Spherical surfaces with conical singularities

October 20, 2023

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I Constant curvature surfaces with conical singularities

Definition

A *spherical/Euclidean/hyperbolic surface* S with conical points $(x_1, \ldots, x_n)$ is a surface with a complete metric of curvature 1, 0, -1 on $S \setminus (x_1, \ldots, x_n)$.

Here x_i's are *conical points*. For any x_i the length of the radius ε circle centred at x_i is $2\pi\theta_i \cdot \sin(\varepsilon)$, $2\pi\theta_i \cdot \varepsilon$, $2\pi\theta_i \cdot \sinh(\varepsilon)$. We call $2\pi\theta_i$ the *conical angle*.

Example

1) $\mathbb{S}^2/\mathbb{Z}_n$ is a sphere with two conical points of angle $2\pi/n$.

2) Degree n cover of $\mathbb{S}^2$ with two ramification is a sphere with two conical points of angle $2\pi n$.

Remark

1) Any spherical/Euclidean/hyperbolic surface with conical singularities can be glued from a collection of geodesic triangles.

2) Any such surface is a compact Riemann surface.

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Gauss-Bonnet. $MSPH(\theta_1, \ldots, \theta_n)$ and main questions.

Gauss-Bonnet. Let S be a surface of constant curvature $K \in \{-1, 0, 1\}$. Then we have the following formula.

$$\frac{1}{2\pi}\mathrm{area}(S) \cdot K = \chi(S) + \sum_i (\theta_i - 1).$$

Moral. The sign of $\chi(S) + \sum_i (\theta_i - 1)$ is the sign of the curvature.

Definition

$M_{g,n}$ - moduli space of Riemann surfaces of genus g with n marked points.

$MSPH_{g,n}(\theta_1, \ldots, \theta_n)$ - moduli space of spherical surfaces of genus g with n conical points of angles $2\pi\theta_i$.

Question (Main questions)

1. *How does $MSPH_{g,n}(\theta_1, \ldots, \theta_n)$ look like?*
2. *How does the forgetful map, $MSPH_{g,n}(\theta_1, \ldots, \theta_n) \to M_{g,n}$ look like?*

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Theorem (MacOwen-Troyanov)

Let $(S, x_1, \ldots, x_n)$ be a Riemann surface. Suppose $\chi(S) + \sum_i \theta_i - 1 < 0$.

Then $\exists!$ conformal hyperbolic metric with conical angles $2\pi\theta_i$ at x_i on S.

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Hyperbolic and Euclidean cases are easy.

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Developing map. Let $(S, x_1, \ldots, x_n)$ be a spherical surface with conical points $x = (x_1, \ldots, x_n)$. The *developing map* $\iota : \dot{S} \to \mathbb{S}^2$ is a multi-valued locally isometric map.

Mondoromy. Take a loop in $\dot{S}$ and extend the developing map along the loop. This defines to us the *monodromy*: $\pi_1(\dot{S}) \to SO(3)$.

Lemma (Troyanov). There is no spherical metric with a unique conical point x on S^2.

Proof. The monodomy is trivial, since $\pi(S^2 \setminus x) = 0$. So the conical angle has to be integer. So the developping map is univalent. But there is no ramified cover of S^2 with one branching point.

Lemma (Troyanov). Let S be a sphere with two conical points of angles $2\pi\theta_1, 2\pi\theta_2$. Then $\theta_1 = \theta_2 = \theta$, moreover,

- If θ is not integer, the metric is S^1-symmetric.
- If θ is integer, the metric is obtained by a ramified cover of unit S^2 with two branching points.

Proof. Cut the sphere by a shortest geodesic joining two conical points to get a spherical digon. Then take the developing map.

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Developing map, monodromy, spheres with ≤ 2 conical points

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Lemma (Felix Klein)

For a spherical triangle with angles $\pi(\theta_1, \theta_2, \theta_3)$, $|\mathbb{Z}^3_{ev}, (\theta_1, \theta_2, \theta_3)|_1 \geq 1$.
$\mathbb{Z}^3_{ev} \subset \mathbb{Z}^3$ *are points* (n_1, n_2, n_3) *with* $n_1 + n_2 + n_3 \in 2\mathbb{Z}$. $|(x, y, z)|_1 = x + y + z$.

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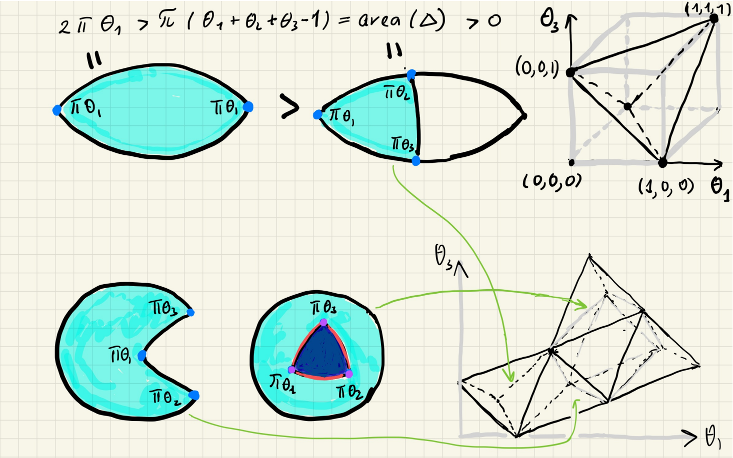

When is $MSPH_{g,n}(\bar{\theta})$ non-empty? $g > 0$

Question

Which connected compact surfaces without boundary can be obtained from a topological polygon by identifying its edges pairwise, so that all corners are identified to one point?

Theorem

(Mondello-P. GAFA 2019). Suppose $g > 0$, and $2 - 2g + \sum_i(\theta_i - 1) > 0$. Then $MSPH_{g,n}(\theta_1, \ldots, \theta_n)$ is non-empty.

Proof. Take a digon with angles $\pi(2 - 2g + \sum_i(\theta_i - 1))$. Make an equilateral $4g$-gon from it and glue into a genus g surface in the standard way. This produces a surface with one conical point of angle $2\pi(\sum_i(\theta_i - 1) + 1)$.

Split the unique conical point into n points with angles $2\pi\theta_i$ (need to work - do this for Euclidean polygons, then perturb to spherical ones).

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When is $MSPH_{0,n}(\bar{\theta})$ non-empty? $g = 0$.

Denote by $||.||_1$ the l^1-norm on $\mathbb{R}^n$ and d_1 the associated l^1 distance.

Let $\mathbb{Z}_o^n$ be the subset of points $(m_1, \ldots, m_n)$ in $\mathbb{Z}^n$, with $||m||_1 = \sum_i m_i$ odd.

Theorem

(Mondello-P. IMRN 2016) Suppose that there is a spherical metric on S^2 with conical angles $2\pi(\theta_1, \ldots, \theta_n)$. Then the following inequalities hold.

$$\sum(\theta_i - 1) > -2. \tag{1}$$

$$d_1(\mathbb{Z}_o, (\theta_1 - 1, \ldots, \theta_n - 1)) \geq 1 \tag{2}$$

If (2) is satisfied strictly, *then on S^2 there is a metric with angles $2\pi\theta_i$.*

Inequality (1) is Gauss-Bonnet. Inequality (2) is deduced from monodromy.

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Denote by $||.||_1$ the l^1-norm on $\mathbb{R}^n$ and d_1 the associated l^1 distance.

Let $\mathbb{Z}^n_o$ be the subset of points $(m_1, \ldots, m_n)$ in $\mathbb{Z}^n$, with $||m||_1 = \sum_i m_i$ odd.

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(Mondello-P. IMRN 2016) Suppose that there is a spherical metric on S^2 with conical angles $2\pi(\theta_1, \ldots, \theta_n)$. Then the following inequalities hold.

$$\sum (\theta_i - 1) > -2. \tag{1}$$

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When is $MSPH_{0,n}(\overline{\theta})$ non-empty? $g = 0$.

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Products of matrices $U_1 \cdot \ldots \cdot U_n = 1$ and spherical polygons.

Definition

A *standard set of matrices for* $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n_+$ is a n-uple $(U_1, \ldots, U_n)$ of elements of $SU(2)$ such that $U_1 \cdot U_2 \cdot \ldots \cdot U_n = I$ and the eigenvalues of U_j are $e^{\pm i\pi(\theta_j - 1)}$ for $j = 1, \ldots, n$.

Theorem

For any spherical metric on S^2 with conical angles $2\pi(\theta_1, \ldots, \theta_n)$ there exists a standard set of matrices.

Sketch proof. Take the monodromy representation $\pi_1(\dot{S}^2) \to SO(3)$. This representation can be canonically lifted to $SU(2)$.

Question

For which $(\theta_1, \ldots, \theta_n)$ a standard set of matrices exists?

Answer. It exists when a spherical polygon with lengths $\pi \cdot (\theta_i - 1)$ exists.

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Spheres with integral angles and Schubert calculus I

Suppose $S = \mathbb{C}P^1$ and θ_i are integers. Then there is a complete answer to main questions. It is given by algebraic geometry.

Lemma

Every spherical metric with cone angles divisible by 2π is a pull-back of a Fubini-Study metric under a ramified cover $\mathbb{C}P^1 \to \mathbb{C}P^1$.

Proof. Take a local isometry from S to unit $\mathbb{S}^2$ and extend it to S.

Question

Let $(x_1, \ldots, x_n)$ be points on $\mathbb{C}P^1$ and $m_1, \ldots, m_n$ be positive integers. What is the number of ramified covers $\mathbb{C}P^1 \to \mathbb{C}P^1$ with ramifications of multiplicities m_i at x_i?

Answer: Consider the rational normal curve $C_d \subset \mathbb{C}P^d$, given by $(y_0 : y_1) \to (y_0^d, y_0^{d-1}y_1, \ldots)$. It is a conic in $\mathbb{C}P^2$, twisted cubic in $\mathbb{C}P^3$. Now, for any point of C_d there is an osculating hyperplane $\mathbb{C}P^{d-1}$. Hence for any line $\mathbb{C}P^1 \subset \mathbb{C}P^d$ we have a degree d map $C_d \to \mathbb{C}P^1$.

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Spheres with integral angles and Schubert calculus II

The number of lines in $\mathbb{C}P^3$, intersecting four given ones is the number of spherical metrics with angles 4π on a given $\mathbb{C}P^1$ with four marked points!

Catalan numbers.
Recall $C_n = \frac{1}{n+1}\binom{2n}{n}$, $C_0 = 1$, $C_2 = 1$, $C_2 = 2$, $C_3 = 5$, 14, 42...

Theorem

(**Goldberg 1991.**) *Fix $2n-2$ generic points $x_1, \dots, x_{2n-2}$ on $\mathbb{C}P^1$ and consider rational degree n maps from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ that have a simple ramification at each x_i. The number of such maps up to projective equivalence is C_{n-1}.*
For example, the number of lines in $\mathbb{C}P^4$ intersecting six given 2-planes is 5.

Remark

This number is the degree of the Plücker embedding of the Grassmanian of lines in $\mathbb{P}^n$.

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II. Systole inequality. Image $F_{g,n,\theta} : MSPH_{g,n}(\theta) \to M_{g,n}$ can be tiny

Recall that $MSPH_{g,n}(\theta_1, \ldots, \theta_n)$ is the moduli space of spherical surfaces of genus g with n conical points of angels $2\pi\theta_i$.

Theorem (Mondello-P)

Suppose $\chi(\dot{S}) \leq -1$ and $(g,n) \neq (0,3)$. Suppose $\theta_i < \varepsilon$. Then, the image $F_{g,n,\theta} : MSPH_{g,n}(\theta) \to M_{g,n}$ is tiny.

Remark. This is in contrast with the theorem of Daniele Bartolucci, Francesca De Marchis, and Andrea Malchiodi:

If θ_i's are not on a bubbling wall, $\theta_i > 1$, $g > 0$, $\frac{1}{2\pi} Area(S) > 2$, then the forgetful map is surjective.

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Aside: $MSPH_{g,n}$ is a topological space

Definition (Lipschitz distance)

Let $(X, x_1, \ldots, x_n; d_X)$ and $(Y, y_1, \ldots, y_n; d_Y)$ be two metric spaces with marked points x_i, y_i. The *Lipschitz distance* between them is defined by

$$d_{\mathcal{L}}((X,x),(Y,y)) = \inf_f \log(\max\{\mathrm{dil}(f), \mathrm{dil}(f^{-1})\}),$$

where

$$\mathrm{dil}(f) = \sup_{p_1 \neq p_2 \in X} \frac{d_Y(f(p_1), f(p_2))}{d_X(p_1, p_2)}$$

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Spherical systole and its properties

Definition (Spherical systole)

Let $(S, x_1, \ldots, x_n)$ be a spherical surface with conical singularities. The spherical systole of S, $Sys(S, x)$ is $\frac{1}{2}*$(the length of shortest geodesic segment of loop based at x).

Lemma (Properness of $Sys(S)^{-1}$)

The function $Sys(S)^{-1}$ is proper on $MSPH_{g,n}(\theta)$.

Lemma (Small angle $\Rightarrow$ small systole)

For any spherical surface S with $\chi(\dot{S}) < 0$ we have $Sys(S, x) \leq \pi \min(\theta_i)$.

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Extremal systole vs spherical systole

Definition (Cylinders, Extremal systole)

The *modulus* of a complex cylinder C of hight M and waist 1 is M.

Let $(S, x_1, \ldots, x_n)$ be a punctured Riemann surface. The extremal systole of $(S, x_1, \ldots, x_n)$ is the infinum of $\frac{1}{M_C}$ over all essential cylinders in $\dot{S}$.

It is denoted $Extsys(\dot{S})$.

Lemma (Small $Extsys(S, x)$ imply small $Sys(S, x)$)

$$Sys(S, x) \leq \sqrt{(\pi/2)Extsys(S, x)|\theta|_1}$$

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The systole inequality

Question. Does small $Sys(S,x)$ imply small $Extsys(S,x)$?

Definition (Non-bubbling parameter)

$$NB_\theta(S,x) = dist(\chi(\dot{S}) + \sum_i \pm(\theta_i), 2\mathbb{Z}_{\geq 0}).$$

Theorem (Main theorem)

Let S be a spherical surface with conical singularities such that $\chi(\dot{S}) < 0$ and S is not a sphere with three conical points. Suppose

$$NB_\theta(S,x) > \varepsilon > 0.$$

Then $Extsys(S,x) \geq \frac{2\pi|\theta|_1}{\log(1/\varepsilon)}$ implies $Sys(S,x) \geq \left(\frac{\varepsilon}{4\pi|\theta|_1}\right)^{-3\chi(\dot{S})+1}$.

Moral. Small $\theta_i \Rightarrow$ small systole spherical systole $\Rightarrow$ small extremal systole $\Rightarrow$ closeness to the boundary of $M_{g,n}$.

The systole inequality

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$$NB_\theta(S,x) = dist(\chi(\dot{S}) + \sum_i \pm(\theta_i), 2\mathbb{Z}_{\geq 0}).$$

Theorem (Main theorem)

Let S be a spherical surface with conical singularities such that $\chi(\dot{S}) < 0$ and S is not a sphere with three conical points. Suppose

$$NB_\theta(S,x) > \varepsilon > 0.$$

Then $Extsys(S,x) \geq \frac{2\pi|\theta|_1}{\log(1/\varepsilon)}$ implies $Sys(S,x) \geq \left(\frac{\varepsilon}{4\pi|\theta|_1}\right)^{-3\chi(\dot{S})+1}$.

Moral. Small $\theta_i \Rightarrow$ small systole spherical systole $\Rightarrow$ small extremal systole $\Rightarrow$ closeness to the boundary of $M_{g,n}$.

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Systole inequality has the following corollary:

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Outside of bubbling walls

$$\chi(\dot{S}) + \sum_i \pm\theta_i = 2b, \quad |b \in \mathbb{Z}_{\geq 0}$$

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- *Voronoi graph* $\Gamma(S)$ *is a graph with at most* $-3\chi(\dot{S})$ *geodesic edges.*
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The space $MSPH_{0,n+3}(\theta_1, \ldots, \theta_n)$ can have $\geq 3^n$ connected components. Moreover, for an appropriate choice of θ_i the image of this moduli space in $M_{0,n+3}$ has at least 3^n components.

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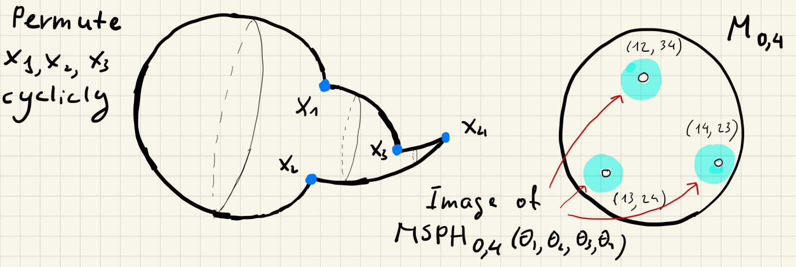

III Moduli of spherical tori with one conical point. Harer-Zagier, Euler characteristics of $M_{g,1}$

$M_{g,1}$ is the moduli space of complex curves of genus g with one marked point.

It is a complex orbifold of dimension $3g - 3 + 1$.

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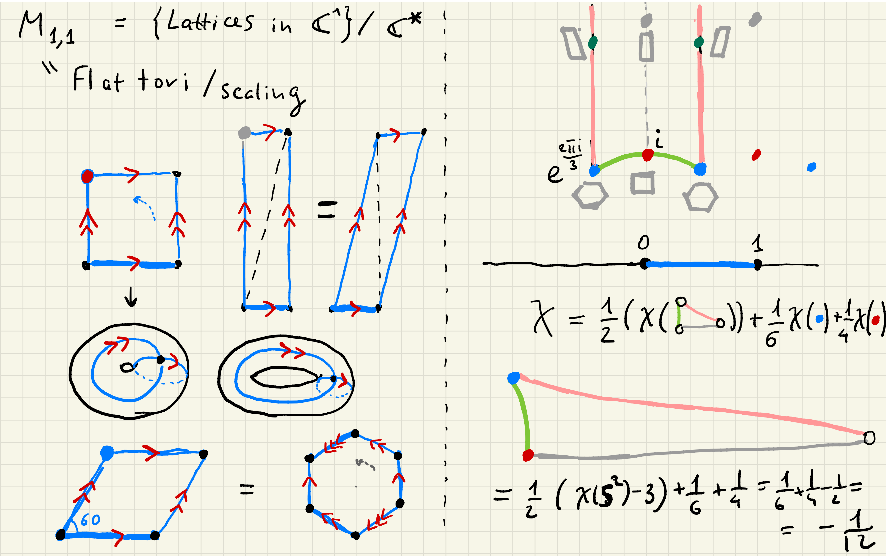
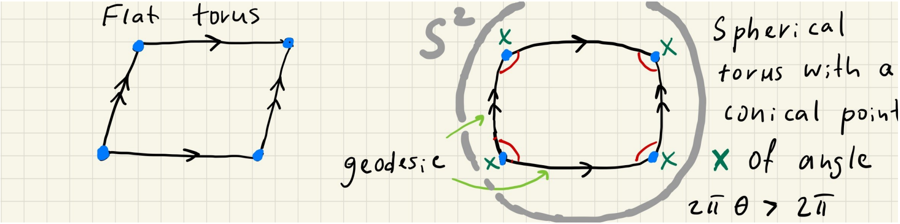


Replace *flat* tori with 1 marked point by *spherical* tori with 1 conical point.



Theorem (Eremenko, Mondello, P, 2020)

Let $\theta \in (2m-1, 2m+1)$. The moduli space $MSPH_{1,1}(\theta)$ of spherical tori with one conical point of angle $2\pi\theta$ is a connected orientable two-dimensional orbifold of Euler characteristic $\frac{-m^2}{12}$.

Note that for $m = 1$ we get $\chi(MSPH_{1,1}(\theta)) = -\frac{1}{12}$, which agrees with $\chi(M_{1,1}) = -\frac{1}{12}$.

Spherical tori - first main result

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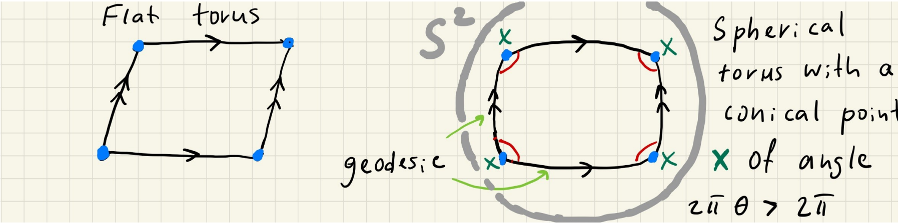

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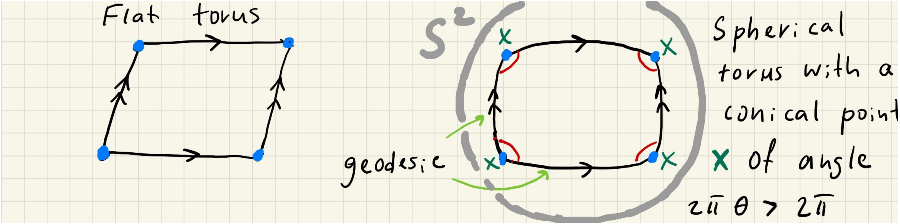

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Angles of spherical triangles

Lemma (Felix Klein)

For a spherical triangle with angles $\pi(\theta_1, \theta_2, \theta_3)$, $|\mathbb{Z}^3_{ev}, (\theta_1, \theta_2, \theta_3)|_1 \geq 1$.

$\mathbb{Z}^3_{ev} \subset \mathbb{Z}^3$ *are points* (n_1, n_2, n_3) *with* $n_1 + n_2 + n_3 \in 2\mathbb{Z}$. $|(x, y, z)|_1 = x + y + z$.

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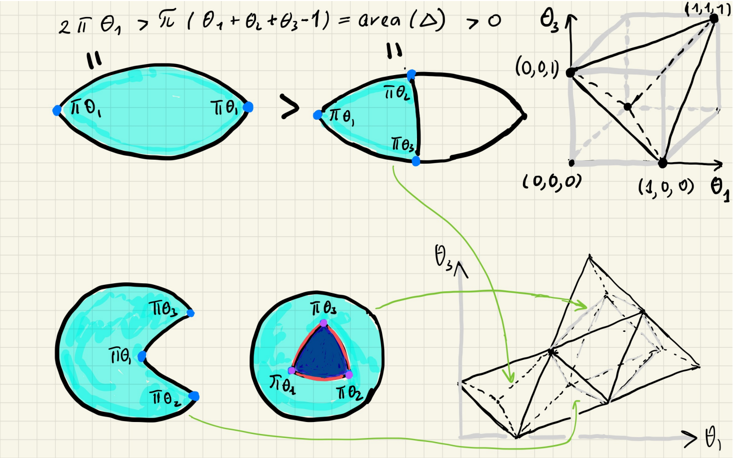

Balanced spherical triangles

Recall the elementary fact. An Euclidean triangle contains the centre of the circumscribed circle if an only if it is not obtuse.

Theorem

A spherical triangle contains a point equidistant from its 3 vertices if and only if it is **balanced***:= its angles* $\pi(\theta_1, \theta_2, \theta_3)$ *satisfy the triangle inequality.*

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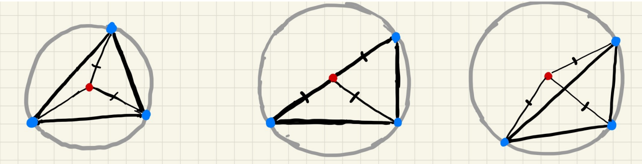

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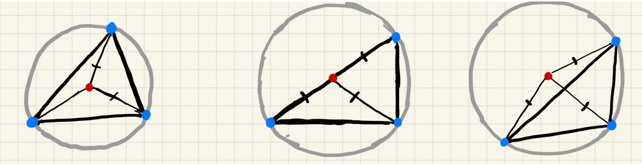

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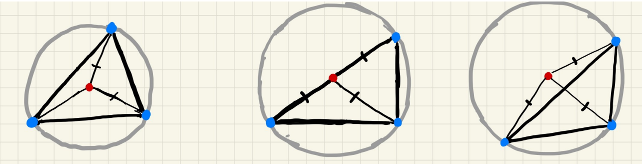

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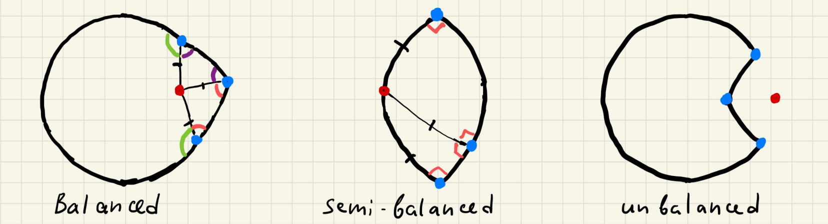
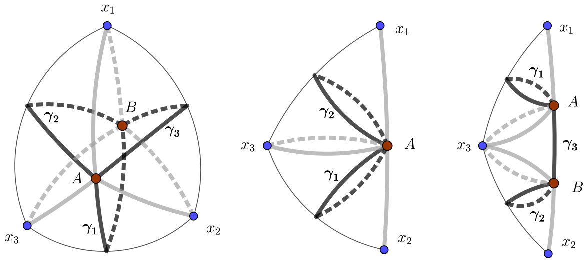


Voronoi graphs and Delaunay decompositions

Voronoi graph $\Gamma(S)$ consists of points $p \in S$ for which there are at least two geodesics of length $d(p, \{x_1, \ldots, x_n\})$ connecting p with conical points x_i.



Delaunay decomposition is dual to the Voronoi graph. 1934, Sur la sphere vide.

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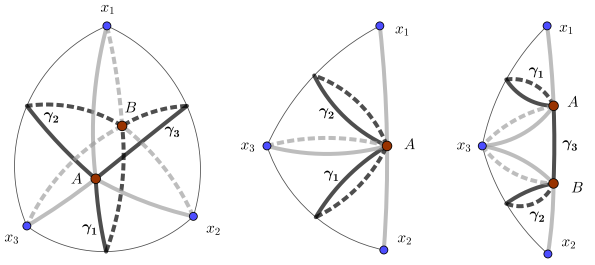

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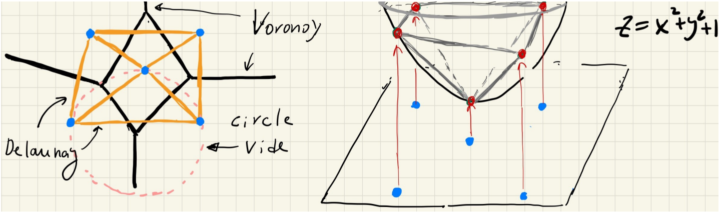

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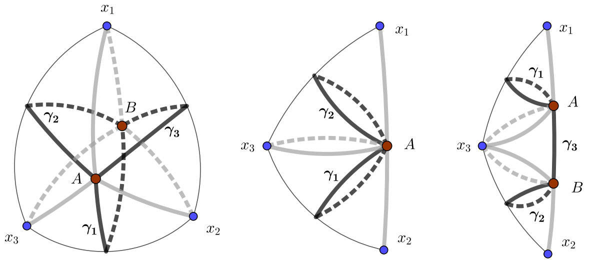

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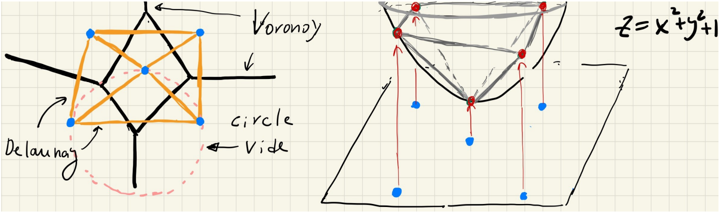

We call a spherical torus T *rectangular*, if it has an isometric orientation reversing involution τ, such that $T^\tau \cong S^1 \cup S^1$.

Theorem

Let (T, x) be a spherical torus with one conical point of angle $2\pi\theta$ such that $\theta \in (1, \infty) \setminus (2\mathbb{Z}+1)$.

- *(T, x) has a unique isometric orientation preserving involution σ.*
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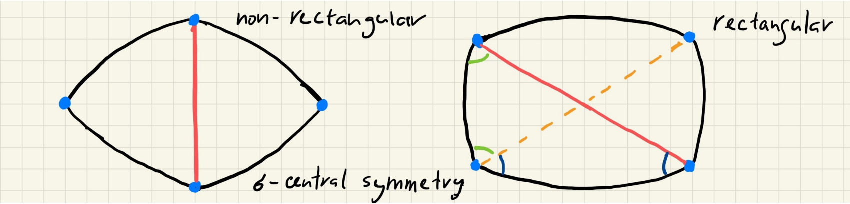

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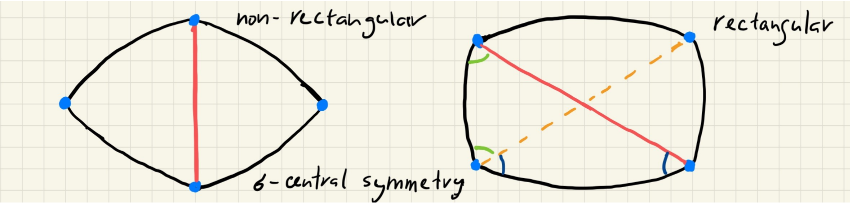

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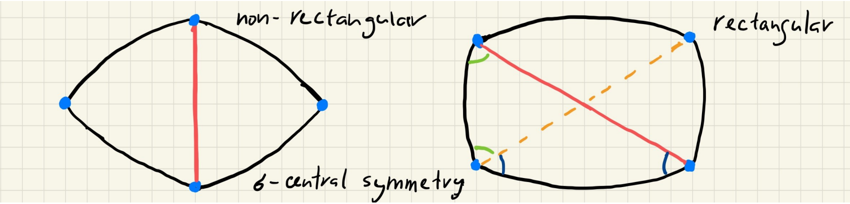

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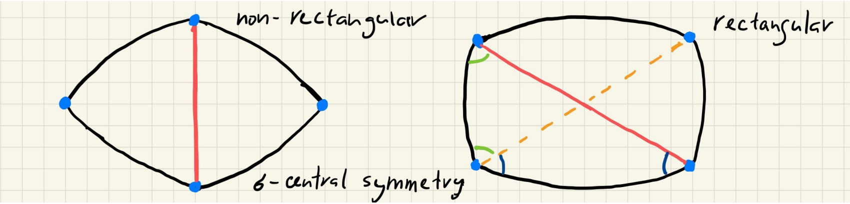

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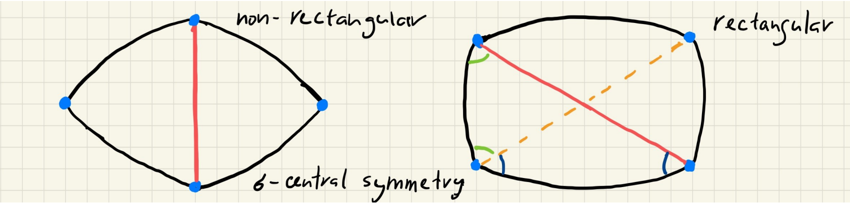
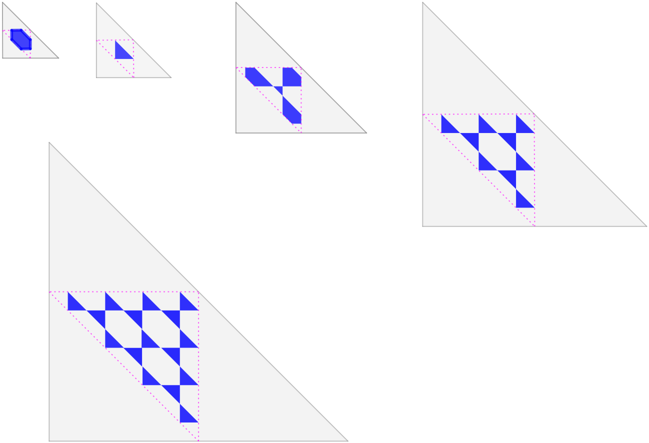


Fix θ and consider the set of all balanced triangles with total angles $\pi \cdot (\theta_1, \theta_2, \theta_3)$, so that $\sum_i \theta_i = \theta$. The projection of this set to the θ_1, θ_2 plane is called the *balanced carpet.*

Below you see balanced carpets for $\theta = 1.5,\ 2,\ 3.5,\ 6,\ 8.$



Proof II. Understanding topology of $MSPH_{1,1}(\theta)$

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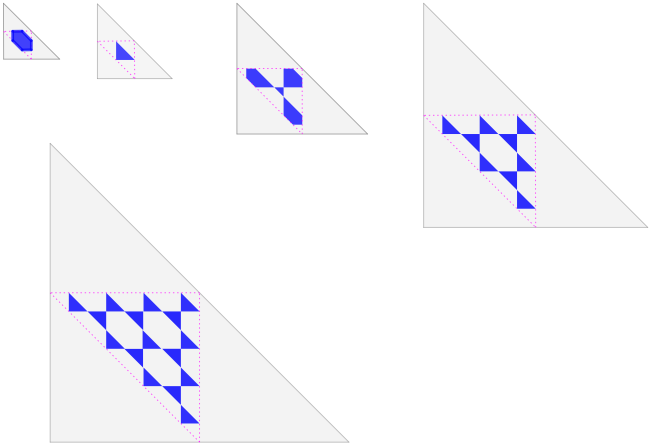

Theorem ($\theta = 2m+1$)

Fix an integer $m \geq 1$ and consider the moduli space $MSPH_{1,1}(2m+1)^\sigma$ of tori with a σ-invariant spherical metric of area $4m\pi$.
As a topological space, $MSPH_{1,1}(2m+1)^\sigma$ is homeomorphic to the disjoint union of $\lceil \frac{m(m+1)}{6} \rceil$ two-dimensional open disks.

Proof. Balanced triangles of integer area a glued from half spheres.

Theorem ($MSPH_{1,1}(2m)$ is a Belyi curve)

For each integer $m > 0$ there exists a subgroup $G_m < \mathrm{SL}(2,\mathbb{Z})$ of index m^2 such that the orbifold $MSPH_{1,1}(2m)$ is biholomorphic to the quotient $\mathbb{H}^2/G_m$. Such G_m is non-normal for $m > 1$. Moreover, the points in $\mathbb{H}^2/G_m$ that project to the geodesic ray $[i,\infty)$ in the modular curve $\mathbb{H}^2/\mathrm{SL}(2,\mathbb{Z})$ correspond to tori T such that the triangle $\Delta(T)$ has one integral angle.

Proof. Quotient by $\mathbb{Z}_2$, and identify with a Hurwitz space.

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