

Flow around nematic liquid crystals due to elastic forces

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1 The fluid equation of motion

The equation of motion for the fluid is given by

$$\rho \frac{\partial v_i}{\partial t} = -\partial_i p + \partial_j (\sigma_{ji}^v + \sigma_{ji}^e) \quad (1)$$

We make the assumption that $\partial v_i / \partial t \approx 0$. Note that we may get rid of the pressure term as well by taking the curl of both sides:

$$\begin{aligned} 0 &= -\epsilon_{lki} \partial_k \partial_i p + \epsilon_{lki} \partial_l \partial_j \sigma_{ji}^v + \epsilon_{lki} \partial_l \partial_j \sigma_{ji}^e \\ &= \epsilon_{lki} \partial_l \partial_j \sigma_{ji}^v + \epsilon_{lki} \partial_l \partial_j \sigma_{ji}^e \end{aligned} \quad (2)$$

where in the second step we have used that $\partial_k \partial_i p$ is symmetric in k and i , so that $\epsilon_{lki} \partial_k \partial_i p = 0$.

2 The viscous stress tensor

The viscous stress tensor is given by:

$$\sigma_{ij}^v = \beta_1 Q_{ij} Q_{kl} A_{kl} + \beta_4 A_{ij} + \beta_5 Q_{ik} A_{ki} + \frac{1}{2} \mu_2 N_{ij} - \mu_1 Q_{ik} N_{kj} + \mu_1 Q_{jk} N_{ki} \quad (3)$$

Each term in the A -tensor and N -tensor involve at least one factor of v_i or Q_{ij} . Since we are taking a linear approximation, we may neglect all but the β_4 and μ_2 terms. A_{ij} is given by

$$A_{ij} \equiv \frac{1}{2} (\partial_i v_j + \partial_j v_i) \quad (4)$$

Also, N_{ij} is given by

$$N_{ij} \equiv \frac{dQ_{ij}}{dt} + W_{ik} Q_{kj} - Q_{ik} W_{kj} \quad (5)$$

where

$$\frac{dQ_{ij}}{dt} \equiv \frac{\partial Q_{ij}}{\partial t} + (v \cdot \nabla) Q_{ij} \quad (6)$$

and

$$W_{ij} \equiv \frac{1}{2} (\partial_i v_j - \partial_j v_i) \quad (7)$$

The only term which is linear in v_i and Q_{ij} is the time evolution term. Hence

$$N_{ij} \approx \frac{\partial Q_{ij}}{\partial t} \quad (8)$$

The viscous stress tensor then reads:

$$\sigma_{ij}^v \approx \frac{1}{2} \beta_4 (\partial_i v_j + \partial_j v_i) + \frac{1}{2} \mu_2 \frac{\partial Q_{ij}}{\partial t} \quad (9)$$

We need to take the divergence and curl to plug it into the equation of motion:

$$\begin{aligned}
\partial_j \sigma_{ji}^v &= \partial_j \left(\frac{1}{2} \beta_4 (\partial_j v_i + \partial_i v_j) + \frac{1}{2} \mu_2 \frac{\partial Q_{ij}}{\partial t} \right) \\
&= \frac{1}{2} \beta_4 (\partial_j^2 v_i + \partial_i \partial_j) + \frac{1}{2} \mu_2 \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \frac{1}{2} \beta_4 \partial_j^2 v_i + \frac{1}{2} \mu_2 \partial_j \frac{\partial Q_{ij}}{\partial t}
\end{aligned} \tag{10}$$

In the last step we have used the incompressibility condition $\partial_j v_j = 0$. Taking the curl and substituting in the stream function $v_i = \epsilon_{imn} \partial_m \psi_n$ yields:

$$\begin{aligned}
\epsilon_{lki} \partial_k \partial_j \sigma_{ji}^v &= \frac{1}{2} \beta_4 \epsilon_{lki} \partial_k \partial_j^2 v_i + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \frac{1}{2} \beta_4 \epsilon_{lki} \epsilon_{imn} \partial_k \partial_j^2 \partial_m \psi_n + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \frac{1}{2} \beta_4 (\delta_{lm} \delta_{kn} - \delta_{ln} \delta_{km}) \partial_k \partial_j^2 \partial_m \psi_n + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= \frac{1}{2} \beta_4 (\partial_j^2 \partial_l \partial_k \psi_k - \partial_j^2 \partial_k^2 \psi_l) + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \\
&= -\frac{1}{2} \beta_4 \partial_j^2 \partial_k^2 \psi_l + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t}
\end{aligned} \tag{11}$$

In the last step we have noted that only the z -component of ψ_k is nonzero. However, we are assuming a quasi-2D system, so that the system is constant in the z -direction. Hence, $\partial_k \psi_k = 0$ so we can get rid of that term.

3 The elastic stress tensor

The elastic stress tensor is given by:

$$\sigma_{ij}^e = - \frac{\partial f}{\partial (\partial_i Q_{kl})} \partial_j Q_{kl} \tag{12}$$

where f is the free energy of the system. We only have to consider the elastic portion of the free energy, because that is the only part dependent on spatial derivatives of Q_{ij} :

$$f_e(Q_{ij}, \partial_k Q_{ij}) = L_1 \partial_k Q_{ij} \partial_k Q_{ij} + L_2 \partial_j Q_{ij} \partial_k Q_{ik} + L_3 Q_{kl} \partial_k Q_{ij} \partial_l Q_{ij} \tag{13}$$

Right now we are only considering the case of isotropic elasticity, so we assume $L_2 = L_3 = 0$. We also define $L \equiv 2L_1$ to be consistent with the Zumer paper. Hence, for our purposes:

$$f_e(Q_{np}, \partial_m Q_{np}) = \frac{1}{2} L \partial_m Q_{np} \partial_m Q_{np} \tag{14}$$

This yields:

$$\begin{aligned}
\frac{\partial f}{\partial (\partial_i Q_{kl})} &= \frac{1}{2} L (\delta_{im} \delta_{kn} \delta_{lp} \partial_m Q_{np} + \partial_m Q_{np} \delta_{im} \delta_{kn} \delta_{lp}) \\
&= \frac{1}{2} L (\partial_i Q_{kl} + \partial_i Q_{kl}) \\
&= L \partial_i Q_{kl}
\end{aligned} \tag{15}$$

Then the isotropic elastic stress tensor reads:

$$\sigma_{ij}^e = -L (\partial_i Q_{kl}) (\partial_j Q_{kl}) \tag{16}$$

Taking the divergence yields:

$$f_i^{L_1} \equiv \partial_j \sigma_{ji}^e = -L \left[(\partial_j^2 Q_{kl}) (\partial_i Q_{kl}) + (\partial_j Q_{kl}) (\partial_i \partial_j Q_{kl}) \right] \quad (17)$$

Note that this is the force which acts on an infinitesimal area of the fluid, and so we have called it f_i^{e, L_1} . Taking the curl of this yields:

$$\begin{aligned} \epsilon_{mni} \partial_n \partial_j \sigma_{ji}^e &= -L \epsilon_{mni} \left[(\partial_n \partial_j^2 Q_{kl}) (\partial_i Q_{kl}) + (\partial_j^2 Q_{kl}) (\partial_n \partial_i Q_{kl}) \right. \\ &\quad \left. + (\partial_n \partial_j Q_{kl}) (\partial_i \partial_j Q_{kl}) + (\partial_j Q_{kl}) (\partial_n \partial_i \partial_j Q_{kl}) \right] \\ &= -\epsilon_{mni} L (\partial_n \partial_j^2 Q_{kl}) (\partial_i Q_{kl}) \end{aligned} \quad (18)$$

In the last step, we have noted that all the terms in brackets, except the first, are symmetric in n and i . Since there is an ϵ_{mni} out front, these go to zero.

4 The stream function equation

Plugging these terms back into the equation of motion yields:

$$\begin{aligned} 0 &= -\frac{1}{2} \beta_4 \partial_j^2 \partial_k^2 \psi_l + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} - \epsilon_{lki} L (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) \\ \implies \frac{1}{2} \beta_4 \partial_j^2 \partial_k^2 \psi_l &= -\epsilon_{lki} L (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) + \frac{1}{2} \mu_2 \epsilon_{lki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \end{aligned} \quad (19)$$

Again, the only nonzero component of ψ_l is the z -component. This lets us define the two-dimensional Levi-Civita tensor $\epsilon_{ki} \equiv \epsilon_{3ki}$. Additionally, we may define the biharmonic operator $\partial^4 \equiv \partial_j^2 \partial_k^2$. The equation then reads:

$$\partial^4 \psi = -\epsilon_{ki} \left[2 \frac{L}{\beta_4} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) - \frac{\mu_2}{\beta_4} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \quad (20)$$

5 Nondimensionalizing the stream function equation

Define a dimensionless length and time:

$$\bar{x} = \frac{x}{\xi}, \quad \bar{t} \equiv \frac{t}{\tau} \quad (21)$$

Further, choose the time scale to be such that

$$\tau = \frac{\mu_1 \xi^2}{L} \quad (22)$$

Finally, define

$$\bar{\psi} \equiv \frac{\tau}{\xi^2} \psi \quad (23)$$

For brevity, we drop the overlines. Plugging this in yields:

$$\begin{aligned} \frac{1}{\xi^4} \partial^4 \frac{\xi^2}{\tau} \psi &= -\epsilon_{ki} \left[2 \frac{L}{\beta_4} \frac{1}{\xi^4} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) - \frac{\mu_2}{\beta_4} \frac{1}{\tau \xi^2} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \\ \implies \partial^4 \psi &= -\epsilon_{ki} \left[2 \frac{1}{\beta_4} \frac{L\tau}{\xi^2} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) - \frac{\mu_2}{\beta_4} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \\ &= -\epsilon_{ki} \left[2 \frac{\mu_1}{\beta_4} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) - \frac{\mu_2}{\beta_4} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \end{aligned} \quad (24)$$

Finally, we define the dimensionless quantities:

$$\begin{aligned} a &\equiv \frac{\mu_2}{\mu_1} \approx -1.92 \\ b &\equiv \frac{\beta_4}{\mu_1} \approx 1.99 \end{aligned} \quad (25)$$

Plugging this in gives:

$$\partial^4 \psi = -\epsilon_{ki} \left[\frac{2}{b} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) - \frac{a}{b} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \right] \quad (26)$$

Finally, define:

$$\Phi_{L_1}(Q) \equiv -\epsilon_{ki} (\partial_k \partial_j^2 Q_{mn}) (\partial_i Q_{mn}) \quad (27)$$

and

$$\Phi_{\mu_2}(Q) \equiv \epsilon_{ki} \partial_k \partial_j \frac{\partial Q_{ij}}{\partial t} \quad (28)$$

where we note from above that:

$$\epsilon_{mni} L (\partial_n \partial_j^2 Q_{kl}) (\partial_i Q_{kl}) = \frac{L}{\xi^4} \Phi_{L_1}(Q) = \frac{2L_1}{\xi^4} \Phi_{L_1}(Q) \quad (29)$$

Now we have that the final equation reads:

$$\partial^4 \psi = \frac{1}{b} [2\Phi_{L_1}(Q) + a\Phi_{\mu_2}(Q)] \quad (30)$$

where we note that a is negative in this case.

6 Writing in terms of auxiliary variables

Since Q_{ij} is necessarily symmetric and symmetric, we would like to choose a representation which takes advantage of the fewer degrees of freedom. Further, since we're in a quasi-2D system, it is necessary that $Q_{31} = Q_{32} = 0$. This gives us three degrees of freedom for the 2×2 sub-matrix, and then no extra degrees of freedom for the Q_{33} entry (since it needs to be traceless). Hence, we define three auxiliary variables η , μ , and ν such that

$$Q = \begin{pmatrix} \frac{2}{\sqrt{3}}\eta & \nu & 0 \\ \nu & -\frac{1}{\sqrt{3}}\eta + \mu & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}}\eta - \mu \end{pmatrix} \quad (31)$$

This is just the sum of three basis elements which are themselves traceless, symmetric matrices:

$$e_1 = \begin{pmatrix} \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}; \quad e_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (32)$$

Note that these are all orthogonal, and have norm $\sqrt{2}$. We want to write our force terms ($\Phi_{L_1}(Q)$ and $\Phi_{\mu_2}(Q)$) in terms of these auxiliary variables. For now, we are only interested in $\Phi_{L_1}(Q)$ because we assume the configurations given are in equilibrium. We can write up a Sympy program to carry out this sum explicitly. The result is given below:

$$\begin{aligned} \Phi_{L_1}(Q) = & -2 [(\partial_x \partial^2 \eta) (\partial_y \eta) - (\partial_y \partial^2 \eta) (\partial_x \eta) + (\partial_x \partial^2 \mu) (\partial_y \mu) \\ & - (\partial_y \partial^2 \mu) (\partial_x \mu) + (\partial_x \partial^2 \nu) (\partial_y \nu) - (\partial_y \partial^2 \nu) (\partial_x \nu)] \end{aligned} \quad (33)$$

where ∂^2 is the Laplacian. The stream function for the equilibrium configurations is just the solution to the biharmonic equation above with the source terms as $\Phi_{L_1}(Q)$ given above and $\Phi_{\mu_2}(Q) = 0$.

7 Force from the elastic stress tensor

We can nondimensionalize the elastic force as follows:

$$\bar{f}_i^{L_1} \equiv \frac{\xi^3}{L} f_i^{L_1} = - [(\partial_j^2 Q_{kl}) (\partial_i Q_{kl}) + (\partial_j Q_{kl}) (\partial_i \partial_j Q_{kl})] \quad (34)$$

We drop the overline for brevity. Now we can plug in for Q_{kl} to get everything in terms of the auxiliary variables. Carrying this out in Sympy yields:

$$\begin{aligned} f_x^{L_1} = & -2 [(\partial_x \eta) (2\partial_x^2 \eta + \partial_y^2 \eta) + (\partial_y \eta) (\partial_x \partial_y \eta) \\ & + (\partial_x \mu) (2\partial_x^2 \mu + \partial_y^2 \mu) + (\partial_y \mu) (\partial_x \partial_y \mu) \\ & + (\partial_x \nu) (2\partial_x^2 \nu + \partial_y^2 \nu) + (\partial_y \nu) (\partial_x \partial_y \nu)] \end{aligned} \quad (35)$$

$$\begin{aligned} f_y^{L_1} = & -2 [(\partial_y \eta) (2\partial_y^2 \eta + \partial_x^2 \eta) + (\partial_x \eta) (\partial_x \partial_y \eta) \\ & + (\partial_y \mu) (2\partial_y^2 \mu + \partial_x^2 \mu) + (\partial_x \mu) (\partial_x \partial_y \mu) \\ & + (\partial_y \nu) (2\partial_y^2 \nu + \partial_x^2 \nu) + (\partial_x \nu) (\partial_x \partial_y \nu)] \end{aligned} \quad (36)$$

8 The elastic stress tensor cont. (anisotropic parts)

Here we explicitly evaluate the anisotropic parts of the elastic stress tensor. First let's consider the L_2 term:

$$\begin{aligned} \frac{\partial}{\partial (\partial_m Q_{np})} \partial_j Q_{ij} \partial_k Q_{ik} &= \delta_{jm} \delta_{ni} \delta_{pj} \partial_k Q_{ik} + \partial_j Q_{ij} \delta_{mk} \delta_{ni} \delta_{pk} \\ &= \delta_{mp} \partial_k Q_{nk} + \delta_{mp} \partial_j Q_{nj} \\ &= 2\delta_{mp} \partial_k Q_{nk} \end{aligned} \quad (37)$$

Then the L_3 term:

$$\begin{aligned} \frac{\partial}{\partial (\partial_m Q_{np})} Q_{kl} \partial_k Q_{ij} \partial_l Q_{ij} &= Q_{kl} (\delta_{mk} \delta_{ni} \delta_{pj} \partial_l Q_{ij} + \delta_{ml} \delta_{ni} \delta_{pj} \partial_k Q_{ij}) \\ &= Q_{ml} \partial_l Q_{np} + Q_{km} \partial_k Q_{np} \\ &= 2Q_{mk} \partial_k Q_{np} \end{aligned} \quad (38)$$

Recall the definition of the elastic stress tensor, using indices as above:

$$\sigma_{mj}^e = - \frac{\partial f}{\partial (\partial_m Q_{np})} \partial_j Q_{np} \quad (39)$$

Substituting these (and the L_1 term) into the above equation yields:

$$\sigma_{mj}^e = -2 [L_1 (\partial_m Q_{np}) (\partial_j Q_{np}) + L_2 (\partial_k Q_{nk}) (\partial_j Q_{nm}) + L_3 Q_{mk} (\partial_k Q_{np}) (\partial_j Q_{np})] \quad (40)$$

Making the substitutions $m \rightarrow j$, $n \rightarrow k$, $p \rightarrow l$, $k \rightarrow m$, and $j \rightarrow i$ yields:

$$\sigma_{ji}^e = -2 [L_1 (\partial_j Q_{kl}) (\partial_i Q_{kl}) + L_2 (\partial_m Q_{km}) (\partial_i Q_{kj}) + L_3 Q_{jm} (\partial_m Q_{kl}) (\partial_i Q_{kl})] \quad (41)$$

Now we need to take the divergence of the two latter terms. For the second term this yields:

$$\partial_j (\partial_m Q_{km}) (\partial_i Q_{kj}) = (\partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) + (\partial_m Q_{km}) (\partial_j \partial_i Q_{kj}) \quad (42)$$

Third term:

$$\begin{aligned} \partial_j [Q_{jm} (\partial_m Q_{kl}) (\partial_i Q_{kl})] &= (\partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + Q_{jm} (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) + Q_{jm} (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl}) \end{aligned} \quad (43)$$

Taking the curl of the L_2 term yields:

$$\begin{aligned} \epsilon_{npi} \partial_p [(\partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) \\ + (\partial_m Q_{km}) (\partial_j \partial_i Q_{kj})] &= \epsilon_{npi} [(\partial_p \partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) + (\partial_j \partial_m Q_{km}) (\partial_p \partial_i Q_{kj}) \\ &\quad + (\partial_p \partial_m Q_{km}) (\partial_j \partial_i Q_{kj}) + (\partial_m Q_{km}) (\partial_j \partial_i \partial_p Q_{kj})] \\ &= \epsilon_{npi} (\partial_p \partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) \end{aligned} \quad (44)$$

where the last step follows from the fact that each term except the first is symmetric in i and p . Give that last expression a name (including the minus sign from (41)):

$$\Phi_{L_2}(Q) \equiv -2L_2 \epsilon_{npi} (\partial_p \partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) \quad (45)$$

Taking the curl of the L_3 term yields:

$$\begin{aligned} \epsilon_{npi} \partial_p [(\partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) \\ + Q_{jm} (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ + Q_{jm} (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl})] &= \epsilon_{npi} [(\partial_p \partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + (\partial_j Q_{jm}) (\partial_p \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + (\partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_p \partial_i Q_{kl}) \\ &\quad + (\partial_p Q_{jm}) (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + Q_{jm} (\partial_p \partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + Q_{jm} (\partial_j \partial_m Q_{kl}) (\partial_p \partial_i Q_{kl}) \\ &\quad + (\partial_p Q_{jm}) (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl}) \\ &\quad + Q_{jm} (\partial_p \partial_m Q_{kl}) (\partial_i \partial_j Q_{kl}) \\ &\quad + Q_{jm} (\partial_m Q_{kl}) (\partial_p \partial_i \partial_j Q_{kl})] \end{aligned} \quad (46)$$

Only a few of these terms are symmetric in p and i , namely terms 3, 6, and 9. But also note that, for the penultimate term:

$$\begin{aligned} \epsilon_{npi} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} &= -\epsilon_{nip} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} \\ &= -\epsilon_{nip} Q_{jm} \partial_i \partial_j Q_{kl} \partial_p \partial_m Q_{kl} \\ &= -\epsilon_{npi} Q_{mj} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} \\ &= -\epsilon_{npi} Q_{jm} \partial_p \partial_m Q_{kl} \partial_i \partial_j Q_{kl} \end{aligned} \quad (47)$$

where in the penultimate line we have relabeled lots of indices, and in the last line we have used the fact that Q_{mj} is symmetric. This then gives:

$$\begin{aligned} \epsilon_{npi} \partial_p [(\partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) \\ + Q_{jm} (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ + Q_{jm} (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl})] &= \epsilon_{npi} [(\partial_p \partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + (\partial_j Q_{jm}) (\partial_p \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + (\partial_p Q_{jm}) (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + Q_{jm} (\partial_p \partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) \\ &\quad + (\partial_p Q_{jm}) (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl})] \end{aligned} \quad (48)$$

Give this last expression a name (including the minus sign from (41)):

$$\begin{aligned}\Phi_{L_3}(Q) = & -2L_3\epsilon_{npi}\big[(\partial_p\partial_jQ_{jm})(\partial_mQ_{kl})(\partial_iQ_{kl}) \\ & + (\partial_jQ_{jm})(\partial_p\partial_mQ_{kl})(\partial_iQ_{kl}) \\ & + (\partial_pQ_{jm})(\partial_j\partial_mQ_{kl})(\partial_iQ_{kl}) \\ & + Q_{jm}(\partial_p\partial_j\partial_mQ_{kl})(\partial_iQ_{kl}) \\ & + (\partial_pQ_{jm})(\partial_mQ_{kl})(\partial_i\partial_jQ_{kl})\big]\end{aligned}\tag{49}$$

Note that in these equations, only the $n = 3$ term is nonzero since we have ∂_p and ∂_i terms in each term. Hence, if we have $p = 3$ or $i = 3$ then everything goes to zero. Thus, we can just use the 2D Levi-Civita. We may also non-dimensionalize these equations by dividing by $L = 2L_1$ and multiplying by ξ^4 . We will consider only the non-dimensionalized equations as follows.

9 Writing anisotropic elastic source terms using auxiliary variables

Plugging the expression for f_{L_2} into Sympy yields:

$$\begin{aligned}\Phi_{L_2} = & -\kappa_2\left[\frac{1}{3}(\partial_x\eta)\left[\sqrt{3}\partial_y^3\mu - \partial_y^3\eta - 4\partial_y\partial_x^2\eta - \sqrt{3}\partial_y^2\partial_x\nu\right] \right. \\ & + \frac{1}{3}(\partial_y\eta)\left[4\partial_x^3\eta + \partial_y^2\partial_x\eta - \sqrt{3}\partial_y^2\partial_x\mu + \sqrt{3}\partial_y\partial_x^2\nu\right] \\ & + (\partial_x\mu)\left[\frac{1}{\sqrt{3}}\partial_y^3\eta - \partial_y^3\mu - \partial_y^2\partial_x\nu\right] \\ & + (\partial_y\mu)\left[-\frac{1}{\sqrt{3}}\partial_y^2\partial_x\eta + \partial_y^2\partial_x\mu + \partial_y\partial_x^2\nu\right] \\ & - (\partial_x\nu)\left[\partial_y^3\nu + \frac{1}{\sqrt{3}}\partial_y^2\partial_x\eta + \partial_y^2\partial_x\mu + \partial_y\partial_x^2\nu\right] \\ & \left. + (\partial_y\nu)\left[\partial_x^3\nu + \frac{1}{\sqrt{3}}\partial_y\partial_x^2\eta + \partial_y\partial_x^2\mu + \partial_y^2\partial_x\nu\right]\right]\end{aligned}\tag{50}$$

where we have defined $\kappa_2 \equiv L_2/L_1$, and in the last line we have simplified a few terms. Now we may plug in for $\Phi_{L_3}(Q)$ in terms of the auxiliary variables:

$$\begin{aligned}
\Phi_{L_3}(Q) = & -\kappa_3 \left[\frac{2}{\sqrt{3}} \eta \left[(\partial_x \eta) (\partial_y^3 \eta - 2\partial_y \partial_x^2 \eta) + (\partial_y \eta) (2\partial_x^3 \eta - \partial_x \partial_y^2 \eta) \right. \right. \\
& + (\partial_x \mu) (\partial_y^3 \mu - 2\partial_y \partial_x^2 \mu) + (\partial_y \mu) (2\partial_x^3 \mu - \partial_x \partial_y^2 \mu) \\
& + (\partial_x \nu) (\partial_y^3 \nu - 2\partial_y \partial_x^2 \nu) + (\partial_y \nu) (2\partial_x^3 \nu - \partial_x \partial_y^2 \nu) \left. \right] \\
& + 2\mu \left[(\partial_y \eta) (\partial_y^2 \partial_x \eta) - (\partial_x \eta) (\partial_y^3 \eta) + (\partial_y \mu) (\partial_y^2 \partial_x \mu) \right. \\
& - (\partial_x \mu) (\partial_y^3 \mu) + (\partial_y \nu) (\partial_y^2 \partial_x \nu) - (\partial_x \nu) (\partial_y^3 \nu) \left. \right] \\
& + 4\nu \left[(\partial_y \eta) (\partial_y \partial_x^2 \eta) - (\partial_x \eta) (\partial_y^2 \partial_x \eta) + (\partial_y \mu) (\partial_y \partial_x^2 \mu) \right. \\
& - (\partial_x \mu) (\partial_y^2 \partial_x \mu) + (\partial_y \nu) (\partial_y \partial_x^2 \nu) - (\partial_x \nu) (\partial_y^2 \partial_x \nu) \left. \right] \\
& + \frac{2}{\sqrt{3}} \partial_x \eta \left[(\partial_y \eta) (2\partial_x^2 \eta + \partial_y^2 \eta - \sqrt{3} \partial_y^2 \mu) \right. \\
& + 2(\partial_y \mu) (2\partial_x^2 \mu - \partial_y^2 \mu - \sqrt{3} \partial_y^2 \eta) + 2(\partial_y \nu) (2\partial_x^2 \nu - \partial_y^2 \nu - 2\sqrt{3} \partial_x \partial_y \eta) \left. \right] \\
& + \frac{4}{\sqrt{3}} \partial_y \eta \left[\partial_x \mu (\sqrt{3} \partial_y^2 \eta - 2\partial_x^2 \mu + \partial_y^2 \mu) + \partial_x \nu (2\sqrt{3} \partial_x \partial_y \eta - 2\partial_x^2 \nu + \partial_y^2 \nu) \right. \\
& - 2 \left[((\partial_x \eta)^2 + (\partial_x \mu)^2 + (\partial_x \nu)^2) (\partial_y^2 \nu + \frac{2}{\sqrt{3}} \partial_y \partial_x \eta) \right] \\
& + 2 \left[((\partial_y \eta)^2 + (\partial_y \mu)^2 + (\partial_y \nu)^2) (\partial_x^2 \nu - \frac{1}{\sqrt{3}} \partial_y \partial_x \eta + \partial_y \partial_x \mu) \right] \\
& + \frac{2}{3} \left[(2\sqrt{3} \partial_x^2 \eta + \sqrt{3} \partial_y^2 \eta - 3\partial_y^2 \mu) (\partial_x \mu \partial_y \mu + \partial_x \nu \partial_y \nu) \right. \\
& \left. \left. + 4 \left[(\partial_y^2 \nu - 2\partial_y \partial_x \mu) (\partial_x \mu \partial_y \nu - \partial_y \mu \partial_x \nu) \right] \right] \right]
\end{aligned} \tag{51}$$

Note that, with these definitions, the anisotropic flow equation becomes:

$$\partial^4 \psi = \frac{1}{b} \left[2(\Phi_{L_1}(Q) + \kappa_2 \Phi_{L_2}(Q) + \kappa_3 \Phi_{L_3}(Q)) + a \Phi_{\mu_2}(Q) \right] \tag{52}$$

10 Anisotropic elastic forces

These forces are just given by the divergence of the corresponding terms in the elastic stress tensor. From section 8, we may read off that:

$$f_i^{L_2} = -2L_2 [(\partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) + (\partial_m Q_{km}) (\partial_j \partial_i Q_{kj})] \tag{53}$$

Define $\overline{f}_i^{L_2} \equiv \frac{\xi^3}{L} f_i^{L_2}$ and then drop the overline for brevity sake. then we get that:

$$f_i^{L_2} = -\kappa_2 [(\partial_j \partial_m Q_{km}) (\partial_i Q_{kj}) + (\partial_m Q_{km}) (\partial_j \partial_i Q_{kj})] \tag{54}$$

where $\kappa_2 \equiv L_2/L_1$. Plugging this into Sympy to get an expression in terms of the auxiliary variables yields:

$$\begin{aligned}
f_x^{L_2} = & -\kappa_2 \left[\frac{1}{3} (\partial_x \eta) [8\partial_x^2 \eta + \partial_y^2 \eta - \sqrt{3}\partial_y^2 \mu + 3\sqrt{3}\partial_y \partial_x \nu] \right. \\
& + \frac{1}{3} (\partial_y \eta) [-\sqrt{3}\partial_x^2 \nu + \partial_x \partial_y \eta - \sqrt{3}\partial_y \partial_x \mu] \\
& + (\partial_x \mu) [-\frac{1}{\sqrt{3}}\partial_y^2 \eta + \partial_y^2 \mu + \partial_x \partial_y \nu] \\
& + (\partial_y \mu) [\partial_x^2 \nu - \frac{1}{\sqrt{3}}\partial_x \partial_y \eta + \partial_x \partial_y \mu] \\
& + (\partial_x \nu) [2\partial_x^2 \nu + \partial_y^2 \nu + 2\partial_x \partial_y \mu] \\
& \left. + (\partial_y \nu) [\partial_x \partial_y \nu + \frac{2}{\sqrt{3}}\partial_x^2 \eta] \right] \\
f_y^{L_2} = & -\kappa_2 \left[\frac{2}{3} (\partial_x \eta) [\sqrt{3}\partial_y^2 \nu + 2\partial_x \partial_y \eta] \right. \\
& + \frac{2}{3} (\partial_y \eta) [2\partial_x^2 \eta + \partial_y^2 \eta - \sqrt{3}\partial_y^2 \mu] \\
& + 2 (\partial_y \mu) [\partial_y^2 \mu + \partial_x \partial_y \nu - \frac{1}{\sqrt{3}}\partial_y^2 \eta] \\
& + (\partial_x \nu) [\partial_y^2 \mu - \frac{1}{\sqrt{3}}\partial_y^2 \eta + \partial_x \partial_y \nu] \\
& \left. + (\partial_y \nu) [\partial_x^2 \nu + 2\partial_y^2 \nu + \sqrt{3}\partial_x \partial_y \eta + \partial_x \partial_y \mu] \right]
\end{aligned} \tag{55}$$

For the third anisotropic elastic force, we may read off:

$$f_i^{L_3} = -2L_3 [(\partial_j Q_{jm}) (\partial_m Q_{kl}) (\partial_i Q_{kl}) + Q_{jm} (\partial_j \partial_m Q_{kl}) (\partial_i Q_{kl}) + Q_{jm} (\partial_m Q_{kl}) (\partial_i \partial_j Q_{kl})] \tag{56}$$

We may define the dimensionless force in the same way as above. Then the third anisotropic elastic force is given explicitly by:

$$\begin{aligned}
f_x^{L_3} = & -\kappa_3 \left[\frac{2}{\sqrt{3}} \eta \left[(\partial_x \eta) (4\partial_x^2 \eta - \partial_y^2 \eta) + (\partial_x \mu) (4\partial_x^2 \mu - \partial_y^2 \mu) + (\partial_x \nu) (4\partial_x^2 \nu - \partial_y^2 \nu) \right. \right. \\
& \quad \left. \left. - \partial_y \mu \partial_x \partial_y \mu - \partial_y \eta \partial_x \partial_y \eta - \partial_y \nu \partial_x \partial_y \nu \right] \right. \\
& \quad + 2\mu \left[\partial_x \eta \partial_y^2 \eta + \partial_y \eta \partial_x \partial_y \eta + \partial_x \mu \partial_y^2 \mu + \partial_y \mu \partial_x \partial_y \mu + \partial_x \nu \partial_y^2 \nu + \partial_y \nu \partial_x \partial_y \nu \right] \\
& \quad + 2\nu \left[3\partial_x \eta \partial_x \partial_y \eta + \partial_x^2 \eta \partial_y \eta + 3\partial_x \mu \partial_x \partial_y \mu + \partial_x^2 \mu \partial_y \mu + 3\partial_x \nu \partial_x \partial_y \nu + \partial_x^2 \nu \partial_y \nu \right] \\
& \quad + 2(\partial_x \eta) \left[(\partial_x \eta) \left(\frac{2}{\sqrt{3}} (\partial_x \eta) + \partial_y \nu \right) + (\partial_y \eta) \left(\partial_y \mu + \partial_x \nu - \frac{1}{\sqrt{3}} (\partial_y \eta) \right) + \frac{2}{\sqrt{3}} (\partial_x \mu)^2 + \frac{2}{\sqrt{3}} (\partial_x \nu)^2 \right] \\
& \quad \left. - \frac{2}{\sqrt{3}} (\partial_y \eta) \left[\partial_x \mu \partial_y \mu + \partial_x \nu \partial_y \nu \right] + 2(\partial_y \nu) \left[(\partial_x \mu)^2 + \partial_y \mu \partial_x \nu + 2(\partial_x \nu)^2 \right] + 2(\partial_x \mu) (\partial_y \mu) \left[\partial_y \mu + \partial_x \nu \right] \right] \\
f_y^{L_3} = & -\kappa_3 \left[\frac{4}{\sqrt{3}} \eta \left[(\partial_y \eta) (\partial_x^2 \eta - \partial_y^2 \eta) + (\partial_y \mu) (\partial_x^2 \mu - \partial_y^2 \mu) + (\partial_y \nu) (\partial_x^2 \nu - \partial_y^2 \nu) \right. \right. \\
& \quad \left. \left. + \partial_x \eta \partial_x \partial_y \eta + \partial_x \mu \partial_x \partial_y \mu + \partial_x \nu \partial_x \partial_y \nu \right] \right. \\
& \quad \left. + 4\mu \left[\partial_y \eta \partial_y^2 \eta + \partial_y \mu \partial_y^2 \mu + \partial_y \nu \partial_y^2 \nu \right] \right. \\
& \quad 2\nu \left[\partial_x \eta \partial_y^2 \eta + 3\partial_y \eta \partial_x \partial_y \eta + \partial_x \mu \partial_y^2 \mu + 3\partial_y \mu \partial_x \partial_y \mu + \partial_x \nu \partial_y^2 \nu + 3\partial_y \nu \partial_x \partial_y \nu \right] \\
& \quad + \frac{4}{\sqrt{3}} (\partial_x \eta) \left[(\partial_x \eta) \partial_y \eta + \frac{\sqrt{3}}{2} \partial_y \eta \partial_y \nu + \partial_x \mu \partial_y \mu + \partial_x \nu \partial_y \nu \right] \\
& \quad 2(\partial_y \eta) \left[(\partial_y \eta) \left(\partial_y \mu + \partial_x \nu - \frac{1}{\sqrt{3}} (\partial_y \eta) \right) - \frac{1}{\sqrt{3}} (\partial_y \mu)^2 - \frac{1}{\sqrt{3}} (\partial_y \nu)^2 \right] \\
& \quad \left. + 2(\partial_y \mu) \left[\partial_x \mu \partial_y \nu + (\partial_y \mu)^2 + (\partial_y \mu) \partial_x \nu + (\partial_y \nu)^2 \right] + 4\partial_x \nu (\partial_y \nu)^2 \right]
\end{aligned} \tag{57}$$