



Decision Support

Impact of storage competition on energy markets

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ABSTRACT

We study how storage, operating as a price maker within a market environment, may be optimally operated over an extended period of time. The optimality criterion may be the maximisation of the profit of the storage itself, where this profit results from the exploitation of the differences in market clearing prices at different times. Alternatively it may be the minimisation of the cost of generation, or the maximisation of consumer surplus or social welfare. In all cases there is calculated for each successive time-step the cost function measuring the total impact of whatever action is taken by the storage. The succession of such cost functions provides the information for the storage to determine how to behave over time, forming the basis of the appropriate optimisation problem.

We study particularly competition between multiple stores, where the objective of each store is to maximise its own income given the activities of the remainder. We show that, at the Cournot Nash equilibrium, multiple stores which between them have market impact collectively erode their own abilities to make profits: essentially each store attempts to increase its own profit over time by overcompeting at the expense of the remainder. We quantify this for linear price functions.

We give examples throughout based on electricity storage and Great Britain electricity spot-price market data.

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1. Introduction

There has been much discussion in recent years on the role of storage in future energy networks. It can be used to buffer the highly variable output of renewable generation such as wind and solar power, and it further has the potential to smooth fluctuations in demand, thereby reducing the need for expensive and carbon-emitting peaking plants. For a discussion of the use of storage in providing multiple buffering and smoothing capabilities, including the ability to integrate renewable generation into energy networks see, for example, the fairly recent review by Denholm, Ela, Kirby, and Milligan (2010), and the many references therein. Within an economic framework much of the value of energy storage may be realised by allowing it to operate in a market environment, provided that the latter is structured in such a way as to allow this to happen. Thus the smoothing of variations in demand between, for example, nighttime when demand is low and daytime when demand is high may be achieved by allowing a store to buy energy at night when the low demand typically means that it is relatively cheap, and to sell it again in the day when it is expensive. Simi-

larly, the use of storage for buffering against shortfalls in renewable generation may – at least in part – be effected by allowing storage to operate in a responsive spot-price market when prices will rise at the times of such shortfall. We remark though that if it is intended that the use of storage should facilitate, for example, a reduction in carbon emissions, then there is of course no guarantee that a market environment will in itself permit this to happen – for some recent insights into the possible unexpected side effects of storage operating in a market, see Virasjoki, Rocha, Siddiqui, and Salo (2015).

It is also the case that price arbitrage as above is not the only way in which storage may compete in the marketplace. In particular much energy storage has the ability to provide power – sometimes in large quantities, as in the case of some pumped storage facilities – at very short notice, i.e. within time scales of the order of seconds or less (see, for example, the recent GB National Grid enhanced frequency response (EFR) auctions National Grid plc (2017)). Thus energy storage typically provides a variety of services, and even those which are concerned with smoothing imbalances in supply and demand on time scales longer than those above may be paid for other than through arbitrage opportunities, for example through fixed “capacity” contracts which cover substantial periods of time and in which stores are paid fees fixed in advance simply to be available to provide energy as needed. Nevertheless the

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use of storage for arbitrage is significant and may become more so in future systems, for example, in the presence of either more nuclear generation or of increased renewable generation, neither of these being easily controllable so as to smooth fluctuations in the supply-demand balance. (Whether the benefits which storage can bring in such situations is paid for through providing arbitrage opportunities to the storage will depend very much on how markets are organised.) A storage facility may well reserve some of its energy capacity and power capabilities for the provision of services such as EFR, and then seek to use the remainder of its resource so as to make money through arbitrage. For some work on the simultaneous use of storage for both arbitrage and buffering against the effects of sudden events see [Cruise and Zachary \(2015\)](#), while for work on a whole systems assessment of the value of energy storage see [Pudjianto, Aunedi, Djapic, and Strbac \(2014\)](#).

When stores buy and sell in the market, it is important to understand the effect on the market itself of the activities of the stores, and to understand also the effect of competition between stores on the profitability of their activities. A small store may be expected to function as a price-taker, buying and selling so as, for example, to maximise its own profit over time. However, a larger store, or a sufficient number of smaller stores, will act as price-makers, perhaps significantly affecting the market in which they operate, and thus also affecting quantities such as generator costs, consumer surplus and social welfare. Further a number of stores which between them possess market impact, by competing with each other, may smooth prices to the point where they are unable to make sufficient profits as to be economically viable – at least via their arbitrage activities, as we demonstrate in the simple example below.

Example 1. Consider a model with two time periods and n perfectly efficient stores. Suppose that each store k buys x_k in time period 1 which it then sells in time period 2, and that this results in a unit price differential (the price at time 2 less that at time 1) of $p - p' \sum_{k=1}^n x_k$ where $p > 0$, $p' > 0$. (This will be the case when, for example, the stores face appropriate linear residual supply functions in each time period – see [Section 2](#).) In what is a model of simple Cournot competition, we suppose that each store k seeks to maximise its profit $(p - p' \sum_{j=1}^n x_j)x_k$ given the quantities x_j , $j \neq k$, traded by the remaining $n - 1$ stores. If the stores are unrestricted in the quantities x_k they may trade, it is easily checked that at the Nash equilibrium we have $x_k = p/(p'(n + 1))$ for all k , and that the price differential between the two time periods is $p/(n + 1)$. Thus in particular each store makes a profit proportional to $1/(n + 1)^2$, and the total profit made by all the stores declines as approximately $1/n$ as n becomes large. Consider also, for reference, the case in which the stores instead cooperate and each trade a fraction $1/n$ of that which a single store would have traded in the case $n = 1$, so that here $x_k = p/(2p'n)$ for all k . Here the total profit made by all the stores remains constant as n increases, a result which would also hold in the competitive case if the capacities of the individual stores were appropriately constrained.

The above example is concerned with the effects of competition between stores themselves, something which we explore in more detail in [Section 4](#), where in particular we study competition over extended periods of time. There are, however, also many more general issues surrounding the effects of storage considered collectively on the market in which it operates. Aspects of many of these issues have been explored in the literature. Recent work on the use of storage in a specifically market environment is given by [Gast, Tomozei, and Le Boudec \(2012\)](#), [Gast, Le Boudec, Proutiere, and Tomozei \(2013\)](#), [Graves, Jenkin, and Murphy \(1999\)](#), [Hu, Chen, and Bak-Jensen \(2010\)](#) and [Secomandi \(2010\)](#). [Sioshansi, Denholm, Jenkin, and Weiss \(2009\)](#) study the effects of storage on producer and consumer surplus and on social welfare. [Sioshansi \(2014\)](#) gives

an example where storage may reduce social welfare. [Gast et al. \(2013\)](#) show how in appropriate circumstances storage may be used to minimise generation costs and thus maximise consumer welfare.

There is also a considerable literature on the economics of hydroelectric power, which may be regarded as storage in which in general only the output process is controllable. Within this literature input flows are given and often modelled as stochastic; then the problem is that of the optimal control of the outflows, something which is frequently approached via stochastic dynamic programming – for recent work see in particular [Löhdorf, Wozabal, and Minner \(2013\)](#) and [Zéphy, Lang, Lamond, and Côté \(2017\)](#) and the references therein. Other work is more applied, focusing on hydroelectric power as it exists today in particular places – see [Fleten and Kristoffersen \(2007\)](#), [Borenstein and Bushnell \(1999\)](#), [Bushnell \(2003\)](#), and the survey by [Rangel \(2008\)](#). The latter three papers are concerned with the market impact of hydroelectric storage in a competitive environment and stores are therefore treated as price makers. The present paper is concerned with competition between more general forms of storage which is sufficiently large as to have market impact, in which both input and output may be controlled, and in which it is necessary to explicitly account for both rate and capacity constraints in the optimisation of the behaviour of each individual store. We make use of a Lagrangian approach ([Proposition 1](#) below) to yield prices for the rate and capacity constraints. The resulting optimality conditions for each agent given the actions of its competitors provide a complementarity problem defining a Nash equilibrium.

In the present paper we therefore aim to develop a more comprehensive mathematical theory of the way in which storage, buying and selling over a possibly extended period of time in such a way as to maximise its overall profit from these activities, interacts with the time-varying market in which it finds itself. Our motivation is to understand both this interaction and also the way in which competing stores, through this interaction, affect each other's abilities to make profits. We study the former in [Section 3](#), looking in particular at the impact of storage on prices and consumer surplus, and providing examples with conclusions which are in some cases counter-intuitive; these results complement those of other authors. Our principal concern, however, is to study the effect of competition between stores. While this is illustrated in [Example 1](#) above, in that example the activity of each store is determined by its decision at a single point in time – since what is bought at time 1 must be sold at time 2. We show in [Section 4](#) that conclusions similar to those of [Example 1](#) continue to hold when stores, unconstrained in their capacities and rates, operate over extended periods of time under a similar model of Cournot competition: within this model each store optimises its own total profit over time given the profiles of quantities traded by the remainder. Notably we show that again a large number of stores severely reduce each other's profitability in a manner which (precisely) quantitatively mirrors that of the earlier example. However, the imposition of capacity and/or rate constraints on the stores reduces their ability to affect each other in this way, to the benefit of all the stores. We discuss also in [Section 4](#) the extent to which other models of competition between stores are possible.

Our fundamental assumption is therefore that each individual store operates over a given period of time in such a way as to optimise its “profit” – or equivalently minimise its costs – with respect to time-varying cost functions presented to it. These may represent either the prevailing costs within a free market, as may be natural when the store is independently owned, or adjusted costs which take into account the wider impact of the store's activities, as would be appropriate when the store was owned, for example, by the generators or by society – see [Section 5](#). Thus if it is desirable that a store should function in a particular way – for example

so as to minimise generation costs – it may be fed the appropriate cost signals and, given those signals, left to perform as an autonomous agent. Such an approach is notably desirable in facilitating distributed control and optimisation within a possibly complex environment.

We outline in Section 2 the model for the market in which storage operates, allowing that it may do so over an extended period of time. In particular, the model allows for supply and demand which are sensitive to price, and hence also for an impact on price of the market activities of the storage itself. We assume for the moment (but see also below) that a single store wishes to optimise its own profit, or minimise its own costs, by trading in the market. We formulate the corresponding optimisation problem faced by the store and we state how it may be solved. Formally the environment is deterministic; however, we discuss in Section 6 one way in which we may proceed similarly in a stochastic environment.

In Section 3, we study the effect of a single profit-maximising store in a market. We look at its effect on both market prices and on consumer surplus and give sensitivity results for the variation of the size of the store. We give examples based on Great Britain market data.

In Section 4, we study a number of competing stores operating in a market. We consider possible models of competition, whereby the stores make bids and clearing prices in the market are determined. For models for two time periods, as in Example 1 above, we show that it is possible to consider not just Cournot competition but also quite general supply function bidding, as in the “single-shot-in-time” models which would hold in the absence of storage. For models for longer time periods, it is difficult to formulate such more general bidding strategies in a realistic manner, and we focus on Cournot models of competition in which stores bid time-indexed vectors of quantities. We identify Nash equilibria for such models, give existence and uniqueness results, and show how equilibria may be determined. We generalise Example 1 to show that, over such extended time periods, an oversupply of storage capacity also leads to a situation in which, with linear price functions, the total profit made by all the stores is approximately inversely proportional to their number. Essentially what happens here is that, as in Example 1 and relative to a cooperative solution, each store over-trades in order to acquire a larger share of total profit, thereby impacting on the market in such a way as to reduce price differentials over time and thus also the profits to be made by other stores. In this section we also give examples of such competition which are again based on GB market data.

In Section 5, we consider variant problems in which storage (instead of consisting of independent profit-maximising entities) is managed, for example, for the optimal benefit of consumers, or for the optimal benefit of generators. We show that, by suitable redefinition of cost functions, these variant problems may be reduced mathematically to those already studied. Finally, in Section 6 and as indicated above, we also make suggestions as to how one might reasonably proceed in a stochastic environment.

We remark also that, although the motivation for the present paper is that of understanding better the contribution storage is able to make in the management of complex energy systems, together with the impact of such storage on the markets in which it is traded, the results of the paper are of course equally applicable to the storage and trading of other commodities.

2. Model

We now formulate our model for a set of $n \geq 1$ stores operating in an energy market. Formally we treat prices and costs as deterministic – however, see also Section 6.

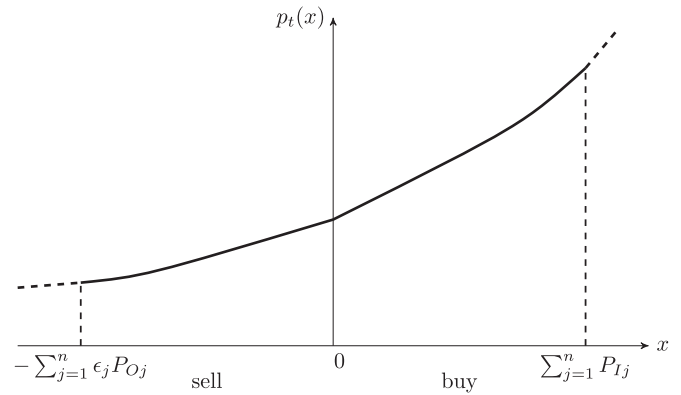


Fig. 1. Illustrative price function p_t . Positive or negative values of its argument correspond respectively to the stores collectively buying or selling. The domain of the function is effectively restricted to the interval $[-\sum_{j=1}^n \epsilon_j P_{Oj}, \sum_{j=1}^n P_{Ij}]$.

We assume that each store j has an energy capacity E_j and input and output rate constraints P_{Ij} and P_{Oj} respectively (the maximum amount of energy which can enter or leave the store per unit time). Each such store j also has an efficiency $\epsilon_j \in (0, 1]$, where ϵ_j is the number of units of energy output which the store can achieve for each unit of energy input. We assume without loss of generality that any loss of energy due to inefficiency occurs immediately after leaving the store (so that the above capacity and rate constraints – both input and output – apply to volume of energy input). For simplicity we also assume that there is no time-dependent leakage of energy from the stores; the simple adjustments required to deal with any such leakage are straightforward.

We work in discrete time $t = 1, \dots, T$ for some finite time horizon T . Associated with each such time t is a price function p_t such that $p_t(x)$ is the price per unit of energy when x is the total amount (positive or negative) of energy bought from the market by all the stores, i.e. $x p_t(x)$ is the total cost to the stores of buying this energy. According to the environment in which the stores operate, the total cost $x p_t(x)$ which they incur may be determined by, for example, prices in both interday and intraday markets: our model is one of competition between stores making profits by price arbitrage over both long and short periods of time. (Each of the functions p_t is of course influenced by everything else that is happening in the market at time t ; it explicitly measures only the further effect on price of the activity of the stores.) We assume throughout that, over the range of possible values of its argument (i.e. the interval $[-\sum_{j=1}^n \epsilon_j P_{Oj}, \sum_{j=1}^n P_{Ij}]$), each of the functions p_t is positive and increasing and is such that, for any constant k , the function of x given by $x p_t(x + k)$ is convex and increasing. (The quantity $x p_t(x + k)$ is the total cost to a store of buying x units of energy – again positive or negative – at time t when the total amount bought by the remaining stores at that time is k .) A typical price function p_t is thus as illustrated in Fig. 1. An important case in which these conditions are satisfied, and which we consider in detail later, is that where the prices are linearised so that

$$p_t(x) = \bar{p}_t + p'_t x \quad (1)$$

where $\bar{p}_t > 0$ and where $p'_t \geq 0$ is such that the function p_t remains positive for all possible values of its argument as above. This should, for example, be a good approximation whenever the total storage capacity is not too large in relation to the total size of the market in which the stores operate. In such a case, we may take $\bar{p}_t = p_t(0)$ (i.e. the price at time t without storage on the system) and $p'_t = p'_t(0)$. More generally, the above conditions on the functions p_t seem likely to be satisfied in many cases, for example when they do not differ too much from the above linear case, and are in all cases readily checkable.

In particular if $s_t(p)$ is the amount externally supplied to the market at time t and price p and $d_t(p)$ is the corresponding total demand at that time and price – and if the functions s_t and d_t are given independently of the activities of any stores – then we may define the residual supply function R_t at that time by $R_t(p) = s_t(p) - d_t(p)$; if R_t is continuous and strictly increasing then we have that p_t is the inverse of the function R_t and is similarly continuous and strictly increasing. If, furthermore, each of the functions R_t is differentiable and prices take the form (1), with $\bar{p}_t = p_t(0)$ and $p'_t = p'_t(0)$, then we may relate p'_t to the point elasticities of supply and demand at price \bar{p}_t , denoted e_s and e_d respectively, in the following way:

$$p'_t = \frac{\bar{p}_t}{e_s s_t(\bar{p}_t) - e_d d_t(\bar{p}_t)}. \quad (2)$$

This method of determining the price functions p_t is especially relevant when the other players in the market make their decisions without taking the stores' actions into account, perhaps due to the relatively small level of storage capacity in relation to the rest of the market.

Obtaining appropriate price functions may be quite challenging in practice. However, appropriate data on the relationships (in many different countries) between supply, demand and price is available from various sources – see in particular the very extensive discussion in the report by Newbery et al. (2013) and the further references therein. It is further to be expected that were storage to be used more extensively in future energy systems (greater use of storage being considered as one option in the management of the supply-demand balance in such systems – again see Newbery et al. (2013)) then the operation of that storage within markets would itself provide data on the market impact of the stores' activities. Indeed with sufficient information, more complex price functions p_t than the linear ones defined by (1) could reasonably be derived.

We denote the successive levels of each store j by a vector $S_j = (S_{j0}, \dots, S_{jT})$ where each S_{jt} is the energy level of the store at time t . It is convenient to assume that the initial and final levels of the store are constrained to fixed values S_{j0}^* and S_{jT}^* respectively. For each such vector S_j and for each $t = 1, \dots, T$, define also $x_t(S_j) = S_{jt} - S_{j,t-1}$ to be the amount (positive or negative) by which the level of the store is increased at time t .

In order to incorporate efficiency, it is helpful to define, for each store j , the function h_j on \mathbb{R} by $h_j(x) = x$ for $x \geq 0$ and $h_j(x) = \epsilon_j x$ for $x < 0$. For each time t such that $x_t(S_j) \geq 0$, store j buys $x_t(S_j)$ units of energy from the market, while for t such that $x_t(S_j) < 0$, it sells $-\epsilon_j x_t(S_j)$ units of energy to the market. For each store j and time t , and given the changes x_{it} , $j \neq i$, (positive or negative) in the levels of the remaining stores at that time, define now the cost function $C_{jt}(\cdot; x_{it}, j \neq i)$ by

$$C_{jt}(x_{jt}; x_{it}, j \neq i) = h_j(x_{jt}) p_t \left(\sum_{i=1}^n h_i(x_{it}) \right); \quad (3)$$

this represents the cost to store j of increasing its level by x_{jt} (again positive or negative) at time t , given the corresponding activities of the remaining stores at that time. Note that the conditions on the function p_t ensure that $C_{jt}(x_{jt}; x_{it}, j \neq i)$ is an increasing convex function of its principal argument x_{jt} and takes the value zero when this argument is zero.

In particular if the objective of store j is to optimise its profit, given the policy over time $S_j = (S_{j0}, \dots, S_{jT})$ of every other store $i \neq j$, then it faces the following optimisation problem:

P_j: Choose $S_j = (S_{j0}, \dots, S_{jT})$ so as to minimise the function of S_j given by

$$\sum_{t=1}^T C_{jt}(x_t(S_j); x_t(S_i), j \neq i) \quad (4)$$

subject to the capacity constraints

$$S_{j0} = S_{j0}^*, \quad S_{jT} = S_{jT}^*, \quad 0 \leq S_{jt} \leq E_j, \quad 1 \leq t \leq T-1. \quad (5)$$

and the rate constraints

$$x_t(S_j) \in X_j, \quad 1 \leq t \leq T, \quad (6)$$

where $X_j = \{x : -P_{0j} \leq x \leq P_{1j}\}$.

Note that the observed convexity of the cost functions $C_{jt}(\cdot; x_{it}, j \neq i)$ ensures that a solution to the optimisation problem **P_j** always exists.

At various points we make use of the following proposition, in which each of the vectors μ_j^* is essentially a vector of (cumulative) Lagrange multipliers. The first part of the result follows from Theorem 1 of Cruise, Gibbens, and Zachary (2014), while the crucial existence assertion at the end of the proposition is a consequence of the algorithm of that paper and is summarised in its Theorem 2. The proof of the entire result is relatively short, and to provide insight and make the present paper self-contained we give a version of it in the Appendix.

Proposition 1. For any store $j = 1, \dots, n$, and for any fixed policies S_i of every other store $i \neq j$, suppose that there exists a vector $\mu_j^* = (\mu_{j1}^*, \dots, \mu_{jT}^*)$ and a value $S_j^* = (S_{j0}^*, \dots, S_{jT}^*)$ of S_j such that

(i) S_j^* is feasible for the stated problem **P_j**;

(ii) for each t with $1 \leq t \leq T$, $x_t(S_j^*)$ minimises

$$C_{jt}(x_{jt}; x_t(S_i), j \neq i) - \mu_{jt}^* x_{jt}$$

in $x_{jt} \in X_j$; and

(iii) the pair (S_j^*, μ_j^*) satisfies the complementary slackness conditions, for $1 \leq t \leq T-1$,

$$\begin{cases} \mu_{j,t+1}^* = \mu_{jt}^* & \text{if } 0 < S_{jt}^* < E_j, \\ \mu_{j,t+1}^* \leq \mu_{jt}^* & \text{if } S_{jt}^* = 0, \\ \mu_{j,t+1}^* \geq \mu_{jt}^* & \text{if } S_{jt}^* = E_j. \end{cases} \quad (7)$$

Then S_j^* solves the above optimisation problem **P_j**. Further, the given convexity of the cost functions $C_{jt}(\cdot; x_{it}, j \neq i)$ guarantees the existence of such a pair (S_j^*, μ_j^*) .

For a single store, Cruise et al. (2014) provides an algorithm for determining a suitable pair (S_j^*, μ_j^*) satisfying the conditions of Proposition 1. (For a discussion of what happens in the absence of the convexity of the cost functions $C_{jt}(\cdot; x_{it}, j \neq i)$ see Flatley, MacKay, and Waterson (2016).)

Remark 1. In cases where the stores are not independent profit maximising entities but are instead owned by, for example, the generators or by society, the above cost functions C_{jt} may be appropriately modified so that the problems **P_j** continue to define optimal behaviour for the stores; see Section 5 for a discussion of how this may be done.

3. The single store in a market

In the case $n = 1$ of a single store it is convenient to drop the subscript j and to write S for S_j , etc. The single-store optimisation problem is then to choose $S = (S_0, \dots, S_T)$ so as to minimise

$$\sum_{t=1}^T C_t(x_t(S))$$

(where the C_t are the cost functions defined by (3)) subject to the capacity constraints (5) and rate constraints (6). For simplicity we assume the strict convexity of the cost functions C_t – as, for example, will be the case when the linear approximation (1) holds with $p'_t > 0$ for each t . This strict convexity is sufficient to guarantee the uniqueness of the solution S^* of the optimisation problem **P**.

3.1. Sensitivity of store activity to capacity and rate constraints

Let (S^*, μ^*) be the pair identified in Proposition 1, defining the solution S^* of the above optimisation problem **P**. Then the market clearing price at each time t is $p_t(h(x_t(S^*)))$. The successive clearing prices then determine such quantities as consumer surplus – in the way we describe later.

As a measure of the sensitivity of the market to variation of the size of the store, we use Proposition 1 to describe briefly how variation of either the capacity or the rate constraints of the store impacts on the solution S^* of **P**. Proposition 1 continues to hold when we allow either the capacity or the rate constraints of the store to depend on the time t . Therefore it is sufficient to consider the effect of variation of these constraints at any single time t_0 .

Consider first the effect of an arbitrarily small increase (positive or negative) δE_{t_0} in the capacity of the store at time t_0 ; since the initial and final levels S_0^* and S_T^* are fixed we assume $0 < t_0 < T$. It is clear from Proposition 1 that this infinitesimal change has no effect on S^* unless $S_{t_0}^* = E$; further if $\delta E_{t_0} > 0$ we also require the strict inequality $\mu_{t_0+1}^* > \mu_{t_0}^*$. Under these conditions there exist times $t_1 < t_0 < t_2$, such that the effect of the increment δE_{t_0} – provided it is indeed sufficiently small – is to change $\mu_{t_1}^*$, and so also $x_t(S^*)$ (via condition (ii) of Proposition 1), for t such that $t_1 < t \leq t_0$, both the original and the new values of μ_t^* being constant over this interval, and to similarly change $\mu_{t_2}^*$ and $x_t(S^*)$ for t such that $t_0 < t \leq t_2$, again both the original and the new values of μ_t^* being constant over this interval; all changes within the second of the above intervals have the opposite sign to those within the first; for all remaining values of t , the parameter μ_t^* remains unchanged. The change in μ_t^* over each of the above intervals is readily determined by the requirement that now $S_{t_0}^* = E + \delta E_{t_0}$. (Thus, for example, for a perfectly efficient store and twice differentiable cost functions C_t , the effect of an increment $\delta E_{t_0} > 0$ – where t_0 is such that $\mu_{t_0+1}^* > \mu_{t_0}^*$ – will be to increase $x_t(S^*)$ in proportion to $1/C_t''(x_t(S^*))$ for times t such that $t_1 < t \leq t_0$ and at which the input rate constraint is nonbinding, and to similarly decrease $x_t(S^*)$ in proportion to $1/C_t''(x_t(S^*))$ for times t such that $t_0 < t \leq t_2$ and at which the output rate constraint is nonbinding.)

Similarly an arbitrarily small change at time t_0 in either the input or the output rate constraint has no effect on (S^*, μ^*) unless $\mu_{t_0}^*$ and $x_{t_0}(S^*)$ are such that that constraint is binding in the solution of the minimisation problem of (ii) of Proposition 1. The effect is then again to change $\mu_{t_1}^*$ and $x_t(S^*)$ for those t in an interval which includes t_0 ; both this interval and the required changes are again readily identifiable from that proposition.

3.2. Impact of a store on prices and consumer surplus

It is helpful to understand the impact of storage on various economic measures. Sioshansi (2014) considers some of these issues in the context of a two-time-period model in which the generating cost functions are quadratic and the same in each time period and in which the demand functions are linear (see also the further references in that paper). Sioshansi studies various market and ownership structures and shows that the introduction of storage can, under certain circumstances, reduce *social welfare* – defined as a sum of consumer surplus, producer (i.e. generator) surplus and storage profit. In this section we discuss briefly, in the

more general setting of the present paper, two related economic problems. The first issue is concerned with the impact of storage, and in particular of varying storage efficiency, on market prices; here we give an example to illustrate possible behaviour. The second issue is that of the impact of storage itself on consumer surplus, where we show that time-varying price sensitivities lead to possibly counter-intuitive behaviour.

3.2.1. Impact on prices

In general we may expect the impact of the store on the market to be that of smoothing prices over time: the store will in general buy at times when prices are low, thereby competing in the market and increasing prices at those times, and similarly sell at times when prices are high, thereby decreasing them at those times. Relaxing the rate or capacity constraints of the store might then be expected to result in further smoothing of the prices, as the store is able to buy and sell more at times of low and high prices respectively, thereby augmenting the above effect. However, such price smoothing by a profit-optimising store also reduces the profits of the store, relative to the situation which would be the case if the store's activities did not have market impact. In particular such smoothing may not occur to the same extent as would be socially optimal – as for example when the store was owned by society. Specifically the store may sell less than is socially optimal at periods of high prices and buy less than is socially optimal at periods of low prices in order to maintain the price differential and hence maximise the store's profit. The situation is exacerbated when price sensitivities are high. A similar phenomenon is identified by Bushnell (2003), who considers hydroelectric storage (in which only the storage output is controllable). He shows that it may be advantageous to the store to reduce output – relative to that which would be socially optimal, or obtain in a regulated market – at times of high prices and relatively inelastic demand, and to correspondingly increase output at other times, so as to benefit from a price profile which is more differentiated over time than that which would obtain were the store to be optimally operated for the benefit of society.

We might also expect that increasing the efficiency of the store will further smooth prices; however, behaviour here is more complex, as we illustrate in the following example.

Example 2. Consider price functions of the linear form (1) and a store which operates over just two time steps ($T = 2$), starting and finishing empty but not otherwise subject to capacity or rate constraints. Suppose further that $\bar{p}_2 > \bar{p}_1$. Given its efficiency ϵ , the store buys $x(\epsilon)$ units of energy at time 1 and sells $\epsilon x(\epsilon)$ units at time 2 so as to minimise its total cost

$$x(\epsilon)(\bar{p}_1 + p'_1 x(\epsilon)) - \epsilon x(\epsilon)(\bar{p}_2 - p'_2 \epsilon x(\epsilon)),$$

so that

$$x(\epsilon) = \begin{cases} 0 & \text{if } \epsilon \bar{p}_2 < \bar{p}_1 \\ \frac{\epsilon \bar{p}_2 - \bar{p}_1}{2(p'_1 + \epsilon^2 p'_2)} & \text{otherwise.} \end{cases} \quad (8)$$

In the presence of the store the difference between the market clearing price at time t_2 and that at time t_1 is given by

$$p_2(-\epsilon x(\epsilon)) - p_1(x(\epsilon)) \quad (9)$$

Suppose now that the efficiency ϵ of the store is increased. For suitable values of the parameters \bar{p}_t , p'_t , $t = 1, 2$, it follows from (8) that the quantity $x(\epsilon)$ bought by the store at time 1 then also increases (increasing the market clearing price at time 1) and so similarly the quantity $\epsilon x(\epsilon)$ sold by the store at time 2 increases (decreasing the market clearing price at time 2), with the overall consequence of reducing the price differential (9) as expected. However, it is straightforward to check – by for example differentiation of (8) – that, for other values of the parameters (specifically

when $\bar{p}_2/\bar{p}_1 > 2p'_2/(p'_2 - p'_1)$, the quantity $x(\epsilon)$ is eventually decreasing in ϵ as the latter increases towards 1, so that increasing the efficiency of the store now decreases the market clearing price at time 1. It is also straightforward (if slightly tedious) to check that under these circumstances the quantity $x(\epsilon)$ continues to be an increasing function of ϵ and that further, for all values of the parameters \bar{p}_t , p'_t , $t = 1, 2$, the price differential (9) is always a decreasing function of ϵ . The minimum price differential, at $\epsilon = 1$, is $(\bar{p}_2 - \bar{p}_1)/2$, i.e. exactly half of the price differential in the absence of storage.

3.2.2. Impact on consumer surplus

The consumer surplus associated with a demand function d and clearing price p_0 is usually defined as $\int_{p_0}^{\infty} d(p) dp$, and so the consumer surplus of the store's optimal strategy S^* is given by

$$\sum_{t=1}^T \int_{p_t(h(x_t(S^*)))}^{\infty} d_t(p) dp, \quad (10)$$

where $d_t(p)$ is the consumer demand associated with price p at time t . If the size or activity level of the store is such that the price changes caused by its introduction are relatively small, and we additionally make the linear approximation (1), then the change in consumer surplus due to the introduction of the store is well approximated by

$$-\sum_{t=1}^T h(x_t(S^*)) p'_t d_t(\bar{p}_t). \quad (11)$$

It might reasonably be expected that, if the store is reasonably efficient (ϵ is close to one) and if prices are well-correlated with demand, then the store will buy ($x_t > 0$) at times of low consumer demand and sell ($x_t < 0$) at times of high consumer demand, and that this will have a beneficial effect on consumer surplus – as suggested by (11) whenever the price sensitivities p'_t are sufficiently similar to each other. However, these price sensitivities p'_t do need to be taken into account. Again we give an example.

Example 3. Consider again a store with linear prices of the form (1), which starts and finishes empty and which operates over just two time steps, i.e. $T = 2$. Assume that the power ratings of the store exceed its capacity and that demand is completely inelastic, so that, for $t = 1, 2$, there exists $d_t^* \geq 0$ such that $d_t(p) = d_t^*$ for all prices p . Then, from (8) and (10), as long as $p_1 < \epsilon p_2$, the change in consumer surplus on introducing the store to the electricity network is

$$\min \left(\frac{\epsilon \bar{p}_2 - \bar{p}_1}{2(p'_1 + \epsilon^2 p'_2)}, E \right) (\epsilon p'_2 d_2^* - p'_1 d_1^*),$$

which is clearly negative whenever $\epsilon p'_2 d_2^* < p'_1 d_1^*$. In the latter case the price sensitivity p'_1 at time 1 is sufficiently high that the decrease in consumer surplus at this time as a result the store buying outweighs the increase in consumer surplus at time 2 as a result of the store selling.

3.3. Example

We consider an example based on half-hourly market electricity prices in Great Britain throughout the year 2014. These are the so-called Market Index Prices as supplied by Elexon Ltd (2017), who are responsible for operating the Balancing and Settlement Code for the Great Britain wholesale electricity market. These are considered to form a good approximation to real-time spot prices.

These prices, given in units of pounds per megawatt-hour, exhibit an approximately cyclical behaviour, being high by day and low by night and, apart from this, are reasonably consistent

throughout the year except for some mild seasonal variation, notably that prices are slightly lower during the summer months.

We take the price functions p_t to be given by

$$p_t(x) = \bar{p}_t(1 + \lambda x), \quad (12)$$

where the \bar{p}_t , $t = 1, \dots, T$, are proportional to the spot market prices referred to above. These price functions are a special case of the linear functions (1), in which the price sensitivity p'_t is proportional to \bar{p}_t , an assumption which is in many circumstances very plausible; the constant of proportionality $\lambda \geq 0$ may then be considered a market impact factor. Relation (12) also implies that λ should be chosen in proportion to the physical size of the unit of energy: for any $k > 0$, the substitution of x/k for x and $k\lambda$ for λ leaves (12) unchanged. We therefore find it convenient to consider a store whose nominal dimensions are generally held constant, and to allow λ to vary: the market impact as λ is increased is equivalent to that which occurs when λ is held constant and the dimensions of the store are allowed to increase instead. The case $\lambda = 0$ corresponds to no market impact (appropriate to a relatively small store). Clearly also there exists λ_{\max} such that, for $\lambda \geq \lambda_{\max}$ both the rate and capacity constraints of the store cease to be binding, so that for all $\lambda \geq \lambda_{\max}$ the market impact of the store is the same, and – again by the above scaling argument – may be regarded as that of an unconstrained store.

We take a storage facility with common input and output rate constraints and, without loss of generality, we choose units of energy such that, on the half-hourly timescale of the spot-price data, this common rate constraint is equal to 1 unit per half-hour. For the numerical example, we in general take the capacity of the store to be given by $E = 10$ units; this corresponds to the assumption that the store empties or fills in a total time of 5 hours. This capacity to rate ratio is fairly typical, being in particular close to that for the Dinorwig pumped storage facility in Snowdonia (First Hydro Company, 2017) (though the charge time and discharge times for Dinorwig are approximately 7 hours and 5 hours respectively). We in general take the round-trip efficiency as $\epsilon = 0.75$, which is again comparable to that of Dinorwig. Thus the effect on market prices given by varying λ , which we discuss below, corresponds to the effect on the market given by considering rescaled versions of a facility not too dissimilar from Dinorwig. We also investigate briefly the effect of varying the capacity constraint E relative to the unit rate constraint, and the effect of varying the round-trip efficiency ϵ .

Fig. 2 shows, for $E = 10$ and $\epsilon = 0.75$, the effect of varying the market impact λ . The control of the store is optimised, as previously discussed, over the entire one-year period for which price data are available (with the store starting and finishing empty). For relatively small values of λ the store fills and empties (or nearly so) on a daily cycle, as it takes advantage of low nighttime and high daytime prices. For significantly larger values of the market impact factor λ , the store no longer fills and empties on a daily basis (as this factor now erodes the day-night price differential as the volume traded increases); however, the level of the store may gradually vary on a much longer time scale as the store remains able to take advantage of even modest seasonal price variations. The first six panels of panels of Fig. 2 show plots of the time-varying levels of the store against selected values of λ . For $\lambda = 0$, $\lambda = 0.1$ and $\lambda = 0.5$ the level of the store is plotted against time for the first two weeks of the year, while for $\lambda = 1$, $\lambda = 5$ and $\lambda = 10$ the level of the store is plotted against time for the entire year. The final panel of Fig. 2 shows a plot against time – for the first two weeks of the year – of the market clearing price corresponding to $\lambda = 0$, $\lambda = 0.5$, and $\lambda = 10$. The erosion of the day/night price differential as λ increases is clearly seen.

For values of λ greater than $\lambda_{\max} \approx 23$ the volumes traded are such that neither the rate nor the capacity constraints of the store

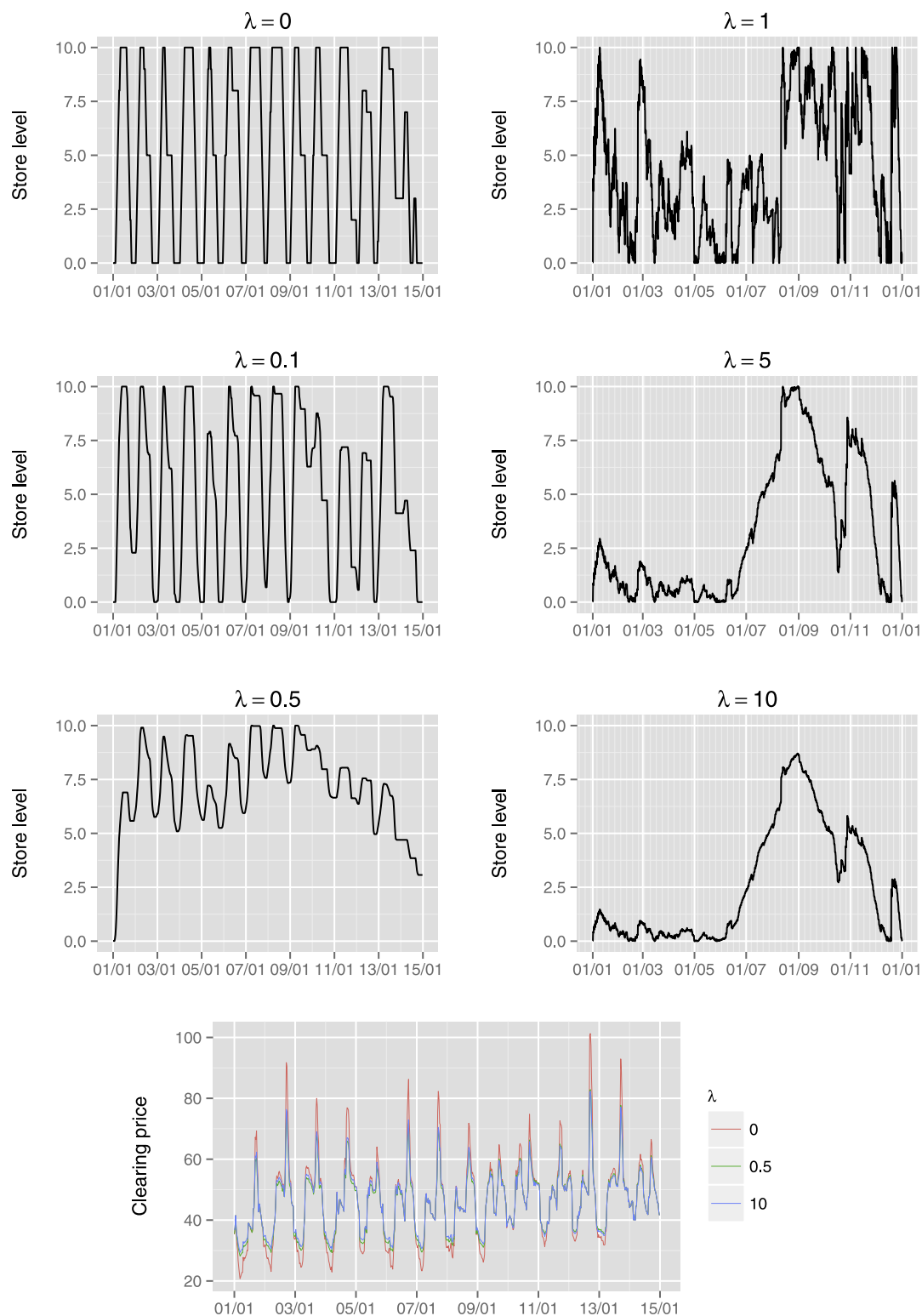


Fig. 2. Single store: behaviour of store level and market clearing price (see text for a discussion of units) as the market impact factor λ is varied – equivalently the size of the store is varied.

are binding, so that for $\lambda > \lambda_{\max}$ volumes traded are simply proportional to $1/\lambda$.

The left panels of Fig. 3 show the effect on store level – over the entire year – of decreasing the efficiency of the store from $\epsilon = 0.75$ (for which the store level is shown in red) to $\epsilon = 0.65$ (for which the store level is shown in blue), for each of the larger values of λ considered above, i.e. for $\lambda = 1$, $\lambda = 5$ and $\lambda = 10$.

The capacity of the store is here kept at our base level of $E = 10$. Decreasing the efficiency of the store reduces its ability to exploit the daily cycle of price variation in a manner not dissimilar from that of increasing the market impact λ , so that again the volumes of daily trading are reduced, while the store may continue to exploit its full capacity on a seasonal basis – again for a very modest further gain. We remark also that reducing the

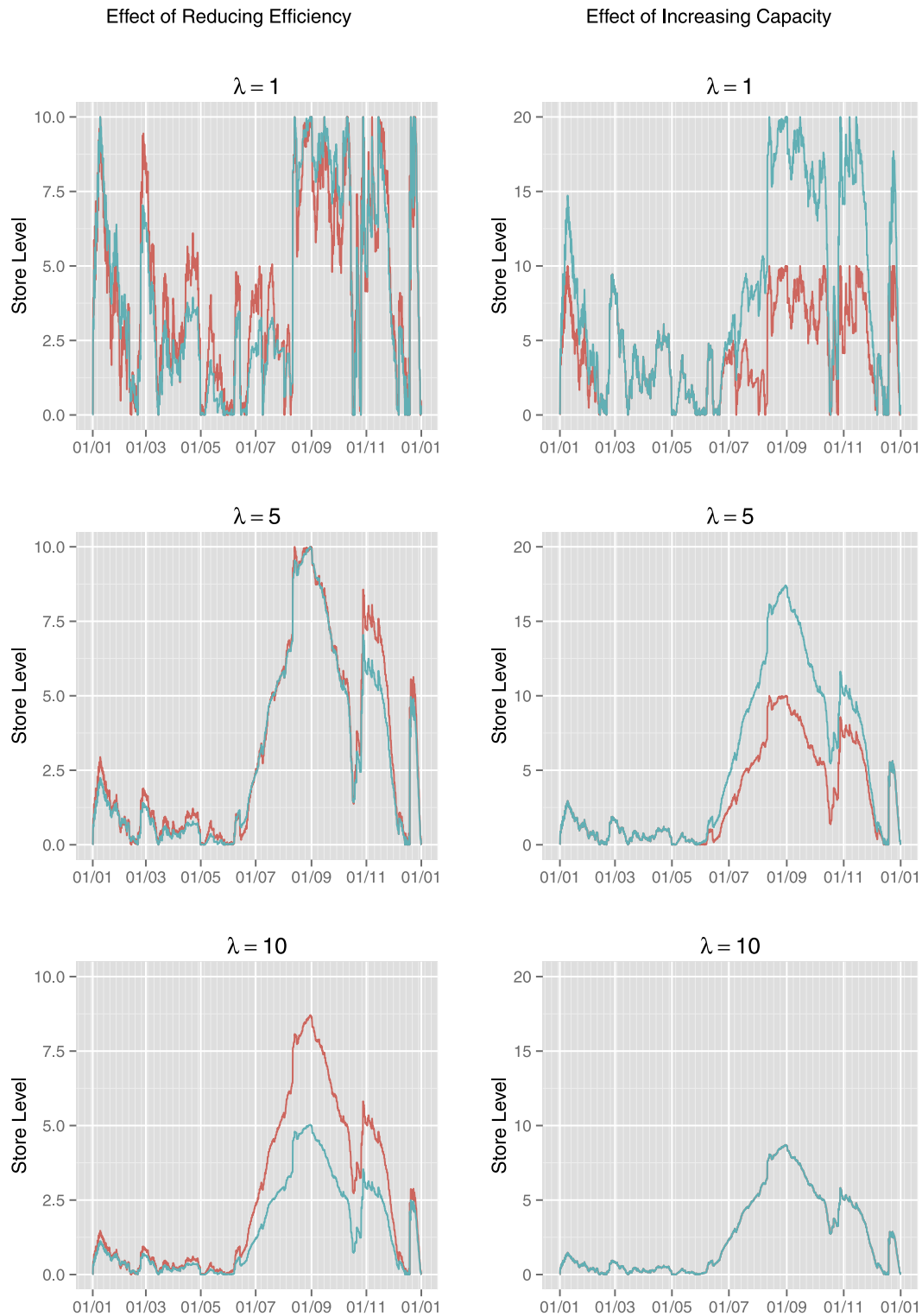


Fig. 3. Single store: behaviour of the store level as the round-trip efficiency ϵ is varied from 0.75 to 0.65 (left panels) and as the capacity E is varied from 10 to 20 (right panels), in each case for $\lambda = 1$, $\lambda = 5$ and $\lambda = 10$. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

efficiency of the store reduces the extent to which it is able to smooth prices.

The right panels of Fig. 3 similarly show the effect – again over the entire year and for the same three values of λ – of increasing the capacity of the store from $E = 10$ (for which the store level is shown in red) to $E = 20$ (for which the store level is shown in blue). The round trip efficiency of the store is kept at $\epsilon = 0.75$. In

each case it is seen that the daily variation in the level of the store remains much the same as E is increased (since for these levels of λ there is too much market impact to make profitable greater volumes of daily trading, except on occasions in the case $\lambda = 1$). However, for $\lambda = 1$ and for $\lambda = 5$, as E is increased the store is able to make some (very modest) additional profit by varying slowly throughout the year the general level at which it operates. For

$\lambda = 10$ the market impact is so great that the capacity constraint $E = 10$ – and so also the capacity constraint $E = 20$ – is never binding, so that in this case the increase in the capacity has no effect.

4. Competing stores in a market

In this section, we discuss n competing stores in a market, where it is assumed that the objective of each store is to maximise its own profit. The optimal strategy of each store in general depends on the activities of the remainder, and what happens depends on the extent to which there is cooperation between the stores. Such cooperation may occur, for example, if a number of stores are owned by the same company. Note, however, that, in the case of energy storage at least, any form of collusion between storage facilities (or indeed any other form of excessive profit maximisation) would inevitably be the subject of regulatory action, so that perfect cooperation between large storage facilities seems unlikely. In the absence of cooperation between stores we might reasonably expect some form of convergence over time to a Nash equilibrium, in which each store's strategy is optimal given those of the others. We nevertheless first discuss briefly the cooperative solution, primarily for the purpose of reference, before considering the effect of market competition.

4.1. The cooperative solution

Here the stores behave cooperatively so as to minimise their combined cost

$$\sum_{i=1}^n \sum_{t=1}^T h_i(x_t(S_i)) p_t \left(\sum_{k=1}^n h_k(x_t(S_k)) \right), \quad (13)$$

subject to the capacity constraints (5) and rate constraints (6). This is a generalisation to higher dimensions of the single-store problem, and we do not discuss a detailed solution here. Note, however, that an iterative approach to the determination of a solution may be possible. Under our assumptions on the price functions, the function of S_1, \dots, S_n given by (13) is convex. For any store j , given the levels S_i of the remaining stores $i \neq j$, the minimisation of (13) in S_j (subject to the above constraints) is an instance of the single-store problem discussed in Section 3 – with cost functions modified so as to reflect the overall cost to all the stores of the actions of the store j . This leads to the obvious iterative algorithm in which (13) is minimised in S_j for successive stores j until convergence is achieved. However, the limiting value of (S_1, \dots, S_n) , while frequently a global minimum, is not guaranteed to be so.

In the case where the stores have identical efficiencies one might also consider the simplified single-store problem in which the individual capacity constraints are summed and individual rate constraints are summed. If the solution to this, suitably divided between the stores (i.e. with a fraction κ_i of the optimal flow assigned to each store i , where $\sum_{i=1}^n \kappa_i = 1$), is feasible for the original problem then it solves that problem. One case where this is true is where additionally the ratios E_j/P_{ij} and E_j/P_{oj} are the same for all stores j ; the solution to the simplified single-store problem is then just divided among the stores in proportion to their capacities to give the cooperative solution to the n -store problem.

The impact of the stores on market prices and consumer surplus is determined in a manner entirely analogous to that of Section 3.2.

4.2. The competitive solution

When stores compete there needs to be a mechanism whereby a clearing price in the market is determined. Here there are in principle various possibilities according to the rules under which

the market is to operate. We discuss some of these in Section 4.2.1, making a formal link with the various classical modes of competition in simple “single shot in time” markets for balancing supply and demand in situations where storage does not operate. In the succeeding sections we look in particular at what happens when stores bid quantities, i.e. at Cournot models of competition.

4.2.1. Possible models of competition

Consider first the case $T = 2$, and assume for simplicity that the stores are perfectly efficient. Suppose that each store k buys and then sells x_k (positive or negative), and that this results in a price differential of p (the clearing price at time 2 less that at time 1) so that each store k makes a profit px_k . We might consider the situation where, in a precise analogue of the supply function bidding of Klemperer and Meyer (1989), each store k declares, for each possible value of p , a value $S_k(p)$ which it contracts to buy at time 1 and then sell at time 2 if the clearing prices at those times are set such that the price differential is p . If each “supply function” S_k is a nondecreasing function of p , the auctioneer then chooses the clearing prices p_1 and p_2 such that

$$R_1(p_1) = \sum_k S_k(p) \quad (14)$$

$$R_2(p_2) = - \sum_k S_k(p) \quad (15)$$

$$p_2 - p_1 = p, \quad (16)$$

where, for $t = 1, 2$, R_t is the residual supply function defined in Section 2.

Assume that the residual supply functions R_t are strictly increasing. The system of Eqs. (14)–(16) is easily seen to have a unique solution (provided the supply functions S_k are such that one exists at all): suppose that, as p varies, p_1 and p_2 are chosen as functions of p such that $p_2 - p_1 = p$ and $R_2(p_2) = -R_1(p_1)$; then, as p increases, $\sum_k S_k(p)$ increases while $R_1(p_1)$ decreases, and at the unique value of p such that we have equality between these two quantities the above system of equations (14)–(16) is satisfied.

Mathematically, this situation is no different from that of the classical “one-shot” supply function bidding of Klemperer and Meyer (1989). This was further studied in applications to energy markets by Green and Newbery (1992) and by Bolle (1992), and subsequently by many others – see in particular Anderson and Philpott (2002), and the very comprehensive review by Holmberg and Newbery (2010). In such supply function bidding suppliers (for example, electricity generators) submit nondecreasing supply functions to a market in which there is also a nonincreasing demand function, the market clearing price being the price (usually unique) at which the total supply equals the total demand. In the above “one-shot” situation a set of supply functions constitute a Nash equilibrium if the resulting clearing price (and corresponding quantity traded) is optimal for each supplier, given the supply functions of the remaining suppliers. Klemperer and Meyer (1989) show that, in a deterministic environment, sets of supply functions constituting Nash equilibria are not in general unique unless additional conditions are imposed on them. In practice one might well wish to do this so as to achieve economically acceptable solutions – see in particular Johari and Tsitsiklis (2011). In a random environment, Klemperer and Meyer (1989) further show that appropriate forms of uncertainty in demand may force the existence of a unique supply function Nash equilibrium.

Two particular cases of such bidding are the classical situations where either suppliers may bid prices at which they are prepared to supply any amount of the commodity to be traded – corresponding to “vertical” supply functions and leading to a Bertrand

equilibrium, or else suppliers may bid quantities which they are prepared to supply at whatever price clears the market – corresponding to horizontal supply functions and leading to a Cournot equilibrium. In the former case, at the Nash equilibrium, the one supplier who is able to offer the lowest price corners the market (and, in the case of symmetric suppliers, makes zero profit). In the latter case, modest profits are to be made, but the total profit of all the suppliers decreases rapidly as their number increases – as is seen also in our results for storage models below.

It is difficult to find a sensible and realistic way of extending the concept of general supply function bidding to competition amongst stores operating over more than two time periods – the dimensionality of the space in which the supply functions would then live is high, and the set of possibilities for market clearing mechanisms is complex. (In principle something might be attempted and the conditions under which a Nash equilibrium existed and was unique would then be a matter for significant further research.) Nor is it realistic to consider the situation where stores bids prices, since as indicated above, profits are then typically too small for stores to be able to recover their set-up costs. We therefore restrict our attention to the case where stores bid quantities – as seems to be the case where elsewhere in the literature market competition between stores is considered (see, for example, Sioshansi (2014)). Here the Nash equilibria are Cournot equilibria and the profits made by the stores at such equilibria may be expected to provide reasonable upper bounds on such profits as might be made in practice – for a review in the context of “one-shot in time” markets again see Holmberg and Newbery (2010).

Finally we remark here that it is also possible to consider the situation in which, given the functions S_k declared by the stores, market players other than the stores, notably generators, choose their own supply functions – one for each point in time – so that the resulting sequence of prices and quantities at which the market clears are optimal for these players, who thus take account of the presence of the stores in formulating their bidding strategies. This generalises the classical situation of supply function bidding by introducing a linkage of markets over time. This linkage means that, as described here, there would be considerable challenges in implementation over anything other than the shortest periods of time. However, one might well consider simpler situations in which, for example, generators, in formulating their successive supply functions, estimate as best they may the predicted response of storage.

4.2.2. General convex cost functions

We consider stores bidding quantities as above and look for Nash (Cournot) equilibria. A (pure strategy) Nash equilibrium is then a set of vectors (S_1, \dots, S_n) such that the strategy S_j of each store j (i.e. the vector of quantities traded over time by that store) is optimal given the strategies S_i , $i \neq j$, of the remaining stores; thus the vector S_j solves the optimisation problem \mathbf{P}_j (defined by the remaining vectors S_i , $i \neq j$) of Section 2. Equivalently, at a Nash equilibrium, the vector S_j minimises function (13) subject to constraints (5) and (6) and with the values of the vectors S_i , $i \neq j$, held constant.

Broadly what happens at such an equilibrium is that stores will buy and sell more than at the cooperative solution, since each store gains for itself the benefits of so doing, while the corresponding costs are shared out among all stores. In particular, in a generalisation of Example 1, consider n identical competing stores with nonbinding capacity and rate constraints, but with common given starting and finishing levels; for the moment assume further that they have round-trip efficiencies $\epsilon = 1$, and that the price functions p_t are differentiable. For each store k and for each time t , write $x_{kt} = x_t(S_k)$. At the symmetric Nash equilibrium, and for each store j , there are equalised over time t the partial derivatives with

respect to x_{jt} of the functions $x_{jt} p_t(\sum_{k=1}^n x_{kt})$. (For $n = 1$ these are just the derivatives of the cost functions seen by the store.) It is straightforward to show that the convexity of these functions ensures that in general unit prices received by the store at those times when it is selling are higher than unit prices paid by the store at those times when it is buying, and so the store is able to make a strictly positive profit. However, as n becomes large the above partial derivatives tend to the price functions $p_t(\sum_{k=1}^n x_{kt})$ so that, in the limit as $n \rightarrow \infty$, prices become equalised over time and the stores no longer make any profit. As earlier, the intuitive explanation is that in the limit the stores become price takers and any individual store is able to exploit any inequality over time in market clearing prices so as to increase its profit; thus at the Nash equilibrium market clearing prices are equalised over time and stores are unable to make any profit. It is easy to see that essentially the same result holds when round-trip efficiencies are less than one.

More generally the impact on prices of competition between stores, in comparison to the cooperative solution, is to further reduce the price variation between the different times over which the stores operate. Arguing as in Section 3.2, one would typically expect such increased competition to lead to a further increase in consumer surplus. However, again this need not always be the case.

4.2.2.1. Existence and uniqueness of Nash equilibria. The following result shows the existence of a (pure strategy) Nash equilibrium.

Theorem 1. Under the assumptions of Section 2 on the price functions p_t , there exists at least one Nash equilibrium.

Proof. The assumptions on the price functions p_t guarantee convexity of the cost functions defined by (4). We assume first that the price functions are such that these cost functions are strictly convex. Write $S = (S_1, \dots, S_n)$ where each S_j is the strategy over time of store j . Let \mathcal{S} be the set of all possible S ; note that \mathcal{S} is convex and compact. Define a function $f: \mathcal{S} \rightarrow \mathcal{S}$ by $f(S) = (f_1(S), \dots, f_n(S))$ where each $f_j(S)$ minimises the function $G_j(\cdot; S_1, \dots, S_{j-1}, S_{j+1}, \dots, S_n)$ given by (4) subject to constraints (5) and (6), i.e. $f_j(S)$ is the best response of store j to $(S_1, \dots, S_{j-1}, S_{j+1}, \dots, S_n)$. It follows from the strict convexity assumption that each $f_j(S)$ is uniquely defined.

Now suppose that a sequence $(S^{(n)})$ in \mathcal{S} is such that $S^{(n)} \rightarrow S$ as $n \rightarrow \infty$. Then, for each j , the functions $G_j(\cdot; S_1^{(n)}, \dots, S_{j-1}^{(n)}, S_{j+1}^{(n)}, \dots, S_n^{(n)})$ (of S_j) converge uniformly to the continuous and strictly convex function $G_j(\cdot; S_1, \dots, S_{j-1}, S_{j+1}, \dots, S_n)$, so that also $f_j(S^{(n)}) \rightarrow f_j(S)$. Hence the function f is itself continuous. Thus by the Brouwer fixed point theorem there exists $S = f(S)$, which by definition is a (Cournot) Nash equilibrium.

In the case where the price functions p_t are such that the cost functions given by (4) are convex but not strictly so, we may consider a sequence of modifications to the former, tending to zero and such that we do have strict convexity of the corresponding cost functions. Compactness ensures that the corresponding Nash equilibria converge, at least in a subsequence, to a limit which straightforward continuity arguments show to be a Nash equilibrium for the problem defined by the unmodified price functions. \square

In general the uniqueness of any Nash equilibrium is unclear. However, we show in Section 4.2.3 that, under a linear approximation to the price functions, the Nash equilibrium is unique.

The proof of Theorem 1 also suggests an iterative algorithm to identify possible Nash equilibria – analogous to the algorithm suggested in Section 4.1. Given any S the determination of each $f_j(S)$ introduced in the above proof requires only the solution of

a single-store optimisation problem, which may be achieved as described in, for example, [Cruise et al. \(2014\)](#). Hence, starting with any $S^{(0)}$, we may construct a sequence $\{S^{(n)}\}_{n \geq 0}$ such that $S^{(n)} = f(S^{(n-1)})$. Then, as in the above proof, any limit S of the sequence $\{S^{(n)}\}$ satisfies $S = f(S)$ and hence constitutes a Nash equilibrium. Different starting points $S^{(0)}$ may be tried, but, in the case of nonuniqueness, there is of course no guarantee that all Nash equilibria will be found.

Even under our given assumptions on the price functions p_t the general characterisation of Nash equilibria seems difficult. The following theorem gives a monotonicity result.

Theorem 2. Consider n competing stores with identical rate constraints and efficiencies and whose starting levels and finishing levels are ordered by their capacity constraints. Then, at any Nash equilibrium $S^* = (S_1^*, \dots, S_n^*)$, the levels of the stores are at all times ordered by their capacity constraints.

Proof. Let $(\mu_1^*, \dots, \mu_n^*)$ be the set of vectors (Lagrange multipliers) associated with the Nash equilibrium $S^* = (S_1^*, \dots, S_n^*)$ as defined by [Proposition 1](#). It follows from (ii) of that proposition that, for any t , and any i, j ,

$$\mu_{it}^* \geq \mu_{jt}^* \iff x_t(S_i^*) \geq x_t(S_j^*). \quad (17)$$

Suppose now that the assertion of the theorem is false. Then there exist i, j with $E_i < E_j$ and some t_0 such that

$$x_{t_0}(S_i^*) > x_{t_0}(S_j^*), \quad S_{it_0}^* > S_{jt_0}^*. \quad (18)$$

It now follows by induction that, for all $t' \geq t_0$,

$$x_{t'}(S_i^*) \geq x_{t'}(S_j^*), \quad S_{it'}^* > S_{jt'}^*, \quad \mu_{it'}^* \geq \mu_{jt'}^*. \quad (19)$$

That (19) is true for $t' = t_0$ follows from (17) and (18). Suppose now that (19) is true for some particular $t' \geq t_0$. It then follows from [Proposition 1](#) that the condition $S_{it'}^* > S_{jt'}^*$ implies $\mu_{i,t'+1}^* \geq \mu_{j,t'+1}^*$; hence, by (17), $x_{t'+1}(S_i^*) \geq x_{t'+1}(S_j^*)$ and so finally $S_{i,t'+1}^* > S_{j,t'+1}^*$. However, this contradicts the assumption $S_{it}^* \leq S_{jt}^*$. \square

4.2.3. Quadratic cost functions (i.e. linearised price functions)

We can make considerably more progress in the case of the linear approximation to the price functions given by [Eq. \(1\)](#), where we again assume that, for each t , we have $\bar{p}_t = p_t(0) > 0$, $p'_t = p'_t(0) \geq 0$, and that the function p_t remains positive over the range of possible values of its argument (so that our standing assumptions on the functions p_t are satisfied). This linearisation (1) is a reasonable approximation when storage facilities are collectively sufficiently large as to have an impact on market prices, but are not so very large as to require a more sophisticated price function. The main reason for greater analytical tractability in this case is that for a set of vectors (S_1, \dots, S_n) to be a Nash equilibrium is then equivalent to the requirement that they minimise a given convex function. In particular we have the following result.

Theorem 3. Given the price functions (1), there always exists a unique Nash equilibrium.

Proof. It follows from (1) and (4) that the requirement that a set of vectors (S_1, \dots, S_n) be a Nash equilibrium is equivalent to the requirement that, for each store j , given the policies S_i , $i \neq j$, being operated by the remaining stores, the vector S_j minimises the total cost

$$\sum_{t=1}^T h(x_t(S_j)) \left(\bar{p}_t + p'_t \sum_{i=1}^n h(x_t(S_i)) \right), \quad (20)$$

subject to the capacity and rate constraints on store j given by (5) and (6). Now note that this is further equivalent to the

requirement that the set of vectors (S_1, \dots, S_n) minimises the strictly convex function

$$\sum_{t=1}^T \left[\bar{p}_t \sum_{i=1}^n h_i(x_t(S_i)) + \frac{1}{2} p'_t \left(\sum_{i=1}^n h_i(x_t(S_i))^2 + \left(\sum_{i=1}^n h_i(x_t(S_i)) \right)^2 \right) \right] \quad (21)$$

subject to constraints (5) and (6) being satisfied for all j . Further since this minimum is also to be taken over a compact set, its existence and uniqueness – and hence that of the Nash equilibrium – follows. \square

[Theorem 4](#) below, which is a scaling result, reduces the optimisation problem (the determination of the Nash equilibrium) for n identical competing stores to that of the corresponding problem for an appropriately redimensioned single store.

Theorem 4. Given the price functions (1) and a common efficiency ϵ , for each $n \geq 1$, consider n identical competing stores with common capacity $E^{(n)}$, common rate input and output constraints $P_1^{(n)}$ and $P_0^{(n)}$, and common starting and finishing levels $S_0^{(n)}$ and $S_T^{(n)}$ respectively, where we have

$$E^{(n)} = 2E^{(1)}/(n+1),$$

$$P_1^{(n)} = 2P_1^{(1)}/(n+1), \quad P_0^{(n)} = 2P_0^{(1)}/(n+1),$$

$$S_0^{(n)} = 2S_0^{(1)}/(n+1), \quad S_T^{(n)} = 2S_T^{(1)}/(n+1).$$

For each n , let $S^{(n)} = (S_1^{(n)}, \dots, S_T^{(n)})$ be the common policy over time of each of the stores at the unique and necessarily symmetric competitive Nash equilibrium. Then, at this equilibrium and at each time t , the quantity traded by each store in the n -store problem is $2/(n+1)$ times the quantity traded in the single store problem, i.e. $h(x_t(S^{(n)})) = 2h(x_t(S^{(1)}))/(n+1)$.

Proof. It follows from [Theorem 3](#) that, for each n , $S^{(n)}$ minimises the strictly convex function

$$n \sum_{t=1}^T \left(\bar{p}_t h(x_t(S^{(n)})) + \frac{1}{2} (n+1) p'_t h(x_t(S^{(n)}))^2 \right) \quad (22)$$

subject to the capacity constraints

$$S_0^{(n)} = S_0^*/(n+1), \quad S_T^{(n)} = S_T^*/(n+1), \quad 0 \leq S_t^{(n)} \leq E/(n+1), \quad 1 \leq t \leq T-1,$$

and the rate constraints

$$-P_1/(n+1) \leq x_t(S^{(n)}) \leq P_0/(n+1), \quad 1 \leq t \leq T.$$

The substitution $z_t = 2(n+1)x_t(S^{(n)})$, for $t = 1, \dots, T$, yields a single store minimisation problem which is independent of n (apart from a factor $2n/(n+1)$ in objective (22)) so that, for each t , $x_t(S^{(n)})$ (and so also $h(x_t(S^{(n)}))$) is proportional to $1/(n+1)$, so that the required result is now immediate. \square

Remark 2. The reduction in [Theorem 4](#) (for linear price functions) of the problem for n identical stores to a single store problem, allows also the application of the various sensitivity results of [Section 3.1](#).

[Theorem 5](#) below generalises [Example 1](#) to quantify the effect of competition between n unconstrained stores with identical efficiencies.

Theorem 5. Given the price functions (1) and a common efficiency ϵ , consider n stores subject to neither capacity nor rate constraints. Suppose further that the stores have a common starting level S_0^* and the same common finishing level $S_T^* = S_0^*$, and that this level is sufficiently large that, at the (unique and necessarily symmetric) Nash equilibrium, the stores never empty. Then, at this equilibrium, the

quantity traded per store is proportional to $1/(n+1)$ and the profit per store is proportional to $1/(n+1)^2$.

Proof. The first assertion of the theorem may be deduced from the scaling result of Theorem 4, and that theorem might be extended to enable also the second assertion of the present theorem to be deduced. However, we use instead the argument below, which also explicitly identifies the behaviour of the stores.

Write $\bar{S} = (\bar{S}_0, \dots, \bar{S}_T)$ (where $\bar{S}_T = \bar{S}_0 = S_0^*$) for the common policy over time of each of the stores at the Nash equilibrium. It now follows from Theorem 3 and the minimisation of the function (21) subject to the constraint

$$\bar{S}_T = \bar{S}_0, \quad (23)$$

that this equilibrium is given by

$$x_t(\bar{S}) = \begin{cases} \frac{\lambda - \bar{p}_t}{(n+1)p'_t}, & \bar{p}_t < \lambda \\ 0, & \lambda \leq \bar{p}_t \leq \frac{\lambda}{\epsilon} \\ \frac{\lambda - \epsilon \bar{p}_t}{(n+1)\epsilon^2 p'_t}, & \bar{p}_t \geq \frac{\lambda}{\epsilon} \end{cases} \quad (24)$$

for some Lagrange multiplier λ such that (23) is satisfied. Note, in particular, that λ is independent of n . Thus, as n varies, we have again that $(x_1(\bar{S}), \dots, x_T(\bar{S}))$ is proportional to $1/(n+1)$ as required. It follows also from (24) (by checking separately each of the three cases there) that, for all t ,

$$h(x_t(\bar{S}))(\bar{p}_t + (n+1)p'_t h(x_t(\bar{S}))) = \lambda x_t(\bar{S}). \quad (25)$$

It follows from (20) and from (25) that, at the Nash equilibrium, each store j incurs a total cost (the negative of its profit) equal to

$$\begin{aligned} \sum_{t=1}^T h(x_t(\bar{S}))(\bar{p}_t + np'_t h(x_t(\bar{S}))) &= \sum_{t=1}^T \lambda x_t(\bar{S}) - p'_t h(x_t(\bar{S}))^2 \\ &= - \sum_{t=1}^T p'_t h(x_t(\bar{S}))^2, \end{aligned}$$

where the first equality above follows from (25) and the second from (23). Since, as n varies, $(h(x_1(\bar{S})), \dots, h(x_T(\bar{S})))$ is proportional to $1/(n+1)$, the required result for the profit of each store follows. \square

Note that, under the conditions of the above theorem, the total quantity traded by the n stores (at each instant in time) is $2n/(n+1)$ times that traded by a single store, while the total profit made by the n stores is $4n/(n+1)^2$ times that made by a single store. Clearly also, were the stores subject to capacity or rate constraints, their ability to negatively impact on each other would be less – as in the example below.

4.3. Example

We consider again the half-hourly Market Index Price data for Great Britain throughout 2014, as introduced in the example of Section 3.3. We again let the price function be as given by (12) and (without loss of generality as explained in Section 3.3) take the market impact factor $\lambda = 1$. We consider $n = 1, 2, 3$ identical stores in competition, each with a round-trip efficiency $\epsilon = 0.75$. For the single-store case $n = 1$, we take $E = 10$ and common input and output rate constraint $P = 1$; for $n = 2$ we take $E = 5$ and $P = 1/2$ for each of the two stores, and for $n = 3$ we take $E = 10/3$ and $P = 1/3$ for each of the three stores. Thus the total storage available in each case is the same. The values of E and P are chosen so that the constraints on the stores are not so severe as to force essentially identical combined behaviour of the stores for each of the three values of n considered; nor are they so lax that the stores

behave as if they were unconstrained as considered in Theorem 5. For each n , we consider the unique Nash equilibrium in which each of the n stores optimises its behaviour (minimises its cost) over the entire year subject to the constraints of starting and finishing empty, and (for $n > 1$) given the behaviour of the remaining store(s).

In the units of the example – for a discussion of which again see Section 3.3 – the total profits made throughout the year by the n stores are 4096 for $n = 1$, 3733 for $n = 2$ and 3267 for $n = 3$. For each of the latter two cases, if the stores were to cooperate instead of competing, they would make the same total profit as in the single store case. Thus the decrease in total profit is again due to the effects of competition. However, note that as n increases through the above three values the total profit decreases at a rate which is slower than that in the case of unconstrained stores, as given by Theorem 5.

Fig. 4 shows the total level of the $n = 1, 2, 3$ stores and the corresponding market clearing prices (again in the units of the example) over the first two weeks of the year. The upper panel of the figure clearly shows that $n = 2$ and $n = 3$ competing stores consistently overtrade in relation to the case $n = 1$ (corresponding to the cooperative solution). The lower panel shows the extent to which competition between multiple stores smooths market clearing prices, which is of course associated with the reduction in overall profits. The times of maximum store activity correspond to the peaks and troughs of the market clearing price and it is these peaks and troughs which are smoothed by the competition. Note also that, because the round-trip efficiency $\epsilon = 0.75$ is significantly less than 1, there are significant periods of during which the stores neither buy nor sell.

5. Variant economic problems

Heretofore we have considered the optimal control of stores where the objective of each has in general been to maximise its own profit, obtained through price arbitrage over time. Such behaviour has a variable effect on both producers (in the case of energy the generators) and consumers. However, a store may alternatively be used to maximise the benefit to any defined group (who we describe here as being the “owner” of the store), whether that group be some set of consumers (e.g. society, if the generators are excluded from the latter), or some subset of generators, or society as a whole. In each case the operation of any single store is described by a vector $S = (S_0, \dots, S_T)$ such that S_t is the chosen level of the store at each time t . The total cost to the intended beneficiaries of the store's operation is then $\sum_{t=1}^T C_t(x_t(S))$ where, for each t , we take $x_t(S) = S_t - S_{t-1}$ and where C_t is some appropriately defined function. The control of the store which is optimal for its intended beneficiaries is then given by minimising $\sum_{t=1}^T C_t(x_t(S))$ subject to capacity and rate constraints analogous to those given by (5) and (6) in Section 2. Thus in each case the mathematical form of the optimisation problem is unchanged from that considered earlier; however, each function C_t now reflects the total cost of the action at time t to those for whose benefit the store is operated. Thus we have the same form of solution as previously, and may obtain the same insights into the effects of competitive behaviour. We give some examples below, in each case identifying the cost functions C_t .

5.1. One or more stores owned by consumers

Suppose that a single store is notionally owned by some set of consumers. For example, this might be a single consumer, a small number of consumers, or society as a whole, if the latter excludes the generators. Here the problem is to use the store so as to maximise the benefit to that group. If at each time t an amount x_t

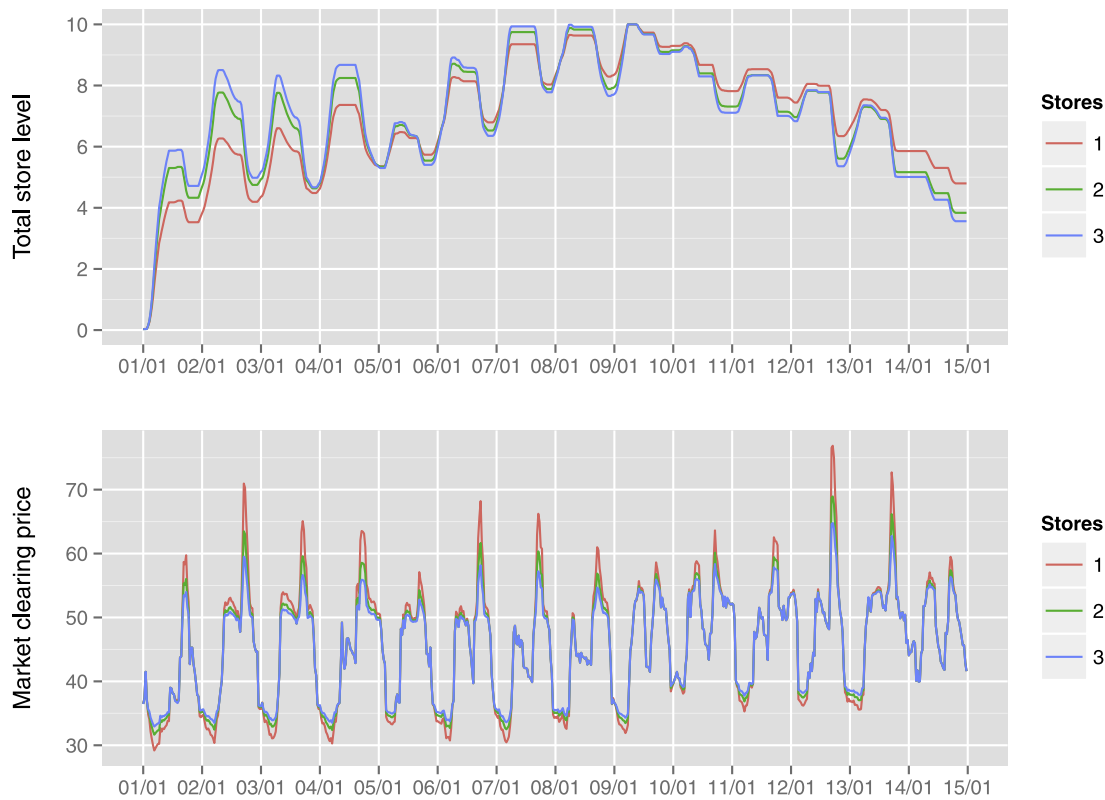


Fig. 4. Total store level and market clearing price for each of $n = 1, 2, 3$ stores in competition.

(positive or negative) is placed in the store, then the corresponding total cost $C_t(x_t)$ (again positive or negative) to the store owner is the sum of the extra payment to the generator plus the reduction in consumer benefit to the owner due to the market impact of the activity of the store – the latter would typically be measured by the reduction in consumer surplus experienced by the store owner. Hence the objective function $\sum_{t=1}^T C_t(x_t(S))$ to be minimised is the sum of these total costs (negative profits) over time. In the case where further benefits flow to the store owner as a result of the store's activities, e.g. through taxation regimes, these may similarly be incorporated in the cost functions C_t so as to define the correct objective function for the associated optimisation problem.

5.2. One or more stores owned by a generator

Now suppose that a store is owned by a generator, and is used by the latter with the intention of maximising its own total profit. Thus if, at each time t , an amount x_t (positive or negative) is placed in the store, then this has a cost to the generator which is simply that of producing it; further, if (at that time) the generator's production costs are nonlinear, the generator will re-optimize the amount supplied to the market, thereby affecting its profit from that activity; hence we may determine the total cost $C_t(x_t)$ to the generator of the action x_t , and the overall objective function to be minimised is again the sum over time of these costs.

5.3. Both generators and stores owned by society

Finally suppose that both the generator(s) and any store are owned by the consumers, i.e. by society, and managed jointly so as to maximise the benefit to society. In the absence of the store, the generator's supply function may be replaced by its (inverse) cost function i.e. that function which gives the amount which may be (just) economically supplied as a (generally increasing) function

of unit price; the point of intersection of this function with the demand function gives the optimal price, and the optimised benefit to society is the consumer surplus at that price. The introduction of the store now modifies this theory in a manner entirely analogous to that in the earlier case where just the store is owned by society.

6. Stochastic environments

The theory of the present paper formally assumes a deterministic cost environment, so that stores are able to plan optimally their future activity with a full knowledge of the cost functions involved. In a stochastic environment the cost function at any future time t is a random function whose distribution evolves so that the cost function typically becomes more precisely known as the time t approaches.

One obvious way in which one might therefore proceed in such an environment is, at any time at which decisions (on quantities to be bought or sold by stores) are required, to replace future cost functions by their expected values at that time and then to proceed as in the deterministic case. This strategy may be improved by revisiting decisions – as to future levels of activity – at each successive point in time and re-optimising those decisions in the light of updated knowledge. For an example of the success of this “re-optimisation” policy in the case of the optimal control of a single store (without market impact) see [Secomandi \(2015\)](#).

While replacing random functions by their expected values may not be optimal given full stochastic descriptions of future cost function evolution, in practice the vagaries of markets are such that these descriptions are rarely available. Further, associated with the optimal control of a store, as described by [Proposition 1](#), is a rolling *planning* or *decision horizon* beyond which it is not necessary to know future cost functions so as to determine successive optimal decisions. The existence of this horizon follows from

the algorithm of Cruise et al. (2014) – see also the appendix to the present paper. When, as in the examples of the present paper based on GB market data, prices exhibit very strong diurnal fluctuations, this horizon is often of the order of only a day or so, and over such short periods of time prices may well be accurately predictable.

7. Conclusions

In the present paper, we have considered how storage, operating as a price maker within a market environment, may be optimally operated over an extended or indefinite period of time. The optimality criterion may be that of maximising the profit over time of the storage itself, where this profit results from the ability of the storage to exploit differences in market clearing prices at different times. Alternatively it may be that of minimising over time the cost of generation, or of maximising consumer surplus or social welfare. In all cases there is calculated for each successive step in time the cost function measuring the total impact of whatever action (amount to buy or sell) is taken by the storage. The succession of such cost functions provides the appropriate information to the storage as to how to behave over time, forming the basis of the appropriate mathematical optimisation problem. We have also studied the various economic impacts – on market clearing prices, consumer surplus and social welfare – of the activities of the storage. Where these impacts are considered undesirable, the remedy is again the modification of the successive cost signals supplied to the storage. We have given examples based on real Great Britain market data.

We have been particularly concerned to study competition between multiple stores, where the objective of each store is to maximise its own income given the activities of the remainder. We have shown that at the Nash equilibrium – with respect to Cournot competition – multiple stores of sufficient size collectively erode their own abilities to make profits: essentially each store attempts to increase its own profit over time by overcompeting at the expense of the remainder. We have quantified this in the case of linear price functions, and again given examples based on market data.

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Appendix. Proof of Proposition 1

Since the result is concerned with the behaviour of a single store j , given the policies S_i of the remaining stores $i \neq j$, it is convenient to drop the subscript j and to write $S^* = (S_0^*, \dots, S_T^*)$ for the policy $S_j^* = (S_{j0}^*, \dots, S_{jT}^*)$ and $\mu^* = (\mu_1^*, \dots, \mu_T^*)$ for the vector $\mu_j^* = (\mu_{j1}^*, \dots, \mu_{jT}^*)$ of the proposition. Similarly, for each time t ,

we write $C_t(x_t)$ for the cost function $C_{jt}(x_{jt}; x_t(S_i), j \neq i)$ of the proposition. We write X , E and \mathbf{P} respectively for the rate constraint region X_j , the capacity constraint E_j and the store's optimisation problem \mathbf{P}_j .

Let S be any vector which is feasible for the problem \mathbf{P} , i.e. satisfies constraints (5) and (6) (with S replacing S_j , etc). Then, from condition (ii) of Proposition 1,

$$\sum_{t=1}^T [C_t(x_t(S^*)) - \mu_t^* x_t(S^*)] \leq \sum_{t=1}^T [C_t(x_t(S)) - \mu_t^* x_t(S)].$$

Recall that since both S and S^* are feasible for the problem \mathbf{P} we have $S_0 = S_0^*$ and $S_T = S_T^*$ and so, rearranging the above inequality,

$$\begin{aligned} \sum_{t=1}^T C_t(x_t(S^*)) - \sum_{t=1}^T C_t(x_t(S)) &\leq \sum_{t=1}^T \mu_t^* (S_t^* - S_{t-1}^* - S_t + S_{t-1}) \\ &= \sum_{t=1}^{T-1} (S_t^* - S_t) (\mu_t^* - \mu_{t+1}^*) \\ &\leq 0, \end{aligned}$$

where the latter inequality follows by condition (iii) of the proposition. Thus S^* solves the problem \mathbf{P} as required.

To show the existence of a pair (S^*, μ^*) satisfying the conditions of the proposition we give an explicit construction. We show how to define a sequence of times $0 < T_1 < \dots < T_k = T$ and successive values of (S_t^*, μ_t^*) such that the constructed pair (S^*, μ^*) satisfies conditions (i) and (ii) of the proposition; further $\mu_t^* = \mu_{t+1}^*$ for all $t < T$ for which we do not have $t = T_i$ for some $1 \leq i \leq k-1$. At each of the times $t = T_1, \dots, T_{k-1}$ it will be the case that either $S_t^* = 0$ or $S_t^* = E$, and it is then only necessary to verify that Eq. (7) of condition (iii) is also satisfied at each of these times.

We show first how to define the time T_1 together with $(S_0^*, \dots, S_{T_1}^*)$ and $(\mu_1^*, \dots, \mu_{T_1}^*)$. Suppose initially that each of the cost functions C_t is strictly convex. For any t satisfying $1 \leq t \leq T$ and for any (scalar) μ , define $x_t^*(\mu)$ to be the unique value of x which minimises $C_t(x) - \mu x$ subject to the rate constraint $x \in X$. Note that $x_t^*(\mu)$ is continuous and increasing (not necessarily strictly) in μ . For each such scalar μ , define a vector $S(\mu) = (S_0(\mu), \dots, S_T(\mu))$ by $S_0(\mu) = S_0^*$ and $S_t(\mu) = S_{t-1}(\mu) + x_t^*(\mu)$ for $1 \leq t \leq T$. Suppose first that there is some value μ^* of μ such that $S(\mu^*)$ satisfies the capacity constraints (5) for the problem \mathbf{P} (i.e. $0 \leq S_t(\mu^*) \leq E$ for $1 \leq t \leq T-1$ and $S_T(\mu^*) = S_T^*$), so that, by construction, the vector $S(\mu^*)$ is feasible for the problem \mathbf{P} ; we may then define $T_1 = T$, and $S_t^* = S_t(\mu^*)$ and $\mu_t^* = \mu^*$ for $1 \leq t \leq T$, and we are done. Otherwise, for every value of μ , there exists a first time $T'_1(\mu)$ such that $S_{T'_1(\mu)}(\mu)$ violates the capacity constraint (5) at this time. Define M_1 to be the set of all μ such that this first capacity constraint violation is below (i.e. if $T'_1(\mu) < T$ then $S_{T'_1(\mu)}(\mu) < 0$ and if $T'_1(\mu) = T$ then $S_{T'_1(\mu)}(\mu) < S_T^*$); similarly define M'_1 to be the set of all μ such that this first capacity constraint violation is above (i.e. if $T'_1(\mu) < T$ then $S_{T'_1(\mu)}(\mu) > E$ and if $T'_1(\mu) = T$ then $S_{T'_1(\mu)}(\mu) > S_T^*$). The assumption that there is at least some vector S which is feasible for the problem \mathbf{P} means that neither of the sets M_1 , M'_1 is empty. Further, since for each t the function $x_t^*(\mu)$ is increasing in μ , so also the successive components of the vector $S(\mu)$ are increasing in μ . Hence there is necessarily some scalar $\mu^* = \sup M_1$ such that either $M_1 = (-\infty, \mu^*]$, $M'_1 = (\mu^*, \infty)$ or else $M_1 = (-\infty, \mu^*)$, $M'_1 = [\mu^*, \infty)$. In the former case (i.e. $\mu^* \in M_1$) there is necessarily some time $T_1 < T'_1(\mu^*)$ (not necessarily uniquely defined) such that $S_{T_1}(\mu^*) = E$, for otherwise, by the continuity of the functions $x_t^*(\mu)$, μ could be increased above μ^* while still remaining within the set M_1 . Similarly in the latter case (i.e. $\mu^* \in M'_1$) there is necessarily some time $T_1 < T'_1(\mu^*)$ such that $S_{T_1}(\mu^*) = 0$. Thus in either case we define the time T_1

as above (in the event of nonuniqueness choosing, for example, the largest possible value), and define $\mu_t^* = \mu^*$ and $S_t^* = S_t(\mu^*)$ for $1 \leq t \leq T_1$.

For $T_1 \neq T$, the above algorithm may be restarted at the time T_1 , and at successive times T_i as necessary, to identify the entire sequence $0 < T_1 < \dots < T_k = T$ together with the entire vectors $\mu^* = (\mu_1^*, \dots, \mu_T^*)$ and $S^* = (S_0^*, \dots, S_T^*)$. It is a consequence of the construction that all the conditions of the proposition are satisfied, except only that, as stated above, for $k > 1$ it is still necessary to check that (7) holds at each of the times $t = T_1, \dots, T_{k-1}$. For this it is sufficient to consider the time T_1 . Suppose that, in the argument above, the scalar μ^* belongs to the set M_1 , so that $S_{T_1}^* = S_{T_1}(\mu^*) = E$. Then the condition $\mu^* \in M_1$ implies that when the vector $S(\mu^*)$ first violates the store capacity constraint (5) (at the $T_1'(\mu^*)$) it violates this constraint below. Recalling that $T_1 < T_1'(\mu^*)$, it follows that, when the above algorithm is restarted at the time T_1 , the same scalar μ^* (with $\mu^* = \mu_{T_1}^*$) continues to belong to the updated set M_2 replacing M_1 in the restarted argument. Hence we obtain that $\sup M_2 \geq \sup M_1$ and so $\mu_{T_1+1}^* \geq \mu_{T_1}^*$ as required. Similarly if the scalar μ^* belongs to the set M_1' , so that $S_{T_1}^* = 0$, we obtain that $\mu_{T_1+1}^* \leq \mu_{T_1}^*$ again as required.

When the cost functions C_t are convex, but not necessarily strictly so for all t , they may be represented as necessary as limits of sequences of strictly convex cost functions and standard continuity arguments used to deduce again the required result.

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