, 1, 2\}$  be a set with three elements. What are the different topologies that  $X$  admits?

As we know every Topology contains  $\emptyset$  and  $\{0, 1, 2\}$ , so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic:  $\{\emptyset, \{0, 1, 2\}\} - \mathcal{P}(\{0, 1, 2\})$ .
- (8) With  $\{0\}$ :  $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\} - \{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With  $\{1\}$ :  $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With  $\{2\}$ :  $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton:  $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

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EXERCISE 2. Let  $\mathbb{Z}$  be the set of integers. Consider the cofinite topology  $\mathcal{T}$  on  $\mathbb{Z}$ , defined as follows: a subset  $O \subset \mathbb{Z}$  is an open set if and only if  $O = \emptyset$  or  ${}^cO$  is finite. Here,  ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$  represents the complementary of  $O$  in  $\mathbb{Z}$

1. Show that  $\mathcal{T}$  is a topology on  $\mathbb{Z}$ .

Let's verify the three axioms:

- (a)  $\emptyset$  is an open set by definition and  $\mathbb{Z}$  is open set because  ${}^c\mathbb{Z} = \emptyset$  is finite.
- (b) Let  $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$ . So  ${}^cO = {}^c\left(\bigcup_{\alpha \in A} O_\alpha\right) = \bigcap_{\alpha \in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall \alpha \in A$ . If  $\forall \alpha, O_\alpha = \emptyset$ , then  ${}^cO = {}^c\emptyset \implies O = \emptyset$  and  $O$  is open. On the other hand, if there exists  $\alpha \in A$  such that  $O_\alpha \neq \emptyset$  we have  ${}^cO_\alpha$  being finite, so is  ${}^cO$ , given the inclusion. We conclude  $O$  is open set.
- (c) Let  $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$ . So  ${}^cO = {}^c\left(\bigcap_{1 \leq i \leq n} O_i\right) = \bigcup_{1 \leq i \leq n} {}^cO_i$ . If  $O_i = \emptyset$  for some  $1 \leq i \leq n$ ,  $O = \emptyset$  because of the intersection. Alternatively, if  $\forall i, O_i \neq \emptyset$  we have that  ${}^cO_i$  is finite and a finite union of finites is finite. We conclude that  $O$  is open set.

By (a), (b) and (c),  $\mathcal{T}$  is a topology on  $\mathbb{Z}$ .

2. Exhibit an sequence of open sets  $\{O_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$  such that  $\bigcap_{n \in \mathbb{N}} O_n$  is not an open set.

Let  $O_n = {}^c\{1, \dots, n\}$ . Thus  ${}^cO_n = \{1, \dots, n\}$  is finite and

$${}^c\left(\bigcap_{n \in \mathbb{N}} O_n\right) = \bigcup_{n \in \mathbb{N}} {}^cO_n = \bigcup_{n \in \mathbb{N}} \{1, \dots, n\} = \mathbb{N},$$

that is not finite. Therefore, this intersection is not an open set.

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EXERCISE 3. Let  $x \in \mathbb{R}^n$ , and  $r > 0$ . Let  $y \in \mathcal{B}(x, r)$ . Show that

$$\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$$

Let  $z \in \mathcal{B}(y, r - \|x - y\|)$ , so  $\|z - y\| < r - \|x - y\| \implies \|z - y\| + \|x - y\| < r$ . We can conclude that, by the triangular inequality,

$$\|x - z\| \leq \|x - y\| + \|z - y\| < r.$$

In that sense,  $z \in \mathcal{B}(x, r)$  and  $\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$ .

*Remark.* In the notes, the exercise is to prove  $\mathcal{B}(y, \|x - y\|) \subset \mathcal{B}(x, r)$ , however, this does not hold, because if we take  $y$  next the border of  $\mathcal{B}(x, r)$ ,  $\|x - y\| \approx r$  and  $\mathcal{B}(y, r - \epsilon) \not\subset \mathcal{B}(x, r)$ .

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EXERCISE 4. Let  $x, y \in \mathbb{R}^n$ , and  $r = \|x - y\|$ . Show that

$$\mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$$

Denote  $m = \frac{x+y}{2}$ . Take  $z \in \mathcal{B}\left(m, \frac{r}{2}\right)$ . Thus, using the triangular inequality,

$$\|x - z\| \leq \|x - m\| + \|m - z\| = \frac{1}{2}\|x - y\| + \|m - z\| < r/2 + r/2 = r$$

$$\|y - z\| \leq \|y - m\| + \|m - z\| = \frac{1}{2}\|y - x\| + \|m - z\| < r/2 + r/2 = r$$

So  $z \in \mathcal{B}(x, r)$ ,  $z \in \mathcal{B}(y, r)$  and  $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$ . Therefore  $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$ .

EXERCISE 5. Show that the open balls  $\mathcal{B}(x, r)$  of  $\mathbb{R}^n$  are open sets (with respect to the Euclidean topology).

We have to prove that for every  $y \in \mathcal{B}(x, r)$ , there exists  $\epsilon > 0$  such that  $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$ . Put  $\epsilon = r - \|x - y\|$ . As we have proved in exercise 3,  $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$ . So  $\mathcal{B}(x, r)$  is open set.

EXERCISE 6. Consider  $X = \mathbb{R}$  endowed with the Euclidean topology. Are the following sets open? Are they closed?

1.  $[0, 1]$ . It's not open set because for every  $\epsilon > 0$ ,  $\mathcal{B}(0, \epsilon) = (-\epsilon, \epsilon) \not\subset [0, 1]$ . It's closed because  $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$  is an union of two open sets, as we prove in item 3.
2.  $[0, 1)$ . It's not open for the same reason as before. It's not closed because  $\mathcal{B}(1, \epsilon) = (1 - \epsilon, 1 + \epsilon) \not\subset [0, 1]$ .
3.  $(-\infty, 1)$ . It's open because: take  $x < 1$ . Put  $r = 1 - x$  and take  $z \in \mathcal{B}(x, r)$ . If  $z > x$ ,  $|x - z| < 1 - x \implies z < 1$ . If  $z < x$ , it follows  $z < 1$ . It proves  $z < 1$  and  $(-\infty, 1)$  is open. It's not closed cause  $\forall \epsilon > 0$ ,  $\mathcal{B}(1, \epsilon) \not\subset (-\infty, 1)$ .
4. the singletons. It's not open cause  $\forall \epsilon > 0$ ,  $x + \epsilon/2 \in \mathcal{B}(x, \epsilon)$ . It's close cause  $(-\infty, x) \cup (x, \infty)$  is union of open sets.
5.  $\mathbb{Q}$ . It's not open because for every open ball around a rational, there are irrationals, that is, let  $x \in \mathbb{Q}$  and take  $\epsilon > 0$ , then there exists  $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x, \epsilon)$ . It's not closed for the same reason, for every irrational, there is rationals for every open ball.

*Remark.* We shall prove the rationals are dense in the reals. Let  $x \in \mathbb{Q}$  and  $\epsilon > 0$ . If  $\epsilon$  is irrational, take  $x - \epsilon/2 \subset (x - \epsilon, x + \epsilon)$ . Suppose  $x - \epsilon/2$  is rational, then  $\frac{2x - \epsilon}{2} = \frac{m}{n}$  for some integers  $m$  and  $n$ , that is,  $2x - \epsilon = 2m/n$  and  $\epsilon = 2(x - \frac{m}{n}) \in \mathbb{Q}$ , contradiction. So there is an irrational in  $\mathcal{B}(x, \epsilon)$ . If  $\epsilon$  is rational, consider

$$y = \frac{1}{\sqrt{2}}(x - \epsilon) + (1 - \frac{1}{\sqrt{2}})(x + \epsilon) = (x + \epsilon) - \epsilon\sqrt{2}$$

That is a convex combination, so  $y \in \mathcal{B}(x, \epsilon)$ . Moreover,  $y$  is irrational, with a similar proof by contradiction. This proves the statement.

On the other hand, we must prove for every two irrationals  $(a, b)$ , there is a rational between them. Denote  $c = b - a > 0$ . Let  $n \in \mathbb{N}$  such that  $n > \frac{1}{c} \implies cn > 1 \implies (bn - an) > 1$ . So exists  $m \in (an, bn) \implies m/n \in (a, b)$ . This proves the second statement.

EXERCISE 7. *A map is continuous if and only if the preimage of closed sets are closed sets.*

First we shall prove that  $f^{-1}({}^cA) = {}^c(f^{-1}(A))$ . Let's prove the double inclusion. Take  $x \in f^{-1}({}^cA)$ . So there exists  $y \in {}^cA$  such that  $f(x) = y$ . Suppose that  $x \in f^{-1}(A)$ . It implies the existence of  $z \in A$  such that  $y = f(x) = z$ , absurd. So  $x \in {}^c(f^{-1}(A))$ .

Now take  $x \in {}^c(f^{-1}(A))$ . Therefore,  $\forall y \in A, f(x) \neq y$ . In that case,  $f(x) \in {}^cA \implies x \in f^{-1}({}^cA)$ . Then we have showed the equality.

Now let's prove the equivalence. Suppose  $f$  is a continuous map and take a closed set  $F$ . We shall prove that  $f^{-1}(F)$  is closed. Well,  ${}^c(f^{-1}(F)) = f^{-1}({}^cF)$  is open, because  ${}^cF$  is open, by the continuity. We conclude that  $f^{-1}(F)$  is closed.

Suppose that for every closed set  $F$ , we have  $f^{-1}(F)$  being closed. We will use that  $A$  is open if  ${}^cA$  is closed. This is true because  ${}^c({}^cA) = A$ . Take an open set  $A$ .  ${}^c(f^{-1}({}^cA)) = f^{-1}({}^c({}^cA))$  is closed, because  ${}^c({}^cA) = A$ . Thus  $f^{-1}({}^cA)$  is open and we have proved the continuity of  $f$ .

## 2 Homeomorphisms

### 2.1 Important definitions

DEFINITION 2.1.1. Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be two topological spaces, and  $f : X \rightarrow Y$  a map. We say that  $f$  is a homeomorphism if

1.  $f$  is a bijection,
2.  $f : X \rightarrow Y$  is continuous,
3.  $f^{-1} : Y \rightarrow X$  is continuous.

If there exists such a homeomorphism, we say that the two topological spaces are homeomorphic.

DEFINITION 2.1.2. Let  $(X, \mathcal{T})$  be a topological space. We say that  $X$  is connected if for every open sets  $O, O' \in \mathcal{T}$  such that  $O \cap O' = \emptyset$  (i.e., they are disjoint), we have

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

DEFINITION 2.1.3. Let  $(X, \mathcal{T})$  be a topological space. Suppose that there exists a collection of  $n$  **non-empty, disjoint and connected open sets**  $(O_1, \dots, O_n)$  such that

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

Then we say that  $X$  admits  $n$  connected components.

DEFINITION 2.1.4. Let  $(X, \mathcal{T})$  be a topological space, and  $n \geq 0$ . We say that it has dimension  $n$  if the following is true: for every  $x \in X$ , there exists an open set  $O$  such that  $x \in O$ , and a homeomorphism  $O \rightarrow \mathbb{R}^n$ .

### 2.2 Exercises

EXERCISE 8. Show that the topological spaces  $\mathbb{R}^n$  and  $\mathcal{B}(0, 1) \subset \mathbb{R}^n$  are homeomorphic.

Let  $f : \mathcal{B}(0, 1) \rightarrow \mathbb{R}^n$  be defined as  $f(x) = \frac{x}{1 - \|x\|}$ . I observe it's well defined because  $\|x\| < 1$ . We shall prove  $f$  is a homeomorphism.

1. **Injective:** Take  $x, y \in \mathcal{B}(0, 1)$  and suppose that

$$\frac{x}{1 - \|x\|} = \frac{y}{1 - \|y\|}.$$

Applying the norm in both sides, we obtain the equation

$$\|x\|(1 - \|y\|) = \|y\|(1 - \|x\|) \implies \|x\| = \|y\|.$$

On the other side  $x$  and  $y$  points to the same direction, given that

$$y = \frac{1 - \|y\|}{1 - \|x\|} x = \alpha x,$$

with  $\alpha = 1$  because of the same norm. We conclude  $x = y$ .

2. **Surjective:** Take  $y \in \mathbb{R}^n$ . We shall prove that there exists  $x \in \mathcal{B}(0, 1)$  such that  $f(x) = y$ , that is,

$$\frac{x}{1 - \|x\|} = y$$

Applying the norm we observe that if that is true,  $\|x\| = \|y\| - \|y\|\|x\| \implies \|x\| = \frac{\|y\|}{1 + \|y\|}$ . And  $x = (1 - \|x\|)y = \frac{1}{1 + \|y\|}y$ . We conclude that for every  $y \in \mathbb{R}^n$ , if we take  $x = \frac{y}{1 + \|y\|}$ ,

$$f(x) = \frac{y/(1 + \|y\|)}{1 - \|y\|/(1 + \|y\|)} = y$$

3. **Continuity of  $f$ :** Consider an open set  $A \subset \mathbb{R}^n$ . Let  $B = f^{-1}(A)$ . We shall prove  $B$  is open, that is, for every  $x \in B$ , exists  $r > 0$  such that  $\mathcal{B}(x, r) \subset B$ . Take  $x = f^{-1}(y) \in B$ . Because  $A$  is open, there is  $\epsilon > 0$  such that  $\mathcal{B}(y, \epsilon) \subset A$ . Take  $\delta$  such that

$$\frac{\delta}{1 - \|x\| - \delta}(1 + \|y\|) < \epsilon$$

and  $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$ .

$$\begin{aligned} \|y - w\| &= \left\| \frac{x}{1 - \|x\|} - \frac{z}{1 - \|z\|} \right\| = \frac{1}{1 - \|x\|} \left\| x - \frac{1 - \|x\|}{1 - \|z\|} z \right\| \\ &= \frac{1}{1 - \|x\|} \left\| x - z + z - \frac{1 - \|x\|}{1 - \|z\|} z \right\| \\ &\leq \frac{\|x - z\|}{1 - \|x\|} + \frac{1}{1 - \|x\|} \left( 1 - \frac{1 - \|x\|}{1 - \|z\|} \|z\| \right) \\ &= \frac{\|x - z\|}{1 - \|x\|} + \frac{\|z\|}{1 - \|x\|} \frac{\|x\| - \|z\|}{1 - \|z\|} \\ &\leq \frac{1}{1 - \|x\|} \|x - z\| (1 + \|w\|) \\ &\leq \frac{1}{1 - \|x\|} \|x - z\| (1 + \|y - w\| + \|y\|) \\ \implies \|y - w\| &\leq \frac{\|x - z\|}{1 - \|x\| - \|x - z\|} (1 + \|y\|) \\ &< \frac{\delta}{1 - \|x\| - \delta} (1 + \|y\|) < \epsilon \end{aligned}$$

So  $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$ , what proves  $B$  is open. It concludes the continuity of  $f$ .

4. **Continuity of  $f^{-1}$ :** The inverse is given by

$$f^{-1}(y) = \frac{y}{1 + \|y\|}$$

The demonstration is quite similar to the previous item, given that the only difference is the signal.

By items (1) - (4), we conclude  $f$  is a homeomorphism and  $\mathcal{B}(0, 1) \simeq \mathbb{R}^n$ .

EXERCISE 9. Show that  $\mathcal{B}(x, r)$  and  $\mathcal{B}(y, s)$  are homeomorphic.

Consider the function  $f : \mathcal{B}(0, 1) \rightarrow \mathcal{B}(c, r)$  given by  $f(x) = r \cdot x + c$ . Let's prove  $f$  is a

homeomorphism.

1. **Injective:** If  $x, y \in \mathcal{B}(0, 1)$  and  $rx + c = ry + c \implies x = y$ , because  $r > 0$  by definition. So  $f$  is injective.
2. **Surjective:** Let  $y \in \mathcal{B}(c, r)$  and  $x = (y - c)/r$ . So  $\|x\| = \|y - c\|/r < 1$ , by definition. So  $x \in \mathcal{B}(0, 1)$  and  $f(x) = y$  what proves this function is surjective.
3. **Continuity of  $f$ :** Let  $A \subset \mathcal{B}(c, r)$  open set and denote  $B = f^{-1}(A)$ . Take  $x = f^{-1}(y) \in B$ . We know there exists  $\epsilon > 0$  such that  $\mathcal{B}(y, \epsilon) \subset A$ . Define  $\delta = \epsilon/r$  and take  $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$ .

$$\|y - w\| = \|rx + c - (rz + c)\| = r\|x - z\| < r\delta = \epsilon$$

Therefore  $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$ . So  $\mathcal{B}(x, \delta) \subset B$ , what proves  $B$  is open. This concludes the continuity of  $f$ .

4. **Continuity of  $f^{-1}$ :** The inverse is given by

$$f^{-1}(y) = \frac{y - c}{r}$$

This function is continuous for the same argument as before.

By items (1) - (4), we conclude  $f$  is a homeomorphism and  $\mathcal{B}(0, 1) \simeq \mathcal{B}(c, r)$ . Since this is an equivalence relation, we have that

$$\mathcal{B}(0, 1) \simeq \mathcal{B}(x, r) \text{ and } \mathcal{B}(0, 1) \simeq \mathcal{B}(y, s) \text{ implies } \mathcal{B}(x, r) \simeq \mathcal{B}(y, s).$$

EXERCISE 10. Show that  $\mathbb{S}(0, 1)$ , the unit circle of  $\mathbb{R}^2$ , is homeomorphic to the ellipse

$$\mathcal{S}(a, b) = \left\{ (x, y) \in \mathbb{R}^2, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \right\},$$

for any  $a, b > 0$ .

Consider the function  $f : \mathbb{S}(0, 1) \rightarrow \mathcal{S}(a, b)$  defined as  $f(x, y) = (ax, by)$ . Let's prove it is a homeomorphism.

1. **Injective:** Let  $(x_1, y_1), (x_2, y_2) \in \mathbb{S}(0, 1)$  such that  $(ax_1, by_1) = (ax_2, by_2)$ . Since  $a, b > 0$ , we have  $x_1 = x_2$  and  $y_1 = y_2$ . It proves  $f$  is injective.
2. **Surjective:** Let  $(z, w) \in \mathcal{S}(a, b)$  and  $(x, y) = \left(\frac{z}{a}, \frac{w}{b}\right)$ . It's clear that  $f(x, y) = (z, w)$  and  $x^2 + y^2 = \frac{z^2}{a^2} + \frac{w^2}{b^2} = 1$ , so  $(x, y) \in \mathbb{S}(0, 1)$ . It proves  $f$  is surjective.
3. **Continuity of  $f$ :** Let  $A \subset \mathcal{S}(a, b)$  open set and denote  $B = f^{-1}(A)$ . Take  $(x, y) = f^{-1}(z, w) \in B$ . We know there exists  $\epsilon > 0$  such that  $\mathcal{B}((z, w), \epsilon) \subset A$ . Put  $\delta$  as defined below and take  $(x', y') = f^{-1}((z', w')) \in \mathcal{B}((x, y), \delta)$ . Consider the norm 1

$$\begin{aligned} \|(z', w') - (z, w)\|_1 &= \|(ax', by') - (ax, by)\|_1 = \|(a(x' - x), b(y' - y))\|_1 \\ &= a|x' - x| + b|y' - y|, \text{ define } c = \max\{a, b\} \\ &\leq c(|x' - x| + |y' - y|) = c\|(x' - x, y' - y)\|_1 \end{aligned}$$

By the equivalence of the norms, there exists constants  $k_1, k_2$  such that

$$\|(z', w') - (z, w)\| \leq k_1 \|(z', w') - (z, w)\|_1 \leq ck_1 \|(x' - x, y' - y)\|_1 \leq ck_1 k_2 \|(x' - x, y' - y)\|$$

Then we need  $\delta = \frac{\epsilon}{ck_1 k_2}$  in order to prove that  $(z', w') \in \mathcal{B}((z, w), \epsilon) \subset A \implies (x', y') \in B$ . So  $\mathcal{B}((x, y), \delta) \subset B$ , what proves  $B$  is open. This concludes the continuity of  $f$ .

4. **Continuity of  $f^{-1}$ :** The inverse is given by

$$f^{-1}((z, w)) = (z/a, w/b)$$

This function is continuous for the same argument as before.

By items (1) - (4), we conclude  $f$  is a homeomorphism and  $\mathbb{S}(0, 1) \simeq \mathcal{S}(a, b)$ .

EXERCISE 11. *Show that  $[0, 1)$  and  $(0, 1)$  are not homeomorphic.*

We shall prove by contradiction. Suppose there exists a homeomorphism  $f : [0, 1) \rightarrow (0, 1)$ . Let  $0 < z = f(0) < 1$  and define the following function

$$\begin{aligned} g : (0, 1) &\rightarrow (0, z) \cup (z, 1) \\ x &\mapsto g(x) = f(x) \end{aligned}$$

This function is well defined given that  $z$  is not image of other point but 0. The function is injective because if  $g(y) = g(x) \implies f(y) = f(x) \implies x = y$ , given that  $f$  is injective. This function is also surjective since  $f$  is and  $0 < w < 1$  and  $w \neq z$ , it's clear that  $f(0) \neq w$ . As  $g$  is an induced map of a continuous function, by Proposition 1.21 from the notes, it's continuous and so is its inverse. We conclude  $g$  is a homeomorphism.

Now I will prove that  $(0, 1)$  admits only 1 connected component, that is, it's connected. Suppose it's not and there exists  $O, O' \subset (0, 1)$  open disjoint sets such that  $(0, 1) = O \cup O'$  and none of them are empty sets. Let  $a \in O, b \in O'$  with  $a < b$  without loss of generality. Define  $\alpha = \sup\{x \in \mathbb{R} : [a, x) \subset O\}$ . It's well defined because this set is not empty, given  $O$  is open and  $b$  is an upper bound. Then  $\alpha \leq b$ . Suppose  $\alpha \in O'$ , then there exists  $r > 0$  such that  $(\alpha - r, \alpha + r) \subset O'$ . We know that for every  $\epsilon > 0$ , there exists  $w \in (\alpha - \epsilon, \alpha]$  such that  $[a, w) \subset O$ . That is a contradiction since there exists  $w \in (\alpha - r, \alpha)$  such that  $[a, w) \subset O$ . So  $\alpha \in O \implies (\alpha - r, \alpha + r) \subset O$ , for some  $r$ . We infer that  $[a, \alpha + r) \subset O$ , what is absurd. Therefore  $(0, 1)$  is connected.

For a similar argument, we prove that  $(0, z)$  and  $(z, 1)$  are connected. This implies that the union admits 2 connected components.

In that sense, we have a homeomorphism between a topological space with 1 connected component and other with 2 connected components, what is a contradiction by Proposition 2.14 from the notes. We conclude that  $[0, 1)$  and  $(0, 1)$  are not homeomorphic.



## 3 Homotopies

### 3.1 Important definitions

DEFINITION 3.1.1. Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be two topological spaces, and  $f, g : X \rightarrow Y$  two continuous maps. A homotopy between  $f$  and  $g$  is a map  $F : X \times [0, 1] \rightarrow Y$  such that:

1.  $F(\cdot, 0)$  is equal to  $f$ ,
2.  $F(\cdot, 1)$  is equal to  $g$ ,
3.  $F : X \times [0, 1] \rightarrow Y$  is continuous.

If such a homotopy exists, we say that the maps  $f$  and  $g$  are homotopic.

Remark. Before asking for  $F : X \times [0, 1] \rightarrow Y$  to be continuous, we have to give  $X \times [0, 1]$  a topology. The topology we choose is the product topology. Consider the topological space  $(X, \mathcal{T})$ , and endow  $[0, 1]$  with the subspace topology of  $\mathbb{R}$ , denoted  $T_{[0,1]}$ . The product topology on  $X \times [0, 1]$ , denoted  $T \otimes T_{[0,1]}$ , is defined as follows: a set  $O \subset X \times [0, 1]$  is open if and only if it can be written as a union  $\bigcup_{\alpha \in A} O_\alpha \times O'_\alpha$  where every  $O_\alpha$  is an open set of  $X$  and  $O'_\alpha$  is an open set of  $[0, 1]$ .

DEFINITION 3.1.2. Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{U})$  be two topological spaces. A homotopy equivalence between  $X$  and  $Y$  is a pair of continuous maps  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  such that:

1.  $g \circ f : X \rightarrow X$  is homotopic to the identity map  $\text{id} : X \rightarrow X$ ,
2.  $f \circ g : Y \rightarrow Y$  is homotopic to the identity map  $\text{id} : Y \rightarrow Y$ ,

If such a homotopy equivalence exists, we say that  $X$  and  $Y$  are homotopy equivalent.

DEFINITION 3.1.3. Let  $(X, \mathcal{T})$  be a topological space and  $Y \subset X$  a subset, endowed with the subspace topology  $T|_Y$ . A retraction is a continuous map  $r : X \rightarrow Y$  such that  $\forall y \in Y, r(y) = y$ .

A deformation retraction is a homotopy  $F : X \times [0, 1] \rightarrow Y$  between the identity map  $\text{id} : X \rightarrow X$  and a retraction  $r : X \rightarrow Y$ .

### 3.2 Exercises

EXERCISE 12. Let  $f : \mathbb{R}^n \rightarrow X$  be a continuous map. Then  $f$  is homotopic to a constant map.

I must prove that there exists a homotopy between  $f$  and a constant map. Consider the function  $F : \mathbb{R}^n \times [0, 1] \rightarrow X$  defined as

$$F(x, t) = f(tx)$$

It's clear that  $F(x, 0) = f(0)$ , for every  $x \in \mathbb{R}^n$ . So it's the constant map  $f(0)$ . We also have that  $F(x, 1) = f(x), \forall x \in \mathbb{R}^n$ . Moreover, let's prove  $F$  is continuous. Denote  $F' : \mathbb{R}^n \times \mathbb{R} \rightarrow X$  the function  $F'(x, t) = f(xt)$  and  $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  the function  $g(x, t) = xt$ . So  $F' = f \circ g$ .

Let's prove  $g$  is a continuous function. As we are dealing with a real-valued function, by Proposition 1.19 from the notes, I can use the  $\epsilon - \delta$  proof. Let  $(x, t) \in \mathbb{R}^{n+1}$  and  $\epsilon > 0$ . In the proof I use the norm 1, without loss of generality because of the equivalence of norms in  $\mathbb{R}^n$ . Put

$\delta = \min\{1, \frac{\epsilon}{\max\{\|x\|, |t|+1\}}\}$  and suppose  $\|(x, t) - (x', t')\| = \|x - x'\| + |t - t'| < \delta$ . So,

$$\begin{aligned}\|xt - x't'\| &= \|xt - xt' + xt' - x't'\| \\ &\leq |t - t'| \|x\| + |t'| \|x - x'\| \\ &\leq |t - t'| \|x\| + (|t| + \delta) \|x - x'\| \\ &< \max\{\|x\|, |t| + \delta\} \delta \\ &\leq \max\{\|x\|, |t| + 1\} \delta \leq \epsilon\end{aligned}$$

By this,  $g$  is a continuous function. Since  $f$  is also continuous, the composition  $F'$  is also continuous, by Proposition 1.18. By Proposition 1.21, when we endow  $F'$  in  $\mathbb{R}^n \times [0, 1]$ , we obtain a continuous function, that is  $F$  is continuous. Then we conclude that  $f$  is homotopic to a constant function.

EXERCISE 13. Let  $f : \mathbb{S}_1 \rightarrow \mathbb{S}_2$  be a continuous map which is not surjective. Prove that it is homotopic to a constant map where the unit sphere  $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, \|x\| = 1\}$ .

Let  $x_0 \in \mathbb{S}_2$  such that  $x_0 \notin f(\mathbb{S}_1)$  and consider the constant map  $g(x) = -x_0$ , for every  $x \in \mathbb{S}_1$ . Let  $F : \mathbb{S}_1 \times [0, 1] \rightarrow \mathbb{S}_2$  be defined as

$$F(x, t) = 2 \frac{(1-t)f(x) - tx_0}{\|(1-t)f(x) - tx_0\|}$$

The first thing we must prove it's well defined. Suppose that  $(1-t)f(x) - tx_0 = 0$ . If  $t = 1$ , then  $x_0 = 0$ , an absurd given that  $\|x_0\| = 2$ . If  $t < 1$ ,  $f(x) = \frac{t}{1-t}x_0$  and applying the norm on both sides  $2 = \|f(x)\| = \frac{t}{1-t}\|x_0\| = 2\frac{t}{1-t} \implies t = 1/2$ . If that is the case,  $f(x) - x_0 = 0 \implies f(x) = x_0$ , contradiction. Moreover, for all  $x$  and  $t$ ,  $\|F(x, t)\| = 2 \implies F(\mathbb{S}_1, [0, 1]) \subset \mathbb{S}_2$ .

Now let's prove it's a homotopy:

1.  $F(x, 0) = 2 \frac{f(x)}{\|f(x)\|} = f(x), \forall x \in \mathbb{S}_1$ .
2.  $F(x, 1) = 2 \frac{-x_0}{\|x_0\|} = -x_0, \forall x \in \mathbb{S}_1$ .
3. Consider the extension of the function  $F' : \mathbb{S}_1 \times [0, 1] \rightarrow \mathbb{R}^3$ . This function is continuous because it's a combination of continuous functions. So  $F$  is continuous because it's a restriction of  $F'$ . I needed to extend the function because  $(1-t)f(x)$  is not necessary in the sphere, so I couldn't prove it's continuous. However, when extended we see each part is continuous.

By (1) - (3), we have proved  $F$  is a homotopy and  $f$  are homotopic to a constant function.

*Remark.* If the function is surjective, it's harder to prove, and I couldn't yet. For instance, this is a reference<sup>1</sup> (but the answers use specialized tools)

EXERCISE 14. Show that being homotopic is a transitive relation between maps: for every triplet of maps  $f, g, h : X \rightarrow Y$ , if  $f, g$  are homotopic and  $g, h$  are homotopic, then  $f, h$  are homotopic.

We shall prove there exist a homotopy  $H$  between  $f$  and  $h$ . By assumption, there exists a homotopy  $F$  between  $f$  and  $g$  and a homotopy  $G$  between  $g$  and  $h$ . Define  $H : X \times [0, 1] \rightarrow Y$

<sup>1</sup><https://math.stackexchange.com/questions/3807715/>

such that

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq 1/2 \\ G(x, 2t - 1), & 1/2 < t \leq 1 \end{cases}$$

that is,  $H$  behaves as  $F$  until it reaches a half. When that occurs,  $H(x, 1/2) = F(x, 1) = g(x) = G(x, 0)$ . After that,  $H$  follows  $G$  until the end of the interval. So, it's clear that  $H(x, 0) = F(x, 0) = f(x), \forall x \in X$  and  $H(x, 1) = G(x, 1) = h(x), \forall x \in X$ . Moreover, since  $F$  and  $G$  are continuous and in the point  $t = 1/2$ , both functions agree,  $H$  is continuous and, therefore,  $f$  and  $h$  are homotopic.

EXERCISE 15. *Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).*

1. (*reflexive*): Consider the identity map  $id : X \rightarrow X$ , that is continuous. We shall prove that this function is homotopic to itself. Consider  $F : X \times [0, 1] \rightarrow X$  given by  $F(x, t) = x$  for every  $x$  and  $t$ . It's clear this is a homotopy because  $F(x, 0) = F(x, 1) = x$  and it's continuous. Moreover  $id \circ id = id$  by definition of identity. Therefore, there exists a homotopy equivalence between  $id$  and itself. We conclude  $X \approx X$ .
2. (*symmetric*): Suppose  $X \approx Y$ . So, there exists continuous functions  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  that form a homotopy equivalence. This means that  $g \circ f : X \rightarrow X$  and  $f \circ g : Y \rightarrow Y$  are a homotopy equivalence as well. So  $Y \approx X$ .
3. (*transitive*): Suppose  $X \approx Y$ , and let  $f_1 : X \rightarrow Y$  and  $g_1 : Y \rightarrow X$  form a homotopy equivalence. Also suppose  $Y \approx Z$  and let  $f_2 : Y \rightarrow Z$  and  $g_2 : Z \rightarrow Y$  form a homotopy equivalence. Define  $f_3 = f_2 \circ f_1$  and  $g_3 = g_1 \circ g_2$ . Let's prove this is a homotopy equivalence. Both functions are continuous given that they are a composition of continuous functions.

- (a)  $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1$  is homotopic to  $id : X \rightarrow X$ .

Let  $F_1$  be a homotopy between  $g_1 \circ f_1$  and  $id$  and  $F_2$  a homotopy between  $g_2 \circ f_2$  and  $id$ . Define

$$F_3(x, t) = \begin{cases} g_1 \circ F_2(\cdot, 2t) \circ f_1(x), & 0 \leq t \leq 1/2 \\ F_1(x, 2t - 1), & 1/2 < t \leq 1 \end{cases}$$

So  $F_3(x, 0) = g_1(F_2(f_1(x), 0)) = g_1(g_2(f_2(f_1(x)))) = g_3 \circ f_3(x)$ , for every  $x$  and  $F_3(x, 1) = F_1(x, 1) = x$ , for every  $x$ . When  $t = 1/2$ ,

$$F_3(x, 1/2) = g_1(F_2(f_1(x), 1)) = g_1(f_1(x)) = F_1(x, 0)$$

By this equality and the fact that composition of continuous functions is a continuous map, we conclude that  $F_3$  is continuous. This implies that  $g_3 \circ f_3$  is homotopic to the identity.

- (b)  $f_3 \circ g_3 = f_1 \circ f_2 \circ g_2 \circ g_1$  is homotopic to  $id : Z \rightarrow Z$ .

This follows a quite similar demonstration and can be omitted.

By the points above  $f_3$  and  $g_3$  is a homotopy equivalence what proves  $X \approx Z$ . Consequently, homotopy equivalence is an equivalence relation.

---

EXERCISE 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

I will consider the upper case alphabet and each letter will be considered as a topological space (a subset from  $\mathbb{R}^2$ ), for example the letter *O* is homotopy equivalent to a circle, while *L* is to an interval, or equivalently, a point. Observe that most of the letters are equivalent to a point, because we can think in a continuous reduction. When we have a hole, such as *A, D, R, O, P, Q*, this continuity is impossible since we'll have a point break. *B* is a special case because we can't deform into a point without breaking points and also we cannot join the holes in one. So there are three classes, given by its representatives

1. *O*
2. *B*
3. *I*

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	1	3	3	3	3	3	3	3	3	3
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
3	1	1	1	1	3	3	3	3	3	3	3	3

## 4 Simplicial complexes

### 4.1 Important definitions

DEFINITION 4.1.1. The standard simplex of dimension  $n$  is the following subset of  $\mathbb{R}^{n+1}$ ,

$$\Delta_n = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \forall i, x_i \geq 0 \text{ and } \sum_{i=1}^{n+1} x_i = 1\}.$$

For any collection of points  $a_1, \dots, a_k \in \mathbb{R}^n$ , we define their convex hull as:

$$\text{conv}(\{a_1, \dots, a_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i a_i \mid \sum_{1 \leq i \leq k} t_i = 1, t_1, \dots, t_k \geq 0 \right\}$$

DEFINITION 4.1.2. Let  $V$  be a set (called the set of vertices). A simplicial complex over  $V$  is a set  $K$  of subsets of  $V$  (called the simplices) such that, for every  $\sigma \in K$  and every non-empty  $\tau \subset \sigma$ , we have  $\tau \in K$ . If  $\sigma \in K$  is a simplex, its non-empty subsets  $\tau \subset \sigma$  are called faces of  $\sigma$ , and  $\sigma$  is called a coface of  $\tau$ . Moreover, its dimension is  $|\sigma| - 1$  and the dimension of a simplicial is the maximum dimension of its simplices.

DEFINITION 4.1.3. Let  $K$  be a simplicial complex, with vertex  $V = \llbracket 1, n \rrbracket$ . In  $\mathbb{R}^n$ , consider, for every  $i \in \llbracket 1, n \rrbracket$ , the vector  $e_i = (0, \dots, 1, 0, \dots, 0)$  ( $i^{\text{th}}$  coordinate 1, the other ones 0). Let  $|K|$  be the subset of  $\mathbb{R}^n$  defined as:

$$|K| = \bigcup_{\sigma \in K} \text{conv}(\{e_j, j \in \sigma\}).$$

Endowed with the subspace topology,  $(|K|, T_{||K|})$  is a topological space, that we call the **topological realization** of  $K$ .

DEFINITION 4.1.4. Let  $(X, \mathcal{T})$  be a topological space. A triangulation of  $X$  is a simplicial complex  $K$  such that its topological realization  $(|K|, T_{||K|})$  is homeomorphic to  $(X, \mathcal{T})$ .

DEFINITION 4.1.5. Let  $K$  be a simplicial complex of dimension  $n$ . Its Euler characteristic is the integer

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

DEFINITION 4.1.6. The Euler characteristic of a topological space is the Euler characteristic of any triangulation of it.

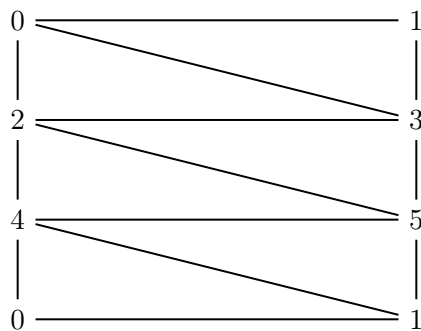
### 4.2 Exercises

EXERCISE 17. Give a triangulation of the cylinder.

We can think a triangulation of the cylinder in that following form: each circular section is mapped into a triangle graph. On the other hand, the line can be mapped to an edge. Since a cylinder can be written as  $\mathbb{S}_1 \times \mathbb{R}$ , the triangulation as well.

Let's write down:

$$K = \{[0, 1, 3], [0, 2, 3], [2, 3, 5], [2, 4, 5], [4.5, 1], [4, 1, 0]\}$$




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EXERCISE 18. What are the Euler characteristics of Examples 4.5 and 4.6? What is the Euler characteristic of the icosahedron?

**Exemplo 4.5:**

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

$$\chi(K) = 4 - 6 + 4 = 2$$

**Exemplo 4.6:**

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [1, 3], [0, 2], [2, 3], [3, 0], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3], [0, 1, 2, 3]\}.$$

$$\chi(K) = 4 - 6 + 4 - 1 = 1$$

**D20:** It has 20 faces (dimension 2), 30 edges (dimension 1) and 12 vertices (dimension 0), its Euler characteristic is  $12 - 30 + 20 = 2$  (Euler relation).

---

EXERCISE 19. Let  $K$  be a simplicial complex (with vertex set  $V$ ). A sub-complex of  $K$  is a set  $M \subset K$  that is a simplicial complex. Suppose that there exists two sub-complexes  $M$  and  $N$  of  $K$  such that  $K = M \cup N$ . Show the inclusion-exclusion principle:

$$\chi(K) = \chi(M) + \chi(N) - \chi(M \cap N)$$

Denote  $k_i$  the number of simplices of dimension  $i$ , that is,  $\chi(K) = \sum_{0 \leq i \leq k} (-1)^i k_i$ , with  $k$  being its dimension. Each simplex of dimension  $i$  belongs to  $M$ , to  $N$  or both, so let, respectively,

$$k_i = k_i^M + k_i^N - k_i^{MN}.$$

Therefore,

$$\chi(K) = \sum_{0 \leq i \leq k} (-1)^i (k_i^M + k_i^N - k_i^{MN}) = \sum_{0 \leq i \leq k} (-1)^i k_i^M + \sum_{0 \leq i \leq k} (-1)^i k_i^N - \sum_{0 \leq i \leq k} (-1)^i k_i^{MN}$$

Let  $m$  and  $n$  be the dimension of  $M$  and  $N$ , respectively. Now I shall prove  $M \cap N$  is a simplicial complex. Take  $\sigma \subset M \cap N$  and  $\tau \subset \sigma$ , then  $\tau \subset \sigma \in M$  and

$\tau \subset \sigma \in N$ , what implies  $\tau \in M \cap N$ . It proves its a simplicial complex with dimension  $p$ . If the dimension of  $M$  is  $m$ ,  $k_i^M = 0, \forall i > m$ . Suppose not. So there is a simplex in  $M$  with dimension  $> m$ , a contraction. This holds for  $N$  and  $M \cap N$ .

$$\chi(K) = \sum_{0 \leq i \leq k} (-1)^i (k_i^M + k_i^N - k_i^{MN}) = \sum_{0 \leq i \leq k} (-1)^i k_i^M + \sum_{0 \leq i \leq k} (-1)^i k_i^N - \sum_{0 \leq i \leq k} (-1)^i k_i^{MN}$$

We conclude that

$$\chi(K) = \sum_{0 \leq i \leq m} (-1)^i k_i^M + \sum_{0 \leq i \leq n} (-1)^i k_i^N - \sum_{0 \leq i \leq p} (-1)^i k_i^{MN} = \chi(M) + \chi(N) - \chi(M \cap N)$$

EXERCISE 20. *What is the Euler characteristic of a sphere of dimension 1? 2? 3?*

We may find the Euler characteristic of one triangulation of the sphere. So we first need to find a triangulation for the sphere  $\mathbb{S}_n \subset \mathbb{R}^{n+1}$ . We can think in the simplex in this space. In  $\mathbb{R}^2$  it's the triangle, in  $\mathbb{R}^3$  it's the tetrahedron, in  $\mathbb{R}^4$  it's the 5-cell and so on. The simplex in  $\mathbb{R}^{n+1}$  has  $n+2$  vertices and each vertex connect to all the other. We have also  $n+2$  simplices of dimension  $n$ , because each has  $n+1$  points, that is,  $\binom{n+2}{n+1} = n+2$ . For each of them, we must include all its subsets.

Now we can calculate the Euler characteristic for each triangulation and therefore each sphere.

$$\chi(\mathbb{S}_1) = -3 + 3 = 0$$

$$\chi(\mathbb{S}_2) = 4 - 6 + 4 = 2$$

$$\chi(\mathbb{S}_3) = -5 + 10 - 10 + 5 = 0$$

$$\chi(\mathbb{S}_4) = 6 - 15 + 20 - 15 + 6 = 2$$

EXERCISE 21. *Using the previous exercise, show that  $\mathbb{R}^3 - \{0\}$  and  $\mathbb{R}^4 - \{0\}$  are not homotopy equivalent.*

Suppose that  $\mathbb{R}^3 - \{0\}$  and  $\mathbb{R}^4 - \{0\}$  are homotopy equivalent. By Example 3.15,  $\mathbb{S}_{n-1}$  is homotopic equivalent to  $\mathbb{R}^n - \{0\}$ . By this and using the transitive property we conclude that  $\mathbb{S}_2$  and  $\mathbb{S}_3$  are homotopic equivalent. If that is true, we infer that they have the same Euler characteristic, what is a contraction by the last exercise. Hence  $\mathbb{R}^3 - \{0\}$  and  $\mathbb{R}^4 - \{0\}$  are not homotopy equivalent.

The computational exercises (22 - 26) can be found in the Github<sup>2</sup>

EXERCISE 22. *Build triangulations of the letters of the alphabet, and compute their Euler characteristic.*

EXERCISE 23. *For every  $n$ , triangulate the bouquet of  $n$  circles (see below). Compute their Euler characteristic.*

EXERCISE 24. *Implement the following triangulation of the torus.*

<sup>2</sup>[github.com/lucasmoschen/topological-data-analysis/blob/main/tutorials/tutorial-1.ipynb](https://github.com/lucasmoschen/topological-data-analysis/blob/main/tutorials/tutorial-1.ipynb)

EXERCISE 25. Consider the following dataset of 30 points  $x_0, \dots, x_{29}$  in  $R^2$ .

Write a function that takes as an input a parameter  $r \geq 0$ , and returns the simplicial complex  $\mathcal{G}(r)$  defined as follows:

1. the vertices of  $\mathcal{G}(r)$  are the points  $x_0, \dots, x_{29}$ ,
2. for all  $i, j \in [[0, 29]]$  with  $i \neq j$ , the edge  $[i, j]$  belongs to  $\mathcal{G}(r)$  if and only if  $\|x_i - x_j\| \leq r$ .

Compute the number of connected components of  $\mathcal{G}(r)$  for several values of  $r$ . What do you observe?

EXERCISE 26. A Erdos–Renyi random graph  $\mathcal{G}(n, p)$  is a simplicial complex obtained as follows:

1. add  $n$  vertices  $1, \dots, n$ ,
2. for every  $a, b \in [[1, n]]$ , add the edge  $[a, b]$  to the complex with probability  $p$ .

Builds a function that, given  $n$  and  $p$ , outputs a simplicial complex  $\mathcal{G}(n, p)$ . Observe the influence of  $p$  on the number of connected components of  $\mathcal{G}(10, p)$  and  $\mathcal{G}(100, p)$ .