Numerically solving Dzyaloshinsky as an initial condition

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1 Definitions

The Dzyaloshinsky solution is a solution for the director $\mathbf{n}$ given the Frank-Oseen elastic free energy. That is:

$$\mathcal{F}_{el} = \tfrac{1}{2} K_1 \left(\nabla \mathbf{n}\right)^2 + \tfrac{1}{2} K_2 \left(\mathbf{n} \cdot \nabla \times \mathbf{n}\right)^2 + \tfrac{1}{2} K_3 \left(\mathbf{n} \times \left(\nabla \times \mathbf{n}\right)\right)^2 \tag{1}$$

Note that the K_2 term is always zero in the 2D case, because $\nabla \times \mathbf{n}$ will always be in the z-direction, while $\mathbf{n}$ is always in the xy-plane. Hence, the anisotropy is characterized by a single parameter:

$$\varepsilon = \frac{K_3 - K_1}{K_3 + K_1} \tag{2}$$

Which ranges from -1 in the bend-dominated case, to 0 in the isotropic case, to 1 in the splay-dominated case. Given that $\mathbf{n}$ is confined to 2-dimensions, we may parameterize it by a single angle $0 \leq \phi \leq \pi$ so that:

$$\mathbf{n} = \left(\cos \phi, \sin \phi\right) \tag{3}$$

This is a function of the polar coordinate θ. The solution is given for $\phi(\theta)$ given the constraint that $\phi(\theta + 2\pi) = \phi(theta) + 2\pi m$ where m is an integer or half-integer value corresopnding to the charge of the defect. Here we only include the $m = +1/2$ solution which gives:

$$\frac{d^2 \phi}{d\theta^2} \left[1 - \varepsilon \cos 2\left(\phi - \theta\right)\right] - \left[2 \frac{d\phi}{d\theta} - \left(\frac{d\phi}{d\theta}\right)^2\right] \varepsilon \sin 2 \left(\phi - \theta\right) = 0 \tag{4}$$