Analytic solution for flow around defect from anisotropic elasticity

Lucas Myers

January 13, 2021

1 Director field configuration

The director field configuration is given by:

$$\mathbf{n} = (\cos k\phi, \sin k\phi, 0) \tag{1}$$

for 3-dimensions. Recall also that the Q-tensor is given by:

$$Q_{ij} = \frac{S}{2}\left(3n_i n_j - \delta_{ij}\right) \tag{2}$$

Plugging this in yields:

$$\begin{aligned} Q &= \frac{S}{2}\begin{bmatrix} \left(3\cos^2 k\phi - 1\right) & \frac{3}{2}\sin 2k\phi & 0 \\ \frac{3}{2}\sin 2k\phi & \left(3\sin^2 k\phi - 1\right) & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ &= \frac{S}{2}\begin{bmatrix} \left(\frac{1}{2} + \frac{3}{2}\cos 2k\phi\right) & \frac{3}{2}\sin 2k\phi & 0 \\ \frac{3}{2}\sin 2k\phi & \left(\frac{1}{2} - \frac{3}{2}\cos 2k\phi\right) & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned} \tag{3}$$

2 Flow equation and source terms

Now, the flow equation for an equilibrium configuration is given by:

$$\partial^4 \psi = \frac{1}{b}\Big[2\big(\Phi_{L_1}(Q) + \kappa_2 \Phi_{L_2}(Q) + \kappa_3 \Phi_{L_3}(Q)\big) + a\Phi_{\mu_2}(Q)\Big] \tag{4}$$

with

$$\Phi_{L_1}(Q) \equiv -\epsilon_{ki}\left(\partial_k \partial_j^2 Q_{mn}\right)\left(\partial_i Q_{mn}\right) \tag{5}$$

and

$$\begin{aligned} \Phi_{L_3}(Q) = \; -2L_3\epsilon_{npi}\Big[&\left(\partial_p \partial_j Q_{jm}\right)\left(\partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) \\ &+ \left(\partial_j Q_{jm}\right)\left(\partial_p \partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) \\ &+ \left(\partial_p Q_{jm}\right)\left(\partial_j \partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) \\ &+ Q_{jm}\left(\partial_p \partial_j \partial_m Q_{kl}\right)\left(\partial_i Q_{kl}\right) \\ &+ \left(\partial_p Q_{jm}\right)\left(\partial_m Q_{kl}\right)\left(\partial_i \partial_j Q_{kl}\right)\Big] \end{aligned} \tag{6}$$

This equation is derived in the "FlowFromElasticForces" file. Note that we are unconcerned with the Φ_{L_2} and Φ_{μ_2} terms, because for this scenario we set $\kappa_2 = 0$ and assume an equilibrium configuration.

If we plug in the Q-configuration from above, this yields the following:

$$\Phi_{L_1}(Q) = 0 \tag{7}$$

Clearly a solution to the Biharmonic equation is $\psi = 0$. See this stack exchange post for an explanation of uniqueness of this solution.

For the anisotropic term, we get that:

$$\begin{aligned}\Phi_{L_3}(Q) = \frac{27S^3}{256r^4}\Big[&64\sin^6\left(\frac{\phi}{2}\right)\sin(2\phi) + 96\sin^4\left(\frac{\phi}{2}\right)\sin(\phi) \\ &- 96\sin^4\left(\frac{\phi}{2}\right)\sin(2\phi) - 64\sin^4\left(\frac{\phi}{2}\right)\sin(3\phi) \\ &- 12\sin(\phi) + 24\sin(2\phi) + 9\sin(3\phi) - 16\sin(4\phi) + 5\sin(5\phi)\Big]\end{aligned} \tag{8}$$

If we simplify this using various trig identities, we're left with the very simple expression:

$$\Phi_{L_3}(Q) = \frac{81S^3\sin(\phi)}{32r^4} \tag{9}$$

3 Solution for the anisotropic source term

This source term is separable in polar coordinates, so we attempt a separable solution:

$$\psi(r,\phi) = R(r)\Phi(\phi) \tag{10}$$

where $\Phi(\phi)$ is not to be confused with $\Phi_{L_3}(Q)$. We also write our source term as

$$\Phi_{L_3}(Q) = f(r)g(\phi) \tag{11}$$

so that our solution may be slightly more general. Plugging this in and considering the form of the biharmonic operator in polar coordinates, the equation becomes:

$$\Delta^2\psi = \Phi(\phi)F(r) + \frac{2}{r^2}\Phi''(\phi)R''(r) + \frac{1}{r^4}\Phi''''(\phi)R(r) - \frac{2}{r^3}\Phi''(\phi)R'(r) + \frac{4}{r^4}\Phi''(\phi)R(r) = f(\phi)g(r) \tag{12}$$

where

$$F(r) \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\right)\right) = R''''(r) + \frac{2}{r}R'''(r) - \frac{1}{r^2}R''(r) + \frac{1}{r^3}R'(r) \tag{13}$$

Now, if we assume the form

$$g(\phi) = \sin m\phi\,;\Phi(\phi) = \sin m\phi \tag{14}$$

or

$$g(\phi) = \cos m\phi\,;\Phi(\phi) = \cos m\phi \tag{15}$$

then the PDE above reduces to an ODE in r:

$$F(r) - \frac{2}{r^2}m^2R''(r) + \frac{1}{r^4}m^4R(r) + \frac{2}{r^3}m^2R'(r) - \frac{4}{r^4}m^2R(r) = g(r) \tag{16}$$

Explicitly writing out $F(r)$ yields.

$$R''''(r) + \frac{2}{r}R'''(r) - \frac{1}{r^2}\left(1+2m^2\right)R''(r) + \frac{1}{r^3}\left(1+2m^2\right)R'(r) + \frac{1}{r^4}\left(m^4-4m^2\right)R(r) = \frac{\lambda}{r^4} \tag{17}$$

We first solve the homogeneous equation. From Wolfram Mathematica, the solution is:

$$R(r) = c_1r^{-m} + c_2r^{2-m} + c_3r^{2+m} + c_4r^m \tag{18}$$

Now we have to consider the source term. Note that the following is a particular solution:

$$R(r) = \frac{\lambda}{m^4 - 4m^2} \tag{19}$$

Note that, since the particular solution is constant, we must have that $c_1 = 0$. Further, if $m > 2$ we need $c_2 = 0$. If $m = 1$ we can absorb c_2 into c_4. We will specify to the $m = 1$ case for now, given that our source term has a $\sin(\phi)$ in it. Our first boundary condition gives:

$$c_3 r_0^3 + c_4 r_0 - \tfrac{1}{3}\lambda = 0 \implies c_4 = \tfrac{1}{3}\frac{\lambda}{r_0} - c_3 r_0^2 \tag{20}$$

The second boundary condition gives:

$$3c_3 r_0^2 + c_4 = 0 \implies c_4 = -3c_3 r_0^2 \tag{21}$$

These two conditions together give:

$$c_3 = -\tfrac{1}{6}\frac{\lambda}{r_0^3} \tag{22}$$

Plugging back in gives:

$$c_4 = \tfrac{1}{2}\frac{\lambda}{r_0} \tag{23}$$

Hence, the solution is given by:

$$R(r) = \lambda\left(-\frac{1}{6}\left(\frac{r}{r_0}\right)^3 + \frac{1}{2}\left(\frac{r}{r_0}\right) - \frac{1}{3}\right) \tag{24}$$

So that the stream function is given by:

$$\psi(r,\phi) = \lambda\left(-\frac{1}{6}\left(\frac{r}{r_0}\right)^3 + \frac{1}{2}\left(\frac{r}{r_0}\right) - \frac{1}{3}\right)\sin(\phi) \tag{25}$$

with $\lambda = 81S^3/32$. Taking the curl yields:

$$\begin{aligned}\nabla\times\psi &= \frac{1}{r}\frac{\partial\psi}{\partial\phi}\hat{r} - \frac{\partial\psi}{\partial r}\hat{\phi} \\ &= \lambda\left(-\frac{1}{6}\left(\frac{r^2}{r_0^3}\right) + \frac{1}{2}\left(\frac{1}{r_0}\right) - \frac{1}{3}\frac{1}{r}\right)\cos(\phi)\hat{r} - \lambda\left(-\frac{1}{2}\left(\frac{r^2}{r_0^3}\right) + \frac{1}{2}\left(\frac{1}{r_0}\right)\right)\sin(\phi)\hat{\phi}\end{aligned} \tag{26}$$