Aula 40 (24/Abr)

Na aula de hoje:

* Resolução de exercícios da Folha 8 - Momento Angular.
  - Ex.2 (Valores esperados do momento angular).
  - Ex.4 (Momento angular orbital).
  - Ex.6 (Rotação molécula diatómica).
* Resolução de exercícios da Folha 9 - Partícula num potencial central e átomo hidrogénio.
  - Ex.2 (Átomo de hidrogénio).
  - Ex.4 (OHQ 3D num campo magnético).

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Folha 8 (2021.1)

Momento Angular

② Valores esperados dos operadores de momento angular

$$j=1 \Rightarrow m=-1,0,1 \Rightarrow \{|-1\rangle, |0\rangle, |1\rangle\}$$

$$|\psi\rangle = \alpha|1\rangle + \beta|0\rangle + \gamma|-1\rangle \Rightarrow N_\psi^2 = |\alpha|^2+|\beta|^2+|\gamma|^2$$

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$$

(a) $\langle \hat{\vec{J}} \rangle = ?$

$$\hat{J}_\pm |k\ell m\rangle = \hbar\sqrt{\ell(\ell+1)-m(m\pm1)}\,|k,\ell,m\pm1\rangle$$

$$\hat{J}_{x\,(y)} = \frac{\hat{J}_+ \pm \hat{J}_-}{2\,(i)}$$

$$\langle \hat{J}_{x\,(y)} \rangle = \left(\langle 1|\alpha^* + \langle 0|\beta^* + \langle -1|\gamma^*\right) \frac{\hat{J}_+ \pm \hat{J}_-}{2\,(i)} \left(\alpha|1\rangle + \beta|0\rangle + \gamma|-1\rangle\right) / N_\psi^2$$

$$= \frac{1}{2(i) N_\psi^2}(\cdots)\left[\left(0 + \beta.\hbar\sqrt{2}|1\rangle + \gamma\hbar\sqrt{2}|0\rangle\right) \pm \left(\alpha.\hbar\sqrt{2}|0\rangle + \beta\hbar\sqrt{2}|-1\rangle + 0\right)\right]$$

$$= \frac{\hbar}{(i)\sqrt{2}}(\cdots)\left(\beta|1\rangle + (\gamma\pm\alpha)|0\rangle \pm \beta|-1\rangle\right)/N_\psi^2$$

$\langle i|j\rangle = \delta_{ij}$

$i,j=-1,0,1$

$$= \frac{\hbar}{(i)\sqrt{2}}\left[\alpha^*.\beta + \beta^*.(\gamma\pm\alpha) \pm \gamma^*.\beta\right]/N_\psi^2$$

$$\Rightarrow \langle \hat{J}_x \rangle = \frac{2\hbar}{N_\psi^2}\, \mathrm{Re}\left[\alpha^*\beta + \beta^*\gamma\right]$$

$$\langle \hat{J}_y \rangle = \frac{2\hbar}{\cancel{i} N_\psi^2} \cdot \cancel{i}\, \mathrm{Im}\left[\alpha^*\beta + \beta^*\alpha\right]$$

$$\langle \hat{J}_x \rangle = (\quad + \quad + \quad)\, \hat{J}_z\, (\quad + \quad + \quad)$$

$$= \left(\langle 1|\alpha^* + \langle 0|\beta^* + \langle -1|\gamma^*\right)\left(\alpha.\hbar|1\rangle + 0 - \gamma\hbar|-1\rangle\right)$$

$$= \left(|\alpha|^2 - |\gamma|^2\right)\hbar$$

e assim temos

$$\langle \hat{\vec{J}} \rangle = \hbar \left(\mathrm{Re}[\alpha^* \beta + \beta^* \gamma],\ \mathrm{Im}[\alpha^* \beta + \beta^* \gamma],\ |\alpha|^2 - |\gamma|^2 \right) \quad //$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z$$

(b)

$$\langle \hat{J}_{\underset{y}{x}}^2 \rangle = \left\langle \frac{\hat{J}_+^2 \pm \hat{J}_+\hat{J}_- \pm \overbrace{\hat{J}_-\hat{J}_+}^{= \hat{J}_+\hat{J}_- - 2\hbar\hat{J}_z} + \hat{J}_-^2}{4} \right\rangle$$

$$= \left\langle \frac{J_+^2 \pm 2\hat{J}_+\hat{J}_- \mp 2\hbar\hat{J}_z + J_-^2}{4} \right\rangle$$

$$= \frac{1}{4}\left[\underbrace{\langle \hat{J}_+^2 \rangle}_{= (\hbar\sqrt{2})^2 \cdot \alpha^*\gamma} \pm 2\langle \hat{J}_+\hat{J}_- \rangle \mp 2\hbar \overbrace{\langle \hat{J}_z \rangle}^{= \hbar(|\alpha|^2 - |\gamma|^2)} + \underbrace{\langle J_-^2 \rangle}_{(\hbar\sqrt{2})^2 \cdot \gamma^*\alpha} \right]$$

$$= \frac{1}{4N_\psi^2}\Big[2\hbar^2 \left(\alpha^*\gamma + \gamma^*\alpha \mp |\alpha|^2 \pm |\gamma|^2 \right) \pm 2\,(.. + \ldots + ..)\, \hat{J}_+ \underbrace{(\alpha\hbar\sqrt{2}|0\rangle + \beta\hbar\sqrt{2}|-1\rangle + 0)}_{2\hbar(\alpha|1\rangle + \beta|0\rangle)} \Big]$$

$$= \frac{\hbar^2}{2N_\psi^2}\left(\alpha^*\gamma + \gamma^*\alpha \mp \cancel{|\alpha|^2} \pm |\gamma|^2 \pm \cancel{|\alpha|^2} \pm |\beta|^2 \right)$$

$$\langle \hat{J}_z^2 \rangle = \frac{1}{N_\psi^2}\left(\ldots \right) \cdot \left(\alpha\hbar^2|1\rangle + 0 + \beta\hbar^2|-1\rangle \right)$$

$$= \frac{\hbar^2}{N_\psi^2}\left(|\alpha|^2 + |\beta|^2 \right).$$

(c)

$$\langle \hat{J}_{\pm} \rangle = \langle \hat{J}_x \pm i \hat{J}_y \rangle$$

$$= \frac{\hbar^2}{2N_\psi^2} \left[(\alpha^* \gamma + \gamma^* \alpha) . (1 \pm i) + (|\gamma|^2 + |\beta|^2) . (1 \mp i) \right]$$

(4) Momento angular orbital

(a)

(b)

(c)

(d) $$\hat{L}^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\text{sen}^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\text{tg}\,\theta} \frac{\partial}{\partial \theta} \right),$$

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}.$$

Então $u_1(\theta, \phi) = e^{i\phi} \text{sen}\,\theta$, quando actuado por $\hat{L}^2$ e $\hat{L}_z$, fica

$$\hat{L}^2 u_1 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\text{sen}^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\text{tg}\,\theta} \frac{\partial}{\partial \theta} \right) e^{i\phi} \text{sen}\,\theta$$

$$= -\hbar^2 e^{i\phi}\left[-\text{sen}\,\theta - \underbrace{\frac{1}{\text{sen}^2\theta}\cdot\text{sen}\,\theta}_{\llcorner\, -1/\cancel{\text{sen}\,\theta}} + \frac{\overbrace{\cos\theta\ \cos\theta}^{\cancel{1}-\text{sen}^2\theta}}{\text{sen}\,\theta}\right]$$

$$= +2\hbar^2\cdot e^{i\phi}\,\text{sen}\,\theta = \underline{\underline{2\hbar^2}}\cdot u_1(\theta,\phi)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}\, e^{i\phi}\,\text{sen}\,\theta = \underline{\underline{\hbar}}\cdot u_1(\theta,\phi)$$

$$\rightarrow |\overset{l}{1},\overset{m}{1}\rangle$$

Fazendo o mesmo para $u_2(\theta,\phi) = e^{-i\phi}\cdot\text{sen}\,\theta$

$$\hat{L}^2 u_2 = \underline{\underline{2\hbar^2}}\cdot u_2(\theta,\phi),$$

$$\hat{L}_z u_2 = \underline{\underline{-\hbar}}\cdot u_2(\theta,\phi).$$

$$\rightarrow |1,-1\rangle$$

Por fim, para $u_3(\theta,\phi) = \cos\theta$

$$\hat{L}^2 u_3 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\text{sen}^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\text{tg}\,\theta}\frac{\partial}{\partial\theta}\right)\cdot\cos\theta$$

$$= -\hbar^2\left(-\cos\theta - \frac{\cos\theta}{\cancel{\text{sen}\,\theta}}\cancel{\text{sen}\,\theta}\right)$$

$$= +2\hbar^2\cdot\cos\theta = \underline{\underline{2\hbar^2}}\,u_3(\theta,\phi)$$

$$\hat{L}_z\cdot u_3(\theta,\phi) = \underline{\underline{0}} \rightarrow |\overset{l}{1},\overset{m}{0}\rangle$$

(e) $\hat{H} = \dfrac{\hat{L}^2}{2I}$, $I > 0$.

$$\psi(t=0,\theta,\phi) = \operatorname{sen}\theta\cos\phi + \cos\theta$$

$$= \frac{u_1 + u_2}{2} + u_3$$

E_ℓ

Então, como $\hat{H}|\varphi_\ell\rangle = \frac{\ell(\ell+1)\hbar^2}{2I}|\varphi_\ell\rangle$. Assim, fica claro que u_1, u_2 e u_3 são auto-estados de $\hat{H}$ com auto-val E_1. Assim,

$$\psi(t,\theta,\phi) = \left[\operatorname{sen}\theta.\cos\phi + \cos\theta\right].e^{-i\hbar^2 t/I\hbar}$$

(b) $N_\psi^2 = \langle\psi|\psi\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \operatorname{sen}\theta \left(\operatorname{sen}\theta\cos\phi + \cos\theta\right)^2$

$$= \cdots = \frac{8\pi}{3}.$$

$$\langle u_1|u_1\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \operatorname{sen}\theta \left|e^{i\phi}\operatorname{sen}\theta\right|^2 = \frac{8\pi}{3}$$

$$\langle u_1|u_2\rangle = \cdots = 0$$

$$\langle u_1|u_3\rangle = \cdots = 0$$

$$\langle u_2|u_2\rangle = \langle u_1|u_1\rangle = \frac{8\pi}{3}$$

$$\langle u_2|u_3\rangle = 0$$

$$\langle u_3|u_3\rangle = \cdots = \frac{4\pi}{3}$$

$$\Rightarrow \langle \psi|\psi\rangle = \left(\frac{\langle u_1| + \langle u_2|}{2} + \langle u_3|\right)\left(\frac{|u_1\rangle + |u_2\rangle}{2} + |u_3\rangle\right)$$

$$= \frac{1}{4}\frac{16\pi}{3} + \frac{4\pi}{3} = \frac{8\pi}{3} = N_\psi^2$$

Assim sendo

$$\langle \hat{L}^2\rangle = \left(\frac{\langle u_1| + \langle u_2|}{2} + \langle u_3|\right) 2\hbar^2 \cdot \left(\frac{|u_1\rangle + |u_2\rangle}{2} + |u_3\rangle\right)\cdot \frac{1}{N_\psi^2}$$

$$= 2\hbar^2$$

$$\langle \hat{L}_z\rangle = \left(\frac{\langle u_1| + \langle u_2|}{2} + \langle u_3|\right) \hbar \cdot \left(\frac{|u_1\rangle - |u_2\rangle}{2} + 0|u_3\rangle\right)\cdot \frac{1}{N_\psi^2}$$

$$= \frac{1}{4N_\psi^2}\left(\frac{8\pi}{3} - \frac{8\pi}{3}\right) = 0 .$$

Isto é válido para qualquer t.

(b) $|k\,l\,m\rangle$ tal que

$$\hat{L}^2|k\,l\,m\rangle = l(l+1)\hbar^2|k\,l\,m\rangle ,$$
$$\hat{L}_z|k\,l\,m\rangle = m\hbar|k\,l\,m\rangle .$$

$$\langle klm|\hat{L}_{\underset{Y}{x}}|klm\rangle = \langle klm|\frac{\hat{L}_{+} \pm \hat{L}_{-}}{2}|klm\rangle = 0$$

Assim, ΔL_x^2 e ΔL_y^2

$$\Delta L_x = \sqrt{\langle \hat{L}_x^2\rangle - \langle \underbrace{\hat{L}_x}_{0}\rangle^2} \iff \Delta \hat{L}_x^2 = \langle \hat{L}_x^2\rangle$$

e o mesmo para $\Delta L_y^2 = \langle \hat{L}_y^2\rangle$.

$$\langle \hat{L}_{\underset{Y}{x}}^2\rangle = \langle klm|\frac{\hat{L}_+^2 \pm \hat{L}_+\hat{L}_- \pm \overbrace{\hat{L}_-\hat{L}_+}^{=\hat{L}_+\hat{L}_- - 2\hbar\hat{L}_z} + \hat{L}_-^2}{4\cdot(i)^2}|klm\rangle$$

$$= \underset{+}{+}\frac{1}{4}\left[\langle klm|2\hat{L}_+\hat{L}_- - 2\hbar\hat{L}_z|klm\rangle\right]$$

$$\hat{L}_{\pm}|klm\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\,|k\,l\,m\pm 1\rangle$$

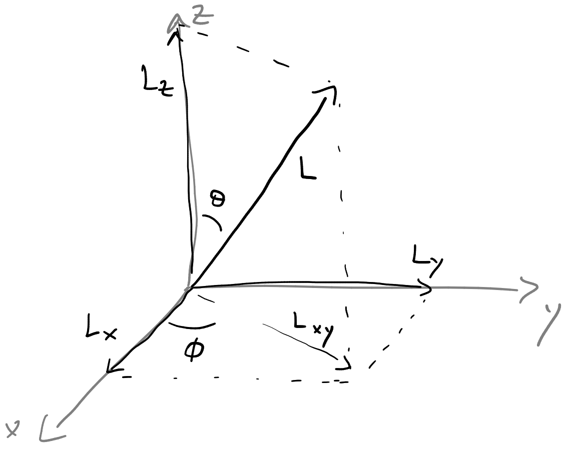
$$= \underset{+}{+}\frac{1}{2}\left[\langle klm|\hbar^2\sqrt{l(l+1) - m(m-1)}\sqrt{l(l+1) - (m-1)m}\,|klm\rangle - \hbar^2 . m\,\langle klm|klm\rangle\right]$$

$$= \underset{+}{+}\frac{\hbar^2}{2}\left[l(l+1) - m^2 + \cancel{m} - \cancel{m}\right]$$

$$\iff \langle \hat{L}_{\underset{Y}{x}}^2\rangle = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

□

Podemos interpretar este resultado pensar do em momentos angulares clássicos



$$L_{xy} = \sqrt{\underbrace{L^2}_{\hbar^2 l(l+1)} - \underbrace{L_z^2}_{= m^2\hbar^2}} = \hbar\sqrt{l(l+1) - m^2}$$

$$\begin{cases} L_x = L_{xy} \cdot \cos\phi \\ L_y = L_{xy} \, \text{sen}\, \phi \end{cases}$$

Se considerarmos $\phi \in [0, 2\pi]$ como variável aleatória, então podemos calcular os valores médios de L_x, L_y, L_x^2 e L_y^2.

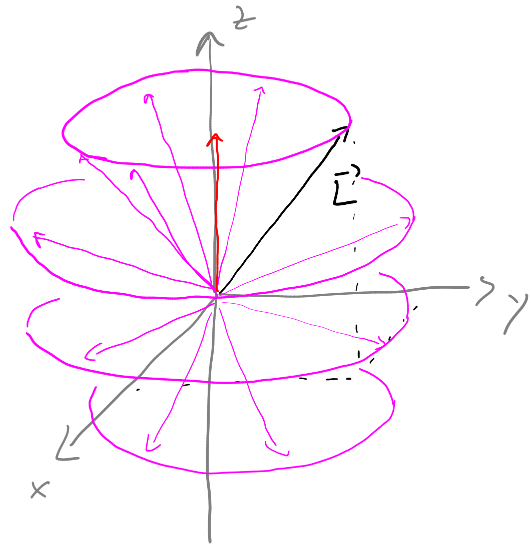
$$\overline{L_x} = \int_0^{2\pi} \overbrace{\frac{d\phi}{2\pi}}^{= dP_\phi} \cdot \cos\phi \cdot L_{xy} = 0$$

$$\overline{L_y} = \dots = 0$$

$$\overline{L_x^2} = \underbrace{\int_0^{2\pi} \frac{d\phi}{2\pi} \cdot \cos^2\phi}_{= \frac{1}{2}} \cdot L_{xy}^2 = \frac{\hbar^2}{2}\left[l(l+1) - m^2\right]$$

que é o mesmo resultado que obtivemos para ΔL_x^2 no caso quântico. O mesmo acontece para $\overline{L_y^2} = \Delta L_y^2$.

Podemos então imaginar a seguinte representação clássica do momento angular

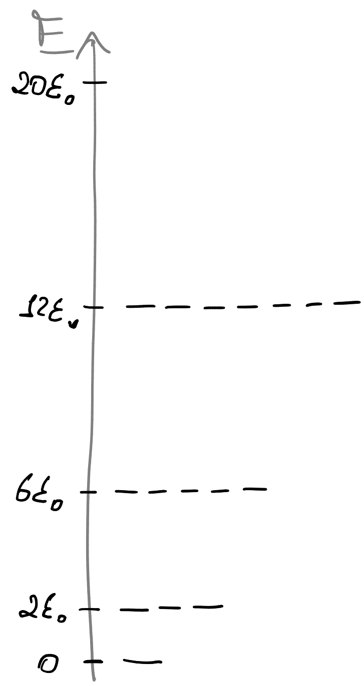


(6) Rotação de molécula diatômica

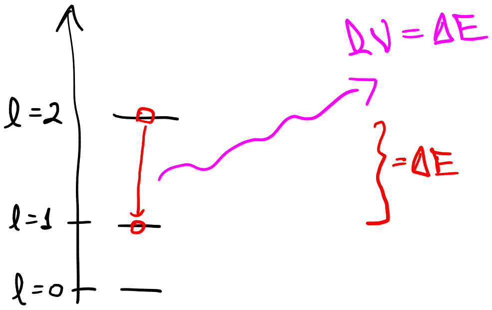
$$\hat{H} = \frac{L^2}{2I}$$

(a) $$E_\ell = \frac{\ell(\ell+1)\hbar^2}{2I} \quad , \quad \ell = 0, 1, 2, \ldots$$

$$= \underbrace{\frac{\hbar^2}{2I}}_{\equiv \varepsilon_0} \underbrace{(\ell^2+\ell)}_{} \rightarrow 0, 2, 6, 12, 20$$



(b)



$$\Delta E = E_2 - E_1 = h\nu$$

$$\Leftrightarrow \frac{\hbar^2}{2I}\cdot(6-2) = h\cdot\nu \Leftrightarrow \nu = \frac{2h^{\cancel{2}}}{4\pi^2\cdot I\cancel{h}} = \frac{h}{2\pi^2\cdot I}$$

(c) Como $m = -l, -l+1, \ldots, -1, 0, 1, \ldots, l$, então degenerescência é $2l+1$, logo degen. do nível $l=2$ será $\underline{\underline{5}}$.

(d)

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{26}}\left[3\,Y_1^1 + 4\,Y_7^3 + Y_7^1\right]$$

(1) $\hat{L}_z \longrightarrow \hbar, 3\hbar.$

$\hat{L}^2 \longrightarrow 2\hbar^2, 56\hbar^2.$

(2) $\langle \psi | \psi \rangle = \frac{1}{26}(9+16+1) = 1 = N_\psi^2$

$\underline{\hat{L}_z}$: $P_\hbar = \frac{9+1}{26} = \frac{10}{26}$

$P_{3\hbar} = \frac{16}{26}$

$\underline{\hat{L}^2}$: $P_{2\hbar^2} = \frac{9}{26}$

$P_{56\hbar^2} = \frac{17}{26}$.

(3) $\psi(t,\theta,\phi) = \frac{1}{\sqrt{26}}\left[3 . Y_1^1 . e^{-iE_1 t/\hbar} + \left(4Y_7^3 + Y_7^1\right) e^{-iE_7 t/\hbar}\right]$

(4) $\langle \hat{H} \rangle = \frac{1}{26}\left[9.E_1 + 16.E_7 + E_7\right]$

$= \frac{\hbar^2}{52I}\left[9.2 + 17.56\right]$

$= \frac{\hbar^2}{I} \frac{970}{52}$

$$I = \frac{\hbar}{4\pi . c} \frac{1}{0{,}304\, m} = 9{,}08 \times 10^{-43}\ J.s^2$$

$$\Rightarrow \langle \hat{H} \rangle = \frac{\hbar^2}{I} \frac{970}{52} \frac{1}{e} = 0{,}000139\ eV.$$

Folha 9 (2021.1)

Partícula num potencial central

② Átomo de hidrogénio

$$|\psi(t=0)\rangle = \frac{1}{6}\left(4|\psi_{200}\rangle + 3|\psi_{211}\rangle + \sqrt{11}\,|\psi_{21,-1}\rangle\right)$$

$N\ \ell\ m$

$$\hat{H}|\psi_{N\ell m}\rangle = \left(-\frac{\hbar^2}{2m a_0^2} \cdot \frac{1}{N^2}\right)|\psi_{N\ell m}\rangle$$

$= E_N$

(a) $$\langle \psi|\psi\rangle = \frac{1}{36}(16 + 9 + 11) = 1 = N_\psi^2$$

$$\langle \hat{H} \rangle = \frac{1}{36}\left[\langle\psi_{200}|4 + \langle\psi_{211}|3 + \langle\psi_{21-1}|\sqrt{11}\right]\left[4|\psi_{200}\rangle + 3|\psi_{211}\rangle + \sqrt{11}\,|\psi_{21-1}\rangle\right]E_2$$

$$= E_2$$

(b) $$\hat{L}_z \rightarrow 0 \rightarrow P_0 = 16/36$$
$$\rightarrow \hbar \rightarrow P_\hbar = 9/36$$
$$\rightarrow -\hbar \rightarrow P_{-\hbar} = 11/36$$

(c) $$\langle \hat{L}^2 \rangle = \frac{1}{36}\left(16.0 + 9.2\hbar^2 + 11.2\hbar^2\right)$$

$$= \frac{40\hbar^2}{36} = \frac{10\hbar^2}{9}.$$

(d) $P_{10^{-10}cm} = ?$ $\qquad r_0 = 10^{-10}\,cm = 10^{-12}\,m$

$$P_{10^{-10}cm} = \int_0^{r_0} dr\, r^2 \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, sen\theta \; |\psi(r,\theta,\phi)|^2$$

$$= \iiint r^2 sen\theta\, dr\, d\theta\, d\phi \left(\frac{16}{36}|\psi_{200}|^2 + \frac{9}{36}|\psi_{211}|^2 + \frac{11}{36}|\psi_{21-1}|^2\right)$$

Y_l^m são ortonorm.

$$= \int_0^{r_0} dr\, r^2 \left[\frac{16}{36} |R_{20}(r)|^2 + \frac{9}{36} |R_{21}(r)|^2 + \frac{11}{36} |R_{21}|^2 \right]$$

$$\longrightarrow 0$$

(e)

$$|\psi(t)\rangle = \frac{1}{6}\left[4|\psi_{200}\rangle + 3|\psi_{211}\rangle + \sqrt{11}\,|\psi_{21-1}\rangle \right] e^{-iE_2 t/\hbar}.$$

④ <u>OHQ 3D na presença de um campo magnético</u>

$$\hat{H} = \frac{1}{2\mu}\left(\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2\right) + \frac{1}{2}\mu\omega_0^2\left(\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2\right)$$

onde $\omega_0 > 0$.

(a) Como temos 3 OHQ 1D, que sabemos resolver usando operadores de criação/destruição, podemos escrever $\hat{H}$ como

$$\hat{H}_0 = \hbar\omega_0 \left(\hat{N}_x + \hat{N}_y + \hat{N}_z + \frac{3}{2}\hat{\mathbb{1}} \right)$$

onde $\hat{N}_i = \hat{a}_i^{+}\hat{a}_i$, $i = x, y, z$. Assim, as energias serão (na base $\{|n_x, n_y, n_z\rangle\}$)

$$E^0_{(n_x, n_y, n_z)} = \hbar\omega_0 \left(n_x + n_y + n_z + 3/2 \right),$$

onde $n_i = 0, 1, 2, \ldots$ com $i = x, y, z$.

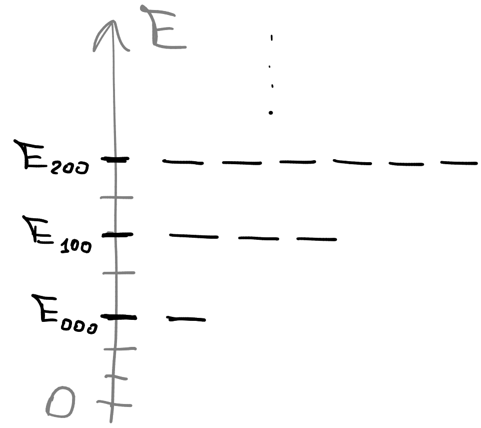
$$E^0_{(0,0,0)} = \frac{3\hbar\omega_0}{2} \longrightarrow \text{deg. } 1$$

$$E^0_{(1,0,0)} = E^0_{(0,1,0)} = E^0_{001} = \frac{5\hbar\omega_0}{2} \longrightarrow \text{deg. } 3$$

$$E_{(2,0,0)} = E_{(020)} = E_{(002)} = E_{(110)}$$

$$= E_{101} = E_{011} = \frac{7\hbar\omega_0}{2}$$

$\hookrightarrow$ deg. 6.



(b) $\vec{B} = \vec{\nabla} \times \vec{A}$ com $\vec{A} = \frac{B}{2}(-y, x, 0)$.

Sabemos que na presença de $\vec{A}$

o momento é transformado

$$\hat{\vec{P}} \longrightarrow \hat{\vec{P}} - q\cdot\hat{\vec{A}}$$

sendo que $\hat{H}$ fica na forma

$$\hat{H} = \frac{\left(\hat{\vec{P}} - q\hat{\vec{A}}\right)^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\cdot\hat{\vec{R}}^2$$

$$= \frac{1}{2\mu}\left[\left(\hat{P}_x + \overset{= -\mu\omega_L}{\frac{qB}{2}}\hat{Y}\right)^2 + \left(\hat{P}_y - \overset{= -\mu\omega_L}{\frac{qB}{2}}\hat{X}\right)^2 + \hat{P}_z^2\right]$$

$$+ \frac{1}{2}\mu\omega_0^2\left(\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2\right)$$

$$= \hat{P}_x^2 - 2\mu\omega_L\hat{P}_x\hat{Y} + \mu^2\omega_L^2\hat{Y}^2$$

$$= \hat{P}_y^2 + 2\mu\omega_L\hat{P}_y\hat{X} + \mu^2\omega_L^2\hat{X}^2$$

Assim,

$$\hat{H} = \frac{1}{2\mu}\Big[\hat{P}_x^2 - 2\mu\omega_L\hat{P}_x\hat{Y} + \mu^2\omega_L^2\hat{Y}^2 +$$

$$+ \hat{P}_y^2 + 2\mu\omega_L \hat{P}_y \hat{X} + \mu^2\omega_L^2 \hat{X}^2 + \hat{P}_z^2 \Big]$$

$$+ \frac{1}{2}\mu\omega_0^2\left(\hat{X}^2 + \hat{Y}^2 + \hat{Z}^2\right)$$

$$= \hat{H}_0 + \underbrace{\omega_L \cdot \overbrace{\left(\hat{X}\hat{P}_y - \hat{Y}\hat{P}_x\right)}^{=\hat{L}_z} + \frac{\mu\omega_L^2}{2}\left(\hat{X}^2 + \hat{Y}^2\right)}_{\equiv \hat{H}_1}$$

e assim $\hat{H}_1$ será

$$\hat{H}_1 = \omega_L \hat{L}_z + \frac{\mu\omega_L^2}{2}\left(\hat{X}^2 + \hat{Y}^2\right)$$

II

(c) Podemos escrever $\hat{H}$ como

$$\hat{H} = \underbrace{\Bigg[\frac{\hat{P}_x^2 + \hat{P}_y^2}{2\mu} + \frac{1}{2}\mu\overbrace{\left(\omega_0^2 + \omega_L^2\right)}^{\equiv\tilde{\omega}^2}\cdot\left(\hat{X}^2 + \hat{Y}^2\right) +}_{= H_{xy}}$$

$$+ \omega_L \cdot \hat{L}_z\Bigg] + \underbrace{\left[\frac{\hat{P}_z^2}{2\mu} + \frac{1}{2}\mu\omega_0^2\hat{Z}^2\right]}_{\equiv \hat{H}_z}$$

Nas aulas vimos que introduzindo quantões circulares

$$\hat{a}_j \longrightarrow \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\mu\tilde{\omega}}{\hbar}}\,\hat{X}_j + \frac{i}{\sqrt{\mu\tilde{\omega}\hbar}}\hat{P}_j\right]$$

onde $j = x, y$. Definindo quantões circulares como $\hat{a}_e \equiv \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y)$ e $\hat{a}_d \equiv \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y)$, sendo $\hat{N}_e \equiv \hat{a}_e^{+}\hat{a}_e$, $\hat{N}_d \equiv \hat{a}_d^{+}\hat{a}_d$, podemos escrever $\hat{H}_{xy}$ e $\hat{L}_z$ como

$$\hat{H}_{xy} = \hbar\tilde{\omega}\left(\hat{N}_d + \hat{N}_e + \hat{\mathbb{1}}\right),$$

$$\hat{L}_z = \hbar\left(\hat{N}_d - \hat{N}_e\right).$$

Assim podemos reescrever $\hat{H}$ em termos de $\{\hat{N}_d, \hat{N}_e, \hat{N}_z\}$ será

$$\hat{H} = \hbar\tilde{\omega}\left(\hat{N}_d + \hat{N}_e + \hat{\mathbb{1}}\right) + \hbar\omega_L\left(\hat{N}_d - \hat{N}_e\right) + \hbar\omega_0\cdot\left(\hat{N}_z + \frac{\hat{\mathbb{1}}}{2}\right)$$

□

(d) Na base de $\{|n_d, n_e, n_z\rangle\}$ de auto-estados comuns a $\hat{N}_d$, $\hat{N}_e$ e $\hat{N}_z$ é fácil ver que as energias são

$$E_{n_d n_e n_z} = \hbar\sqrt{\omega_0^2+\omega_L^2}\,(n_d+n_e+1) + $$
$$+\hbar\omega_L\,(n_d-n_e) + \hbar\omega_0\,(n_z+1/2)$$

Para $\omega_L/\omega_0 \equiv \delta \ll 1$, então

$$\sqrt{\omega_0^2+\omega_L^2} = \omega_0\sqrt{1+\delta^2} \simeq \omega_0\left(1+\frac{\delta^2}{2}+\cdots\right)$$
$$\simeq \omega_0 + \frac{\omega_L^2}{2\omega_0} + \cdots$$

Podemos então escrever $E_{(n_d n_e n_z)}$ como

$$E_{n_d n_e n_z} \simeq \hbar\omega_0\,(n_d+n_e+n_z+3/2) +$$
$$+\hbar\omega_L\left[n_d\cdot\left(1+\frac{\omega_L}{2\omega_0}\right) - n_e\left(1-\frac{\omega_L}{2\omega_0}\right)\right] +$$
$$+\frac{\hbar\omega_L^2}{2\omega_0}$$

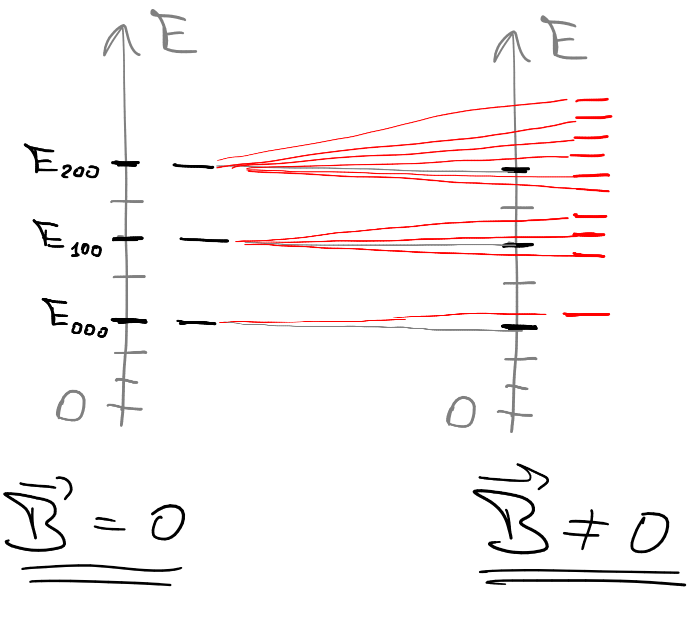
$$E_{000} \simeq \frac{3\hbar\omega_0}{2} + \frac{\hbar\omega_L^2}{2\omega_0}$$

$$E_{100} \simeq \frac{5\hbar\omega_0}{2} + \hbar\omega_L\left(1 + \frac{\omega_L}{2\omega_0}\right) + \frac{\hbar\omega_L^2}{2\omega_0}$$

$$E_{010} \simeq \frac{5\hbar\omega_0}{2} - \hbar\omega_L\left(1 - \frac{\omega_L}{2\omega_0}\right) + \frac{\hbar\omega_L^2}{2\omega_0}$$

$$E_{001} \simeq \frac{5\hbar\omega_0}{2} + \frac{\hbar\omega_L^2}{2\omega_0}$$

$$\vdots$$



É efeito de Zeeman.