Double helix director

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1 Selinger free energy

To investigate this system, we start with the Frank free energy as written by Selinger:

$$F = \frac{1}{2}\left(K_{11} - K_{24}\right) S^2 + \frac{1}{2}\left(K_{22} - K_{24}\right) T^2 + \frac{1}{2} K_{33} \left|\mathbf{B}\right|^2 + K_{24} \text{Tr}\left(\Delta^2\right) \tag{1}$$

with the following definitions for the distortion modes:

$$\begin{aligned} S &= \nabla \cdot \hat{\mathbf{n}} && (2) \\ T &= \hat{\mathbf{n}} \cdot \left(\nabla \times \hat{\mathbf{n}}\right) && (3) \\ B &= \hat{\mathbf{n}} \times \left(\nabla \times \hat{\mathbf{n}}\right) && (4) \\ \Delta_{ij} &= \frac{1}{2}\big[\partial_i n_j + \partial_j n_i && (5) \\ &\qquad - n_i n_k \partial_k n_j - n_j n_k \partial_k n_i \\ &\qquad\; - \delta_{ij} \partial_k n_k + n_i n_j \partial_k n_k\big] \end{aligned}$$

2 Rotated system

We make two assumptions about the system: i) the z-dependence of the director corresponds to a rotation of a plane perpendicular to the cylindrical axis about the cylindrical axis by some angle αz, and ii) the director stays in a plane perpendicular to the cylindrical axis. We note that in an infinitely long cylindrical system, i) is true by translational symmetry. In this case, we may write:

$$\hat{\mathbf{n}} = R(\alpha z)\begin{bmatrix}\cos\theta & \sin\theta & 0\end{bmatrix}^T \tag{6}$$

for some director angle θ as measured from the x-axis in the x-y-plane, and $R(z)$ a rotation about the z-axis and a function of z. We note that θ must be a function of x, y, and z, with the z-dependence corresponding to an *inverse* rotation of angle αz about the z-axis. [1] This gives:

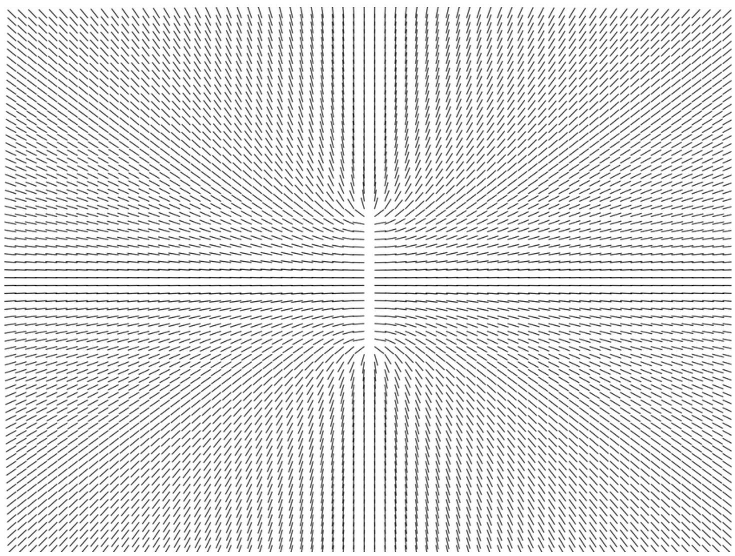
$$\begin{aligned} \begin{bmatrix} x' \\ y' \end{bmatrix} &= R^T(\alpha z)\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha z)\, x + \sin(\alpha z)\, y \\ -\sin(\alpha z)\, x + \cos(\alpha z)\, y \end{bmatrix} \end{aligned} \tag{7}$$

3 Rotated isotropic solution free energy

Taking θ to be the standard isotropic solution for two $+1/2$ disclinations gives:

$$\begin{aligned}\theta(x', y') &= \frac{1}{2}\tan^{-1}\left(\frac{y'}{x' - \frac{d}{2}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{y'}{x' + \frac{d}{2}}\right) \\ &= \frac{1}{2}\tan^{-1}\left(\frac{-\sin(\alpha z)\,x + \cos(\alpha z)\,y}{\cos(\alpha z)\,x + \sin(\alpha z)\,y - \frac{d}{2}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{-\sin(\alpha z)\,x + \cos(\alpha z)\,y}{\cos(\alpha z)\,x + \sin(\alpha z)\,y + \frac{d}{2}}\right)\end{aligned} \tag{8}$$

where here d is the disclination spacing. The result of plotting $\theta + \alpha z$ for $\alpha z = \pi/2$ gives the following rotated configuration:



One may explicitly calculate the free energy density for such a configuration. By symmetry, the free energy density at every z-value should be the same, so we evaluate at $z = 0$ to simplify the expressions. What we find is that (expectedly) only the twist and saddle splay terms depend on α. These give:

$$T^2(\alpha) = \alpha^2 f(x, y)\cos^4\theta \tag{9}$$

$$|\Delta|^2(\alpha) = \alpha^2 f(x, y) + g(x, y) \tag{10}$$

with

$$f(x, y) = \frac{d^2\left(d^2 - 4x^2 + 4y^2\right)^2}{\left(d^4 - 8d^2x^2 + 8d^2y^2 + 16x^4 + 32x^2y^2 + 16y^4\right)^2} \tag{11}$$

and $g(x, y)$ some function independent of α. Then the entire free energy goes as:

$$F = \left(K_{22} + (B - A)K_{24}\right)\alpha^2 + C \tag{12}$$

with

$$\begin{aligned}B &= \int_\Omega f(x, y)dV \\ A &= \int_\Omega f(x, y)\cos^4\theta dV\end{aligned} \tag{13}$$

Clearly $B > A$ always, and so a twisted configuration will never be the minimum, at least for the configuration that we've written down.

4 Selinger Euler-Lagrange equation

To find the equilibrium state, we must minimize the Selinger free energy subject to the constraint that $\hat{\mathbf{n}}$ be a unit vector everywhere. The constraint can be written:

$$g(\mathbf{x}, \hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} - 1 = 0 \tag{14}$$

Then the corresponding Lagrangian to be minimized is:

$$L = F - \lambda(\mathbf{x}) g(\mathbf{x}, \hat{\mathbf{n}}) \tag{15}$$

The resulting functional derivative is is:

$$\delta L = \delta F - (2\lambda(\mathbf{x})\hat{\mathbf{n}}) \cdot \delta\hat{\mathbf{n}} \tag{16}$$

The resulting Euler-Lagrange equation then reads:

$$\frac{\delta F}{\delta \hat{\mathbf{n}}} = \lambda(\mathbf{x})\hat{\mathbf{n}} \tag{17}$$

where we have absorbed the factor of 2 into $\lambda(\mathbf{x})$ since it is arbitrary anyways. If we operate on both sides with $(I - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}})$ we get the following:

$$(I - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) \frac{\delta F}{\delta \hat{\mathbf{n}}} = \lambda(\mathbf{x}) \left[\hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})\, \hat{\mathbf{n}}\right] = 0 \tag{18}$$

where we have used the constraint on the left-hand side.

Below we calculate the functional derivative of F. This will simplify when we restrict the director to only polar-planar configurations. We do this one term at a time:

$$\begin{aligned}
\delta(S^2) &= \int_\Omega 2S\,(\delta S)\; dV \\
&= \int_\Omega 2S\; (\nabla \cdot \delta\hat{\mathbf{n}})\; dV \\
&= -\int_\Omega 2\,(\nabla S) \cdot \delta\hat{\mathbf{n}}\, dV + \int_{\partial\Omega} 2\,(S\boldsymbol{\nu}) \cdot \delta\hat{\mathbf{n}}\, dS
\end{aligned} \tag{19}$$

$$\begin{aligned}
\delta(T^2) &= \int_\Omega 2T\; (\delta T)\; dV \\
&= \int_\Omega 2T\; (\delta\hat{\mathbf{n}} \cdot (\nabla \times \hat{\mathbf{n}}) + \hat{\mathbf{n}} \cdot (\nabla \times \delta\hat{\mathbf{n}}))\; dV \\
&= \int_\Omega 4T\; (\nabla \times \hat{\mathbf{n}}) \cdot \delta\hat{\mathbf{n}}\, dV - \int_{\partial\Omega} 2T\boldsymbol{\nu} \cdot (\hat{\mathbf{n}} \times \delta\hat{\mathbf{n}})\, dS
\end{aligned} \tag{20}$$

where we have used the following identity[2]:

$$A \cdot (\nabla \times B) = -\nabla \cdot (A \times B) + B \cdot (\nabla \times A) \tag{21}$$

Also:

$$\begin{aligned}
\delta\,|\mathbf{B}|^2 &= \int_\Omega 2\mathbf{B} \cdot (\delta\mathbf{B})\; dV \\
&= \int_\Omega 2\mathbf{B} \cdot (\delta\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}}) + \hat{\mathbf{n}} \times (\nabla \times \delta\hat{\mathbf{n}}))\; dV \\
&= \int_\Omega 2\left[\delta\hat{\mathbf{n}} \cdot ((\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}) + (\nabla \times \delta\hat{\mathbf{n}}) \cdot (\mathbf{B} \times \hat{\mathbf{n}})\right]\; dV \\
&= \int_\Omega 2\left[\delta\hat{\mathbf{n}} \cdot ((\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}) + \nabla \cdot (\delta\hat{\mathbf{n}} \times (\mathbf{B} \times \hat{\mathbf{n}})) + \delta\hat{\mathbf{n}} \cdot (\nabla \times (\mathbf{B} \times \hat{\mathbf{n}}))\right]\; dV \\
&= \int_\Omega 2\left[\nabla \times (\mathbf{B} \times \hat{\mathbf{n}}) + (\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}\right] \cdot \delta\hat{\mathbf{n}}\, dV + \int_{\partial\Omega} 2\left[\boldsymbol{\nu} \cdot (\delta\hat{\mathbf{n}} \times (\mathbf{B} \times \hat{\mathbf{n}}))\right]\; dS
\end{aligned} \tag{22}$$

where we have used the following identities:

$$A \cdot (B \times C) = C \cdot (A \times B) = B \cdot (C \times A) \tag{23}$$

and

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - (\nabla \times B) \cdot A \tag{24}$$

And finally, we look at the Δ term:

$$\begin{aligned}
\delta \left(\Delta_{ij} \Delta_{ji} \right) &= \int_\Omega 2\Delta_{ij} (\delta \Delta_{ij})\, dV \\
&= \int_\Omega 2\Delta_{ij} \big[2\partial_i \delta n_j - 2\delta n_i \, n_k \partial_k n_j - 2 n_i \delta n_k \partial_k n_j - 2 n_i n_k \partial_k \delta n_j \\
&\qquad + \delta n_i n_j \partial_k n_k + n_i \delta n_j \partial_k n_k + n_i n_j \partial_k \delta n_k \big]\, dV \\
&= \int_\Omega 2 \big[-2\Delta_{ij} \delta n_i \, n_k \partial_k n_j - 2\Delta_{ij} n_i \delta n_k \partial_k n_j + \Delta_{ij} \delta n_i n_j \partial_k n_k + \Delta_{ij} n_i \delta n_j \partial_k n_k \\
&\qquad 2\Delta_{ij} \partial_i \delta n_j - 2\Delta_{ij} n_i n_k \partial_k \delta n_j + \Delta_{ij} n_i n_j \partial_k \delta n_k \big]\, dV \\
&= \int_\Omega 2 \big[-2\Delta_{kj} n_i \partial_i n_j - 2\Delta_{ij} n_i \partial_k n_j + \Delta_{kj} n_j \partial_i n_i + \Delta_{ik} n_i \partial_j n_j \big] \delta n_k \, dV \\
&\qquad + \int_\Omega 2 \big[-2\partial_i \Delta_{ik} + 2\partial_j \left(\Delta_{ik} n_i n_j \right) - \partial_k \left(\Delta_{ij} n_i n_j \right) \big] \delta n_k \, dV \\
&\qquad + \int_{\partial\Omega} 2 \big[2\Delta_{ij} \nu_i \delta n_j - 2\Delta_{ij} n_i n_k \nu_k \delta n_j + \Delta_{ij} n_i n_j \nu_k \delta n_k \big]\, dV
\end{aligned} \tag{25}$$

For now, we assume that the boundaries are fixed so that the surface terms vanish. Putting all of these terms together gives the following Euler-Lagrange equation:

$$\begin{aligned}
0 = (I - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) \Big[&- \left(K_{11} - K_{24} \right) \nabla S \\
&+ 2 \left(K_{22} - K_{24} \right) T \left(\nabla \times \hat{\mathbf{n}} \right) \\
&+ K_{33} \left[\nabla \times (\mathbf{B} \times \hat{\mathbf{n}}) + (\nabla \times \hat{\mathbf{n}}) \times \mathbf{B} \right] \\
&+ 4K_{24} \big[\Delta \cdot \hat{\mathbf{n}} \left(\nabla \cdot \hat{\mathbf{n}} \right) - (\hat{\mathbf{n}} \cdot \nabla \hat{\mathbf{n}}) \cdot \Delta - (\nabla \hat{\mathbf{n}}) \cdot \Delta \cdot \hat{\mathbf{n}} \\
&- \nabla \cdot \Delta + \nabla \cdot (\hat{\mathbf{n}} \otimes (\hat{\mathbf{n}} \cdot \Delta)) - \tfrac{1}{2} \nabla \left(\hat{\mathbf{n}} \cdot \Delta \cdot \hat{\mathbf{n}} \right) \big] \Big]
\end{aligned} \tag{26}$$

The idea here is to look at a general expression for a twisted planar configuration:

$$\hat{\mathbf{n}}' = R(\alpha z) \begin{bmatrix} \cos\big(\theta(x', y')\big) \\ \sin\big(\theta(x', y')\big) \\ 0 \end{bmatrix} \tag{27}$$

with

$$\begin{aligned}
x' &= \cos(\alpha z) x + \sin(\alpha z) y \\
y' &= -\sin(\alpha z) x + \cos(\alpha z) y
\end{aligned} \tag{28}$$

If we plug into eq. (26) and set $z = 0$ we will get a PDE in x and y. Imposing homeotropic boundary conditions gives us a minimum-energy configuration for a fixed α. Presumably we will have to solve this perturbatively with a regular and non-regular part. The non-regular part will have to be the two-defect configuration separated by a distance d. We may map out the free energy landscape for these two parameters, at the very least.

5 Bend-Splay Euler-Lagrange, as a check

To check our calculation, we just consider Bend-Splay terms. Taking $\epsilon = (K_{33} - K_{11})/(K_{33} + K_{11})$ gives:

$$(I - \hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) \left[-(1+\epsilon)\nabla S + (1-\epsilon)\left[\nabla \times (\mathbf{B} \times \hat{\mathbf{n}}) + (\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}\right] \right] = 0 \tag{29}$$

We calculate each term separately:

$$\nabla S - (\hat{\mathbf{n}} \cdot \nabla S)\, \hat{\mathbf{n}} \tag{30}$$

$$\hat{\mathbf{n}} \cdot \nabla \times (\mathbf{B} \times \hat{\mathbf{n}}) = \nabla \cdot (\hat{\mathbf{n}} \times (\mathbf{B} \times \hat{\mathbf{n}})) + (\nabla \times \hat{\mathbf{n}}) \cdot (\mathbf{B} \times \hat{\mathbf{n}}) \tag{31}$$

$$\begin{aligned} \hat{\mathbf{n}} \cdot [(\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}] &= (\nabla \times \hat{\mathbf{n}}) \cdot (\mathbf{B} \times \hat{\mathbf{n}}) \\ &= (\nabla \times \hat{\mathbf{n}}) \cdot ((\hat{\mathbf{n}} \times (\nabla \times \hat{\mathbf{n}})) \times \hat{\mathbf{n}}) \\ &= (\nabla \times \hat{\mathbf{n}}) \cdot ((-(\hat{\mathbf{n}} \cdot \nabla)\, \hat{\mathbf{n}}) \times \hat{\mathbf{n}}) \end{aligned} \tag{32}$$

6 Simplified free energy

6.1 Free energy and Euler-Lagrange equation

For sake of ease, we assume $K_{24} = 0$ (in an infinite system we assume it does not matter) and $K_{11} = K_{33} = K$. We take ζ to be our twist elastic constant:

$$\zeta = \frac{K - K_{22}}{K + K_{22}} \tag{33}$$

This gives:

$$K_{22} = K\, \frac{1-\zeta}{1+\zeta} \tag{34}$$

Then the free energy is given by:

$$F = \int_\Omega (1+\zeta)S^2 + (1-\zeta)T^2 + (1+\zeta)\, |\mathbf{B}|^2 \, dV \tag{35}$$

We may plug in for a planar $\hat{\mathbf{n}}$ to get an expression in terms of θ. Calculating explicitly gives:

$$F = \int_\Omega (1+\zeta)\, |\nabla\theta|^2 - 2\zeta\, \theta_z^2 dV \tag{36}$$

The differential is:

$$\begin{aligned} \delta F &= \int_\Omega 2(1+\zeta)\,(\nabla\theta)\cdot(\nabla\delta\theta) - 4\zeta \left(\frac{d\theta}{dz}\right)\left(\frac{d\delta\theta}{dz}\right) dV \\ &= \int_\Omega 2(1+\zeta)\left[\nabla\cdot(\delta\theta\nabla\theta) - \left(\nabla^2\theta\right)\delta\theta\right] - 4\zeta\left[\frac{d}{dz}\left(\delta\theta\frac{d\theta}{dz}\right) - \left(\frac{d^2\theta}{dz^2}\right)\delta\theta\right] dV \\ &= \int_\Omega -2\left[(1+\zeta)\nabla^2\theta - 2\zeta\frac{d^2\theta}{dz^2}\right]\delta\theta dV \end{aligned} \tag{37}$$

where we have assumed that the variation goes to zero at all the boundaries. Then the Euler-Lagrange equation reads:

$$(1+\zeta)\nabla^2\theta - 2\zeta\frac{d^2\theta}{dz^2} = 0 \tag{38}$$

6.2 Constant twist angular velocity configuration

We now consider a director given by:

$$\theta(x,y,z) = \theta(x',y') + \alpha z \tag{39}$$

where

$$\begin{aligned} x' &= \cos(\alpha z)x + \sin(\alpha z)y \\ y' &= -\sin(\alpha z)x + \cos(\alpha z)y \end{aligned} \tag{40}$$

We calculate this explicitly as follows:

$$\frac{\partial\theta}{\partial x} = \frac{\partial\theta}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial\theta}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial\theta}{\partial x'}\cos(\alpha z) - \frac{\partial\theta}{\partial y'}\sin(\alpha z) \tag{41}$$

so that

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial^2\theta}{\partial x'^2}\cos^2(\alpha z) - 2\frac{\partial^2\theta}{\partial x'\partial y'}\cos(\alpha z)\sin(\alpha z) + \frac{\partial^2\theta}{\partial y'^2}\sin^2(\alpha z) \tag{42}$$

similarly for y:

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\theta}{\partial x'^2}\sin^2(\alpha z) + 2\frac{\partial^2\theta}{\partial x'\partial y'}\cos(\alpha z)\sin(\alpha z) + \frac{\partial^2\theta}{\partial y'^2}\cos^2(\alpha z) \tag{43}$$

and finally:

$$\frac{\partial\theta}{\partial z} = \frac{\partial\theta}{\partial x'}\frac{\partial x'}{\partial z} + \frac{\partial\theta}{\partial y'}\frac{\partial y'}{\partial z} = \frac{\partial\theta}{\partial x'}\alpha y' - \frac{\partial\theta}{\partial y'}\alpha x' \tag{44}$$

so that:

$$\begin{aligned} \frac{\partial^2\theta}{\partial z^2} &= \frac{\partial^2\theta}{\partial x'\partial z}\alpha y' + \frac{\partial\theta}{\partial x'}\alpha\frac{\partial y'}{\partial z} - \frac{\partial^2\theta}{\partial y'\partial z}\alpha x' - \frac{\partial\theta}{\partial y'}\alpha\frac{\partial x'}{\partial z} \\ &= \frac{\partial^2\theta}{\partial x'^2}\alpha^2 y'^2 - \frac{\partial^2\theta}{\partial x'\partial y'}\alpha^2 y'x' - \frac{\partial\theta}{\partial x'}\alpha^2 x' + \frac{\partial^2\theta}{\partial y'^2}\alpha^2 x'^2 - \frac{\partial^2\theta}{\partial y'\partial x'}\alpha^2 x'y' - \frac{\partial\theta}{\partial y'}\alpha^2 y' \end{aligned} \tag{45}$$

For an infinite system with cylindrical symmetry, every z-axis is the same, so we consider the slice for $z=0$. Altogether the Euler-Lagrange equation reads:

$$(1+\zeta)\,\nabla^2\theta + (1-\zeta)\,\alpha^2\,(\mathbf{x}\times\nabla)^2\,\theta = 0 \tag{46}$$

where here $\nabla = \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}}$ and $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$[3]. We may decompose θ into $\theta_{\text{iso}} + \theta_c$ where $\theta_{\text{iso}} = \frac{1}{2}(\varphi_1+\varphi_2)$ with φ_1, φ_2 the polar coordinates centered at $\left(\pm\frac{d}{2}, 0\right)$ and θ_c is a regular correction. Note that d is the spacing of the two $+1/2$ disclinations. Since $\nabla^2\theta_{\text{iso}} = 0$ this gives the following linear equation for θ_c:

$$\left[(1+\zeta)\nabla^2 + (1-\zeta)\alpha^2\,(\mathbf{x}\times\nabla)^2\right]\theta_c = -(1-\zeta)\alpha^2\,(\mathbf{x}\times\nabla)^2\,\theta_{\text{iso}} \tag{47}$$

6.3 Constant twist weak form

The $(\mathbf{x}\times\nabla)$ operator also has a nice product rule[4]. Also note that, in two dimensions, it has a nice integration formula[5] Taking the inner product with some test function η then yields the following weak form:

$$(1+\zeta)\left\langle\eta, \nabla^2\theta_c\right\rangle + (1-\zeta)\alpha^2\left\langle\eta, (\mathbf{x}\times\nabla)^2\,\theta_c\right\rangle = -(1-\zeta)\alpha^2\left\langle\eta, (\mathbf{x}\times\nabla)^2\,\theta_{\text{iso}}\right\rangle \tag{48}$$

which finally yields:

$$\begin{aligned} &-(1+\zeta)\left\langle\nabla\eta, \nabla\theta_c\right\rangle \\ &\quad-(1-\zeta)\alpha^2\left\langle(\mathbf{x}\times\nabla)\,\eta, (\mathbf{x}\times\nabla)\,\theta_c\right\rangle = (1-\zeta)\alpha^2\left\langle(\mathbf{x}\times\nabla)\,\eta, (\mathbf{x}\times\nabla)\,\theta_{\text{iso}}\right\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (1+\zeta)\left\langle\eta, \boldsymbol{\nu}\cdot\nabla\theta_c\right\rangle_{\partial\Omega} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (1-\zeta)\alpha^2\left\langle\eta, (\mathbf{x}\times\boldsymbol{\nu})\,(\mathbf{x}\times\nabla)\,\theta_c\right\rangle_{\partial\Omega} \end{aligned} \tag{49}$$

6.4 Variable twist angular velocity

Here we assume a director configuration whose twist angular velocity α is a function of z. In this case, the x- and y-derivatives are unchanged, but the z derivatives vary considerably:

$$\frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial x'}(\alpha + z\alpha')y' - \frac{\partial \theta}{\partial y'}(\alpha + z\alpha')x' \tag{50}$$

and then

$$\begin{aligned}\frac{\partial^2 \theta}{\partial z^2} = \frac{\partial^2 \theta}{\partial x'^2}(\alpha + z\alpha')^2 y'^2 - \frac{\partial^2 \theta}{\partial x' \partial y'}(\alpha + z\alpha')^2 y' x' + \frac{\partial \theta}{\partial x'}(2\alpha' + z\alpha'')y' - \frac{\partial \theta}{\partial x'}(\alpha + z\alpha')^2 x' \\ + \frac{\partial^2 \theta}{\partial y'^2}(\alpha + z\alpha')^2 x'^2 - \frac{\partial^2 \theta}{\partial y' \partial x'}(\alpha + z\alpha')^2 x' y' - \frac{\partial \theta}{\partial y'}(2\alpha' + z\alpha'')x' - \frac{\partial \theta}{\partial y'}(\alpha + z\alpha')^2 y'\end{aligned} \tag{51}$$

Finally we have:

$$\frac{\partial^2}{\partial z^2}\alpha z = \frac{\partial}{\partial z}\left[\alpha' z + \alpha\right] = \alpha'' z + 2\alpha' \tag{52}$$

Notes

[1] Suppose $\mathbf{v}(\mathbf{x})$ is a vector field. Take L to be a linear transformation. We would like to act on $\mathbf{v}$ by L in an *active* way. This means that, if L rotates a plane by some angle θ, then we are imagining taking $\mathbf{v}$ (say on a piece of paper) and rotating the whole thing by the angle θ. There's two pieces to this: i) is that we must act on each of the vectors outputted by $\mathbf{v}$ by L (again, think of rotating a vector field printed on a piece of paper). ii) is that, if we want to get the correct vector field at $\mathbf{x}$, we must actually sample $\mathbf{v}$ at a point $L^{-1}\mathbf{x}$. This is because $L^{-1}\mathbf{x}$ is the point that will get mapped to $\mathbf{x}$ by L.

[2]

$$\begin{aligned} A \cdot (\nabla \times B) &= A_i \epsilon_{ijk} \partial_j B_k \\ &= \epsilon_{ijk}\left(\partial_j (A_i B_k) - B_k \partial_j A_i\right) \\ &= -\partial_j \left(\epsilon_{jik} A_i B_k\right) + B_k \epsilon_{kji} \partial_j A_i \\ &= -\nabla \cdot (A \times B) + B \cdot (\nabla \times A) \end{aligned} \tag{53}$$

[3]

$$\begin{aligned} (\mathbf{x} \times \nabla)^2 &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \\ &= \left(x^2\frac{\partial}{\partial y^2} - x\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - yx\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2\frac{\partial}{\partial x^2}\right) \end{aligned} \tag{54}$$

[4]

$$\begin{aligned} (\mathbf{x} \times \nabla)(fg) &= \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)(fg) \\ &= x\left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right) - y\left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right) \\ &= g\left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right) + f\left(x\frac{\partial g}{\partial y} - y\frac{\partial g}{\partial x}\right) \\ &= g(\mathbf{x} \times \nabla) f + f(\mathbf{x} \times \nabla) g \end{aligned} \tag{55}$$

[5] We first rewrite the operator as a divergence: $(\mathbf{x} \times \nabla)f = \nabla \cdot \mathbf{g}$ with $\mathbf{g} = f[-y\; x]^T$. Then we may explicitly calculate the integration formula using the divergence theorem:

$$\begin{aligned} \int_\Omega (\mathbf{x} \times \nabla) f\, dV &= \int_\Omega \nabla \cdot \mathbf{g}\, dV \\ &= \int_{\partial\Omega} (\hat{\boldsymbol{\nu}} \cdot \mathbf{g})\, dS \\ &= \int_{\partial\Omega} f(-\nu_x y + \nu_y x)\, dS \\ &= \int_{\partial\Omega} f(\mathbf{x} \times \hat{\boldsymbol{\nu}})\, dS \end{aligned} \tag{56}$$

Then the simplified Euler-Lagrange equation is:

$$0 = -(1+\zeta)\nabla S + 2(1-\zeta)T\ (\nabla \times \hat{\mathbf{n}}) + (1+\zeta)\left[\nabla \times (\mathbf{B} \times \hat{\mathbf{n}}) + (\nabla \times \hat{\mathbf{n}}) \times \mathbf{B}\right] \tag{57}$$