

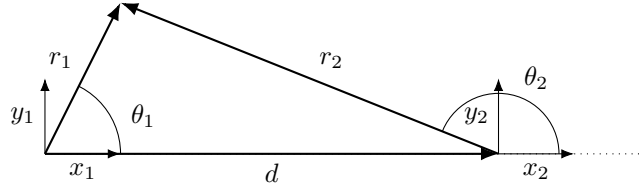
Two defect Dzyaloshinskii approximation

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1 Expression of director around one defect from interaction with another (isotropic)

The configuration is as below: Here we consider the director angle as a function of θ_1 and r_1 . Note



that the director angle ϕ is given by:

$$\phi(x, y) = q_1 \theta_1(x, y) + q_2 \theta_2(x, y) \quad (1)$$

If we call coordinates (x_2, y_2) centered on q_2 then we get:

$$\theta_2(x_2, y_2) = \arctan\left(\frac{y_2}{x_2}\right) \quad (2)$$

Writing these in terms of (x_1, y_1) coordinates centered at q_1 we get:

$$\begin{aligned} x_1 &= x_2 + d \\ y_1 &= y_2 \end{aligned} \quad (3)$$

Substituting in yields:

$$\theta_2(x_1, y_1) = \arctan\left(\frac{y_1}{x_1 - d}\right) \quad (4)$$

Then, considering polar coordinates (θ_1, r_1) we get that:

$$\begin{aligned} x_1 &= r_1 \cos(\theta_1) \\ y_1 &= r_1 \sin(\theta_1) \end{aligned} \quad (5)$$

Substituting yields:

$$\theta_2(\theta_1, r_1) = \arctan\left(\frac{r_1 \sin(\theta_1)}{r_1 \cos(\theta_1) - d}\right) \quad (6)$$

Hence, the isotropic contribution to the director field at the location of q_1 from the defect pair is:

$$\phi_{\text{iso}}(\theta_1, r_1) = q_1 \theta_1 + q_2 \arctan\left(\frac{r_1 \sin(\theta_1)}{r_1 \cos(\theta_1) - d}\right) \quad (7)$$

If we consider isomorph (a), then add $\pi/2$, otherwise don't add anything.

Finally, if we would like to write out the isotropic director field from two defects at q_2 , everything is the same except d changes sign:

$$\phi_{\text{iso}}(\theta_2, r_2) = q_1 \arctan\left(\frac{r_2 \sin(\theta_2)}{r_2 \cos(\theta_2) + d}\right) + q_2 \theta_2 \quad (8)$$

2 Checking Fourier series

We need to check that the analysis of the director Fourier modes is correct. To do this, we note that the director ϕ as a function of θ for a $-1/2$ in isomorph (a) is given by:

$$\phi(\theta) = -\frac{1}{2}\theta + \frac{\pi}{2} \quad (9)$$

on the interval $[0, 2\pi]$. We would like to find the Fourier series of this. This is given by:

$$s(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \quad (10)$$

We calculate these coefficients as follows:

$$\begin{aligned} A_0 &= \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) d\theta \\ &= \frac{1}{2\pi} \left[-\frac{1}{4}\theta^2 + \frac{\pi}{2}\theta \right]_0^{2\pi} \\ &= 0 \end{aligned} \quad (11)$$

$$\begin{aligned} B_n &= \frac{1}{\pi} \int_0^{2\pi} \phi(\theta) \sin(n\theta) d\theta \\ &= -\frac{1}{2\pi} \left[\frac{-2\pi}{n} \right] \\ &= \frac{1}{n} \end{aligned} \quad (12)$$

3 Verifying DFT identities

From the numpy documentation, we have that the inverse DFT is given by:

$$a_m = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp\left\{2\pi i \frac{mk}{n}\right\} \quad (13)$$

Here a_m is a set of values of our function f taken at some discrete points x_m . If $x \in [0, L)$ then:

$$\frac{x_m}{L} = \frac{m}{n} \implies m = \frac{x_m n}{L} \quad (14)$$

Substituting, we get:

$$f(x_m) = \frac{1}{n} \sum_{k=0}^{n-1} A_k \exp \left\{ \frac{2\pi}{L} i k x_m \right\} \quad (15)$$

Expanding into trigonometric functions using Euler's formula, and splitting A_k into real and imaginary components, we get:

$$\begin{aligned} f(x_m) &= \frac{1}{n} \sum_{k=0}^{n-1} (B_k + iC_k) \left[\cos \left(\frac{2\pi}{L} k x_m \right) + i \sin \left(\frac{2\pi}{L} k x_m \right) \right] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} B_k \cos \left(\frac{2\pi}{L} k x_m \right) - C_k \sin \left(\frac{2\pi}{L} k x_m \right) + i \left[B_k \sin \left(\frac{2\pi}{L} k x_m \right) + C_k \cos \left(\frac{2\pi}{L} k x_m \right) \right] \\ &= \frac{1}{n} \sum_{k=0}^{n-1} B_k \cos \left(\frac{2\pi}{L} k x_m \right) - C_k \sin \left(\frac{2\pi}{L} k x_m \right) \end{aligned} \quad (16)$$

Where the last line follows from the fact that $f(x)$ is real. Hence, the trigonometric Fourier coefficients should correspond with B_k/n and $-C_k/n$ respectively.