LC Flow Velocity Eqs

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1 Elastic generalized force

From Svensek and Zumer, the free energy density is given by

$$f = \phi(Q) + \frac{1}{2} L \partial_i Q_{jk} \partial_i Q_{jk} \tag{1}$$

so that the homogeneous elastic part of the generalized force is given by:

$$h_{ij}^{he} = L \partial_k^2 Q_{ij} - \frac{\partial \phi}{\partial Q_{ij}} + \lambda \delta_{ij} + \lambda_k \epsilon_{kij} \tag{2}$$

where ϕ is the Landau de Gennes bulk free energy:

$$\phi(Q) = \frac{1}{2} A Q_{ij} Q_{ji} + \frac{1}{3} B Q_{ij} Q_{jk} Q_{ki} + \frac{1}{4} C (Q_{ij} Q_{ji})^2 \tag{3}$$

We begin by explicitly calculating the second term in terms of Q_{ij}, one term at a time in ϕ:

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{2} A Q_{ij} Q_{ji} \right) &= \frac{1}{2} A \left(\delta_{im} \delta_{jn} Q_{ji} + Q_{ij} \delta_{jm} \delta_{in} \right) \\ &= \frac{1}{2} A (Q_{nm} + Q_{nm}) \\ &= A Q_{mn} \end{aligned} \tag{4}$$

where in the last step we've used symmetry of Q_{ij}.

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{3} B Q_{ij} Q_{jk} Q_{ki} \right) &= \frac{1}{3} B (\delta_{im} \delta_{jn} Q_{jk} Q_{ki} + Q_{ij} \delta_{jm} \delta_{kn} Q_{ki} + Q_{ij} Q_{jk} \delta_{mk} \delta_{ni}) \\ &= \frac{1}{3} B (Q_{nk} Q_{km} + Q_{im} Q_{ni} + Q_{nj} Q_{jm}) \\ &= B Q_{ni} Q_{im} \end{aligned} \tag{5}$$

And the final term gives:

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{4} C (Q_{ij} Q_{ji})^2 \right) &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \frac{\partial (Q_{kl} Q_{lk})}{\partial Q_{mn}} \\ &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \cdot (\delta_{mk} \delta_{nl} Q_{lk} + Q_{kl} \delta_{lm} \delta_{kn}) \\ &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \cdot (Q_{nm} + Q_{nm}) \\ &= C Q_{mn} (Q_{ij} Q_{ji}) \end{aligned} \tag{6}$$

Thus, the total homogeneous and elastic force reads:

$$h_{ij}^{he} = L \partial^2 Q_{ij} - A Q_{ij} - B Q_{ik} Q_{kj} - C Q_{ij} (Q_{kl} Q_{lk}) + \lambda \delta_{ij} + \lambda_k \epsilon_{kij} \tag{7}$$

2 Viscous generalized force

Now we need the viscous force on the liquid crystals. From Svensek and Zumer, the viscous force is given by:

$$-h^v_{ij} = \frac{1}{2}\mu_2 A_{ij} + \mu_1 N_{ij} \tag{8}$$

with

$$N_{ij} = \frac{dQ_{ij}}{dt} + W_{ik}Q_{kj} - Q_{ik}W_{kj} \tag{9}$$

and

$$\frac{dQ_{ij}}{dt} = \frac{\partial Q_{ij}}{\partial t} + (v \cdot \nabla)Q_{ij} \tag{10}$$

where

$$W_{ij} = \tfrac{1}{2}\left(\partial_i v_j - \partial_j v_i\right) \tag{11}$$

The second two terms in the expression for N_{ij} are quadratic in v_i and Q_{ij} so we may drop them, and $(v \cdot \nabla)Q_{ij}$ is clearly quadratic. Hence, we make the approximation

$$N_{ij} \approx \frac{\partial Q_{ij}}{\partial t} \tag{12}$$

We also have the definition

$$A_{ij} = (\partial_i v_j + \partial_j v_i) \tag{13}$$

Plugging this into the viscous force yields:

$$-h^v_{ij} = \frac{1}{2}\mu_2(\partial_i v_j + \partial_j v_i) + \mu_1 \frac{\partial Q_{ij}}{\partial t} \tag{14}$$

Balancing the forces gives the equation:

$$h^e_{ij} = -h^v_{ij} \tag{15}$$

Which yields:

$$\mu_1 \frac{\partial Q_{ij}}{\partial t} = L\partial^2 Q_{ij} - AQ_{ij} - BQ_{ik}Q_{kj} - CQ_{ij}\left(Q_{kl}Q_{lk}\right) - \frac{1}{2}\mu_2\left(\partial_i v_j + \partial_j v_i\right) \tag{16}$$

(here we have dropped the Lagrange multipliers because we are going to explicitly ensure that Q_{ij} is traceless and symmetric).

3 Elastic stress tensor

The elastic stress tensor is obtained via

$$\sigma^e_{ij} = -\frac{\partial f}{\partial(\partial_i Q_{kl})}\partial_j Q_{kl} \tag{17}$$

Note that only the elastic part of the free energy make references to derivatives:

$$\begin{aligned}
\frac{\partial f}{\partial(\partial_i Q_{kl})} &= \frac{\partial}{\partial(\partial_i Q_{kl})}\frac{1}{2}L\partial_j Q_{mn}\partial_j Q_{mn} \\
&= \frac{1}{2}L(\delta_{ij}\delta_{km}\delta_{ln}\partial_j Q_{mn} + \partial_j Q_{mn}\delta_{ij}\delta_{km}\delta_{ln}) \\
&= \frac{1}{2}L(\partial_i Q_{kl} + \partial_i Q_{kl}) \\
&= L\partial_i Q_{kl}
\end{aligned} \tag{18}$$

Then the elastic stress tensor is given by

$$\sigma^e_{ij} = -L\partial_i Q_{kl}\partial_j Q_{kl} \tag{19}$$

4 Viscous stress tensor

The viscous stress tensor is given by

$$\sigma_{ij}^v = \beta_1 Q_{ij} Q_{kl} A_{kl} + \beta_4 A_{ij} + \beta_5 Q_{ik} A_{ki} + \frac{1}{2}\mu_2 N_{ij} - \mu_1 Q_{ik} N_{kj} + \mu_1 Q_{jk} N_{ki} \tag{20}$$

However, only the β_4 and μ_2 are linear in Q_{ij} and v_i. Hence, this simplifies to

$$\sigma_{ij}^v \approx \beta_4 A_{ij} + \frac{1}{2}\mu_2 N_{ij} \tag{21}$$

Again, plugging in for A_{ij} and N_{ij} as we did for the viscous force, we get

$$\sigma_{ij}^v \approx \beta_4(\partial_i v_j + \partial_j v_i) + \frac{1}{2}\mu_2 \frac{\partial Q_{ij}}{\partial t} \tag{22}$$

Plugging in for the time evolution of Q_{ij} gives:

$$\sigma_{ij}^v \approx \beta_4 \left(\partial_i v_j + \partial_j v_i\right) + \frac{1}{2}\frac{\mu_2}{\mu_1}\left[L\partial^2 Q_{ij} - AQ_{ij} - BQ_{ik}Q_{kj} - CQ_{ij}\left(Q_{kl}Q_{lk}\right) - \frac{1}{2}\mu_2\left(\partial_i v_j + \partial_j v_i\right)\right] \tag{23}$$

5 Fluid equation of motion

The equation of motion for the fluid is given by

$$\rho \frac{\partial v_i}{\partial t} = -\partial_i p + \partial_j (\sigma_{ji}^v + \sigma_{ji}^e) \tag{24}$$

We can make the assumption that $\partial v_i / \partial t \approx 0$. Using this and plugging in for σ_{ji}^v and σ_{ji}^e, we get:

$$\begin{aligned}\partial_i p = &\left(\beta_4 - \frac{1}{4}\frac{\mu_2^2}{\mu_1}\right)\left(\partial_i \partial_j v_j + \partial^2 v_i\right) \\ &+ \frac{1}{2}\frac{\mu_2}{\mu_1}\left[L\partial^2 \partial_j Q_{ij} - A\partial_j Q_{ij} - B\partial_j\left(Q_{ik}Q_{kj}\right) - C\partial_j\left[Q_{ij}\left(Q_{kl}Q_{lk}\right)\right]\right] \\ &- L\partial_j\left(\partial_i Q_{kl}\partial_j Q_{kl}\right)\end{aligned} \tag{25}$$

Note that, by incompressibility, $\partial_i v_i = 0$ so a term in parentheses goes away. Now define:

$$f_{\mu_2,i}(Q) = L\partial^2 \partial_j Q_{ij} - A\partial_j Q_{ij} - B\partial_j\left(Q_{ik}Q_{kj}\right) - C\partial_j\left[Q_{ij}\left(Q_{kl}Q_{lk}\right)\right] \tag{26}$$

$$f_{L,i}(Q) = \partial_j\left(\partial_i Q_{kl}\partial_j Q_{kl}\right) \tag{27}$$

$$f_i(Q) = -\frac{1}{2}\frac{\mu_2}{\mu_1} f_{\mu_2,i}(Q) + L f_{L,i}(Q) \tag{28}$$

$$\alpha = \left(\beta_4 - \frac{1}{4}\frac{\mu_2^2}{\mu_1}\right)^{-1} \tag{29}$$

Then the fluid equation of motion becomes:

$$\partial^2 v_i = \alpha\left(\partial_i p + f_i\right) \tag{30}$$

6 Choosing a specific Q_{ij}

Now, we would like to choose f_i to be something which is just a sum of sines. These terminate at the endpoints, so it happens that we will fulfill our no slip condition ($v_i = 0$ at the boundary). Since we need to actually choose Q_{ij} to do that (so that we can use the same scenario for the stream function formulation), we will have to just guess and check. Note that many of the terms involved in f_i have third derivatives, so we will want to choose a cos to be the important term. Recall that, in the uniaxial case:

$$Q_{ij} = \frac{S}{2}\left(3n_i n_j - \delta_{ij}\right) \tag{31}$$

Choose S to be constant. Plugging Q_{ij} into the expression for $f_{\mu_2,i}$ yields:

$$\begin{aligned} f_{\mu_2,i}(Q) = \frac{3S}{2} L\partial^2\partial_j\left(n_i n_j\right) - \frac{3S}{2} A\partial_j\left(n_i n_j\right) - \frac{S^2}{4} B\partial_j\left(3n_i n_k - \delta_{ik}\right)\left(3n_k n_j - \delta_{kj}\right) \\ - \frac{S^3}{8} C\partial_j\left[\left(3n_i n_j - \delta_{ij}\right)\left(3n_k n_l - \delta_{kl}\right)\left(3n_l n_k - \delta_{lk}\right)\right] \end{aligned} \tag{32}$$

This simplifies to:

$$\begin{aligned} f_{\mu_2,i}(Q) &= \frac{3S}{2} L\partial^2\partial_j\left(n_i n_j\right) - \frac{3S}{2} A\partial_j\left(n_i n_j\right) - \frac{3S^2}{2} B\partial_j\left(n_i n_j\right) - \frac{9S^3}{4} C\partial_j\left(n_i n_j\right) \\ &= \frac{3S}{2}\left(L\partial^2 - A - SB - \frac{3S^2}{2} C\right)\partial_j\left(n_i n_j\right) \end{aligned} \tag{33}$$

Do the same for $f_{L,i}$ to get:

$$f_{L,i}(Q) = \frac{9S^2}{4}\partial_j\left[\partial_i\left(n_k n_l\right)\partial_j\left(n_l n_k\right)\right] \tag{34}$$

Our first guess will be:

$$\hat{n} = \left(\cos k_x x, \sin k_x x, 0\right) \tag{35}$$

Note that this expression is independent of y so that any y-derivatives will be zero by default. We will do pieces at a time:

$$\begin{aligned} \partial_j\left(n_x n_j\right) &= \partial_x \cos^2 k_x x \\ &= 2\cos k_x x\left(-k_x \sin k_x x\right) \\ &= -k_x \sin\left(2k_x x\right) \end{aligned} \tag{36}$$

Great, this still terminates at the endpoints. Now taking the Laplacian of this:

$$\begin{aligned} \partial^2\partial_j\left(n_x n_j\right) &= \partial_x^2\left(-k_x \sin\left(2k_x x\right)\right) \\ &= -4k_x^2\partial_x \cos\left(2k_x x\right) \\ &= 8k_x^3 \sin\left(2k_x x\right) \end{aligned} \tag{37}$$

Cool, this also terminates at the endpoints. So far so good. Let's look at the y-component:

$$\begin{aligned} \partial_j\left(n_y n_j\right) &= \partial_x\left(\cos k_x x \sin k_x x\right) \\ &= \partial_x \tfrac{1}{2}\sin 2k_x x \\ &= -k_x \cos 2k_x x \end{aligned} \tag{38}$$

This certainly does not go to zero at the endpoints, so we have not met our no-slip condition. Dang. Now for the last term:

$$\begin{aligned}\partial_j\left[\partial_x\left(n_k n_l\right)\partial_j\left(n_l n_k\right)\right] &= \partial_x\left[\left(\partial_x \cos^2 k_x x\right)^2 + \left(\partial_x \cos k_x x \sin k_x x\right)^2 + \left(\partial_x \cos k_x x \sin k_x x\right)^2 + \left(\partial_x \sin^2 k_x x\right)^2\right] \\ &= \partial_x\left[k_x^2 \sin^2\left(2k_x x\right) + 2\left(\partial_x \frac{1}{2}\sin 2k_x x\right)^2 + k_x^2 \sin^2\left(2k_x x\right)\right] \\ &= \partial_x\left[2k_x^2 \sin^2\left(2k_x x\right) + 2k_x^2 \cos^2\left(2k_x x\right)\right] \\ &= 0\end{aligned} \tag{39}$$

Well this is not actually helpful to understanding why the elastic piece is so small – here it is actually zero. However, we can check the rest of the configuration with this scheme. In this case, we get:

$$f_i(x,y) = -\frac{\mu_2}{\mu_1}\frac{3S}{4}\left[8k_x^3 L + \left(A + SB + \frac{3S^2}{2}C\right)k_x\right]\sin\left(2k_x x\right) \tag{40}$$

Clearly we have that for all $k_y \neq 0$, we have $\hat{f}_{i,k_x k_y} = 0$ (since there is no y-dependence). Additionally, the only nonzero Fourier term is for $k_x = 4\pi/L_x$, and is completely imaginary since only sine is involved. This comes out to be:

$$f_{x,4\pi/L_x,0} = -i\frac{\mu_2}{\mu_1}\frac{3S}{4}\left[8k_x^3 L + \left(A + SB + \frac{3S^2}{2}C\right)k_x\right] \tag{41}$$

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This simplifies to:

$$\begin{aligned}f_{\mu_2,i}(Q) &= \frac{3S}{2}L\partial^2\partial_j\left(n_i n_j\right) - \frac{3S}{2}A\partial_j\left(n_i n_j\right) - \frac{3S^2}{2}B\partial_j\left(n_i n_j\right) - \frac{9S^3}{4}C\partial_j\left(n_i n_j\right) \\ &= \frac{3S}{2}\left(L\partial^2 - A - SB - \frac{3S^2}{2}C\right)\partial_j\left(n_i n_j\right)\end{aligned} \tag{44}$$

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Cool, this also terminates at the endpoints. So far so good. Now for the last term:

$$\begin{aligned}
\partial_j \left[\partial_x \left(n_k n_l\right) \partial_j \left(n_l n_k\right)\right] &= \partial_x \left[\left(\partial_x \cos^2 k_x x\right)^2 + \left(\partial_x \cos k_x x \sin k_x x\right)^2 + \left(\partial_x \cos k_x x \sin k_x x\right)^2 + \left(\partial_x \sin^2 k_x x\right)^2\right] \\
&= \partial_x \left[k_x^2 \sin^2\left(2k_x x\right) + 2\left(\partial_x \frac{1}{2} \sin 2k_x x\right)^2 + k_x^2 \sin^2\left(2k_x x\right)\right] \\
&= \partial_x \left[2k_x^2 \sin^2\left(2k_x x\right) + 2k_x^2 \cos^2\left(2k_x x\right)\right] \\
&= 0
\end{aligned} \tag{49}$$

Well this is not actually helpful to understanding why the elastic piece is so small – here it is actually zero. However, we can check the rest of the configuration with this scheme. In this case, we get:

$$f_i(x,y) = -\frac{\mu_2}{\mu_1} \frac{3S}{4} \left[8k_x^3 L + \left(A + SB + \frac{3S^2}{2} C\right) k_x\right] \sin\left(2k_x x\right) \tag{50}$$

Clearly we have that for all $k_y \neq 0$, we have $\hat{f}_{i,k_x k_y} = 0$ (since there is no y-dependence). Additionally, the only nonzero Fourier term is for $k_x = 4\pi/L_x$, and is completely imaginary since only sine is involved. This comes out to be:

$$f_{x,4\pi/L_x,0} = -i \frac{\mu_2}{\mu_1} \frac{3S}{4} \left[8k_x^3 L + \left(A + SB + \frac{3S^2}{2} C\right) k_x\right] \tag{51}$$

Note that we only have nonzero $\hat{f}$ for $i = x$ – we see this in previous calculations.