Topological Data Analysis - Exercises

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1 General topology

1.1 Important definitions

Definition 1.1.1. *A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:*

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. *for every infinite collection* $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, *we have* $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. *for every finite collection* $\{O_i\}_{1 \le i \le n} \subset \mathcal{T}$, *we have* $\bigcap_{1 \le i \le n} O_i \in \mathcal{T}$.

Definition 1.1.2. *Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r, denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, ||x - y|| < r\}$.*

Definition 1.1.3. *Let $A \subset \mathbb{R}$ and $x \in A$. We say that A is open around x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is open if for every $x \in A$, A is open around x.*

Definition 1.1.4. *Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. We define the subspace topology on Y as the following set:*

$$T_{|Y} = \{O \cap Y, O \in \mathcal{T}\}$$

Definition 1.1.5. *Let $f : X \to Y$ be a map. We say that f is continuous if for every $O \in U$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.*

1.2 Exercises

Exercise 1. *Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?*

As we know every Topology contains $\emptyset$ and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\}$ - $\mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\}$ $\{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\}$ $\{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\}$ $\{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

EXERCISE 2. *Let $\mathbb{Z}$ be the set of integers. Consider the cofinite topology $\mathcal{T}$ on $\mathbb{Z}$, defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or ${}^cO$ is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in $\mathbb{Z}$*

1. Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

   Let's verify the three axioms:

   (a) $\emptyset$ is an open set by definition and $\mathbb{Z}$ is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.

   (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcup_{\alpha\in A} O_\alpha\right) = \bigcap_{\alpha\in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall\alpha \in A$. If $\forall\alpha, O_\alpha = \emptyset$, then ${}^cO = {}^c\emptyset \implies O = \emptyset$ and O is open. On the other hand, if there exists $\alpha \in A$ such that $O_\alpha \neq \emptyset$ we have ${}^cO_\alpha$ being finite, so is ${}^cO$, given the inclusion. We conclude O is open set.

   (c) Let $\{O_i\}_{1\leq i\leq n} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcap_{1\leq i\leq n} O_i\right) = \bigcup_{1\leq i\leq n} {}^cO_i$. If $O_i = \emptyset$ for some $1 \leq i \leq n$, $O = \emptyset$ because of the intersection. Alternatively, if $\forall i, O_i \neq \emptyset$ we have that ${}^cO_i$ is finite and a finite union of finites is finite. We conclude that O is open set.

   By (a), (b) and (c), $\mathcal{T}$ is a topology on $\mathbb{Z}$.

2. Exhibit an sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not an open set.

   Let $O_n = {}^c\{1, ..., n\}$. Thus ${}^cO_n = \{1, ..., n\}$ is finite and

   $${}^c\left(\bigcap_{n\in\mathbb{N}} O_n\right) = \bigcup_{n\in\mathbb{N}} {}^cO_n = \bigcup_{n\in\mathbb{N}} \{1, ..., n\} = \mathbb{N},$$

   that is not finite. Therefore, this intersection is not an open set.

EXERCISE 3. *Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that*

$$\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$$

Let $z \in \mathcal{B}(y, r - ||x - y||)$, so $||z - y|| < r - ||x - y|| \implies ||z - y|| + ||x - y|| < r$. We can conclude that, by the triangular inequality,

$$||x - z|| \leq ||x - y|| + ||z - y|| < r.$$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$.

Remark. In the notes, the exercise is to prove $\mathcal{B}(y, ||x - y||) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take y next the border of $\mathcal{B}(x, r)$, $||x - y|| \approx r$ and $B(y, r - \epsilon) \not\subset B(x, r)$.

EXERCISE 4. *Let $x, y \in \mathbb{R}^n$, and $r = ||x - y||$. Show that*

$$\mathcal{B}\left(\frac{x + y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$$

Denote $m = \dfrac{x+y}{2}$. Take $z \in \mathcal{B}\left(m, \dfrac{r}{2}\right)$. Thus, using the triangular inequality,

$$||x-z|| \leq ||x-m|| + ||m-z|| = \frac{1}{2}||x-y|| + ||m-z|| < r/2 + r/2 = r$$

$$||y-z|| \leq ||y-m|| + ||m-z|| = \frac{1}{2}||y-x|| + ||m-z|| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x,r)$, $z \in \mathcal{B}(y,r)$ and $z \in \mathcal{B}(x,r) \cap \mathcal{B}(y,r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x,r) \cap \mathcal{B}(y,r)$.

EXERCISE 5. *Show that the open balls $\mathcal{B}(x,r)$ of R^n are open sets (with respect to the Euclidean topology).*

We have to prove that for every $y \in \mathcal{B}(x,r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y,\epsilon) \subset \mathcal{B}(x,r)$. Put $\epsilon = r - ||x-y||$. As we have proved in exercise 3, $\mathcal{B}(y,\epsilon) \subset \mathcal{B}(x,r)$. So $\mathcal{B}(x,r)$ is open set.

EXERCISE 6. *Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?*

1. $[0,1]$. It's not open set because for every $\epsilon > 0, \mathcal{B}(0,\epsilon) = (-\epsilon,\epsilon) \not\subset [0,1]$. It's closed because $[0,1]^c = (-\infty,0) \cup (1,\infty)$ is an union of two open sets, as we prove in item 3.
2. $[0,1)$. It's not open for the same reason as before. It's not closed because $B(1,\epsilon) = (1-\epsilon, 1+\epsilon) \not\subset (-\infty,0) \cup [1,\infty]$.
3. $(-\infty,1)$. It's open because: take $x < 1$. Put $r = 1-x$ and take $z \in \mathcal{B}(x,r)$. If $z > x$, $|x-z| < 1-x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty,1)$ is open. It's not closed cause $\forall\epsilon > 0, \mathcal{B}(1,\epsilon) \not\subset [1,\infty)$.
4. the singletons. It's not open cause $\forall \epsilon > 0, x+\epsilon/2 \in \mathcal{B}(x,\epsilon)$. It's close cause $(-\infty,x) \cup (x,\infty)$ is union of open sets.
5. $\mathbb{Q}$. It's not open because for every open ball around a rational, there is irrationals, that is, for $x \in \mathbb{Q}$ and $\forall\epsilon > 0$, exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x,\epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

EXERCISE 7. *A map is continuous if and only if the preimage of closed sets are closed sets.*

First we shall prove that $f^{-1}({}^cA) = {}^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}({}^cA)$. So there exists $y \in {}^cA$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in {}^c(f^{-1}(A))$.

Now take $x \in {}^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in {}^cA \implies x \in f^{-1}({}^cA)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose f is a continuos map and take a closed set F. We shall prove that $f^{-1}(F)$ is closed. Well, ${}^c(f^{-1}(F)) = f^{-1}({}^cF)$ is open, because ${}^cF$ is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set F, we have $f^{-1}(F)$ being closed. We will use that A is open if ${}^cA$ is closed. This is true because ${}^c({}^c(A)) = A$. Take an open set A. ${}^c(f^{-1}(A)) = f^{-1}({}^cA)$ is closed, because ${}^cA$ is. Thus $f^{-1}(A)$ is open and we have proved the continuity of f.

2 Homeomorphisms

2.1 Important definitions

DEFINITION 2.1.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f : X \to Y$ a map. We say that f is a homeomorphism if*

1. *f is a bijection,*
2. *$f : X \to Y$ is continuos,*
3. *$f^{-1} : Y \to X$ is continuos.*

If there exists such a homeomorphism, we say that the two topological spaces are homeomorphic.

DEFINITION 2.1.2. *Let $(X, \mathcal{T})$ be a topological space. We say that X is connected if for every open sets $O, O' \in \mathcal{T}$ such that $O \cap O' = \emptyset$ (i.e., they are disjoint), we have*

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

DEFINITION 2.1.3. *Let $(X, \mathcal{T})$ be a topological space. Suppose that there exists a collection of n* ***non-empty, disjoint*** *and* ***connected open sets*** *$(O_1, ..., O_n)$ such that*

$$\bigcup_{1 \le i \le n} O_i = X.$$

Then we say that X admits n connected components.

DEFINITION 2.1.4. *Let $(X, \mathcal{T})$ be a topological space, and $n \ge 0$. We say that it has dimension n if the following is true: for every $x \in X$, there exists an open set O such that $x \in O$, and a homeomorphism $O \to \mathbb{R}^n$.*

2.2 Exercises

EXERCISE 8. *Show that the topological spaces $\mathbb{R}^n$ and $\mathcal{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.*

Let $f : \mathcal{B}(0,1) \to \mathbb{R}^n$ be defined as $f(x) = \dfrac{x}{1 - ||x||}$. I observe it's well defined because $||x|| < 1$. We shall prove f is a homeomorphism.

1. **Injective:** Take $x, y \in \mathcal{B}(0,1)$ and suppose that

$$\frac{x}{1 - ||x||} = \frac{y}{1 - ||y||}.$$

   Applying the norm in both sides, we obtain the equation

$$||x||(1 - ||y|||) = ||y||(1 - ||x||) \implies ||x|| = ||y||.$$

   On the other side x and y points to the same direction, given that

$$y = \frac{1 - ||y||}{1 - ||x||} x = \alpha x,$$

   with $\alpha = 1$ because of the same norm. We conclude $x = y$.

2. **Surjective:** Take $y \in \mathbb{R}^n$. We shall prove that there exists $x \in \mathcal{B}(0,1)$ such that $f(x) = y$, that is,

$$\frac{x}{1 - ||x||} = y$$

Applying the norm we observe that if that is true, $||x|| = ||y|| - ||y|| ||x|| \implies ||x|| = \frac{||y||}{1+||y||}$. And $x = (1 - ||x||)y = \frac{1}{1+||y||}y$. We conclude that for every $y \in \mathbb{R}^n$, if we take $x = \frac{y}{1+||y||}$,

$$f(x) = \frac{y/(1 + ||y||)}{1 - ||y||/(1 + ||y||)} = y$$

3. **Continuity of f:** Consider an open set $A \subset \mathbb{R}^n$. Let $B = f^{-1}(A)$. We shall prove B is open, that is, for every $x \in B$, exists $r > 0$ such that $\mathcal{B}(x, r) \subset B$. Take $x = f^{-1}(y) \in B$. Because A is open, there is $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset A$. Take δ such that

$$\frac{\delta}{1 - ||x|| - \delta}(1 + ||y||) < \epsilon$$

and $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$.

$$\begin{aligned}
||y - w|| &= \left|\left| \frac{x}{1 - ||x||} - \frac{z}{1 - ||z||} \right|\right| = \frac{1}{1 - ||x||} \left|\left| x - \frac{1 - ||x||}{1 - ||z||} z \right|\right| \\
&= \frac{1}{1 - ||x||} \left|\left| x - z + z - \frac{1 - ||x||}{1 - ||z||} z \right|\right| \\
&\leq \frac{||x - z||}{1 - ||x||} + \frac{1}{1 - ||x||} \left(1 - \frac{1 - ||x||}{1 - ||z||} ||z||\right) \\
&= \frac{||x - z||}{1 - ||x||} + \frac{||z||}{1 - ||x||} \frac{||x|| - ||z||}{1 - ||z||} \\
&\leq \frac{1}{1 - ||x||} ||x - z|| (1 + ||w||) \\
&\leq \frac{1}{1 - ||x||} ||x - z|| (1 + ||y - w|| + ||y||) \\
\implies ||y - w|| &\leq \frac{||x - z||}{1 - ||x|| - ||x - z||} (1 + ||y||) \\
&< \frac{\delta}{1 - ||x|| - \delta} (1 + ||y||) < \epsilon
\end{aligned}$$

So $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$, what proves B is open. It concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by

$$f^{-1}(y) = \frac{y}{1 + ||y||}$$

The demonstration is quite similar to the previous item, given that the only difference is the signal.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0,1) \simeq \mathbb{R}^n$.

EXERCISE 9. *Show that $\mathcal{B}(x, r)$ and $\mathcal{B}(y, s)$ are homeomorphic.*

Consider the function $f : \mathcal{B}(0,1) \to \mathcal{B}(c, r)$ given by $f(x) = r \cdot x + c$. Let's prove f is a homeomorphism.

1. **Injective:** If $x, y \in \mathcal{B}(0,1)$ and $rx + c = ry + c \implies x = y$, because $r > 0$ by definition. So f is injective.

2. **Surjective:** Let $y \in \mathcal{B}(c,r)$ and $x = (y-c)/r$. So $||x|| = ||y-c||/r < 1$, by definition. So $x \in \mathcal{B}(0,1)$ and $f(x) = y$ what proves this function is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{B}(c,r)$ open set and denote $B = f^{-1}(A)$. Take $x = f^{-1}(y) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}(y,\epsilon) \subset A$. Define $\delta = \epsilon/r$ and take $z = f^{-1}(w) \in \mathcal{B}(x,\delta)$.
$$||y - w|| = ||rx + c - (rz + c)|| = r||x - z|| < r\delta = \epsilon$$
Therefore $w \in \mathcal{B}(y,\epsilon) \subset A \implies z \in B$. So $\mathcal{B}(x,\delta) \subset B$, what proves B is open. This concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by
$$f^{-1}(y) = \frac{y-c}{r}$$
This function is continuos for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0,1) \simeq \mathcal{B}(c,r)$. Since this is a equivalence relation, we have that

$$\mathcal{B}(0,1) \simeq \mathcal{B}(x,r) \text{ and } \mathcal{B}(0,1) \simeq \mathcal{B}(y,s) \text{ implies } \mathcal{B}(x,r) \simeq \mathcal{B}(y,s).$$

EXERCISE 10. *Show that $\mathbb{S}(0,1)$, the unit circle of $\mathbb{R}^2$, is homeomorphic to the ellipse*

$$\mathcal{S}(a,b) = \left\{(x,y) \in \mathbb{R}^2, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right\},$$

for any $a, b > 0$.

Consider the function $f : \mathbb{S}(0,1) \to \mathcal{S}(a,b)$ such that $f(x,y) = (ax, by)$. Let's prove it is a homeomorphism.

1. **Injective:** Let $(x_1,y_1), (x_2,y_2) \in \mathbb{S}(0,1)$ such that $(ax_1, by_1) = (ax_2, by_2)$. Since $a, b > 0$, we have $x_1 = x_2$ and $y_1 = y_2$. It proves f is injective.

2. **Surjective:** Let $(z,w) \in \mathcal{S}(a,b)$ and $(x,y) = \left(\frac{z}{a}, \frac{w}{b}\right)$. It's clear that $f(x,y) = (z,w)$ and $x^2 + y^2 = \frac{z^2}{a^2} + \frac{w^2}{b^2} = 1$, so $(x,y) \in \mathbb{S}(0,1)$. It proves f is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{S}(a,b)$ open set and denote $B = f^{-1}(A)$. Take $(x,y) = f^{-1}(z,w) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}((z,w),\epsilon) \subset A$. Put δ as defined below and take $(x',y') = f^{-1}((z',w')) \in \mathcal{B}((x,y),\delta)$. Consider the norm 1
$$\begin{aligned} ||(z',w') - (z,w)||_1 &= ||(ax',by') - (ax,by)||_1 = ||\left(a(x'-x), b(y'-y)\right)||_1 \\ &= a|x'-x| + b|y'-y|, \text{ define } c = \max\{a,b\} \\ &\le c(|x'-x| + |y'-y|) = c||(x'-x, y'-y)||_1 \end{aligned}$$

By the equivalente of the norms, there exists constants k_1, k_2 such that

$$||(z', w') - (z, w)|| \leq k_1||(z', w') - (z, w)||_1 \leq ck_1||(x' - x, y' - y)||_1 \leq ck_1k_2||(x' - x, y' - y)||$$

Then we need $\delta = \frac{\epsilon}{ck_1k_2}$ in order to prove that $(z', w') \in \mathcal{B}((z, w), \epsilon) \subset A \implies (x', y') \in B$. So $\mathcal{B}((x, y), \delta) \subset B$, what proves B is open. This concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by

$$f^{-1}((z, w)) = (z/a, w/b)$$

This function is continuos for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathbb{S}(0, 1) \simeq \mathcal{S}(a, b)$.

EXERCISE 11. *Show that* $[0, 1)$ *and* $(0, 1)$ *are not homeomorphic.*

We shall prove by contradiction. Suppose these exists a homeomorphism $f : [0, 1) \to (0, 1)$. Let $0 < z = f(0) < 1$ and define the following function

$$\begin{aligned} g : (0, 1) &\to (0, z) \cup (z, 1) \\ x &\mapsto g(x) = f(x) \end{aligned}$$

This function is well defined given that z is not image of other point but 0. The function is injective because if $g(y) = g(x) \implies f(y) = f(x) \implies x = y$, given that f is injective. This function is also surjective since f is and $0 < w < 1$ and $w \neq z$, it's clear that $f(0) \neq w$. As g is an induced map of a continuos function, by Proposition 1.21 from the notes, it's continuos and so is its inverse. We conclude g is a homeomorphism.

Now I will prove that $(0, 1)$ admits only 1 connected component, that is, it's connected. Suppose it's not and there exists $O, O' \subset (0, 1)$ open disjoint sets such that $(0, 1) = O \cup O'$ and none of them are empty sets. Let $a \in O, b \in O'$ with $a < b$ without loss of generality. Define $\alpha = \sup\{x \in \mathbb{R} : [a, x) \subset O\}$. It's well defined because this set is not empty, given O is open and b is an upper bound. Then $\alpha \leq b$. Suppose $\alpha \in O'$, then there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subset O'$. We know that for every $\epsilon > 0$, there exists $w \in (\alpha - \epsilon, \alpha]$ such that $[a, w) \subset O$. That is a contradiction since there exists $w \in (\alpha - r, \alpha)$ such that $[a, w) \subset O$. So $\alpha \in O \implies (\alpha - r, \alpha + r) \subset O$, for some r. We infer that $[a, \alpha + r) \subset O$, what is an absurd. Therefore $(0, 1)$ is connected.

For a similar argument, we prove that $(0, z)$ and $(z, 1)$ are connected. This implies that the union admits 2 connected components.

In that sense, we have a homeomorphism between a topological space with 1 connected component and other with 2 connected components, what is a contradiction by Proposition 2.14 from the notes. We conclude that $[0, 1)$ and $(0, 1)$ are not homeomorphic.

3 Homotopies

3.1 Important definitions

DEFINITION 3.1.1. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be two topological spaces, and* $f, g : X \to Y$ *two continuous maps. A homotopy between* f *and* g *is a map* $F : X \times [0,1] \to Y$ *such that:*

1. $F(\cdot, 0)$ *is equal to* f,

2. $F(\cdot, 1)$ *is equal to* g,

3. $F : X \times [0,1] \to Y$ *is continuous.*

If such a homotopy exists, we say that the maps f *and* g *are homotopic.*

Remark. Before asking for $F : X \times [0,1] \to Y$ to be continuous, we have to give $X \times [0,1]$ a topology. The topology we choose is the product topology. Consider the topological space $(X, \mathcal{T})$, and endow $[0,1]$ with the subspace topology of $\mathbb{R}$, denoted $T_{|[0,1]}$. The product topology on $X \times [0,1]$, denoted $T \otimes T_{|[0,1]}$, is defined as follows: a set $O \subset X \times [0,1]$ is open if and only if it can be written as a union $U_{\alpha \in A} O_\alpha \times O'_\alpha$ where every O_α is an open set of X and O'_α is an open set of $[0,1]$.

DEFINITION 3.1.2. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be two topological spaces. A homotopy equivalence between* X *and* Y *is a pair of continuous maps* $f : X \to Y$ *and* $g : Y \to X$ *such that:*

1. $g \circ f : X \to X$ *is homotopic to the identity map* $id : X \to X$,

2. $f \circ g : Y \to Y$ *is homotopic to the identity map* $id : Y \to Y$,

If such a homotopy equivalence exists, we say that X *and* Y *are homotopy equivalent.*

DEFINITION 3.1.3. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subset X$ *a subset, endowed with the subspace topology* $T_{|Y}$. *A retraction is a continuous map* $r : X \to Y$ *such that* $\forall y \in Y, r(y) = y$.

A deformation retraction is a homotopy $F : X \times [0,1] \to Y$ *between the identity map* $id : X \to X$ *and a retraction* $r : X \to Y$.

3.2 Exercises

EXERCISE 12. *Let* $f : \mathbb{R}^n \to X$ *be a continuous map. Then* f *is homotopic to a constant map.*

I must prove that there exists a homotopy between f and a constant map. Consider the function $F : \mathbb{R}^n \times [0,1] \to X$ defined as

$$F(x,t) = f(tx)$$

It's clear that $F(x, 0) = f(0)$, for every $x \in \mathbb{R}^n$. So it's the constant map $f(0)$. We also have that $F(x, 1) = f(x), \forall x \in \mathbb{R}^n$. Moreover, let's prove F is continuos. Denote $F' : \mathbb{R}^n \times \mathbb{R} \to X$ the function $F'(x,t) = f(xt)$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the function $g(x,t) = xt$. So $F' = f \circ g$.

Let's prove g is a continuous function. As we are dealing with a real-valued function, by Proposition 1.19 from the notes, I can use the $\epsilon - \delta$ proof. Let $(x,t) \in \mathbb{R}^{n+1}$ and $\epsilon > 0$. In the proof I use the norm 1, without loss of generality because of the equivalence of norms in $\mathbb{R}^n$. Put

$\delta = \min\{1, \frac{\epsilon}{\max\{||x||,|t|+1\}}\}$ and suppose $||(x,t)-(x',t')|| = ||x-x'|| + |t-t'| < \delta$. So,

$$\begin{aligned}||xt - x't'|| &= ||xt - xt' + xt' - x't'|| \\ &\leq |t-t'|||x|| + |t'|||x-x'|| \\ &\leq |t-t'|||x|| + (|t|+\delta)||x-x'|| \\ &< \max\{||x||, |t|+\delta\}\delta \\ &\leq \max\{||x||, |t|+1\}\delta \leq \epsilon\end{aligned}$$

By this, g is a continuos function. Since f is also continuos, the composition F' is also continuos, by Proposition 1.18. By Proposition 1.21, when we endow F' in $\mathbb{R}^n \times [0,1]$, we obtain a continuos function, that is F is continuos. Then we conclude that f is homotopic to a constant function.

EXERCISE 13. *Show that every map $f : \mathbb{S}_1 \to \mathbb{S}_2$ is homotopic to a constant map, where the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, ||x|| = 1\}$.*

Remark. I will suppose f is continuous, otherwise I think it's not possible to prove there is a homotopy.

EXERCISE 14. *Show that being homotopic is a transitive relation between maps: for every triplet of maps $f, g, h : X \to Y$, if f, g are homotopic and g, h are homotopic, then f, h are homotopic.*

We shall prove there exist a homotopy H between f and h. By assumption, there exists a homotopy F between f and g and a homotopy G between g and h. Define $H : X \times [0,1] \to Y$ such that

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leq t \leq 1/2 \\ G(x,2t-1), & 1/2 < t \leq 1 \end{cases}$$

that is, H behaves as F until it reaches a half. When that occurs, $H(x,1/2) = F(x,1) = g(x) = G(x,0)$. After that, H follows G until the end of the interval. So, it's clear that $H(x,0) = F(x,0) = f(x), \forall x \in X$ and $H(x,1) = G(x,1) = h(x), \forall x \in X$. Moreover, since F and G are continuos and in the point $t = 1/2$, both functions agree, H is continuos and, therefore, f and h are homotopic.

EXERCISE 15. *Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).*

1. (*reflexive*): Consider the identity map $id : X \to X$, that is continuos. We shall prove that this function is homotopic to itself. Consider $F : X \times [0,1] \to X$ given by $F(x,t) = x$ for every x and t. It's clear this is a homotopy because $F(x,0) = F(x,1) = x$ and it's continuos. Moreover $id \circ id = id$ by definition of identity. Therefore, there exists a homotopy equivalence between id and itself. We conclude $X \approx X$.

2. (*symmetric*): Suppose $X \approx Y$. So, there exists continuos functions $f : X \to Y$ and $g : Y \to X$ that form a homotopy equivalence. This means that $g : Y \to X$ and $f : X \to Y$ are a homotopy equivalence as well. So $Y \approx X$.

3. (*transitive*): Suppose $X \approx Y$, and let $f_1 : X \to Y$ and $g_1 : Y \to X$ form a homotopy equivalence. Also suppose $Y \approx Z$ and let $f_2 : Y \to Z$ and $g_2 : Z \to Y$ form a homotopy equivalence. Define $f_3 = f_2 \circ f_1$ and $g_3 = g_1 \circ g_2$. Let's proof this is a homotopy equivalence. Both functions are continuos given that they are a composition of continuos functions.

   (a) $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1$ is homotopic to $id : X \to X$.
   Let F_1 be a homotopy between $g_1 \circ f_1$ and id and F_2 a homotopy between $g_2 \circ f_2$ and id. Define
   $$F_3(x,t) = \begin{cases} g_1 \circ F_2(\cdot, 2t) \circ f_1(x), & 0 \le t \le 1/2 \\ F_1(x, 2t-1), & 1/2 < t \le 1 \end{cases}$$
   So $F_3(x,0) = g_1(F_2(f_1(x),0)) = g_1(g_2(f_2(f_1(x)))) = g_3 \circ f_3(x)$, for every x and $F_3(x,1) = F_1(x,1) = x$, for every x. When $t = 1/2$,
   $$F_3(x,1/2) = g_1(F_2(f_1(x),1)) = g_1(f_1(x)) = F_1(x,0)$$
   By this equality and the fact that composition of continuos functions is a continuos map, we conclude that F_3 is continuos. This implies that $g_3 \circ f_3$ is homotopic to the identity.

   (b) $f_3 \circ g_3 = f_1 \circ f_2 \circ g_2 \circ g_1$ is homotopic to $id : Z \to Z$.
   This follows a quite similar demonstration and can be omitted.

   By the points above f_3 and g_3 is a homotopy equivalence what proves $X \approx Z$.

Consequently, homotopy equivalence is a equivalence relation.

EXERCISE 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

I will consider the upper case alphabet and each letter will be considered as a topological space (a subset from $\mathbb{R}^2$), for example the letter O is homotopy equivalent to a circle, while L is to an interval, or equivalently, a point. Observe that most of the letters are equivalent to a point, because we can think in a continuos reduction. When we have a hole, such as A, D, R, O, P, Q, this continuity is impossible since we'll have a point break. B is a special case because we can't deform into a point without breaking points and also we cannot join the holes in one. So there is three classes, given by its representatives

1. O
2. B
3. I

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	1	3	3	3	3	3	3	3	3	3

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
3	1	1	1	1	3	3	3	3	3	3	3	3