Twisted disclination velocity

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February 23, 2024

1 Single disclination with added twist

To begin, we consider an isolated disclination which has an added twist. This corresponds to $\hat{\mathbf{\Omega}}$ making an angle β with the tangent vector $\hat{\mathbf{T}}$. In our simulations, it appears that the plane which $\hat{\mathbf{\Omega}}$ is confined to is perpendicular to the vector between the two disclinations:

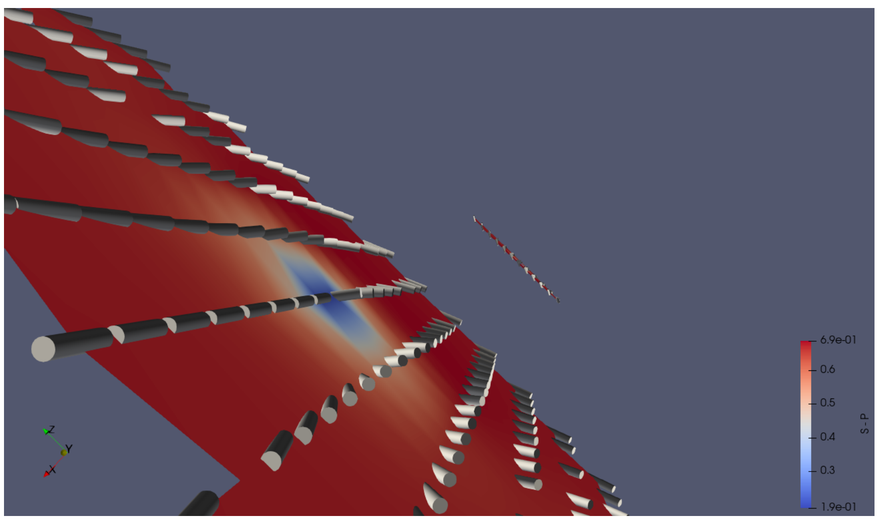

Figure 1: Close-up of a cross-section of a twisted disclination. The axes in the image are different than what is in this note.value(), so take out of the page as $\hat{\mathbf{x}}$ and upward normal to the cross-sectional plane to be $\hat{\mathbf{z}}$. Here $\beta < 0$ which corresponds to a positive rotation of the director about the $\hat{\mathbf{x}}$ axis.

For concreteness, we choose $\hat{\mathbf{T}} = \hat{\mathbf{z}}$ and $\hat{\mathbf{\Omega}} = \sin\beta \hat{\mathbf{y}} + \cos\beta \hat{\mathbf{z}}$. We note that, to get from a $+1/2$ wedge disclination to the twist disclination described by this $\hat{\mathbf{T}}$ and $\hat{\mathbf{\Omega}}$, one must rotate by β in the $-\hat{\mathbf{x}}$ direction. Hence, in Cody's parlance we have that:

$$\tilde{\varphi}(z)\,\hat{\mathbf{q}} = -\beta(z)\,\hat{\mathbf{x}} \tag{1}$$

From Eq. (7.8) in Cody's thesis, it's clear that the disclination velocity is always zero if we only consider the isotropic elasticity contribution to the equations of motion (since $\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{x}} = 0$).

1.1 Calculating L_2 contribution to velocity

Note that:

$$\tilde{\mathbf{n}}_k = \hat{\mathbf{n}}_k + \tilde{\varphi}\,\mathbf{p}_k \tag{2}$$

with

$$\mathbf{p}_k = (\hat{\mathbf{q}} \times \hat{\mathbf{n}}_k) \tag{3}$$

This gives:

$$\nabla \tilde{\mathbf{n}}_k = \nabla \tilde{\varphi}\, \mathbf{p}_k \tag{4}$$

Then, from Eq. (7.3) in the thesis we get:

$$Q_{\mu\nu} \approx S_N \left[\frac{1}{6} \delta_{\mu\nu} - \frac{1}{2} \hat{\Omega}_\mu \hat{\Omega}_\nu + \frac{x}{2a} \left(\tilde{n}_{0\mu} \tilde{n}_{0\nu} - \tilde{n}_{1\mu} \tilde{n}_{1\nu} \right) + \frac{y}{2a} \left(\tilde{n}_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{1\mu} \tilde{n}_{0\nu} \right) \right] \tag{5}$$

We compute the gradients as follows:

$$\begin{aligned} \partial_k Q_{\mu\nu} \approx \frac{S_N}{2a} \Big[& \left(\tilde{n}_{0\mu} \tilde{n}_{0\nu} - \tilde{n}_{1\mu} \tilde{n}_{1\nu} \right) \delta_{kx} + x \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \\ & + \left(\tilde{n}_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{1\mu} \tilde{n}_{0\nu} \right) \delta_{ky} + y \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \Big] \end{aligned} \tag{6}$$

and higher order derivatives:

$$\begin{aligned} \partial_l \partial_k Q_{\mu\nu} \approx \frac{S_N}{2a} \Big[& \partial_l \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \delta_{kx} \\ & + \left(\partial_k \tilde{\varphi}\, \delta_{lx} + x\, \partial_l \partial_k \tilde{\varphi} \right) \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \\ & + 2x \left(\partial_l \tilde{\varphi} \right) \left(\partial_k \tilde{\varphi} \right) \left(p_{0\mu} p_{0\nu} - p_{1\mu} p_{1\nu} \right) \\ & + \partial_l \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \delta_{ky} \\ & + \left(\partial_k \tilde{\varphi}\, \delta_{ly} + y\, \partial_l \partial_k \tilde{\varphi} \right) \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \\ & + 2y \left(\partial_l \tilde{\varphi} \right) \left(\partial_k \tilde{\varphi} \right) \left(p_{0\mu} p_{1\nu} + p_{1\mu} p_{0\nu} \right) \Big] \end{aligned} \tag{7}$$

Evaluated at $x = y = 0$ (i.e. the disclination core) this becomes:

$$\partial_k Q_{\mu\nu} \approx \frac{S_N}{2a} \left[\left(\tilde{n}_{0\mu} \tilde{n}_{0\nu} - \tilde{n}_{1\mu} \tilde{n}_{1\nu} \right) \delta_{kx} + \left(\tilde{n}_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{1\mu} \tilde{n}_{0\nu} \right) \delta_{ky} \right] \tag{8}$$

and for the higher order derivatives:

$$\begin{aligned} \partial_l \partial_k Q_{\mu\nu} \big|_{x=y=0} \approx \frac{S_N}{2a} \Big[& \partial_l \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \delta_{kx} \\ & + \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \delta_{lx} \\ & + \partial_l \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \delta_{ky} \\ & + \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \delta_{ly} \Big] \end{aligned} \tag{9}$$

Note that this matches Cody's Eq. (7.5) for $k = l$:

$$\begin{aligned} \partial_k \partial_k Q_{\mu\nu} \big|_{x=y=0} \approx \frac{S_N}{a} \Big[& \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{0\nu} + \tilde{n}_{0\mu} p_{0\nu} - p_{1\mu} \tilde{n}_{1\nu} - \tilde{n}_{1\mu} p_{1\nu} \right) \delta_{kx} \\ & + \partial_k \tilde{\varphi} \left(p_{0\mu} \tilde{n}_{1\nu} + \tilde{n}_{0\mu} p_{1\nu} + p_{1\mu} \tilde{n}_{0\nu} + \tilde{n}_{1\mu} p_{0\nu} \right) \delta_{ky} \Big] \end{aligned} \tag{10}$$

Before calculating the L_2 term (which is ostensibly harder), we redo Cody's isotropic calculation to make sure everything works correctly. To simplify things, we note that Eq. (10) already has an explicit factor of $\partial_k \tilde{\varphi}$ on every term, and so when we calculate $\mathbf{g}$ to $\mathcal{O}(\tilde{\varphi})$ we may take approximate all other factors to $\mathcal{O}(1)$. In particular, this implies $\tilde{\mathbf{n}} \approx \hat{\mathbf{n}}$. We use the following identities:

$$
\begin{aligned}
\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}}_1 &= 0 \\
\hat{\mathbf{n}}_0 \cdot \hat{\mathbf{n}}_0 &= \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_1 = 1 \\
\mathbf{p}_0 \cdot \hat{\mathbf{n}}_0 &= \mathbf{p}_1 \cdot \hat{\mathbf{n}}_1 = 0 \\
\mathbf{p}_0 \cdot \hat{\mathbf{n}}_1 &= -\mathbf{p}_1 \cdot \hat{\mathbf{n}}_0 = \hat{\mathbf{q}} \cdot \hat{\mathbf{\Omega}} \\
\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{n}}_0 &= \hat{\mathbf{\Omega}} \cdot \hat{\mathbf{n}}_1 = 0 \\
\hat{\mathbf{\Omega}} \cdot \hat{\mathbf{\Omega}} &= 1 \\
\hat{\mathbf{n}}_0 \times \hat{\mathbf{n}}_1 &= \hat{\mathbf{\Omega}} \\
\hat{\mathbf{\Omega}} \times \hat{\mathbf{n}}_0 &= \hat{\mathbf{n}}_1 \\
\hat{\mathbf{\Omega}} \times \hat{\mathbf{n}}_1 &= -\hat{\mathbf{n}}_0 \\
\mathbf{p}_0 \times \hat{\mathbf{n}}_0 &= -\hat{\mathbf{q}} + \hat{\mathbf{n}}_0 \left(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_0\right) \\
\mathbf{p}_1 \times \hat{\mathbf{n}}_1 &= -\hat{\mathbf{q}} + \hat{\mathbf{n}}_1 \left(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_1\right) \\
\mathbf{p}_0 \times \hat{\mathbf{n}}_1 &= \hat{\mathbf{n}}_0 \left(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_1\right) \\
\mathbf{p}_1 \times \hat{\mathbf{n}}_0 &= \hat{\mathbf{n}}_1 \left(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}}_0\right)
\end{aligned} \tag{11}
$$

We calculate $\hat{\mathbf{\Omega}} \cdot \mathbf{g}$ for the isotropic case in a Jupyter notebook and end up with Eq. (7.7) from Cody's thesis.

Now for the L_2 terms we calculate:

$$
\begin{aligned}
\partial_i \partial_k Q_{kj}\big|_{x=y=0} \approx \frac{S_N}{2a} \Big[&\partial_i \tilde{\varphi} \left(p_{0x} \tilde{n}_{0j} + \tilde{n}_{0x} p_{0j} - p_{1x} \tilde{n}_{1j} - \tilde{n}_{1x} p_{1j}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{0j} + \tilde{n}_{0k} p_{0j} - p_{1k} \tilde{n}_{1j} - \tilde{n}_{1k} p_{1j}\right) \delta_{ix} \\
&+ \partial_i \tilde{\varphi} \left(p_{0y} \tilde{n}_{1j} + \tilde{n}_{0y} p_{1j} + p_{1y} \tilde{n}_{0j} + \tilde{n}_{1y} p_{0j}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{1j} + \tilde{n}_{0k} p_{1j} + p_{1k} \tilde{n}_{0j} + \tilde{n}_{1k} p_{0j}\right) \delta_{iy} \Big]
\end{aligned} \tag{12}
$$

We may find $\partial_j \partial_k Q_{ki}$ by just taking the transpose. The last term that we need is:

$$
\begin{aligned}
\partial_l \partial_k Q_{kl}\big|_{x=y=0} \approx \frac{S_N}{2a} \Big[&\partial_l \tilde{\varphi} \left(p_{0x} \tilde{n}_{0l} + \tilde{n}_{0x} p_{0l} - p_{1x} \tilde{n}_{1l} - \tilde{n}_{1x} p_{1l}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{0x} + \tilde{n}_{0k} p_{0x} - p_{1k} \tilde{n}_{1x} - \tilde{n}_{1k} p_{1x}\right) \\
&+ \partial_l \tilde{\varphi} \left(p_{0y} \tilde{n}_{1l} + \tilde{n}_{0y} p_{1l} + p_{1y} \tilde{n}_{0l} + \tilde{n}_{1y} p_{0l}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{1y} + \tilde{n}_{0k} p_{1y} + p_{1k} \tilde{n}_{0y} + \tilde{n}_{1k} p_{0y}\right) \Big]
\end{aligned} \tag{13}
$$

Now we have to compute $\hat{\mathbf{\Omega}} \cdot \mathbf{g}$. Cody has already done this for the isotropic medium, we need to do it for the L_2 term. Luckily $\mathbf{g}$ is linear in $\partial_t Q$ terms, so we first calculate:

$$
\begin{aligned}
\left(\partial_i \partial_k Q_{kj}\right) \left(\partial_l Q_{mj}\right) = \frac{S_N^2}{4a^2} \Big[&\partial_i \tilde{\varphi} \left(p_{0x} \tilde{n}_{0j} + \tilde{n}_{0x} p_{0j} - p_{1x} \tilde{n}_{1j} - \tilde{n}_{1x} p_{1j}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{0j} + \tilde{n}_{0k} p_{0j} - p_{1k} \tilde{n}_{1j} - \tilde{n}_{1k} p_{1j}\right) \delta_{ix} \\
&+ \partial_i \tilde{\varphi} \left(p_{0y} \tilde{n}_{1j} + \tilde{n}_{0y} p_{1j} + p_{1y} \tilde{n}_{0j} + \tilde{n}_{1y} p_{0j}\right) \\
&+ \partial_k \tilde{\varphi} \left(p_{0k} \tilde{n}_{1j} + \tilde{n}_{0k} p_{1j} + p_{1k} \tilde{n}_{0j} + \tilde{n}_{1k} p_{0j}\right) \delta_{iy} \Big] \\
&\cdot \left[\left(\tilde{n}_{0m} \tilde{n}_{0j} - \tilde{n}_{1m} \tilde{n}_{1j}\right) \delta_{lx} + \left(\tilde{n}_{0m} \tilde{n}_{1j} + \tilde{n}_{1m} \tilde{n}_{0j}\right) \delta_{ly} \right] \\
=&
\end{aligned} \tag{14}
$$

We note the following properties:

$$\begin{aligned}
\epsilon_{\gamma im}\delta_{ix} = \epsilon_{\gamma xm} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \delta_{\gamma z}\delta_{my} - \delta_{\gamma y}\delta_{mz} \\
\epsilon_{\gamma im}\delta_{iy} = \epsilon_{\gamma ym} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \delta_{\gamma x}\delta_{mz} - \delta_{\gamma z}\delta_{mx}
\end{aligned} \tag{15}$$