Flow around nematic liquid crystals due to elastic forces

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November 30, 2020

1 The fluid equation of motion

The equation of motion for the fluid is given by

$$\rho \frac{\partial v_i}{\partial t} = -\partial_i p + \partial_j (\sigma^v_{ji} + \sigma^e_{ji}) \tag{1}$$

We make the assumption that $\partial v_i / \partial t \approx 0$. Note that we may get rid of the pressure term as well by taking the curl of both sides:

$$\begin{aligned} 0 &= -\epsilon_{lki} \partial_k \partial_i p + \epsilon_{lki} \partial_l \partial_j \sigma^v_{ji} + \epsilon_{lki} \partial_l \partial_j \sigma^e_{ji} \\ &= \epsilon_{lki} \partial_l \partial_j \sigma^v_{ji} + \epsilon_{lki} \partial_l \partial_j \sigma^e_{ji} \end{aligned} \tag{2}$$

where in the second step we have used that $\partial_k \partial_i p$ is symmetric in k and i, so that $\epsilon_{lki} \partial_k \partial_i p = 0$.

2 The viscous stress tensor

The viscous stress tensor is given by:

$$\sigma^v_{ij} = \beta_1 Q_{ij} Q_{kl} A_{kl} + \beta_4 A_{ij} + \beta_5 Q_{ik} A_{ki} + \frac{1}{2} \mu_2 N_{ij} - \mu_1 Q_{ik} N_{kj} + \mu_1 Q_{jk} N_{ki} \tag{3}$$

Each term in the A-tensor and N-tensor involve at least one factor of v_i or Q_{ij}. Since we are taking a linear approximation, we may neglect all but the β_4 and μ_2 terms. A_{ij} is given by

$$A_{ij} \equiv \tfrac{1}{2} \left(\partial_i v_j + \partial_j v_i \right) \tag{4}$$

Also, N_{ij} is given by

$$N_{ij} \equiv \frac{dQ_{ij}}{dt} + W_{ik} Q_{kj} - Q_{ik} W_{kj} \tag{5}$$

where

$$\frac{dQ_{ij}}{dt} \equiv \frac{\partial Q_{ij}}{\partial t} + \left(v \cdot \nabla \right) Q_{ij} \tag{6}$$

and

$$W_{ij} \equiv \tfrac{1}{2} \left(\partial_i v_j - \partial_j v_i \right) \tag{7}$$

The only term which is linear in v_i and Q_{ij} is the time evolution term. Hence

$$N_{ij} \approx \frac{\partial Q_{ij}}{\partial t} \tag{8}$$

The viscous stress tensor then reads:

$$\sigma^v_{ij} \approx \tfrac{1}{2} \beta_4 \left(\partial_i v_j + \partial_j v_i \right) + \tfrac{1}{2} \mu_2 \frac{\partial Q_{ij}}{\partial t} \tag{9}$$

We need to take the divergence and curl to plug it into the equation of motion:

$$\begin{aligned}
\partial_j \sigma^v_{ji} &= \partial_j \left(\tfrac{1}{2}\beta_4 \left(\partial_j v_i + \partial_i v_j\right) + \tfrac{1}{2}\mu_2 \frac{\partial Q_{ij}}{\partial t} \right) \\
&= \tfrac{1}{2}\beta_4 \left(\partial_j^2 v_i + \partial_i \partial_j\right) + \tfrac{1}{2}\mu_2 \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \tfrac{1}{2}\beta_4 \partial_j^2 v_i + \tfrac{1}{2}\mu_2 \partial_j \frac{\partial Q_{ij}}{\partial t}
\end{aligned} \tag{10}$$

In the last step we have used the incompressibility condition $\partial_j v_j = 0$. Taking the curl and substituting in the stream function $v_i = \epsilon_{imn}\partial_m \psi_n$ yields:

$$\begin{aligned}
\epsilon_{lki}\partial_k \partial_j \sigma^v_{ji} &= \tfrac{1}{2}\beta_4 \epsilon_{lki}\partial_k \partial_j^2 v_i + \tfrac{1}{2}\mu_2 \epsilon_{lki}\partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \tfrac{1}{2}\beta_4 \epsilon_{lki}\epsilon_{imn}\partial_k \partial_j^2 \partial_m \psi_n + \tfrac{1}{2}\mu_2 \epsilon_{lki}\partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \tfrac{1}{2}\beta_4 \left(\delta_{lm}\delta_{kn} - \delta_{ln}\delta_{km}\right) \partial_k \partial_j^2 \partial_m \psi_n + \tfrac{1}{2}\mu_2 \epsilon_{lki}\partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \tfrac{1}{2}\beta_4 \left(\partial_j^2 \partial_l \partial_k \psi_k - \partial_j^2 \partial_k^2 \psi_l\right) + \tfrac{1}{2}\mu_2 \epsilon_{lki}\partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= -\tfrac{1}{2}\beta_4 \partial_j^2 \partial_k^2 \psi_l + \tfrac{1}{2}\mu_2 \epsilon_{lki}\partial_k \partial_j \frac{\partial Q_{ij}}{\partial t}
\end{aligned} \tag{11}$$

In the last step we have noted that only the z-component of ψ_k is nonzero. However, we are assuming a quasi-2D system, so that the system is constant in the z-direction. Hence, $\partial_k \psi_k = 0$ so we can get rid of that term.

3 The elastic stress tensor

The elastic stress tensor is given by:

$$\sigma^e_{ij} = -\frac{\partial f}{\partial \left(\partial_i Q_{kl}\right)} \partial_j Q_{kl} \tag{12}$$

where f is the free energy of the system. We only have to consider the elastic portion of the free energy, because that is the only part dependent on spatial derivatives of Q_{ij}:

$$f_e \left(Q_{ij}, \partial_k Q_{ij}\right) = L_1 \partial_k Q_{ij} \partial_k Q_{ij} + L_2 \partial_j Q_{ij} \partial_k Q_{ik} + L_3 Q_{kl} \partial_k Q_{ij} \partial_l Q_{ij} \tag{13}$$

Right now we are only considering the case of isotropic elasticity, so we assume $L_2 = L_3 = 0$. We also define $L \equiv 2L_1$ to be consistent with the Zumer paper. Hence, for our purposes:

$$f_e \left(Q_{np}, \partial_m Q_{np}\right) = \tfrac{1}{2} L \partial_m Q_{np} \partial_m Q_{np} \tag{14}$$

This yields:

$$\begin{aligned}
\frac{\partial f}{\partial \left(\partial_i Q_{kl}\right)} &= \tfrac{1}{2} L \left(\delta_{im}\delta_{kn}\delta_{lp}\partial_m Q_{np} + \partial_m Q_{np}\delta_{im}\delta_{kn}\delta_{lp}\right) \\
&= \tfrac{1}{2} L \left(\partial_i Q_{kl} + \partial_i Q_{kl}\right) \\
&= L \partial_i Q_{kl}
\end{aligned} \tag{15}$$

Then the isotropic elastic stress tensor reads:

$$\sigma^e_{ij} = -L \left(\partial_i Q_{kl}\right) \left(\partial_j Q_{kl}\right) \tag{16}$$

Taking the divergence yields:

$$f_i^{L_1} \equiv \partial_j \sigma^e_{ji} = -L\left[\left(\partial_j^2 Q_{kl}\right)\left(\partial_i Q_{kl}\right) + \left(\partial_j Q_{kl}\right)\left(\partial_i \partial_j Q_{kl}\right)\right] \tag{17}$$

Note that this is the force which acts on an infinitessimal area of the fluid, and so we have called it f_i^{e,L_1}. Taking the curl of this yields:

$$\begin{aligned}\epsilon_{mni}\partial_n\partial_j\sigma^e_{ji} = -L\epsilon_{mni}\Big[&\left(\partial_n\partial_j^2 Q_{kl}\right)\left(\partial_i Q_{kl}\right) + \left(\partial_j^2 Q_{kl}\right)\left(\partial_n\partial_i Q_{kl}\right)\\ &+ \left(\partial_n\partial_j Q_{kl}\right)\left(\partial_i\partial_j Q_{kl}\right) + \left(\partial_j Q_{kl}\right)\left(\partial_n\partial_i\partial_j Q_{kl}\right)\Big]\\ = -\epsilon_{mni}L&\left(\partial_n\partial_j^2 Q_{kl}\right)\left(\partial_i Q_{kl}\right)\end{aligned} \tag{18}$$

In the last step, we have noted that all the terms in brackets, except the first, are symmetric in n and i. Since there is an ϵ_{mni} out front, these go to zero.

4 The stream function equation

Plugging these terms back into the equation of motion yields:

$$\begin{aligned}0 &= -\tfrac{1}{2}\beta_4\partial_j^2\partial_k^2\psi_l + \tfrac{1}{2}\mu_2\epsilon_{lki}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t} - \epsilon_{lki}L\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right)\\ \implies \tfrac{1}{2}\beta_4\partial_j^2\partial_k^2\psi_l &= -\epsilon_{lki}L\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) + \tfrac{1}{2}\mu_2\epsilon_{lki}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t}\end{aligned} \tag{19}$$

Again, the only nonzero component of ψ_l is the z-component. This lets us define the two-dimensional Levi-Civita tensor $\epsilon_{ki} \equiv \epsilon_{3ki}$. Additionally, we may define the biharmonic operator $\partial^4 \equiv \partial_j^2\partial_k^2$. The equation then reads:

$$\partial^4\psi = -\epsilon_{ki}\left[2\frac{L}{\beta_4}\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) - \frac{\mu_2}{\beta_4}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t}\right] \tag{20}$$

5 Nondimensionalizing the stream function equation

Define a dimensionless length and time:

$$\overline{x} = \frac{x}{\xi},\ \bar{t} \equiv \frac{t}{\tau} \tag{21}$$

Further, choose the time scale to be such that

$$\tau = \frac{\mu_1\xi^2}{L} \tag{22}$$

Finally, define

$$\overline{\psi} \equiv \frac{\tau}{\xi^2}\psi \tag{23}$$

For brevity, we drop the overlines. Plugging this in yields:

$$\begin{aligned}\frac{1}{\xi^4}\partial^4\frac{\xi^2}{\tau}\psi &= -\epsilon_{ki}\left[2\frac{L}{\beta_4}\frac{1}{\xi^4}\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) - \frac{\mu_2}{\beta_4}\frac{1}{\tau\xi^2}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t}\right]\\ \implies \partial^4\psi &= -\epsilon_{ki}\left[2\frac{1}{\beta_4}\frac{L\tau}{\xi^2}\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) - \frac{\mu_2}{\beta_4}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t}\right]\\ &= -\epsilon_{ki}\left[2\frac{\mu_1}{\beta_4}\left(\partial_k\partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) - \frac{\mu_2}{\beta_4}\partial_k\partial_j\frac{\partial Q_{ij}}{\partial t}\right]\end{aligned} \tag{24}$$

Finally, we define the dimensionless quantities:

$$\begin{aligned} a &\equiv \frac{\mu_2}{\mu_1} \approx -1.92 \\ b &\equiv \frac{\beta_4}{\mu_1} \approx 1.99 \end{aligned} \tag{25}$$

Plugging this in gives:

$$\partial^4 \psi = -\epsilon_{ki} \left[\frac{2}{b} \left(\partial_k \partial_j^2 Q_{mn} \right) \left(\partial_i Q_{mn} \right) - \frac{a}{b} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \tag{26}$$

Finally, define:

$$\Phi_{L_1}(Q) \equiv -\epsilon_{ki} \left(\partial_k \partial_j^2 Q_{mn} \right) \left(\partial_i Q_{mn} \right) \tag{27}$$

and

$$\Phi_{\mu_2}(Q) \equiv \epsilon_{ki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \tag{28}$$

where we note from above that:

$$\epsilon_{mni} L \left(\partial_n \partial_j^2 Q_{kl} \right) \left(\partial_i Q_{kl} \right) = \frac{L}{\xi^4} \Phi_{L_1}(Q) = \frac{2L_1}{\xi^4} \Phi_{L_1}(Q) \tag{29}$$

Now we have that the final equation reads:

$$\partial^4 \psi = \frac{1}{b} \left[2\Phi_{L_1}(Q) + a\Phi_{\mu_2}(Q) \right] \tag{30}$$

where we note that a is negative in this case.

6 Writing in terms of auxiliary variables

Since Q_{ij} is necessarily symmetric and symmetric, we would like to choose a representation which takes advantage of the fewer degrees of freedom. Further, since we're in a quasi-2D system, it is necessary that $Q_{31} = Q_{32} = 0$. This gives us three degrees of freedom for the 2×2 sub-matrix, and then no extra degrees of freedom for the Q_{33} entry (since it needs to be traceless). Hence, we define three auxiliary variables η, μ, and ν such that

$$Q = \begin{pmatrix} \frac{2}{\sqrt{3}}\eta & \nu & 0 \\ \nu & -\frac{1}{\sqrt{3}}\eta + \mu & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}}\eta - \mu \end{pmatrix} \tag{31}$$

This is just the sum of three basis elements which are themselves tracless, symmetric matrices:

$$e_1 = \begin{pmatrix} \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}; \qquad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{32}$$

Note that these are all orthogonal, and have norm $\sqrt{2}$. We want to write our force terms ($\Phi_{L_1}(Q)$ and $\Phi_{\mu_2}(Q)$) in terms of these auxiliary variables. For now, we are only interested in $\Phi_{L_1}(Q)$ because we assume the configurations given are in equilibrium. We can write up a Sympy program to carry out this sum explicitly. The result is given below:

$$\begin{aligned} \Phi_{L_1}(Q) = &-2\big[\left(\partial_x \partial^2 \eta \right) \left(\partial_y \eta \right) - \left(\partial_y \partial^2 \eta \right) \left(\partial_x \eta \right) + \left(\partial_x \partial^2 \mu \right) \left(\partial_y \mu \right) \\ &- \left(\partial_y \partial^2 \mu \right) \left(\partial_x \mu \right) + \left(\partial_x \partial^2 \nu \right) \left(\partial_y \nu \right) - \left(\partial_y \partial^2 \nu \right) \left(\partial_x \nu \right) \big] \end{aligned} \tag{33}$$

where ∂^2 is the Laplacian. The stream function for the equilibrium configurations is just the solution to the biharmonic equation above with the source terms as $\Phi_{L_1}(Q)$ given above and $\Phi_{\mu_2}(Q) = 0$.

7 Force from the elastic stress tensor

We can nondimensionalize the elastic force as follows:

$$\overline{f}_i^{L_1} \equiv \frac{\xi^3}{L} f_i^{L_1} = -\left[\left(\partial_j^2 Q_{kl}\right)\left(\partial_i Q_{kl}\right) + \left(\partial_j Q_{kl}\right)\left(\partial_i \partial_j Q_{kl}\right)\right] \tag{34}$$

We drop the overline for brevity. Now we can plug in for Q_{kl} to get everything in terms of the auxiliary variables. Carrying this out in Sympy yields:

$$\begin{aligned} f_x^{L_1} = &-2\big[\left(\partial_x \eta\right)\left(2\partial_x^2 \eta + \partial_y^2 \eta\right) + \left(\partial_y \eta\right)\left(\partial_x \partial_y \eta\right) \\ &+ \left(\partial_x \mu\right)\left(2\partial_x^2 \mu + \partial_y^2 \mu\right) + \left(\partial_y \mu\right)\left(\partial_x \partial_y \mu\right) \\ &+ \left(\partial_x \nu\right)\left(2\partial_x^2 \nu + \partial_y^2 \nu\right) + \left(\partial_y \nu\right)\left(\partial_x \partial_y \nu\right)\big] \end{aligned} \tag{35}$$

$$\begin{aligned} f_y^{L_1} = &-2\big[\left(\partial_y \eta\right)\left(2\partial_y^2 \eta + \partial_x^2 \eta\right) + \left(\partial_x \eta\right)\left(\partial_x \partial_y \eta\right) \\ &+ \left(\partial_y \mu\right)\left(2\partial_y^2 \mu + \partial_x^2 \mu\right) + \left(\partial_x \mu\right)\left(\partial_x \partial_y \mu\right) \\ &+ \left(\partial_y \nu\right)\left(2\partial_y^2 \nu + \partial_x^2 \nu\right) + \left(\partial_y \nu\right)\left(\partial_x \partial_y \nu\right)\big] \end{aligned} \tag{36}$$

8 The elastic stress tensor cont. (anisotropic parts)

Here we explicitly evaluate the anisotropic parts of the elastic stress tensor. First let's consider the L_2 term:

$$\begin{aligned} \frac{\partial}{\partial\left(\partial_m Q_{np}\right)} \partial_j Q_{ij} \partial_k Q_{ik} &= \delta_{jm}\delta_{ni}\delta_{pj}\partial_k Q_{ik} + \partial_j Q_{ij}\delta_{mk}\delta_{ni}\delta_{pk} \\ &= \delta_{mp}\partial_k Q_{nk} + \delta_{mp}\partial_j Q_{nj} \\ &= 2\delta_{mp}\partial_k Q_{nk} \end{aligned} \tag{37}$$

Then the L_3 term:

$$\begin{aligned} \frac{\partial}{\partial\left(\partial_m Q_{np}\right)} Q_{kl}\partial_k Q_{ij}\partial_l Q_{ij} &= Q_{kl}\left(\delta_{mk}\delta_{ni}\delta_{pj}\partial_l Q_{ij} + \delta_{ml}\delta_{ni}\delta_{pj}\partial_k Q_{ij}\right) \\ &= Q_{ml}\partial_l Q_{np} + Q_{km}\partial_k Q_{np} \\ &= 2Q_{mk}\partial_k Q_{np} \end{aligned} \tag{38}$$

Recall the definition of the elastic stress tensor, using indices as above:

$$\sigma_{mj}^e = -\frac{\partial f}{\partial\left(\partial_m Q_{np}\right)} \partial_j Q_{np} \tag{39}$$

Substituting these (and the L_1 term) into the above equation yields:

$$\sigma_{mj}^e = -2\left[L_1\left(\partial_m Q_{np}\right)\left(\partial_j Q_{np}\right) + L_2\left(\partial_k Q_{nk}\right)\left(\partial_j Q_{nm}\right) + L_3 Q_{mk}\left(\partial_k Q_{np}\right)\left(\partial_j Q_{np}\right)\right] \tag{40}$$

Making the substitutions $m \to j$, $n \to k$, $p \to l$, $k \to m$, and $j \to i$ yields:

$$\sigma_{ji}^e = -2\left[L_1\left(\partial_j Q_{kl}\right)\left(\partial_i Q_{kl}\right) + L_2\left(\partial_m Q_{km}\right)\left(\partial_i Q_{kj}\right) + L_3 Q_{jm}\left(\partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right)\right] \tag{41}$$

Now we need to take the divergence of the two latter terms. For the second term this yields:

$$\partial_j\left(\partial_m Q_{km}\right)\left(\partial_i Q_{kj}\right) = \left(\partial_j \partial_m Q_{km}\right)\left(\partial_i Q_{kj}\right) + \left(\partial_m Q_{km}\right)\left(\partial_j \partial_i Q_{kj}\right) \tag{42}$$

Third term:

$$
\begin{aligned}
\partial_j \left[Q_{jm} \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \right] = &\left(\partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&+ Q_{jm} \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) + Q_{jm} \left(\partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right)
\end{aligned}
\tag{43}
$$

Taking the curl of the L_2 term yields:

$$
\begin{aligned}
&\epsilon_{npi} \partial_p \big[\left(\partial_j \partial_m Q_{km} \right) \left(\partial_i Q_{kj} \right) \\
&\qquad + \left(\partial_m Q_{km} \right) \left(\partial_j \partial_i Q_{kj} \right) \big] = \epsilon_{npi} \big[\left(\partial_p \partial_j \partial_m Q_{km} \right) \left(\partial_i Q_{kj} \right) + \left(\partial_j \partial_m Q_{km} \right) \left(\partial_p \partial_i Q_{kj} \right) \\
&\qquad\qquad + \left(\partial_p \partial_m Q_{km} \right) \left(\partial_j \partial_i Q_{kj} \right) + \left(\partial_m Q_{km} \right) \left(\partial_j \partial_i \partial_p Q_{kj} \right) \big] \\
&\qquad = \epsilon_{npi} \left(\partial_p \partial_j \partial_m Q_{km} \right) \left(\partial_i Q_{kj} \right)
\end{aligned}
\tag{44}
$$

where the last step follows from the fact that each term except the first is symmetric in i and p. Give that last expression a name (including the minus sign from (41)):

$$
\Phi_{L_2}(Q) \equiv -2 L_2 \epsilon_{npi} \left(\partial_p \partial_j \partial_m Q_{km} \right) \left(\partial_i Q_{kj} \right)
\tag{45}
$$

Taking the curl of the L_3 term yields:

$$
\begin{aligned}
&\epsilon_{npi} \partial_p \big[\left(\partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad + Q_{jm} \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\quad + Q_{jm} \left(\partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right) \big] = \epsilon_{npi} \big[\left(\partial_p \partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_j Q_{jm} \right) \left(\partial_p \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_p \partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_p Q_{jm} \right) \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + Q_{jm} \left(\partial_p \partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + Q_{jm} \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_p \partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_p Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right) \\
&\qquad\qquad + Q_{jm} \left(\partial_p \partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right) \\
&\qquad\qquad + Q_{jm} \left(\partial_m Q_{kl} \right) \left(\partial_p \partial_i \partial_j Q_{kl} \right) \big]
\end{aligned}
\tag{46}
$$

Only a few of these terms are symmetric in p and i, namely terms 3, 6, and 9. But also note that, for the penultimate term:

$$
\begin{aligned}
\epsilon_{npi} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} &= -\epsilon_{nip} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} \\
&= -\epsilon_{nip} Q_{jm} \partial_i \partial_j Q_{kl} \partial_p \partial_m Q_{kl} \\
&= -\epsilon_{npi} Q_{mj} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} \\
&= -\epsilon_{npi} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl}
\end{aligned}
\tag{47}
$$

where in the penultimate line we have relabeled lots of indices, and in the last line we have used the fact that Q_{mj} is symmetric. This then gives:

$$
\begin{aligned}
&\epsilon_{npi} \partial_p \big[\left(\partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad + Q_{jm} \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\quad + Q_{jm} \left(\partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right) \big] = \epsilon_{npi} \big[\left(\partial_p \partial_j Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_j Q_{jm} \right) \left(\partial_p \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_p Q_{jm} \right) \left(\partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + Q_{jm} \left(\partial_p \partial_j \partial_m Q_{kl} \right) \left(\partial_i Q_{kl} \right) \\
&\qquad\qquad + \left(\partial_p Q_{jm} \right) \left(\partial_m Q_{kl} \right) \left(\partial_i \partial_j Q_{kl} \right) \big]
\end{aligned}
\tag{48}
$$

Give this last expression a name (including the minus sign from (41)):

$$
\begin{aligned}
\Phi_{L_3}(Q) = & -2L_3\epsilon_{npi}\big[(\partial_p\partial_j Q_{jm})\,(\partial_m Q_{kl})\,(\partial_i Q_{kl}) \\
& \qquad + (\partial_j Q_{jm})\,(\partial_p\partial_m Q_{kl})\,(\partial_i Q_{kl}) \\
& \qquad + (\partial_p Q_{jm})\,(\partial_j\partial_m Q_{kl})\,(\partial_i Q_{kl}) \\
& \qquad + Q_{jm}\,(\partial_p\partial_j\partial_m Q_{kl})\,(\partial_i Q_{kl}) \\
& \qquad + (\partial_p Q_{jm})\,(\partial_m Q_{kl})\,(\partial_i\partial_j Q_{kl})\big]
\end{aligned} \tag{49}
$$

Note that in these equations, only the $n = 3$ term is nonzero since we have ∂_p and ∂_i terms in each term. Hence, if we have $p = 3$ or $i = 3$ then everything goes to zero. Thus, we can just use the 2D Levi-Civita. We may also non-dimensionalize these equations by dividing by $L = 2L_1$ and multiplying by ξ^4. We will consider only the non-dimensionalized equations as follows.

9 Writing anisotropic elastic source terms using auxiliary variables

Plugging the expression for f_{L_2} into Sympy yields:

$$
\begin{aligned}
\Phi_{L_2} = & -\kappa_2\Bigg[\tfrac{1}{3}\,(\partial_x\eta)\left[\sqrt{3}\partial_y^3\mu - \partial_y^3\eta - 4\partial_y\partial_x^2\eta - \sqrt{3}\partial_y^2\partial_x\nu\right] \\
& + \tfrac{1}{3}\,(\partial_y\eta)\left[4\partial_x^3\eta + \partial_y^2\partial_x\eta - \sqrt{3}\partial_y^2\partial_x\mu + \sqrt{3}\partial_y\partial_x^2\nu\right] \\
& + (\partial_x\mu)\left[\tfrac{1}{\sqrt{3}}\partial_y^3\eta - \partial_y^3\mu - \partial_y^2\partial_x\nu\right] \\
& + (\partial_y\mu)\left[-\tfrac{1}{\sqrt{3}}\partial_y^2\partial_x\eta + \partial_y^2\partial_x\mu + \partial_y\partial_x^2\nu\right] \\
& - (\partial_x\nu)\left[\partial_y^3\nu + \tfrac{1}{\sqrt{3}}\partial_y^2\partial_x\eta + \partial_y^2\partial_x\mu + \partial_y\partial_x^2\nu\right] \\
& + (\partial_y\nu)\left[\partial_x^3\nu + \tfrac{1}{\sqrt{3}}\partial_y\partial_x^2\eta + \partial_y\partial_x^2\mu + \partial_y^2\partial_x\nu\right]\Bigg]
\end{aligned} \tag{50}
$$

where we have defined $\kappa_2 \equiv L_2/L_1$, and in the last line we have simplified a few terms. Now we may plug in for $\Phi_{L_3}(Q)$ in terms of the auxiliary variables:

$$
\begin{aligned}
\Phi_{L_3}(Q) = & -\kappa_3 \Bigg[\tfrac{2}{\sqrt{3}}\eta \Big[(\partial_x \eta)\left(\partial_y^3 \eta - 2\partial_y \partial_x^2 \eta\right) + (\partial_y \eta)\left(2\partial_x^3 \eta - \partial_x \partial_y^2 \eta\right) \\
& + (\partial_x \mu)\left(\partial_y^3 \mu - 2\partial_y \partial_x^2 \mu\right) + (\partial_y \mu)\left(2\partial_x^3 \mu - \partial_x \partial_y^2 \mu\right) \\
& + (\partial_x \nu)\left(\partial_y^3 \nu - 2\partial_y \partial_x^2 \nu\right) + (\partial_y \nu)\left(2\partial_x^3 \nu - \partial_x \partial_y^2 \nu\right) \Big] \\
& + 2\mu \Big[(\partial_y \eta)\left(\partial_y^2 \partial_x \eta\right) - (\partial_x \eta)\left(\partial_y^3 \eta\right) + (\partial_y \mu)\left(\partial_y^2 \partial_x \mu\right) \\
& - (\partial_x \mu)\left(\partial_y^3 \mu\right) + (\partial_y \nu)\left(\partial_y^2 \partial_x \nu\right) - (\partial_x \nu)\left(\partial_y^3 \nu\right) \Big] \\
& + 4\nu \Big[(\partial_y \eta)\left(\partial_y \partial_x^2 \eta\right) - (\partial_x \eta)\left(\partial_y^2 \partial_x \eta\right) + (\partial_y \mu)\left(\partial_y \partial_x^2 \mu\right) \\
& - (\partial_x \mu)\left(\partial_y^2 \partial_x \mu\right) + (\partial_y \nu)\left(\partial_y \partial_x^2 \nu\right) - (\partial_x \nu)\left(\partial_y^2 \partial_x \nu\right) \Big] \\
& + \tfrac{2}{\sqrt{3}}\partial_x \eta \Big[\underbrace{\tfrac{1}{3}(\partial_y \eta)\left(2\partial_x^2 \eta + \partial_y^2 \eta - \sqrt{3}\partial_y^2 \mu\right)} \quad 1/3 \to -1 \\
& + 2(\partial_y \mu)\left(2\partial_x^2 \mu - \partial_y^2 \mu - \sqrt{3}\partial_y^2 \eta\right) + 2(\partial_y \nu)\left(2\partial_x^2 \nu - \partial_y^2 \nu - 2\sqrt{3}\partial_x \partial_y \eta\right) \Big] \\
& + \tfrac{4}{\sqrt{3}}\partial_y \eta \Big[\partial_x \mu \left(\sqrt{3}\partial_y^2 \eta - 2\partial_x^2 \mu + \partial_y^2 \mu\right) + \partial_x \nu \left(2\sqrt{3}\partial_x \partial_y \eta - 2\partial_x^2 \nu + \partial_y^2 \nu\right) \Big] \\
& - 2 \Big[\left((\partial_x \eta)^2 + (\partial_x \mu)^2 + (\partial_x \nu)^2\right)\left(\partial_y^2 \nu + \tfrac{2}{\sqrt{3}}\partial_y \partial_x \eta\right) \Big] \\
& + 2 \Big[\left((\partial_y \eta)^2 + (\partial_y \mu)^2 + (\partial_y \nu)^2\right)\left(\partial_x^2 \nu - \tfrac{1}{\sqrt{3}}\partial_y \partial_x \eta + \partial_y \partial_x \mu\right) \Big] \\
& + \tfrac{2}{3} \Big[\left(2\sqrt{3}\partial_x^2 \eta + \sqrt{3}\partial_y^2 \eta - 3\partial_y^2 \mu\right)\left(\partial_x \mu \partial_y \mu + \partial_x \nu \partial_y \nu\right) \Big] \\
& + 4 \Big[\left(\partial_y^2 \nu - 2\partial_y \partial_x \mu\right)\left(\partial_x \mu \partial_y \nu - \partial_y \mu \partial_x \nu\right) \Big] \Bigg]
\end{aligned}
\tag{51}
$$

$-2 \to +2$

$2 \to -2$

$2/3 \to -2/3$

Note that, with these definitions, the anisotropic flow equation becomes:

$$
\partial^4 \psi = \frac{1}{b}\Big[2\big(\Phi_{L_1}(Q) + \kappa_2 \Phi_{L_2}(Q) + \kappa_3 \Phi_{L_3}(Q)\big) + a\Phi_{\mu_2}(Q) \Big] \tag{52}
$$

10 Anisotropic elastic forces

These forces are just given by the divergence of the corresponding terms in the elastic stress tensor. From section 8, we may read off that:

$$
f_i^{L_2} = -2L_2 \left[(\partial_j \partial_m Q_{km})(\partial_i Q_{kj}) + (\partial_m Q_{km})(\partial_j \partial_i Q_{kj}) \right] \tag{53}
$$

Define $\overline{f}_i^{L_2} \equiv \frac{\xi^3}{L} f_i^{L_2}$ and then drop the overline for brevity sake. then we get that:

$$
f_i^{L_2} = -\kappa_2 \left[(\partial_j \partial_m Q_{km})(\partial_i Q_{kj}) + (\partial_m Q_{km})(\partial_j \partial_i Q_{kj}) \right] \tag{54}
$$

where $\kappa_2 \equiv L_2/L_1$. Plugging this into Sympy to get an expression in terms of the auxiliary variables yields:

$$\begin{aligned}
f_x^{L_2} &= -\kappa_2 \Bigg[\tfrac{1}{3}\left(\partial_x \eta\right)\left[8\partial_x^2\eta + \partial_y^2\eta - \sqrt{3}\partial_y^2\mu + 3\sqrt{3}\partial_y\partial_x\nu\right] \\
&\quad + \tfrac{1}{3}\left(\partial_y \nu\right)\left[2\sqrt{3}\partial_x^2\eta - \sqrt{3}\partial_x^2\nu + \partial_x\partial_y\eta - \sqrt{3}\partial_y\partial_x\mu\right] \\
&\quad + \left(\partial_x \mu\right)\left[-\tfrac{1}{\sqrt{3}}\partial_y^2\eta + \partial_y^2\mu + \partial_x\partial_y\nu\right] \\
&\quad + \left(\partial_y \mu\right)\left[\partial_x^2\nu - \tfrac{1}{\sqrt{3}}\partial_x\partial_y\eta + \partial_x\partial_y\mu\right] \\
&\quad + \left(\partial_x \nu\right)\left[2\partial_x^2\nu + \partial_y^2\nu + 2\partial_x\partial_y\mu\right] + \left(\partial_y\nu\right)\left(\partial_x\partial_y\nu\right)\Bigg] \\
f_y^{L_2} &= -\kappa_2 \Bigg[\tfrac{2}{3}\left(\partial_x \eta\right)\left[\sqrt{3}\partial_y^2\nu + 2\partial_x\partial_y\eta\right] \\
&\quad + \tfrac{2}{3}\left(\partial_y \eta\right)\left[2\partial_x^2\eta + \partial_y^2\eta - \sqrt{3}\partial_y^2\mu\right] \\
&\quad + 2\left(\partial_y \mu\right)\left[\partial_y^2\mu + \partial_x\partial_y\nu - \tfrac{1}{\sqrt{3}}\partial_y^2\eta\right] \\
&\quad + \left(\partial_x \nu\right)\left[\partial_y^2\mu - \tfrac{1}{\sqrt{3}}\partial_y^2\eta + \partial_x\partial_y\nu\right] \\
&\quad + \left(\partial_y \nu\right)\left[\partial_x^2\nu + 2\partial_y^2\nu + \sqrt{3}\partial_x\partial_y\eta + \partial_x\partial_y\mu\right]\Bigg]
\end{aligned} \tag{55}$$

For the third anisotropic elastic force, we may read off:

$$f_i^{L_3} = -2L_3\left[\left(\partial_j Q_{jm}\right)\left(\partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) + Q_{jm}\left(\partial_j\partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) + Q_{jm}\left(\partial_m Q_{kl}\right)\left(\partial_i\partial_j Q_{kl}\right)\right] \tag{56}$$

We may define the dimensionless force in the same way as above. Then the third anisotropic elastic force is given explicitly by:

$$\begin{aligned}
f_x^{L_3} &= -\kappa_3 \Bigg[\tfrac{2}{\sqrt{3}}\eta\Big[\left(\partial_x\eta\right)\left(4\partial_x^2\eta - \partial_y^2\eta\right) + \left(\partial_x\mu\right)\left(4\partial_x^2\mu - \partial_y^2\mu\right) + \left(\partial_x\nu\right)\left(4\partial_x^2\nu - \partial_y^2\nu\right) \\
&\qquad - \partial_y\mu\partial_x\partial_y\mu - \partial_y\eta\partial_x\partial_y\eta - \partial_y\nu\partial_x\partial_y\nu\Big] \\
&\quad + 2\mu\Big[\partial_x\eta\partial_y^2\eta + \partial_y\eta\partial_x\partial_y\eta + \partial_x\mu\partial_y^2\mu + \partial_y\mu\partial_x\partial_y\mu + \partial_x\nu\partial_y^2\nu + \partial_y\nu\partial_x\partial_y\nu\Big] \\
&\quad + 2\nu\Big[3\partial_x\eta\partial_x\partial_y\eta + \partial_x^2\eta\partial_y\eta + 3\partial_x\mu\partial_x\partial_y\mu + \partial_x^2\mu\partial_y\mu + 3\partial_x\nu\partial_x\partial_y\nu + \partial_x^2\nu\partial_y\nu\Big] \\
&\quad + 2\left(\partial_x\eta\right)\Big[\left(\partial_x\eta\right)\Big(\tfrac{2}{\sqrt{3}}\left(\partial_x\eta\right) + \partial_y\nu\Big) + \left(\partial_y\eta\right)\Big(\partial_y\mu + \partial_x\nu - \tfrac{1}{\sqrt{3}}\left(\partial_y\eta\right)\Big) + \tfrac{2}{\sqrt{3}}\left(\partial_x\mu\right)^2 + \tfrac{2}{\sqrt{3}}\left(\partial_x\nu\right)^2\Big] \\
&\quad - \tfrac{2}{\sqrt{3}}\left(\partial_y\eta\right)\Big[\partial_x\mu\partial_y\mu + \partial_x\nu\partial_y\nu\Big] + 2\left(\partial_y\nu\right)\Big[\left(\partial_x\mu\right)^2 + \partial_y\mu\partial_x\nu + 2\left(\partial_x\nu\right)^2\Big] + 2\left(\partial_x\mu\right)\left(\partial_y\mu\right)\Big[\partial_y\mu + \partial_x\nu\Big]\Bigg] \\
f_y^{L_3} &= -\kappa_3 \Bigg[\tfrac{4}{\sqrt{3}}\eta\Big[\left(\partial_y\eta\right)\left(\partial_x^2\eta - \partial_y^2\eta\right) + \left(\partial_y\mu\right)\left(\partial_x^2\mu - \partial_y^2\mu\right) + \left(\partial_y\nu\right)\left(\partial_x^2\nu - \partial_y^2\nu\right) \\
&\qquad + \partial_x\eta\partial_x\partial_y\eta + \partial_x\mu\partial_x\partial_y\mu + \partial_x\nu\partial_x\partial_y\nu\Big] \\
&\quad + 4\mu\Big[\partial_y\eta\partial_y^2\eta + \partial_y\mu\partial_y^2\mu + \partial_y\nu\partial_y^2\nu\Big] \\
&\quad 2\nu\Big[\partial_x\eta\partial_y^2\eta + 3\partial_y\eta\partial_x\partial_y\eta + \partial_x\mu\partial_y^2\mu + 3\partial_y\mu\partial_x\partial_y\mu + \partial_x\nu\partial_y^2\nu + 3\partial_y\nu\partial_x\partial_y\nu\Big] \\
&\quad + \tfrac{4}{\sqrt{3}}\left(\partial_x\eta\right)\Big[\left(\partial_x\eta\right)\partial_y\eta + \tfrac{\sqrt{3}}{2}\partial_y\eta\partial_y\nu + \partial_x\mu\partial_y\mu + \partial_x\nu\partial_y\nu\Big] \\
&\quad 2\left(\partial_y\eta\right)\Big[\left(\partial_y\eta\right)\Big(\partial_y\mu + \partial_x\nu - \tfrac{1}{\sqrt{3}}\left(\partial_y\eta\right)\Big) - \tfrac{1}{\sqrt{3}}\left(\partial_y\mu\right)^2 - \tfrac{1}{\sqrt{3}}\left(\partial_y\nu\right)^2\Big] \\
&\quad + 2\left(\partial_y\mu\right)\Big[\partial_x\mu\partial_y\nu + \left(\partial_y\mu\right)^2 + \left(\partial_y\mu\right)\partial_x\nu + \left(\partial_y\nu\right)^2\Big] + 4\partial_x\nu\left(\partial_y\nu\right)^2\Bigg]
\end{aligned} \tag{57}$$