LC Flow Velocity Eqs

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1 Elastic generalized force

From Svensek and Zumer, the free energy density is given by

$$f = \phi(Q) + \frac{1}{2} L \partial_i Q_{jk} \partial_i Q_{jk} \tag{1}$$

so that the homogeneous elastic part of the generalized force is given by:

$$h_{ij}^{he} = L \partial_k^2 Q_{ij} - \frac{\partial \phi}{\partial Q_{ij}} + \lambda \delta_{ij} + \lambda_k \epsilon_{kij} \tag{2}$$

where ϕ is the Landau de Gennes bulk free energy:

$$\phi(Q) = \frac{1}{2} A Q_{ij} Q_{ji} + \frac{1}{3} B Q_{ij} Q_{jk} Q_{ki} + \frac{1}{4} C (Q_{ij} Q_{ji})^2 \tag{3}$$

We begin by explicitly calculating the second term in terms of Q_{ij}, one term at a time in ϕ:

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{2} A Q_{ij} Q_{ji} \right) &= \frac{1}{2} A \left(\delta_{im} \delta_{jn} Q_{ji} + Q_{ij} \delta_{jm} \delta_{in} \right) \\ &= \frac{1}{2} A (Q_{nm} + Q_{nm}) \\ &= A Q_{mn} \end{aligned} \tag{4}$$

where in the last step we've used symmetry of Q_{ij}.

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{3} B Q_{ij} Q_{jk} Q_{ki} \right) &= \frac{1}{3} B (\delta_{im} \delta_{jn} Q_{jk} Q_{ki} + Q_{ij} \delta_{jm} \delta_{kn} Q_{ki} + Q_{ij} Q_{jk} \delta_{mk} \delta_{ni}) \\ &= \frac{1}{3} B (Q_{nk} Q_{km} + Q_{im} Q_{ni} + Q_{nj} Q_{jm}) \\ &= B Q_{ni} Q_{im} \end{aligned} \tag{5}$$

And the final term gives:

$$\begin{aligned} \frac{\partial}{\partial Q_{mn}} \left(\frac{1}{4} C (Q_{ij} Q_{ji})^2 \right) &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \frac{\partial (Q_{kl} Q_{lk})}{\partial Q_{mn}} \\ &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \cdot (\delta_{mk} \delta_{nl} Q_{lk} + Q_{kl} \delta_{lm} \delta_{kn}) \\ &= \frac{1}{4} C \cdot 2 (Q_{ij} Q_{ji}) \cdot (Q_{nm} + Q_{nm}) \\ &= C Q_{mn} (Q_{ij} Q_{ji}) \end{aligned} \tag{6}$$

Thus, the total homogeneous and elastic force reads:

$$h_{ij}^{he} = L \partial^2 Q_{ij} - A Q_{ij} - B Q_{ik} Q_{kj} - C Q_{ij} (Q_{kl} Q_{lk}) + \lambda \delta_{ij} + \lambda_k \epsilon_{kij} \tag{7}$$

2 Viscous generalized force

Now we need the viscous force on the liquid crystals. From Svensek and Zumer, the viscous force is given by:

$$-h^v_{ij} = \frac{1}{2}\mu_2 A_{ij} + \mu_1 N_{ij} \tag{8}$$

with

$$N_{ij} = \frac{dQ_{ij}}{dt} + W_{ik}Q_{kj} - Q_{ik}W_{kj} \tag{9}$$

and

$$\frac{dQ_{ij}}{dt} = \frac{\partial Q_{ij}}{\partial t} + (v\cdot\nabla)Q_{ij} \tag{10}$$

where

$$W_{ij} = \tfrac{1}{2}\left(\partial_i v_j - \partial_j v_i\right) \tag{11}$$

The second two terms in the expression for N_{ij} are quadratic in v_i and Q_{ij} so we may drop them, and $(v\cdot\nabla)Q_{ij}$ is clearly quadratic. Hence, we make the approximation

$$N_{ij} \approx \frac{\partial Q_{ij}}{\partial t} \tag{12}$$

We also have the definition

$$A_{ij} = (\partial_i v_j + \partial_j v_i) \tag{13}$$

Plugging this into the viscous force yields:

$$-h^v_{ij} = \frac{1}{2}\mu_2(\partial_i v_j + \partial_j v_i) + \mu_1\frac{\partial Q_{ij}}{\partial t} \tag{14}$$

Balancing the forces gives the equation:

$$h^e_{ij} = -h^v_{ij} \tag{15}$$

Which yields:

$$\mu_1\frac{\partial Q_{ij}}{\partial t} = L\partial^2 Q_{ij} - AQ_{ij} - BQ_{ik}Q_{kj} - CQ_{ij}\left(Q_{kl}Q_{lk}\right) - \frac{1}{2}\mu_2\left(\partial_i v_j + \partial_j v_i\right) \tag{16}$$

(here we have dropped the Lagrange multipliers because we are going to explicitly ensure that Q_{ij} is traceless and symmetric).

3 Elastic stress tensor

The elastic stress tensor is obtained via

$$\sigma^e_{ij} = -\frac{\partial f}{\partial(\partial_i Q_{kl})}\partial_j Q_{kl} \tag{17}$$

Note that only the elastic part of the free energy make references to derivatives:

$$\begin{aligned}\frac{\partial f}{\partial(\partial_i Q_{kl})} &= \frac{\partial}{\partial(\partial_i Q_{kl})}\frac{1}{2}L\partial_j Q_{mn}\partial_j Q_{mn} \\ &= \frac{1}{2}L(\delta_{ij}\delta_{km}\delta_{ln}\partial_j Q_{mn} + \partial_j Q_{mn}\delta_{ij}\delta_{km}\delta_{ln}) \\ &= \frac{1}{2}L(\partial_i Q_{kl} + \partial_i Q_{kl}) \\ &= L\partial_i Q_{kl}\end{aligned} \tag{18}$$

Then the elastic stress tensor is given by

$$\sigma^e_{ij} = -L\partial_i Q_{kl}\partial_j Q_{kl} \tag{19}$$

4 Viscous stress tensor

The viscous stress tensor is given by

$$\sigma_{ij}^v = \beta_1 Q_{ij} Q_{kl} A_{kl} + \beta_4 A_{ij} + \beta_5 Q_{ik} A_{ki} + \frac{1}{2}\mu_2 N_{ij} - \mu_1 Q_{ik} N_{kj} + \mu_1 Q_{jk} N_{ki} \tag{20}$$

However, only the β_4 and μ_2 are linear in Q_{ij} and v_i. Hence, this simplifies to

$$\sigma_{ij}^v \approx \beta_4 A_{ij} + \frac{1}{2}\mu_2 N_{ij} \tag{21}$$

Again, plugging in for A_{ij} and N_{ij} as we did for the viscous force, we get

$$\sigma_{ij}^v \approx \beta_4 (\partial_i v_j + \partial_j v_i) + \frac{1}{2}\mu_2 \frac{\partial Q_{ij}}{\partial t} \tag{22}$$

Plugging in for the time evolution of Q_{ij} gives:

$$\sigma_{ij}^v \approx \beta_4 \left(\partial_i v_j + \partial_j v_i\right) + \frac{1}{2}\frac{\mu_2}{\mu_1}\left[L\partial^2 Q_{ij} - A Q_{ij} - B Q_{ik} Q_{kj} - C Q_{ij}\left(Q_{kl} Q_{lk}\right) - \frac{1}{2}\mu_2\left(\partial_i v_j + \partial_j v_i\right)\right] \tag{23}$$

5 Fluid equation of motion

The equation of motion for the fluid is given by

$$\rho \frac{\partial v_i}{\partial t} = -\partial_i p + \partial_j (\sigma_{ji}^v + \sigma_{ji}^e) \tag{24}$$

We can make the assumption that $\partial v_i / \partial t \approx 0$. Using this and plugging in for σ_{ji}^v and σ_{ji}^e, we get:

$$\begin{aligned}\partial_i p = \left(\beta_4 - \frac{1}{4}\frac{\mu_2^2}{\mu_1}\right)\left(\partial_i \partial_j v_j + \partial^2 v_i\right) \\ + \frac{1}{2}\frac{\mu_2}{\mu_1}\left[L\partial^2 \partial_j Q_{ij} - A\partial_j Q_{ij} - B\partial_j\left(Q_{ik} Q_{kj}\right) - C\partial_j\left[Q_{ij}\left(Q_{kl} Q_{lk}\right)\right]\right] \\ - L\partial_j\left(\partial_i Q_{kl} \partial_j Q_{kl}\right)\end{aligned} \tag{25}$$

Note that, by incompressibility, $\partial_i v_i = 0$ so a term in parentheses goes away. Now define:

$$f_{\mu_2,i}(Q) = L\partial^2 \partial_j Q_{ij} - A\partial_j Q_{ij} - B\partial_j\left(Q_{ik} Q_{kj}\right) - C\partial_j\left[Q_{ij}\left(Q_{kl} Q_{lk}\right)\right] \tag{26}$$

$$f_{L,i}(Q) = \partial_j\left(\partial_i Q_{kl} \partial_j Q_{kl}\right) \tag{27}$$

$$f_i(Q) = -\frac{1}{2}\frac{\mu_2}{\mu_1} f_{\mu_2,i}(Q) + L f_{L,i}(Q) \tag{28}$$

$$\alpha = \left(\beta_4 - \frac{1}{4}\frac{\mu_2^2}{\mu_1}\right)^{-1} \tag{29}$$

Then the fluid equation of motion becomes:

$$\partial^2 v_i = \alpha\left(\partial_i p + f_i\right) \tag{30}$$

6 Solving the fluid equation of motion with periodic boundary conditions

In order to solve this, we'll assume periodic boundary conditions. To actually use this to check our stream function formulation, we'll have to choose f_i such that v_i turns out to be zero along the boundaries. Given periodic boundary conditions, we may write:

$$v_i(x,y) = \sum_{k_x,k_y} \hat{v}_{i,k_xk_y} e^{i(k_xx+k_yy)} \tag{31}$$

$$f_i(x,y) = \sum_{k_x,k_y} \hat{f}_{i,k_xk_y} e^{i(k_xx+k_yy)} \tag{32}$$

$$p(x,y) = \sum_{k_x,k_y} \hat{p}_{k_xk_y} e^{i(k_xx+k_yy)} \tag{33}$$

where $k_x = 2\pi m/L_x$ and $k_y = 2\pi n/L_y$ with $m = ..., -1, 0, 1, ...$ and $n = ..., -1, 0, 1, ...$ Plugging this into the equation of motion yields:

$$\sum_{k_x,k_y} -\left(k_x^2+k_y^2\right)\hat{v}_{i,k_xk_y} = \alpha\left(\sum_{k_x,k_y} ik_i\hat{p}_{k_xk_y}e^{i(k_xx+k_yy)} + \sum_{k_x,k_y}\hat{f}_{i,k_xk_y}e^{i(k_xx+k_yy)}\right) \tag{34}$$

By orthogonality of the exponentials, we get that:

$$\begin{aligned} -\left(k_x^2+k_y^2\right)\hat{v}_{i,k_xk_y} &= \alpha\left(ik_i\hat{p}_{k_xk_y}e^{i(k_xx+k_yy)} + \hat{f}_{i,k_xk_y}e^{i(k_xx+k_yy)}\right) \\ \implies \hat{v}_{i,k_xk_y} &= \frac{\alpha\left(ik_i\hat{p}_{k_xk_y}+\hat{f}_{i,k_xk_y}\right)}{-\left(k_x^2+k_y^2\right)} \end{aligned} \tag{35}$$

for all k_x and k_y. In order to get rid of the contribution from pressure, we need to take the divergence of the original equation. This yields:

$$\partial^2\partial_i v_i = \alpha\left(\partial^2 p + \partial_i f_i\right) = 0 \tag{36}$$

This is just a Poisson equation in p with periodic boundary conditions. Again rewriting as a Fourier series and using the orthogonality condition, we get:

$$\begin{aligned} -\left(k_x^2+k_y^2\right)\hat{p}_{k_xk_y} &= -ik_i\hat{f}_{i,k_xk_y} \\ \implies \hat{p}_{k_xk_y} &= \frac{ik_i}{k_x^2+k_y^2}\hat{f}_{i,k_xk_y} \end{aligned} \tag{37}$$

Plugging this back into the original equation yields:

$$v_i(x,y) = \sum_{k_x,k_y}\frac{\alpha}{k^2}\left(\frac{k_ik_j\hat{f}_{j,k_xk_y}}{k^2} - \hat{f}_{i,k_xk_y}\right)e^{i(k_xx+k_yy)} \tag{38}$$

7 Choosing a specific Q_{ij}

Now, we would like to choose f_i to be something which is just a sum of sines. These terminate at the endpoints, so it happens that we will fulfill our no slip condition ($v_i = 0$ at the boundary). Since we need to actually choose Q_{ij} to do that (so that we can use the same scenario for the stream function formulation), we will have to just guess and check. Note that many of the terms involved in f_i have third derivatives, so we will want to choose a cos to be the important term. Recall that, in the uniaxial case:

$$Q_{ij} = \frac{S}{2}\left(3n_in_j - \delta_{ij}\right) \tag{39}$$

Choose S to be constant. Plugging Q_{ij} into the expression for $f_{\mu_2,i}$ yields:

$$\begin{aligned} f_{\mu_2,i}(Q) = \frac{3S}{2}L\partial^2\partial_j\left(n_in_j\right) - \frac{3S}{2}A\partial_j\left(n_in_j\right) - \frac{S^2}{4}B\partial_j\left(3n_in_k - \delta_{ik}\right)\left(3n_kn_j - \delta_{kj}\right) \\ - \frac{S^3}{8}C\partial_j\left[\left(3n_in_j - \delta_{ij}\right)\left(3n_kn_l - \delta_{kl}\right)\left(3n_ln_k - \delta_{lk}\right)\right] \end{aligned} \tag{40}$$

This simplifies to:

$$\begin{aligned} f_{\mu_2,i}(Q) &= \frac{3S}{2}L\partial^2\partial_j\left(n_in_j\right) - \frac{3S}{2}A\partial_j\left(n_in_j\right) - \frac{3S^2}{2}B\partial_j\left(n_in_j\right) - \frac{9S^3}{4}C\partial_j\left(n_in_j\right) \\ &= \frac{3S}{2}\left(L\partial^2 - A - SB - \frac{3S^2}{2}C\right)\partial_j\left(n_in_j\right) \end{aligned} \tag{41}$$

Do the same for $f_{L,i}$ to get:

$$f_{L,i}(Q) = \frac{9S^2}{4}\partial_j\left[\partial_i\left(n_kn_l\right)\partial_j\left(n_ln_k\right)\right] \tag{42}$$

Our first guess will be:

$$\hat{n} = \left(\epsilon\cos k_x'x, 1, 0\right) \tag{43}$$

for some small ϵ. Note that this expression is independent of y so that any y-derivatives will be zero by default. We will do pieces at a time:

$$\begin{aligned} \partial_j\left(n_xn_j\right) &= \partial_x\epsilon^2\cos^2 k_x'x \\ &= 2\epsilon^2\cos k_x'x\left(-k_x'\sin k_x'x\right) \\ &= -k_x'\epsilon^2\sin\left(2k_x'x\right) \end{aligned} \tag{44}$$

Great, this still terminates at the endpoints. Now taking the Laplacian of this:

$$\begin{aligned} \partial^2\partial_j\left(n_xn_j\right) &= \partial_x^2\left(-k_x'\epsilon^2\sin\left(2k_x'x\right)\right) \\ &= -2k_x'^2\epsilon^2\partial_x\cos\left(2k_x'x\right) \\ &= 4k_x'^3\epsilon^2\sin\left(2k_x'x\right) \end{aligned} \tag{45}$$

Cool, this also terminates at the endpoints. So far so good. Now for the y-component:

$$\begin{aligned} \partial^2\partial_j\left(n_yn_j\right) &= \partial_x^2\partial_x\left(\epsilon\cos k_x'x\right) \\ &= \partial_x^2\left(-k_x'\epsilon\sin k_x'x\right) \\ &= \partial_x\left(-k_x'^2\epsilon\cos k_x'x\right) \\ &= k_x'^3\epsilon\sin k_x'x \end{aligned} \tag{46}$$

Now for the last term:

$$\begin{aligned} \partial_j\left[\partial_x\left(n_kn_l\right)\partial_j\left(n_ln_k\right)\right] &= \partial_x\left[\left(\partial_x\epsilon^2\cos^2 k_x'x\right)^2 + \left(\partial_x\epsilon\cos k_x'x\right)^2 + \left(\partial_x\epsilon\cos k_x'x\right)^2\right] \\ &= \partial_x\left[k_x'^2\epsilon^4\sin^2\left(2k_x'x\right) + 2k_x'^2\epsilon^2\sin^2\left(k_x'x\right)\right] \\ &= 2k_x'^2\epsilon^4\sin\left(2k_x'x\right)\cos\left(2k_x'x\right)2k_x' + 4k_x'^2\epsilon^2\sin\left(k_x'x\right)\cos\left(k_x'x\right)k_x' \\ &= 2k_x'^3\epsilon^4\sin\left(4k_x'x\right) + 2k_x'^2\epsilon^2\sin\left(2k_x'x\right) \end{aligned} \tag{47}$$

This also terminates at the endpoints, so it works with our no-slip condition. Note that, again, there is no y-component because the director field is independent of y.

7.1 Nondimensionalizing velocity equations of motion

We want to nondimensionalize the velocity field in the same way that we nondimmensionalized the stream function. Note that the velocity is derived from the stream function as

$$v_i = \epsilon_{ij} \partial_j \psi \tag{48}$$

Now we will rewrite ψ in terms of $\overline{\psi}$ and rewrite $\partial / \partial x_j$ in terms of $\partial / \partial \overline{x}_j$. This yields:

$$v_i = \frac{1}{\xi} \epsilon_{ij} \partial_j \frac{\xi^2}{\tau} \overline{\psi} = \frac{L}{\mu_1 \xi} \epsilon_{ij} \partial_j \overline{\psi} \tag{49}$$

From this, we define the dimensionless variable $\overline{v}_i$ to be

$$\overline{v}_i = \frac{\mu_1 \xi}{L} v_i \tag{50}$$

We can plug this back into the Poisson equation for the velocity to yield:

$$\begin{aligned}
\frac{L}{\mu_1 \xi^3} \partial^2 \overline{v}_i &= \alpha \left(\frac{1}{\xi} \partial_i p + f_i \right) \\
\implies \partial^2 \overline{v}_i &= \frac{\alpha \mu_1 \xi^3}{L} \left(\frac{1}{\xi} \partial_i p + f_i \right)
\end{aligned} \tag{51}$$

We may define $\overline{\alpha} = \mu_1 \alpha$. Additionally, we define:

$$\overline{p} \equiv \frac{\xi^2}{L} p, \quad \overline{f}_i \equiv \frac{\xi^3}{L} f_i \tag{52}$$

Let's find explicit expressions for $\overline{f}_i$ and $\overline{p}$. From (28) we get:

$$\begin{aligned}
\overline{f}_i &= \frac{\xi^3}{L} f_i \\
&= \frac{\xi^3}{L} \left(-\frac{1}{2} \frac{\mu_2}{\mu_1} f_{\mu_2, i}(Q) \right) + \frac{\xi^3}{L} L f_{L,i}(Q)
\end{aligned} \tag{53}$$

Going term by term and using (26) and (27) yields:

$$\begin{aligned}
\frac{\xi^3}{L} f_{\mu_2, i}(Q) &= \frac{\xi^3}{L} \left(L \partial^2 \partial_j Q_{ij} - A \partial_j Q_{ij} - B \partial_j \left(Q_{ik} Q_{kj} \right) - C \partial_j \left[Q_{ij} \left(Q_{kl} Q_{lk} \right) \right] \right) \\
&= \partial^2 \partial_j Q_{ij} - \frac{A \xi^2}{L} \partial_j Q_{ij} - \frac{B \xi^2}{L} \partial_j \left(Q_{ik} Q_{kj} \right) - \frac{C \xi^2}{L} \partial_j \left[Q_{ij} \left(Q_{kl} Q_{lk} \right) \right] \\
&= \partial^2 \partial_j Q_{ij} - \overline{A} \partial_j Q_{ij} - \overline{B} \partial_j \left(Q_{ik} Q_{kj} \right) - \overline{C} \partial_j \left[Q_{ij} \left(Q_{kl} Q_{lk} \right) \right] \\
&\equiv \overline{f}_{\mu_2, i}(Q)
\end{aligned} \tag{54}$$

Where in the second equality we have invoked the dimensionless derivatives. Additionally, we get

$$\begin{aligned}
\xi^3 f_{L,i}(Q) &= \xi^3 \partial_j \left(\partial_i Q_{kl} \partial_j Q_{kl} \right) \\
&= \partial_j \left(\partial_i Q_{kl} \partial_j Q_{kl} \right) \\
&\equiv \overline{f}_{L,i}
\end{aligned} \tag{55}$$

where in the penultimate step we have invoked the dimensionless derivatives. Hence, we arrive at:

$$\overline{f}_i = -\tfrac{1}{2} a \overline{f}_{\mu_2, i} + \overline{f}_{L,i} \tag{56}$$

where we have defined $a = \mu_2/\mu_1$. Now for p. Since the only explicit expression we have for p is in terms of Fourier components, define $\overline{k}_x \equiv k_x/\xi$ and $\overline{k}_y \equiv k_y/\xi$. Then (38) becomes:

$$
\begin{aligned}
\overline{p} &= \frac{\xi^2}{L} p \\
&= \sum_{\overline{k}_x \overline{k}_y} \frac{-i\overline{k}_i}{\overline{k}_x^2 + \overline{k}_y^2} \xi \frac{\xi^2}{L} \hat{f}_{i,\overline{k}_x \overline{k}_y} e^{i\left(\overline{k}_x \overline{x} + \overline{k}_y \overline{y}\right)} \\
&= \sum_{\overline{k}_x \overline{k}_y} \frac{-i\overline{k}_i}{\overline{k}_x^2 + \overline{k}_y^2} \hat{\overline{f}}_{i,\overline{k}_x \overline{k}_y} e^{i\left(\overline{k}_x \overline{x} + \overline{k}_y \overline{y}\right)}
\end{aligned} \tag{57}
$$

where we have defined

$$
\hat{\overline{f}}_{i,\overline{k}_x \overline{k}_y} \equiv \frac{\xi^3}{L} \hat{f}_{i,\overline{k}_x \overline{k}_y} \tag{58}
$$

Let's check consistency with the definition of $\overline{f}_i$:

$$
\begin{aligned}
\overline{f}_i &= \frac{\xi^3}{L} f_i \\
&= \frac{\xi^3}{L} \sum_{k_x k_y} \hat{f}_{i,k_x k_y} e^{i(k_x x + k_y y)} \\
&= \sum_{\overline{k}_x \overline{k}_y} \hat{\overline{f}}_{i,\overline{k}_x \overline{k}_y} e^{i\left(\overline{k}_x \overline{x} + \overline{k}_y \overline{y}\right)}
\end{aligned} \tag{59}
$$

Great, everything works out. Then, from above (with our chosen configuration of $\hat{n}$ and fixed S) we get

$$
\begin{aligned}
\overline{f}_x &= -\tfrac{3}{4} a \overline{k}_x \epsilon^2 S \left(4\overline{k}_x'^2 + \overline{A} + S\overline{B} + \frac{3S^2}{2}\overline{C}\right) \sin\left(2\overline{k}_x' \overline{x}\right) + \frac{9S^2}{2} \overline{k}_x'^2 \epsilon^2 \left(\overline{k}_x' \epsilon^2 \sin\left(4\overline{k}_x' \overline{x}\right) + \sin\left(\overline{k}_x' \overline{x}\right)\right) \\
\overline{f}_y &= -a \tfrac{3}{4} \overline{k}_x' \epsilon S \left(\overline{A} + S\overline{B} + \frac{3S^2}{2}\overline{C} + \overline{k}_x'^2\right) \sin \overline{k}_x' \overline{x}
\end{aligned} \tag{60}
$$

In this case, we clearly have that $\hat{\overline{f}}_{x,\overline{k}_x \overline{k}_y} = 0$ for $\overline{k}_y \neq 0$. Additionally, since $\overline{f}_x$ only involves sums of sines, we know that all the Fourier components are purely imaginary. Explicitly:

$$
\begin{aligned}
\hat{\overline{f}}_{x,\pm\overline{k}_x',0} &= \pm \frac{9S^2}{4i} \overline{k}_x'^2 \epsilon^2 \\
\hat{\overline{f}}_{x,\pm 2\overline{k}_x',0} &= \mp \frac{3}{8i} a \overline{k}_x' \epsilon^2 S \left(4\overline{k}_x'^2 + \overline{A} + S\overline{B} + \frac{3S^2}{2}\overline{C}\right) \\
\hat{\overline{f}}_{x,\pm 4\overline{k}_x',0} &= \pm \frac{9S^2}{4i} \overline{k}_x'^3 \epsilon^4 \\
\hat{\overline{f}}_{y,\pm\overline{k}_x',0} &= \mp \frac{3}{8i} a \overline{k}_x' \epsilon S \left(\overline{A} + S\overline{B} + \frac{3S^2}{2}\overline{C} + \overline{k}_x'^2\right) \sin \overline{k}_x' \overline{x}
\end{aligned} \tag{61}
$$

Now, since $\overline{f}_y = 0$ and $\hat{\overline{f}}_{x,\overline{k}_x \overline{k}_y} = 0$ for all $\overline{k}_y \neq 0$ then (39) reduces to:

$$
\begin{aligned}
v_x &= \sum_{\overline{k}_x} \frac{\overline{\alpha}}{\overline{k}_x^2} \left(\hat{\overline{f}}_{x,\overline{k}_x,0} - \hat{\overline{f}}_{x,\overline{k}_x,0}\right) e^{i\overline{k}_x \overline{x}} = 0 \\
v_y &= \sum_{\overline{k}_x} \frac{\overline{\alpha}}{\overline{k}^2} \hat{\overline{f}}_{y,\overline{k}_x,0} e^{i\overline{k}_x \overline{x}}
\end{aligned} \tag{62}
$$

Plugging in yields:

$$\begin{aligned} v_x &= 0 \\ v_y &= -\frac{3\overline{\alpha} a S}{4\overline{k}'_x}\left(\overline{A} + S\overline{B} + \frac{3S^2}{2}\overline{C} + \overline{k}'^2_x\right)\epsilon \sin \overline{k}'_x \overline{x} \end{aligned} \tag{63}$$

Now we should go back to make sure that $\partial_i \overline{v}_i = 0$ and that the original velocity equation is satisfied. For this, we will need to explicitly write out what $\overline{p}$ is.