# Topological Data Analysis - Exercises

Lucas Moschen, EMAp/FGV January 29, 2021

## 1 General topology

### 1.1 Important definitions

Definition 1.1.1. *A topological space is a pair $(X, \mathcal{T})$ where $X$ is a set and $\mathcal{T}$ is a collection of subsets of $X$ such that:*

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. *for every infinite collection* $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, *we have* $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. *for every finite collection* $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$, *we have* $\bigcap_{1 \leq i \leq n} O_i \in \mathcal{T}$.

Definition 1.1.2. *Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center $x$ and radius $r$, denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, ||x - y|| < r\}$.*

Definition 1.1.3. *Let $A \subset \mathbb{R}$ and $x \in A$. We say that $A$ is open around $x$ if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that $A$ is open if for every $x \in A$, $A$ is open around $x$.*

Definition 1.1.4. *Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. We define the subspace topology on $Y$ as the following set:*

$$T_{|Y} = \{O \cap Y, O \in \mathcal{T}\}$$

Definition 1.1.5. *Let $f : X \to Y$ be a map. We say that $f$ is continuous if for every $O \in U$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.*

### 1.2 Exercises

Exercise 1. *Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that $X$ admits?*

As we know every Topology contains $\emptyset$ and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\}$ - $\mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\}$ $\{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\}$ $\{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\}$ $\{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

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Exercise 2. *Let $\mathbb{Z}$ be the set of integers. Consider the cofinite topology $\mathcal{T}$ on $\mathbb{Z}$, defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or ${}^cO$ is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of $O$ in $\mathbb{Z}$*

1. Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

   Let's verify the three axioms:

   (a) $\emptyset$ is an open set by definition and $\mathbb{Z}$ is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.

   (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcup_{\alpha \in A} O_\alpha\right) = \bigcap_{\alpha \in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall \alpha \in A$. If $\forall \alpha, O_\alpha = \emptyset$, then ${}^cO = {}^c\emptyset \implies O = \emptyset$ and $O$ is open. On the other hand, if there exists $\alpha \in A$ such that $O_\alpha \neq \emptyset$ we have ${}^cO_\alpha$ being finite, so is ${}^cO$, given the inclusion. We conclude $O$ is open set.

   (c) Let $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcap_{1 \leq i \leq n} O_i\right) = \bigcup_{1 \leq i \leq n} {}^cO_i$. If $O_i = \emptyset$ for some $1 \leq i \leq n$, $O = \emptyset$ because of the intersection. Alternatively, if $\forall i, O_i \neq \emptyset$ we have that ${}^cO_i$ is finite and a finite union of finites is finite. We conclude that $O$ is open set.

   By (a), (b) and (c), $\mathcal{T}$ is a topology on $\mathbb{Z}$.

2. Exhibit an sequence of open sets $\{O_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n \in \mathbb{N}} O_n$ is not an open set.

   Let $O_n = {}^c\{1, ..., n\}$. Thus ${}^cO_n = \{1, ..., n\}$ is finite and

   $$ {}^c\left(\bigcap_{n \in \mathbb{N}} O_n\right) = \bigcup_{n \in \mathbb{N}} {}^cO_n = \bigcup_{n \in \mathbb{N}} \{1, ..., n\} = \mathbb{N}, $$

   that is not finite. Therefore, this intersection is not an open set.

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Exercise 3. *Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that*

$$ \mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r) $$

Let $z \in \mathcal{B}(y, r - ||x - y||)$, so $||z - y|| < r - ||x - y|| \implies ||z - y|| + ||x - y|| < r$. We can conclude that, by the triangular inequality,

$$ ||x - z|| \leq ||x - y|| + ||z - y|| < r. $$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$.

*Remark.* In the notes, the exercise is to prove $\mathcal{B}(y, ||x - y||) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take $y$ next the border of $\mathcal{B}(x, r)$, $||x - y|| \approx r$ and $B(y, r - \epsilon) \not\subset B(x, r)$.

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Exercise 4. *Let $x, y \in \mathbb{R}^n$, and $r = ||x - y||$. Show that*

$$ \mathcal{B}\left(\frac{x + y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r) $$

Denote $m = \dfrac{x+y}{2}$. Take $z \in \mathcal{B}\left(m, \dfrac{r}{2}\right)$. Thus, using the triangular inequality,

$$||x - z|| \leq ||x - m|| + ||m - z|| = \frac{1}{2}||x - y|| + ||m - z|| < r/2 + r/2 = r$$

$$||y - z|| \leq ||y - m|| + ||m - z|| = \frac{1}{2}||y - x|| + ||m - z|| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x, r)$, $z \in \mathcal{B}(y, r)$ and $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$.

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Exercise 5. *Show that the open balls $\mathcal{B}(x, r)$ of $R^n$ are open sets (with respect to the Euclidean topology).*

We have to prove that for every $y \in \mathcal{B}(x, r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. Put $\epsilon = r - ||x - y||$. As we have proved in exercise 3, $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. So $\mathcal{B}(x, r)$ is open set.

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Exercise 6. *Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?*

1. $[0, 1]$. It's not open set because for every $\epsilon > 0, \mathcal{B}(0, \epsilon) = (-\epsilon, \epsilon) \not\subset [0, 1]$. It's closed because $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$ is an union of two open sets, as we prove in item 3.

2. $[0, 1)$. It's not open for the same reason as before. It's not closed because $B(1, \epsilon) = (1 - \epsilon, 1 + \epsilon) \not\subset (-\infty, 0) \cup [1, \infty]$.

3. $(-\infty, 1)$. It's open because: take $x < 1$. Put $r = 1 - x$ and take $z \in \mathcal{B}(x, r)$. If $z > x$, $|x - z| < 1 - x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty, 1)$ is open. It's not closed cause $\forall \epsilon > 0, \mathcal{B}(1, \epsilon) \not\subset [1, \infty)$.

4. the singletons. It's not open cause $\forall \epsilon > 0, x + \epsilon/2 \in \mathcal{B}(x, \epsilon)$. It's close cause $(-\infty, x) \cup (x, \infty)$ is union of open sets.

5. $\mathbb{Q}$. It's not open because for every open ball around a rational, there are irrationals, that is, let $x \in \mathbb{Q}$ and take $\epsilon > 0$, then there exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x, \epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

   *Remark.* We shall prove the rationals are dense in the reals. Let $x \in \mathbb{Q}$ and $\epsilon > 0$. If $\epsilon$ is irrational, take $x - \epsilon/2 \subset (x - \epsilon, x + \epsilon)$. Suppose $x - \epsilon/2$ is rational, then $\frac{2x - \epsilon}{2} = \frac{m}{n}$ for some integers $m$ and $n$, that is, $2x - \epsilon = 2m/n$ and $\epsilon = 2(x - \frac{m}{n}) \in \mathbb{Q}$, contradiction. So there is an irrational in $\mathcal{B}(x, \epsilon)$. If $\epsilon$ is rational, consider

   $$y = \frac{1}{\sqrt{2}}(x - \epsilon) + (1 - \frac{1}{\sqrt{2}})(x + \epsilon) = (x + \epsilon) - \epsilon\sqrt{2}$$

   That is a convex combination, so $y \in \mathcal{B}(x, \epsilon)$. Moreover, $y$ is irrational, with a similar proof by contradiction. This proves the statement.

   On the other hand, we must prove for every two irrationals $(a, b)$, there is a rational between them. Denote $c = b - a > 0$. Let $n \in \mathbb{N}$ such that $n > \frac{1}{c} \implies cn > 1 \implies (bn - an) > 1$. So exists $m \in (an, bn) \implies m/n \in (a, b)$. This proves the second statement.

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Exercise 7. *A map is continuous if and only if the preimage of closed sets are closed sets.*

First we shall prove that $f^{-1}( {}^cA) = {}^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}( {}^cA)$. So there exists $y \in {}^cA$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in {}^c(f^{-1}(A))$.

Now take $x \in {}^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in {}^cA \implies x \in f^{-1}( {}^cA)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose $f$ is a continuos map and take a closed set $F$. We shall prove that $f^{-1}(F)$ is closed. Well, ${}^c(f^{-1}(F)) = f^{-1}( {}^cF)$ is open, because ${}^cF$ is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set $F$, we have $f^{-1}(F)$ being closed. We will use that $A$ is open if ${}^cA$ is closed. This is true because ${}^c({}^c(A)) = A$. Take an open set $A$. ${}^c(f^{-1}(A)) = f^{-1}( {}^cA)$ is closed, because ${}^cA$ is. Thus $f^{-1}(A)$ is open and we have proved the continuity of $f$.

# 2 Homeomorphisms

## 2.1 Important definitions

Definition 2.1.1. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be two topological spaces, and* $f : X \to Y$ *a map. We say that* $f$ *is a homeomorphism if*

1. $f$ *is a bijection,*
2. $f : X \to Y$ *is continuos,*
3. $f^{-1} : Y \to X$ *is continuos.*

*If there exists such a homeomorphism, we say that the two topological spaces are homeomorphic.*

Definition 2.1.2. *Let* $(X, \mathcal{T})$ *be a topological space. We say that* $X$ *is connected if for every open sets* $O, O' \in \mathcal{T}$ *such that* $O \cap O' = \emptyset$ *(i.e., they are disjoint), we have*

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

Definition 2.1.3. *Let* $(X, \mathcal{T})$ *be a topological space. Suppose that there exists a collection of* $n$ ***non-empty, disjoint*** *and* ***connected open sets*** $(O_1, ..., O_n)$ *such that*

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

*Then we say that* $X$ *admits* $n$ *connected components.*

Definition 2.1.4. *Let* $(X, \mathcal{T})$ *be a topological space, and* $n \geq 0$. *We say that it has dimension* $n$ *if the following is true: for every* $x \in X$, *there exists an open set* $O$ *such that* $x \in O$, *and a homeomorphism* $O \to \mathbb{R}^n$.

## 2.2 Exercises

Exercise 8. *Show that the topological spaces* $\mathbb{R}^n$ *and* $\mathcal{B}(0,1) \subset \mathbb{R}^n$ *are homeomorphic.*

Let $f : \mathcal{B}(0,1) \to \mathbb{R}^n$ be defined as $f(x) = \dfrac{x}{1 - ||x||}$. I observe it's well defined because $||x|| < 1$. We shall prove $f$ is a homeomorphism.

1. **Injective:** Take $x, y \in \mathcal{B}(0,1)$ and suppose that

   $$\frac{x}{1 - ||x||} = \frac{y}{1 - ||y||}.$$

   Applying the norm in both sides, we obtain the equation

   $$||x||(1 - ||y|||) = ||y||(1 - ||x||) \implies ||x|| = ||y||.$$

   On the other side $x$ and $y$ points to the same direction, given that

   $$y = \frac{1 - ||y||}{1 - ||x||} x = \alpha x,$$

   with $\alpha = 1$ because of the same norm. We conclude $x = y$.

2. **Surjective:** Take $y \in \mathbb{R}^n$. We shall prove that there exists $x \in \mathcal{B}(0,1)$ such that $f(x) = y$, that is,
$$\frac{x}{1 - ||x||} = y$$
Applying the norm we observe that if that is true, $||x|| = ||y|| - ||y|| ||x|| \implies ||x|| = \frac{||y||}{1+||y||}$. And $x = (1 - ||x||)y = \frac{1}{1+||y||} y$. We conclude that for every $y \in \mathbb{R}^n$, if we take $x = \frac{y}{1+||y||}$,
$$f(x) = \frac{y/(1 + ||y||)}{1 - ||y||/(1 + ||y||)} = y$$

3. **Continuity of f:** Consider an open set $A \subset \mathbb{R}^n$. Let $B = f^{-1}(A)$. We shall prove $B$ is open, that is, for every $x \in B$, exists $r > 0$ such that $\mathcal{B}(x, r) \subset B$. Take $x = f^{-1}(y) \in B$. Because $A$ is open, there is $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset A$. Take $\delta$ such that
$$\frac{\delta}{1 - ||x|| - \delta}(1 + ||y||) < \epsilon$$
and $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$.
$$\begin{aligned}
||y - w|| &= \left|\left| \frac{x}{1 - ||x||} - \frac{z}{1 - ||z||} \right|\right| = \frac{1}{1 - ||x||} \left|\left| x - \frac{1 - ||x||}{1 - ||z||} z \right|\right| \\
&= \frac{1}{1 - ||x||} \left|\left| x - z + z - \frac{1 - ||x||}{1 - ||z||} z \right|\right| \\
&\leq \frac{||x - z||}{1 - ||x||} + \frac{1}{1 - ||x||} \left( 1 - \frac{1 - ||x||}{1 - ||z||} ||z|| \right) \\
&= \frac{||x - z||}{1 - ||x||} + \frac{||z||}{1 - ||x||} \frac{||x|| - ||z||}{1 - ||z||} \\
&\leq \frac{1}{1 - ||x||} ||x - z|| (1 + ||w||) \\
&\leq \frac{1}{1 - ||x||} ||x - z|| (1 + ||y - w|| + ||y||) \\
\implies ||y - w|| &\leq \frac{||x - z||}{1 - ||x|| - ||x - z||} (1 + ||y||) \\
&< \frac{\delta}{1 - ||x|| - \delta} (1 + ||y||) < \epsilon
\end{aligned}$$
So $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$, what proves $B$ is open. It concludes the continuity of $f$.

4. **Continuity of $f^{-1}$:** The inverse is given by
$$f^{-1}(y) = \frac{y}{1 + ||y||}$$
The demonstration is quite similar to the previous item, given that the only difference is the signal.

By items (1) - (4), we conclude $f$ is a homeomorphism and $\mathcal{B}(0,1) \simeq \mathbb{R}^n$.

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Exercise 9. *Show that $\mathcal{B}(x, r)$ and $\mathcal{B}(y, s)$ are homeomorphic.*

Consider the function $f : \mathcal{B}(0,1) \to \mathcal{B}(c, r)$ given by $f(x) = r \cdot x + c$. Let's prove $f$ is a homeomorphism.

1. **Injective:** If $x, y \in \mathcal{B}(0,1)$ and $rx + c = ry + c \implies x = y$, because $r > 0$ by definition. So $f$ is injective.

2. **Surjective:** Let $y \in \mathcal{B}(c,r)$ and $x = (y-c)/r$. So $||x|| = ||y - c||/r < 1$, by definition. So $x \in \mathcal{B}(0,1)$ and $f(x) = y$ what proves this function is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{B}(c,r)$ open set and denote $B = f^{-1}(A)$. Take $x = f^{-1}(y) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}(y,\epsilon) \subset A$. Define $\delta = \epsilon/r$ and take $z = f^{-1}(w) \in \mathcal{B}(x,\delta)$.
$$||y - w|| = ||rx + c - (rz + c)|| = r||x - z|| < r\delta = \epsilon$$
Therefore $w \in \mathcal{B}(y,\epsilon) \subset A \implies z \in B$. So $\mathcal{B}(x,\delta) \subset B$, what proves $B$ is open. This concludes the continuity of $f$.

4. **Continuity of $f^{-1}$:** The inverse is given by
$$f^{-1}(y) = \frac{y - c}{r}$$
This function is continuos for the same argument as before.

By items (1) - (4), we conclude $f$ is a homeomorphism and $\mathcal{B}(0,1) \simeq \mathcal{B}(c,r)$. Since this is an equivalence relation, we have that
$$\mathcal{B}(0,1) \simeq \mathcal{B}(x,r) \text{ and } \mathcal{B}(0,1) \simeq \mathcal{B}(y,s) \text{ implies } \mathcal{B}(x,r) \simeq \mathcal{B}(y,s).$$

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Exercise 10. *Show that $\mathbb{S}(0,1)$, the unit circle of $\mathbb{R}^2$, is homeomorphic to the ellipse*
$$\mathcal{S}(a,b) = \left\{(x,y) \in \mathbb{R}^2, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right\},$$
*for any $a, b > 0$.*

Consider the function $f : \mathbb{S}(0,1) \to \mathcal{S}(a,b)$ defined as $f(x,y) = (ax, by)$. Let's prove it is a homeomorphism.

1. **Injective:** Let $(x_1,y_1),(x_2,y_2) \in \mathbb{S}(0,1)$ such that $(ax_1, by_1) = (ax_2, by_2)$. Since $a, b > 0$, we have $x_1 = x_2$ and $y_1 = y_2$. It proves $f$ is injective.

2. **Surjective:** Let $(z,w) \in \mathcal{S}(a,b)$ and $(x,y) = \left(\frac{z}{a}, \frac{w}{b}\right)$. It's clear that $f(x,y) = (z,w)$ and $x^2 + y^2 = \frac{z^2}{a^2} + \frac{w^2}{b^2} = 1$, so $(x,y) \in \mathbb{S}(0,1)$. It proves $f$ is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{S}(a,b)$ open set and denote $B = f^{-1}(A)$. Take $(x,y) = f^{-1}(z,w) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}((z,w),\epsilon) \subset A$. Put $\delta$ as defined below and take $(x',y') = f^{-1}((z',w')) \in \mathcal{B}((x,y),\delta)$. Consider the norm 1
$$\begin{aligned} ||(z',w') - (z,w)||_1 &= ||(ax',by') - (ax,by)||_1 = ||\left(a(x'-x), b(y'-y)\right)||_1 \\ &= a|x'-x| + b|y'-y|, \text{ define } c = \max\{a,b\} \\ &\leq c(|x'-x| + |y'-y|) = c||(x'-x, y'-y)||_1 \end{aligned}$$

By the equivalente of the norms, there exists constants $k_1, k_2$ such that

$$||(z', w') - (z, w)|| \leq k_1||(z', w') - (z, w)||_1 \leq ck_1||(x' - x, y' - y)||_1 \leq ck_1k_2||(x' - x, y' - y)||$$

Then we need $\delta = \frac{\epsilon}{ck_1k_2}$ in order to prove that $(z', w') \in \mathcal{B}((z, w), \epsilon) \subset A \implies (x', y') \in B$. So $\mathcal{B}((x, y), \delta) \subset B$, what proves $B$ is open. This concludes the continuity of $f$.

4. **Continuity of $f^{-1}$:** The inverse is given by

$$f^{-1}((z, w)) = (z/a, w/b)$$

This function is continuos for the same argument as before.

By items (1) - (4), we conclude $f$ is a homeomorphism and $\mathbb{S}(0, 1) \simeq \mathcal{S}(a, b)$.

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Exercise 11. *Show that* $[0, 1)$ *and* $(0, 1)$ *are not homeomorphic.*

We shall prove by contradiction. Suppose these exists a homeomorphism $f : [0, 1) \to (0, 1)$. Let $0 < z = f(0) < 1$ and define the following function

$$\begin{aligned} g : (0, 1) &\to (0, z) \cup (z, 1) \\ x &\mapsto g(x) = f(x) \end{aligned}$$

This function is well defined given that $z$ is not image of other point but 0. The function is injective because if $g(y) = g(x) \implies f(y) = f(x) \implies x = y$, given that $f$ is injective. This function is also surjective since $f$ is and $0 < w < 1$ and $w \neq z$, it's clear that $f(0) \neq w$. As $g$ is an induced map of a continuos function, by Proposition 1.21 from the notes, it's continuos and so is its inverse. We conclude $g$ is a homeomorphism.

Now I will prove that $(0, 1)$ admits only 1 connected component, that is, it's connected. Suppose it's not and there exists $O, O' \subset (0, 1)$ open disjoint sets such that $(0, 1) = O \cup O'$ and none of them are empty sets. Let $a \in O, b \in O'$ with $a < b$ without loss of generality. Define $\alpha = \sup\{x \in \mathbb{R} : [a, x) \subset O\}$. It's well defined because this set is not empty, given $O$ is open and $b$ is an upper bound. Then $\alpha \leq b$. Suppose $\alpha \in O'$, then there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subset O'$. We know that for every $\epsilon > 0$, there exists $w \in (\alpha - \epsilon, \alpha]$ such that $[a, w) \subset O$. That is a contradiction since there exists $w \in (\alpha - r, \alpha)$ such that $[a, w) \subset O$. So $\alpha \in O \implies (\alpha - r, \alpha + r) \subset O$, for some $r$. We infer that $[a, \alpha + r) \subset O$, what is an absurd. Therefore $(0, 1)$ is connected.

For a similar argument, we prove that $(0, z)$ and $(z, 1)$ are connected. This implies that the union admits 2 connected components.

In that sense, we have a homeomorphism between a topological space with 1 connected component and other with 2 connected components, what is a contradiction by Proposition 2.14 from the notes. We conclude that $[0, 1)$ and $(0, 1)$ are not homeomorphic.

# 3 Homotopies

## 3.1 Important definitions

Definition 3.1.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f, g : X \to Y$ two continuous maps. A homotopy between $f$ and $g$ is a map $F : X \times [0,1] \to Y$ such that:*

1. *$F(\cdot, 0)$ is equal to $f$,*
2. *$F(\cdot, 1)$ is equal to $g$,*
3. *$F : X \times [0,1] \to Y$ is continuous.*

*If such a homotopy exists, we say that the maps $f$ and $g$ are homotopic.*

*Remark.* Before asking for $F : X \times [0,1] \to Y$ to be continuous, we have to give $X \times [0,1]$ a topology. The topology we choose is the product topology. Consider the topological space $(X, \mathcal{T})$, and endow $[0,1]$ with the subspace topology of $\mathbb{R}$, denoted $T_{|[0,1]}$. The product topology on $X \times [0,1]$, denoted $T \otimes T_{|[0,1]}$, is defined as follows: a set $O \subset X \times [0,1]$ is open if and only if it can be written as a union $U_{\alpha \in A} O_\alpha \times O'_\alpha$ where every $O_\alpha$ is an open set of $X$ and $O'_\alpha$ is an open set of $[0,1]$.

Definition 3.1.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces. A homotopy equivalence between $X$ and $Y$ is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that:*

1. *$g \circ f : X \to X$ is homotopic to the identity map $id : X \to X$,*
2. *$f \circ g : Y \to Y$ is homotopic to the identity map $id : Y \to Y$,*

*If such a homotopy equivalence exists, we say that $X$ and $Y$ are homotopy equivalent.*

Definition 3.1.3. *Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset, endowed with the subspace topology $T_{|Y}$. A retraction is a continuous map $r : X \to Y$ such that $\forall y \in Y, r(y) = y$.*

*A deformation retraction is a homotopy $F : X \times [0,1] \to Y$ between the identity map $id : X \to X$ and a retraction $r : X \to Y$.*

## 3.2 Exercises

Exercise 12. *Let $f : \mathbb{R}^n \to X$ be a continuous map. Then $f$ is homotopic to a constant map.*

I must prove that there exists a homotopy between $f$ and a constant map. Consider the function $F : \mathbb{R}^n \times [0,1] \to X$ defined as

$$F(x, t) = f(tx)$$

It's clear that $F(x, 0) = f(0)$, for every $x \in \mathbb{R}^n$. So it's the constant map $f(0)$. We also have that $F(x, 1) = f(x), \forall x \in \mathbb{R}^n$. Moreover, let's prove $F$ is continuos. Denote $F' : \mathbb{R}^n \times \mathbb{R} \to X$ the function $F'(x, t) = f(xt)$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the function $g(x, t) = xt$. So $F' = f \circ g$.

Let's prove $g$ is a continuous function. As we are dealing with a real-valued function, by Proposition 1.19 from the notes, I can use the $\epsilon - \delta$ proof. Let $(x, t) \in \mathbb{R}^{n+1}$ and $\epsilon > 0$. In the proof I use the norm 1, without loss of generality because of the equivalence of norms in $\mathbb{R}^n$. Put

$\delta = \min\{1, \frac{\epsilon}{\max\{||x||, |t|+1\}}\}$ and suppose $||(x,t)-(x',t')|| = ||x-x'|| + |t-t'| < \delta$. So,

$$\begin{aligned}
||xt - x't'|| &= ||xt - xt' + xt' - x't'|| \\
&\leq |t-t'|||x|| + |t'|||x-x'|| \\
&\leq |t-t'|||x|| + (|t|+\delta)||x-x'|| \\
&< \max\{||x||, |t|+\delta\}\delta \\
&\leq \max\{||x||, |t|+1\}\delta \leq \epsilon
\end{aligned}$$

By this, $g$ is a continuos function. Since $f$ is also continuos, the composition $F'$ is also continuos, by Proposition 1.18. By Proposition 1.21, when we endow $F'$ in $\mathbb{R}^n \times [0,1]$, we obtain a continuos function, that is $F$ is continuos. Then we conclude that $f$ is homotopic to a constant function.

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EXERCISE 13. *Let $f : \mathbb{S}_1 \to \mathbb{S}_2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map where the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, ||x|| = 1\}$.*

Let $x_0 \in \mathbb{S}_2$ such that $x_0 \notin f(\mathbb{S}_1)$ and consider the constant map $g(x) = -x_0$, for every $x \in \mathbb{S}_1$. Let $F : \mathbb{S}_1 \times [0,1] \to \mathbb{S}_2$ be defined as

$$F(x,t) = 2\frac{(1-t)f(x) - tx_0}{||(1-t)f(x) - tx_0||}$$

The first thing we must prove it's well defined. Suppose that $(1-t)f(x) - tx_0 = 0$. If $t = 1$, then $x_0 = 0$, an absurd given that $||x_0|| = 2$. If $t < 1$, $f(x) = \frac{t}{1-t}x_0$ and applying the norm on both sides $2 = ||f(x)|| = \frac{t}{1-t}||x_0|| = 2\frac{t}{1-t} \implies t = 1/2$. If that is the case, $f(x) - x_0 = 0 \implies f(x) = x_0$, contradiction. Moreover, for all $x$ and $t$, $||F(x,t)|| = 2 \implies F(\mathbb{S}_1, [0,1]) \subset \mathbb{S}_2$.

Now let's prove it's a homotopy:

1. $F(x,0) = 2\frac{f(x)}{||f(x)||} = f(x), \forall x \in \mathbb{S}_1$.
2. $F(x,1) = 2\frac{-x_0}{||x_0||} = -x_0, \forall x \in \mathbb{S}_1$.
3. Consider the extension of the function $F' : \mathbb{S}_1 \times [0,1] \to \mathbb{R}^3$. This function is continuous because it's a combination os continuos function. So $F$ is continuos because it's a restriction of $F'$. I needed to extend the function because $(1-t)f(x)$ is not necessary in the sphere, so I couldn't prove it's continuos. However, when extended we see each part is continuos.

By (1) - (3), we have proved $F$ is a homotopy and $f$ are homotopic to a constant function.

*Remark.* If the functions is surjective, it's harder to prove, and I couldn't yet. For instance, this is a reference[1] (but the answers use specialized tools)

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EXERCISE 14. *Show that being homotopic is a transitive relation between maps: for every triplet of maps $f, g, h : X \to Y$, if $f$, $g$ are homotopic and $g$, $h$ are homotopic, then $f$, $h$ are homotopic.*

We shall prove there exist a homotopy $H$ between $f$ and $h$. By assumption, there exists a homotopy $F$ between $f$ and $g$ and a homotopy $G$ between $g$ and $h$. Define $H : X \times [0,1] \to Y$

[1] https://math.stackexchange.com/questions/3807715/

such that

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leq t \leq 1/2 \\ G(x,2t-1), & 1/2 < t \leq 1 \end{cases}$$

that is, $H$ behaves as $F$ until it reaches a half. When that occurs, $H(x,1/2) = F(x,1) = g(x) = G(x,0)$. After that, $H$ follows $G$ until the end of the interval. So, it's clear that $H(x,0) = F(x,0) = f(x), \forall x \in X$ and $H(x,1) = G(x,1) = h(x), \forall x \in X$. Moreover, since $F$ and $G$ are continuos and in the point $t = 1/2$, both functions agree, $H$ is continuos and, therefore, $f$ and $h$ are homotopic.

---

EXERCISE 15. *Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).*

1. (*reflexive*): Consider the identity map $id : X \to X$, that is continuos. We shall prove that this function is homotopic to itself. Consider $F : X \times [0,1] \to X$ given by $F(x,t) = x$ for every $x$ and $t$. It's clear this is a homotopy because $F(x,0) = F(x,1) = x$ and it's continuos. Moreover $id \circ id = id$ by definition of identity. Therefore, there exists a homotopy equivalence between $id$ and itself. We conclude $X \approx X$.

2. (*symmetric*): Suppose $X \approx Y$. So, there exists continuos functions $f : X \to Y$ and $g : Y \to X$ that form a homotopy equivalence. This means that $g : Y \to X$ and $f : X \to Y$ are a homotopy equivalence as well. So $Y \approx X$.

3. (*transitive*): Suppose $X \approx Y$, and let $f_1 : X \to Y$ and $g_1 : Y \to X$ form a homotopy equivalence. Also suppose $Y \approx Z$ and let $f_2 : Y \to Z$ and $g_2 : Z \to Y$ form a homotopy equivalence. Define $f_3 = f_2 \circ f_1$ and $g_3 = g_1 \circ g_2$. Let's proof this is a homotopy equivalence. Both functions are continuos given that they are a composition of continuos functions.

   (a) $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1$ is homotopic to $id : X \to X$.
   Let $F_1$ be a homotopy between $g_1 \circ f_1$ and $id$ and $F_2$ a homotopy between $g_2 \circ f_2$ and $id$. Define

   $$F_3(x,t) = \begin{cases} g_1 \circ F_2(\cdot, 2t) \circ f_1(x), & 0 \leq t \leq 1/2 \\ F_1(x, 2t-1), & 1/2 < t \leq 1 \end{cases}$$

   So $F_3(x,0) = g_1(F_2(f_1(x),0)) = g_1(g_2(f_2(f_1(x)))) = g_3 \circ f_3(x)$, for every $x$ and $F_3(x,1) = F_1(x,1) = x$, for every $x$. When $t = 1/2$,

   $$F_3(x,1/2) = g_1(F_2(f_1(x),1)) = g_1(f_1(x)) = F_1(x,0)$$

   By this equality and the fact that composition of continuos functions is a continuos map, we conclude that $F_3$ is continuos. This implies that $g_3 \circ f_3$ is homotopic to the identity.

   (b) $f_3 \circ g_3 = f_1 \circ f_2 \circ g_2 \circ g_1$ is homotopic to $id : Z \to Z$.
   This follows a quite similar demonstration and can be omitted.

   By the points above $f_3$ and $g_3$ is a homotopy equivalence what proves $X \approx Z$. Consequently, homotopy equivalence is an equivalence relation.

---

EXERCISE 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

I will consider the upper case alphabet and each letter will be considered as a topological space (a subset from $\mathbb{R}^2$), for example the letter $O$ is homotopy equivalent to a circle, while $L$ is to an interval, or equivalently, a point. Observe that most of the letters are equivalent to a point, because we can think in a continuos reduction. When we have a hole, such as $A, D, R, O, P, Q$, this continuity is impossible since we'll have a point break. $B$ is a special case because we can't deform into a point without breaking points and also we cannot join the holes in one. So there is three classes, given by its representatives

1. O
2. B
3. I

| A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

| N | O | P | Q | R | S | T | U | V | W | X | Y | Z |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

# 4 Simplicial complexes

## 4.1 Important definitions

Definition 4.1.1. *The standard simplex of dimension $n$ is the following subset of $R^{n+1}$,*

$$\Delta_n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \forall i, x_i \geq 0 \text{ and } \sum_{i=1}^{n+1} x_1 = 1\}.$$

*For any collection of points $a_1, ..., a_k \in \mathbb{R}^n$, we define their convex hull as:*

$$conv(\{a_1, ..., a_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i a_i | \sum_{1 \leq i \leq k} t_i = 1, t_1, ..., t_k \geq 0 \right\}$$

Definition 4.1.2. *Let $V$ be a set (called the set of vertices). A simplicial complex over $V$ is a set $K$ of subsets of $V$ (called the simplices) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$. If $\sigma \in K$ is a simplex, its non-empty subsets $\tau \subset \sigma$ are called faces of $\sigma$, and $\sigma$ is called a coface of $\tau$. Moreover, its dimension is $|\sigma| - 1$ and the dimension of a simplicial is the maximum dimension of its simplices.*

Definition 4.1.3. *Let $K$ be a simplicial complex, with vertex $V = [\![1, n]\!]$. In $\mathbb{R}^n$, consider, for every $i \in [\![1, n]\!]$, the vector $e_i = (0, ..., 1, 0, ..., 0)$ ($i^{th}$ coordinate 1, the other ones 0). Let $|K|$ be the subset of $\mathbb{R}^n$ defined as:*

$$|K| = \bigcup_{\sigma \in K} conv(\{e_j, j \in \sigma\}).$$

*Endowed with the subspace topology, $(|K|, T_{||K|})$ is a topological space, that we call the* ***topological realization*** *of $K$.*

Definition 4.1.4. *Let $(X, \mathcal{T})$ be a topological space. A triangulation of $X$ is a simplicial complex $K$ such that its topological realization $(|K|, T_{||K|})$ is homeomorphic to $(X, \mathcal{T})$.*

Definition 4.1.5. *Let $K$ be a simplicial complex of dimension $n$. Its Euler characteristic is the integer*

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

Definition 4.1.6. *The Euler characteristic of a topological space is the Euler characteristic of any triangulation of it.*

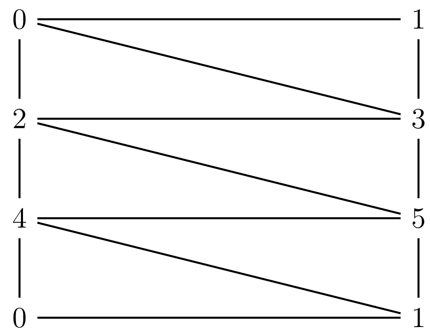
## 4.2 Exercises

Exercise 17. *Give a triangulation of the cylinder.*

We can think a triangulation of the cylinder in that following form: each circular section is mapped into a triangle graph. On the other hand, the line can be mapped to an edge. Since a cylinder can be written as $\mathbb{S}_1 \times \mathbb{R}$, the triangulation as well.

Let's write down:

$$K = \{[0, 1, 3], [0, 2, 3], [2, 3, 5], [2, 4, 5], [4.5, 1], [4, 1, 0]\}$$




EXERCISE 18. *What are the Euler characteristics of Examples 4.5 and 4.6? What is the Euler characteristic of the icosahedron?*

**Exemplo 4.5:**

$$K = \{[0], [1], [2], [3], [0,1], [1,2], [2,3], [3,0], [0,2], [1,3], [0,1,2], [0,1,3], [0,2,3], [1,2,3]\}.$$

$$\chi(K) = 4 - 6 + 4 = 2$$

**Exemplo 4.6:**

$$K = \{[0], [1], [2], [3], [0,1], [1,2], [1,3], [0,2], [2,3], [3,0], \\ [0,1,2], [0,1,3], [0,2,3], [1,2,3], [0,1,2,3]\}.$$

$$\chi(K) = 4 - 6 + 4 - 1 = 1$$

**D20:** It has 20 faces (dimension 2), 30 edges (dimension 1) and 12 vertices (dimension 0), its Euler characteristic is $12 - 30 + 20 = 2$ (Euler relation).

EXERCISE 19. *Let $K$ be a simplicial complex (with vertex set $V$). A sub-complex of $K$ is a set $M \subset K$ that is a simplicial complex. Suppose that there exists two sub-complexes $M$ and $N$ of $K$ such that $K = M \cup N$. Show the inclusion-exclusion principle:*

$$\chi(K) = \chi(M) + \chi(N) - \chi(M \cap N)$$

Denote $k_i$ the number of simplices of dimension $i$, that is, $\chi(K) = \sum_{0 \le i \le k} (-1)^i k_i$, with $k$ being its dimension. Each simplex of dimension $i$ belongs to $M$, to $N$ or both, so let, respectively,

$$k_i = k_i^M + k_i^N - k_i^{MN}.$$

Therefore,

$$\chi(K) = \sum_{0 \le i \le k} (-1)^i (k_i^M + k_i^N - k_i^{MN}) = \sum_{0 \le i \le k} (-1)^i k_i^M + \sum_{0 \le i \le k} (-1)^i k_i^N - \sum_{0 \le i \le k} (-1)^i k_i^{MN}$$

Let $m$ and $n$ be the dimension of $m$ and $n$ the dimension of $M$ and $N$, respectively. Now I shall prove $M \cap N$ is a simplicial complex. Take $\sigma \subset M \cap N$ and $\tau \subset \sigma$, then $\tau \subset \sigma \in M$ and

$\tau \subset \sigma \in N$, what implies $\tau \in M \cap N$. It proves its a simplicial complex with dimension $p$. If the dimension of $M$ is $m$, $k_i^M = 0, \forall i > m$. Suppose not. So there is a simplex in $M$ with dimension $> m$, a contraction. This holds for $N$ and $M \cap N$.

$$\chi(K) = \sum_{0 \leq i \leq k} (-1)^i (k_i^M + k_i^N - k_i^{MN}) = \sum_{0 \leq i \leq k} (-1)^i k_i^M + \sum_{0 \leq i \leq k} (-1)^i k_i^N - \sum_{0 \leq i \leq k} (-1)^i k_i^{MN}$$

We conclude that

$$\chi(K) = \sum_{0 \leq i \leq m} (-1)^i k_i^M + \sum_{0 \leq i \leq n} (-1)^i k_i^N - \sum_{0 \leq i \leq p} (-1)^i k_i^{MN} = \chi(M) + \chi(N) - \chi(M \cap N)$$

---

Exercise 20. *What is the Euler characteristic of a sphere of dimension 1? 2? 3?*

We may find the Euler characteristic of one triangulation of the sphere. So we first need to find a triangulation for the sphere $\mathbb{S}_n \subset \mathbb{R}^{n+1}$. We can think in the simplex in this space. In $\mathbb{R}^2$ it's the triangle, in $\mathbb{R}^3$ it's the tetrahedron, in $\mathbb{R}^4$ it's the 5-cell and so on. The simplex in $\mathbb{R}^{n+1}$ has $n+2$ vertices and each vertex connect to all the other. We have also $n+2$ simplices of dimension $n$, because each has $n+1$ points, that is, $\binom{n+2}{n+1} = n+2$. For each of them, we must include all its subsets.

Now we can calculate the Euler characteristic for each triangulation and therefore each sphere.

$$\chi(\mathbb{S}_1) = -3 + 3 = 0$$

$$\chi(\mathbb{S}_2) = 4 - 6 + 4 = 2$$

$$\chi(\mathbb{S}_3) = -5 + 10 - 10 + 5 = 0$$

$$\chi(\mathbb{S}_4) = 6 - 15 + 20 - 15 + 6 = 2$$

---

Exercise 21. *Using the previous exercise, show that $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are not homotopy equivalent.*

Suppose that $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are homotopy equivalent. By Example 3.15, $\mathbb{S}_{n-1}$ is homotopic equivalent to $\mathbb{R}^n - \{0\}$. By this and using the transitive property we conclude that $\mathbb{S}_2$ and $\mathbb{S}_3$ are homotopic equivalent. If that is true, we infer that they have the same Euler characteristic, what is a contraction by the last exercise. Hence $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are not homotopy equivalent.

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The computational exercises can be found in the Github[2]

[2] a