Topological Data Analysis - Exercises

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Contents

1 General topology

1.1 Important definitions

Definition 1.1.1. *A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a collection of subsets of X such that:*

1. $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
2. *for every infinite collection* $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$, *we have* $\bigcup_{\alpha \in A} O_\alpha \in \mathcal{T}$.
3. *for every finite collection* $\{O_i\}_{1 \le i \le n} \subset \mathcal{T}$, *we have* $\bigcap_{1 \le i \le n} O_i \in \mathcal{T}$.

Definition 1.1.2. *Let $x \in \mathbb{R}^n$ and $r > 0$. The open ball of center x and radius r, denoted $\mathcal{B}(x, r)$, is defined as: $\mathcal{B}(x, r) = \{y \in \mathbb{R}^n, ||x - y|| < r\}$.*

Definition 1.1.3. *Let $A \subset \mathbb{R}$ and $x \in A$. We say that A is open around x if there exists $r > 0$ such that $\mathcal{B}(x, r) \subset A$. We say that A is open if for every $x \in A$, A is open around x.*

DEFINITION 1.1.4. *Let $(X, \mathcal{T})$ be a topological space, and $Y \subset X$. We define the subspace topology on Y as the following set:*

$$T_{|Y} = \{O \cap Y, O \in \mathcal{T}\}$$

DEFINITION 1.1.5. *Let $f : X \to Y$ be a map. We say that f is continuous if for every $O \in U$, the preimage $f^{-1}(O) = \{x \in X, f(x) \in O\}$ is in $\mathcal{T}$.*

1.2 Exercises

EXERCISE 1. *Let $X = \{0, 1, 2\}$ be a set with three elements. What are the different topologies that X admits?*

As we know every Topology contains $\emptyset$ and $\{0, 1, 2\}$, so we can disconsider when writing the topologies, that is, all below contain these subsets.

- (2) Basic: $\{\emptyset, \{0, 1, 2\}\}$ - $\mathcal{P}(\{0, 1, 2\})$.
- (8) With $\{0\}$: $\{\{0\}\} - \{\{0\}, \{0, 1\}\} - \{\{0\}, \{1, 2\}\} - \{\{0\}, \{0, 2\}\} - \{\{0\}, \{0, 2\}, \{0, 1\}\}$ $\{\{0\}, \{2\}, \{0, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{1, 2\}\} - \{\{0\}, \{2\}, \{0, 2\}, \{0, 1\}\}$
- (8) With $\{1\}$: $\{\{1\}\} - \{\{1\}, \{0, 1\}\} - \{\{1\}, \{1, 2\}\} - \{\{1\}, \{0, 2\}\} - \{\{1\}, \{1, 2\}, \{0, 1\}\}$ $\{\{0\}, \{1\}, \{0, 1\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{1, 2\}\} - \{\{0\}, \{1\}, \{0, 1\}, \{0, 2\}\}$
- (8) With $\{2\}$: $\{\{2\}\} - \{\{2\}, \{0, 1\}\} - \{\{2\}, \{1, 2\}\} - \{\{2\}, \{0, 2\}\} - \{\{2\}, \{0, 2\}, \{1, 2\}\}$ $\{\{1\}, \{2\}, \{1, 2\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 1\}\} - \{\{1\}, \{2\}, \{1, 2\}, \{0, 2\}\}$
- (3) No singleton: $\{\{0, 1\}\} - \{\{1, 2\}\} - \{\{0, 2\}\}$

EXERCISE 2. *Let $\mathbb{Z}$ be the set of integers. Consider the cofinite topology $\mathcal{T}$ on $\mathbb{Z}$, defined as follows: a subset $O \subset \mathbb{Z}$ is an open set if and only if $O = \emptyset$ or ${}^cO$ is finite. Here, ${}^cO = \{x \in \mathbb{Z}, x \notin O\}$ represents the complementary of O in $\mathbb{Z}$*

1. Show that $\mathcal{T}$ is a topology on $\mathbb{Z}$.

   Let's verify the three axioms:

   (a) $\emptyset$ is an open set by definition and $\mathbb{Z}$ is open set because ${}^c\mathbb{Z} = \emptyset$ is finite.

   (b) Let $\{O_\alpha\}_{\alpha \in A} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcup_{\alpha \in A} O_\alpha\right) = \bigcap_{\alpha \in A} {}^cO_\alpha \implies {}^cO \subset {}^cO_\alpha, \forall \alpha \in A$. If $\forall \alpha, O_\alpha = \emptyset$, then ${}^cO = {}^c\emptyset \implies O = \emptyset$ and O is open. On the other hand, if there exists $\alpha \in A$ such that $O_\alpha \neq \emptyset$ we have ${}^cO_\alpha$ being finite, so is ${}^cO$, given the inclusion. We conclude O is open set.

   (c) Let $\{O_i\}_{1 \leq i \leq n} \subset \mathcal{T}$. So ${}^cO = {}^c\left(\bigcap_{1 \leq i \leq n} O_i\right) = \bigcup_{1 \leq i \leq n} {}^cO_i$. If $O_i = \emptyset$ for some $1 \leq i \leq n$, $O = \emptyset$ because of the intersection. Alternatively, if $\forall i, O_i \neq \emptyset$ we have that ${}^cO_i$ is finite and a finite union of finites is finite. We conclude that O is open set.

   By (a), (b) and (c), $\mathcal{T}$ is a topology on $\mathbb{Z}$.

2. Exhibit an sequence of open sets $\{O_n\}_{n\in\mathbb{N}} \subset \mathcal{T}$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not an open set.

   Let $O_n = {}^c\{1, ..., n\}$. Thus ${}^cO_n = \{1, ..., n\}$ is finite and

$$^c\left(\bigcap_{n\in\mathbb{N}} O_n\right) = \bigcup_{n\in\mathbb{N}} {}^cO_n = \bigcup_{n\in\mathbb{N}} \{1, ..., n\} = \mathbb{N},$$

   that is not finite. Therefore, this intersection is not an open set.

EXERCISE 3. *Let $x \in \mathbb{R}^n$, and $r > 0$. Let $y \in \mathcal{B}(x, r)$. Show that*

$$\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$$

Let $z \in \mathcal{B}(y, r - ||x - y||)$, so $||z - y|| < r - ||x - y|| \implies ||z - y|| + ||x - y|| < r$. We can conclude that, by the triangular inequality,

$$||x - z|| \leq ||x - y|| + ||z - y|| < r.$$

In that sense, $z \in \mathcal{B}(x, r)$ and $\mathcal{B}(y, r - ||x - y||) \subset \mathcal{B}(x, r)$.

Remark. In the notes, the exercise is to prove $\mathcal{B}(y, ||x - y||) \subset \mathcal{B}(x, r)$, however, this does not hold, because if we take y next the border of $\mathcal{B}(x, r)$, $||x - y|| \approx r$ and $B(y, r - \epsilon) \not\subset B(x, r)$.

EXERCISE 4. *Let $x, y \in \mathbb{R}^n$, and $r = ||x - y||$. Show that*

$$\mathcal{B}\left(\frac{x + y}{2}, \frac{r}{2}\right) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$$

Denote $m = \dfrac{x + y}{2}$. Take $z \in \mathcal{B}\left(m, \dfrac{r}{2}\right)$. Thus, using the triangular inequality,

$$||x - z|| \leq ||x - m|| + ||m - z|| = \frac{1}{2}||x - y|| + ||m - z|| < r/2 + r/2 = r$$

$$||y - z|| \leq ||y - m|| + ||m - z|| = \frac{1}{2}||y - x|| + ||m - z|| < r/2 + r/2 = r$$

So $z \in \mathcal{B}(x, r)$, $z \in \mathcal{B}(y, r)$ and $z \in \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$. Therefore $\mathcal{B}(m, \frac{r}{2}) \subset \mathcal{B}(x, r) \cap \mathcal{B}(y, r)$.

EXERCISE 5. *Show that the open balls $\mathcal{B}(x, r)$ of R^n are open sets (with respect to the Euclidean topology).*

We have to prove that for every $y \in \mathcal{B}(x, r)$, there exists $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. Put $\epsilon = r - ||x - y||$. As we have proved in exercise 3, $\mathcal{B}(y, \epsilon) \subset \mathcal{B}(x, r)$. So $\mathcal{B}(x, r)$ is open set.

EXERCISE 6. *Consider $X = \mathbb{R}$ endowed with the Euclidean topology. Are the following sets open? Are they closed?*

1. $[0, 1]$. It's not open set because for every $\epsilon > 0, \mathcal{B}(0, \epsilon) = (-\epsilon, \epsilon) \not\subset [0, 1]$. It's closed because $[0, 1]^c = (-\infty, 0) \cup (1, \infty)$ is an union of two open sets, as we prove in item 3.

2. $[0,1)$. It's not open for the same reason as before. It's not closed because $B(1,\epsilon) = (1-\epsilon, 1+\epsilon) \not\subset (-\infty, 0) \cup [1, \infty]$.

3. $(-\infty, 1)$. It's open because: take $x < 1$. Put $r = 1 - x$ and take $z \in \mathcal{B}(x, r)$. If $z > x$, $|x - z| < 1 - x \implies z < 1$. If $z < x$, it follows $z < 1$. It proves $z < 1$ and $(-\infty, 1)$ is open. It's not closed cause $\forall \epsilon > 0, \mathcal{B}(1, \epsilon) \not\subset [1, \infty)$.

4. the singletons. It's not open cause $\forall \epsilon > 0, x+\epsilon/2 \in \mathcal{B}(x, \epsilon)$. It's close cause $(-\infty, x) \cup (x, \infty)$ is union of open sets.

5. $\mathbb{Q}$. It's not open because for every open ball around a rational, there are irrationals, that is, let $x \in \mathbb{Q}$ and take $\epsilon > 0$, then there exists $y \in (\mathbb{R} - \mathbb{Q}) \cap \mathcal{B}(x, \epsilon)$. It's not closed for the same reason, for every irrational, there is rationals for every open ball.

   Remark. We shall prove the rationals are dense in the reals. Let $x \in \mathbb{Q}$ and $\epsilon > 0$. If ϵ is irrational, take $x - \epsilon/2 \subset (x - \epsilon, x + \epsilon)$. Suppose $x - \epsilon/2$ is rational, then $\frac{2x-\epsilon}{2} = \frac{m}{n}$ for some integers m and n, that is, $2x - \epsilon = 2m/n$ and $\epsilon = 2(x - \frac{m}{n}) \in \mathbb{Q}$, contradiction. So there is an irrational in $\mathcal{B}(x, \epsilon)$. If ϵ is rational, consider

   $$y = \frac{1}{\sqrt{2}}(x - \epsilon) + (1 - \frac{1}{\sqrt{2}})(x + \epsilon) = (x + \epsilon) - \epsilon\sqrt{2}$$

   That is a convex combination, so $y \in \mathcal{B}(x, \epsilon)$. Moreover, y is irrational, with a similar proof by contradiction. This proves the statement.

   On the other hand, we must prove for every two irrationals (a, b), there is a rational between them. Denote $c = b - a > 0$. Let $n \in \mathbb{N}$ such that $n > \frac{1}{c} \implies cn > 1 \implies (bn - an) > 1$. So exists $m \in (an, bn) \implies m/n \in (a, b)$. This proves the second statement.

EXERCISE 7. *A map is continuous if and only if the preimage of closed sets are closed sets.*

First we shall prove that $f^{-1}({}^cA) = {}^c(f^{-1}(A))$. Let's prove the double inclusion. Take $x \in f^{-1}({}^cA)$. So there exists $y \in {}^cA$ such that $f(x) = y$. Suppose that $x \in f^{-1}(A)$. It implies the existence of $z \in A$ such that $y = f(x) = z$, absurd. So $x \in {}^c(f^{-1}(A))$.

Now take $x \in {}^c(f^{-1}(A))$. Therefore, $\forall y \in A, f(x) \neq y$. In that case, $f(x) \in {}^cA \implies x \in f^{-1}({}^cA)$. Then we have showed the equality.

Now let's prove the equivalence. Suppose f is a continuos map and take a closed set F. We shall prove that $f^{-1}(F)$ is closed. Well, ${}^c(f^{-1}(F)) = f^{-1}({}^cF)$ is open, because ${}^cF$ is open, by the continuity. We conclude that $f^{-1}(F)$ is closed.

Suppose that for every closed set F, we have $f^{-1}(F)$ being closed. We will use that A is open if ${}^cA$ is closed. This is true because ${}^c({}^c(A)) = A$. Take an open set A. ${}^c(f^{-1}(A)) = f^{-1}({}^cA)$ is closed, because ${}^cA$ is. Thus $f^{-1}(A)$ is open and we have proved the continuity of f.

2 Homeomorphisms

2.1 Important definitions

DEFINITION 2.1.1. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be two topological spaces, and $f : X \to Y$ a map. We say that f is a homeomorphism if*

1. *f is a bijection,*

2. *$f : X \to Y$ is continuos,*

3. *$f^{-1} : Y \to X$ is continuos.*

If there exists such a homeomorphism, we say that the two topological spaces are homeomorphic.

DEFINITION 2.1.2. *Let $(X, \mathcal{T})$ be a topological space. We say that X is connected if for every open sets $O, O' \in \mathcal{T}$ such that $O \cap O' = \emptyset$ (i.e., they are disjoint), we have*

$$X = O \cup O' \implies O = \emptyset \text{ or } O' = \emptyset.$$

DEFINITION 2.1.3. *Let $(X, \mathcal{T})$ be a topological space. Suppose that there exists a collection of n* ***non-empty, disjoint*** *and* ***connected open sets*** *$(O_1, ..., O_n)$ such that*

$$\bigcup_{1 \leq i \leq n} O_i = X.$$

Then we say that X admits n connected components.

DEFINITION 2.1.4. *Let $(X, \mathcal{T})$ be a topological space, and $n \geq 0$. We say that it has dimension n if the following is true: for every $x \in X$, there exists an open set O such that $x \in O$, and a homeomorphism $O \to \mathbb{R}^n$.*

2.2 Exercises

EXERCISE 8. *Show that the topological spaces $\mathbb{R}^n$ and $\mathcal{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.*

Let $f : \mathcal{B}(0,1) \to \mathbb{R}^n$ be defined as $f(x) = \dfrac{x}{1 - ||x||}$. I observe it's well defined because $||x|| < 1$. We shall prove f is a homeomorphism.

1. **Injective:** Take $x, y \in \mathcal{B}(0,1)$ and suppose that

$$\frac{x}{1 - ||x||} = \frac{y}{1 - ||y||}.$$

Applying the norm in both sides, we obtain the equation

$$||x||(1 - ||y|||) = ||y||(1 - ||x||) \implies ||x|| = ||y||.$$

On the other side x and y points to the same direction, given that

$$y = \frac{1 - ||y||}{1 - ||x||} x = \alpha x,$$

with $\alpha = 1$ because of the same norm. We conclude $x = y$.

2. **Surjective:** Take $y \in \mathbb{R}^n$. We shall prove that there exists $x \in \mathcal{B}(0,1)$ such that $f(x) = y$, that is,
$$\frac{x}{1 - ||x||} = y$$
Applying the norm we observe that if that is true, $||x|| = ||y|| - ||y||||x|| \implies ||x|| = \frac{||y||}{1+||y||}$. And $x = (1 - ||x||)y = \frac{1}{1+||y||}y$. We conclude that for every $y \in \mathbb{R}^n$, if we take $x = \frac{y}{1+||y||}$,
$$f(x) = \frac{y/(1 + ||y||)}{1 - ||y||/(1 + ||y||)} = y$$

3. **Continuity of f:** Consider an open set $A \subset \mathbb{R}^n$. Let $B = f^{-1}(A)$. We shall prove B is open, that is, for every $x \in B$, exists $r > 0$ such that $\mathcal{B}(x, r) \subset B$. Take $x = f^{-1}(y) \in B$. Because A is open, there is $\epsilon > 0$ such that $\mathcal{B}(y, \epsilon) \subset A$. Take δ such that
$$\frac{\delta}{1 - ||x|| - \delta}(1 + ||y||) < \epsilon$$
and $z = f^{-1}(w) \in \mathcal{B}(x, \delta)$.
$$\begin{aligned}
||y - w|| &= \left|\left|\frac{x}{1 - ||x||} - \frac{z}{1 - ||z||}\right|\right| = \frac{1}{1 - ||x||}\left|\left|x - \frac{1 - ||x||}{1 - ||z||}z\right|\right| \\
&= \frac{1}{1 - ||x||}\left|\left|x - z + z - \frac{1 - ||x||}{1 - ||z||}z\right|\right| \\
&\leq \frac{||x - z||}{1 - ||x||} + \frac{1}{1 - ||x||}\left(1 - \frac{1 - ||x||}{1 - ||z||}||z||\right) \\
&= \frac{||x - z||}{1 - ||x||} + \frac{||z||}{1 - ||x||}\frac{||x|| - ||z||}{1 - ||z||} \\
&\leq \frac{1}{1 - ||x||}||x - z||(1 + ||w||) \\
&\leq \frac{1}{1 - ||x||}||x - z||(1 + ||y - w|| + ||y||) \\
\implies ||y - w|| &\leq \frac{||x - z||}{1 - ||x|| - ||x - z||}(1 + ||y||) \\
&< \frac{\delta}{1 - ||x|| - \delta}(1 + ||y||) < \epsilon
\end{aligned}$$
So $w \in \mathcal{B}(y, \epsilon) \subset A \implies z \in B$, what proves B is open. It concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by
$$f^{-1}(y) = \frac{y}{1 + ||y||}$$
The demonstration is quite similar to the previous item, given that the only difference is the signal.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0, 1) \simeq \mathbb{R}^n$.

EXERCISE 9. *Show that $\mathcal{B}(x, r)$ and $\mathcal{B}(y, s)$ are homeomorphic.*

Consider the function $f : \mathcal{B}(0, 1) \to \mathcal{B}(c, r)$ given by $f(x) = r \cdot x + c$. Let's prove f is a homeomorphism.

1. **Injective:** If $x, y \in \mathcal{B}(0,1)$ and $rx + c = ry + c \implies x = y$, because $r > 0$ by definition. So f is injective.

2. **Surjective:** Let $y \in \mathcal{B}(c,r)$ and $x = (y-c)/r$. So $||x|| = ||y-c||/r < 1$, by definition. So $x \in \mathcal{B}(0,1)$ and $f(x) = y$ what proves this function is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{B}(c,r)$ open set and denote $B = f^{-1}(A)$. Take $x = f^{-1}(y) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}(y,\epsilon) \subset A$. Define $\delta = \epsilon/r$ and take $z = f^{-1}(w) \in \mathcal{B}(x,\delta)$.
$$||y - w|| = ||rx + c - (rz + c)|| = r||x - z|| < r\delta = \epsilon$$
Therefore $w \in \mathcal{B}(y,\epsilon) \subset A \implies z \in B$. So $\mathcal{B}(x,\delta) \subset B$, what proves B is open. This concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by
$$f^{-1}(y) = \frac{y - c}{r}$$
This function is continuos for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathcal{B}(0,1) \simeq \mathcal{B}(c,r)$. Since this is an equivalence relation, we have that

$$\mathcal{B}(0,1) \simeq \mathcal{B}(x,r) \text{ and } \mathcal{B}(0,1) \simeq \mathcal{B}(y,s) \text{ implies } \mathcal{B}(x,r) \simeq \mathcal{B}(y,s).$$

Exercise 10. *Show that $\mathbb{S}(0,1)$, the unit circle of $\mathbb{R}^2$, is homeomorphic to the ellipse*

$$\mathcal{S}(a,b) = \left\{(x,y) \in \mathbb{R}^2, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1\right\},$$

for any $a, b > 0$.

Consider the function $f : \mathbb{S}(0,1) \to \mathcal{S}(a,b)$ defined as $f(x,y) = (ax, by)$. Let's prove it is a homeomorphism.

1. **Injective:** Let $(x_1,y_1),(x_2,y_2) \in \mathbb{S}(0,1)$ such that $(ax_1, by_1) = (ax_2, by_2)$. Since $a, b > 0$, we have $x_1 = x_2$ and $y_1 = y_2$. It proves f is injective.

2. **Surjective:** Let $(z,w) \in \mathcal{S}(a,b)$ and $(x,y) = \left(\frac{z}{a}, \frac{w}{b}\right)$. It's clear that $f(x,y) = (z,w)$ and $x^2 + y^2 = \frac{z^2}{a^2} + \frac{w^2}{b^2} = 1$, so $(x,y) \in \mathbb{S}(0,1)$. It proves f is surjective.

3. **Continuity of f:** Let $A \subset \mathcal{S}(a,b)$ open set and denote $B = f^{-1}(A)$. Take $(x,y) = f^{-1}(z,w) \in B$. We know there exists $\epsilon > 0$ such that $\mathcal{B}((z,w),\epsilon) \subset A$. Put δ as defined below and take $(x',y') = f^{-1}((z',w')) \in \mathcal{B}((x,y),\delta)$. Consider the norm 1

$$\begin{aligned} ||(z',w') - (z,w)||_1 &= ||(ax',by') - (ax,by)||_1 = ||\left(a(x'-x), b(y'-y)\right)||_1 \\ &= a|x'-x| + b|y'-y|, \text{ define } c = \max\{a,b\} \\ &\leq c(|x'-x| + |y'-y|) = c||(x'-x, y'-y)||_1 \end{aligned}$$

By the equivalente of the norms, there exists constants k_1, k_2 such that

$$||(z', w') - (z, w)|| \leq k_1||(z', w') - (z, w)||_1 \leq ck_1||(x' - x, y' - y)||_1 \leq ck_1k_2||(x' - x, y' - y)||$$

Then we need $\delta = \frac{\epsilon}{ck_1k_2}$ in order to prove that $(z', w') \in \mathcal{B}((z, w), \epsilon) \subset A \implies (x', y') \in B$. So $\mathcal{B}((x, y), \delta) \subset B$, what proves B is open. This concludes the continuity of f.

4. **Continuity of f^{-1}:** The inverse is given by

$$f^{-1}((z, w)) = (z/a, w/b)$$

This function is continuos for the same argument as before.

By items (1) - (4), we conclude f is a homeomorphism and $\mathbb{S}(0, 1) \simeq \mathcal{S}(a, b)$.

Exercise 11. *Show that* $[0, 1)$ *and* $(0, 1)$ *are not homeomorphic.*

We shall prove by contradiction. Suppose these exists a homeomorphism $f : [0, 1) \to (0, 1)$. Let $0 < z = f(0) < 1$ and define the following function

$$\begin{aligned} g : (0, 1) &\to (0, z) \cup (z, 1) \\ x &\mapsto g(x) = f(x) \end{aligned}$$

This function is well defined given that z is not image of other point but 0. The function is injective because if $g(y) = g(x) \implies f(y) = f(x) \implies x = y$, given that f is injective. This function is also surjective since f is and $0 < w < 1$ and $w \neq z$, it's clear that $f(0) \neq w$. As g is an induced map of a continuos function, by Proposition 1.21 from the notes, it's continuos and so is its inverse. We conclude g is a homeomorphism.

Now I will prove that $(0, 1)$ admits only 1 connected component, that is, it's connected. Suppose it's not and there exists $O, O' \subset (0, 1)$ open disjoint sets such that $(0, 1) = O \cup O'$ and none of them are empty sets. Let $a \in O, b \in O'$ with $a < b$ without loss of generality. Define $\alpha = \sup\{x \in \mathbb{R} : [a, x) \subset O\}$. It's well defined because this set is not empty, given O is open and b is an upper bound. Then $\alpha \leq b$. Suppose $\alpha \in O'$, then there exists $r > 0$ such that $(\alpha - r, \alpha + r) \subset O'$. We know that for every $\epsilon > 0$, there exists $w \in (\alpha - \epsilon, \alpha]$ such that $[a, w) \subset O$. That is a contradiction since there exists $w \in (\alpha - r, \alpha)$ such that $[a, w) \subset O$. So $\alpha \in O \implies (\alpha - r, \alpha + r) \subset O$, for some r. We infer that $[a, \alpha + r) \subset O$, what is an absurd. Therefore $(0, 1)$ is connected.

For a similar argument, we prove that $(0, z)$ and $(z, 1)$ are connected. This implies that the union admits 2 connected components.

In that sense, we have a homeomorphism between a topological space with 1 connected component and other with 2 connected components, what is a contradiction by Proposition 2.14 from the notes. We conclude that $[0, 1)$ and $(0, 1)$ are not homeomorphic.

3 Homotopies

3.1 Important definitions

DEFINITION 3.1.1. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be two topological spaces, and* $f, g : X \to Y$ *two continuous maps. A homotopy between* f *and* g *is a map* $F : X \times [0,1] \to Y$ *such that:*

1. $F(\cdot, 0)$ *is equal to* f*,*
2. $F(\cdot, 1)$ *is equal to* g*,*
3. $F : X \times [0,1] \to Y$ *is continuous.*

If such a homotopy exists, we say that the maps f *and* g *are homotopic.*

Remark. Before asking for $F : X \times [0,1] \to Y$ to be continuous, we have to give $X \times [0,1]$ a topology. The topology we choose is the product topology. Consider the topological space $(X, \mathcal{T})$, and endow $[0,1]$ with the subspace topology of $\mathbb{R}$, denoted $T_{|[0,1]}$. The product topology on $X \times [0,1]$, denoted $T \otimes T_{|[0,1]}$, is defined as follows: a set $O \subset X \times [0,1]$ is open if and only if it can be written as a union $U_{\alpha \in A} O_\alpha \times O'_\alpha$ where every O_α is an open set of X and O'_α is an open set of $[0,1]$.

DEFINITION 3.1.2. *Let* $(X, \mathcal{T})$ *and* $(Y, \mathcal{U})$ *be two topological spaces. A homotopy equivalence between* X *and* Y *is a pair of continuous maps* $f : X \to Y$ *and* $g : Y \to X$ *such that:*

1. $g \circ f : X \to X$ *is homotopic to the identity map* $id : X \to X$*,*
2. $f \circ g : Y \to Y$ *is homotopic to the identity map* $id : Y \to Y$*,*

If such a homotopy equivalence exists, we say that X *and* Y *are homotopy equivalent.*

DEFINITION 3.1.3. *Let* $(X, \mathcal{T})$ *be a topological space and* $Y \subset X$ *a subset, endowed with the subspace topology* $T_{|Y}$*. A retraction is a continuous map* $r : X \to Y$ *such that* $\forall y \in Y, r(y) = y$*.*

A deformation retraction is a homotopy $F : X \times [0,1] \to Y$ *between the identity map* $id : X \to X$ *and a retraction* $r : X \to Y$*.*

3.2 Exercises

EXERCISE 12. *Let* $f : \mathbb{R}^n \to X$ *be a continuous map. Then* f *is homotopic to a constant map.*

I must prove that there exists a homotopy between f and a constant map. Consider the function $F : \mathbb{R}^n \times [0,1] \to X$ defined as

$$F(x, t) = f(tx)$$

It's clear that $F(x, 0) = f(0)$, for every $x \in \mathbb{R}^n$. So it's the constant map $f(0)$. We also have that $F(x, 1) = f(x), \forall x \in \mathbb{R}^n$. Moreover, let's prove F is continuos. Denote $F' : \mathbb{R}^n \times \mathbb{R} \to X$ the function $F'(x, t) = f(xt)$ and $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ the function $g(x, t) = xt$. So $F' = f \circ g$.

Let's prove g is a continuous function. As we are dealing with a real-valued function, by Proposition 1.19 from the notes, I can use the $\epsilon - \delta$ proof. Let $(x, t) \in \mathbb{R}^{n+1}$ and $\epsilon > 0$. In the proof I use the norm 1, without loss of generality because of the equivalence of norms in $\mathbb{R}^n$. Put

$\delta = \min\{1, \frac{\epsilon}{\max\{||x||,|t|+1\}}\}$ and suppose $||(x,t)-(x',t')|| = ||x-x'|| + |t-t'| < \delta$. So,

$$\begin{aligned} ||xt - x't'|| &= ||xt - xt' + xt' - x't'|| \\ &\leq |t-t'|||x|| + |t'|||x-x'|| \\ &\leq |t-t'|||x|| + (|t|+\delta)||x-x'|| \\ &< \max\{||x||, |t|+\delta\}\delta \\ &\leq \max\{||x||, |t|+1\}\delta \leq \epsilon \end{aligned}$$

By this, g is a continuos function. Since f is also continuos, the composition F' is also continuos, by Proposition 1.18. By Proposition 1.21, when we endow F' in $\mathbb{R}^n \times [0,1]$, we obtain a continuos function, that is F is continuos. Then we conclude that f is homotopic to a constant function.

Exercise 13. *Let $f : \mathbb{S}_1 \to \mathbb{S}_2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map where the unit sphere $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n, ||x|| = 1\}$.*

Let $x_0 \in \mathbb{S}_2$ such that $x_0 \notin f(\mathbb{S}_1)$ and consider the constant map $g(x) = -x_0$, for every $x \in \mathbb{S}_1$. Let $F : \mathbb{S}_1 \times [0,1] \to \mathbb{S}_2$ be defined as

$$F(x,t) = \frac{(1-t)f(x) - tx_0}{||(1-t)f(x) - tx_0||}$$

The first thing we must prove it's well defined. Suppose that $(1-t)f(x) - tx_0 = 0$. If $t = 1$, then $x_0 = 0$, an absurd given that $||x_0|| = 1$. If $t < 1$, $f(x) = \frac{t}{1-t}x_0$ and applying the norm on both sides $1 = ||f(x)|| = \frac{t}{1-t}||x_0|| = \frac{t}{1-t} \implies t = 1/2$. If that is the case, $f(x) - x_0 = 0 \implies f(x) = x_0$, contradiction. Moreover, for all x and t, $||F(x,t)|| = 1 \implies F(\mathbb{S}_1, [0,1]) \subset \mathbb{S}_2$.

Now let's prove it's a homotopy:

1. $F(x,0) = \frac{f(x)}{||f(x)||} = f(x), \forall x \in \mathbb{S}_1$.
2. $F(x,1) = \frac{-x_0}{||x_0||} = -x_0, \forall x \in \mathbb{S}_1$.
3. Consider the extension of the function $F' : \mathbb{S}_1 \times [0,1] \to \mathbb{R}^3$. This function is continuous because it's a combination os continuos function. So F is continuos because it's a restriction of F'. I needed to extend the function because $(1-t)f(x)$ is not necessary in the sphere, so I couldn't prove it's continuos. However, when extended we see each part is continuos.

By (1) - (3), we have proved F is a homotopy and f are homotopic to a constant function.

Remark. If the function is not surjective, it's harder to prove, and I couldn't yet. For instance, this is a reference[1] (but the answers use specialized tools)

Exercise 14. *Show that being homotopic is a transitive relation between maps: for every triplet of maps $f, g, h : X \to Y$, if f, g are homotopic and g, h are homotopic, then f, h are homotopic.*

We shall prove there exist a homotopy H between f and h. By assumption, there exists a homotopy F between f and g and a homotopy G between g and h. Define $H : X \times [0,1] \to Y$

[1] https://math.stackexchange.com/questions/3807715/

such that

$$H(x,t) = \begin{cases} F(x,2t), & 0 \leq t \leq 1/2 \\ G(x,2t-1), & 1/2 < t \leq 1 \end{cases}$$

that is, H behaves as F until it reaches a half. When that occurs, $H(x,1/2) = F(x,1) = g(x) = G(x,0)$. After that, H follows G until the end of the interval. So, it's clear that $H(x,0) = F(x,0) = f(x), \forall x \in X$ and $H(x,1) = G(x,1) = h(x), \forall x \in X$. Moreover, let's prove H is continuos. Let $A \subset Y$ closed set. So

$$\begin{aligned} H^{-1}(A) &= \{(x,t) : F(x,2t) \in A, t \leq 1/2\} \cup \{(x,t) : G(x,2t-1) \in A, t \geq 1/2\} \\ &= \tilde{F}^{-1}(A) \cup \tilde{G}^{-1}(A), \end{aligned}$$

where $\tilde{F}(x,t) = F(x,2t)$ and $\tilde{G}(x,t) = G(x,2t-1)$, with domain being, respectively, $X \times [0,1/2]$ and $X \times [1/2,1]$. As we see in the following remark, both functions are continuos, so $\tilde{F}^{-1}(A) \cup \tilde{G}^{-1}(A)$ is closed, what proves H is continuos and, therefore, f and h are homotopic.

Remark. The map $(x,t) \mapsto (x,2t)$ is continuos. In order to see that, take $A \subset X \times [0,1]$ open. So it can be written as $\bigcup_a O_a \times I_a$, where $I_a \subset [0,1]$ is an open interval and $O_a \subset X$ is open. The pre-image of A is given by $\bigcup_a O_a \times \frac{1}{2} I_a$, that is still open. With that and using the fact that combination of continuos functions is continuos, $\tilde{F}$ is continuos. For a similar argument, $\tilde{G}$ is continuos.

EXERCISE 15. *Show that being homotopy equivalent is an equivalence relation (reflexive, symmetric and transitive).*

1. (*reflexive*): Consider the identity map $id : X \to X$, that is continuos. We shall prove that this function is homotopic to itself. Consider $F : X \times [0,1] \to X$ given by $F(x,t) = x$ for every x and t. It's clear this is a homotopy because $F(x,0) = F(x,1) = x$ and it's continuos. Moreover $id \circ id = id$ by definition of identity. Therefore, there exists a homotopy equivalence between id and itself. We conclude $X \approx X$.

2. (*symmetric*): Suppose $X \approx Y$. So, there exists continuos functions $f : X \to Y$ and $g : Y \to X$ that form a homotopy equivalence. This means that $g : Y \to X$ and $f : X \to Y$ are a homotopy equivalence as well. So $Y \approx X$.

3. (*transitive*): Suppose $X \approx Y$, and let $f_1 : X \to Y$ and $g_1 : Y \to X$ form a homotopy equivalence. Also suppose $Y \approx Z$ and let $f_2 : Y \to Z$ and $g_2 : Z \to Y$ form a homotopy equivalence. Define $f_3 = f_2 \circ f_1$ and $g_3 = g_1 \circ g_2$. Let's proof this is a homotopy equivalence. Both functions are continuos given that they are a composition of continuos functions.

   (a) $g_3 \circ f_3 = g_1 \circ g_2 \circ f_2 \circ f_1$ is homotopic to $id : X \to X$.
   Let F_1 be a homotopy between $g_1 \circ f_1$ and id and F_2 a homotopy between $g_2 \circ f_2$ and id. Define

   $$F_3(x,t) = \begin{cases} g_1 \circ F_2(\cdot, 2t) \circ f_1(x), & 0 \leq t \leq 1/2 \\ F_1(x, 2t-1), & 1/2 < t \leq 1 \end{cases}$$

   So $F_3(x,0) = g_1(F_2(f_1(x),0)) = g_1(g_2(f_2(f_1(x)))) = g_3 \circ f_3(x)$, for every x and

$F_3(x,1) = F_1(x,1) = x$, for every x. When $t = 1/2$,

$$F_3(x,1/2) = g_1(F_2(f_1(x),1)) = g_1(f_1(x)) = F_1(x,0)$$

F_3 is continuos by a similar proof written in the last exercise. This implies that $g_3 \circ f_3$ is homotopic to the identity.

(b) $f_3 \circ g_3 = f_1 \circ f_2 \circ g_2 \circ g_1$ is homotopic to $id : Z \to Z$.
This follows a quite similar demonstration and can be omitted.

By the points above f_3 and g_3 is a homotopy equivalence what proves $X \approx Z$. Consequently, homotopy equivalence is an equivalence relation.

Exercise 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

I will consider the upper case alphabet and each letter will be considered as a topological space (a subset from $\mathbb{R}^2$), for example the letter O is homotopy equivalent to a circle, while L is to an interval, or equivalently, a point. Observe that most of the letters are equivalent to a point, because we can think in a continuos reduction. When we have a hole, such as A, D, R, O, P, Q, this continuity is impossible since we'll have a point break. B is a special case because we can't deform into a point without breaking points and also we cannot join the holes in one. So there is three classes, given by its representatives

1. O
2. B
3. I

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	1	3	3	3	3	3	3	3	3	3

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
3	1	1	1	1	3	3	3	3	3	3	3	3

4 Simplicial complexes

4.1 Important definitions

DEFINITION 4.1.1. *The standard simplex of dimension n is the following subset of R^{n+1},*

$$\Delta_n = \{x = (x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} | \forall i, x_i \geq 0 \text{ and } \sum_{i=1}^{n+1} x_1 = 1\}.$$

For any collection of points $a_1, ..., a_k \in \mathbb{R}^n$, we define their convex hull as:

$$conv(\{a_1, ..., a_k\}) = \left\{ \sum_{1 \leq i \leq k} t_i a_i | \sum_{1 \leq i \leq k} t_i = 1, t_1, ..., t_k \geq 0 \right\}$$

DEFINITION 4.1.2. *Let V be a set (called the set of vertices). A simplicial complex over V is a set K of subsets of V (called the simplices) such that, for every $\sigma \in K$ and every non-empty $\tau \subset \sigma$, we have $\tau \in K$. If $\sigma \in K$ is a simplex, its non-empty subsets $\tau \subset \sigma$ are called faces of σ, and σ is called a coface of τ. Moreover, its dimension is $|\sigma| - 1$ and the dimension of a simplicial is the maximum dimension of its simplices.*

DEFINITION 4.1.3. *Let K be a simplicial complex, with vertex $V = [\![1, n]\!]$. In $\mathbb{R}^n$, consider, for every $i \in [\![1, n]\!]$, the vector $e_i = (0, ..., 1, 0, ..., 0)$ (i^{th} coordinate 1, the other ones 0). Let $|K|$ be the subset of $\mathbb{R}^n$ defined as:*

$$|K| = \bigcup_{\sigma \in K} conv(\{e_j, j \in \sigma\}).$$

*Endowed with the subspace topology, $(|K|, T_{||K|})$ is a topological space, that we call the **topological realization** of K.*

DEFINITION 4.1.4. *Let $(X, \mathcal{T})$ be a topological space. A triangulation of X is a simplicial complex K such that its topological realization $(|K|, T_{||K|})$ is homeomorphic to $(X, \mathcal{T})$.*

DEFINITION 4.1.5. *Let K be a simplicial complex of dimension n. Its Euler characteristic is the integer*

$$\chi(K) = \sum_{0 \leq i \leq n} (-1)^i \cdot (\text{number of simplices of dimension } i).$$

DEFINITION 4.1.6. *The Euler characteristic of a topological space is the Euler characteristic of any triangulation of it.*

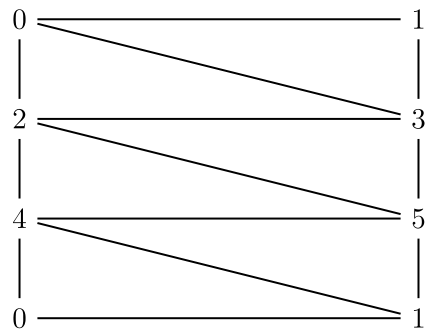
4.2 Exercises

EXERCISE 17. *Give a triangulation of the cylinder.*

We can think a triangulation of the cylinder in that following form: each circular section is mapped into a triangle graph. On the other hand, the line can be mapped to an edge. Since a cylinder can be written as $\mathbb{S}_1 \times \mathbb{R}$, the triangulation as well.

Let's write down:

$$K = \{[0, 1, 3], [0, 2, 3], [2, 3, 5], [2, 4, 5], [4.5, 1], [4, 1, 0]\}$$



EXERCISE 18. *What are the Euler characteristics of Examples 4.5 and 4.6? What is the Euler characteristic of the icosahedron?*

Exemplo 4.5:

$$K = \{[0],[1],[2],[3],[0,1],[1,2],[2,3],[3,0],[0,2],[1,3],[0,1,2],[0,1,3],[0,2,3],[1,2,3]\}.$$

$$\chi(K) = 4 - 6 + 4 = 2$$

Exemplo 4.6:

$$K = \{[0],[1],[2],[3],[0,1],[1,2],[1,3],[0,2],[2,3],[3,0], \\ [0,1,2],[0,1,3],[0,2,3],[1,2,3],[0,1,2,3]\}.$$

$$\chi(K) = 4 - 6 + 4 - 1 = 1$$

D20: It has 20 faces (dimension 2), 30 edges (dimension 1) and 12 vertices (dimension 0), its Euler characteristic is $12 - 30 + 20 = 2$ (Euler relation).

EXERCISE 19. *Let K be a simplicial complex (with vertex set V). A sub-complex of K is a set $M \subset K$ that is a simplicial complex. Suppose that there exists two sub-complexes M and N of K such that $K = M \cup N$. Show the inclusion-exclusion principle:*

$$\chi(K) = \chi(M) + \chi(N) - \chi(M \cap N)$$

Denote k_i the number of simplices of dimension i, that is, $\chi(K) = \sum_{0\leq i\leq k}(-1)^i k_i$, with k being its dimension. Each simplex of dimension i belongs to M, to N or both. Denote k_i^M, k_i^N and k_i^{MN} the number od simplices of dimension i in M, N and $M \cap N$ respectively. So

$$k_i = k_i^M + k_i^N - k_i^{MN}.$$

Therefore,

$$\chi(K) = \sum_{0\leq i\leq k}(-1)^i(k_i^M + k_i^N - k_i^{MN}) = \sum_{0\leq i\leq k}(-1)^i k_i^M + \sum_{0\leq i\leq k}(-1)^i k_i^N - \sum_{0\leq i\leq k}(-1)^i k_i^{MN}$$

Let m and n be the dimension of M and N, respectively. Now I shall prove $M \cap N$ is a simplicial complex. Take $\sigma \subset M \cap N$ and $\tau \subset \sigma$, then $\tau \subset \sigma \in M$ and $\tau \subset \sigma \in N$, what implies $\tau \in M \cap N$. It proves its a simplicial complex with dimension p. We know the dimension of M is m, so it implies that for all $i > m$, $k_i^M = 0$. Suppose not and let $k_j^M > 0$ for some $j > m$, we would have a simplex with dimension greater or equal than m. This is a contradiction, because the the dimension of M is the maximum dimension of its simplices. This holds for N and $M \cap N$.

$$\chi(K) = \sum_{0 \leq i \leq k} (-1)^i (k_i^M + k_i^N - k_i^{MN}) = \sum_{0 \leq i \leq k} (-1)^i k_i^M + \sum_{0 \leq i \leq k} (-1)^i k_i^N - \sum_{0 \leq i \leq k} (-1)^i k_i^{MN}$$

We conclude that

$$\chi(K) = \sum_{0 \leq i \leq m} (-1)^i k_i^M + \sum_{0 \leq i \leq n} (-1)^i k_i^N - \sum_{0 \leq i \leq p} (-1)^i k_i^{MN} = \chi(M) + \chi(N) - \chi(M \cap N)$$

EXERCISE 20. *What is the Euler characteristic of a sphere of dimension 1? 2? 3?*

We may find the Euler characteristic of one triangulation of the sphere. So we first need to find a triangulation for the sphere $\mathbb{S}_n \subset \mathbb{R}^{n+1}$. We can think in the simplex in this space. In $\mathbb{R}^2$ it's the triangle, in $\mathbb{R}^3$ it's the tetrahedron, in $\mathbb{R}^4$ it's the 5-cell and so on. The simplex in $\mathbb{R}^{n+1}$ has $n+2$ vertices and each vertex connect to all the other. We have also $n+2$ simplices of dimension n, because each has $n+1$ points, that is, $\binom{n+2}{n+1} = n+2$. For each of them, we must include all its subsets.

Now we can calculate the Euler characteristic for each triangulation and therefore each sphere.

$$\chi(\mathbb{S}_1) = -3 + 3 = 0$$

$$\chi(\mathbb{S}_2) = 4 - 6 + 4 = 2$$

$$\chi(\mathbb{S}_3) = -5 + 10 - 10 + 5 = 0$$

$$\chi(\mathbb{S}_4) = 6 - 15 + 20 - 15 + 6 = 2$$

EXERCISE 21. *Using the previous exercise, show that $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are not homotopy equivalent.*

Suppose that $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are homotopy equivalent. By Example 3.15, $\mathbb{S}_{n-1}$ is homotopic equivalent to $\mathbb{R}^n - \{0\}$. By this and using the transitive property we conclude that $\mathbb{S}_2$ and $\mathbb{S}_3$ are homotopic equivalent. If that is true, we infer that they have the same Euler characteristic, what is a contraction by the last exercise. Hence $\mathbb{R}^3 - \{0\}$ and $\mathbb{R}^4 - \{0\}$ are not homotopy equivalent.

The computational exercises (22 - 26) can be found in the Github[2]

EXERCISE 22. *Build triangulations of the letters of the alphabet, and compute their Euler characteristic.*

EXERCISE 23. *For every n, triangulate the bouquet of n circles (see below). Compute their Euler characteristic.*

[2] `github.com/lucasmoschen/topological-data-analysis/blob/main/tutorials/tutorial-1.ipynb`

EXERCISE 24. *Implement the following triangulation of the torus.*

EXERCISE 25. *Consider the following dataset of 30 points $x_0, ..., x_{29}$ in R^2.*

Write a function that takes as an input a parameter $r \geq 0$, and returns the simplicial complex $\mathcal{G}(r)$ defined as follows:

1. *the vertices of $\mathcal{G}(r)$ are the points $x_0, ..., x_{29}$,*

2. *for all $i, j \in [[0, 29]]$ with $i \neq j$, the edge $[i, j]$ belongs to $\mathcal{G}(r)$ if and only if $||x_i - x_j|| \leq r$.*

Compute the number of connected components of $\mathcal{G}(r)$ for several values of r. What do you observe?

EXERCISE 26. *A Erdos–Renyi random graph $\mathcal{G}(n, p)$ is a simplicial complex obtained as follows:*

1. *add n vertices $1, ..., n$,*

2. *for every $a, b \in [[1, n]]$, add the edge $[a, b]$ to the complex with probability p.*

Builds a function that, given n and p, outputs a simplicial complex $\mathcal{G}(n, p)$. Observe the influence of p on the number of connected components of $\mathcal{G}(10, p)$ and $\mathcal{G}(100, p)$.

5 Homological algebra

5.1 Important Definitions

DEFINITION 5.1.1. *Let K be a simplicial complex. For any $n \geq 0$, define the sets*

$$K_n = \{\sigma \in K, dim(\sigma) \leq n\}$$

$$K_{(n)} = \{\sigma \in K, dim(\sigma) = n\}.$$

The first set is a simplicial complex, called the n-skeleton of K.

DEFINITION 5.1.2. *Let $n \geq 0$. The n-chains of K is the set $C_n(K)$ whose elements are the formal sums*

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma \text{ where } \forall \sigma \in K_{(n)}, \epsilon_\sigma \in \mathbb{Z}/2\mathbb{Z}.$$

We can give $C_n(K)$ a group structure via

$$\sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma + \sum_{\sigma \in K_{(n)}} \eta_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} (\epsilon_\sigma + \eta_\sigma) \cdot \sigma.$$

DEFINITION 5.1.3. *Let $n \geq 1$, and $\sigma \in K_{(n)}$ a simplex of dimension n. We define its boundary as the following element of $C_{n-1}(K)$:*

$$\partial_n \sigma = \sum_{\tau \subset \sigma, |\tau| = |\sigma| - 1} \tau$$

and

$$\partial_n \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \sigma = \sum_{\sigma \in K_{(n)}} \epsilon_\sigma \cdot \partial_n \sigma$$

DEFINITION 5.1.4. *We define*

1. *The n-cycles: $Z_n(K) = Ker(\partial_n)$,*
2. *The n-boundaries: $B_n(K) = Im(\partial_{n+1})$.*

We say that two chains $c, c' \in C_n(K)$ are homologous if there exists $b \in B_n(K)$ such that $c = c' + b$.

DEFINITION 5.1.5. *The n^{th} homology group of K is $H_n(K) = Z_n(K)/B_n(K)$.*

DEFINITION 5.1.6. *Let K be a simplicial complex and $n \geq 0$. Its n^{th} Betti number is the integer $\beta_n(K) = dim\ H_n(K)$.*

DEFINITION 5.1.7. *The homology groups of a topological space are the homology groups of any triangulation of it. We define their Betti numbers similarly.*

5.2 Exercises

EXERCISE 27. *Let V be a $\mathbb{Z}/2\mathbb{Z}$-vector space, and W a linear subspace. Prove that*

$$dim\ V/W = dim\ V - dim\ W.$$

Suppose V is finite-dimensional, such that $dim\ V = n + m$ and $dim\ W = m$. As we have a finite basis and a finite field, we only have finite number of combinations from the vector of this basis. In special V is finite. By Proposition 5.2, V has cardinal 2^{m+n} and W has cardinal 2^m. Let's prove that V/W has cardinal 2^n. We know the quotient space has finite dimension k, because it's finite. So it has 2^k elements. Let $V/W = \{[v_1], ..., [v_{2^k}]\}$ where v_i represents an equivalence class. We know for each $v \in V$ there exists i such that $v \in v_i + W$ and there is $w \in W$ with $x = v_i + w$. As each class is disjoint, we first choose one of the 2^k classes v_i. After we pick out one element of this class. We have 2^m elements in W. Let's see $|v_i + W| = 2^m$. Take $w, w' \in W$ and suppose $v_i + w = v_i + w'$. Summing v_i in both sides we obtaing that $w = w'$. So we have $2^k 2^m$ ways of forming elements os V. We conclude k must be n, what finishes our proof.

Remark. Consider the following proof slightly more general.

Suppose V is finite-dimensional and let $\{w_1, ..., w_m\}$ be a basis of W. So we can extend this to a basis on V, namely, $\{w_1, ..., w_m, v_1, ..., v_n\}$, where, dim $V = m + n$. Consider the set $\{v_1 + W, ..., v_n + W\}$. First, let's prove it's free.

$$0 + W = \sum_{j=1}^{n} \lambda_j (v_j + W) = \left(\sum_{j=1}^{n} \lambda_j v_j \right) + W,$$

So $\left(\sum_{j=1}^{n} \lambda_j v_j \right) \in W$. But it implies it can be written as a linear combination of $\{w_1, ..., w_m\}$, what contradicts the fact that $\{w_1, ..., w_m, v_1, ..., v_n\}$ is linear independent.

Now take $v + W \in V/W$, where $v = \sum_{j=1}^{m} \lambda_j w_j + \sum_{j=1}^{n} \lambda_{j+m} v_j$. So

$$\begin{aligned} v + W &= \left[\sum_{j=1}^{m} \lambda_j w_j + \sum_{j=1}^{n} \lambda_{j+m} v_j \right] + W \\ &= \sum_{j=1}^{m} \lambda_j (w_j + W) + \sum_{j=1}^{n} \lambda_{j+m} (v_j + W) \\ &= \sum_{j=1}^{n} \lambda_{j+m} (v_j + W) \end{aligned}$$

We conclude $\{v_1, ..., v_n\}$ is a basis what implies dim $V/W = n = m + n - m$.

EXERCISE 28. *Let $(G, +)$ be a group, potentially non-commutative. Prove that*

$$\forall g \in G, g + g = 0 \implies G \textit{ is commutative.}$$

Let $u, v \in G$. So $u + v + v = u + (v + v) = u + 0 = u$. With that in mind, we see that

$$u + v + v + u = 0 \implies (u + v) + (v + u) = 0$$

We know $v + u$ and $u + v$ are elements of G. So we can add to each side and obtain

$$(u + v) + (v + u) + (v + u) = (v + u) \implies u + v = v + u$$

As u and v are arbitrary, G is commutative.

Exercise 29. *Compute the Betti numbers $\beta_0(K), \beta_1(K)$ and $\beta_2(K)$ of the following simplicial complex:*

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0]\}.$$

$Z_0(K) = C_0(K)$
$B_0(K) = \{[0] + [1], [1] + [2], [2] + [3], [3] + [0], [0] + [2], [1] + [3], [0] + [1] + [2] + [3], 0\}$
As B_0 have 2^3 elements and Z_0 has 2^4, we deduce that

$$\beta_0(K) = dim\ Z_0(K) - dim\ B_0(K) = 4 - 3 = 1$$

$Z_1(K) = \{[0, 1] + [1, 2] + [2, 3] + [3, 0], 0\}$
$B_1(K) = \{0\}$
Nesse caso, observamos que, utilizando a mesma ideia do ponto anterior

$$\beta_1(K) = 1 - 0 = 1.$$

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0]\}.$$

$Z_2(K) = \{0\}$
$B_2(K) = \{0\}$
Portanto $\beta_2(K) = 0$.

Exercise 30. *Compute the Betti numbers $\beta_0(K), \beta_1(K)$ and $\beta_2(K)$ of the following simplicial complex:*

$$K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}.$$

$Z_0(K) = C_0(K)$
$B_0(K) = \{[0] + [1], [1] + [2], [2] + [3], [3] + [0], [0] + [2], [1] + [3], [0] + [1] + [2] + [3], 0\}$
As B_0 have 2^3 elements and Z_0 has 2^4, we deduce that

$$\beta_0(K) = dim\ Z_0(K) - dim\ B_0(K) = 4 - 3 = 1$$

$$\begin{aligned} Z_1(K) = \{&[0, 1] + [1, 2] + [0, 2]; [0, 1] + [0, 3] + [1, 3]; [0, 2] + [2, 3] + [0, 3]; \\ &[1, 2] + [2, 3] + [1, 3]; [0, 1] + [1, 2] + [2, 3] + [0, 3]; [0, 1] + [1, 3] + [2, 3] + [2, 0]; \\ &[0, 2] + [1, 2] + [1, 3] + [0, 3]; 0\} \end{aligned} \tag{1}$$

$$\begin{aligned} B_1(K) = \{&[0, 1] + [1, 2] + [0, 2]; [0, 1] + [0, 3] + [1, 3]; [0, 2] + [2, 3] + [0, 3]; \\ &[1, 2] + [2, 3] + [1, 3]; [0, 1] + [1, 2] + [2, 3] + [0, 3]; [0, 1] + [1, 3] + [2, 3] + [0, 2]; \\ &[0, 2] + [1, 2] + [1, 3] + [0, 3]; 0\} \end{aligned} \tag{2}$$

Observe que os conjuntos são iguais e, portanto,

$$\beta_1(K) = 0.$$

$Z_2(K) = \{[0,1,2] + [0,1,3] + [0,2,3] + [1,2,3], 0\}$
$B_2(K) = \{0\}$
Portanto $\beta_2(K) = 1$.

6 Incremental algorithm

6.1 Important definitions

DEFINITION 6.1.1. *Let $i \in [[1,n]]$, and $d = dim(\sigma_i)$. The simplex σ_i is positive if there exists a cycle $c \in Z_d(K^i\ )$ that contains σ_i. Otherwise, σ_i is negative.*

DEFINITION 6.1.2. *Defines for a set $V = \{0, 1, ..., n\}$ a simplicial complex*

$$\Delta_n = \{S \subset C, S \neq \emptyset\}$$

and call it simplicial standard n-simplex with boundary

$$\partial\Delta_n = \Delta_n/V$$

DEFINITION 6.1.3. *Define the boundary matrix of K, denoted Δ as follows: Δ is a $n \times n$ matrix, whose*

$$\Delta_{i,j} = \begin{cases} 1, & \text{if } \sigma_i \subset \sigma_j \text{ and } |\sigma^i| = |\sigma^j| - 1 \\ 0, & \text{else} \end{cases}$$

6.2 Exercise

EXERCISE 31. *Compute again the Betti numbers of the simplicial complexes of Exercises 29 and 30, using the incremental algorithm.*

1. $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0]\}$.

   First we determine the ordering to be as placed in the set. It fulfills the required property. After, we find the signals for it σ^i. The first four elements are positive, because, ∂_0 has $C_0(K^i)$ as kernel. On the other hand $[0, 1]$, $[1, 2]$ and $[2, 3]$ are negatives. At last $[3, 0]$ is positive, because $[0, 1] + [1, 2] + [2, 3] + [3, 0]$ belongs to $Z_1(K^8)$. Now we can follow the algorithm thorough a table:

-	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8
Signal	+	+	+	+	-	-	-	+
$\beta_0(K)$	1	2	3	4	3	2	1	1
$\beta_1(K)$	0	0	0	0	0	0	0	1
$\beta_2(K)$	0	0	0	0	0	0	0	0

2. $K = \{[0], [1], [2], [3], [0, 1], [1, 2], [2, 3], [3, 0], [0, 2], [1, 3], [0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]\}$.

   First we determine the ordering to be as placed in the set. It fulfills the required property. After, we find the signals for it σ^i. The vertices have positive signal. The following three edges cannot form any cycle, so they are negative. The last three edges are part of a cycle considering three other already placed of its dimension. When we achieve the simplices with dimension 2, the first three must be negative, because when we sum every combination of them, the boundary has image different from 0. The last, however will be positive. Now we can follow the algorithm thorough a table:

-	σ^1	σ^2	σ^3	σ^4	σ^5	σ^6	σ^7	σ^8	σ^9	σ^{10}	σ^{11}	σ^{12}	σ^{13}	σ^{14}
Signal	+	+	+	+	-	-	-	+	+	+	-	-	-	+
$\beta_0(K)$	1	2	3	4	3	2	1	1	1	1	1	1	1	1
$\beta_1(K)$	0	0	0	0	0	0	0	1	2	3	2	1	0	0
$\beta_2(K)$	0	0	0	0	0	0	0	0	0	0	0	0	0	1

The result corroborates with those found previously.

EXERCISE 32. *Prove that $\partial\Delta_n$ is a triangulation of the $(n-1)$-sphere $\mathbb{S}_{n-1} \subset \mathbb{R}^n$.*

It's clear that $\partial\Delta_n$ is a simplicial complex, because, for every simplex $\sigma \in \partial\Delta_n$ and $\tau \subset \sigma$, $\tau \in \Delta_n$, because its a simplicial complex and $\tau \neq V$, then $\tau \in \partial\Delta_n$. I shall prove that $|\partial\Delta_n|$, the topological realization of the set, is homeomorphic to the sphere. We can describe the convex hull of boundary of the simplex as

$$B_n := |\partial\Delta_n| = \{(\alpha_0, ..., \alpha_n) \in [0,1]^{n+1}, \sum_{i=0}^{n} \alpha_i = 1 \text{ and for some } i, \alpha_i = 0\}$$

Let $H = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} x_i = 1\}$. It's clear that $B_n \subset H$. I claim H is homeomorphic to $\mathbb{R}^n$. Define $f : \mathbb{R}^n \to H$ to be $f(x_1, ..., x_n) = (1 - \sum_{i=1}^{n} x_i, x_1, ..., x_n)$. Let's prove it is injective. Take two points in the hyperplane such that

$$(1 - \sum_{i=1}^{n} x_i, x_1, ..., x_n) = (1 - \sum_{i=1}^{n} y_i, y_1, ..., y_n) \implies y_1 = x_1, ..., y_n = x_n$$

what proves f is injective. Take $y = (y_0, ..., y_n) \in H$ and define $x = (y_1, ..., y_n)$, then $f(x) = (1 - \sum_{i=1}^{n} y_i, y_1, ..., y_n) = (y_0, ..., y_n) = y$ and f is surjective. Extend the H to $\mathbb{R}^{n+1}$ in order to prove f is continuos with ϵ-δ. Suppose $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\epsilon > 0$. Let $\delta = \epsilon/2$ and suppose $||x - y|| < \delta$. By the equivalence of the norms, I can use the norm 1 to prove it. Then

$$\begin{aligned}
||f(x) - f(y)|| &= |\sum_{i=1}^{n} (y_i - x_i)| + |x_1 - y_1| + ... + |x_n - y_n| \\
&\leq \sum_{i=1}^{n} |y_i - x_i| + \sum_{i=1}^{n} |y_i - x_i| \\
&= 2||x - y|| < 2\delta = \epsilon
\end{aligned}$$

and f is continuos. Because of that, when we restrict the image of f, it will remain continuos. Let's do it with the inverse extended defined as

$$g(x_0, ..., x_n) = (x_1, ..., x_n)$$

Take $x = (x_0, ..., x_n) \in R^{n+1}$ and $\epsilon > 0$. Let $\delta = \epsilon$ and suppose $||x - y|| < \delta$. So

$$\begin{aligned}
||g(x) - g(y)|| &= |x_1 - y_1| + ... + |x_n - y_n| \\
&\leq |x_1 - y_1| + ... + |x_n - y_n| + |x_0 - y_0| \\
&= ||x - y|| < \epsilon
\end{aligned}$$

So g is also continuos. In particular g restricted to H is also continuos and it's equal to f^{-1}. Therefore f is a Homeomorphism. Because of that $\mathbb{S}_{n-1}$ is homeomorphic to $f(\mathbb{S}_{n-1})$. Let's prove $f(\mathbb{S}_{n-1})$ is homeomorphic to $\mathbb{S}_n \cap H$.

Exercise 33. *Show that the algorithm stops after a finite number of steps.*

Consider the algorithm
Algorithm: Reduction of the boundary matrix
Input: a boundary matrix Δ
Output: a reduced matrix $\tilde{\Delta}$

```
for i <- 1 to n do
    while there exists i < j com delta(i) = delta(j) do
        add column i to column j
```

Exercise 34. *Apply Algorithm to solve Exercise 31.*