

TOPOLOGICAL DATA ANALYSIS - EXERCISES

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1 General topology

1.1 Important definitions

DEFINITION 1.1.1. A topological space is a pair (X, \mathcal{T}) where X is a set and \mathcal{T} is a collection of subsets of X such that:

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. for every infinite collection $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, we have $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. for every finite collection $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$, we have $\bigcap_{1 \leq i \leq n} O_i \in \mathcal{T}$.

DEFINITION 1.1.2. Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r , denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, \|x - y\| < r\}$.

DEFINITION 1.1.3. Let $A \subset \mathbb{R}$ and $x \in A$. We say that A is open around x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is open if for every $x \in A$, A is open around x .

DEFINITION 1.1.4. Let (X, \mathcal{T}) be a topological space, and $Y \subset X$. We define the subspace topology on Y as the following set:

$$T|_Y = \{O \cap Y, O \in \mathcal{T}\}$$

DEFINITION 1.1.5. Let $f : X \rightarrow Y$ be a map. We say that f is continuous if for every $O \in \mathcal{U}$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in \mathcal{T} .

1.2 Exercises

EXERCISE 1. Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?

As we know every Topology contains \emptyset and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\} - \mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\} - \{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

EXERCISE 2. Let \mathbb{Z} be the set of integers. Consider the cofinite topology \mathcal{T} on \mathbb{Z} , defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or cO is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in \mathbb{Z}

1. Show that \mathcal{T} is a topology on \mathbb{Z} .

Let's verify the three axioms:

- (a) \emptyset is an open set by definition and \mathbb{Z} is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.
- (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcup_{\alpha \in A} O_\alpha\right) = \bigcap_{\alpha \in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall \alpha \in A$. If $\forall \alpha, O_\alpha = \emptyset$, then ${}^cO = {}^c\emptyset \implies O = \emptyset$ and O is open. On the other hand, if there exists $\alpha \in A$ such that $O_\alpha \neq \emptyset$ we have ${}^cO_\alpha$ being finite, so is cO , given the inclusion. We conclude O is open set.
- (c) Let $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcap_{1 \leq i \leq n} O_i\right) = \bigcup_{1 \leq i \leq n} {}^cO_i$. If $O_i = \emptyset$ for some $1 \leq i \leq n$, $O = \emptyset$ because of the intersection. Alternatively, if $\forall i, O_i \neq \emptyset$ we have that cO_i is finite and a finite union of finites is finite. We conclude that O is open set.

By (a), (b) and (c), \mathcal{T} is a topology on \mathbb{Z} .

2. Exhibit an sequence of open sets $\{O_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n \in \mathbb{N}} O_n$ is not an open set.

Let $O_n = {}^c\{1, \dots, n\}$. Thus ${}^cO_n = \{1, \dots, n\}$ is finite and

$${}^c\left(\bigcap_{n \in \mathbb{N}} O_n\right) = \bigcup_{n \in \mathbb{N}} {}^cO_n = \bigcup_{n \in \mathbb{N}} \{1, \dots, n\} = \mathbb{N},$$

that is not finite. Therefore, this intersection is not an open set.

EXERCISE 3. Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that

$$\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$$

Let $z \in \mathcal{B}(y, r - \|x - y\|)$, so $\|z - y\| < r - \|x - y\| \implies \|z - y\| + \|x - y\| < r$. We can conclude that, by the triangular inequality,

$$\|x - z\| \leq \|x - y\| + \|z - y\| < r.$$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - \|x - y\|) \subset \mathcal{B}(x, r)$.

Remark. In the notes, the exercise is to prove $\mathcal{B}(y, \|x - y\|) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take y next the border of $\mathcal{B}(x, r)$, $\|x - y\| \approx r$ and $\mathcal{B}(y, r - \epsilon) \not\subset \mathcal{B}(x, r)$.

EXERCISE 4. Let $x, y \in \mathbb{R}^n$, and $r = \|x - y\|$. Show that

$$\mathcal{B}\left(\frac{x+y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$$

Denote $m = \frac{x+y}{2}$. Take $z \in \mathcal{B}\left(m, \frac{r}{2}\right)$. Thus, using the triangular inequality,

$$\|x - z\| \leq \|x - m\| + \|m - z\| = \frac{1}{2}\|x - y\| + \|m - z\| < r/2 + r/2 = r$$

$$\|y - z\| \leq \|y - m\| + \|m - z\| = \frac{1}{2}\|y - x\| + \|m - z\| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x, r)$, $z \in \mathcal{B}(y, r)$ and $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$.

EXERCISE 5. Show that the open balls $\mathcal{B}(x, r)$ of \mathbb{R}^n are open sets (with respect to the Euclidean topology).

We have to prove that for every $y \in \mathcal{B}(x, r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. Put $\epsilon = r - \|x - y\|$. As we have proved in exercise 3, $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. So $\mathcal{B}(x, r)$ is open set.

EXERCISE 6. Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?

1. $[0, 1]$. It's not open set because for every $\epsilon > 0$, $\mathcal{B}(0, \epsilon) = (-\epsilon, \epsilon) \not\subset [0, 1]$. It's closed because $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$ is an union of two open sets, as we prove in item 3.
2. $[0, 1)$. It's not open for the same reason as before. It's not closed because $\mathcal{B}(1, \epsilon) = (1 - \epsilon, 1 + \epsilon) \not\subset [0, 1]$.
3. $(-\infty, 1)$. It's open because: take $x < 1$. Put $r = 1 - x$ and take $z \in \mathcal{B}(x, r)$. If $z > x$, $|x - z| < 1 - x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty, 1)$ is open. It's not closed cause $\forall \epsilon > 0$, $\mathcal{B}(1, \epsilon) \not\subset (-\infty, 1)$.
4. the singletons. It's not open cause $\forall \epsilon > 0$, $x + \epsilon/2 \in \mathcal{B}(x, \epsilon)$. It's close cause $(-\infty, x) \cup (x, \infty)$ is union of open sets.
5. \mathbb{Q} . It's not open because for every open ball around a rational, there are irrationals, that is, let $x \in \mathbb{Q}$ and take $\epsilon > 0$, then there exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x, \epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

Remark. We shall prove the rationals are dense in the reals. Let $x \in \mathbb{Q}$ and $\epsilon > 0$. If ϵ is irrational, take $x - \epsilon/2 \subset (x - \epsilon, x + \epsilon)$. Suppose $x - \epsilon/2$ is rational, then $\frac{2x - \epsilon}{2} = \frac{m}{n}$ for some integers m and n , that is, $2x - \epsilon = 2m/n$ and $\epsilon = 2(x - \frac{m}{n}) \in \mathbb{Q}$, contradiction. So there is an irrational in $\mathcal{B}(x, \epsilon)$. If ϵ is rational, consider

$$y = \frac{1}{\sqrt{2}}(x - \epsilon) + (1 - \frac{1}{\sqrt{2}})(x + \epsilon) = (x + \epsilon) - \epsilon\sqrt{2}$$

That is a convex combination, so $y \in \mathcal{B}(x, \epsilon)$. Moreover, y is irrational, with a similar proof by contradiction. This proves the statement.

On the other hand, we must prove for every two irrationals (a, b) , there is a rational between them. Denote $c = b - a > 0$. Let $n \in \mathbb{N}$ such that $n > \frac{1}{c} \implies cn > 1 \implies (bn - an) > 1$. So exists $m \in (an, bn) \implies m/n \in (a, b)$. This proves the second statement.

EXERCISE 7. *A map is continuous if and only if the preimage of closed sets are closed sets.*

First we shall prove that $f^{-1}({}^cA) = {}^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}({}^cA)$. So there exists $y \in {}^cA$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in {}^c(f^{-1}(A))$.

Now take $x \in {}^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in {}^cA \implies x \in f^{-1}({}^cA)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose f is a continuous map and take a closed set F . We shall prove that $f^{-1}(F)$ is closed. Well, ${}^c(f^{-1}(F)) = f^{-1}({}^cF)$ is open, because cF is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set F , we have $f^{-1}(F)$ being closed. We will use that A is open if cA is closed. This is true because ${}^c({}^c(A)) = A$. Take an open set A . ${}^c(f^{-1}(A)) = f^{-1}({}^cA)$ is closed, because cA is. Thus $f^{-1}(A)$ is open and we have proved the continuity of f .

2 Homeomorphisms

2.1 Important definitions

DEFINITION 2.1.1. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces, and $f : X \rightarrow Y$ a map. We say that f is a homeomorphism if

1. f is a bijection,
2. $f : X \rightarrow Y$ is continuous,
3. $f^{-1} : Y \rightarrow X$ is continuous.

If there exists such a homeomorphism, we say that the two topological spaces are homeomorphic.

DEFINITION 2.1.2. Let (X, \mathcal{T}) be a topological space. We say that X is connected if for every open sets $O, O' \in \mathcal{T}$ such that $O \cap O' = \emptyset$ (i.e., they are disjoint), we have

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

DEFINITION 2.1.3. Let (X, \mathcal{T}) be a topological space. Suppose that there exists a collection of n **non-empty, disjoint and connected open sets** (O_1, \dots, O_n) such that

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

Then we say that X admits n connected components.

DEFINITION 2.1.4. Let (X, \mathcal{T}) be a topological space, and $n \geq 0$. We say that it has dimension n if the following is true: for every $x \in X$, there exists an open set O such that $x \in O$, and a homeomorphism $O \rightarrow \mathbb{R}^n$.

2.2 Exercises

EXERCISE 8. Show that the topological spaces \mathbb{R}^n and $\mathcal{B}(0, 1) \subset \mathbb{R}^n$ are homeomorphic.

Let $f : \mathcal{B}(0, 1) \rightarrow \mathbb{R}^n$ be defined as $f(x) = \frac{x}{1 - \|x\|}$. I observe it's well defined because $\|x\| < 1$. We shall prove f is a homeomorphism.

1. **Injective:** Take $x, y \in \mathcal{B}(0, 1)$ and suppose that

$$\frac{x}{1 - \|x\|} = \frac{y}{1 - \|y\|}.$$

Applying the norm in both sides, we obtain the equation

$$\|x\|(1 - \|y\|) = \|y\|(1 - \|x\|) \implies \|x\| = \|y\|.$$

On the other side x and y points to the same direction, given that

$$y = \frac{1 - \|y\|}{1 - \|x\|} x = \alpha x,$$

with $\alpha = 1$ because of the same norm. We conclude $x = y$.

2. **Surjective:** Take $y \in \mathbb{R}^n$. We shall prove that there exists $x \in \mathcal{B}(0, 1)$ such that $f(x) = y$, that is,

$$\frac{x}{1 - \|x\|} = y$$

Applying the norm we observe that if that is true, $\|x\| = \|y\| - \|y\|\|x\| \implies \|x\| = \frac{\|y\|}{1 + \|y\|}$. And $x = (1 - \|x\|)y = \frac{1}{1 + \|y\|}y$. We conclude that for every $y \in \mathbb{R}^n$, if we take $x = \frac{y}{1 + \|y\|}$,

$$f(x) = \frac{y/(1 + \|y\|)}{1 - \|y\|/(1 + \|y\|)} = y$$

3. **Continuity of f :** Consider an open set $A \subset \mathbb{R}^n$. Let $B = f^{-1}(A)$. We shall prove B is open, that is, for every $x \in B$, exists $r > 0$ such that $\mathcal{B}(x, r) \subset B$. Take $x = f^{-1}(y) \in B$. Because A is open, there is $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset A$. Take δ such that

$$\frac{\delta}{1 - \|x\| - \delta}(1 + \|y\|) < \epsilon$$

and $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$.

$$\begin{aligned} \|y - w\| &= \left\| \frac{x}{1 - \|x\|} - \frac{z}{1 - \|z\|} \right\| = \frac{1}{1 - \|x\|} \left\| x - \frac{1 - \|x\|}{1 - \|z\|} z \right\| \\ &= \frac{1}{1 - \|x\|} \left\| x - z + z - \frac{1 - \|x\|}{1 - \|z\|} z \right\| \\ &\leq \frac{\|x - z\|}{1 - \|x\|} + \frac{1}{1 - \|x\|} \left(1 - \frac{1 - \|x\|}{1 - \|z\|} \|z\| \right) \\ &= \frac{\|x - z\|}{1 - \|x\|} + \frac{\|z\|}{1 - \|x\|} \frac{\|x\| - \|z\|}{1 - \|z\|} \\ &\leq \frac{1}{1 - \|x\|} \|x - z\| (1 + \|w\|) \\ &\leq \frac{1}{1 - \|x\|} \|x - z\| (1 + \|y - w\| + \|y\|) \\ \implies \|y - w\| &\leq \frac{\|x - z\|}{1 - \|x\| - \|x - z\|} (1 + \|y\|) \\ &< \frac{\delta}{1 - \|x\| - \delta} (1 + \|y\|) < \epsilon \end{aligned}$$

So $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$, what proves B is open. It concludes the continuity of f .

4. **Continuity of f^{-1} :** The inverse is given by

$$f^{-1}(y) = \frac{y}{1 + \|y\|}$$

The demonstration is quite similar to the previous item, given that the only difference is the signal.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0, 1) \simeq \mathbb{R}^n$.

EXERCISE 9. Show that $\mathcal{B}(x, r)$ and $\mathcal{B}(y, s)$ are homeomorphic.

Consider the function $f : \mathcal{B}(0, 1) \rightarrow \mathcal{B}(c, r)$ given by $f(x) = r \cdot x + c$. Let's prove f is a

homeomorphism.

1. **Injective:** If $x, y \in \mathcal{B}(0, 1)$ and $rx + c = ry + c \implies x = y$, because $r > 0$ by definition. So f is injective.
2. **Surjective:** Let $y \in \mathcal{B}(c, r)$ and $x = (y - c)/r$. So $\|x\| = \|y - c\|/r < 1$, by definition. So $x \in \mathcal{B}(0, 1)$ and $f(x) = y$ what proves this function is surjective.
3. **Continuity of f :** Let $A \subset \mathcal{B}(c, r)$ open set and denote $B = f^{-1}(A)$. Take $x = f^{-1}(y) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset A$. Define $\delta = \epsilon/r$ and take $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$.

$$\|y - w\| = \|rx + c - (rz + c)\| = r\|x - z\| < r\delta = \epsilon$$

Therefore $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$. So $\mathcal{B}(x, \delta) \subset B$, what proves B is open. This concludes the continuity of f .

4. **Continuity of f^{-1} :** The inverse is given by

$$f^{-1}(y) = \frac{y - c}{r}$$

This function is continuous for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0, 1) \simeq \mathcal{B}(c, r)$. Since this is an equivalence relation, we have that

$$\mathcal{B}(0, 1) \simeq \mathcal{B}(x, r) \text{ and } \mathcal{B}(0, 1) \simeq \mathcal{B}(y, s) \text{ implies } \mathcal{B}(x, r) \simeq \mathcal{B}(y, s).$$

EXERCISE 10. Show that $\mathbb{S}(0, 1)$, the unit circle of \mathbb{R}^2 , is homeomorphic to the ellipse

$$\mathcal{S}(a, b) = \left\{ (x, y) \in \mathbb{R}^2, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \right\},$$

for any $a, b > 0$.

Consider the function $f : \mathbb{S}(0, 1) \rightarrow \mathcal{S}(a, b)$ defined as $f(x, y) = (ax, by)$. Let's prove it is a homeomorphism.

1. **Injective:** Let $(x_1, y_1), (x_2, y_2) \in \mathbb{S}(0, 1)$ such that $(ax_1, by_1) = (ax_2, by_2)$. Since $a, b > 0$, we have $x_1 = x_2$ and $y_1 = y_2$. It proves f is injective.
2. **Surjective:** Let $(z, w) \in \mathcal{S}(a, b)$ and $(x, y) = \left(\frac{z}{a}, \frac{w}{b}\right)$. It's clear that $f(x, y) = (z, w)$ and $x^2 + y^2 = \frac{z^2}{a^2} + \frac{w^2}{b^2} = 1$, so $(x, y) \in \mathbb{S}(0, 1)$. It proves f is surjective.
3. **Continuity of f :** Let $A \subset \mathcal{S}(a, b)$ open set and denote $B = f^{-1}(A)$. Take $(x, y) = f^{-1}(z, w) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}((z, w), \epsilon) \subset A$. Put δ as defined below and take $(x', y') = f^{-1}((z', w')) \in \mathcal{B}((x, y), \delta)$. Consider the norm 1

$$\begin{aligned} \| (z', w') - (z, w) \|_1 &= \| (ax', by') - (ax, by) \|_1 = \| (a(x' - x), b(y' - y)) \|_1 \\ &= a|x' - x| + b|y' - y|, \text{ define } c = \max\{a, b\} \\ &\leq c(|x' - x| + |y' - y|) = c\|(x' - x, y' - y)\|_1 \end{aligned}$$

By the equivalence of the norms, there exists constants k_1, k_2 such that

$$\|(z', w') - (z, w)\| \leq k_1 \|(z', w') - (z, w)\|_1 \leq ck_1 \|(x' - x, y' - y)\|_1 \leq ck_1 k_2 \|(x' - x, y' - y)\|$$

Then we need $\delta = \frac{\epsilon}{ck_1 k_2}$ in order to prove that $(z', w') \in \mathcal{B}((z, w), \epsilon) \subset A \implies (x', y') \in B$. So $\mathcal{B}((x, y), \delta) \subset B$, what proves B is open. This concludes the continuity of f .

4. **Continuity of f^{-1} :** The inverse is given by

$$f^{-1}((z, w)) = (z/a, w/b)$$

This function is continuous for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathbb{S}(0, 1) \simeq \mathcal{S}(a, b)$.

EXERCISE 11. *Show that $[0, 1)$ and $(0, 1)$ are not homeomorphic.*

We shall prove by contradiction. Suppose there exists a homeomorphism $f : [0, 1) \rightarrow (0, 1)$. Let $0 < z = f(0) < 1$ and define the following function

$$\begin{aligned} g : (0, 1) &\rightarrow (0, z) \cup (z, 1) \\ x &\mapsto g(x) = f(x) \end{aligned}$$

This function is well defined given that z is not image of other point but 0. The function is injective because if $g(y) = g(x) \implies f(y) = f(x) \implies x = y$, given that f is injective. This function is also surjective since f is and $0 < w < 1$ and $w \neq z$, it's clear that $f(0) \neq w$. As g is an induced map of a continuous function, by Proposition 1.21 from the notes, it's continuous and so is its inverse. We conclude g is a homeomorphism.

Now I will prove that $(0, 1)$ admits only 1 connected component, that is, it's connected. Suppose it's not and there exists $O, O' \subset (0, 1)$ open disjoint sets such that $(0, 1) = O \cup O'$ and none of them are empty sets. Let $a \in O, b \in O'$ with $a < b$ without loss of generality. Define $\alpha = \sup\{x \in \mathbb{R} : [a, x) \subset O\}$. It's well defined because this set is not empty, given O is open and b is an upper bound. Then $\alpha \leq b$. Suppose $\alpha \in O'$, then there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subset O'$. We know that for every $\epsilon > 0$, there exists $w \in (\alpha - \epsilon, \alpha]$ such that $[a, w) \subset O$. That is a contradiction since there exists $w \in (\alpha - r, \alpha)$ such that $[a, w) \subset O$. So $\alpha \in O \implies (\alpha - r, \alpha + r) \subset O$, for some r . We infer that $[a, \alpha + r) \subset O$, what is absurd. Therefore $(0, 1)$ is connected.

For a similar argument, we prove that $(0, z)$ and $(z, 1)$ are connected. This implies that the union admits 2 connected components.

In that sense, we have a homeomorphism between a topological space with 1 connected component and other with 2 connected components, what is a contradiction by Proposition 2.14 from the notes. We conclude that $[0, 1)$ and $(0, 1)$ are not homeomorphic.

3 Homotopies

3.1 Important definitions

DEFINITION 3.1.1. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces, and $f, g : X \rightarrow Y$ two continuous maps. A homotopy between f and g is a map $F : X \times [0, 1] \rightarrow Y$ such that:

1. $F(\cdot, 0)$ is equal to f ,
2. $F(\cdot, 1)$ is equal to g ,
3. $F : X \times [0, 1] \rightarrow Y$ is continuous.

If such a homotopy exists, we say that the maps f and g are homotopic.

Remark. Before asking for $F : X \times [0, 1] \rightarrow Y$ to be continuous, we have to give $X \times [0, 1]$ a topology. The topology we choose is the product topology. Consider the topological space (X, \mathcal{T}) , and endow $[0, 1]$ with the subspace topology of \mathbb{R} , denoted $T_{[0,1]}$. The product topology on $X \times [0, 1]$, denoted $T \otimes T_{[0,1]}$, is defined as follows: a set $O \subset X \times [0, 1]$ is open if and only if it can be written as a union $\bigcup_{\alpha \in A} O_\alpha \times O'_\alpha$ where every O_α is an open set of X and O'_α is an open set of $[0, 1]$.

DEFINITION 3.1.2. Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two topological spaces. A homotopy equivalence between X and Y is a pair of continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that:

1. $g \circ f : X \rightarrow X$ is homotopic to the identity map $\text{id} : X \rightarrow X$,
2. $f \circ g : Y \rightarrow Y$ is homotopic to the identity map $\text{id} : Y \rightarrow Y$,

If such a homotopy equivalence exists, we say that X and Y are homotopy equivalent.

DEFINITION 3.1.3. Let (X, \mathcal{T}) be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $T|_Y$. A retraction is a continuous map $r : X \rightarrow Y$ such that $\forall y \in Y, r(y) = y$.

A deformation retraction is a homotopy $F : X \times [0, 1] \rightarrow Y$ between the identity map $\text{id} : X \rightarrow X$ and a retraction $r : X \rightarrow Y$.

3.2 Exercises

EXERCISE 12. Let $f : \mathbb{R}^n \rightarrow X$ be a continuous map. Then f is homotopic to a constant map.

I must prove that there exists a homotopy between f and a constant map. Consider the function $F : \mathbb{R}^n \times [0, 1] \rightarrow X$ defined as

$$F(x, t) = f(tx)$$

It's clear that $F(x, 0) = f(0)$, for every $x \in \mathbb{R}^n$. So it's the constant map $f(0)$. We also have that $F(x, 1) = f(x), \forall x \in \mathbb{R}^n$. Moreover, let's prove F is continuous. Denote $F' : \mathbb{R}^n \times \mathbb{R} \rightarrow X$ the function $F'(x, t) = f(xt)$ and $g : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ the function $g(x, t) = xt$. So $F' = f \circ g$.

Let's prove g is a continuous function. As we are dealing with a real-valued function, by Proposition 1.19 from the notes, I can use the $\epsilon - \delta$ proof. Let $(x, t) \in \mathbb{R}^{n+1}$ and $\epsilon > 0$. In the proof I use the norm 1, without loss of generality because of the equivalence of norms in \mathbb{R}^n . Put

$\delta = \min\{1, \frac{\epsilon}{\max\{\|x\|, |t|+1\}}\}$ and suppose $\|(x, t) - (x', t')\| = \|x - x'\| + |t - t'| < \delta$. So,

$$\begin{aligned} \|xt - x't'\| &= \|xt - xt' + xt' - x't'\| \\ &\leq |t - t'| \|x\| + |t'| \|x - x'\| \\ &\leq |t - t'| \|x\| + (|t| + \delta) \|x - x'\| \\ &< \max\{\|x\|, |t| + \delta\} \delta \\ &\leq \max\{\|x\|, |t| + 1\} \delta \leq \epsilon \end{aligned}$$

By this, g is a continuous function. Since f is also continuous, the composition F' is also continuous, by Proposition 1.18. By Proposition 1.21, when we endow F' in $\mathbb{R}^n \times [0, 1]$, we obtain a continuous function, that is F is continuous. Then we conclude that f is homotopic to a constant function.

EXERCISE 13. Show that every map $f : \mathbb{S}_1 \rightarrow \mathbb{S}_2$ is homotopic to a constant map, where the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, \|x\| = 1\}$.

Remark. I will suppose f is continuous, otherwise I think it's not possible to prove there is a homotopy.

EXERCISE 14. Show that being homotopic is a transitive relation between maps: for every triplet of maps $f, g, h : X \rightarrow Y$, if f, g are homotopic and g, h are homotopic, then f, h are homotopic.

We shall prove there exist a homotopy H between f and h . By assumption, there exists a homotopy F between f and g and a homotopy G between g and h . Define $H : X \times [0, 1] \rightarrow Y$ such that

$$H(x, t) = \begin{cases} F(x, 2t), & 0 \leq t \leq 1/2 \\ G(x, 2t - 1), & 1/2 < t \leq 1 \end{cases}$$

that is, H behaves as F until it reaches a half. When that occurs, $H(x, 1/2) = F(x, 1) = g(x) = G(x, 0)$. After that, H follows G until the end of the interval. So, it's clear that $H(x, 0) = F(x, 0) = f(x), \forall x \in X$ and $H(x, 1) = G(x, 1) = h(x), \forall x \in X$. Moreover, since F and G are continuous and in the point $t = 1/2$, both functions agree, H is continuous and, therefore, f and h are homotopic.

EXERCISE 15. Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).

1. (*reflexive*): Consider the identity map $id : X \rightarrow X$, that is continuous. We shall prove that this function is homotopic to itself. Consider $F : X \times [0, 1] \rightarrow X$ given by $F(x, t) = x$ for every x and t . It's clear this is a homotopy because $F(x, 0) = F(x, 1) = x$ and it's continuous. Moreover $id \circ id = id$ by definition of identity. Therefore, there exists a homotopy equivalence between id and itself. We conclude $X \approx X$.
2. (*symmetric*): Suppose $X \approx Y$. So, there exists continuous functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ that form a homotopy equivalence. This means that $g : Y \rightarrow X$ and $f : X \rightarrow Y$ are a homotopy equivalence as well. So $Y \approx X$.

3. (*transitive*): Suppose $X \approx Y$, and let $f_1 : X \rightarrow Y$ and $g_1 : Y \rightarrow X$ form a homotopy equivalence. Also suppose $Y \approx Z$ and let $f_2 : Y \rightarrow Z$ and $g_2 : Z \rightarrow Y$ form a homotopy equivalence. Define $f_3 = f_2 \circ f_1$ and $g_3 = g_1 \circ g_2$. Let's prove this is a homotopy equivalence. Both functions are continuous given that they are a composition of continuous functions.

(a) $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1$ is homotopic to $id : X \rightarrow X$.

Let F_1 be a homotopy between $g_1 \circ f_1$ and id and F_2 a homotopy between $g_2 \circ f_2$ and id . Define

$$F_3(x, t) = \begin{cases} g_1 \circ F_2(\cdot, 2t) \circ f_1(x), & 0 \leq t \leq 1/2 \\ F_1(x, 2t - 1), & 1/2 < t \leq 1 \end{cases}$$

So $F_3(x, 0) = g_1(F_2(f_1(x), 0)) = g_1(g_2(f_1(x))) = g_3 \circ f_3(x)$, for every x and $F_3(x, 1) = F_1(x, 1) = x$, for every x . When $t = 1/2$,

$$F_3(x, 1/2) = g_1(F_2(f_1(x), 1)) = g_1(f_1(x)) = F_1(x, 0)$$

By this equality and the fact that composition of continuous functions is a continuous map, we conclude that F_3 is continuous. This implies that $g_3 \circ f_3$ is homotopic to the identity.

(b) $f_3 \circ g_3 = f_1 \circ f_2 \circ g_2 \circ g_1$ is homotopic to $id : Z \rightarrow Z$.

This follows a quite similar demonstration and can be omitted.

By the points above f_3 and g_3 is a homotopy equivalence what proves $X \approx Z$.

Consequently, homotopy equivalence is an equivalence relation.

EXERCISE 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

I will consider the upper case alphabet and each letter will be considered as a topological space (a subset from \mathbb{R}^2), for example the letter O is homotopy equivalent to a circle, while L is to an interval, or equivalently, a point. Observe that most of the letters are equivalent to a point, because we can think in a continuous reduction. When we have a hole, such as A, D, R, O, P, Q , this continuity is impossible since we'll have a point break. B is a special case because we can't deform into a point without breaking points and also we cannot join the holes in one. So there is three classes, given by its representatives

1. O
2. B
3. I

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	1	3	3	3	3	3	3	3	3	3
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
3	1	1	1	1	3	3	3	3	3	3	3	3