Aula 35 (14/Abr)

Na aula de hoje:

* Resolução de exercícios da Folha 6 - Osciladores harmónicos.
  - Exercício 1 (OHQ 1D da eq. Schrödinger).
  - Exercício 2 (Valores esperados OHQ 1D).
  - Exercício 4 (OHQ 1D num campo eléctrico).
  - Exercício 8 (OHQ anisotrópico em 2D).

Folha de Problemas 6

Osciladores Harmónicos

(1) Resolução do OHQ 1D directamente da eq. de Schrödinger

(a) Usando o ansatz

$$\Phi(x) = \phi(x) \cdot e^{-\frac{m\omega}{2\hbar} x^2}$$

na eq. Schr. independente do tempo

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{m\omega^2}{2}x^2\right]\Phi(x)=E\,\Phi(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\underbrace{\frac{d^2}{dx^2}\left(f(x)\cdot e^{-\dots x^2}\right)}_{\parallel}+\left[\frac{m\omega^2}{2}x^2-E\right]f(x)\cdot e^{-\dots x^2}=0$$

$$\frac{d}{dx}\left(f'(x)\cdot e^{-\dots x^2}+f(x)\cdot\left(-\frac{2m\omega}{2\hbar}x\right)\cdot e^{-\dots x^2}\right)$$

$$=f''(x)\cdot e^{-\dots x^2}+f'(x)\cdot\left(-\frac{\cancel{2}m\omega}{\cancel{2}\hbar}x\right)e^{-\dots x^2}+f'(x)\cdot\left(-\frac{m\omega}{\hbar}x\right)e^{-\dots x^2}$$

$$-\frac{m\omega}{\hbar}f(x)\cdot e^{-\dots x^2}+f(x)\left(-\frac{m\omega}{\hbar}x\right)^2\cdot e^{-\dots x^2}$$

$$=\left[f''(x)-2\frac{m\omega}{\hbar}x\cdot f'(x)-\frac{m\omega}{\hbar}\left(1-\frac{m\omega}{\hbar}x^2\right)\cdot f(x)\right]e^{-\dots x^2}$$

$$\Leftrightarrow -\frac{\hbar^2}{2m}\left[f''(x)-\frac{2m\omega}{\hbar}x\,f'(x)-\frac{m\omega}{\hbar}\left(1-\frac{m\omega}{\hbar}x^2\right)f(x)\right]+\left(\frac{m\omega^2}{2}x^2-E\right)f(x)=0$$

$$\Leftrightarrow f''(x)-\frac{2m\omega}{\hbar}x\cdot f'(x)+\left[\cancel{-\frac{m^2\omega^2}{\hbar^2}x^2}+\frac{2mE}{\hbar^2}-\frac{m\omega}{\hbar}+\cancel{\frac{m^2\omega^2}{\hbar^2}x^2}\right]f(x)=0$$

$$\Leftrightarrow f''(x)-\frac{2m\omega}{\hbar}x\,f'(x)+\frac{2m}{\hbar^2}\left(E-\frac{\hbar\omega}{2}\right)\cdot f(x)=0$$

II

(b) Admitindo que $f(x)=\sum_{n=0}^{\infty}a_n\cdot x^n$

$$f''(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1) \cdot x^{n-2} \overset{n=n-2}{=} \sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) \cdot x^n$$

$$f'(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot x^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2} (n+2)(n+1) \underbrace{- \frac{2m\omega}{\hbar} \cdot a_n \cdot n + \frac{2m}{\hbar^2}\left(E - \frac{\hbar\omega}{2}\right) a_n}_{} \right] x^n = 0$$

$$= -\frac{2m}{\hbar^2}\left(n \cdot \omega\hbar - E + \frac{\hbar\omega}{2}\right) a_n$$

$$\Leftrightarrow \frac{2m}{\hbar^2}\left(E - \hbar\omega\left(n + \frac{1}{2}\right)\right) \cdot a_n$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \left[a_{n+2} (n+2)(n+1) + \frac{2m}{\hbar^2}\left[E - \hbar\omega\left(n + \frac{1}{2}\right)\right] a_n \right] x^n = 0$$

□

(c) Usando $\chi \equiv \sqrt{\frac{\hbar}{m\omega}}\, x$ e $b_n = \left(\frac{\hbar}{m\omega}\right)^{n/2} \cdot a_n$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1) \cdot b_{n+2} \left(\frac{m\omega}{\hbar}\right) + \frac{2m}{\hbar^2}\left(E - \hbar\omega\left(n + \tfrac{1}{2}\right)\right) \cdot b_n \right] \chi^n = 0$$

A série é zero se todos os coeficientes forem zero

$$(n+2)(n+1)\cdot b_{n+2} = -\frac{\cancel{\hbar}}{\cancel{m}\omega}\frac{2\cancel{m}}{\hbar^{\cancel{2}}}\left(E-\hbar\omega\left(n+1/2\right)\right)\cdot b_n$$

$$= -\frac{2}{\hbar\omega}\left(E-\hbar\omega\left(n+1/2\right)\right) b_n$$

$$\Longleftrightarrow (n+2)(n+1)\cdot b_{n+2} = \left(2n+1-\frac{2E}{\hbar\omega}\right)\cdot b_n$$

$\square$

(d) Assumindo que a série é finita, teremos que ter que b_n seja zero para algum n, pois daí em diante todos os b_n's serão zero, já que são obtidos por recorrência

Assim, teremos que ter $n_0 \in \mathbb{N}_0$ tal que

$$2n_0 + 1 - \frac{2E}{\hbar\omega} = 0$$

$$\Longleftrightarrow E = \hbar\omega\left(n_0 + 1/2\right),$$

$\square$

que é a expressão dos níveis de energia do OHQ 1D.

(e) Podemos escrever

$$(n+2)(n+1)\, b_{n+2} = 2(n-n_0)\, b_n . \quad \square$$

que relaciona coeficientes da série com a mesma paridade, pois b_n é coeficiente x^n.

(1) $b_0 = 1$ e $n_0 = 0$

$$\Rightarrow b_2 \cdot 2 \cdot 1 = 2(0-0) \cdot \overbrace{b_0}^{=1} = 0$$

$$\Rightarrow p(x) = 1 . \quad \square$$

(2) $b_1 = 2$ e $n_0 = 1$

$$\Rightarrow b_3 \cdot 3 \cdot 2 = 2(1-1) \cdot \overbrace{b_1}^{=2} = 0$$

$$\Rightarrow p(x) = 2x \quad \square$$

(3) $b_0 = 1$ e $n_0 = 2$

$\underline{n=0}$: $b_2 \cdot 2 \cdot 1 = 2(0-2) \cdot \overbrace{b_0}^{=1} = -4$

$$\Leftrightarrow b_2 = -2$$

$n=2$: $\quad b_4 . 4 . 3 = 2(2-2)\overbrace{b_2}^{=-2} = 0$

$$\Rightarrow \phi(x) = 1 - 2x^2$$

$\square$

(4) $\quad b_1 = 2$ e $n_0 = 3$

$n=1$: $\quad b_3 . 3 . 2 = 2.(1-3).\overbrace{b_1}^{=2}$

$$\Rightarrow b_3 = -\frac{4}{3}$$

$n=3$: $\quad b_5 . 5 . 4 = 2.(3-3)\overbrace{b_3}^{=-4/3} = 0$

$$\Rightarrow b_5 = 0$$

$$\Rightarrow \phi(x) = 2x - \frac{4}{3}x^3$$

(B) A f.o. total é dada

$$\psi(t,x) = e^{-iEt/\hbar} \frac{\sum_{n=0}^{\infty} e_n x^n}{e^{m\omega x^2/2\hbar}}$$

(1) $$e^{\frac{m\omega}{2\hbar}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{m\omega}{2\hbar}x^2\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{m\omega}{2\hbar}\right)^n . x^{2n} = \sum_{n=0}^{\infty} c_n . x^n$$

logo os $c_{2n+1} = 0$ e $c_{2n} = \frac{1}{n!}\left(\frac{m\omega}{2\hbar}\right)^n$.

(2) $$\lim_{n\to\infty} \frac{c_{2n+2}}{c_{2n}} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}\left(\frac{m\omega}{2\hbar}\right)^{n+1}}{\frac{1}{n!}\left(\frac{m\omega}{2\hbar}\right)^{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n+1}\,\frac{m\omega}{2\hbar} \longrightarrow \frac{m\omega}{2n\hbar}$$

$\square$

(3) Em termos dos a_n

$$(n+2)(n+1)\cdot a_{n+2}\left(\frac{\hbar}{m\omega}\right)^{\cancel{\frac{n}{2}+1}} = \left(2n+1-\frac{2E}{\hbar\omega}\right)\cancel{\left(\frac{\hbar}{m\omega}\right)^{\frac{n}{2}}}\cdot a_n$$

$$\Leftrightarrow \frac{a_{n+2}}{a_n} = \frac{2n+1-\frac{2E}{\hbar\omega}}{(n+2)(n+1)}\cdot\frac{m\omega}{\hbar}$$

$$\lim_{n\to\infty} \frac{2n+1-2E/\hbar\omega}{n^2+3n+2}\,\frac{m\omega}{\hbar} \longrightarrow \frac{2m\omega}{n.\hbar}$$

(4) Concluimos que $n\to\infty$ teremos

$$\frac{a_{n+2}}{a_n} > \frac{c_{2n+2}}{c_{2n}}$$

e assim numerador maior do que o denominador e $\psi(t,x)$ qd $x \to \infty$ não cairá para zero, logo não normalizável se a série for infinito.

$\Rightarrow$ f.o. normalizável $\Rightarrow$ série finito de $f(x)$ $\Longrightarrow$ níveis de energia.

② Valores esperados e o OHQ 1D

(a) O valor esperado de $\hat{K}$ é

$$\langle \varphi_n | \hat{K} | \varphi_n \rangle = \langle \varphi_n | \frac{\hat{P}^2}{2m} | \varphi_n \rangle = ?$$

Como $\hat{P} = i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^+ - \hat{a})$ então

$[a, a^+] = 1$

$a^+a + 1$

$$\langle \varphi_n | \hat{K} | \varphi_n \rangle = \frac{1}{2m}\langle \varphi_n | -\frac{m\hbar\omega}{2} \cdot \left((\hat{a}^+)^2 - a^+a - \underbrace{aa^+}_{a^+a+1} + \hat{a}^2\right) | \varphi_n \rangle$$

$$= -\frac{\hbar\omega}{4}\langle \varphi_n | (a^+)^2 - 2a^+a - \hat{\mathbb{1}} + a^2 | \varphi_n \rangle$$

que como $\hat{a}|\varphi_n\rangle = \sqrt{n}\,|\varphi_{n-1}\rangle$ e $\hat{a}^+|\varphi_n\rangle = \sqrt{n+1}\,|\varphi_{n+1}\rangle$

e $\langle \varphi_m | \varphi_n \rangle = \delta_{mn}$, então

$$\Longrightarrow \langle \varphi_n | \hat{K} | \varphi_n \rangle = -\frac{\hbar\omega}{4}(-2n-1) = \frac{\hbar\omega}{2}(n+1/2)$$

Fazendo o mesmo para $\hat{V} = \frac{m\omega^2}{2}\hat{x}^2$ e como

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + \hat{a})$$

então

$$\langle \hat{V} \rangle = \frac{\cancel{m}\omega^{\cancel{2}}}{2}\frac{\hbar}{2\cancel{m\omega}}\langle \varphi_n | (a^+)^2 + \overbrace{a^+a + aa^+}^{\overset{2a^+a+\hat{1}}{//}} + a^2 | \varphi_n \rangle$$

$$= \frac{\hbar\omega}{4}(2n+1) = \frac{\hbar\omega}{2}(n+1/2) = \langle \hat{K} \rangle$$

$\square$

(b) Preparamos

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|\varphi_0\rangle + |\varphi_2\rangle)$$

(1) $\langle \hat{x} \rangle = \langle \psi | \hat{x} | \psi \rangle = \frac{1}{2}(\langle \varphi_0| + \langle \varphi_2|)\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + \hat{a})(|\varphi_0\rangle + |\varphi_2\rangle)$

$$= \frac{1}{2}\sqrt{}(\langle \varphi_0| + \langle \varphi_2|)(|\varphi_1\rangle + \sqrt{3}|\varphi_3\rangle + 0 + \sqrt{2}|\varphi_1\rangle)$$

$\Leftrightarrow \langle \hat{x} \rangle = 0$

(2) Sabemos que

$$\begin{cases} n \text{ par} \longrightarrow \varphi_n(x) \text{ é par} \\ n \text{ ímpar} \longrightarrow \varphi_{n+1}(x) \text{ é ímpar} \end{cases}$$

Então é claro que $|\psi\rangle$ é par pois combino duas pares. $\Rightarrow$ Paridade será conservada pois $[\hat{\Pi}, \hat{H}] = 0$ já que $V(-x) = V(x)$.

(3)
$$\langle \hat{H} \rangle = \frac{1}{2}\left(\langle\varphi_0| + \langle\varphi_2|\right)\left(E_0|\varphi_0\rangle + E_2|\varphi_2\rangle\right)$$

$$= \frac{1}{2}\left(\underbrace{E_0}_{=\frac{\hbar\omega}{2}} + \underbrace{E_2}_{=\hbar\omega\frac{5}{2}}\right) = \frac{3\hbar\omega}{2}$$

que será constante no tempo.

④ OHQ 1D num campo eléctrico

$$V(x) = \frac{m\omega^2}{2}x^2$$

$$\vec{E} = \mathcal{E}_0 \cdot \vec{e_x}$$

(a) Se temos $\vec{E} = \mathcal{E}_0 \vec{e_x}$ então potencial eléctrico $V_e(x) = -\mathcal{E}_0 \cdot q \cdot x$ pois $\vec{F} = q \cdot \mathcal{E}_0 = -\frac{dV_e}{dx}$.

Assim o Hamiltoniano será

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2} x^2 - q\,\mathcal{E}_0 \cdot \hat{x}$$

Podemos completar o quadrado

$$(x-a)^2 = x^2 - 2ax + a^2$$

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\left(\hat{x}^2 - \underbrace{\frac{2q\mathcal{E}_0}{m\omega^2}}_{= a}\cdot\hat{x} + \frac{q^2\mathcal{E}_0^2}{m^2\omega^4} - \frac{q^2\mathcal{E}_0^2}{m^2\omega^4}\right)$$

$$= \frac{\hat{P}^2}{2m} + \frac{m\omega^2}{2}\underbrace{\left(\hat{x} - \frac{q\mathcal{E}_0}{m\omega^2}\right)}_{\equiv \hat{\bar{x}}}^2 - \frac{q^2 \cdot \mathcal{E}_0^2}{2m\omega^2}$$

e assim sabemos as energias pois essencialmente igual ao OHQ 1D comum,

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) - \frac{q^2 \cdot \mathcal{E}_0^2}{2m\omega^2}$$

$\blacksquare$

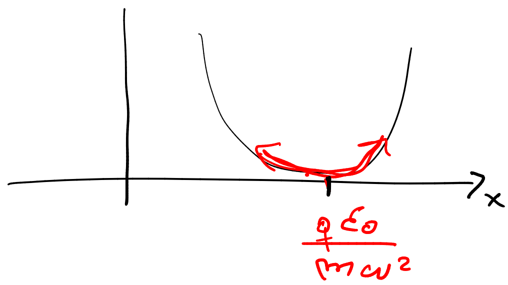
(b) Para o estado fundamental do OHQ 1D comum sabemos que

$$\phi_0(\bar{x}) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega}{2\hbar}\cdot\bar{x}^2}$$

mas como $\bar{x} = x - \frac{q\mathcal{E}_0}{m\omega^2}$, então

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega}{2\hbar}\cdot\left(x-\frac{q\mathcal{E}_0}{m\omega^2}\right)^2}$$

□



(c) O momento dipolar eléctrico é dado

$$\hat{p}_e = q\cdot\hat{x} \Rightarrow \langle\varphi_0|\hat{p}_e|\varphi_0\rangle = q\langle\varphi_0|\hat{x}|\varphi_0\rangle$$

$$\Rightarrow \langle\hat{p}_e\rangle = \frac{q^2\cdot\mathcal{E}_0}{m\omega^2}$$

□

(8) OHQ anisotrópico em 2D

$$\hat{V}(\hat{x}) = \frac{m}{2}\left(\omega_0^2\hat{x}^2+\omega_1^2\hat{y}^2\right)$$

onde $\omega_1^2 > \omega_0^2$.

(a) Podemos usar quantões lineares para resolver o problema.

$$\hat{H} = \frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{m}{2}\left(\omega_0^2 \hat{x}^2 + \omega_1^2 \hat{y}^2\right)$$

$$= \left[\frac{\hat{P}_x^2}{2m} + \frac{m\omega_0^2}{2}\hat{x}^2\right] + \left[\frac{\hat{P}_y^2}{2m} + \frac{m\omega_1^2}{2}\hat{y}^2\right]$$

$$= \hbar\omega_0\left(\hat{N}_x + \frac{\hat{1}}{2}\right) + \hbar\omega_1\left(\hat{N}_y + \frac{\hat{1}}{2}\right)$$

onde $\hat{N}_x = \hat{a}_x^+ \hat{a}_x$ e $\hat{N}_y = \hat{a}_y^+ \hat{a}_y$.

As energias: $E_{n_x, n_y} = \hbar\omega_0\left(n_x + \frac{1}{2}\right) + \hbar\omega_1\left(n_y + \frac{1}{2}\right)$

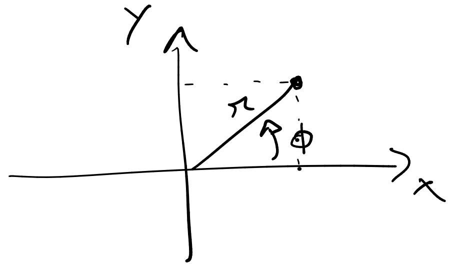
↳ Estado fundamental: $E_{0,0} = \frac{\hbar\omega_0}{2} + \frac{\hbar\omega_1}{2}$

↳ Primeiro estado excitado: $E_{1,0} = \frac{\hbar}{2}(3\omega_0 + \omega_1)$, pois $\omega_0 < \omega_1$.

Ambos são não degenerados pois $\omega_0 \neq \omega_1$.

(b) Coordenadas polares

$$\begin{cases} x = r\cos\phi \\ y = r\,\text{sen}\,\phi \end{cases}$$



O lagrangeano,

$$\mathcal{L} = T - V$$

$$= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - \frac{m}{2}(\omega_0^2 x^2 + \omega_1^2 y^2)$$

$$= \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - \frac{m}{2} r^2 (\omega_0^2 \cos^2\phi + \omega_1^2 \operatorname{sen}^2\phi)$$

Pelo T. Noether sabemos que se $\mathcal{L}$ é invariante por $q \to q + \delta q$ então temos quantidade conservada associada a essa invariância

$$\dot{p}_r = \frac{\partial \mathcal{L}}{\partial r} = m r \dot{\phi}^2 - m r (\omega_0^2 \cos^2\phi + \omega_1^2 \operatorname{sen}^2\phi)$$

$$\dot{p}_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = m r^2 \underbrace{(-\omega_0^2 + \omega_1^2)}_{\text{se } \omega_1 = \omega_0 \Rightarrow \dot{p}_\phi = 0} \operatorname{sen}\phi \cos\phi$$

Sabemos que $l_z = x m \dot{y} - y m \dot{x} = \ldots = m r^2 \dot{\phi}$, que é igual a p_ϕ

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \phi = l_z$$

Assim concluimos

$$\dot{l}_z = \dot{p}_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = m r^2 (\omega_1^2 - \omega_0^2) \operatorname{sen}\phi \cos\phi \neq 0$$

Logo se $\omega_0 \neq \omega_1$ L_z não é constante do movimento.

(c) Equivale a verificar que $[\hat{H}, \hat{L}_z] \neq 0$.

$$[\hat{H}, \hat{L}_z] = \left[\frac{\hat{P}_x^2 + \hat{P}_y^2}{2m} + \frac{m}{2}\left(\omega_0^2 \hat{X}^2 + \omega_1^2 \hat{Y}^2\right), \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right]$$

$$= P_x P_x X - X P_x P_x + P_x X P_x - P_x X P_x = P_x[P_x, X] + [P_x, X]P_x$$

$$= \frac{1}{2m}\underbrace{[\hat{P}_x^2, \hat{X}]}_{= -2i\hbar \hat{P}_x}\hat{P}_y - \frac{1}{2m}\underbrace{[\hat{P}_y^2, \hat{Y}]}_{= -2i\hbar \hat{P}_y}\hat{P}_x - \frac{m}{2}\omega_0^2 \hat{Y}\underbrace{[\hat{X}^2, \hat{P}_x]}_{+2i\hbar \hat{X}}$$

$$+ \frac{m}{2}\omega_1^2 \hat{X}\underbrace{[\hat{Y}^2, \hat{P}_y]}_{= 2i\hbar \hat{Y}}$$

$$= \frac{1}{2m}(-2i\hbar)\underbrace{\left(\hat{P}_x\hat{P}_y - \hat{P}_y\hat{P}_x\right)}_{\to 0} - \frac{m}{2}(2i\hbar)\left(\omega_0^2 \hat{Y}\hat{X} - \omega_1^2 \hat{X}\hat{Y}\right)$$

$$= -i\hbar m \hat{Y}\hat{X}\left(\omega_0^2 - \omega_1^2\right) \neq 0$$

$\hookrightarrow$ se $\omega_0^2 \neq \omega_1^2$