

CS 188: Artificial Intelligence

Markov Decision Processes



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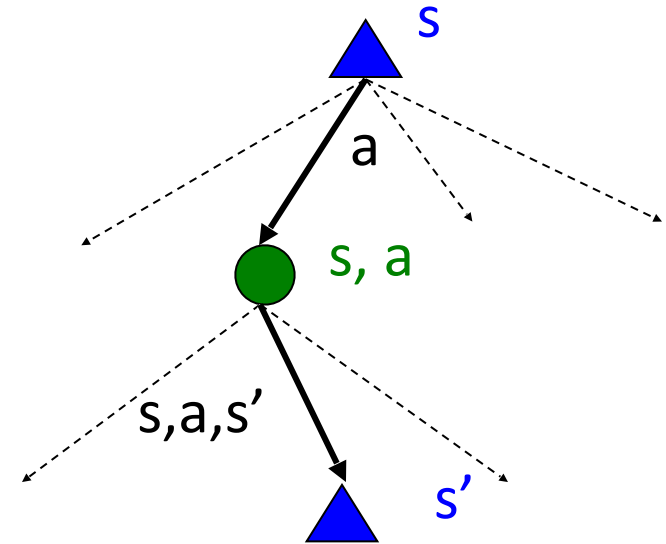
Recap: Defining MDPs

■ Markov decision processes:

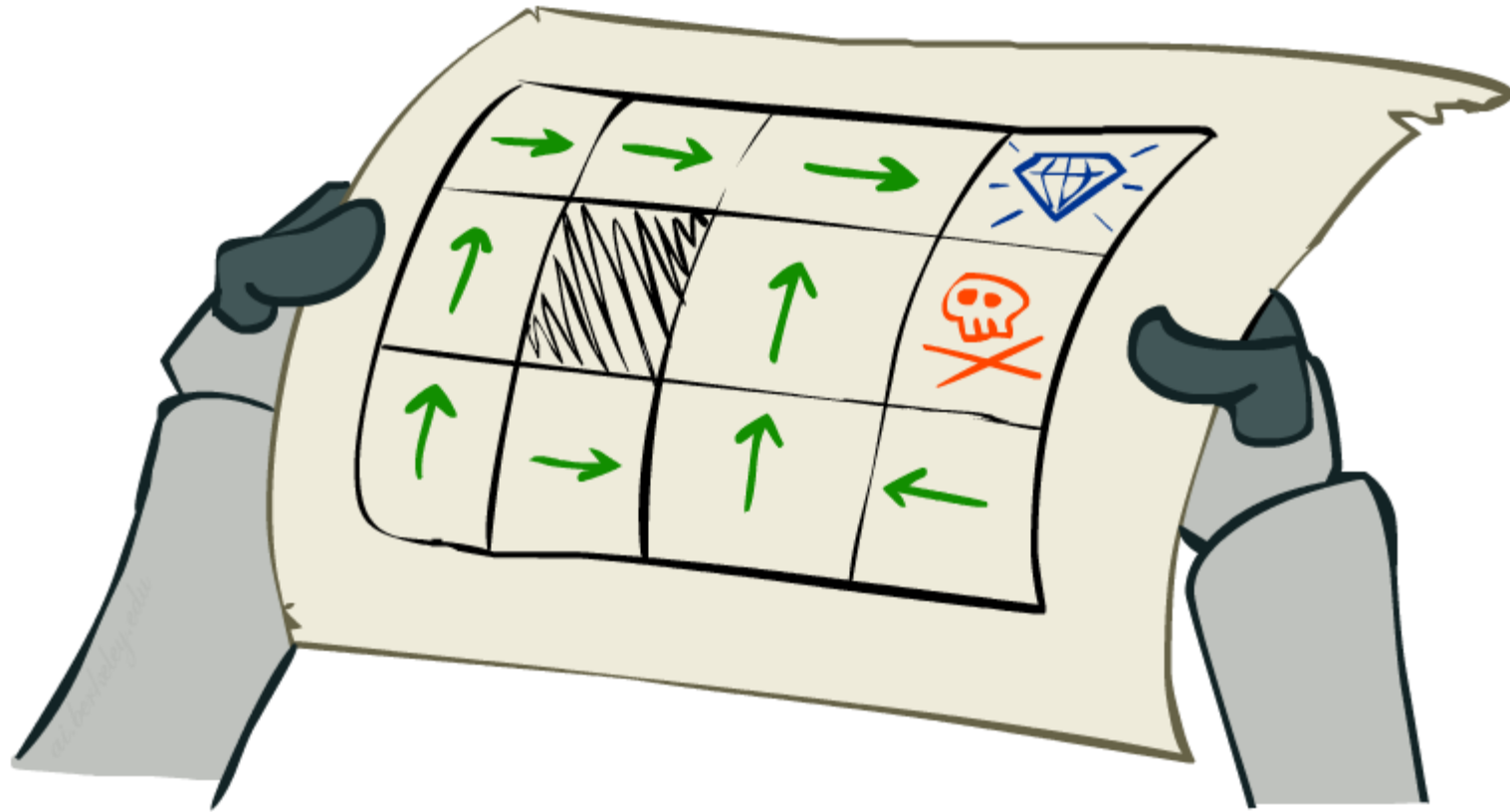
- Set of states S
- Start state s_0
- Set of actions A
- Transitions $P(s' | s, a)$ (or $T(s, a, s')$)
- Rewards $R(s, a, s')$ (and discount γ)

■ MDP quantities so far:

- Policy = Choice of action for each state
- Utility = sum of (discounted) rewards

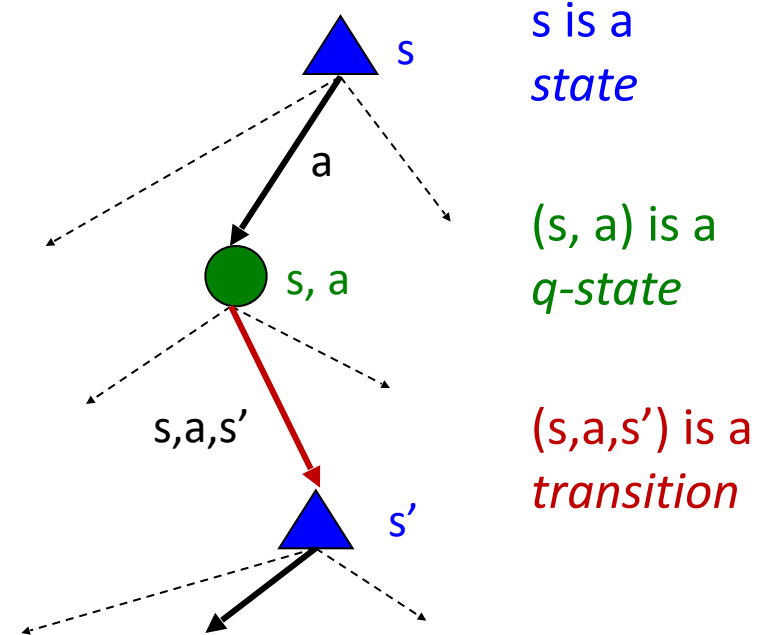


Solving MDPs



Optimal Quantities

- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s

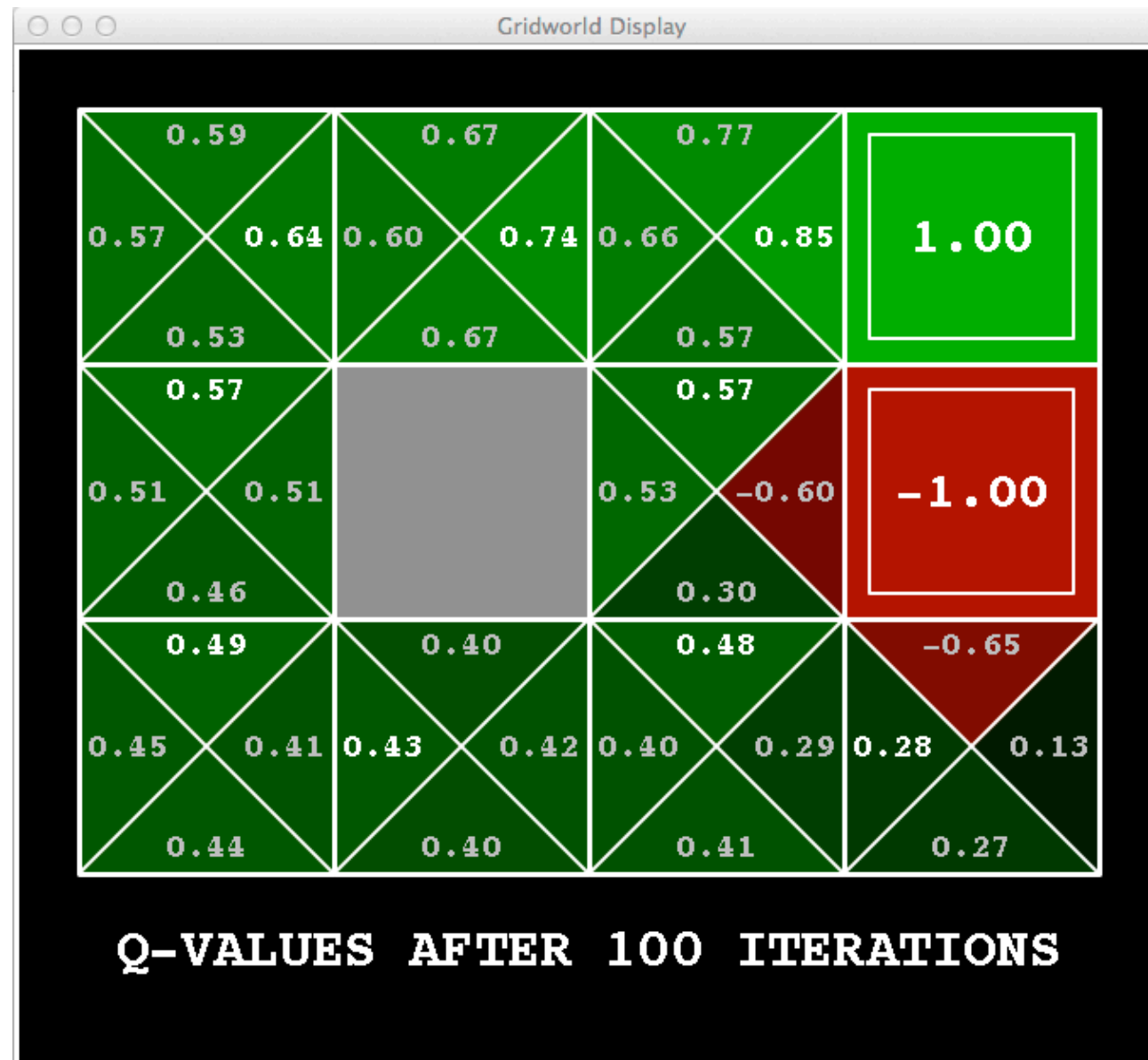


Snapshot of Demo – Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Snapshot of Demo – Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = 0

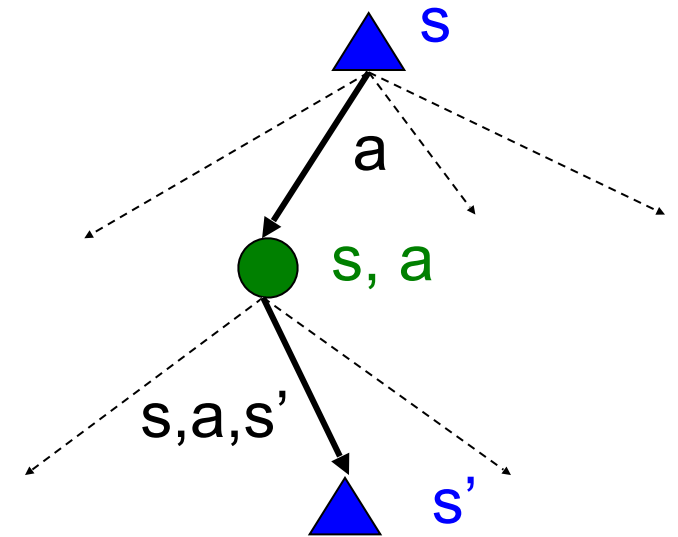
Values of States

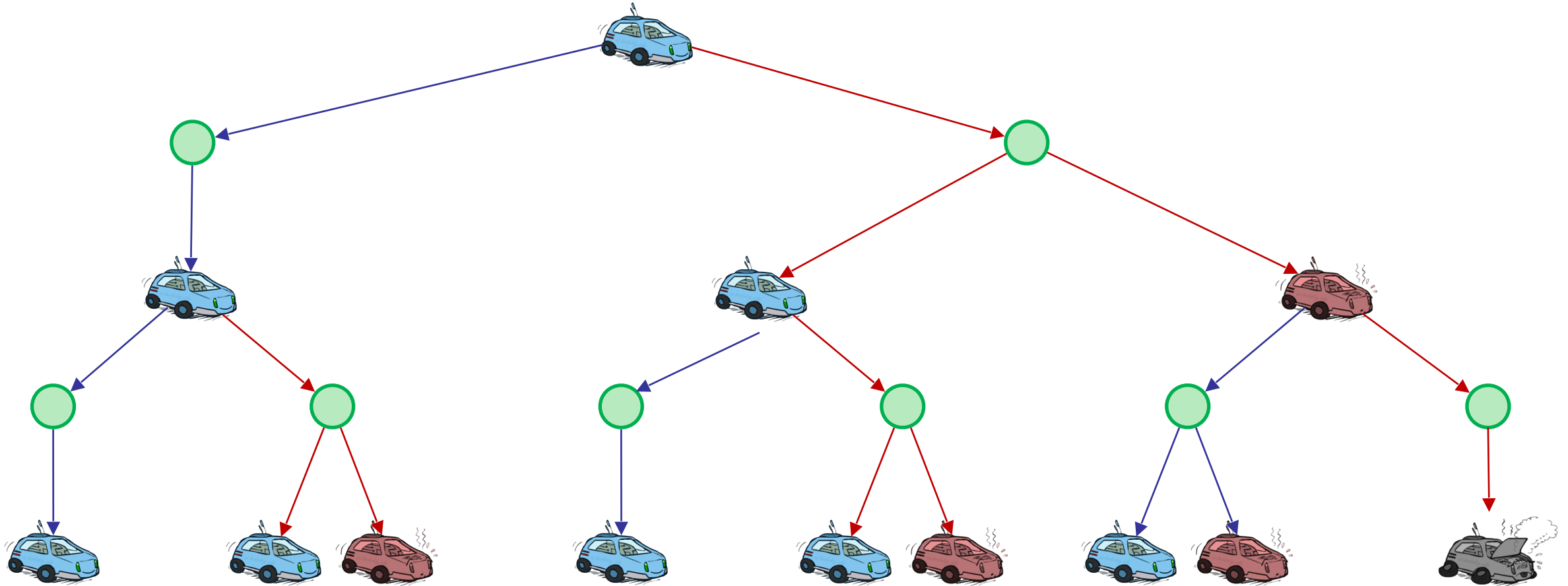
- Fundamental operation: compute the (expectimax) value of a state
 - Expected utility under optimal action
 - Average sum of (discounted) rewards
 - This is just what expectimax computed!
- Recursive definition of value:

$$V^*(s) = \max_a Q^*(s, a)$$

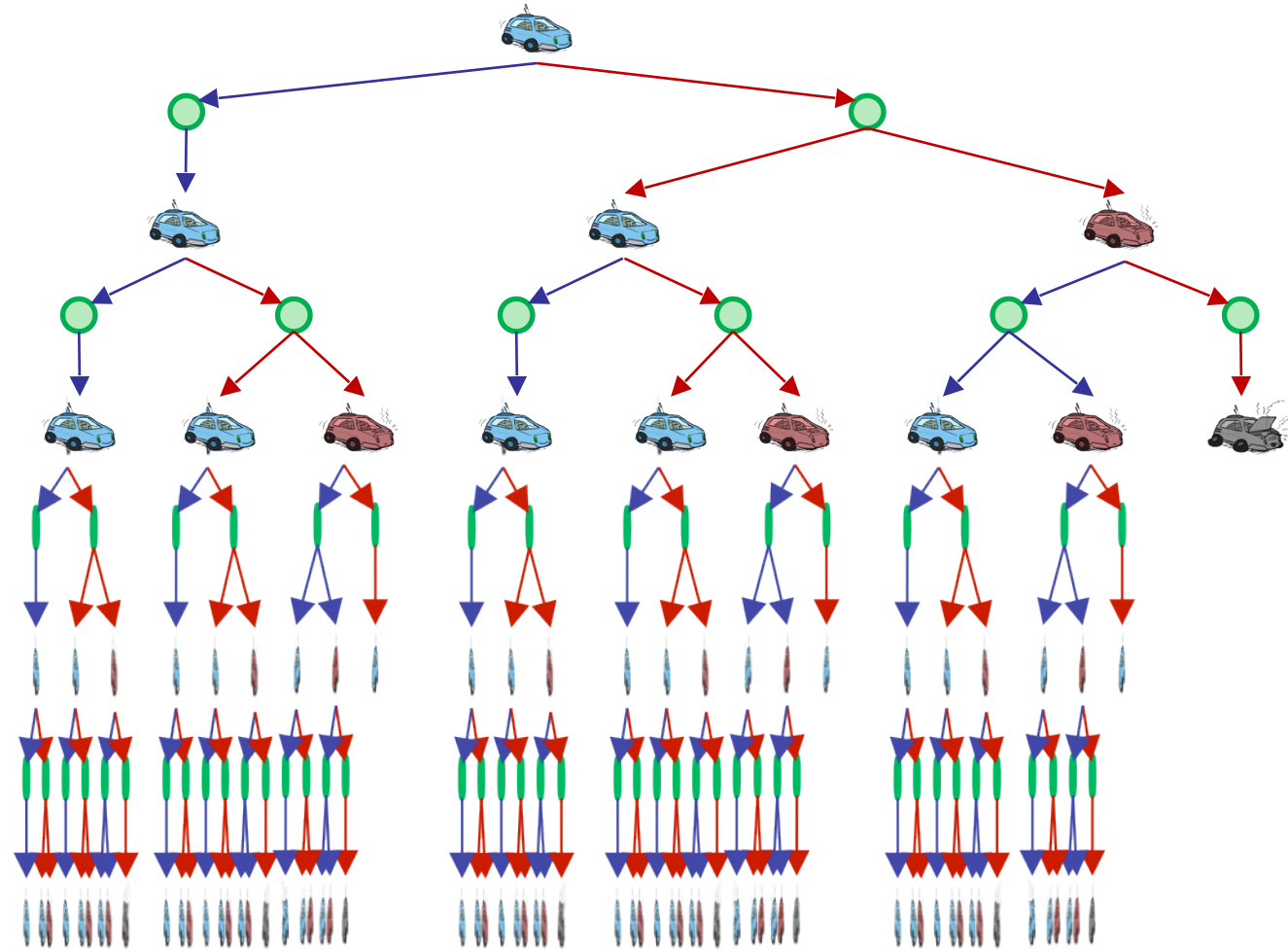
$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



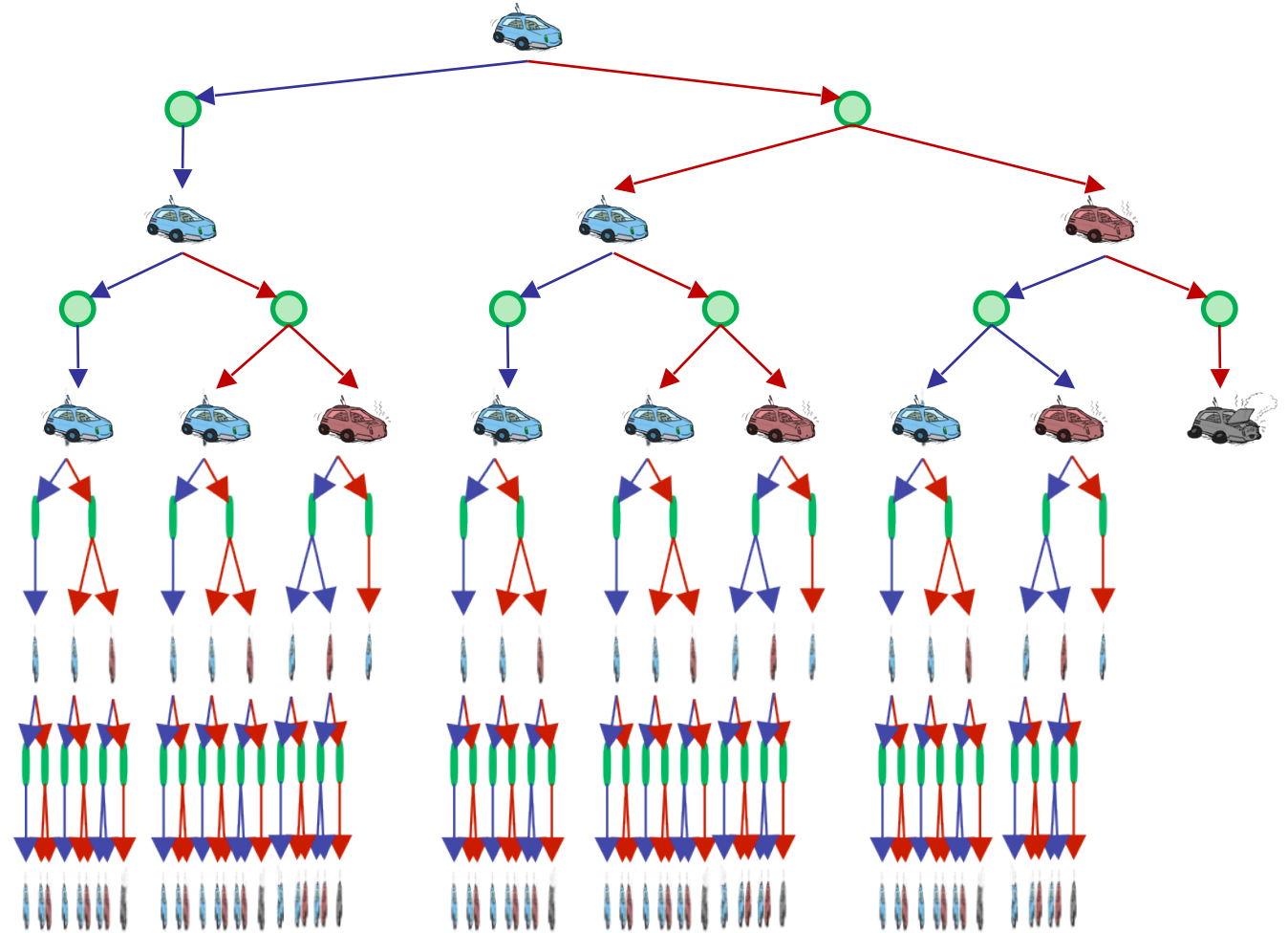


Racing Search Tree



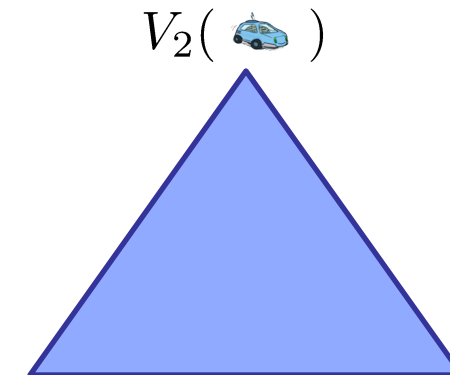
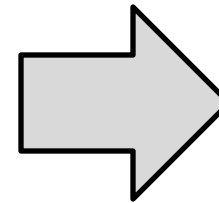
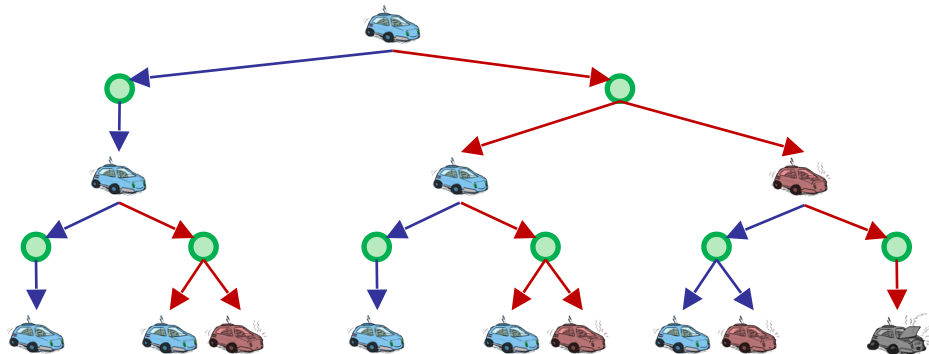
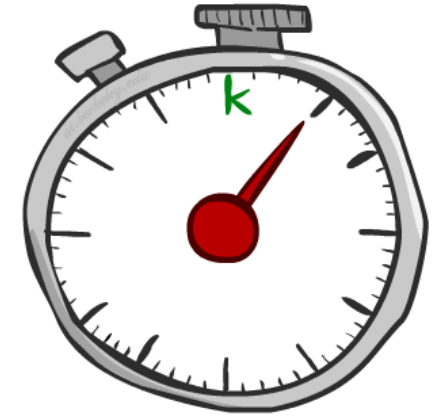
Racing Search Tree

- We're doing way too much work with expectimax!
- Problem: States are repeated
 - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
 - Idea: Do a depth-limited computation, but with increasing depths until change is small
 - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

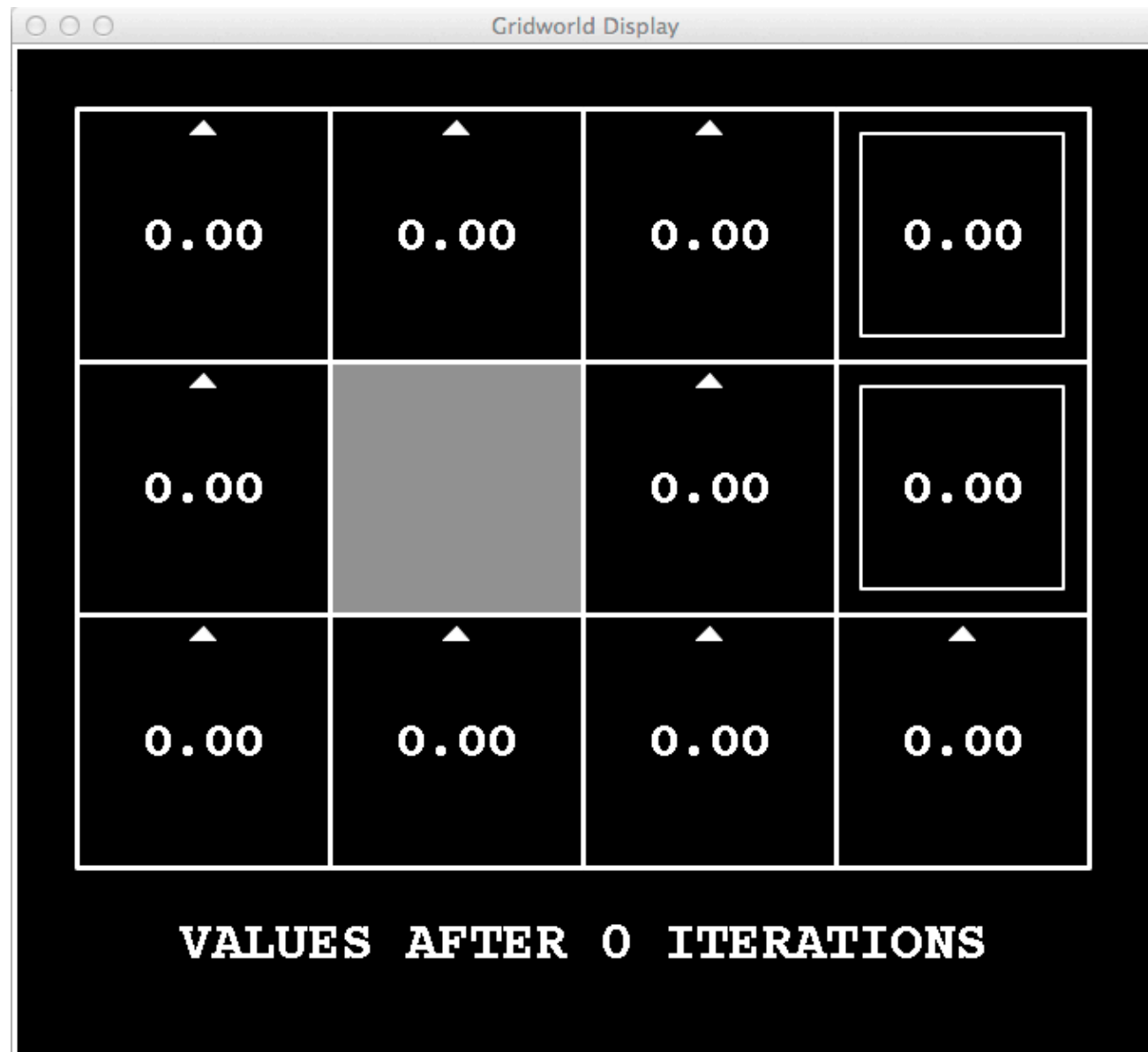


Time-Limited Values

- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
 - Equivalently, it's what a depth- k expectimax would give from s

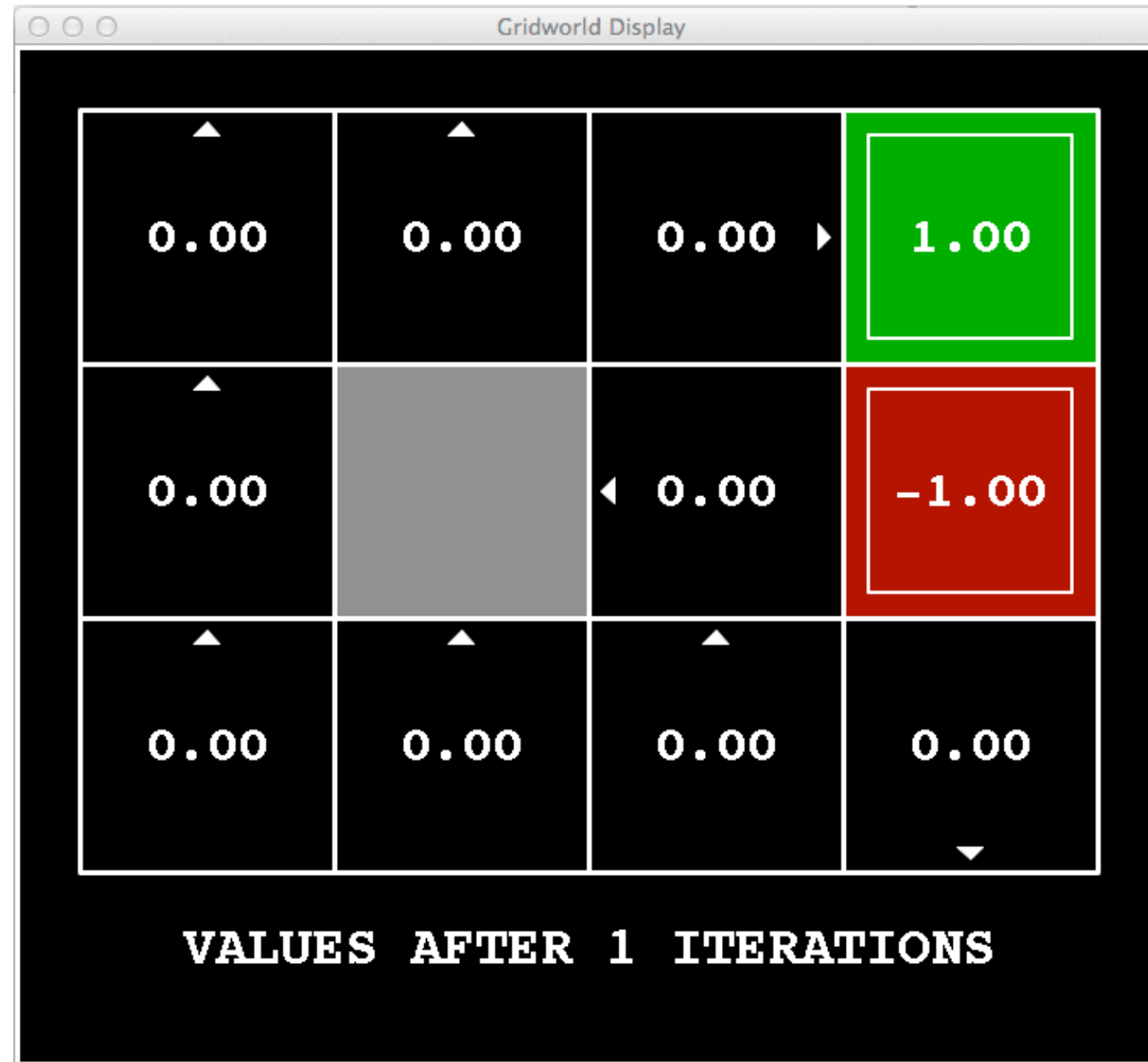


k=0



Noise = 0.2
Discount = 0.9
Living reward = 0

k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

k=2



Noise = 0.2
Discount = 0.9
Living reward = 0

k=3



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=4$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=7$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



Noise = 0.2
Discount = 0.9
Living reward = 0

k=9



Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



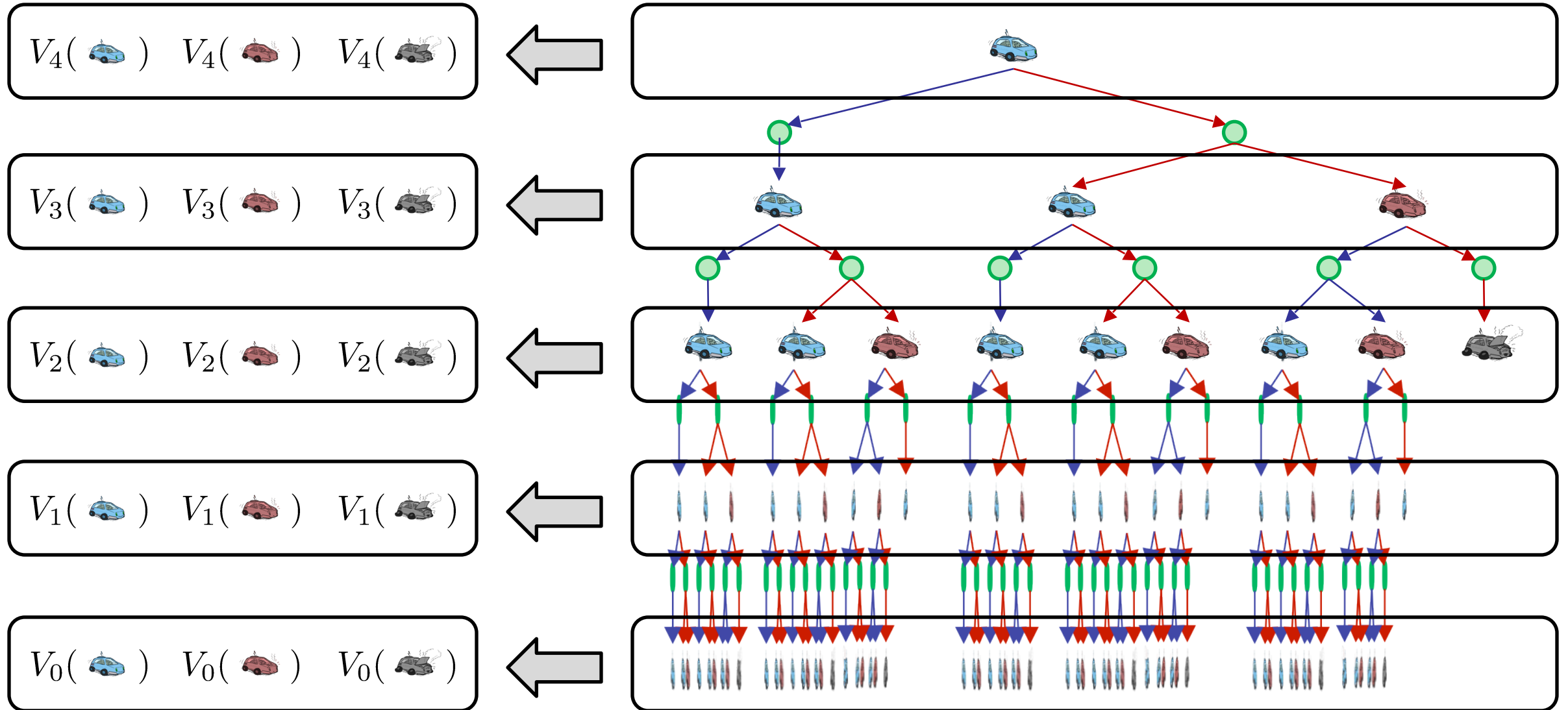
Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

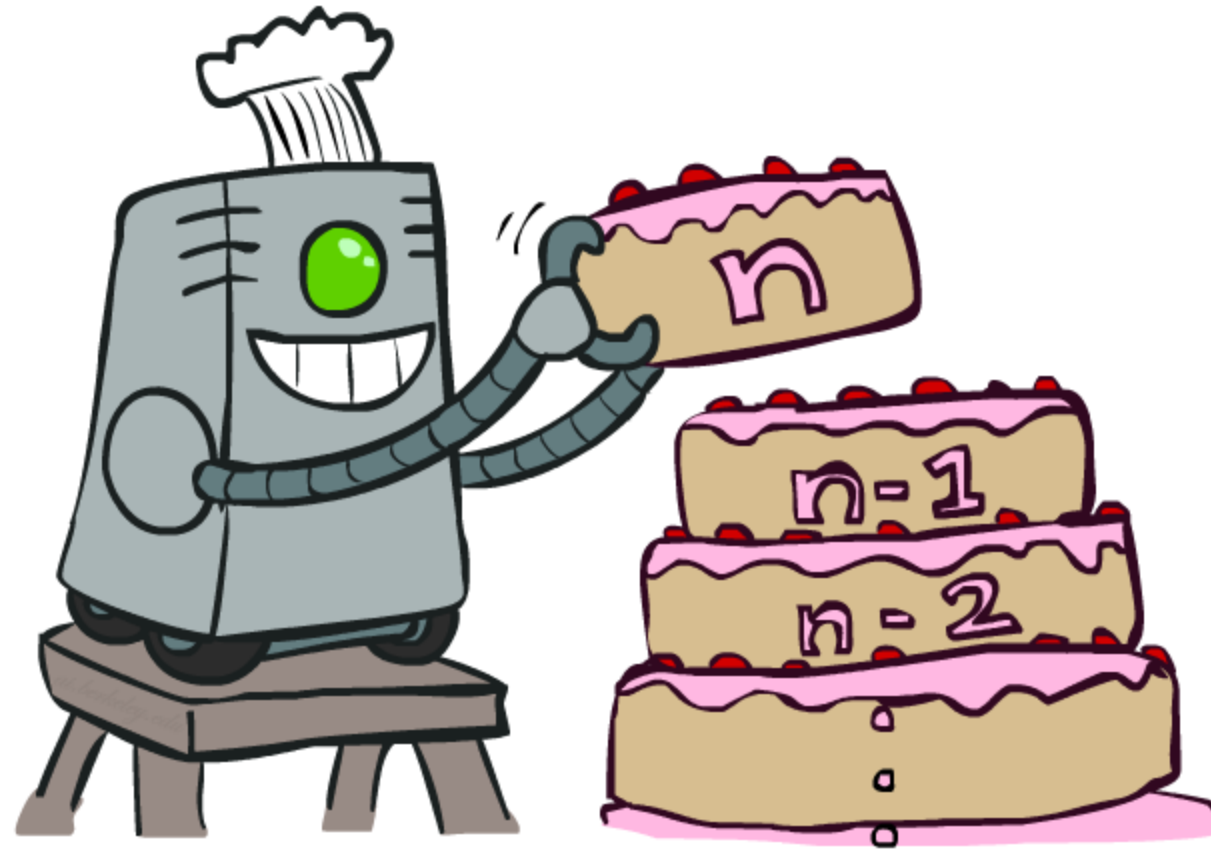


Noise = 0.2
Discount = 0.9
Living reward = 0

Computing Time-Limited Values



Value Iteration

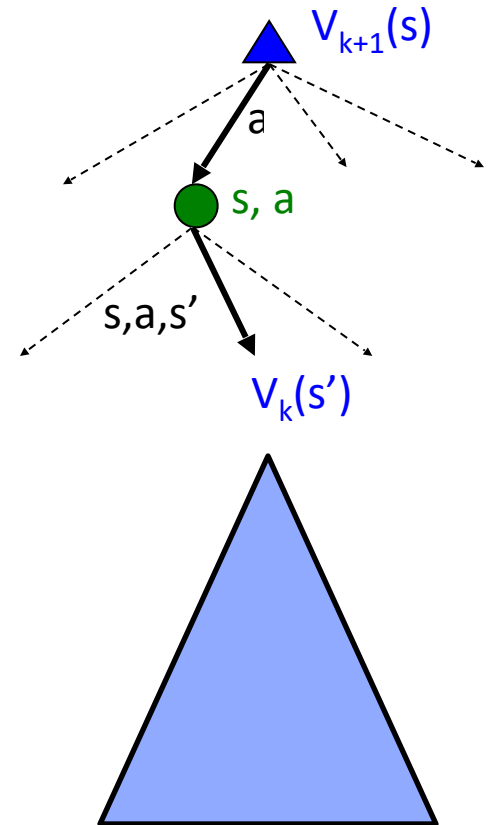


Value Iteration



- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

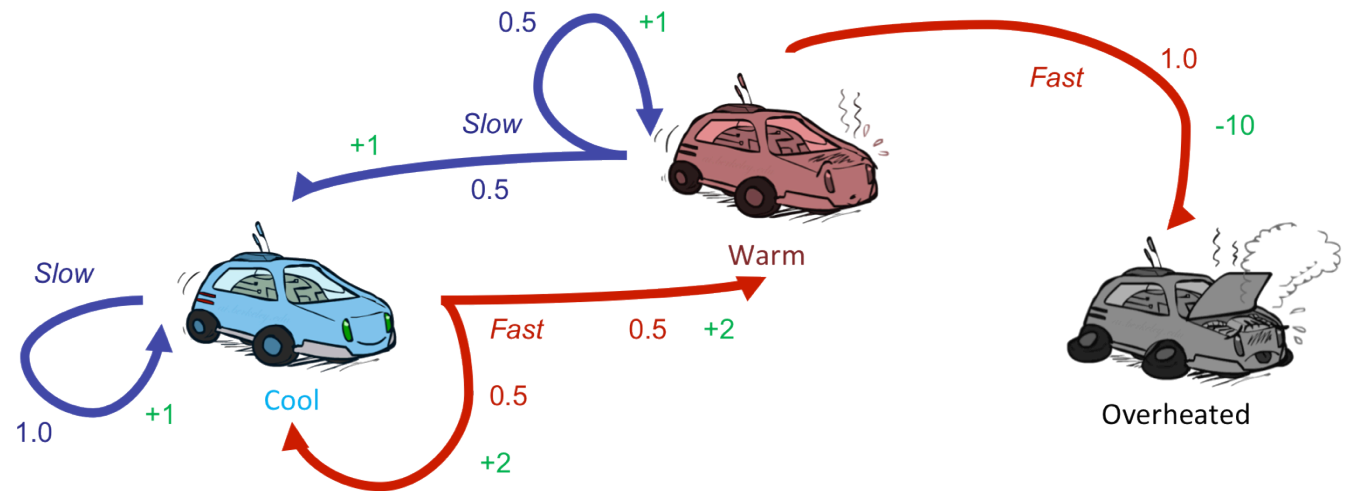
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Repeat until convergence
- Complexity of each iteration: $O(S^2A)$
- Theorem: will converge to unique optimal values
 - Basic idea: approximations get refined towards optimal values
 - Policy may converge long before values do



Example: Value Iteration

			
V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0

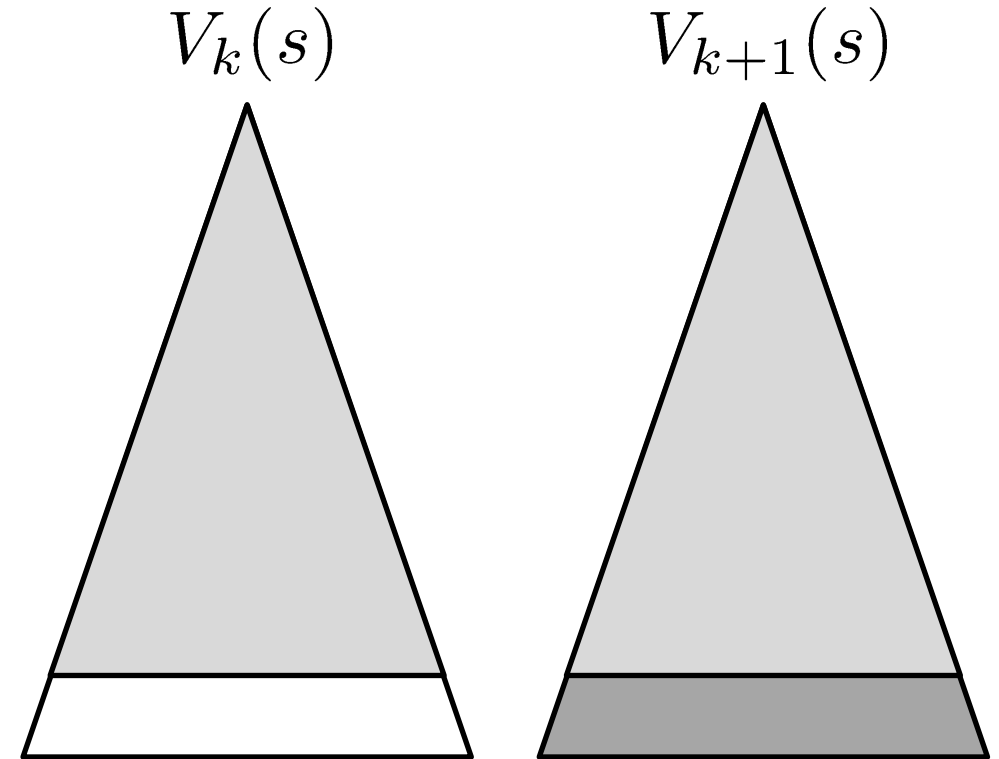


Assume no discount!

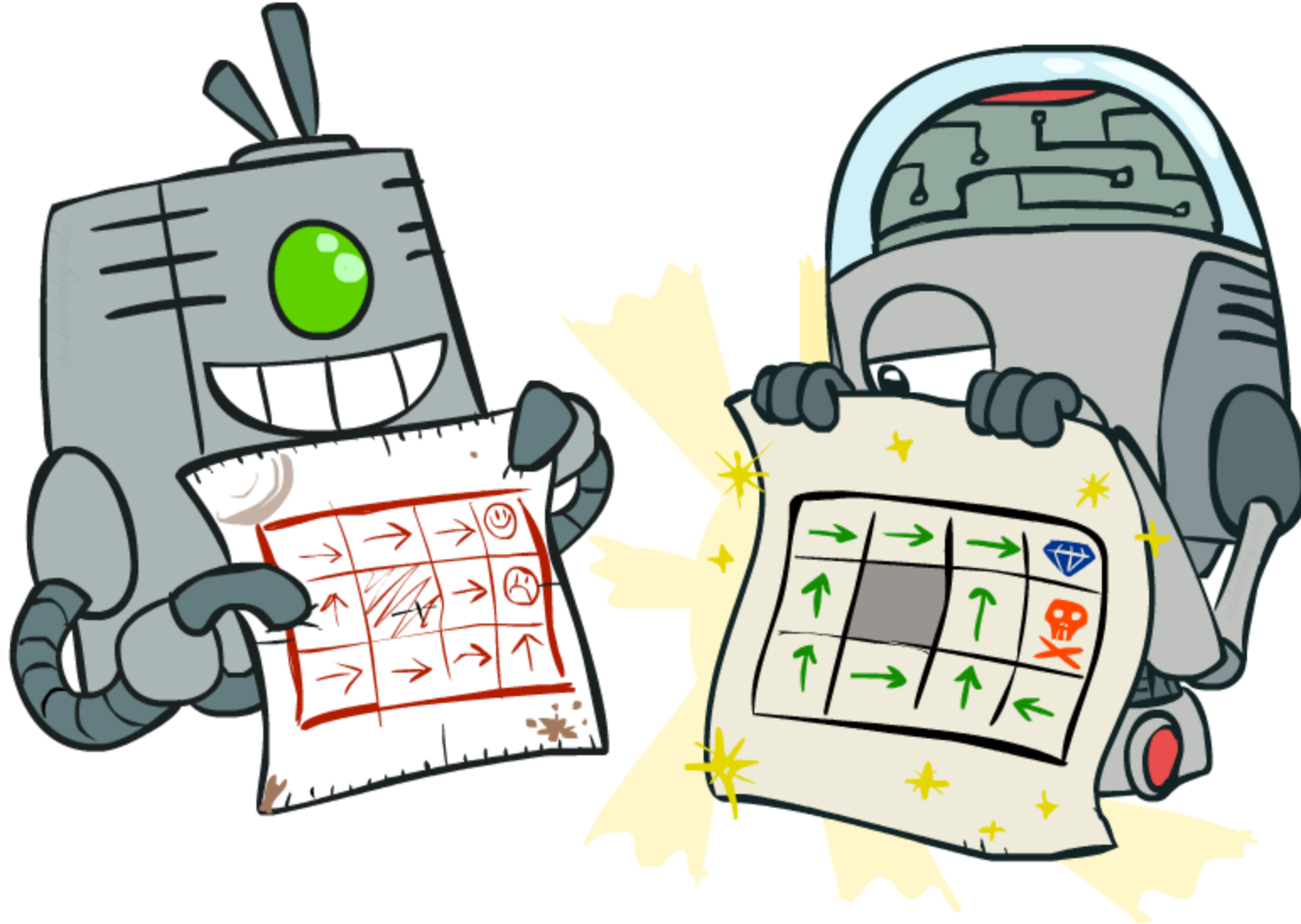
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Convergence*

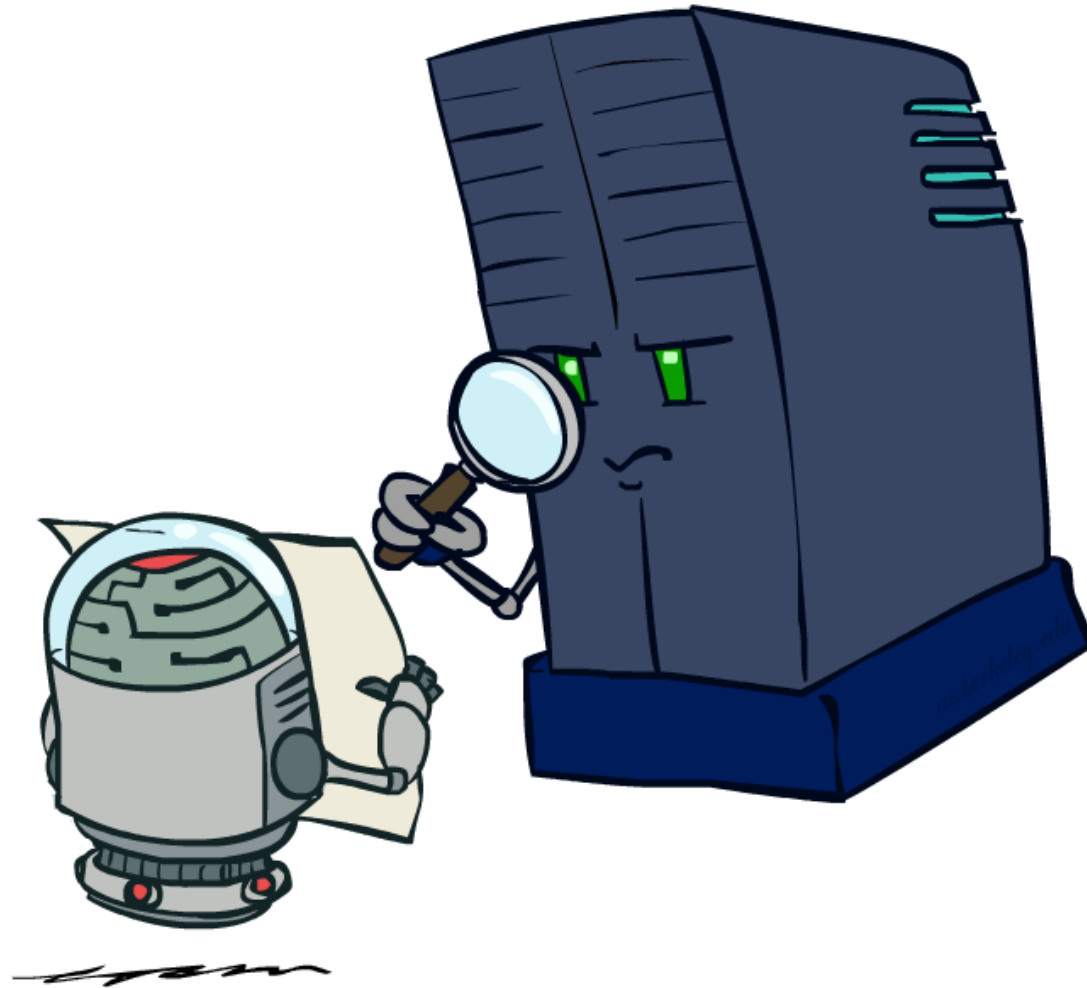
- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
 - Sketch: For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - That last layer is at best all R_{MAX}
 - It is at worst R_{MIN}
 - But everything is discounted by γ^k that far out
 - So V_k and V_{k+1} are at most $\gamma^k \max |R|$ different
 - So as k increases, the values converge



Policy Methods

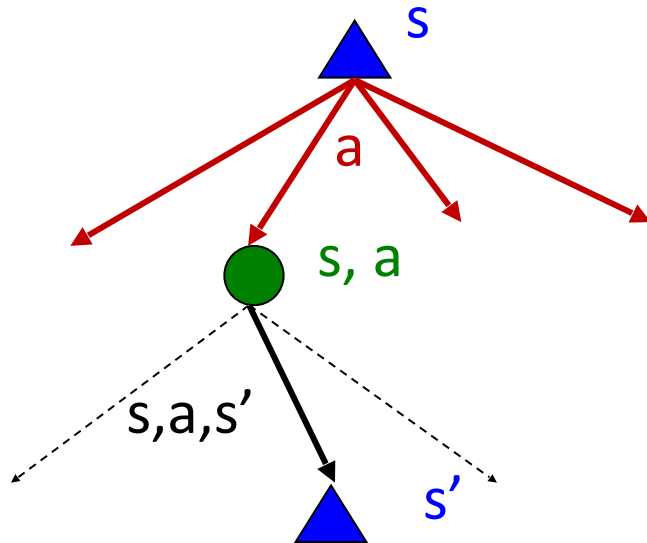


Policy Evaluation

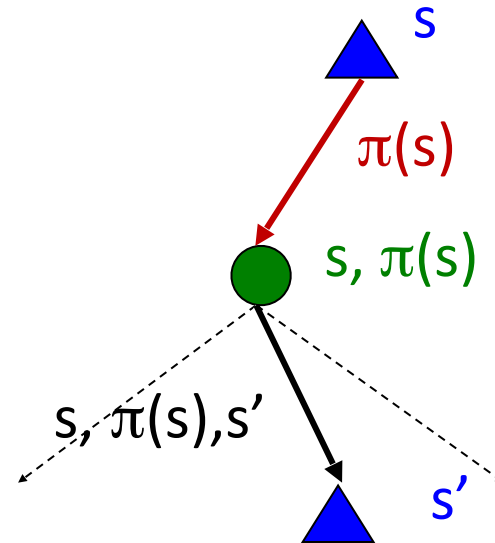


Fixed Policies

Do the optimal action



Do what π says to do

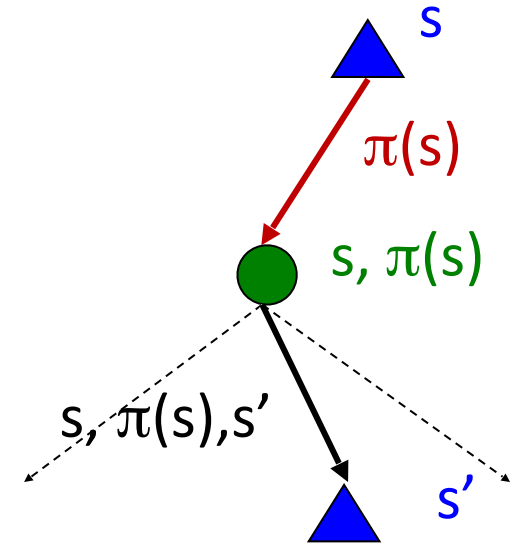


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
 - ... though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

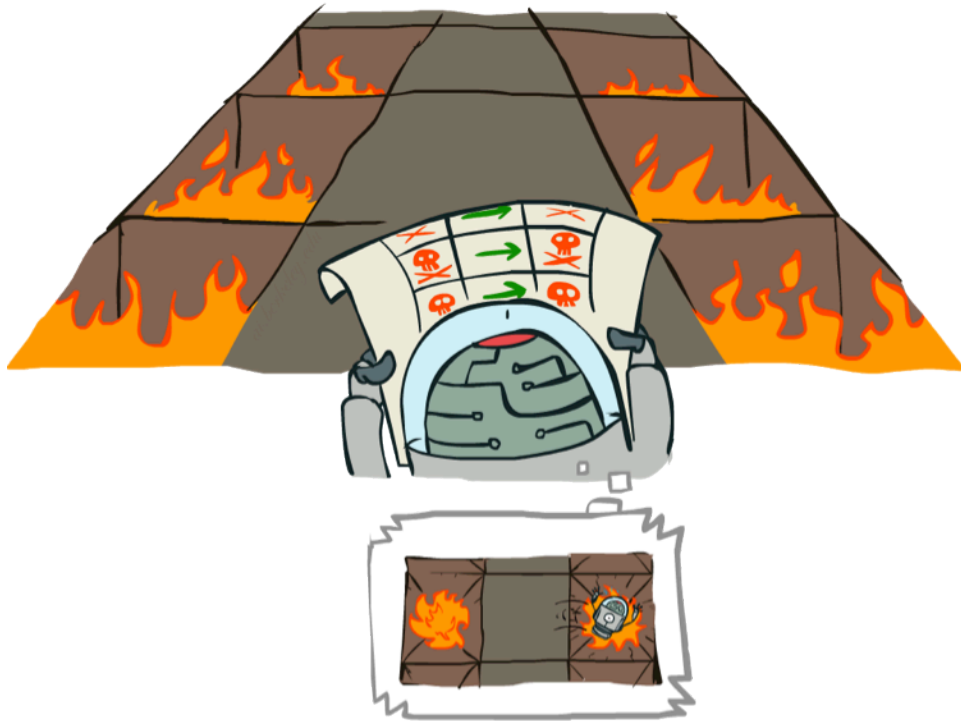
- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s , under a fixed policy π :
 $V^\pi(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):

$$V^\pi(s) = \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V^\pi(s')]$$

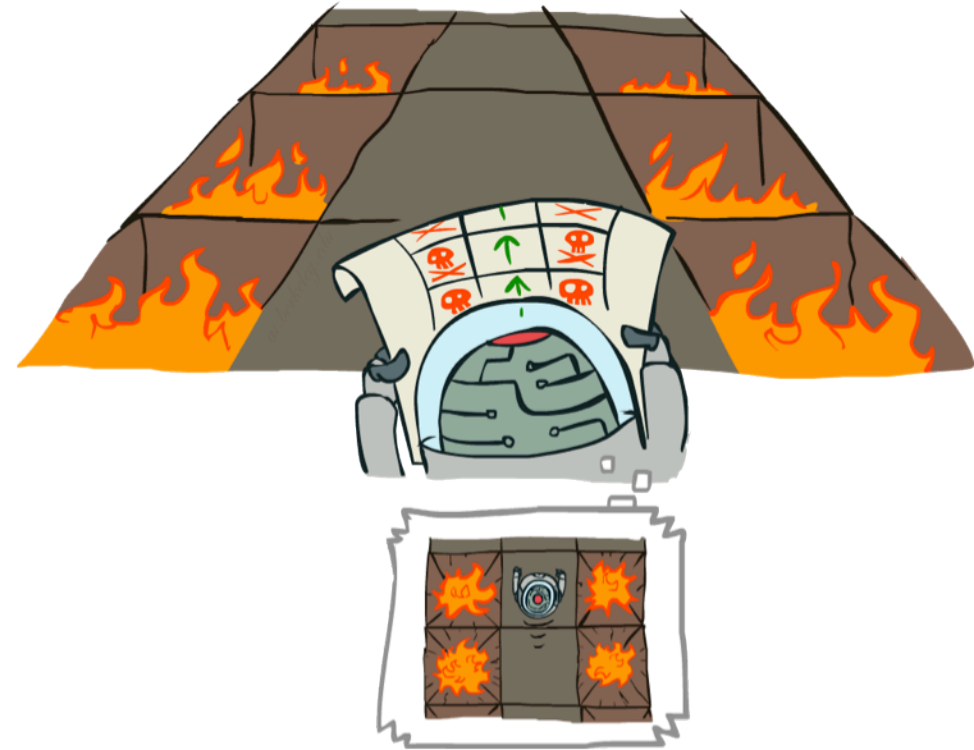


Example: Policy Evaluation

Always Go Right



Always Go Forward

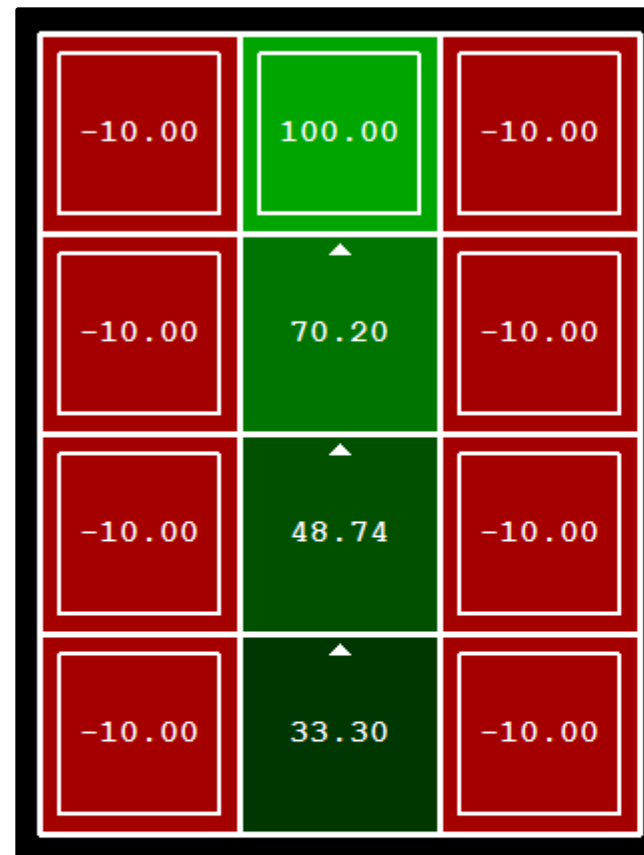


Example: Policy Evaluation

Always Go Right



Always Go Forward



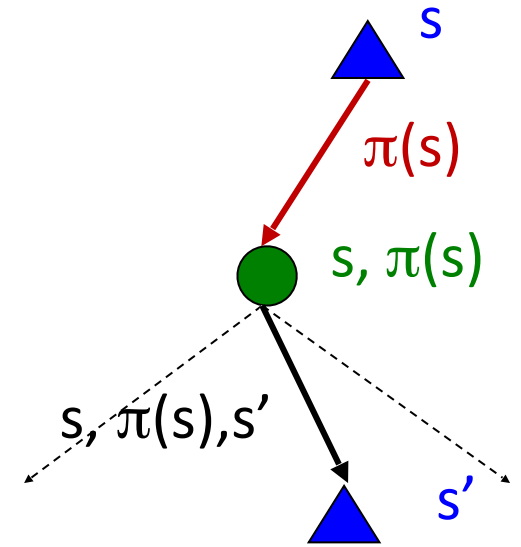
Policy Evaluation

- How do we calculate the V 's for a fixed policy π ?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

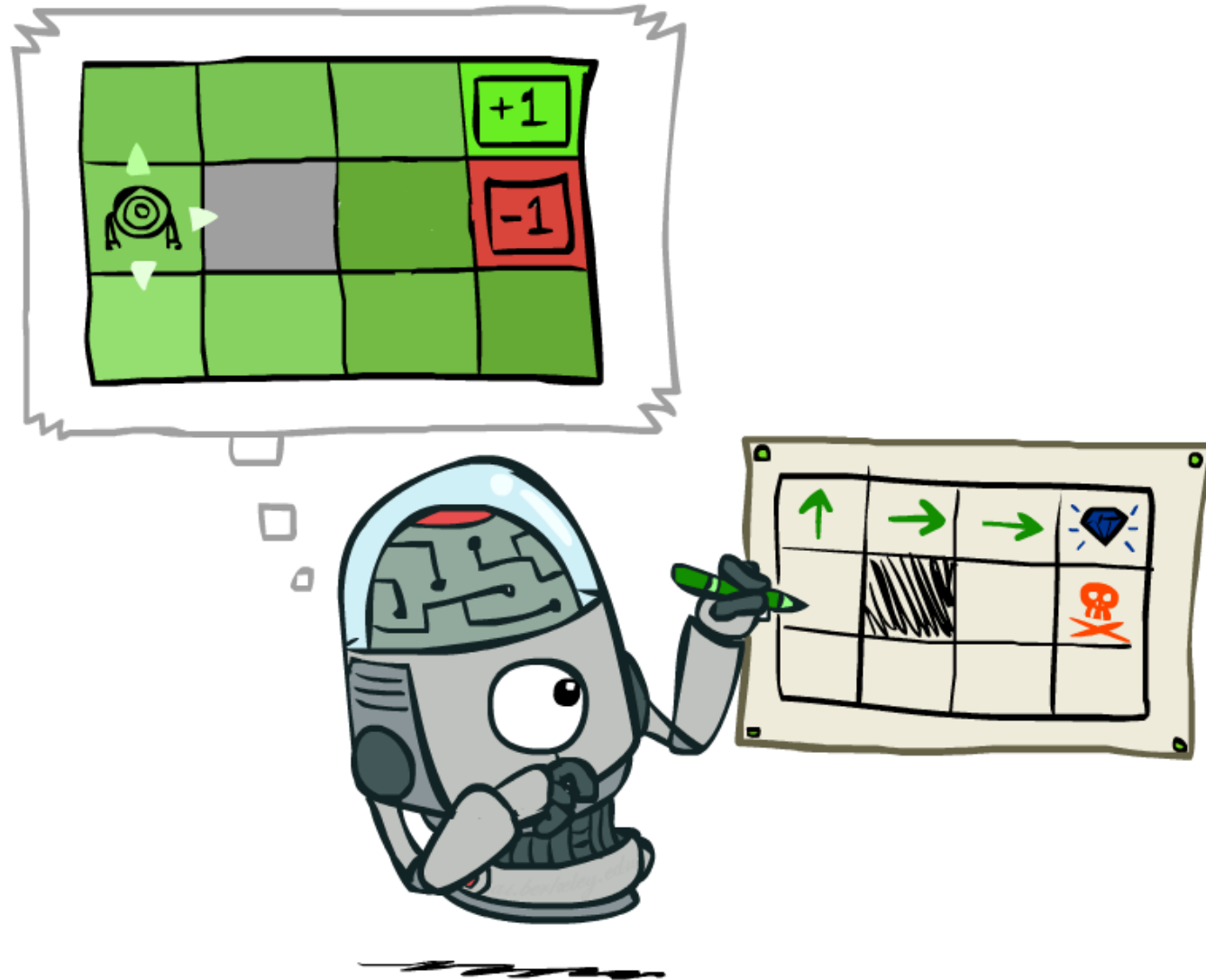
$$V_0^\pi(s) = 0$$

$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s, \pi(s), s') [R(s, \pi(s), s') + \gamma V_k^\pi(s')]$$

- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
 - Solve with Matlab (or your favorite linear system solver)

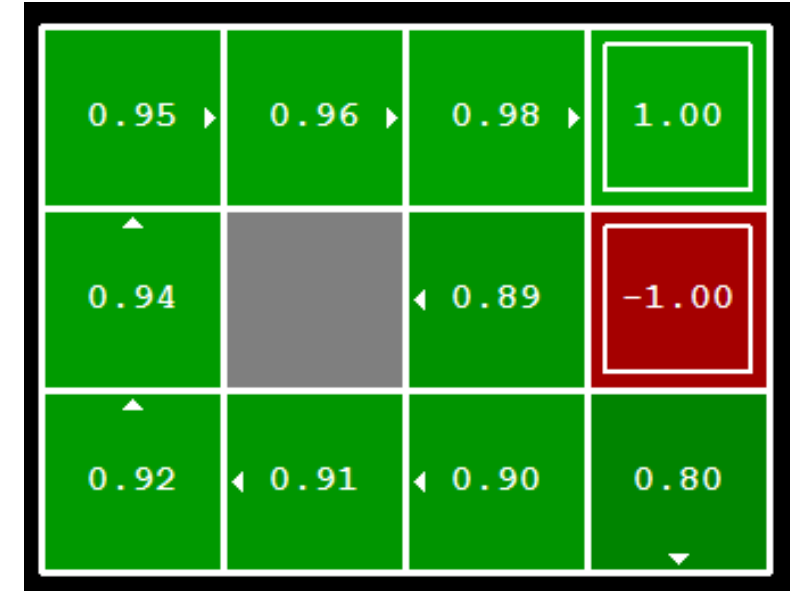


Policy Extraction



Computing Actions from Values

- Let's imagine we have the optimal values $V^*(s)$
- How should we act?
 - It's not obvious!
- We need to do a mini-expectimax (one step)



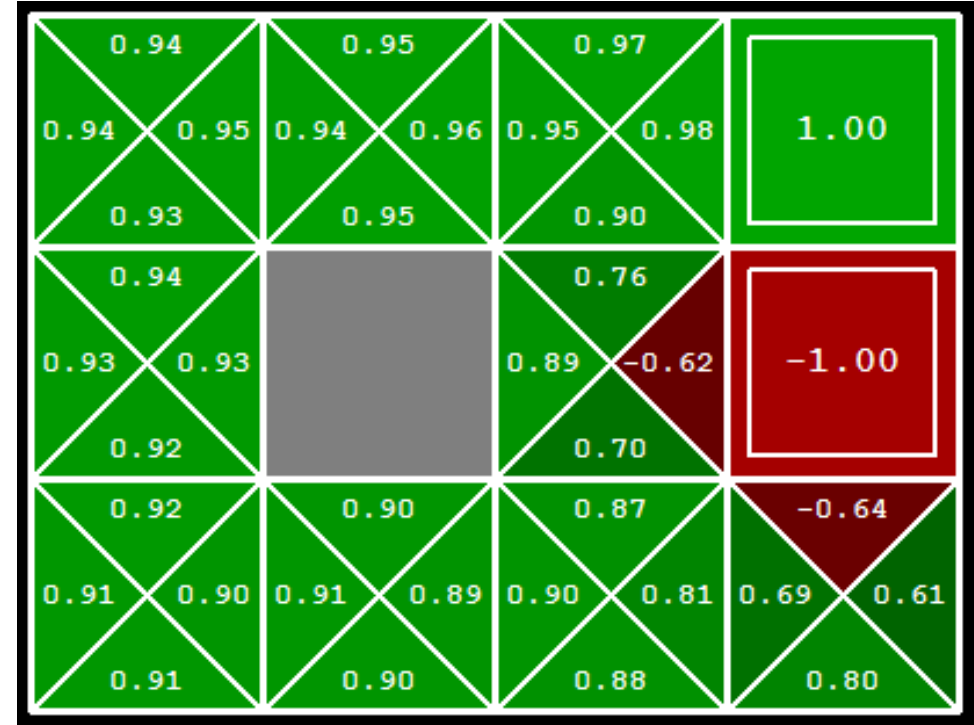
$$\pi^*(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

- This is called **policy extraction**, since it gets the policy implied by the values

Computing Actions from Q-Values

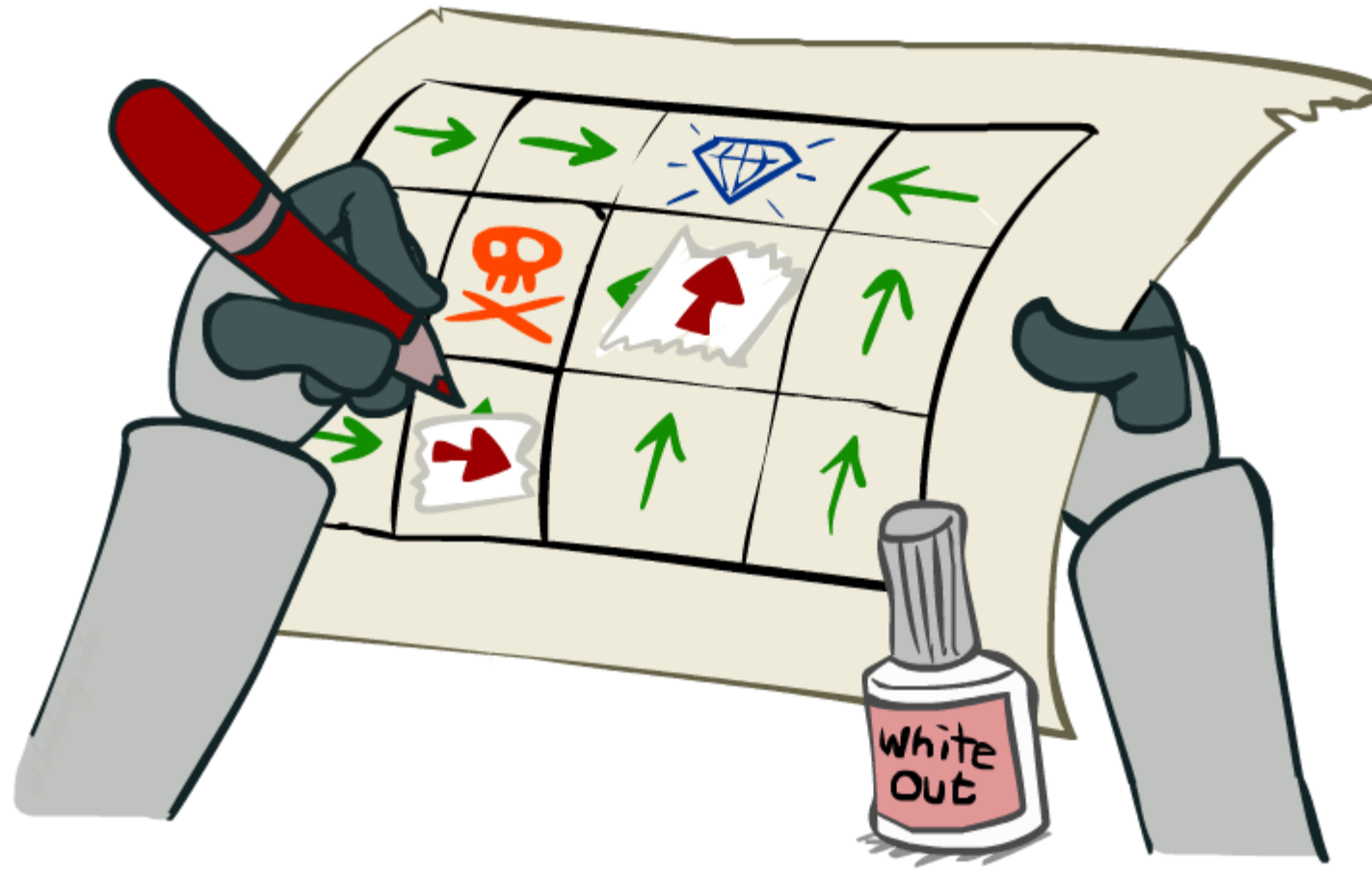
- Let's imagine we have the optimal q-values:
- How should we act?
 - Completely trivial to decide!

$$\pi^*(s) = \arg \max_a Q^*(s, a)$$



- Important lesson: actions are easier to select from q-values than values!

Policy Iteration

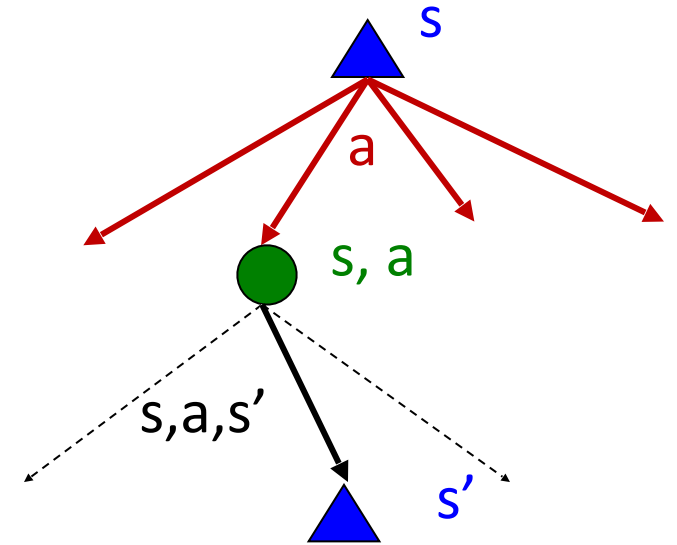


Problems with Value Iteration

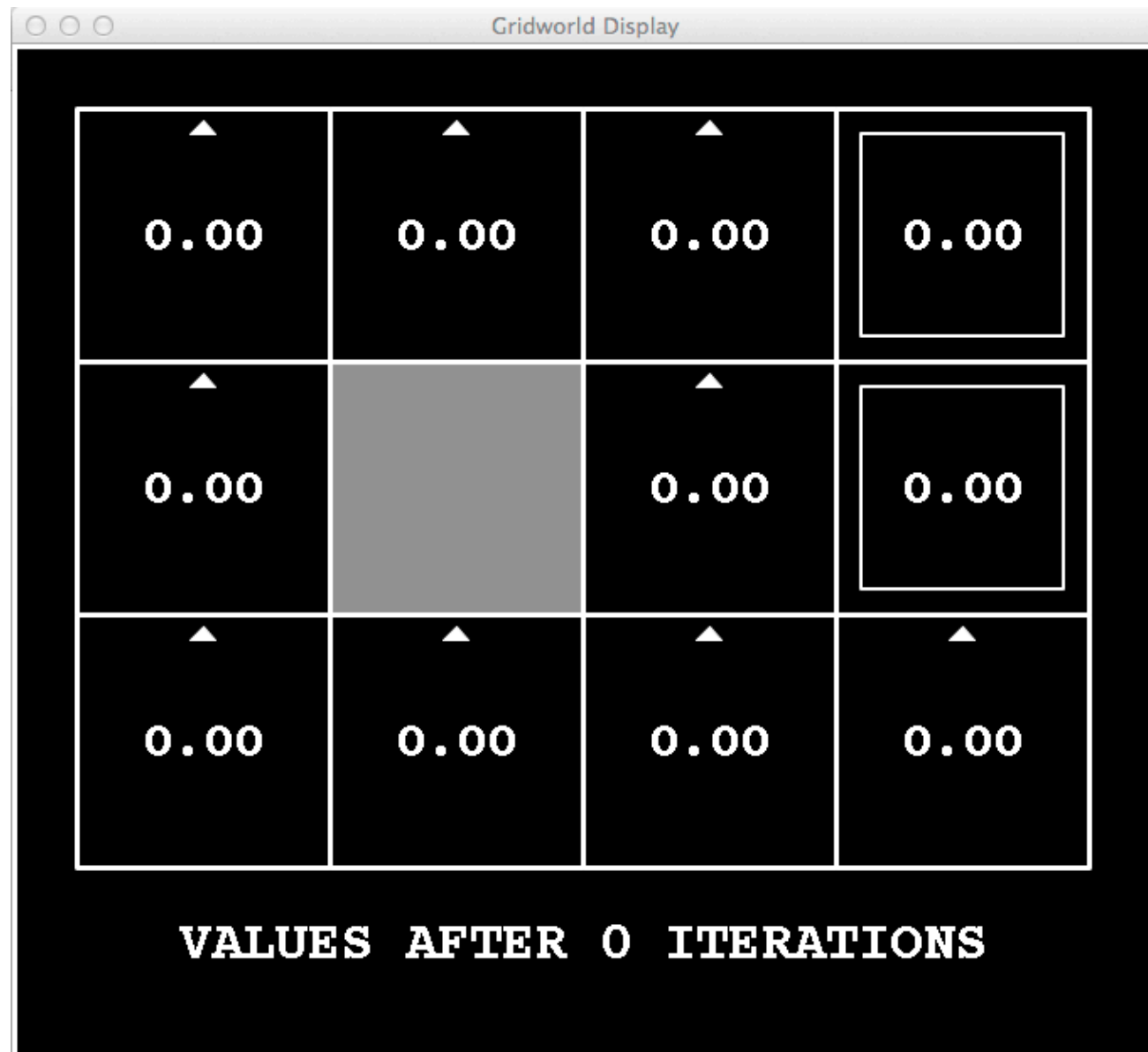
- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- Problem 1: It's slow – $O(S^2A)$ per iteration
- Problem 2: The “max” at each state rarely changes
- Problem 3: The policy often converges long before the values

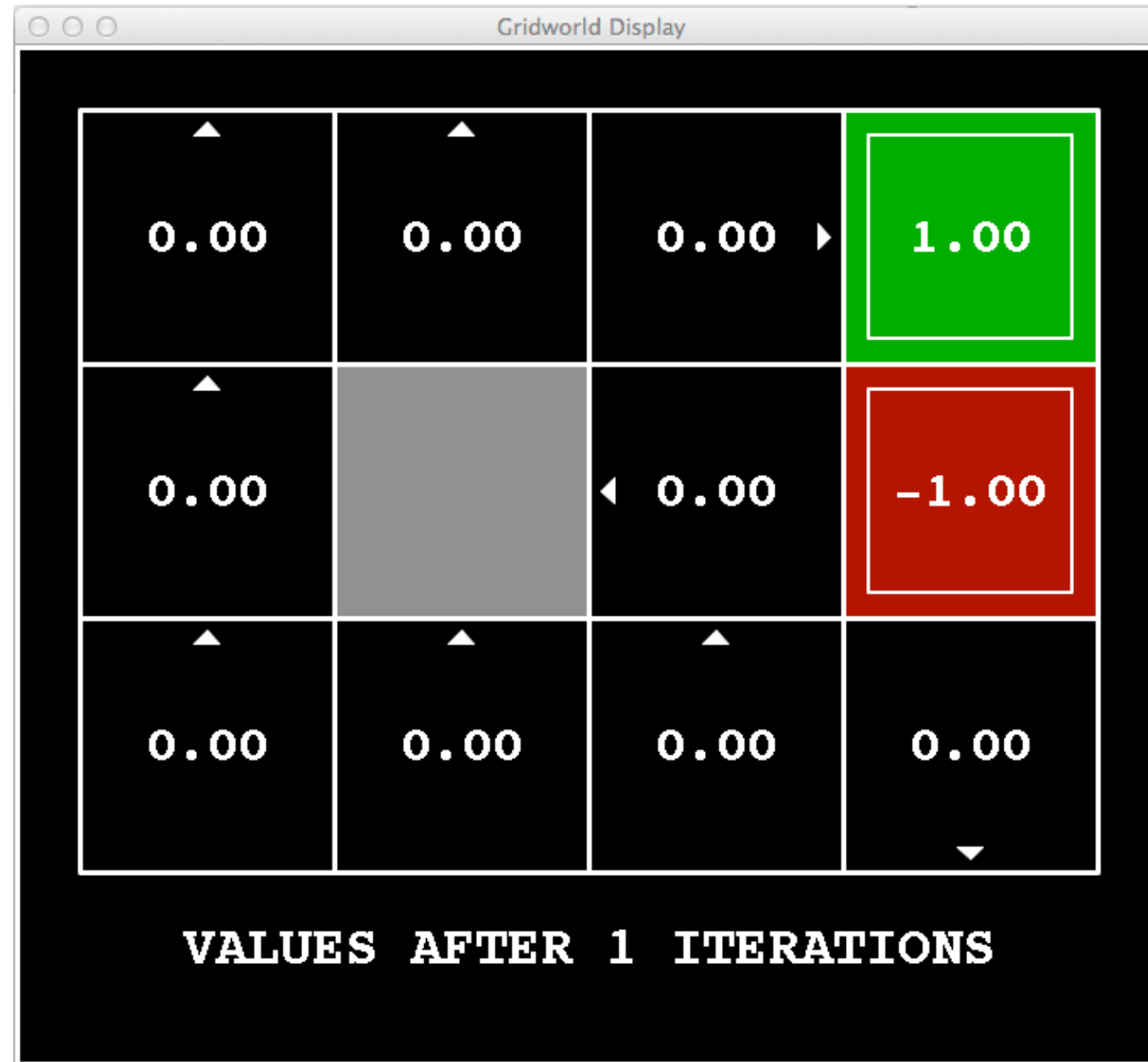


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Noise = 0.2
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k=9



Noise = 0.2
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Living reward = 0

k=10



Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Iteration

- Alternative approach for optimal values:
 - **Step 1: Policy evaluation:** calculate utilities for some fixed policy (not optimal utilities!) until convergence
 - **Step 2: Policy improvement:** update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
 - Repeat steps until policy converges
- This is **policy iteration**
 - It's still optimal!
 - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π , find values with policy evaluation:
 - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') [R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')]$$

- Improvement: For fixed values, get a better policy using policy extraction
 - One-step look-ahead:

$$\pi_{i+1}(s) = \arg \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^{\pi_i}(s')]$$

Comparison

- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
 - Every iteration updates both the values and (implicitly) the policy
 - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
 - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
 - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
 - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

- So you want to....

- Compute optimal values: use value iteration or policy iteration
- Compute values for a particular policy: use policy evaluation
- Turn your values into a policy: use policy extraction (one-step lookahead)

- These all look the same!

- They basically are – they are all variations of Bellman updates
- They all use one-step lookahead expectimax fragments
- They differ only in whether we plug in a fixed policy or max over actions