

TOPOLOGICAL DATA ANALYSIS - EXERCISES

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1 General topology

1.1 Important definitions

DEFINITION 1. A topological space is a pair (X, \mathcal{T}) where X is a set and \mathcal{T} is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. for every infinite collection $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, we have $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. for every finite collection $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$, we have $\bigcap_{1 \leq i \leq n} O_i \in \mathcal{T}$.

DEFINITION 2. Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r , denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$.

DEFINITION 3. Let $A \subset \mathbb{R}$ and $x \in A$. We say that A is open around x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is open if for every $x \in A$, A is open around x .

DEFINITION 4. Let (X, \mathcal{T}) be a topological space, and $Y \subset X$. We define the subspace topology on Y as the following set:

$$T_Y = \{O \cap Y, O \in \mathcal{T}\}$$

DEFINITION 5. Let $f : X \rightarrow Y$ be a map. We say that f is continuous if for every $O \in \mathcal{U}$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in \mathcal{T} .

1.2 Exercises

EXERCISE 1. Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?

As we know every Topology contains \emptyset and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that it, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\} - \mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\} - \{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

EXERCISE 2. Let \mathbb{Z} be the set of integers. Consider the *cofinite topology* \mathcal{T} on \mathbb{Z} , defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or cO is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in \mathbb{Z}

1. Show that \mathcal{T} is a topology on \mathbb{Z} .

Let's verify the three axioms:

- (a) \emptyset is an open set by definition and \mathbb{Z} is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.
- (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c(\bigcup_{\alpha \in A} O_\alpha) = \bigcap_{\alpha \in A} {}^cO_\alpha$. As ${}^cO \subset {}^cO_\alpha$ and ${}^cO_\alpha$ is finite, so is cO . We conclude O is open set.
- (c) Let $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$. So ${}^cO = {}^c(\bigcap_{1 \leq i \leq n} O_i) = \bigcup_{1 \leq i \leq n} {}^cO_i$, where cO_i is finite. A finite union of finite sets is finite, so O is an open set.

By (a), (b) and (c), \mathcal{T} is a topology on \mathbb{Z} .

2. Exhibit an sequence of open sets $\{O_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n \in \mathbb{N}} O_n$ is not an open set.

Let $O_n = {}^c\{1, \dots, n\}$. Thus ${}^cO_n = \{1, \dots, n\}$ is finite and

$${}^c\left(\bigcap_{n \in \mathbb{N}} O_n\right) = \bigcup_{n \in \mathbb{N}} {}^cO_n = \bigcup_{n \in \mathbb{N}} \{1, \dots, n\} = \mathbb{N},$$

que não é finito. Portanto a intersecção não é conjunto aberto.

EXERCISE 3. Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that

$$\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$$

Let $z \in \mathcal{B}(y, r - \|x - y\|)$, so $\|z - y\| < r - \|x - y\| \implies \|z - y\| + \|x - y\| < r$. We can conclude that, by the triangular inequality,

$$\|x - z\| \leq \|x - y\| + \|z - y\| < r.$$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$.

Remark. In the notes, the exercise is to prove $\mathcal{B}(y, \|x - y\|) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take y next the border of $\mathcal{B}(x, r)$, $\|x - y\| \approx r$ and $\mathcal{B}(y, r - \epsilon) \not\subset \mathcal{B}(x, r)$.

EXERCISE 4. Let $x, y \in \mathbb{R}^n$, and $r = \|x - y\|$. Show that

$$\mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$$

Define $m = \frac{x+y}{2}$. Take $z \in \mathcal{B}\left(m, \frac{r}{2}\right)$. Thus, using the triangular inequality,

$$\|x - z\| \leq \|x - m\| + \|m - z\| = \frac{1}{2}\|x - y\| + \|m - z\| < r/2 + r/2 = r$$

$$\|y - z\| \leq \|y - m\| + \|m - z\| = \frac{1}{2}\|y - x\| + \|m - z\| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x, r)$, $z \in \mathcal{B}(y, r)$ and $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$.

EXERCISE 5. Show that the open balls $\mathcal{B}(x, r)$ of \mathbb{R}^n are open sets (with respect to the Euclidean topology).

We have to prove that for every $y \in \mathcal{B}(x, r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. Put $\epsilon = r - \|x - y\|$. As we have proved in exercise 3, $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. So $\mathcal{B}(x, r)$ is open set.

EXERCISE 6. Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?

1. $[0, 1]$. It's not open set because for every $\epsilon > 0$, $\mathcal{B}(0, \epsilon) = (-\epsilon, \epsilon) \not\subset [0, 1]$. It's closed because $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$ is an union of two open sets, as we prove in item 3.

2. $[0, 1)$. It's not open for the same reason as before. It's not closed because $B(1, \epsilon) = (1 - \epsilon, 1 + \epsilon) \not\subset (-\infty, 0) \cup [1, \infty]$.
3. $(-\infty, 1)$. It's open because: take $x < 1$. Put $r = 1 - x$ and take $z \in \mathcal{B}(x, r)$. If $z > x$, $|x - z| < 1 - x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty, 1)$ is open. It's not closed cause $\forall \epsilon > 0, \mathcal{B}(1, \epsilon) \not\subset [1, \infty)$.
4. the singletons. It's not open cause $\forall \epsilon > 0, x + \epsilon/2 \in \mathcal{B}(x, \epsilon)$. It's closed cause $(-\infty, x) \cup (x, \infty)$ is union of open sets.
5. \mathbb{Q} . It's not open because for every open ball around a rational, there is irrationals, that is, for $x \in \mathbb{Q}$ and $\forall \epsilon > 0$, exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x, \epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

EXERCISE 7. A map is continuous if and only if the preimage of closed sets are closed sets.

First we shall prove that $f^{-1}({}^c A) = {}^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}({}^c A)$. So there exists $y \in {}^c A$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in {}^c(f^{-1}(A))$.

Now take $x \in {}^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in {}^c A \implies x \in f^{-1}({}^c A)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose f is a continuous map and take a closed set F . We shall prove that $f^{-1}(F)$ is closed. Well, ${}^c(f^{-1}(F)) = f^{-1}({}^c F)$ is open, because ${}^c F$ is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set F , we have $f^{-1}(F)$ being closed. We will use that A is open if ${}^c A$ is closed. This is true because ${}^c({}^c A) = A$. Take an open set A . ${}^c(f^{-1}(A)) = f^{-1}({}^c A)$ is closed, because ${}^c A$ is. Thus $f^{-1}(A)$ is open and we have proved the continuity of f .

2 Homeomorphisms

2.1 Important definitions

2.2 Exercises