Dzyaloshinskii integral perturbation

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The Dzyaloshinskii solution is given by:

$$\varphi = p \int_0^{\theta - \varphi} \sqrt{\frac{1 + \epsilon \cos 2x}{1 + p^2 \epsilon \cos 2x}} dx \tag{1}$$

with $p^2 < 1/|\epsilon|$ and is defined so that θ is single-valued:

$$\pi = (q-1) p \int_0^{\pi} \sqrt{\frac{1 + \epsilon \cos 2x}{1 + p^2 \epsilon \cos 2x}} dx \tag{2}$$

Now take $\mu = \theta - \varphi$:

$$\varphi = p \int_0^{\mu} \sqrt{\frac{1 + \epsilon \cos 2x}{1 + p^2 \epsilon \cos 2x}} dx \tag{3}$$

Then the fundamental theorem of calculus gives:

$$\frac{d\varphi}{d\mu} = p \sqrt{\frac{1 + \epsilon \cos 2\mu}{1 + p^2 \epsilon \cos 2\mu}} \tag{4}$$

For $|\epsilon| < 1$ we have that $\frac{d\varphi}{d\mu} \neq 0$. If $|\epsilon| = 1$ the solution is a step function which is well-known and may be handled separately, so we take $|\epsilon| < 1$. Then the inverse function theorem gives us:

$$\frac{d\mu}{d\varphi} = \frac{1}{p} \sqrt{\frac{1 + p^2 \epsilon \cos 2\mu}{1 + \epsilon \cos 2\mu}} \tag{5}$$

We may perturbatively expand θ as:

$$\theta = q\varphi + \epsilon \theta_c + \mathcal{O}(\epsilon^2) \tag{6}$$

so that μ is given by:

$$\mu = m\varphi + \epsilon \theta_c + \mathcal{O}(\epsilon^2) \tag{7}$$

with $m = q - 1$. Then we may substitute into (5) and expand to get:

$$\frac{d\theta_c}{d\varphi} = \frac{1 - mp}{\epsilon p} - \frac{p^2 - 1}{2p} \cos 2m\phi \tag{8}$$

The solution is then:

$$\theta_c = \frac{1 - mp}{\epsilon p} \varphi - \frac{p^2 - 1}{4mp} \sin 2m\phi \tag{9}$$

To find p we enforce that $\theta_c(0) = \theta_c(2\pi) = 0$. This yields:

$$p = \frac{1}{m} \tag{10}$$

Plugging this back in for θ_c yields:

$$\theta_c = \frac{q(2-1)}{4(1-q)^2} \sin 2(1-q)\varphi \tag{11}$$