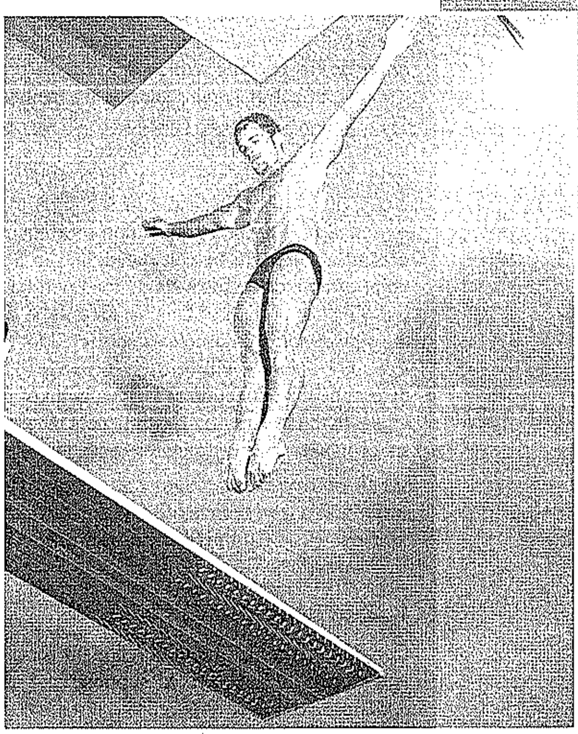
CHAPTER 11

Energy Methods



As the diver comes down on the diving board the potential energy due to his elevation above the board will be converted into strain energy due to the bending of the board. The normal and shearing stresses resulting from energy loadings will be determined in this chapter.

11.1. INTRODUCTION

In the previous chapter we were concerned with the relations existing between forces and deformations under various loading conditions. Our analysis was based on two fundamental concepts, the concept of stress (Chap. 1) and the concept of strain (Chap. 2). A third important concept, the concept of *strain energy*, will now be introduced.

In Sec. 11.2, the *strain energy* of a member will be defined as the increase in energy associated with the deformation of the member. You will see that the strain energy is equal to the work done by a slowly increasing load applied to the member. The *strain-energy density* of a material will be defined as the strain energy per unit volume; it will be seen that it is equal to the area under the stress-strain diagram of the material (Sec. 11.3). From the stress-strain diagram of a material two additional properties will be defined, namely, the *modulus of toughness* and the *modulus of resilience* of the material.

In Sec. 11.4 the elastic strain energy associated with *normal stresses* will be discussed, first in members under axial loading and then in members in bending. Later you will consider the elastic strain energy associated with shearing stresses such as occur in torsional loadings of shafts and in transverse loadings of beams (Sec. 11.5). Strain energy for a *general state of stress* will be considered in Sec. 11.6, where the *maximum-distortion-energy criterion* for yielding will be derived.

The effect of *impact loading* on members will be considered in Sec. 11.7. You will learn to calculate both the *maximum stress* and the *maximum deflection* caused by a moving mass impacting on a member. Properties that increase the ability of a structure to withstand impact loads effectively will be discussed in Sec. 11.8.

In Sec. 11.9 the elastic strain of a member subjected to a *single concentrated load* will be calculated, and in Sec. 11.10 the deflection at the point of application of a single load will be determined.

The last portion of the chapter will be devoted to the determination of the strain energy of structures subjected to *several loads* (Sec. 11.11). *Castigliano's theorem* will be derived in Sec. 11.12 and used in Sec. 11.13 to determine the deflection at a given point of a structure subjected to several loads. In the last section Castigliano's theorem will be applied to the analysis of indeterminate structures (Sec. 11.14).

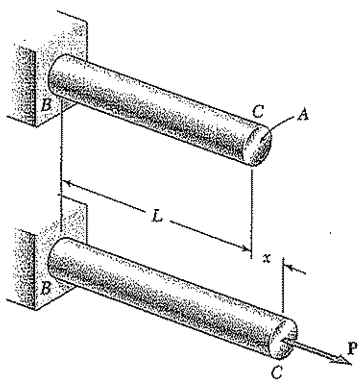


Fig. 11.1

11.2. STRAIN ENERGY

Consider a rod BC of length L and uniform cross-sectional area A, which is attached at B to a fixed support, and subjected at C to a *slowly increasing* axial load $\mathbf{P}$ (Fig. 11.1). As we noted in Sec. 2.2, by plotting the magnitude P of the load against the deformation x of the rod, we obtain a certain load-deformation diagram (Fig. 11.2) that is characteristic of the rod BC.

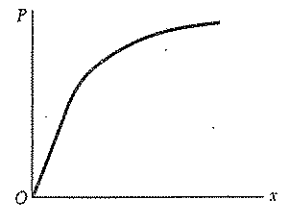


Fig. 11.2

Let us now consider the work dU done by the load $\mathbf{P}$ as the rod elongates by a small amount dx. This *elementary work* is equal to the product of the magnitude P of the load and of the small elongation dx. We write

$$dU = P\,dx \tag{11.1}$$

and note that the expression obtained is equal to the element of area of width dx located under the load-deformation diagram (Fig. 11.3). The *total work* U done by the load as the rod undergoes a deformation x_1 is thus

$$U = \int_0^{x_1} P\,dx$$

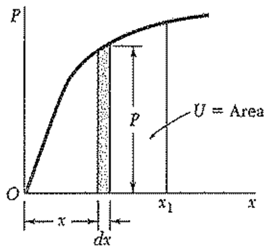


Fig. 11.3

and is equal to the area under the load-deformation diagram between $x = 0$ and $x = x_1$.

The work done by the load $\mathbf{P}$ as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the *strain energy* of the rod. We have, by definition,

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \tag{11.2}$$

We recall that work and energy should be expressed in units obtained by multiplying units of length by units of force. Thus, work and energy are expressed in N · m; this unit is called a *joule* (J).

In the case of a linear and elastic deformation, the portion of the load-deformation diagram involved can be represented by a straight line of equation $P = kx$ (Fig. 11.4). Substituting for P in Eq. (11.2), we have

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2$$

or

$$U = \tfrac{1}{2}P_1x_1 \tag{11.3}$$

where P_1 is the value of the load corresponding to the deformation x_1.

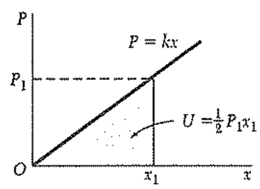


Fig. 11.4

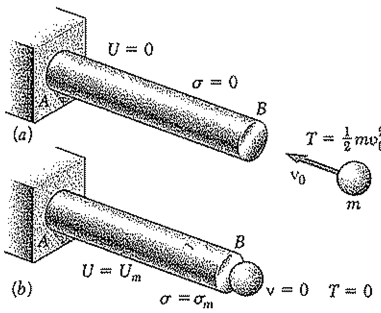


Fig. 11.5

The concept of strain energy is particularly useful in the determination of the effects of impact loadings on structures or machine components. Consider, for example, a body of mass m moving with a velocity $\mathbf{v}_0$ which strikes the end B of a rod AB (Fig. 11.5*a*). Neglecting the inertia of the elements of the rod, and assuming no dissipation of energy during the impact, we find that the maximum strain energy U_m acquired by the rod (Fig. 11.5*b*) is equal to the original kinetic energy $T = \frac{1}{2}mv_0^2$ of the moving body. We then determine the value P_m of the static load which would have produced the same strain energy in the rod, and obtain the value σ_m of the largest stress occurring in the rod by dividing P_m by the cross-sectional area of the rod.

11.3. STRAIN-ENERGY DENSITY

As we noted in Sec. 2.2, the load-deformation diagram for a rod BC depends upon the length L and the cross-sectional area A of the rod. The strain energy U defined by Eq. (11.2), therefore, will also depend upon the dimensions of the rod. In order to eliminate the effect of size from our discussion and direct our attention to the properties of the material, the strain energy per unit volume will be considered. Dividing the strain energy U by the volume $V = AL$ of the rod (Fig. 11.1), and using Eq. (11.2), we have

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

Recalling that P/A represents the normal stress σ_x in the rod, and x/L the normal strain ϵ_x, we write

$$\frac{U}{V} = \int_0^{\epsilon_1} \sigma_x \, d\epsilon_x$$

where ϵ_1 denotes the value of the strain corresponding to the elongation x_1. The strain energy per unit volume, U/V, is referred to as the *strain-energy density* and will be denoted by the letter u. We have, therefore,

$$\text{Strain-energy density} = u = \int_0^{\epsilon_1} \sigma_x \, d\epsilon_x \tag{11.4}$$

The strain-energy density u is expressed in units obtained by dividing units of energy by units of volume. Thus, the strain-energy density is expressed in J/m^3 or its multiples kJ/m^3 and MJ/m^3.

†We note that 1 J/m^3 and 1 Pa are both equal to 1 N/m^2. Thus, strain-energy density and stress are dimensionally equal and could be expressed in the same units.

Referring to Fig. 11.6, we note that the strain-energy density u is equal to the area under the stress-strain curve, measured from $\epsilon_x = 0$ to $\epsilon_x = \epsilon_1$. If the material is unloaded, the stress returns to zero, but there is a permanent deformation represented by the strain ϵ_p, and only the portion of the strain energy per unit volume corresponding to the triangular area is recovered. The remainder of the energy spent in deforming the material is dissipated in the form of heat.

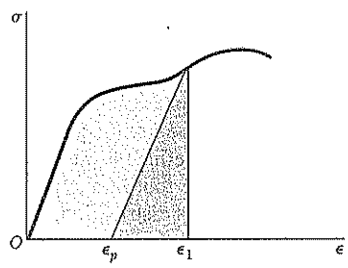


Fig. 11.6

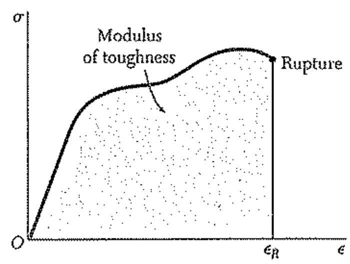


Fig. 11.7

The value of the strain-energy density obtained by setting $\epsilon_1 = \epsilon_R$ in Eq. (11.4), where ϵ_R is the strain at rupture, is known as the *modulus of toughness* of the material. It is equal to the area under the entire stress-strain diagram (Fig. 11.7) and represents the energy per unit volume required to cause the material to rupture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength (Sec. 2.3), and that the capacity of a structure to withstand an impact load depends upon the toughness of the material used (Fig. 11.8).

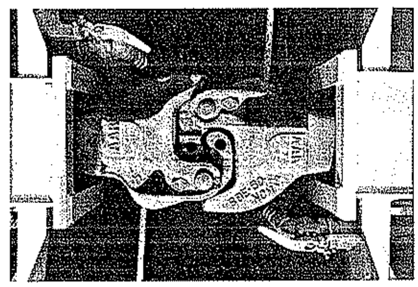


Fig. 11.8 The railroad coupler is made of a ductile steel which has a large modulus of toughness.

If the stress σ_x remains within the proportional limit of the material, Hooke's law applies and we write

$$\sigma_x = E\epsilon_x \tag{11.5}$$

Substituting for σ_x from (11.5) into (11.4), we have

$$u = \int_0^{\epsilon_1} E\epsilon_x \, d\epsilon_x = \frac{E\epsilon_1^2}{2} \tag{11.6}$$

or, using Eq. (11.5) to express ϵ_1 in terms of the corresponding stress σ_1,

$$u = \frac{\sigma_1^2}{2E} \tag{11.7}$$

The value u_Y of the strain-energy density obtained by setting $\sigma_1 = \sigma_Y$ in Eq. (11.7), where σ_Y is the yield strength, is called the *modulus of resilience* of the material. We have

$$u_Y = \frac{\sigma_Y^2}{2E} \tag{11.8}$$

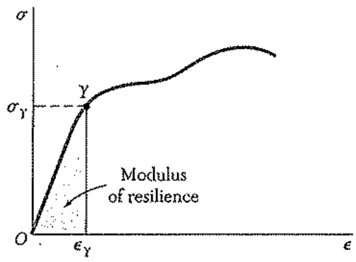


Fig. 11.9

The modulus of resilience is equal to the area under the straight-line portion OY of the stress-strain diagram (Fig. 11.9) and represents the energy per unit volume that the material can absorb without yielding. The capacity of a structure to withstand an impact load without being permanently deformed clearly depends upon the resilience of the material used.

Since the modulus of toughness and the modulus of resilience represent characteristic values of the strain-energy density of the material considered, they are both expressed in J/m^3.†

11.4. ELASTIC STRAIN ENERGY FOR NORMAL STRESSES

Since the rod considered in the preceding section was subjected to uniformly distributed stresses σ_x, the strain-energy density was constant throughout the rod and could be defined as the ratio U/V of the strain energy U and the volume V of the rod. In a structural element or machine part with a nonuniform stress distribution, the strain-energy density u can be defined by considering the strain energy of a small element of material of volume ΔV and writing

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V}$$

or

$$u = \frac{dU}{dV} \tag{11.9}$$

The expression obtained for u in Sec. 11.3 in terms of σ_x and ϵ_x remains valid, i.e., we still have

$$u = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x \tag{11.10}$$

but the stress σ_x, the strain ϵ_x, and the strain-energy density u will generally vary from point to point.

For values of σ_x within the proportional limit, we may set $\sigma_x = E\epsilon_x$ in Eq. (11.10) and write

$$u = \frac{1}{2}E\epsilon_x^2 = \frac{1}{2}\sigma_x\epsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} \tag{11.11}$$

The value of the strain energy U of a body subjected to uniaxial normal stresses can be obtained by substituting for u from Eq. (11.11) into Eq. (11.9) and integrating both members. We have

$$U = \int \frac{\sigma_x^2}{2E} dV \tag{11.12}$$

The expression obtained is valid only for elastic deformations and is referred to as the *elastic strain energy* of the body.

†However, referring to the footnote on page 672, we note that the modulus of toughness and the modulus of resilience could be expressed in the same units as stress.

Strain Energy under Axial Loading. We recall from Sec. 2.17 that, when a rod is subjected to a centric axial loading, the normal stresses σ_x can be assumed uniformly distributed in any given transverse section. Denoting by A the area of the section located at a distance x from the end B of the rod (Fig. 11.10), and by P the internal force in that section, we write $\sigma_x = P/A$. Substituting for σ_x into Eq. (11.12), we have

$$U = \int \frac{P^2}{2EA^2} dV$$

or, setting $dV = A\,dx$,

$$U = \int_0^L \frac{P^2}{2AE} dx \tag{11.13}$$

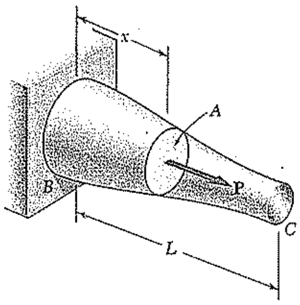


Fig. 11.10

In the case of a rod of uniform cross section subjected at its ends to equal and opposite forces of magnitude P (Fig. 11.11), Eq. (11.13) yields

$$U = \frac{P^2L}{2AE} \tag{11.14}$$

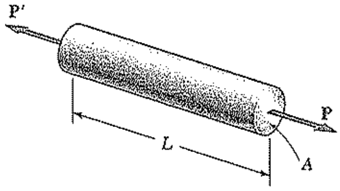


Fig. 11.11

EXAMPLE 11.01

A rod consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 11.12). Determine the strain energy of the rod when it is subjected to a centric axial load **P**, expressing the result in terms of P, L, E, the cross-sectional area A of portion CD, and the ratio n of the two diameters.

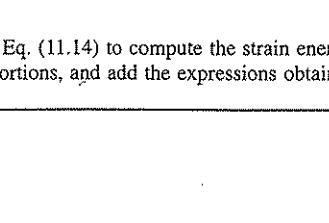


Fig. 11.12

We use Eq. (11.14) to compute the strain energy of each of the two portions, and add the expressions obtained:

$$U_n = \frac{P^2(\frac{1}{2}L)}{2AE} + \frac{P^2(\frac{1}{2}L)}{2(n^2A)E} = \frac{P^2L}{4AE}\left(1 + \frac{1}{n^2}\right)$$

or

$$U_n = \frac{1+n^2}{2n^2}\frac{P^2L}{2AE} \tag{11.15}$$

We check that, for $n = 1$, we have

$$U_1 = \frac{P^2L}{2AE}$$

which is the expression given in Eq. (11.14) for a rod of length L and uniform cross section of area A. We also note that, for $n > 1$, we have $U_n < U_1$; for example, when $n = 2$, we have $U_2 = (\frac{5}{8})U_1$. Since the maximum stress occurs in portion CD of the rod and is equal to $\sigma_{max} = P/A$, it follows that, for a given allowable stress, increasing the diameter of portion BC of the rod results in a *decrease* of the overall energy-absorbing capacity of the rod. Unnecessary changes in cross-sectional area should therefore be avoided in the design of members that may be subjected to loadings, such as impact loadings, where the energy-absorbing capacity of the member is critical.

EXAMPLE 11.02

A load **P** is supported at B by two rods of the same material and of the same uniform cross section of area A (Fig. 11.13). Determine the strain energy of the system.

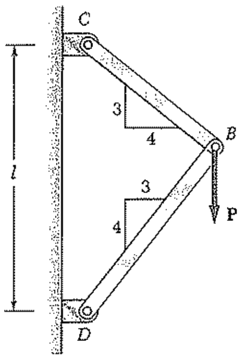


Fig. 11.13

Denoting by F_{BC} and F_{BD}, respectively, the forces in members BC and BD, and recalling Eq. (11.14), we express the strain energy of the system as

$$U = \frac{F_{BC}^2(BC)}{2AE} + \frac{F_{BD}^2(BD)}{2AE} \qquad (11.16)$$

But we note from Fig. 11.13 that

$$BC = 0.6l \qquad BD = 0.8l$$

and from the free-body diagram of pin B and the corresponding force triangle (Fig. 11.14) that

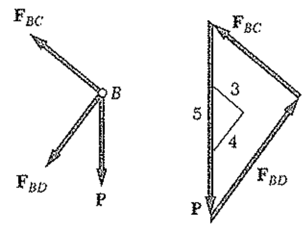


Fig. 11.14

$$F_{BC} = +0.6P \qquad F_{BD} = -0.8P$$

Substituting into Eq. (11.16), we have

$$U = \frac{P^2l[(0.6)^3 + (0.8)^3]}{2AE} = 0.364\frac{P^2l}{AE}$$

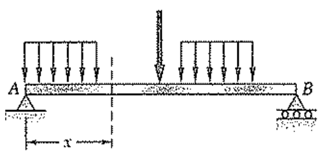


Fig. 11.15

Strain Energy in Bending. Consider a beam AB subjected to a given loading (Fig. 11.15), and let M be the bending moment at a distance x from end A. Neglecting for the time being the effect of shear, and taking into account only the normal stresses $\sigma_x = My/I$, we substitute this expression into Eq. (11.12) and write

$$U = \int \frac{\sigma_x^2}{2E}dV = \int \frac{M^2y^2}{2EI^2}dV$$

Setting $dV = dA\,dx$, where dA represents an element of the cross-sectional area, and recalling that $M^2/2EI^2$ is a function of x alone, we have

$$U = \int_0^L \frac{M^2}{2EI^2}\left(\int y^2\,dA\right)dx$$

Recalling that the integral within the parentheses represents the moment of inertia I of the cross section about its neutral axis, we write

$$U = \int_0^L \frac{M^2}{2EI}dx \qquad (11.17)$$

EXAMPLE 11.03

Determine the strain energy of the prismatic cantilever beam AB (Fig. 11.16), taking into account only the effect of the normal stresses.

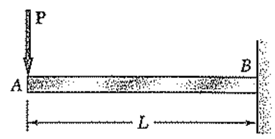


Fig. 11.16

The bending moment at a distance x from end A is $M = -Px$. Substituting this expression into Eq. (11.17), we write

$$U = \int_0^L \frac{P^2x^2}{2EI}dx = \frac{P^2L^3}{6EI}$$

11.5. ELASTIC STRAIN ENERGY FOR SHEARING STRESSES

When a material is subjected to plane shearing stresses τ_{xy}, the strain-energy density at a given point can be expressed as

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\, d\gamma_{xy} \tag{11.18}$$

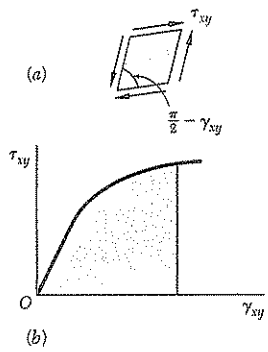


Fig. 11.17

where γ_{xy} is the shearing strain corresponding to τ_{xy} (Fig. 11.17*a*). We note that the strain-energy density u is equal to the area under the shearing-stress-strain diagram (Fig. 11.17*b*).

For values of τ_{xy} within the proportional limit, we have $\tau_{xy} = G\gamma_{xy}$, where G is the modulus of rigidity of the material. Substituting for τ_{xy} into Eq. (11.18) and performing the integration, we write

$$u = \frac{1}{2}G\gamma_{xy}^2 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G} \tag{11.19}$$

The value of the strain energy U of a body subjected to plane shearing stresses can be obtained by recalling from Sec. 11.4 that

$$u = \frac{dU}{dV} \tag{11.9}$$

Substituting for u from Eq. (11.19) into Eq. (11.9) and integrating both members, we have

$$U = \int \frac{\tau_{xy}^2}{2G}dV \tag{11.20}$$

This expression defines the elastic strain associated with the shear deformations of the body. Like the similar expression obtained in Sec. 11.4 for uniaxial normal stresses, it is valid only for elastic deformations.

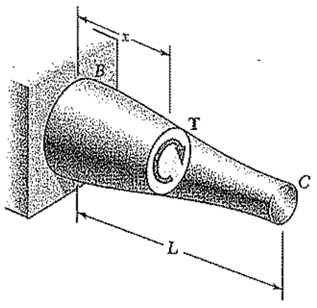


Fig. 11.18

Strain Energy in Torsion. Consider a shaft BC of length L subjected to one or several twisting couples. Denoting by J the polar moment of inertia of the cross section located at a distance x from B (Fig. 11.18), and by T the internal torque in that section, we recall that the shearing stresses in the section are $\tau_{xy} = T\rho/J$. Substituting for τ_{xy} into Eq. (11.20), we have

$$U = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2\rho^2}{2GJ^2} dV$$

Setting $dV = dA\,dx$, where dA represents an element of the cross-sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

$$U = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2\,dA\right) dx$$

Recalling that the integral within the parentheses represents the polar moment of inertia J of the cross section, we have

$$U = \int_0^L \frac{T^2}{2GJ} dx \qquad (11.21)$$

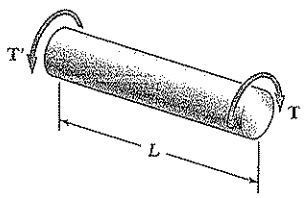


Fig. 11.19

In the case of a shaft of uniform cross section subjected at its ends to equal and opposite couples of magnitude T (Fig. 11.19), Eq. (11.21) yields

$$U = \frac{T^2L}{2GJ} \qquad (11.22)$$

EXAMPLE 11.04

A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 11.20). Determine the strain energy of the shaft when it is subjected to a twisting couple **T** at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio n of the two diameters.

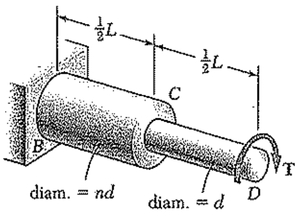


Fig. 11.20

We use Eq. (11.22) to compute the strain energy of each of the two portions of shaft, and add the expressions obtained. Noting that the polar moment of inertia of portion BC is equal to n^4J, we write

$$U_n = \frac{T^2(\frac{1}{2}L)}{2GJ} + \frac{T^2(\frac{1}{2}L)}{2G(n^4J)} = \frac{T^2L}{4GJ}\left(1 + \frac{1}{n^4}\right)$$

or

$$U_n = \frac{1 + n^4}{2n^4}\frac{T^2L}{2GJ} \qquad (11.23)$$

We check that, for $n = 1$, we have

$$U_1 = \frac{T^2L}{2GJ}$$

which is the expression given in Eq. (11.22) for a shaft of length L and uniform cross section. We also note that, for $n > 1$, we have $U_n < U_1$; for example, when $n = 2$, we have $U_2 = (\frac{17}{32})\,U_1$. Since the maximum shearing stress occurs in the portion CD of the shaft and is proportional to the torque T, we note as we did earlier in the case of the axial loading of a rod that, for a given allowable stress, increasing the diameter of portion BC of the shaft results in a *decrease* of the overall energy-absorbing capacity of the shaft.

Strain Energy under Transverse Loading. In Sec. 11.4 we obtained an expression for the strain energy of a beam subjected to a transverse loading. However, in deriving that expression we took into account only the effect of the normal stresses due to bending and neglected the effect of the shearing stresses. In Example 11.05 both types of stresses will be taken into account.

EXAMPLE 11.05

Determine the strain energy of the rectangular cantilever beam AB (Fig. 11.21), taking into account the effect of both normal and shearing stresses.

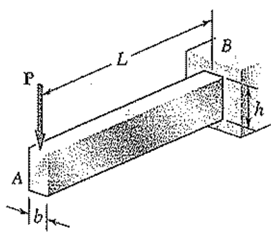


Fig. 11.21

We first recall from Example 11.03 that the strain energy due to the normal stresses σ_x is

$$U_\sigma = \frac{P^2L^3}{6EI}$$

To determine the strain energy U_τ due to the shearing stresses τ_{xy}, we recall Eq. (6.9) of Sec. 6.4 and find that, for a beam with a rectangular cross section of width b and depth h,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) = \frac{3}{2}\frac{P}{bh}\left(1 - \frac{y^2}{c^2}\right)$$

Substituting for τ_{xy} into Eq. (11.20), we write

$$U_\tau = \frac{1}{2G}\left(\frac{3}{2}\frac{P}{bh}\right)^2 \int \left(1 - \frac{y^2}{c^2}\right)^2 dV$$

or, setting $dV = b\,dy\,dx$, and after reductions,

$$U_\tau = \frac{9P^2}{8Gbh^2}\int_{-c}^{c}\left(1 - 2\frac{y^2}{c^2} + \frac{y^4}{c^4}\right)dy \int_0^L dx$$

Performing the integrations, and recalling that $c = h/2$, we have

$$U_\tau = \frac{9P^2L}{8Gbh^2}\left[y - \frac{2}{3}\frac{y^3}{c^2} + \frac{1}{5}\frac{y^5}{c^4}\right]_{-c}^{+c} = \frac{3P^2L}{5Gbh} = \frac{3P^2L}{5GA}$$

The total strain energy of the beam is thus

$$U = U_\sigma + U_\tau = \frac{P^2L^3}{6EI} + \frac{3P^2L}{5GA}$$

or, noting that $I/A = h^2/12$ and factoring the expression for U_σ,

$$U = \frac{P^2L^3}{6EI}\left(1 + \frac{3Eh^2}{10GL^2}\right) = U_\sigma\left(1 + \frac{3Eh^2}{10GL^2}\right) \qquad (11.24)$$

Recalling from Sec. 2.14 that $G \gtrsim E/3$, we conclude that the parenthesis in the expression obtained is less than $1 + 0.9(h/L)^2$ and, thus, that the relative error is less than $0.9(h/L)^2$ when the effect of shear is neglected. For a beam with a ratio h/L less than $\frac{1}{10}$, the percentage error is less than 0.9%. It is therefore customary in engineering practice to neglect the effect of shear in computing the strain energy of slender beams.

11.6. STRAIN ENERGY FOR A GENERAL STATE OF STRESS

In the preceding sections, we determined the strain energy of a body in a state of uniaxial stress (Sec. 11.4) and in a state of plane shearing stress (Sec. 11.5). In the case of a body in a general state of stress characterized by the six stress components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}, the strain-energy density can be obtained by adding the expressions given in Eqs. (11.10) and (11.18), as well as the four other expressions obtained through a permutation of the subscripts.

In the case of the elastic deformation of an isotropic body, each of the six stress-strain relations involved is linear, and the strain-energy density can be expressed as

$$u = \tfrac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) \qquad (11.25)$$

Recalling the relations (2.38) obtained in Sec. 2.14, and substituting for the strain components into (11.25), we have, for the most general state of stress at a given point of an elastic isotropic body,

$$u = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)] + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \qquad (11.26)$$

If the principal axes at the given point are used as coordinate axes, the shearing stresses become zero and Eq. (11.26) reduces to

$$u = \frac{1}{2E}[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a)] \qquad (11.27)$$

where σ_a, σ_b, and σ_c are the principal stresses at the given point.

We now recall from Sec. 7.7 that one of the criteria used to predict whether a given state of stress will cause a ductile material to yield, namely, the maximum-distortion-energy criterion, is based on the determination of the energy per unit volume associated with the distortion, or change in shape, of that material. Let us, therefore, attempt to separate the strain-energy density u at a given point into two parts, a part u_v associated with a change in volume of the material at that point, and a part u_d associated with a distortion, or change in shape, of the material at the same point. We write

$$u = u_v + u_d \qquad (11.28)$$

In order to determine u_v and u_d, we introduce the *average value* $\bar{\sigma}$ of the principal stresses at the point considered,

$$\bar{\sigma} = \frac{\sigma_a + \sigma_b + \sigma_c}{3} \qquad (11.29)$$

and set

$$\sigma_a = \bar{\sigma} + \sigma_a' \qquad \sigma_b = \bar{\sigma} + \sigma_b' \qquad \sigma_c = \bar{\sigma} + \sigma_c' \qquad (11.30)$$

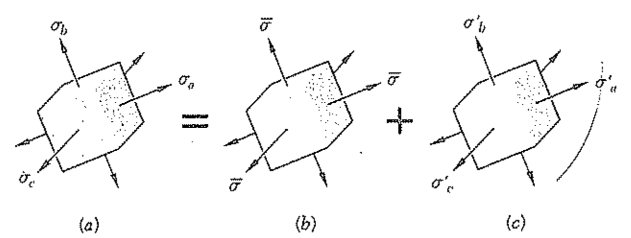


Fig. 11.22

Thus, the given state of stress (Fig. 11.22*a*) can be obtained by superposing the states of stress shown in Fig. 11.22*b* and *c*. We note that the state of stress described in Fig. 11.22*b* tends to change the volume of the element of material, but not its shape, since all the faces of the element are subjected to the same stress $\bar{\sigma}$. On the other hand, it follows from Eqs. (11.29) and (11.30) that

$$\sigma'_a + \sigma'_b + \sigma'_c = 0 \qquad (11.31)$$

which indicates that some of the stresses shown in Fig. 11.22*c* are tensile and others compressive. Thus, this state of stress tends to change the shape of the element. However, it does not tend to change its volume. Indeed, recalling Eq. (2.31) of Sec. 2.13, we note that the dilatation e (i.e., the change in volume per unit volume) caused by this state of stress is

$$e = \frac{1 - 2\nu}{E}(\sigma'_a + \sigma'_b + \sigma'_c)$$

or $e = 0$, in view of Eq. (11.31). We conclude from these observations that the portion u_v of the strain-energy density must be associated with the state of stress shown in Fig. 11.22*b*, while the portion u_d must be associated with the state of stress shown in Fig. 11.22*c*.

It follows that the portion u_v of the strain-energy density corresponding to a change in volume of the element can be obtained by substituting $\bar{\sigma}$ for each of the principal stresses in Eq. (11.27). We have

$$u_v = \frac{1}{2E}[3\bar{\sigma}^2 - 2\nu(3\bar{\sigma}^2)] = \frac{3(1 - 2\nu)}{2E}\bar{\sigma}^2$$

or, recalling Eq. (11.29),

$$u_v = \frac{1 - 2\nu}{6E}(\sigma_a + \sigma_b + \sigma_c)^2 \qquad (11.32)$$

The portion of the strain-energy density corresponding to the distortion of the element is obtained by solving Eq. (11.28) for u_d and substituting for u and u_v from Eqs. (11.27) and (11.32), respectively. We write

$$u_d = u - u_v = \frac{1}{6E}[3(\sigma_a^2 + \sigma_b^2 + \sigma_c^2) - 6\nu(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a) - (1 - 2\nu)(\sigma_a + \sigma_b + \sigma_c)^2]$$

Expanding the square and rearranging terms, we have

$$u_d = \frac{1+\nu}{6E}[(\sigma_a^2 - 2\sigma_a\sigma_b + \sigma_b^2) + (\sigma_b^2 - 2\sigma_b\sigma_c + \sigma_c^2) + (\sigma_c^2 - 2\sigma_c\sigma_a + \sigma_a^2)]$$

Noting that each of the parentheses inside the bracket is a perfect square, and recalling from Eq. (2.43) of Sec. 2.15 that the coefficient in front of the bracket is equal to $1/12G$, we obtain the following expression for the portion u_d of the strain-energy density, i.e., for the distortion energy per unit volume,

$$u_d = \frac{1}{12G}[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2] \qquad (11.33)$$

In the case of *plane stress*, and assuming that the c axis is perpendicular to the plane of stress, we have $\sigma_c = 0$ and Eq. (11.33) reduces to

$$u_d = \frac{1}{6G}(\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2) \qquad (11.34)$$

Considering the particular case of a tensile-test specimen, we note that, at yield, we have $\sigma_a = \sigma_Y$, $\sigma_b = 0$, and thus $(u_d)_Y = \sigma_Y^2/6G$. The maximum-distortion-energy criterion for plane stress indicates that a given state of stress is safe as long as $u_d < (u_d)_Y$ or, substituting for u_d from Eq. (11.34), as long as

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_Y^2 \qquad (7.26)$$

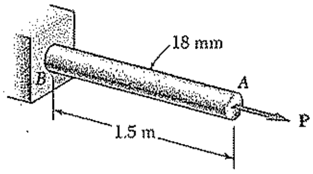
which is the condition stated in Sec. 7.7 and represented graphically by the ellipse of Fig. 7.41. In the case of a general state of stress, the expression (11.33) obtained for u_d should be used. The maximum-distortion-energy criterion is then expressed by the condition.

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 < 2\sigma_Y^2 \qquad (11.35)$$

which indicates that a given state of stress is safe if the point of coordinates σ_a, σ_b, σ_c is located within the surface defined by the equation

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 = 2\sigma_Y^2 \qquad (11.36)$$

This surface is a circular cylinder of radius $\sqrt{2/3}\,\sigma_Y$ with an axis of symmetry forming equal angles with the three principal axes of stress.



SAMPLE PROBLEM 11.1

During a routine manufacturing operation, rod AB must acquire an elastic strain energy of $13.6\ \text{N} \cdot \text{m}$. Using $E = 200$ GPa, determine the required yield strength of the steel if the factor of safety with respect to permanent deformation is to be five.

SOLUTION

Factor of Safety. Since a factor of safety of five is required, the rod should be designed for a strain energy of

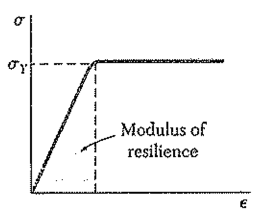
$$U = 5(13.6\ \text{N} \cdot \text{m}) = 68\ \text{N} \cdot \text{m}$$

Strain-Energy Density. The volume of the rod is

$$V = AL = \frac{\pi}{4}(18\ \text{mm})^2(1500\ \text{mm}) = 381700\ \text{mm}^3$$

Since the rod is of uniform cross section, the required strain-energy density is

$$u = \frac{U}{V} = \frac{68000\ \text{N} \cdot \text{mm}}{381700\ \text{mm}^3} = 0.178\ \text{N} \cdot \text{mm/mm}^3$$



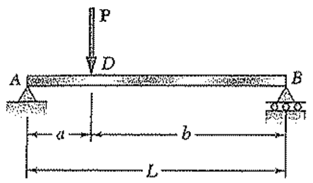
Yield Strength. We recall that the modulus of resilience is equal to the strain-energy density when the maximum stress is equal to σ_Y. Using Eq. (11.8), we write

$$u = \frac{\sigma_Y^2}{2E}$$

$$0.178\ \text{N} \cdot \text{mm/mm}^3 = \frac{\sigma_Y^2}{2(200 \times 10^3\ \text{N/mm}^2)} \qquad \sigma_Y = 266.8\ \text{MPa} \blacktriangleleft$$

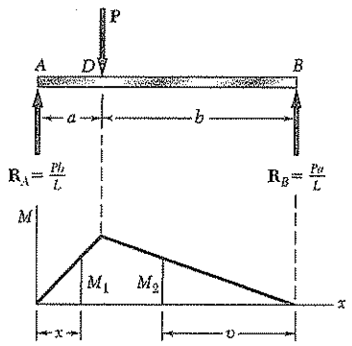
Comment. It is important to note that, since energy loads are not linearly related to the stresses they produce, factors of safety associated with energy loads should be applied to the energy loads and not to the stresses.

SAMPLE PROBLEM 11.2



(*a*) Taking into account only the effect of normal stresses due to bending, determine the strain energy of the prismatic beam AB for the loading shown. (*b*) Evaluate the strain energy, knowing that the beam is a W250 × 67, $P = 160$ kN, $L = 3.6$ m, $a = 0.9$ m, $b = 2.7$ m and $E = 200$ GPa.

SOLUTION



Bending Moment. Using the free-body diagram of the entire beam, we determine the reactions

$$\mathbf{R}_A = \frac{Pb}{L}\uparrow \qquad \mathbf{R}_B = \frac{Pa}{L}\uparrow$$

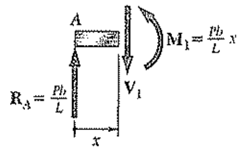
For portion AD of the beam, the bending moment is

$$M_1 = \frac{Pb}{L}x$$

For portion DB, the bending moment at a distance v from end B is

$$M_2 = \frac{Pa}{L}v$$

From A to D:



***a*. Strain Energy.** Since strain energy is a scalar quantity, we add the strain energy of portion AD to that of portion DB to obtain the total strain energy of the beam. Using Eq. (11.17), we write

$$\begin{aligned} U &= U_{AD} + U_{DB} \\ &= \int_0^a \frac{M_1^2}{2EI}\,dx + \int_0^b \frac{M_2^2}{2EI}\,dv \\ &= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}v\right)^2 dv \\ &= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b) \end{aligned}$$

From B to D:

or, since $(a + b) = L$,

$$U = \frac{P^2a^2b^2}{6EIL} \blacktriangleleft$$

***b*. Evaluation of the Strain Energy.** The moment of inertia of a W250 × 67 rolled-steel shape is obtained from Appendix C and the given data is restated.

$P = 160$ kN	$L = 3.6$ m
$a = 0.9$ m	$b = 2.7$ m
$E = 200$ GPa	$I = 104 \times 10^6\ \text{mm}^4$

Substituting into the expression for U, we have

$$U = \frac{(160 \times 10^3\ \text{N})^2(0.9\ \text{m})^2(2.7\ \text{m})^2}{6(200 \times 10^9\ \text{Pa})(104 \times 10^{-6}\ \text{m}^4)(3.6\ \text{m})} \qquad U = 336\ \text{N}\cdot\text{m} \blacktriangleleft$$

PROBLEMS

11.1 Determine the modulus of resilience for each of the following grades of structural steel:

(*a*) ASTM A709 Grade 50: $\sigma_Y = 350$ MPa
(*b*) ASTM A913 Grade 65: $\sigma_Y = 450$ MPa
(*c*) ASTM A709 Grade 100: $\sigma_Y = 700$ MPa

11.2 Determine the modulus of resilience for each of the following aluminum alloys:

(*a*) 1100-H14: $E = 70$ GPa, $\sigma_Y = 55$ MPa
(*b*) 2014-T6: $E = 75$ GPa, $\sigma_Y = 220$ MPa
(*c*) 6061-T6: $E = 70$ GPa, $\sigma_Y = 150$ MPa

11.3 Determine the modulus of resilience for each of the following metals:

(*a*) Stainless steel AISI 302 (annealed): $E = 190$ GPa, $\sigma_Y = 260$ MPa
(*b*) Stainless steel AISI 302 (cold-rolled): $E = 190$ GPa, $\sigma_Y = 520$ MPa
(*c*) Malleable cast iron: $E = 165$ GPa, $\sigma_Y = 230$ MPa

11.4 Determine the modulus of resilience for each of the following alloys:

(*a*) Titanium: $E = 115$ GPa, $\sigma_Y = 875$ MPa
(*b*) Magnesium: $E = 45$ GPa, $\sigma_Y = 200$ MPa
(*c*) Cupronickel (annealed): $E = 140$ GPa, $\sigma_Y = 125$ MPa

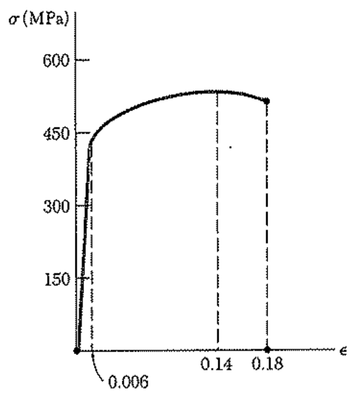


Fig. P11.5

11.5 The stress-strain diagram shown has been drawn from data obtained during a tensile test of an aluminum alloy. Using $E = 72$ GPa, determine (*a*) the modulus of resilience of the alloy, (*b*) the modulus of toughness of the alloy.

11.6 The stress-strain diagram shown has been drawn from data obtained during a tensile test of a specimen of structural steel. Using $E = 200$ GPa, determine (*a*) the modulus of resilience of the steel, (*b*) the modulus of toughness of the steel.

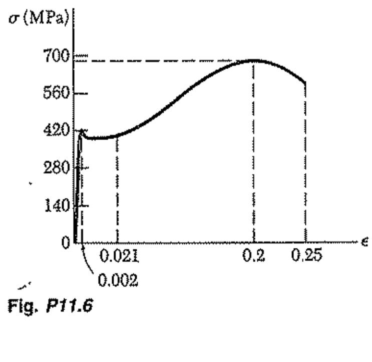


Fig. *P11.6*

11.7 The load-deformation diagram shown has been drawn from data obtained during a tensile test of a 18-mm-diameter rod of an aluminum alloy. Knowing that the deformation was measured using a 400-mm gage length, determine (*a*) the modulus of resilience of the alloy, (*b*) the modulus of toughness of the alloy.

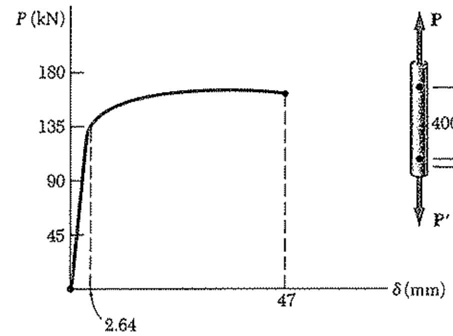


Fig. P11.7

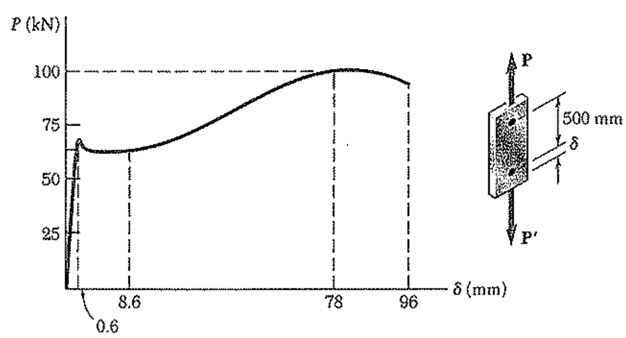


Fig. P11.8

11.8 The load-deformation diagram shown has been drawn from data obtained during a tensile test of structural steel. Knowing that the cross-sectional area of the specimen is 250 mm^2 and that the deformation was measured using a 500-mm gage length, determine (*a*) the modulus of resilience of the steel, (*b*) the modulus of toughness of the steel.

11.9 Using $E = 200$ GPa, determine (*a*) the strain energy of the steel rod ABC when $P = 35$ kN, (*b*) the corresponding strain energy density in portions AB and BC of the rod.

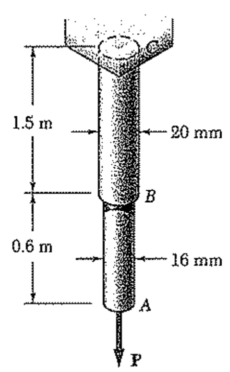


Fig. P11.9

11.10 Using $E = 200$ GPa, determine (*a*) the strain energy of the steel rod ABC when $P = 25$ kN, (*b*) the corresponding strain-energy density in portions AB and BC of the rod.

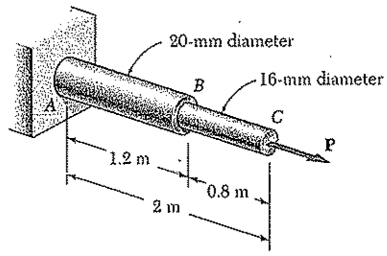


Fig. P11.10 and *P11.11*

11.11 Rod ABC is made of a steel for which the yield strength is $\sigma_Y = 250$ MPa and the modulus of elasticity is $E = 200$ GPa. Determine, for the loading shown, the maximum strain energy that can be acquired by the rod without causing any permanent deformation.

11.12 Rods AB and BC are made of a steel for which the yield strength is $\sigma_Y = 300$ MPa and the modulus of elasticity is $E = 200$ GPa. Determine the maximum strain energy that can be acquired by the assembly without causing permanent deformation when the length a of rod AB is (*a*) 2 m, (*b*) 4 m.

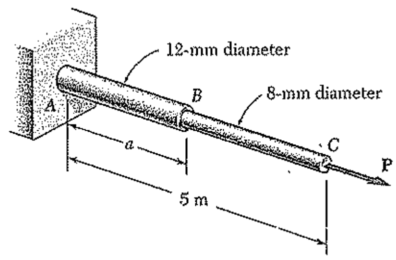


Fig. *P11.12*

11.13 Rod BC is made of a steel for which the yield strength is $\sigma_Y = 300$ MPa and the modulus of elasticity is $E = 200$ GPa. Knowing that a strain energy of 10 J must be acquired by the rod when the axial load **P** is applied, determine the diameter of the rod for which the factor of safety with respect to permanent deformation is six.

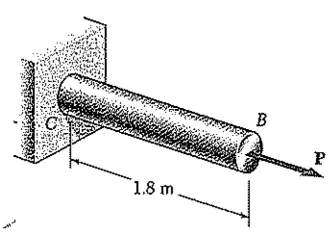


Fig. P11.13

11.14 Rod ABC is made of a steel for which the yield strength is $\sigma_Y = 450$ MPa and the modulus of elasticity is $E = 200$ GPa. Knowing that a strain energy of 7 N · m must be acquired by the rod as the axial load **P** is applied, determine the factor of safety of the rod with respect to permanent deformation.

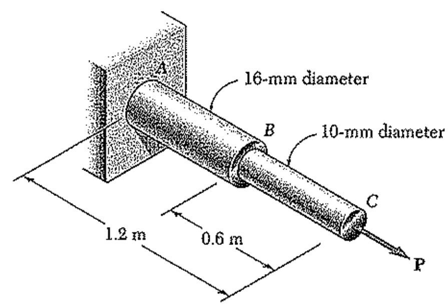


Fig. P11.14

11.15 Show by integration that the strain energy of the tapered rod AB is

$$U = \frac{1}{4}\frac{P^2L}{EA_{min}}$$

where A_{min} is the cross-sectional area at end B.

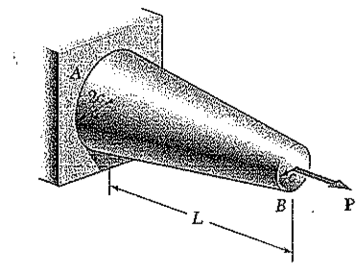


Fig. P11.15

11.16 Solve Prob. 11.15 using the stepped rod shown as an approximation of the tapered rod. What is the percentage error in the answer obtained?

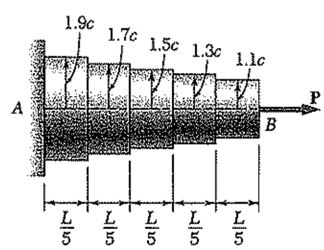


Fig. P11.16

11.17 through 11.20 In the truss shown, all members are made of the same material and have the uniform cross-sectional area indicated. Determine the strain energy of the truss when the load **P** is applied.

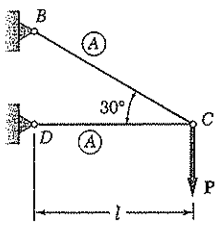


Fig. P11.17

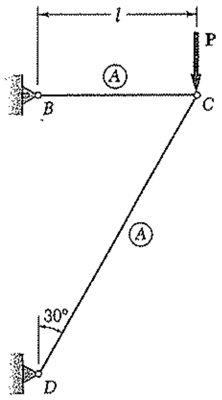


Fig. P11.18

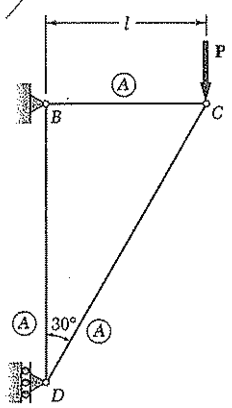


Fig. P11.19

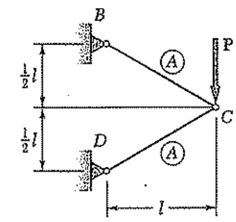


Fig. P11.20

11.21 Each member of the truss shown is made of aluminum and has the cross-sectional area shown. Using $E = 72$ GPa, determine the strain energy of the truss for the loading shown.

11.22 Solve Prob. 11.21, assuming that the 100-kN load is removed.

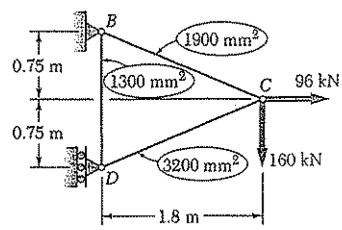


Fig. P11.21

11.23 through 11.26 Taking into account only the effect of normal stresses, determine the strain energy of the prismatic beam AB for the loading shown.

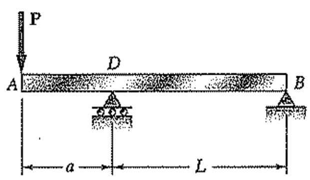


Fig. P11.23

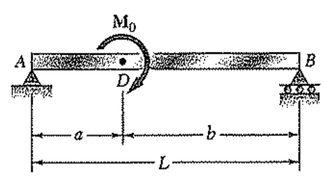


Fig. P11.24

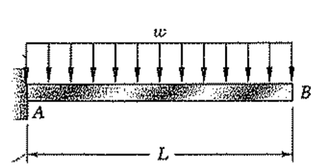


Fig. P11.25

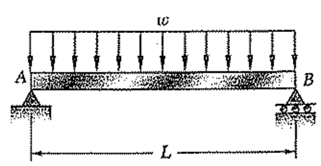


Fig. P11.26

11.27 Assuming that the prismatic beam AB has a rectangular cross section, show that for the given loading the maximum value of the strain-energy density in the beam is

$$u_{max} = 15\frac{U}{V}$$

where U is the strain energy of the beam and V is its volume.

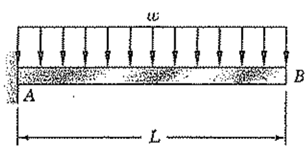


Fig. P11.27

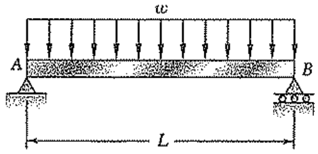


Fig. P11.28

11.28 Assuming that the prismatic beam AB has a rectangular cross section, show that for the given loading the maximum value of the strain-energy density in the beam is

$$u_{max} = \frac{45}{8}\frac{U}{V}$$

where U is the strain energy of the beam and V is its volume.

11.29 and 11.30 Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.

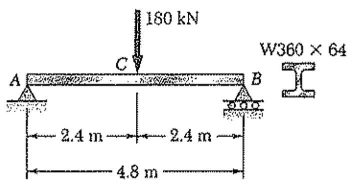


Fig. P11.29

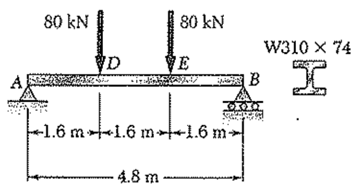


Fig. P11.30

11.31 **and 11.32** Using $E = 200$ GPa, determine the strain energy due to bending for the steel beam and loading shown.

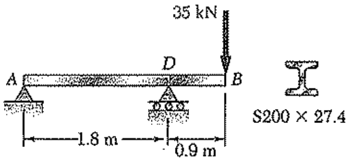


Fig. *P11.31*

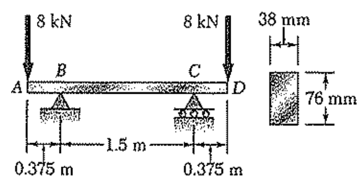


Fig. P11.32

11.33 In the assembly shown torques $\mathbf{T}_A$ and $\mathbf{T}_B$ are exerted on disks A and B, respectively. Knowing that both shafts are solid and made of aluminum ($G = 73$ GPa), determine the total strain energy acquired by the assembly.

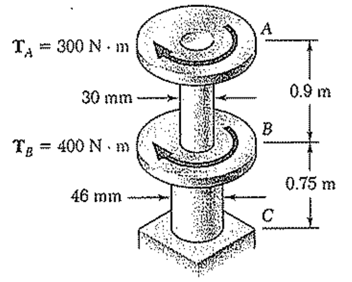


Fig. P11.33

11.34 The design specifications for the steel shaft AB require that the shaft acquire a strain energy of 45 N · m as the 28 kN · m torque is applied. Using $G = 77$ GPa, determine (*a*) the largest inner diameter of the shaft that can be used, (*b*) the corresponding maximum shearing stress in the shaft.

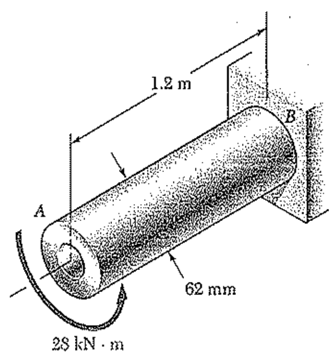


Fig. P11.34

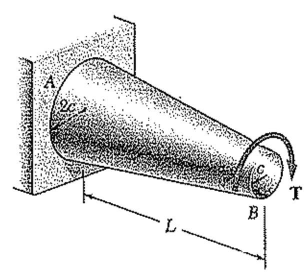


Fig. P11.35

11.35 Show by integration that the strain energy in the tapered rod AB is

$$U = \frac{7}{48}\frac{T^2L}{GJ_{\min}}$$

where $J_{\min}$ is the polar moment of inertia of the rod at end B.

11.36 The state of stress shown occurs in a machine component made of a grade of steel for which $\sigma_Y = 450$ MPa. Using the maximum-distortion-energy criterion, determine the range of values of σ_y for which the factor of safety associated with the yield strength is equal to or larger than 2.2.

11.37 The state of stress shown occurs in a machine component made of a grade of steel for which $\sigma_Y = 450$ MPa. Using the maximum-distortion-energy criterion, determine the factor of safety associated with the yield strength when (*a*) $\sigma_Y = +110$ MPa, (*b*) $\sigma_Y = -110$ MPa.

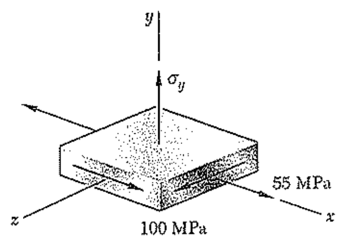


Fig. P11.36 and P11.37

11.38 The state of stress shown occurs in a machine component made of a brass for which $\sigma_Y = 160$ MPa. Using the maximum-distortion-energy criterion, determine whether yield occurs when (*a*) $\sigma_z = +45$ MPa, (*b*) $\sigma_z = -45$ MPa.

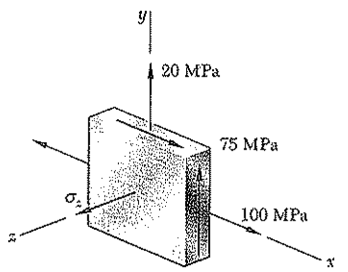
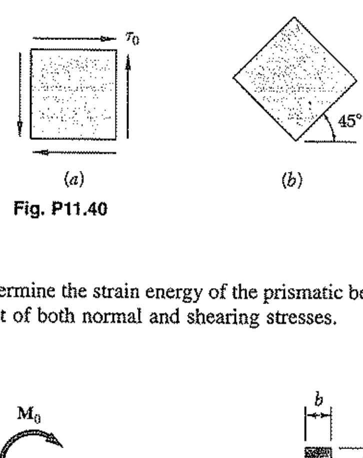


Fig. P11.38 and *P11.39*

11.39 The state of stress shown occurs in a machine component made of a brass for which $\sigma_Y = 160$ MPa. Using the maximum-distortion-energy criterion, determine the range of values of σ_z for which yield does not occur.

11.40 For the state of stress shown in Fig. *a*, determine the stresses in an element oriented as shown in Fig. *b*. Compare the strain energy density in the given state by first using Fig. *a* and then by using Fig. *b*. Equating the two results obtained, show that

$$G = \frac{E}{2(1 + \nu)}$$



(*a*) (*b*)

Fig. P11.40

11.41 Determine the strain energy of the prismatic beam *AB*, taking into account the effect of both normal and shearing stresses.

Fig. P11.41

11.7. IMPACT LOADING

Consider a rod BD of uniform cross section which is hit at its end B by a body of mass m moving with a velocity $\mathbf{v}_0$ (Fig. 11.23*a*). As the rod deforms under the impact (Fig. 11.23b), stresses develop within the rod and reach a maximum value σ_m. After vibrating for a while, the rod will come to rest, and all stresses will disappear. Such a sequence of events is referred to as an *impact loading* (Fig. 11.24).

In order to determine the maximum value σ_m of the stress occurring at a given point of a structure subjected to an impact loading, we are going to make several simplifying assumptions.

First, we assume that the kinetic energy $T = \frac{1}{2}mv_0^2$ of the striking body is transferred entirely to the structure and, thus, that the strain energy U_m corresponding to the maximum deformation x_m is

$$U_m = \tfrac{1}{2}mv_0^2 \tag{11.37}$$

This assumption leads to the following two specific requirements:

1. No energy should be dissipated during the impact.
2. The striking body should not bounce off the structure and retain part of its energy. This, in turn, necessitates that the inertia of the structure be negligible, compared to the inertia of the striking body.

In practice, neither of these requirements is satisfied, and only part of the kinetic energy of the striking body is actually transferred to the structure. Thus, assuming that all of the kinetic energy of the striking body is transferred to the structure leads to a conservative design of that structure.

We further assume that the stress-strain diagram obtained from a static test of the material is also valid under impact loading. Thus, for an elastic deformation of the structure, we can express the maximum value of the strain energy as

$$U_m = \int \frac{\sigma_m^2}{2E}dV \tag{11.38}$$

In the case of the uniform rod of Fig. 11.23, the maximum stress σ_m has the same value throughout the rod, and we write $U_m = \sigma_m^2 V/2E$. Solving for σ_m and substituting for U_m from Eq. (11.37), we write

$$\sigma_m = \sqrt{\frac{2U_mE}{V}} = \sqrt{\frac{mv_0^2E}{V}} \tag{11.39}$$

We note from the expression obtained that selecting a rod with a large volume V and a low modulus of elasticity E will result in a smaller value of the maximum stress σ_m for a given impact loading.

In most problems, the distribution of stresses in the structure is not uniform, and formula (11.39) does not apply. It is then convenient to determine the static load $\mathbf{P}_m$, which would produce the same strain energy as the impact loading, and compute from P_m the corresponding value σ_m of the largest stress occurring in the structure.

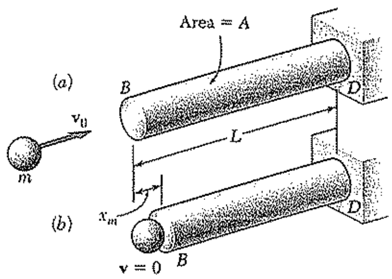


Fig. 11.23

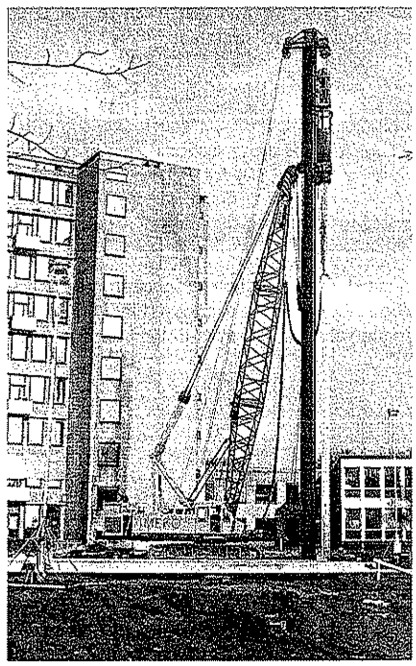


Fig. 11.24 Steam alternately lifts a weight inside the pile driver and then propels it downward. This delivers a large impact load to the pile which is being driven into the ground.

EXAMPLE 11.06

A body of mass m moving with a velocity $\mathbf{v}_0$ hits the end B of the nonuniform rod BCD (Fig. 11.25). Knowing that the diameter of portion BC is twice the diameter of portion CD, determine the maximum value σ_m of the stress in the rod.

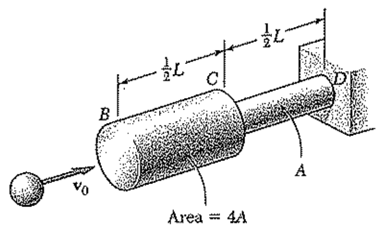


Fig. P11.25

Making $n = 2$ in the expression (11.15) obtained in Example 11.01, we find that when rod BCD is subjected to a static load P_m, its strain energy is

$$U_m = \frac{5P_m^2L}{16AE} \tag{11.40}$$

where A is the cross-sectional area of portion CD of the rod. Solving Eq. (11.40) for P_m, we find that the static load that produces in the rod the same strain energy as the given impact loading is

$$P_m = \sqrt{\frac{16}{5}\frac{U_mAE}{L}}$$

where U_m is given by Eq. (11.37). The largest stress occurs in portion CD of the rod. Dividing P_m by the area A of that portion, we have

$$\sigma_m = \frac{P_m}{A} = \sqrt{\frac{16}{5}\frac{U_mE}{AL}} \tag{11.41}$$

or, substituting for U_m from Eq. (11.37),

$$\sigma_m = \sqrt{\frac{8}{5}\frac{mv_0^2E}{AL}} = 1.265\sqrt{\frac{mv_0^2E}{AL}}$$

Comparing this value with the value obtained for σ_m in the case of the uniform rod of Fig. 11.24 and making $V = AL$ in Eq. (11.39), we note that the maximum stress in the rod of variable cross section is 26.5% larger than in the lighter uniform rod. Thus, as we observed earlier in our discussion of Example 11.01, increasing the diameter of portion BC of the rod results in a *decrease* of the energy-absorbing capacity of the rod.

EXAMPLE 11.07

A block of weight W is dropped from a height h onto the free end of the cantilever beam AB (Fig. 11.26). Determine the maximum value of the stress in the beam.

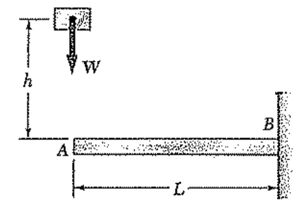


Fig. 11.26

As it falls through the distance h, the potential energy Wh of the block is transformed into kinetic energy. As a result of the impact, the kinetic energy in turn is transformed into strain energy. We have, therefore,†

$$U_m = Wh \tag{11.42}$$

†The total distance through which the block drops is actually $h + y_m$, where y_m is the maximum deflection of the end of the beam. Thus, a more accurate expression for U_m (see Sample Prob. 11.3) is

$$U_m = W(h + y_m) \tag{11.42'}$$

However, when $h \gg y_m$, we may neglect y_m and use Eq. (11.42).

Recalling the expression obtained for the strain energy of the cantilever beam AB in Example 11.03 and neglecting the effect of shear, we write

$$U_m = \frac{P_m^2L^3}{6EI}$$

Solving this equation for P_m, we find that the static force that produces in the beam the same strain energy is

$$P_m = \sqrt{\frac{6U_mEI}{L^3}} \tag{11.43}$$

The maximum stress σ_m occurs at the fixed end B and is

$$\sigma_m = \frac{|M|c}{I} = \frac{P_mLc}{I}$$

Substituting for P_m from (11.43), we write

$$\sigma_m = \sqrt{\frac{6U_mE}{L(I/c^2)}} \tag{11.44}$$

or, recalling (11.42),

$$\sigma_m = \sqrt{\frac{6WhE}{L(I/c^2)}}$$

11.8. DESIGN FOR IMPACT LOADS

Let us now compare the values obtained in the preceding section for the maximum stress σ_m (*a*) in the rod of uniform cross section of Fig. 11.23, (*b*) in the rod of variable cross section of Example 11.06, and (*c*) in the cantilever beam of Example 11.07, assuming that the last has a circular cross section of radius *c*.

(*a*) We first recall from Eq. (11.39) that, if U_m denotes the amount of energy transferred to the rod as a result of the impact loading, the maximum stress in the rod of uniform cross section is

$$\sigma_m = \sqrt{\frac{2U_mE}{V}} \qquad (11.45a)$$

where V is the volume of the rod.

(*b*) Considering next the rod of Example 11.06 and observing that the volume of the rod is

$$V = 4A(L/2) + A(L/2) = 5AL/2$$

we substitute $AL = 2V/5$ into Eq. (11.41) and write

$$\sigma_m = \sqrt{\frac{8U_mE}{V}} \qquad (11.45b)$$

(*c*) Finally, recalling that $I = \frac{1}{4}\pi c^4$ for a beam of circular cross section, we note that

$$L(I/c^2) = L(\tfrac{1}{4}\pi c^4/c^2) = \tfrac{1}{4}(\pi c^2 L) = \tfrac{1}{4}V$$

where V denotes the volume of the beam. Substituting into Eq. (11.44), we express the maximum stress in the cantilever beam of Example 11.07 as

$$\sigma_m = \sqrt{\frac{24U_mE}{V}} \qquad (11.45c)$$

We note that, in each case, the maximum stress σ_m is proportional to the square root of the modulus of elasticity of the material and inversely proportional to the square root of the volume of the member. Assuming all three members to have the same volume and to be of the same material, we also note that, for a given value of the absorbed energy, the uniform rod will experience the lowest maximum stress, and the cantilever beam the highest one.

This observation can be explained by the fact that, the distribution of stresses being uniform in case *a*, the strain energy will be uniformly distributed throughout the rod. In case *b*, on the other hand, the stresses in portion *BC* of the rod are only 25% as large as the stresses in portion *CD*. This uneven distribution of the stresses and of the strain energy results in a maximum stress σ_m twice as large as the corresponding stress in the uniform rod. Finally, in case *c*, where the cantilever beam is subjected to a transverse impact loading, the stresses vary linearly along the beam as well as across a transverse section. The very uneven resulting distribution of strain energy causes the maximum stress σ_m to be 3.46 times larger than if the same member had been loaded axially as in case *a*.

The properties noted in the three specific cases discussed in this section are quite general and can be observed in all types of structures and impact loadings. We thus conclude that a structure designed to withstand effectively an impact load should

1. Have a large volume
2. Be made of a material with a low modulus of elasticity and a high yield strength
3. Be shaped so that the stresses are distributed as evenly as possible throughout the structure

11.9. WORK AND ENERGY UNDER A SINGLE LOAD

When we first introduced the concept of strain energy at the beginning of this chapter, we considered the work done by an axial load **P** applied to the end of a rod of uniform cross section (Fig. 11.1). We defined the strain energy of the rod for an elongation x_1 as the work of the load **P** as it is slowly increased from 0 to the value P_1 corresponding to x_1. We wrote

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \tag{11.2}$$

In the case of an elastic deformation, the work of the load **P**, and thus the strain energy of the rod, were expressed as

$$U = \tfrac{1}{2}P_1x_1 \tag{11.3}$$

Later, in Secs. 11.4 and 11.5, we computed the strain energy of structural members under various loading conditions by determining the strain-energy density u at every point of the member and integrating u over the entire member.

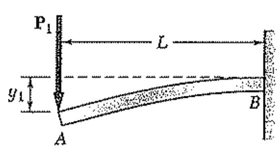


Fig. 11.27

However, when a structure or member is subjected to a *single concentrated load*, it is possible to use Eq. (11.3) to evaluate its elastic strain energy, provided, of course, that the relation between the load and the resulting deformation is known. For instance, in the case of the cantilever beam of Example 11.03 (Fig. 11.27), we write

$$U = \tfrac{1}{2}P_1y_1$$

and, substituting for y_1 the value obtained from the table of *Beam Deflections and Slopes* of Appendix D,

$$U = \frac{1}{2}P_1\left(\frac{P_1L^3}{3EI}\right) = \frac{P_1^2L^3}{6EI} \tag{11.46}$$

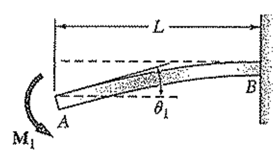


Fig. 11.28

A similar approach can be used to determine the strain energy of a structure or member subjected to a *single couple*. Recalling that the elementary work of a couple of moment M is $M\,d\theta$, where $d\theta$ is a small angle, we find, since M and θ are linearly related, that the elastic strain energy of a cantilever beam AB subjected to a single couple $\mathbf{M}_1$ at its end A (Fig. 11.28) can be expressed as

$$U = \int_0^{\theta_1} M\,d\theta = \tfrac{1}{2}M_1\theta_1 \tag{11.47}$$

where θ_1 is the slope of the beam at A. Substituting for θ_1 the value obtained from Appendix D, we write

$$U = \frac{1}{2}M_1\left(\frac{M_1L}{EI}\right) = \frac{M_1^2L}{2EI} \tag{11.48}$$

In a similar way, the elastic strain energy of a uniform circular shaft AB of length L subjected at its end B to a single torque $\mathbf{T}_1$ (Fig. 11.29) can be expressed as

$$U = \int_0^{\phi_1} T\,d\phi = \tfrac{1}{2}T_1\phi_1 \tag{11.49}$$

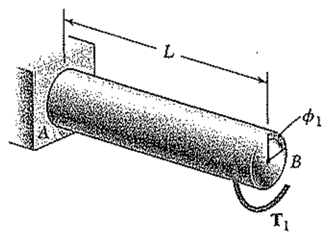


Fig. 11.29

Substituting for the angle of twist ϕ_1 from Eq. (3.16), we verify that

$$U = \frac{1}{2}T_1\left(\frac{T_1L}{JG}\right) = \frac{T_1^2L}{2JG}$$

as previously obtained in Sec. 11.5.

The method presented in this section may simplify the solution of many impact-loading problems. In Example 11.08, the crash of an automobile into a barrier (Fig. 11.30) is considered by using a simplified model consisting of a block and a simple beam.

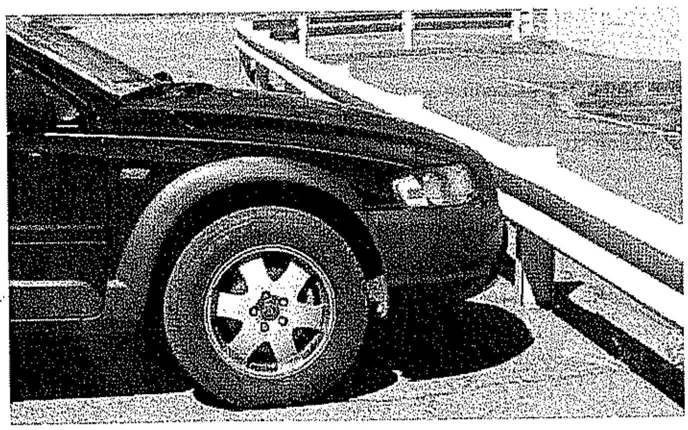


Fig. 11.30 As the automobile crashes into the barrier considerable energy will be dissipated as heat during the permanent deformation of the automobile and the barrier.

EXAMPLE 11.08

A block of mass m moving with a velocity $\mathbf{v}_0$ hits squarely the prismatic member AB at its midpoint C (Fig. 11.31). Determine (a) the equivalent static load P_m, (b) the maximum stress σ_m in the member, and (c) the maximum deflection x_m at point C.

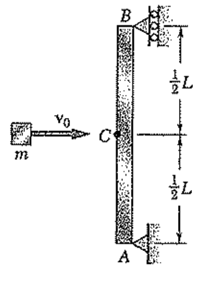


Fig. 11.31

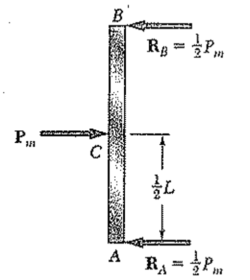


Fig. 11.32

(a) Equivalent Static Load. The maximum strain energy of the member is equal to the kinetic energy of the block before impact. We have

$$U_m = \tfrac{1}{2}mv_0^2 \tag{11.50}$$

On the other hand, expressing U_m as the work of the equivalent horizontal static load as it is slowly applied at the midpoint C of the member, we write

$$U_m = \tfrac{1}{2}P_m x_m \tag{11.51}$$

where x_m is the deflection of C corresponding to the static load P_m. From the table of *Beam Deflections and Slopes* of Appendix D, we find that

$$x_m = \frac{P_m L^3}{48EI} \tag{11.52}$$

Substituting for x_m from (11.52) into (11.51), we write

$$U_m = \frac{1}{2}\frac{P_m^2 L^3}{48EI}$$

Solving for P_m and recalling Eq. (11.50), we find that the static load equivalent to the given impact loading is

$$P_m = \sqrt{\frac{96U_m EI}{L^3}} = \sqrt{\frac{48mv_0^2 EI}{L^3}} \tag{11.53}$$

(b) Maximum Stress. Drawing the free-body diagram of the member (Fig. 11.32), we find that the maximum value of the bending moment occurs at C and is $M_{max} = P_m L/4$. The maximum stress, therefore, occurs in a transverse section through C and is equal to

$$\sigma_m = \frac{M_{max}c}{I} = \frac{P_m Lc}{4I}$$

Substituting for P_m from (11.53), we write

$$\sigma_m = \sqrt{\frac{3mv_0^2 EI}{L(I/c)^2}}$$

(c) Maximum Deflection. Substituting into Eq. (11.52) the expression obtained for P_m in (11.53), we have

$$x_m = \frac{L^3}{48EI}\sqrt{\frac{48mv_0^2 EI}{L^3}} = \sqrt{\frac{mv_0^2 L^3}{48EI}}$$

11.10. DEFLECTION UNDER A SINGLE LOAD BY THE WORK-ENERGY METHOD

We saw in the preceding section that, if the deflection x_1 of a structure or member under a single concentrated load $\mathbf{P}_1$ is known, the corresponding strain energy U is obtained by writing

$$U = \tfrac{1}{2}P_1 x_1 \tag{11.3}$$

A similar expression for the strain energy of a structural member under a single couple $\mathbf{M}_1$ is:

$$U = \tfrac{1}{2}M_1\theta_1 \tag{11.47}$$

Conversely, if the strain energy U of a structure or member subjected to a single concentrated load $\mathbf{P}_1$ or couple $\mathbf{M}_1$ is known, Eq. (11.3) or (11.47) can be used to determine the corresponding deflection x_1 or angle θ_1. In order to determine the deflection under a single load applied to a structure consisting of several component parts, it is easier, rather than use one of the methods of Chap. 9, to first compute the strain energy of the structure by integrating the strain-energy density over its various parts, as was done in Secs. 11.4 and 11.5, and then use either Eq. (11.3) or Eq. (11.47) to obtain the desired deflection. Similarly, the angle of twist ϕ_1 of a composite shaft can be obtained by integrating the strain-energy density over the various parts of the shaft and solving Eq. (11.49) for ϕ_1.

It should be kept in mind that the method presented in this section can be used *only if the given structure is subjected to a single concentrated load or couple.* The strain energy of a structure subjected to several loads *cannot* be determined by computing the work of each load as if it were applied independently to the structure (see Sec. 11.11). We can also observe that, even if it were possible to compute the strain energy of the structure in this manner, only one equation would be available to determine the deflections corresponding to the various loads. In Secs. 11.12 and 11.13, another method based on the concept of strain energy is presented, one that can be used to determine the deflection or slope at a given point of a structure, even when that structure is subjected simultaneously to several concentrated loads, distributed loads, or couples.

EXAMPLE 11.09

A load $\mathbf{P}$ is supported at B by two uniform rods of the same cross-sectional area A (Fig. 11.33). Determine the vertical deflection of point B.

The strain energy of the system under the given load was determined in Example 11.02. Equating the expression obtained for U to the work of the load, we write

$$U = 0.364\frac{P^2 l}{AE} = \frac{1}{2}Py_B$$

and, solving for the vertical deflection of B,

$$y_B = 0.728\frac{Pl}{AE}$$

Remark. We should note that, once the forces in the two rods have been obtained (see Example 11.02), the deformations $\delta_{B/C}$ and $\delta_{B/D}$ of the rods could be obtained by the method of Chap. 2. Determining the vertical deflection of point B from these deformations, however, would require a careful geometric analysis of the various displacements involved. The strain-energy method used here makes such an analysis unnecessary.

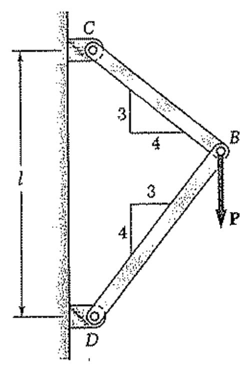


Fig. 11.33

EXAMPLE 11.10

Determine the deflection of end A of the cantilever beam AB (Fig. 11.34), taking into account the effect of (a) the normal stresses only, (b) both the normal and shearing stresses.

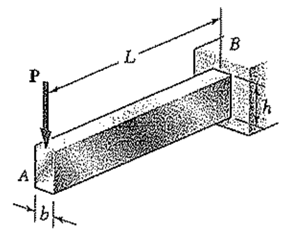


Fig. 11.34

(a) Effect of Normal Stresses. The work of the force **P** as it is slowly applied to A is

$$U = \tfrac{1}{2}Py_A$$

Substituting for U the expression obtained for the strain energy of the beam in Example 11.03, where only the effect of the normal stresses was considered, we write

$$\frac{P^2L^3}{6EI} = \frac{1}{2}Py_A$$

and, solving for y_A,

$$y_A = \frac{PL^3}{3EI}$$

(b) Effect of Normal and Shearing Stresses. We now substitute for U the expression (11.24) obtained in Example 11.05, where the effects of both the normal and shearing stresses were taken into account. We have

$$\frac{P^2L^3}{6EI}\left(1 + \frac{3Eh^2}{10GL^2}\right) = \frac{1}{2}Py_A$$

and, solving for y_A,

$$y_A = \frac{PL^3}{3EI}\left(1 + \frac{3Eh^2}{10GL^2}\right)$$

We note that the relative error when the effect of shear is neglected is the same that was obtained in Example 11.05, i.e., less than $0.9(h/L)^2$. As we indicated then, this is less than 0.9% for a beam with a ratio h/L less than $\frac{1}{10}$.

EXAMPLE 11.11

A torque **T** is applied at the end D of shaft BCD (Fig. 11.35). Knowing that both portions of the shaft are of the same material and same length, but that the diameter of BC is twice the diameter of CD, determine the angle of twist for the entire shaft.

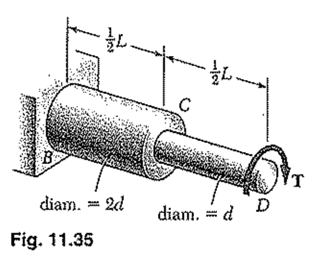


Fig. 11.35

The strain energy of a similar shaft was determined in Example 11.04 by breaking the shaft into its component parts BC and CD. Making $n = 2$ in Eq. (11.23), we have

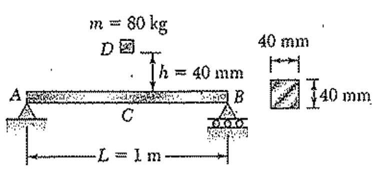
$$U = \frac{17}{32}\frac{T^2L}{2GJ}$$

where G is the modulus of rigidity of the material and J the polar moment of inertia of portion CD of the shaft. Setting U equal to the work of the torque as it is slowly applied to end D, and recalling Eq. (11.49), we write

$$\frac{17}{32}\frac{T^2L}{2GJ} = \frac{1}{2}T\phi_{D/B}$$

and, solving for the angle of twist $\phi_{D/B}$,

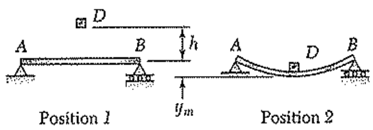
$$\phi_{D/B} = \frac{17TL}{32GJ}$$



SAMPLE PROBLEM 11.3

The block D of mass m is released from rest and falls a distance h before it strikes the midpoint C of the aluminum beam AB. Using $E = 73$ GPa, determine (*a*) the maximum deflection of point C, (*b*) the maximum stress that occurs in the beam.

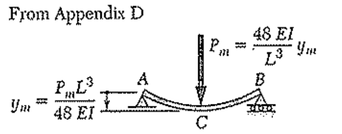
SOLUTION



Principle of Work and Energy. Since the block is released from rest, we note that in position *1* both the kinetic energy and the strain energy are zero. In position *2*, where the maximum deflection y_m occurs, the kinetic energy is again zero. Referring to the table of *Beam Deflections and Slopes* of Appendix D, we find the expression for y_m shown. The strain energy of the beam in position *2* is

$$U_2 = \frac{1}{2}P_m y_m = \frac{1}{2}\frac{48EI}{L^3}y_m^2 \qquad U_2 = \frac{24EI}{L^3}y_m^2$$

From Appendix D



We observe that the work done by the weight $\mathbf{W}$ of the block is $W(h + y_m)$. Equating the strain energy of the beam to the work done by $\mathbf{W}$, we have

$$\frac{24EI}{L^3}y_m^2 = W(h + y_m) \tag{1}$$

***a*. Maximum Deflection of Point *C*.** From the given data we have

$$EI = (73 \times 10^9 \text{ Pa})\tfrac{1}{12}(0.04 \text{ m})^4 = 15.573 \times 10^3 \text{ N} \cdot \text{m}^2$$
$$L = 1 \text{ m} \qquad h = 0.040 \text{ m} \qquad W = mg = (80 \text{ kg})(9.81 \text{ m/s}^2) = 784.8 \text{ N}$$

Substituting into Eq. (1), we obtain and solve the quadratic equation

$$(373.8 \times 10^3)y_m^2 - 784.8y_m - 31.39 = 0 \qquad y_m = 10.27 \text{ mm} \ \triangleleft$$

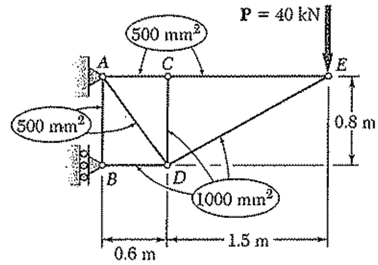
***b*. Maximum Stress.** The value of P_m is

$$P_m = \frac{48EI}{L^3}y_m = \frac{48(15.573 \times 10^3 \text{ N} \cdot \text{m})}{(1 \text{ m})^3}(0.01027 \text{ m}) \qquad P_m = 7677 \text{ N}$$

Recalling that $\sigma_m = M_{max}c/I$ and $M_{max} = \frac{1}{4}P_m L$, we write

$$\sigma_m = \frac{(\frac{1}{4}P_m L)c}{I} = \frac{\frac{1}{4}(7677 \text{ N})(1 \text{ m})(0.020 \text{ m})}{\frac{1}{12}(0.040 \text{ m})^4} \qquad \sigma_m = 179.9 \text{ MPa} \ \triangleleft$$

An approximation for the work done by the weight of the block can be obtained by omitting y_m from the expression for the work and from the right-hand member of Eq. (1), as was done in Example 11.07. If this approximation is used here, we find $y_m = 9.16$ mm; the error is 10.8%. However, if an 8-kg block is dropped from a height of 400 mm, producing the same value of Wh, omitting y_m from the right-hand member of Eq. (1) results in an error of only 1.2%. A further discussion of this approximation is given in Prob. 11.70.

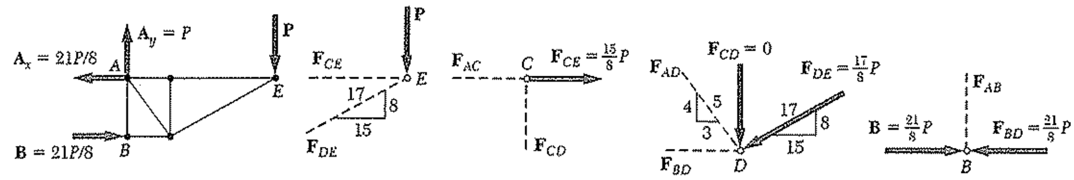


SAMPLE PROBLEM 11.4

Members of the truss shown consist of sections of aluminum pipe with the cross-sectional areas indicated. Using $E = 73$ GPa, determine the vertical deflection of point E caused by the load **P**.

SOLUTION

Axial Forces in Truss Members. The reactions are found by using the free-body diagram of the entire truss. We then consider in sequence the equilibrium of joints, E, C, D, and B. At each joint we determine the forces indicated by dashed lines. At joint B, the equation $\Sigma F_x = 0$ provides a check of our computations.



$\Sigma F_y = 0{:}\ F_{DE} = -\frac{17}{8}P \qquad \Sigma F_x = 0{:}\ F_{AC} = +\frac{15}{8}P \qquad \Sigma F_y = 0{:}\ F_{AD} = +\frac{5}{4}P \qquad \Sigma F_y = 0{:}\ F_{AB} = 0$

$\Sigma F_x = 0{:}\ F_{CE} = +\frac{15}{8}P \qquad \Sigma F_y = 0{:}\ F_{CD} = 0 \qquad \Sigma F_x = 0{:}\ F_{BD} = -\frac{21}{8}P \qquad \Sigma F_x = 0{:}$ (Checks)

Strain Energy. Noting that E is the same for all members, we express the strain energy of the truss as follows

$$U = \sum \frac{F_i^2 L_i}{2A_i E} = \frac{1}{2E} \sum \frac{F_i^2 L_i}{A_i} \tag{1}$$

where F_i is the force in a given member as indicated in the following table and where the summation is extended over all members of the truss.

Member	F_i	L_i, m	A_i, m²	$\frac{F_i^2 L_i}{A_i}$
AB	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0.6	500×10^{-6}	$4\,219P^2$
AD	$+5P/4$	1.0	500×10^{-6}	$3\,125P^2$
BD	$-21P/8$	0.6	1000×10^{-6}	$4\,134P^2$
CD	0	0.8	1000×10^{-6}	0
CE	$+15P/8$	1.5	500×10^{-6}	$10\,547P^2$
DE	$-17P/8$	1.7	1000×10^{-6}	$7\,677P^2$

$$\sum \frac{F_i^2 L_i}{A_i} = 29\,700P^2$$

Returning to Eq. (1), we have

$U = (1/2E)(29.7 \times 10^3 P^2)$.

Principle of Work-Energy. We recall that the work done by the load **P** as it is gradually applied is $\frac{1}{2}Py_E$. Equating the work done by **P** to the strain energy U and recalling that $E = 73$ GPa and $P = 40$ kN, we have

$$\frac{1}{2}Py_E = U \qquad \frac{1}{2}Py_E = \frac{1}{2E}(29.7 \times 10^3 P^2)$$

$$y_E = \frac{1}{E}(29.7 \times 10^3 P) = \frac{(29.7 \times 10^3)(40 \times 10^3)}{73 \times 10^9}$$

$$y_E = 16.27 \times 10^{-3}\ \text{m} \qquad\qquad y_E = 16.27\ \text{mm} \downarrow$$

PROBLEMS

11.42 A 5-kg collar D moves along the uniform rod AB and has a speed $v_0 = 6$ m/s when it strikes a small plate attached to end A of the rod. Using $E = 200$ GPa and knowing that the allowable stress in the rod is 250 MPa, determine the smallest diameter that can be used for the rod.

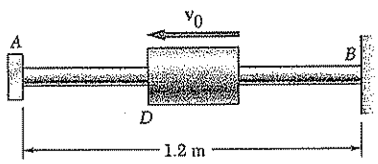


Fig. P11.42 and *P11.43*

11.43 A 6-kg collar has a speed $v_0 = 4.5$ m/s when it strikes a small plate attached to end A of the 20-mm-diameter rod AB. Using $E = 200$ GPa, determine (*a*) the equivalent static load, (*b*) the maximum stress in the rod, (*c*) the maximum deflection of the A.

11.44 The cylindrical block E has a speed $v_0 = 4.8$ m/s when it strikes squarely the yoke BD that is attached to the 22-mm-diameter rods AB and CD. Knowing that the rods are made of a steel for which $\sigma_Y = 350$ MPa and $E = 200$ GPa, determine the weight of block E for which the factor of safety is 5 with respect to permanent deformation of the rods.

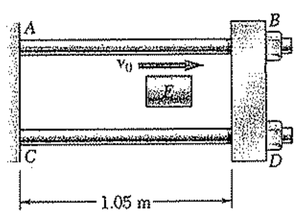


Fig. P11.44 and P11.45

11.45 The 8-kg cylindrical block E has a horizontal velocity v_0 when it strikes squarely the yoke BD that is attached to the 22-mm-diameter rods AB and CD. Knowing that the rods are made of a steel for which $\sigma_Y = 350$ MPa and $E = 200$ GPa, determine the maximum allowable speed v_0 if the rods are not to be permanently deformed.

11.46 Collar D is released from rest in the position shown and is stopped by a small plate attached at end C of the vertical rod ABC. Determine the mass of the collar for which the maximum normal stress in portion BC is 125 MPa.

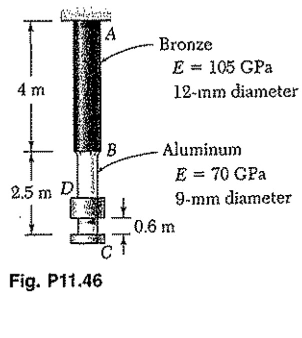


Fig. P11.46

11.47 Solve Prob. 11.46, assuming that both portions of rod *ABC* are made of aluminum.

11.48 The steel beam *AB* is struck squarely at its midpoint *C* by a 4.5-kg block moving horizontally with a speed $v_0 = 2$ m/s. Using $E = 200$ GPa, determine (*a*) the equivalent static load, (*b*) the maximum normal stress in the beam, (*c*) the maximum deflection of the midpoint *C* of the beam.

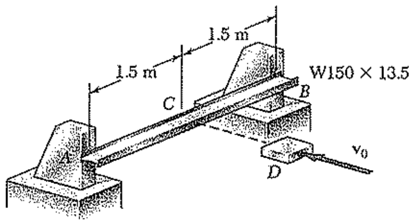


Fig. P11.48

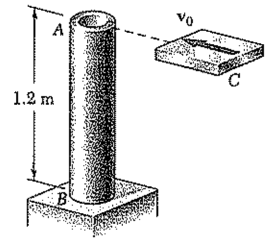


Fig. P11.49

11.49 The post *AB* consists of a steel pipe of 80-mm diameter and 6-mm wall thickness. A 6-kg block *C* moving horizontally with at velocity $\mathbf{v}_0$ hits the post squarely at *A*. Using $E = 200$ GPa, determine the largest speed v_0 for which the maximum normal stress in the pipe does not exceed 180 MPa.

11.50 Solve Prob. 11.49, assuming that the post *AB* consists of a solid steel rod of 80-mm diameter.

11.51 The 20-kg block *D* is dropped from a height $h = 0.18$ m onto the steel beam *AB*. Knowing that $E = 200$ GPa, determine (*a*) the maximum deflection at point *E*, (*b*) the maximum normal stress in the beam.

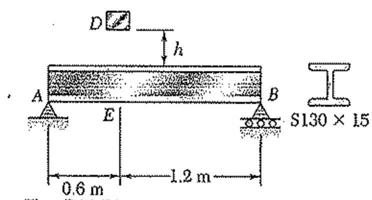


Fig. *P11.51*

11.52 and 11.53 The 2-kg block *D* is dropped from the position shown onto the end of a 16-mm-diameter rod. Knowing that $E = 200$ GPa, determine (*a*) the maximum deflection of end *A*, (*b*) the maximum bending moment in the rod, (*c*) the maximum normal stress in the rod.

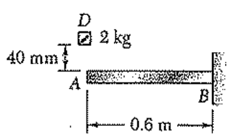


Fig. P11.52

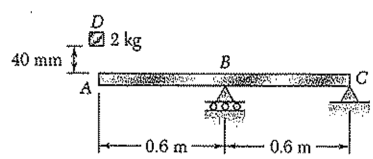


Fig. P11.53

11.54 A block of weight W is placed in contact with a beam at some given point D and released. Show that the resulting maximum deflection at point D is twice as large as the deflection due to a static load W applied at D.

11.55 A block of weight W is dropped from a height h onto the horizontal beam AB and hits it at point D. (*a*) Show that the maximum deflection y_m at point D can be expressed as

$$y_m = y_{st}\left(1 + \sqrt{1 + \frac{2h}{y_{st}}}\right)$$

where y_m represents the deflection at D caused by a static load W applied at that point and where the quantity in parenthesis is referred to at the *impact factor.* (*b*) Compute the impact factor for the beam and the impact of Prob. 11.52.

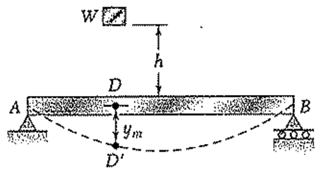


Fig. P11.55 and P11.56

11.56 A block of weight W is dropped from a height h onto the horizontal beam AB and hits it point D. (*a*) Denoting by y_m the exact value of the maximum deflection at D and by y'_m the value obtained by neglecting the effect of this deflection on the change in potential energy of the block, show that the absolute value of the relative error is $(y'_m - y_m)/y_m$ never exceeds $y'_m/2h$. (*b*) Check the result obtained in part *a* by solving part *a* of Prob. 11.52 without taking y_m into account when determining the change in potential energy of the load, and comparing the answer obtained in this way with the exact answer to that problem.

11.57 and 11.58 Using the method of work and energy, determine the deflection at point D caused by the load **P**.

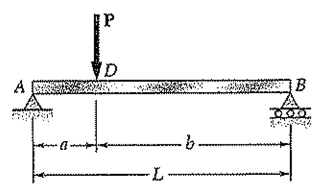


Fig. P11.57

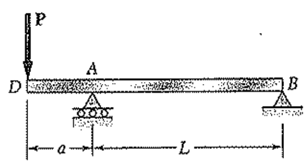


Fig. P11.58

11.59 and *11.60* Using the method of work and energy, determine the slope at point D caused by the couple $\mathbf{M}_0$.

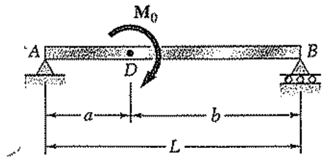


Fig. P11.59

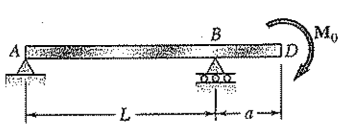


Fig. *P11.60*

***11.61* and 11.62** Using the method of work and energy, determine the deflection at point C caused by the load **P**.

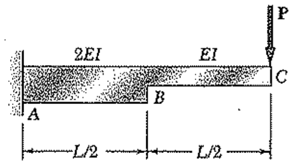


Fig. *P11.61*

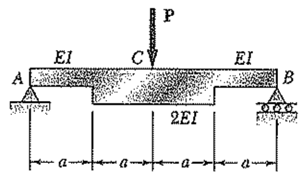


Fig. P11.62

11.63 Using the method of work and energy, determine the slope at point B caused by the couple $\mathbf{M}_0$.

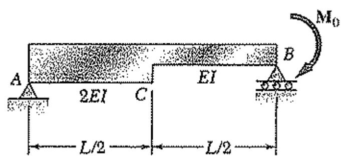


Fig. P11.63

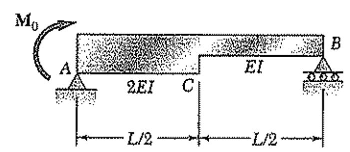


Fig. P11.64

11.64 Using the method of work and energy, determine the slope at point A caused by the couple $\mathbf{M}_0$.

11.65 The 20-mm-diameter steel rod BC is attached to the lever AB and to the fixed support C. The uniform steel lever is 10 mm thick and 30 mm deep. Using the method of work and energy, determine the deflection of point A when $L = 600$ mm. Use $E = 200$ GPa and $G = 77.2$ GPa.

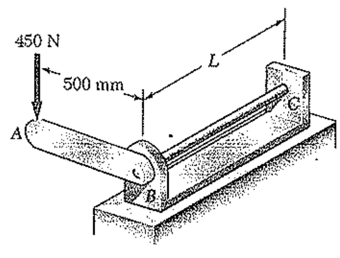


Fig. P11.65 and *P11.66*

11.66 The 20-mm-diameter steel rod BC is attached to the lever AB and to the fixed support C. The uniform steel lever is 10 mm thick and 30 mm deep. Using the method of work and energy, determine the length L of the rod BC for which the deflection at point A is 40 mm. Use $E = 200$ GPa and $G = 77.2$ GPa.

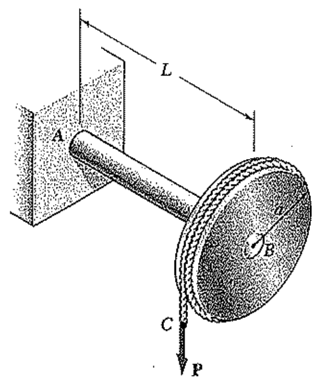


Fig. P11.67

11.67 A disk of radius a has been welded to end B of the solid steel shaft AB. A cable is the wrapped around the disk and a vertical force **P** is applied to end C of the cable. Knowing that the radius of the shaft is r and neglecting the deformations of the disk and of the cable, show that the deflection of point C caused by the application of **P** is

$$\delta_C = \frac{PL^3}{3EI}\left(1 + 1.5\frac{Er^2}{GL^2}\right)$$

11.68 The 12-mm-diameter steel rod ABC has been bent into the shape shown. Knowing that $E = 200$ GPa and $G = 77.2$ GPa, determine the deflection of end C caused by the 150-N force.

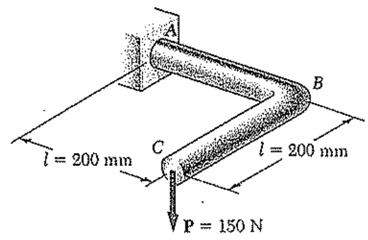


Fig. P11.68

11.69 Two steel shafts, each of 22-mm diameter, are connected by the gears shown. Knowing that $G = 77$ GPa and that shaft DF is fixed at F, determine the angle through which end A rotates when a 135-N · m torque is applied at A. Ignore the strain energy due to the bending of the shafts.

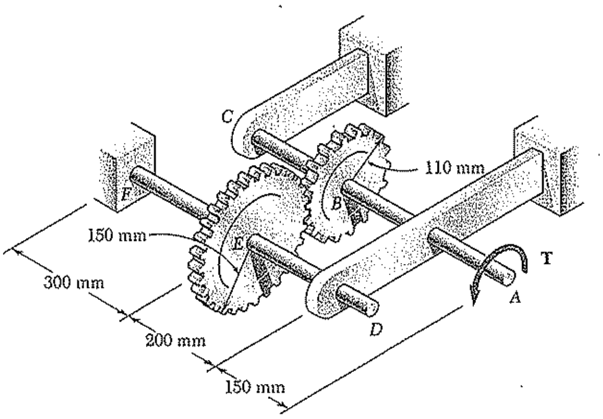


Fig. P11.69

11.70 The thin-walled hollow cylindrical member AB has a noncircular cross section of nonuniform thickness. Using the expression given in Eq. (3.53) of Sec. 3.13, and the expression for the strain-energy density given in Eq. (11.19), show that the angle of twist of member AB is

$$\phi = \frac{TL}{4\mathcal{C}^2 G}\oint \frac{ds}{t}$$

where ds is an element of the center line of the wall cross section and $\mathcal{C}$ is the area enclosed by that center line.

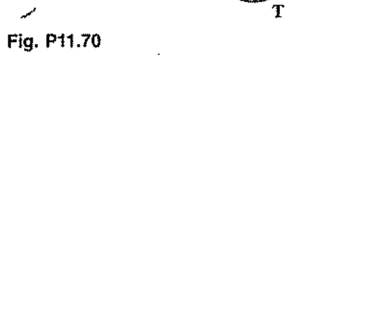


Fig. P11.70

11.71 and 11.72 Each member of the truss shown has a uniform cross-sectional area A. Using the method of work and energy, determine the horizontal deflection of the point of application of the load **P**.

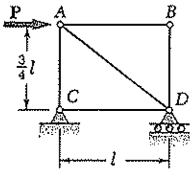


Fig. P11.71

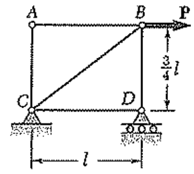


Fig. P11.72

11.73 Each member of the truss shown is made of steel; the cross-sectional area of member BC is 800 mm^2 and for all other members the cross-sectional area is 400 mm^2. Using E = 200 GPa, determine the deflection of point D caused by the 60-kN load.

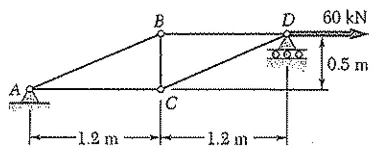


Fig. P11.73

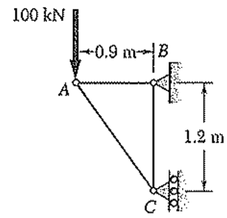


Fig. P11.74

11.74 Each member of the truss shown is made of steel and has a uniform cross-sectional area of 1945 mm^2. Using E = 200 GPa, determine the vertical deflection of joint A caused by the application of the 100-kN load.

11.75 Each member of the truss shown is made of steel and has a cross-sectional area of 3220 mm^2. Using E = 200 GPa, determine the vertical deflection of point B caused by the 80-kN load.

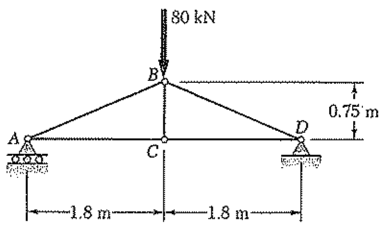


Fig. P11.75

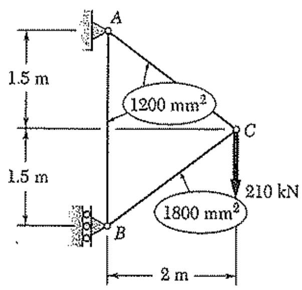


Fig. P11.76

11.76 Members of the truss shown are made of steel and have the cross-sectional areas shown. Using E = 200 GPa, determine the vertical deflection of joint C caused by the application of the 210-kN load.

*11.11. WORK AND ENERGY UNDER SEVERAL LOADS

In this section, the strain energy of a structure subjected to several loads will be considered and will be expressed in terms of the loads and the resulting deflections.

Consider an elastic beam AB subjected to two concentrated loads $\mathbf{P}_1$ and $\mathbf{P}_2$. The strain energy of the beam is equal to the work of $\mathbf{P}_1$ and $\mathbf{P}_2$ as they are slowly applied to the beam at C_1 and C_2, respectively (Fig. 11.36). However, in order to evaluate this work, we must first express the deflections x_1 and x_2 in terms of the loads $\mathbf{P}_1$ and $\mathbf{P}_2$.

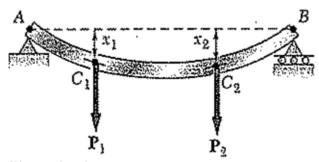


Fig. 11.36

Let us assume that only $\mathbf{P}_1$ is applied to the beam (Fig. 11.37). We note that both C_1 and C_2 are deflected and that their deflections are proportional to the load P_1. Denoting these deflections by x_{11} and x_{21}, respectively, we write

$$x_{11} = \alpha_{11}P_1 \qquad x_{21} = \alpha_{21}P_1 \tag{11.54}$$

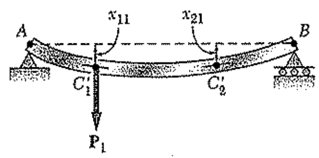


Fig. 11.37

where α_{11} and α_{21} are constants called *influence coefficients*. These constants represent the deflections of C_1 and C_2, respectively, when a unit load is applied at C_1 and are characteristics of the beam AB.

Let us now assume that only $\mathbf{P}_2$ is applied to the beam (Fig. 11.38). Denoting by x_{12} and x_{22}, respectively, the resulting deflections of C_1 and C_2, we write

$$x_{12} = \alpha_{12}P_2 \qquad x_{22} = \alpha_{22}P_2 \tag{11.55}$$

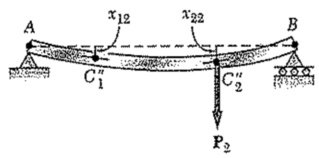


Fig. 11.38

where α_{12} and α_{22} are the influence coefficients representing the deflections of C_1 and C_2, respectively, when a unit load is applied at C_2. Applying the principle of superposition, we express the deflections x_1 and x_2 of C_1 and C_2 when both loads are applied (Fig. 11.36) as

$$x_1 = x_{11} + x_{12} = \alpha_{11}P_1 + \alpha_{12}P_2 \tag{11.56}$$
$$x_2 = x_{21} + x_{22} = \alpha_{21}P_1 + \alpha_{22}P_2 \tag{11.57}$$

To compute the work done by $\mathbf{P}_1$ and $\mathbf{P}_2$, and thus the strain energy of the beam, it is convenient to assume that $\mathbf{P}_1$ is first applied slowly at C_1 (Fig. 11.39*a*). Recalling the first of Eqs. (11.54), we express the work of $\mathbf{P}_1$ as

$$\tfrac{1}{2}P_1x_{11} = \tfrac{1}{2}P_1(\alpha_{11}P_1) = \tfrac{1}{2}\alpha_{11}P_1^2 \tag{11.58}$$

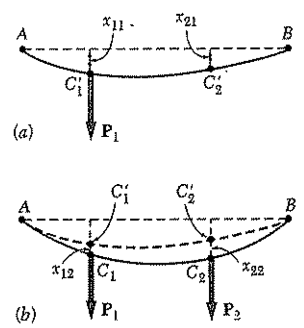


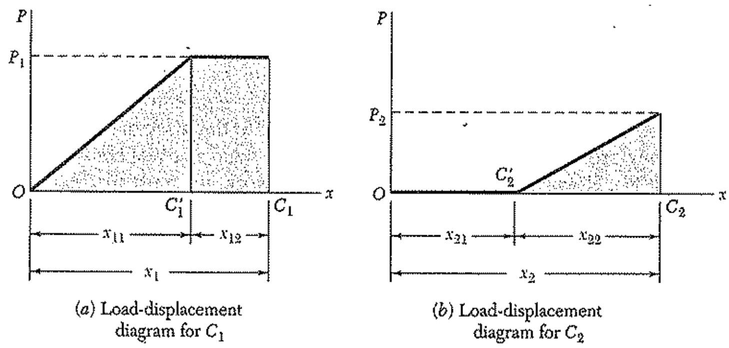
Fig. 11.39

and note that $\mathbf{P}_2$ does no work while C_2 moves through x_{21}, since it has not yet been applied to the beam.

Now we slowly apply $\mathbf{P}_2$ at C_2 (Fig. 11.39*b*); recalling the second of Eqs. (11.55), we express the work of $\mathbf{P}_2$ as

$$\tfrac{1}{2}P_2x_{22} = \tfrac{1}{2}P_2(\alpha_{22}P_2) = \tfrac{1}{2}\alpha_{22}P_2^2 \tag{11.59}$$

But, as $\mathbf{P}_2$ is slowly applied at C_2, the point of application of $\mathbf{P}_1$ moves through x_{12} from C_1' to C_1, and the load $\mathbf{P}_1$ does work. Since $\mathbf{P}_1$ is *fully*



(a) Load-displacement diagram for C_1

(b) Load-displacement diagram for C_2

Fig. 11.40

applied during this displacement (Fig. 11.40), its work is equal to P_1x_{12} or, recalling the first of Eqs. (11.55),

$$P_1x_{12} = P_1(\alpha_{12}P_2) = \alpha_{12}P_1P_2 \tag{11.60}$$

Adding the expressions obtained in (11.58), (11.59), and (11.60), we express the strain energy of the beam under the loads $\mathbf{P}_1$ and $\mathbf{P}_2$ as

$$U = \tfrac{1}{2}(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2) \tag{11.61}$$

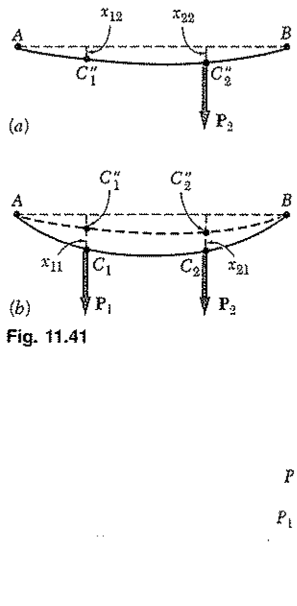
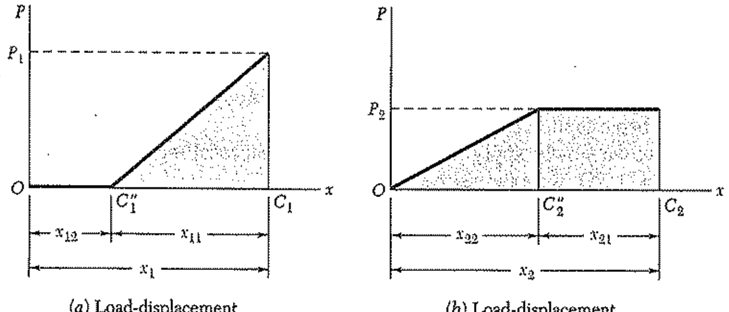


Fig. 11.41

If the load $\mathbf{P}_2$ had first been applied to the beam (Fig. 11.41*a*), and then the load $\mathbf{P}_1$ (Fig. 11.41*b*), the work done by each load would have been as shown in Fig. 11.42. Calculations similar to those we have just carried out would lead to the following alternative expression for the strain energy of the beam:

$$U = \tfrac{1}{2}(\alpha_{22}P_2^2 + 2\alpha_{21}P_2P_1 + \alpha_{11}P_1^2) \tag{11.62}$$

Equating the right-hand members of Eqs. (11.61) and (11.62), we find that $\alpha_{12} = \alpha_{21}$, and thus conclude that the deflection produced at C_1 by a unit load applied at C_2 is equal to the deflection produced at C_2 by a unit load applied at C_1. This is known as *Maxwell's reciprocal theorem,* after the British physicist James Clerk Maxwell (1831–1879).



(a) Load-displacement diagram for C_1

(b) Load-displacement diagram for C_2

Fig. 11.42

While we are now able to express the strain energy U of a structure subjected to several loads as a function of these loads, we cannot use the method of Sec. 11.10 to determine the deflection of such a structure. Indeed, computing the strain energy U by integrating the strain-energy density u over the structure and substituting the expression obtained into (11.61) would yield only one equation, which clearly could not be solved for the various coefficients α.

*11.12. CASTIGLIANO'S THEOREM

We recall the expression obtained in the preceding section for the strain energy of an elastic structure subjected to two loads $\mathbf{P}_1$ and $\mathbf{P}_2$:

$$U = \tfrac{1}{2}(\alpha_{11}P_1^2 + 2\alpha_{12}P_1P_2 + \alpha_{22}P_2^2) \tag{11.61}$$

where α_{11}, α_{12}, and α_{22} are the influence coefficients associated with the points of application C_1 and C_2 of the two loads. Differentiating both members of Eq. (11.61) with respect to P_1 and recalling Eq. (11.56), we write

$$\frac{\partial U}{\partial P_1} = \alpha_{11}P_1 + \alpha_{12}P_2 = x_1 \tag{11.63}$$

Differentiating both members of Eq. (11.61) with respect to P_2, recalling Eq. (11.57), and keeping in mind that $\alpha_{12} = \alpha_{21}$, we have

$$\frac{\partial U}{\partial P_2} = \alpha_{12}P_1 + \alpha_{22}P_2 = x_2 \tag{11.64}$$

More generally, if an elastic structure is subjected to n loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, the deflection x_j of the point of application of $\mathbf{P}_j$, measured along the line of action of $\mathbf{P}_j$, can be expressed as the partial derivative of the strain energy of the structure with respect to the load $\mathbf{P}_j$. We write

$$x_j = \frac{\partial U}{\partial P_j} \tag{11.65}$$

This is *Castigliano's theorem*, named after the Italian engineer Alberto Castigliano (1847–1884) who first stated it.†

†In the case of an elastic structure subjected to n loads $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$, the deflection of the point of application of $\mathbf{P}_j$, measured along the line of action of $\mathbf{P}_j$, can be expressed as

$$x_j = \sum_k \alpha_{jk}P_k \tag{11.66}$$

and the strain energy of the structure is found to be

$$U = \tfrac{1}{2}\sum_i \sum_k \alpha_{ik}P_iP_k \tag{11.67}$$

Differentiating U with respect to P_j, and observing that P_j is found in terms corresponding to either $i = j$ or $k = j$, we write

$$\frac{\partial U}{\partial P_j} = \frac{1}{2}\sum_k \alpha_{jk}P_k + \frac{1}{2}\sum_i \alpha_{ij}P_i$$

or, since $\alpha_{ij} = \alpha_{ji}$,

$$\frac{\partial U}{\partial P_j} = \frac{1}{2}\sum_k \alpha_{jk}P_k + \frac{1}{2}\sum_i \alpha_{ji}P_i = \sum_k \alpha_{jk}P_k$$

Recalling Eq. (11.66), we verify that

$$x_j = \frac{\partial U}{\partial P_j} \tag{11.65}$$

Recalling that the work of a couple **M** is $\frac{1}{2}M\theta$, where θ is the angle of rotation at the point where the couple is slowly applied, we note that Castigliano's theorem may be used to determine the slope of a beam at the point of application of a couple $\mathbf{M}_j$. We have

$$\theta_j = \frac{\partial U}{\partial M_j} \tag{11.68}$$

Similarly, the angle of twist ϕ_j in a section of a shaft where a torque $\mathbf{T}_j$ is slowly applied is obtained by differentiating the strain energy of the shaft with respect to T_j:

$$\phi_j = \frac{\partial U}{\partial T_j} \tag{11.69}$$

*11.13. DEFLECTIONS BY CASTIGLIANO'S THEOREM

We saw in the preceding section that the deflection x_j of a structure at the point of application of a load $\mathbf{P}_j$ can be determined by computing the partial derivative $\partial U/\partial P_j$ of the strain energy U of the structure. As we recall from Secs. 11.4 and 11.5, the strain energy U is obtained by integrating or summing over the structure the strain energy of each element of the structure. The calculation by Castigliano's theorem of the deflection x_j is simplified if the differentiation with respect to the load P_j is carried out before the integration or summation.

In the case of a beam, for example, we recall from Sec. 11.4 that

$$U = \int_0^L \frac{M^2}{2EI}dx \tag{11.17}$$

and determine the deflection x_j of the point of application of the load $\mathbf{P}_j$ by writing

$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j}dx \tag{11.70}$$

In the case of a truss consisting of n uniform members of length L_i, cross-sectional area A_i, and internal force F_i, we recall Eq. (11.14) and express the strain energy U of the truss as

$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E} \tag{11.71}$$

The deflection x_j of the point of application of the load $\mathbf{P}_j$ is obtained by differentiating with respect to P_j each term of the sum. We write

$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E}\frac{\partial F_i}{\partial P_j} \tag{11.72}$$

EXAMPLE 11.12

The cantilever beam AB supports a uniformly distributed load w and a concentrated load $\mathbf{P}$ as shown (Fig. 11.43). Knowing that $L = 2$ m, $w = 4$ kN/m, $P = 6$ kN, and $EI = 5$ MN $\cdot$ m^2, determine the deflection at A.

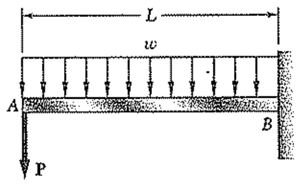


Fig. 11.43

The deflection y_A of the point A where the load $\mathbf{P}$ is applied is obtained from Eq. (11.70). Since $\mathbf{P}$ is vertical and directed downward, y_A represents a vertical deflection and is positive downward. We have

$$y_A = \frac{\partial U}{\partial P} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P}dx \tag{11.73}$$

The bending moment M at a distance x from A is

$$M = -(Px + \tfrac{1}{2}wx^2) \tag{11.74}$$

and its derivative with respect to P is

$$\frac{\partial M}{\partial P} = -x$$

Substituting for M and $\partial M/\partial P$ into Eq. (11.73), we write

$$y_A = \frac{1}{EI}\int_0^L \left(Px^2 + \frac{1}{2}wx^3\right)dx$$

$$y_A = \frac{1}{EI}\left(\frac{PL^3}{3} + \frac{wL^4}{8}\right) \tag{11.75}$$

Substituting the given data, we have

$$y_A = \frac{1}{5 \times 10^6 \text{ N} \cdot \text{m}^2}\left[\frac{(6 \times 10^3 \text{ N})(2 \text{ m})^3}{3} + \frac{(4 \times 10^3 \text{ N/m})(2 \text{ m})^4}{8}\right]$$

$$y_A = 4.8 \times 10^{-3} \text{ m} \qquad y_A = 4.8 \text{ mm} \downarrow$$

We note that the computation of the partial derivative $\partial M/\partial P$ *could not have been carried out* if the numerical value of P had been substituted for P in the expression (11.74) for the bending moment.

We can observe that the deflection x_j of a structure at a given point C_j can be obtained by the direct application of Castigliano's theorem only if a load $\mathbf{P}_j$ happens to be applied at C_j in the direction in which x_j is to be determined. When no load is applied at C_j, or when a load is applied in a direction other than the desired one, we can still obtain the deflection x_j by Castigliano's theorem if we use the following procedure: We apply a fictitious or "dummy" load $\mathbf{Q}_j$ at C_j in the direction in which the deflection x_j is to be determined and use Castigliano's theorem to obtain the deflection

$$x_j = \frac{\partial U}{\partial Q_j} \tag{11.76}$$

due to $\mathbf{Q}_j$ and the actual loads. Making $Q_j = 0$ in Eq. (11.76) yields the deflection at C_j in the desired direction under the given loading.

The slope θ_j of a beam at a point C_j can be determined in a similar manner by applying a fictitious couple $\mathbf{M}_j$ at C_j, computing the partial derivative $\partial U/\partial M_j$, and making $M_j = 0$ in the expression obtained.

EXAMPLE 11.13

The cantilever beam AB supports a uniformly distributed load w (Fig. 11.44). Determine the deflection and slope at A.

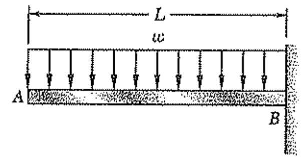


Fig. 11.44

Deflection at A. We apply a dummy downward load $\mathbf{Q}_A$ at A (Fig. 11.45) and write

$$y_A = \frac{\partial U}{\partial Q_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q_A}\,dx \tag{11.77}$$

The bending moment M at a distance x from A is

$$M = -Q_A x - \tfrac{1}{2}wx^2 \tag{11.78}$$

and its derivative with respect to Q_A is

$$\frac{\partial M}{\partial Q_A} = -x \tag{11.79}$$

Substituting for M and $\partial M/\partial Q_A$ from (11.78) and (11.79) into (11.77), and making $Q_A = 0$, we obtain the deflection at A for the given loading:

$$y_A = \frac{1}{EI}\int_0^L (-\tfrac{1}{2}wx^2)(-x)\,dx = +\frac{wL^4}{8EI}$$

Since the dummy load was directed downward, the positive sign indicates that

$$y_A = \frac{wL^4}{8EI}\downarrow$$

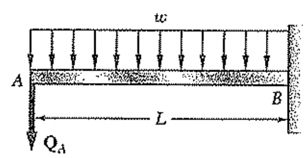


Fig. 11.45

Slope at A. We apply a dummy counterclockwise couple $\mathbf{M}_A$ at A (Fig. 11.46) and write

$$\theta_A = \frac{\partial U}{\partial M_A}$$

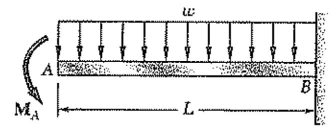


Fig. 11.46

Recalling Eq. (11.17), we have

$$\theta_A = \frac{\partial}{\partial M_A}\int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial M_A}\,dx \tag{11.80}$$

The bending moment M at a distance x from A is

$$M = -M_A - \tfrac{1}{2}wx^2 \tag{11.81}$$

and its derivative with respect to M_A is

$$\frac{\partial M}{\partial M_A} = -1 \tag{11.82}$$

Substituting for M and $\partial M/\partial M_A$ from (11.81) and (11.82) into (11.80), and making $M_A = 0$, we obtain the slope at A for the given loading:

$$\theta_A = \frac{1}{EI}\int_0^L (-\tfrac{1}{2}wx^2)(-1)\,dx = +\frac{wL^3}{6EI}$$

Since the dummy couple was counterclockwise, the positive sign indicates that the angle θ_A is also counterclockwise:

$$\theta_A = \frac{wL^3}{6EI} \measuredangle$$

EXAMPLE 11.14

A load **P** is supported at B by two rods of the same material and of the same cross-sectional area A (Fig. 11.47). Determine the horizontal and vertical deflection of point B.

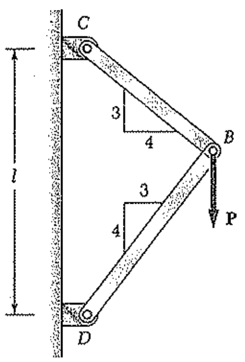


Fig. 11.47

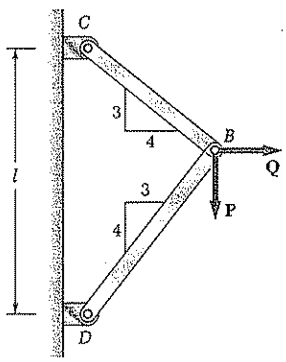


Fig. 11.48

We apply a dummy horizontal load **Q** at B (Fig. 11.48). From Castigliano's theorem we have

$$x_B = \frac{\partial U}{\partial Q} \qquad y_B = \frac{\partial U}{\partial P}$$

Recalling from Sec. 11.4 the expression (11.14) for the strain energy of a rod, we write

$$U = \frac{F_{BC}^2(BC)}{2AE} + \frac{F_{BD}^2(BD)}{2AE}$$

where F_{BC} and F_{BD} represent the forces in BC and BD, respectively. We have, therefore,

$$x_B = \frac{\partial U}{\partial Q} = \frac{F_{BC}(BC)}{AE}\frac{\partial F_{BC}}{\partial Q} + \frac{F_{BD}(BD)}{AE}\frac{\partial F_{BD}}{\partial Q} \qquad (11.83)$$

and

$$y_B = \frac{\partial U}{\partial P} = \frac{F_{BC}(BC)}{AE}\frac{\partial F_{BC}}{\partial P} + \frac{F_{BD}(BD)}{AE}\frac{\partial F_{BD}}{\partial P} \qquad (11.84)$$

From the free-body diagram of pin B (Fig. 11.49), we obtain

$$F_{BC} = 0.6P + 0.8Q \qquad F_{BD} = -0.8P + 0.6Q \qquad (11.85)$$

Differentiating these expressions with respect to Q and P, we write

$$\frac{\partial F_{BC}}{\partial Q} = 0.8 \qquad \frac{\partial F_{BD}}{\partial Q} = 0.6$$
$$\frac{\partial F_{BC}}{\partial P} = 0.6 \qquad \frac{\partial F_{BD}}{\partial P} = -0.8 \qquad (11.86)$$

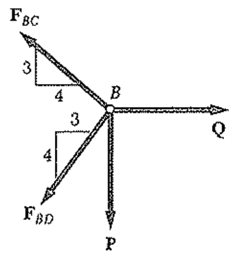


Fig. 11.49

Substituting from (11.85) and (11.86) into both (11.83) and (11.84), making $Q = 0$, and noting that $BC = 0.6l$ and $BD = 0.8l$, we obtain the horizontal and vertical deflections of point B under the given load **P**:

$$x_B = \frac{(0.6P)(0.6l)}{AE}(0.8) + \frac{(-0.8P)(0.8l)}{AE}(0.6)$$
$$= -0.096\frac{Pl}{AE}$$

$$y_B = \frac{(0.6P)(0.6l)}{AE}(0.6) + \frac{(-0.8P)(0.8l)}{AE}(-0.8)$$
$$= +0.728\frac{Pl}{AE}$$

Referring to the directions of the loads **Q** and **P**, we conclude that

$$x_B = 0.096\frac{Pl}{AE} \leftarrow \qquad y_B = 0.728\frac{Pl}{AE} \downarrow$$

We check that the expression obtained for the vertical deflection of B is the same that was found in Example 11.09.

*11.14. STATICALLY INDETERMINATE STRUCTURES

The reactions at the supports of a statically indeterminate elastic structure can be determined by Castigliano's theorem. In the case of a structure indeterminate to the first degree, for example, we designate one of the reactions as redundant and eliminate or modify accordingly the corresponding support. The redundant reaction is then treated as an unknown load that, together with the other loads, must produce deformations that are compatible with the original supports. We first calculate the strain energy U of the structure due to the combined action of the given loads and the redundant reaction. Observing that the partial derivative of U with respect to the redundant reaction represents the deflection (or slope) at the support that has been eliminated or modified, we then set this derivative equal to zero and solve the equation obtained for the redundant reaction.† The remaining reactions can be obtained from the equations of statics.

†This is in the case of a rigid support allowing no deflection. For other types of support, the partial derivative of U should be set equal to the allowed deflection.

EXAMPLE 11.15

Determine the reactions at the supports for the prismatic beam and loading shown (Fig. 11.50).

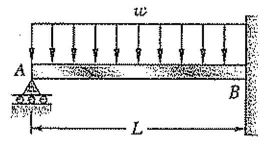


Fig. 11.50

The beam is statically indeterminate to the first degree. We consider the reaction at A as redundant and release the beam from that support. The reaction $\mathbf{R}_A$ is now considered as an unknown load (Fig. 11.51) and will be determined from the condition that the deflection y_A at A must be zero. By Castigliano's theorem $y_A = \partial U/\partial R_A$, where U is the strain energy of the beam under the distributed load and the redundant reaction. Recalling Eq. (11.70), we write

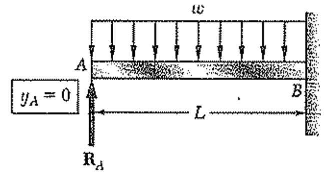


Fig. 11.51

$$y_A = \frac{\partial U}{\partial R_A} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial R_A}\,dx \tag{11.87}$$

We now express the bending moment M for the loading of Fig. 11.51.

The bending moment at a distance x from A is

$$M = R_A x - \tfrac{1}{2}wx^2 \tag{11.88}$$

and its derivative with respect to R_A is

$$\frac{\partial M}{\partial R_A} = x \tag{11.89}$$

Substituting for M and $\partial M/\partial R_A$ from (11.88) and (11.89) into (11.87), we write

$$y_A = \frac{1}{EI}\int_0^L \left(R_A x^2 - \frac{1}{2}wx^3\right)dx = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right)$$

Setting $y_A = 0$ and solving for R_A, we have

$$R_A = \tfrac{3}{8}wL \qquad \mathbf{R}_A = \tfrac{3}{8}wL\uparrow$$

From the conditions of equilibrium for the beam, we find that the reaction at B consists of the following force and couple:

$$\mathbf{R}_B = \tfrac{5}{8}wL\uparrow \qquad \mathbf{M}_B = \tfrac{1}{8}wL^2 \circlearrowright$$

EXAMPLE 11.16

A load **P** is supported at B by three rods of the same material and the same cross-sectional area A (Fig. 11.52). Determine the force in each rod.

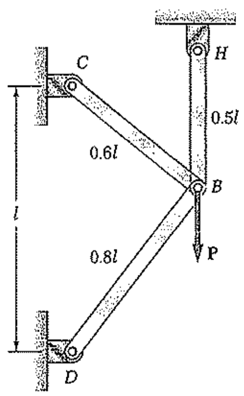


Fig. 11.52

The structure is statically indeterminate to the first degree. We consider the reaction at H as redundant and release rod BH from its support at H. The reaction $\mathbf{R}_H$ is now considered as an unknown load (Fig. 11.53) and will be determined from the condition that the deflection y_H of point H must be zero. By Castigliano's theorem $y_H = \partial U/\partial R_H$, where U is the strain energy of the three-rod system under the load **P** and the redundant reaction $\mathbf{R}_H$. Recalling Eq. (11.72), we write

$$y_H = \frac{F_{BC}(BC)}{AE}\frac{\partial F_{BC}}{\partial R_H} + \frac{F_{BD}(BD)}{AE}\frac{\partial F_{BD}}{\partial R_H} + \frac{F_{BH}(BH)}{AE}\frac{\partial F_{BH}}{\partial R_H} \qquad (11.90)$$

We note that the force in rod BH is equal to R_H and write

$$F_{BH} = R_H \qquad (11.91)$$

Then, from the free-body diagram of pin B (Fig. 11.54), we obtain

$$F_{BC} = 0.6P - 0.6R_H \qquad F_{BD} = 0.8R_H - 0.8P \qquad (11.92)$$

Differentiating with respect to R_H the force in each rod, we write

$$\frac{\partial F_{BC}}{\partial R_H} = -0.6 \qquad \frac{\partial F_{BD}}{\partial R_H} = 0.8 \qquad \frac{\partial F_{BH}}{\partial R_H} = 1 \qquad (11.93)$$

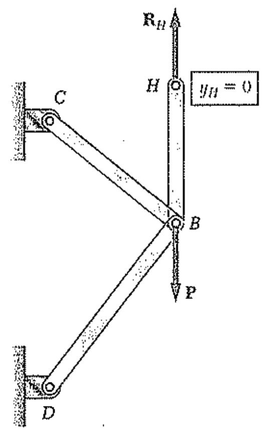


Fig. 11.53

Substituting from (11.91), (11.92), and (11.93) into (11.90), and noting that the lengths BC, BD, and BH are, respectively, equal to $0.6l$, $0.8l$, and $0.5l$, we write

$$y_H = \frac{1}{AE}[(0.6P - 0.6R_H)(0.6l)(-0.6) + (0.8R_H - 0.8P)(0.8l)(0.8) + R_H(0.5l)(1)]$$

Setting $y_H = 0$, we obtain

$$1.228R_H - 0.728P = 0$$

and, solving for R_H,

$$R_H = 0.593P$$

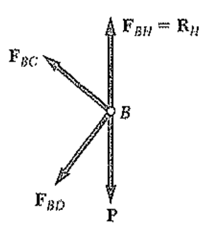
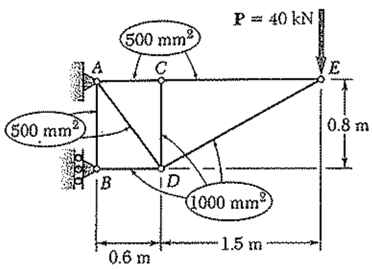


Fig. 11.54

Carrying this value into Eqs. (11.91) and (11.92), we obtain the forces in the three rods:

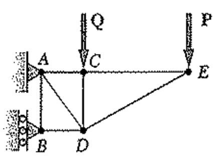
$$F_{BC} = +0.244P \qquad F_{BD} = -0.326P \qquad F_{BH} = +0.593P$$



SAMPLE PROBLEM 11.5

For the truss and loading of Sample Prob. 11.4, determine the vertical deflection of joint C.

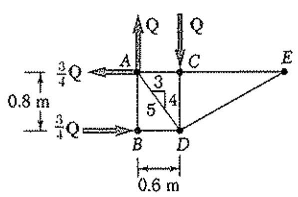
SOLUTION



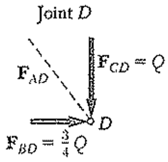
Castigliano's Theorem. Since no vertical load is applied at joint C, we introduce the dummy load $\mathbf{Q}$ as shown. Using Castigliano's theorem, and denoting by F_i the force in a given member i caused by the combined loading of $\mathbf{P}$ and $\mathbf{Q}$, we have, since E = constant,

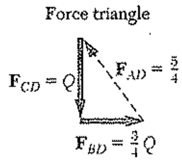
$$y_C = \sum \left(\frac{F_i L_i}{A_i E}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}\sum\left(\frac{F_i L_i}{A_i}\right)\frac{\partial F_i}{\partial Q} \qquad (1)$$

Force in Members. Considering in sequence the equilibrium of joints E, C, B, and D, we determine the force in each member caused by load $\mathbf{Q}$.



Joint E: $F_{CE} = F_{DE} = 0$
Joint C: $F_{AC} = 0$; $F_{CD} = -Q$
Joint B: $F_{AB} = 0$; $F_{BD} = -\frac{3}{4}Q$





The force in each member caused by the load $\mathbf{P}$ was previously found in Sample Prob. 11.4. The total force in each member under the combined action of $\mathbf{Q}$ and $\mathbf{P}$ is shown in the following table. Forming $\partial F_i/\partial Q$ for each member, we then compute $(F_iL_i/A_i)(\partial F_i/\partial Q)$ as indicated in the table.

Member	F_i	$\partial F_i/\partial Q$	L_i, m	A_i, m²	$\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q}$
AB	0	0	0.8	500×10^{-6}	0
AC	$+15P/8$	0	0.6	500×10^{-6}	0
AD	$+5P/4 + 5Q/4$	$\frac{5}{4}$	1.0	500×10^{-6}	$+3125P + 3125Q$
BD	$-21P/8 - 3Q/4$	$-\frac{3}{4}$	0.6	1000×10^{-6}	$+1181P + 338Q$
CD	$-Q$	-1	0.8	1000×10^{-6}	$+ 800Q$
CE	$+15P/8$	0	1.5	500×10^{-6}	0
DE	$-17P/8$	0	1.7	1000×10^{-6}	0

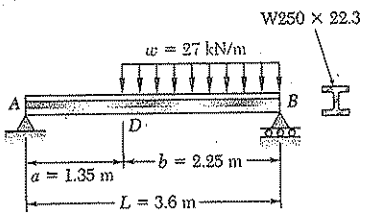
$$\sum\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q} = 4306P + 4263Q$$

Deflection of C. Substituting into Eq. (1), we have

$$y_C = \frac{1}{E}\sum\left(\frac{F_iL_i}{A_i}\right)\frac{\partial F_i}{\partial Q} = \frac{1}{E}(4306P + 4263Q)$$

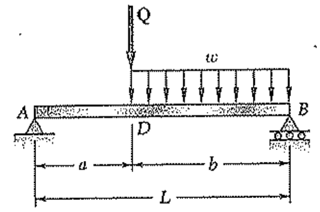
Since the load $\mathbf{Q}$ is not part of the original loading, we set $Q = 0$. Substituting the given data, $P = 40$ kN and $E = 73$ GPa, we find

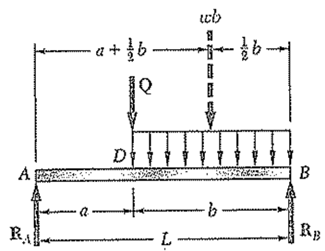
$$y_C = \frac{4306(40 \times 10^3 \text{ N})}{73 \times 10^9 \text{ Pa}} = 2.36 \times 10^{-3} \text{ m} \qquad y_C = 2.36 \text{ mm} \downarrow \blacktriangleleft$$



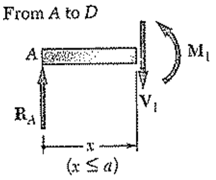
SAMPLE PROBLEM 11.6

For the beam and loading shown, determine the deflection at point D. Use $E = 200$ GPa.

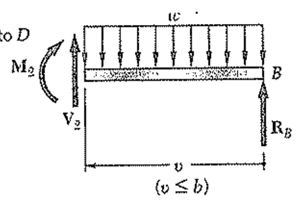




From A to D



From B to D



SOLUTION

Castigliano's Theorem. Since the given loading does not include a vertical load at point D, we introduce the dummy load $\mathbf{Q}$ as shown. Using Castigliano's theorem and noting that EI is constant, we write

$$y_D = \int \frac{M}{EI}\left(\frac{\partial M}{\partial Q}\right)dx = \frac{1}{EI}\int M\left(\frac{\partial M}{\partial Q}\right)dx \qquad (1)$$

The integration will be performed separately for portions AD and DB.

Reactions. Using the free-body diagram of the entire beam, we find

$$\mathbf{R}_A = \frac{wb^2}{2L} + Q\frac{b}{L}\uparrow \qquad \mathbf{R}_B = \frac{wb(a + \frac{1}{2}b)}{L} + Q\frac{a}{L}\uparrow$$

Portion AD of Beam. Using the free body shown, we find

$$M_1 = R_A x = \left(\frac{wb^2}{2L} + Q\frac{b}{L}\right)x \qquad \frac{\partial M_1}{\partial Q} = +\frac{bx}{L}$$

Substituting into Eq. (1) and integrating from A to D gives

$$\frac{1}{EI}\int M_1\frac{\partial M_1}{\partial Q}dx = \frac{1}{EI}\int_0^a R_A x\left(\frac{bx}{L}\right)dx = \frac{R_A a^3 b}{3EIL}$$

We substitute for R_A and then set the dummy load Q equal to zero.

$$\frac{1}{EI}\int M_1\frac{\partial M_1}{\partial Q}dx = \frac{wa^3b^3}{6EIL^2} \qquad (2)$$

Portion DB of Beam. Using the free body shown, we find that the bending moment at a distance v from end B is

$$M_2 = R_B v - \frac{wv^2}{2} = \left[\frac{wb(a + \frac{1}{2}b)}{L} + Q\frac{a}{L}\right]v - \frac{wv^2}{2} \qquad \frac{\partial M_2}{\partial Q} = +\frac{av}{L}$$

Substituting into Eq. (1) and integrating from point B where $v = 0$, to point D where $v = b$, we write

$$\frac{1}{EI}\int M_2\frac{\partial M_2}{\partial Q}dv = \frac{1}{EI}\int_0^b\left(R_B v - \frac{wv^2}{2}\right)\left(\frac{av}{L}\right)dv = \frac{R_B ab^3}{3EIL} - \frac{wab^4}{8EIL}$$

Substituting for R_B and setting $Q = 0$,

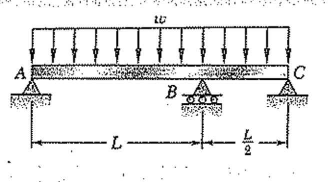
$$\frac{1}{EI}\int M_2\frac{\partial M_2}{\partial Q}dv = \left[\frac{wb(a + \frac{1}{2}b)}{L}\right]\frac{ab^3}{3EIL} - \frac{wab^4}{8EIL} = \frac{5a^2b^4 + ab^5}{24EIL^2}w \qquad (3)$$

Deflection at Point D. Recalling Eqs. (1), (2), and (3), we have

$$y_D = \frac{wab^3}{24EIL^2}(4a^2 + 5ab + b^2) = \frac{wab^3}{24EIL^2}(4a + b)(a + b) = \frac{wab^3}{24EIL}(4a + b)$$

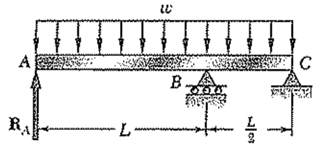
From Appendix C we find that $I = 28.9 \times 10^6\ \text{mm}^4$ for a W250 × 22.3. Substituting for I, w, a, b, and L their numerical values, we obtain

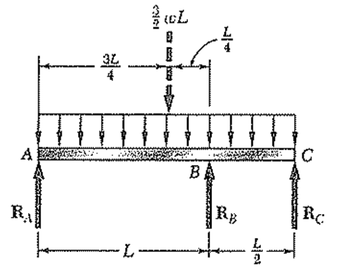
$$y_D = 6.36\ \text{mm}\downarrow$$



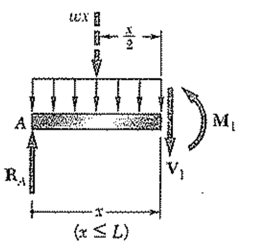
SAMPLE PROBLEM 11.7

For the uniform beam and loading shown, determine the reactions at the supports.

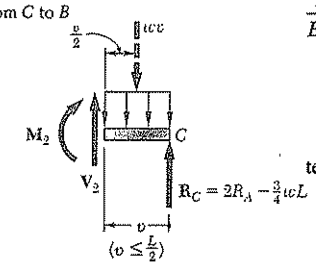




From A to B



From C to B



SOLUTION

Castigliano's Theorem. The beam is indeterminate to the first degree and we choose the reaction $\mathbf{R}_A$ as redundant. Using Castigliano's theorem, we determine the deflection at A due to the combined action of $\mathbf{R}_A$ and the distributed load. Since EI is constant, we write

$$y_A = \int \frac{M}{EI}\left(\frac{\partial M}{\partial R_A}\right) dx = \frac{1}{EI}\int M \frac{\partial M}{\partial R_A}\, dx \qquad (1)$$

The integration will be performed separately for portions AB and BC of the beam. Finally, $\mathbf{R}_A$ is obtained by setting y_A equal to zero.

Free Body: Entire Beam. We express the reactions at B and C in terms of R_A and the distributed load

$$R_B = \tfrac{9}{4}wL - 3R_A \qquad R_C = 2R_A - \tfrac{3}{4}wL \qquad (2)$$

Portion AB of Beam. Using the free-body diagram shown, we find

$$M_1 = R_A x - \frac{wx^2}{2} \qquad \frac{\partial M_1}{\partial R_A} = x$$

Substituting into Eq. (1) and integrating from A to B, we have

$$\frac{1}{EI}\int M_1 \frac{\partial M}{\partial R_A}\, dx = \frac{1}{EI}\int_0^L \left(R_A x^2 - \frac{wx^3}{2}\right) dx = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right) \qquad (3)$$

Portion BC of Beam. We have

$$M_2 = \left(2R_A - \frac{3}{4}wL\right)v - \frac{wv^2}{2} \qquad \frac{\partial M_2}{\partial R_A} = 2v$$

Substituting into Eq. (1) and integrating from C, where $v = 0$, to B, where $v = \frac{1}{2}L$, we have

$$\frac{1}{EI}\int M_2 \frac{\partial M_2}{\partial R_A}\, dv = \frac{1}{EI}\int_0^{L/2} \left(4R_A v^2 - \frac{3}{2}wLv^2 - wv^3\right) dv$$

$$= \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{wL^4}{16} - \frac{wL^4}{64}\right) = \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{5wL^4}{64}\right) \qquad (4)$$

Reaction at A. Adding the expressions obtained in (3) and (4), we determine y_A and set it equal to zero

$$y_A = \frac{1}{EI}\left(\frac{R_A L^3}{3} - \frac{wL^4}{8}\right) + \frac{1}{EI}\left(\frac{R_A L^3}{6} - \frac{5wL^4}{64}\right) = 0$$

Solving for R_A, $\quad R_A = \frac{13}{32}wL \qquad R_A = \frac{13}{32}wL\uparrow$ ◀

Reactions at B and C. Substituting for R_A into Eqs. (2), we obtain

$$R_B = \frac{33}{32}wL\uparrow \qquad R_C = \frac{wL}{16}\uparrow \text{ ◀}$$

PROBLEMS

11.77 through 11.79 Using the information in Appendix D, compute the work of the loads as they are applied to the beam (*a*) if the load **P** is applied first, (*b*) if the couple **M** is applied first.

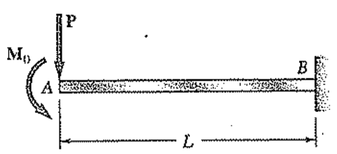


Fig. P11.77

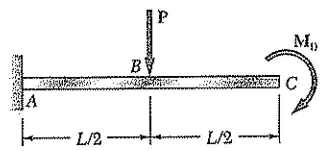


Fig. *P11.78*

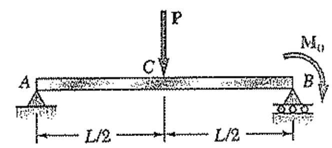


Fig. P11.79

***11.80* through 11.82** For the beam and loading shown, (*a*) compute the work of the loads as they are applied successively to the beam, using the information provided in Appendix D, (*b*) compute the strain energy of the beam by the method of Sec. 11.4 and show that it is equal to the work obtained in part *a*.

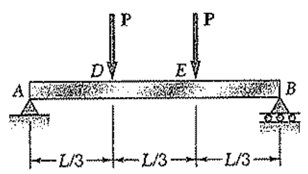


Fig. *P11.80*

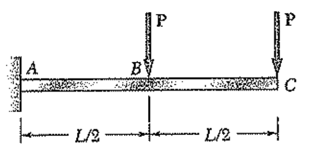


Fig. P11.81

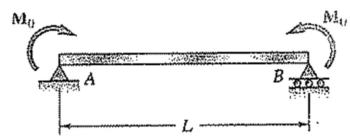


Fig. P11.82

11.83 and 11.85 For the prismatic beam shown, determine the deflection of point *D*.

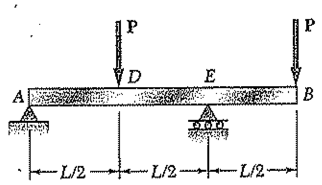


Fig. P11.83 and P11.84

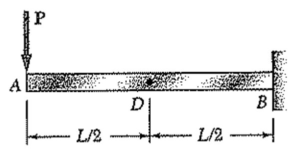


Fig. P11.85 and *P11.86*

11.84 and *11.86* For the prismatic beam shown, determine the slope at point *D*.

11.87 For the prismatic beam shown, determine the slope at point A.

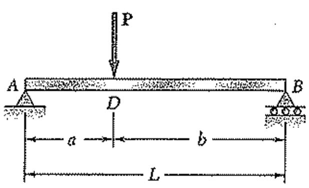


Fig. P11.87

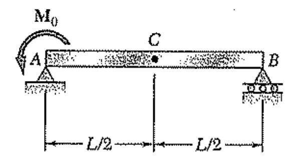


Fig. P11.88

11.88 For the prismatic beam shown, determine the slope at point B.

11.89 and 11.90 For the prismatic beam shown, determine the deflection at point D.

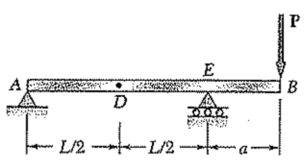


Fig. P11.89 and *P11.91*

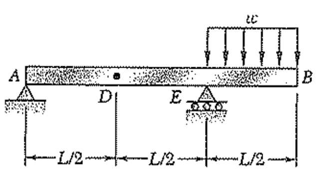


Fig. P11.90 and P11.92

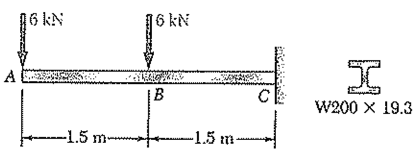


Fig. P11.93 and *P11.94*

11.91 **and 11.92** For the prismatic beam shown, determine the slope at point D.

11.93 For the beam and loading shown, determine the deflection of point B. Use $E = 200$ GPa.

11.94 For the beam and loading shown, determine the deflection of point A. Use $E = 200$ GPa.

11.95 For the beam and loading shown, determine the deflection at point B. Use $E = 200$ GPa.

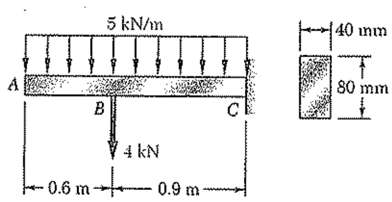


Fig. P11.95

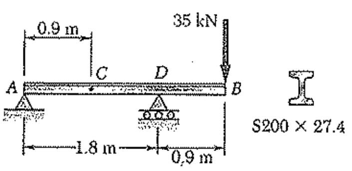


Fig. P11.96 and *P11.97*

11.96 For the beam and loading shown, determine the deflection at point C. Use $E = 200$ GPa.

11.97 For the beam and loading shown, determine the slope at end A. Use $E = 200$ GPa.

11.98 For the beam and loading shown, determine the slope at end A. Use $E = 200$ GPa.

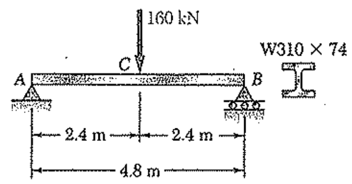


Fig. P11.98

11.99 and 11.100 Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 200$ GPa, determine the deflection indicated.

11.99 Vertical deflection of joint C.

11.100 Horizontal deflection of point C.

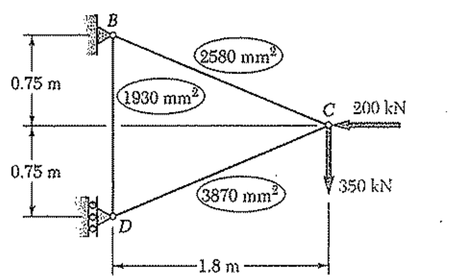


Fig. P11.99 and P11.100

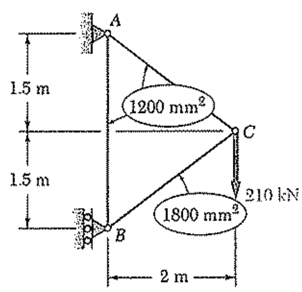


Fig. P11.101 and *P11.102*

11.101 and *11.102* Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 200$ GPa, determine the deflection indicated.

11.101 Vertical deflection of joint C.

11.102 Horizontal deflection of point C.

***11.103* and 11.104** Each member of the truss shown is made of steel and has the cross-sectional area shown. Using $E = 200$ GPa, determine the deflection indicated.

11.103 Vertical deflection of joint C.

11.104 Horizontal deflection of point C.

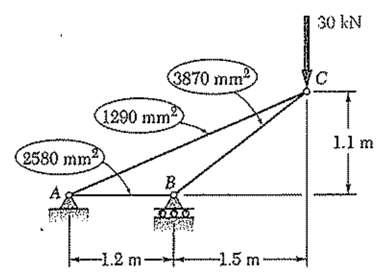


Fig. *P11.103* and P11.104

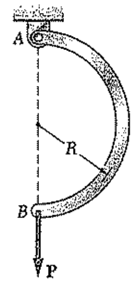


Fig. P11.105

11.105 For the uniform rod and loading shown and using Castigliano's theorem, determine the deflection of point B.

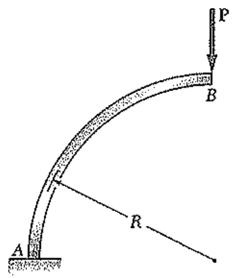


Fig. P11.106

11.106 For the beam and loading shown and using Castigliano's theorem, determine (*a*) the horizontal deflection of point *B*, (*b*) the vertical deflection of point *B*.

11.107 Three rods, each of the cams flexural rigidity *EI*, are welded to form the frame *ABCD*. For the loading shown, determine the deflection at point *D*.

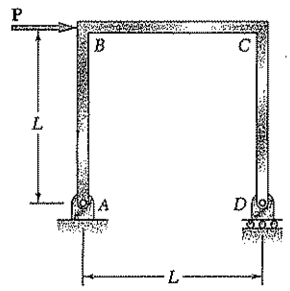


Fig. P11.107 and *P11.108*

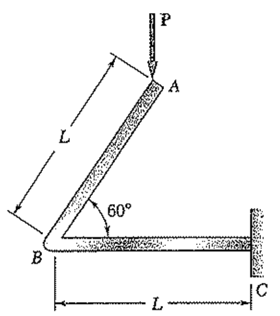


Fig. *P11.109* and *P11.110*

11.108 Three rods, each of the cams flexural rigidity *EI*, are welded to form the frame *ABCD*. For the loading shown, determine the angle formed by the frame at point *D*.

11.109 A uniform rod of flexural rigidity *EI* is bent and loaded as shown. Determine (*a*) the vertical deflection of point *A*, (*b*) the horizontal deflection of point *A*.

11.110 A uniform rod of flexural rigidity *EI* is bent and loaded as shown. Determine (*a*) the vertical deflection of point *B*, (*b*) the slope of *BC* at point *B*.

11.111 through 11.114 Determine the reaction at the roller support and the draw the bending-moment diagram for the beam and loading shown.

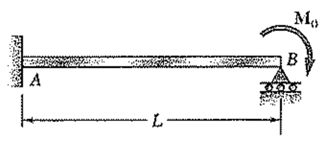


Fig. P11.111

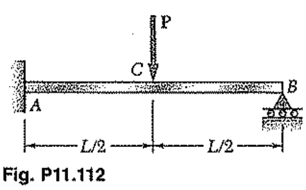


Fig. P11.112

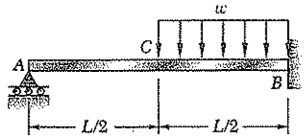


Fig. *P11.113*

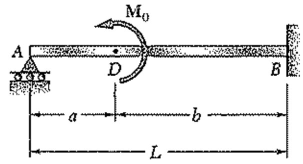
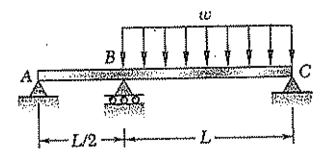


Fig. P11.114

11.115 For the uniform beam and loading shown, determine the reaction at each support.



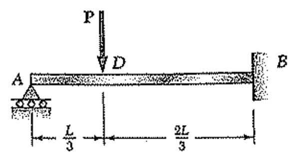


Fig. P11.116

11.116 Determine the reaction at the roller support and draw the bending-moment diagram for the beam and load shown.

11.117 through *11.120* Three members of the same material and same cross-sectional area are used to support the load **P**. Determine the force in member *BC*.

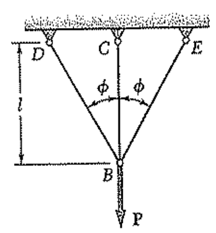


Fig. P11.117

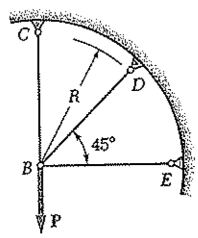


Fig. P11.118

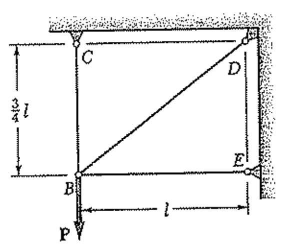


Fig. P11.119

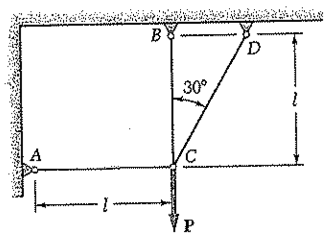


Fig. *P11.120*

11.121 and 11.122 Knowing that the eight members of the indeterminate truss shown had the same uniform cross-sectional area, determine the force in member *AB*.

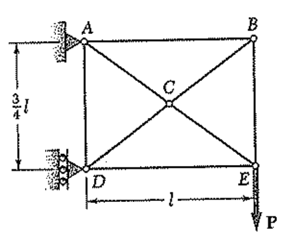


Fig. P11.121

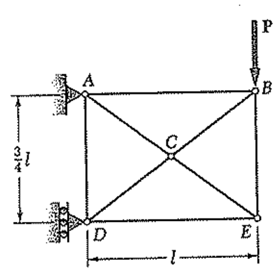


Fig. P11.122

REVIEW AND SUMMARY FOR CHAPTER 11

This chapter was devoted to the study of strain energy and to the ways in which it can be used to determine the stresses and deformations in structures subjected to both static and impact loadings.

In Sec. 11.2 we considered a uniform rod subjected to a slowly increasing axial load **P** (Fig 11.1). We noted that the area under the load-deformation diagram (Fig 11.3) represents the work done by **P**. This work is equal to the *strain energy* of the rod associated with the deformation caused by the load **P**:

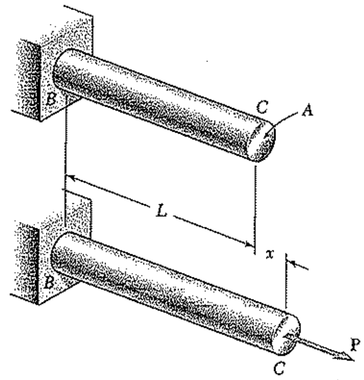


Fig. 11.1

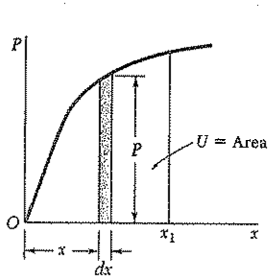


Fig. 11.3

Strain energy

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \qquad (11.2)$$

Since the stress is uniform throughout the rod, we were able to divide the strain energy by the volume of the rod and obtain the strain energy per unit volume, which we defined as the *strain-energy density* of the material [Sec. 11.3]. We found that

Strain-energy density

$$\text{Strain-energy density} = u = \int_0^{\epsilon_1} \sigma_x\,d\epsilon_x \qquad (11.4)$$

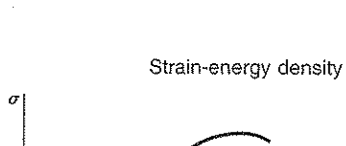


Fig. 11.6

and noted that the strain-energy density is equal to the area under the stress-strain diagram of the material (Fig. 11.6). As we saw in Sec. 11.4, Eq. (11.4) remains valid when the stresses are not uniformly distributed, but the strain-energy density will then vary from point to point. If the material is unloaded, there is a permanent strain ϵ_p and only the strain-energy density corresponding to the triangular area is recovered, the remainder of the energy having been dissipated in the form of heat during the deformation of the material.

Modulus of toughness

The area under the entire stress-strain diagram was defined as the *modulus of toughness* and is a measure of the total energy that can be acquired by the material.

If the normal stress σ remains within the proportional limit of the material, the strain-energy density u is expressed as

$$u = \frac{\sigma^2}{2E}$$

Modulus of resilience

The area under the stress-strain curve from zero strain to the strain ϵ_Y at yield (Fig. 11.9) is referred to as the *modulus of resilience* of the material and represents the energy per unit volume that the material can absorb without yielding. We wrote

$$u_Y = \frac{\sigma_Y^2}{2E} \tag{11.8}$$

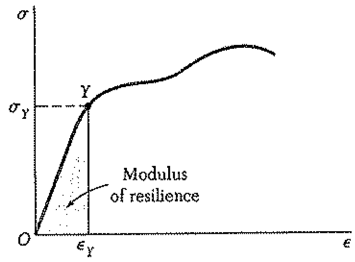


Fig. 11.9

Strain energy under axial load

In Sec. 11.4 we considered the strain energy associated with *normal stresses.* We saw that if a rod of length L and *variable cross-sectional area A* is subjected at its end to a centric axial load $\mathbf{P}$, the strain energy of the rod is

$$U = \int_0^L \frac{P^2}{2AE}\,dx \tag{11.13}$$

If the rod is of *uniform cross section* of area A, the strain energy is

$$U = \frac{P^2L}{2AE} \tag{11.14}$$

Strain energy due to bending

We saw that for a beam subjected to transverse loads (Fig. 11.15) the strain energy associated with the normal stresses is

$$U = \int_0^L \frac{M^2}{2EI}\,dx \tag{11.17}$$

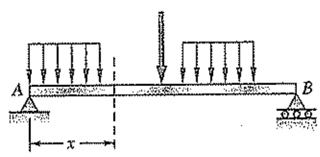


Fig. 11.15

where M is the bending moment and EI the flexural rigidity of the beam.

Strain energy due to shearing stresses

The strain energy associated with *shearing stresses* was considered in Sec. 11.5. We found that the strain-energy density for a material in pure shear is

$$u = \frac{\tau_{xy}^2}{2G} \tag{11.19}$$

where τ_{xy} is the shearing stress and G the modulus of rigidity of the material.

Strain energy due to torsion

For a shaft of length L and uniform cross section subjected at its ends to couples of magnitude T (Fig. 11.19) the strain energy was found to be

$$U = \frac{T^2L}{2GJ} \tag{11.22}$$

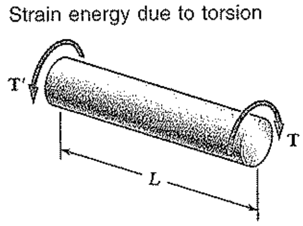


Fig. 11.19

where J is the polar moment of inertia of the cross-sectional area of the shaft.

General state of stress

In Sec. 11.6 we considered the strain energy of an elastic isotropic material under a general state of stress and expressed the strain-energy density at a given point in terms of the principal stresses σ_a, σ_b, and σ_c at that point:

$$u = \frac{1}{2E}\left[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu\left(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a\right)\right] \quad (11.27)$$

The strain-energy density at a given point was divided into two parts: u_v, associated with a change in volume of the material at that point, and u_d, associated with a distortion of the material at the same point. We wrote $u = u_v + u_d$, where

$$u_v = \frac{1-2\nu}{6E}(\sigma_a + \sigma_b + \sigma_c)^2 \quad (11.32)$$

and

$$u_d = \frac{1}{12G}\left[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2\right] \quad (11.33)$$

Using the expression obtained for u_d, we derived the maximum-distortion-energy criterion, which was used in Sec. 7.7 to predict whether a ductile material would yield under a given state of plane stress.

Impact loading

Equivalent static load

In Sec. 11.7 we considered the *impact loading* of an elastic structure being hit by a mass moving with a given velocity. We assumed that the kinetic energy of the mass is transferred entirely to the structure and defined the *equivalent static load* as the load that would cause the same deformations and stresses as are caused by the impact loading.

After discussing several examples, we noted that a structure designed to withstand effectively an impact load should be shaped in such a way that stresses are evenly distributed throughout the structure, and that the material used should have a low modulus of elasticity and a high yield strength [Sec. 11.8].

Members subjected to a single load

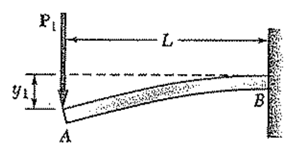


Fig. 11.27

The strain energy of structural members subjected to a *single load* was considered in Sec. 11.9. In the case of the beam and loading of Fig. 11.27 we found that the strain energy of the beam is

$$U = \frac{P_1^2 L^3}{6EI} \quad (11.46)$$

Observing that the work done by the load **P** is equal to $\frac{1}{2}P_1 y_1$, we equated the work of the load and the strain energy of the beam and determined the deflection y_1 at the point of application of the load [Sec. 11.10 and Example 11.10].

The method just described is of limited value, since it is restricted to structures subjected to a single concentrated load and to the determination of the deflection at the point of application of that load. In the remaining sections of the chapter, we presented a more general method, which can be used to determine deflections at various points of structures subjected to several loads.

Castigliano's theorem

In Sec. 11.11 we discussed the strain energy of a structure subjected to several loads, and in Sec. 11.12 introduced *Castigliano's theorem*, which states that the deflection x_j, of the point of application of a load $\mathbf{P}_j$ measured along the line of action of $\mathbf{P}_j$ is equal to the partial derivative of the strain energy of the structure with respect to the load $\mathbf{P}_j$. We wrote

$$x_j = \frac{\partial U}{\partial P_j} \qquad (11.65)$$

We also found that we could use Castigliano's theorem to determine the *slope* of a beam at the point of application of a couple $\mathbf{M}_j$ by writing

$$\theta_j = \frac{\partial U}{\partial M_j} \qquad (11.68)$$

and the *angle of twist* in a section of a shaft where a torque $\mathbf{T}_j$ is applied by writing

$$\phi_j = \frac{\partial U}{\partial T_j} \qquad (11.69)$$

In Sec. 11.13, Castigliano's theorem was applied to the determination of deflections and slopes at various points of a given structure. The use of "dummy" loads enabled us to include points where no actual load was applied. We also observed that the calculation of a deflection x_j was simplified if the differentiation with respect to the load P_j was carried out before the integration. In the case of a beam, recalling Eq. (11.17), we wrote

$$x_j = \frac{\partial U}{\partial P_j} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial P_j}\,dx \qquad (11.70)$$

Similarly, for a truss consisting of n members, the deflection x_j at the point of application of the load $\mathbf{P}_j$ was found by writing

$$x_j = \frac{\partial U}{\partial P_j} = \sum_{i=1}^{n} \frac{F_i L_i}{A_i E}\frac{\partial F_i}{\partial P_j} \qquad (11.72)$$

Indeterminate structures

The chapter concluded [Sec. 11.14] with the application of Castigliano's theorem to the analysis of *statically indeterminate structures* [Sample Prob. 11.7, Examples 11.15 and 11.16].

REVIEW PROBLEMS

11.123 For the beam and loading shown determine the deflection at point B. Use $E = 200$ GPa.

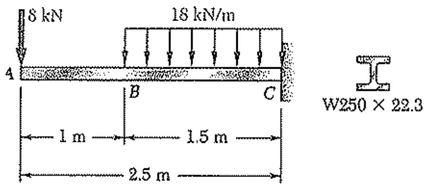


Fig. P11.123

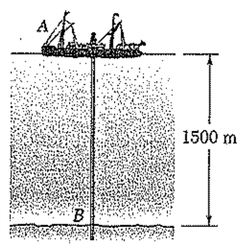


Fig. P11.124

11.124 The ship at A has just started to drill for oil on the ocean floor at a depth of 1500 m. The steel drill pipe has an outer diameter of 200 mm and a uniform wall thickness of 12 mm. Knowing that the top of the drill pipe rotates through two complete revolutions before the drill bit at B starts to operate and using $G = 77$ GPa, determine the maximum strain energy acquired by the drill pipe.

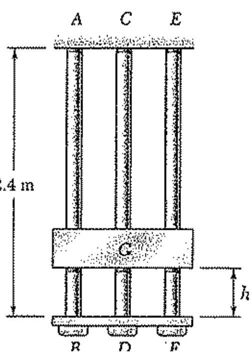


Fig. P11.125

11.125 The 45-kg collar C is released from rest in the position shown and is stopped by Plate BDF that is attached the 22-mm-diameter steel rod CD and to the 16-mm-diameter steel rods AB and EF. Knowing that for the grade of steel used $\sigma_{all} = 165$ MPa and $E = 200$ GPa, determine the largest allowable distance h.

11.126 Solve Prob. 11.125, assuming that the 22-mm-diameter steel rod CD is replaced by a 22-mm-diameter rod made of an aluminum alloy for which $\sigma_{all} = 140$ MPa and $E = 73$ GPa.

11.127 Rod AB is made of a steel for which the yield strength is $\sigma_Y = 450$ MPa and $E = 200$ GPa; rod BC is made of an aluminum alloy for which $\sigma_Y = 280$ MPa and $E = 73$ GPa. Determine the maximum strain energy that can be acquired by the composite rod ABC without causing any permanent deformations.

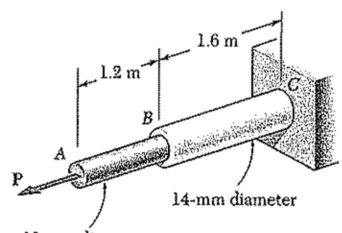


Fig. P11.127

11.128 Each member of the truss shown is made of steel and has a uniform cross-sectional area of 3220 mm^2. Using $E = 200$ GPa, determine the vertical deflection of joint C caused by the application of the 60-kN load.

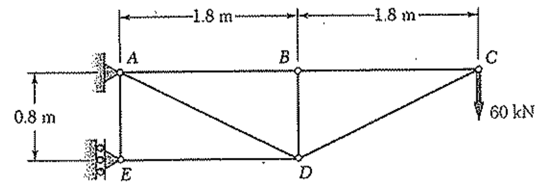


Fig. *P11.128*

11.129 The 1.35-kg block D is released from rest in the position shown and strikes a steel bar AB having the uniform cross section shown. The bar is supported at each end by springs of constant 3500 kN/m. Using $E = 200$ GPa, determine the maximum deflection at the midpoint of the bar.

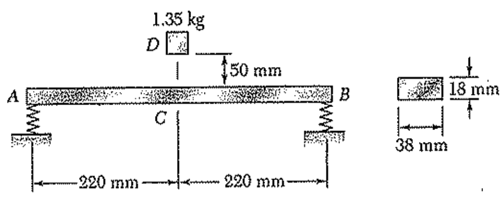


Fig. P11.129

11.130 Solve Prob. 11.129, assuming that the constant of each spring is 7000 kN/m.

11.131 Using $E = 12$ GPa, determine the strain energy due to bending for the timber beam and loading shown.

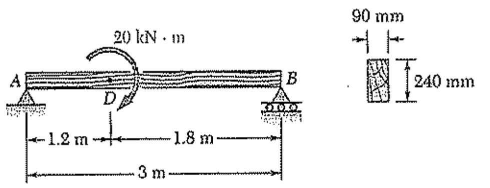


Fig. *P11.131*

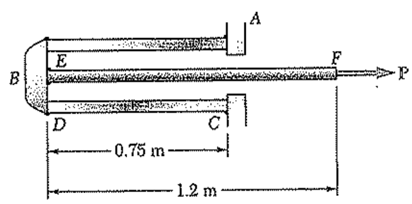


Fig. P11.132

11.132 A 0.75-m length of aluminum pipe of cross-sectional area 1190 mm^2 is welded to a fixed support A and to a rigid cap B. The steel rod EF, of 18-mm diameter, is welded to cap B. Knowing that the modulus of elasticity is 200 GPa for the steel and 73 GPa for the aluminum, determine (a) the total strain of the system when $P = 40$ kN, (b) the corresponding strain-energy density of the pipe CD and in the rod EF.

11.133 Solve Prob. 11.132, when $P = 35$ kN.

11.134 Rod AC is made of aluminum and is subjected to a torque **T** applied at C. Knowing that $G = 73$ GPa and that portion BC of the rod is hollow and has an inner diameter of 16 mm, determine the strain energy of the rod for a maximum shearing stress of 120 MPa.

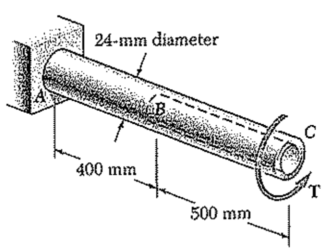


Fig. P11.134

COMPUTER PROBLEMS

The following problems are designed to be solved with a computer.

11.C1 A rod consisting of n elements, each of which is homogeneous and of uniform cross section, is subjected to a load **P** applied at its free end. The length of element i is denoted by L_i and its diameter by d_i. (*a*) Denoting by E the modulus of elasticity of the material used in the rod, write a computer program that can be used to determine the strain energy acquired by the rod and the deformation measured at its free end. (*b*) Use this program to determine the strain energy and deformation for the rods of Probs. 11.9 and 11.10.

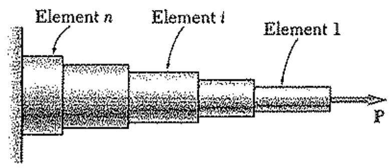


Fig. P11.C1

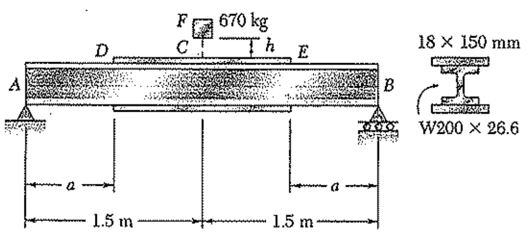


Fig. P11.C2

11.C2 Two 18 × 150-mm cover plates are welded to a W200 × 26.6 rolled-steel beam as shown. The 670-kg block is to be dropped from a height $h = 50$ mm onto the beam. (*a*) Write a computer program to calculate the maximum normal stress on transverse sections just to the left of D and at the center of the beam for values of a from 0 to 1.5 m using 125-mm increments. (*b*) From the values considered in part *a*, select the distance a for which the maximum normal stress is a small as possible. Use $E = 200$ GPa.

11.C3 The 16-kg block D is dropped from a height h onto the free end of the steel bar AB. For the steel used $\sigma_{all} = 120$ MPa and $E = 200$ GPa. (*a*) Write a computer program to calculate the maximum allowable height h for values of the length L from 100 mm to 1.2 m, using 100-mm increments. (*b*) From the values considered in part *a*, select the length corresponding to the largest allowable height.

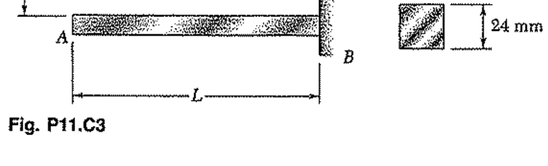


Fig. P11.C3

11.C4 The block D of mass $m = 8$ kg is dropped from a height $h = 750$ mm onto the rolled-steel beam AB. Knowing that $E = 200$ GPa, write a computer program to calculate the maximum deflection of point E and the maximum normal stress in the beam for values of a from 100 to 900 m, using 100-mm increments.

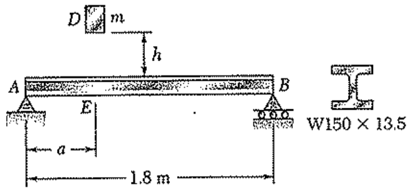


Fig. P11.C4

11.C5 The steel rods AB and BC are made of a steel for which $\sigma_Y = 300$ MPa and $E = 200$ GPa. (a) Write a computer program to calculate for values of a from 0 to 6 m, using 1-m increments, the maximum strain energy that can be acquired by the assembly without causing any permanent deformation. (b) For each value of a considered, calculate the diameter of a uniform rod of length 6 m and of the same mass as the original assembly, and the maximum strain energy that could be acquired by this uniform rod without causing permanent deformation.

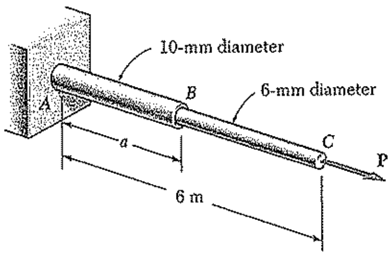


Fig. P11.C5

11.C6 A 72-kg diver jumps from a height of 0.5 m onto end C of a diving board having the uniform cross section shown. Write a computer program to calculate for values of a from 250 to 1250 mm, using 250-mm increments, (a) the maximum defection of point C, (b) the maximum bending moment in the board, (c) the equivalent static load. Assume that the diver's legs remain rigid and use $E = 12$ GPa.

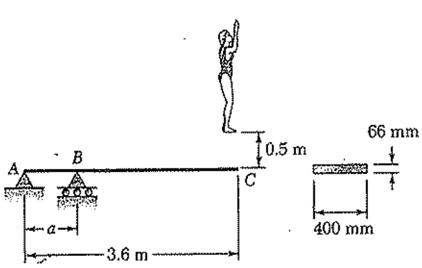


Fig. P11.C6