# Perturbation Methods in Derivatives Pricing under Stochastic Volatility

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# Perturbation Methods in Derivatives Pricing under Stochastic Volatility

by

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# Declaration

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# Abstract

This work employs perturbation techniques to price and hedge financial derivatives in a stochastic volatility framework. Fouque et al. [43] model volatility as a function of two processes operating on different time-scales. One process is responsible for the fast-fluctuating feature of volatility and corresponds to the slow time-scale and the second is for slow-fluctuations or fast time-scale. The former is an Ergodic Markov process and the latter is a strong solution to a Lipschitz stochastic differential equation. This work mainly involves modelling, analysis and estimation techniques, exploiting the concept of mean reversion of volatility. The approach used is robust in the sense that it does not assume a specific volatility model. Using singular and regular perturbation techniques on the resulting PDE a first-order price correction to Black-Scholes option pricing model is derived. Vital groupings of market parameters are identified and their estimation from market data is extremely efficient and stable. The implied volatility is expressed as a linear (affine) function of log-moneyness-to-maturity ratio, and can be easily calibrated by estimating the grouped market parameters from the observed implied volatility surface. Importantly, the same grouped parameters can be used to price other complex derivatives beyond the European and American options, which include Barrier, Asian, Basket and Forward options. However, this semi-analytic perturbative approach is effective for longer maturities and unstable when pricing is done close to maturity. As a result a more accurate technique, the decomposition pricing approach that gives explicit analytic first- and second-order pricing and implied volatility formulae is discussed as one of the current alternatives. Here, the method is only employed for European options but an extension to other options could be an idea for further research. The only requirements for this method are integrability and regularity of the stochastic volatility process. Corrections to [3] remarkable work are discussed here.

# Opsomming

Hierdie werk gebruik steuringstegnieke om finansiële afgeleide instrumente in ’n stogastiese wisselvalligheid raamwerk te prys en te verskans. Fouque et al. [43] gemodelleer wisselvalligheid as ’n funksie van twee prosesse wat op verskillende tyd-skale werk. Een proses is verantwoordelik vir die vinnig-wisselende eienskap van die wisselvalligheid en stem ooreen met die stadiger tyd-skaal en die tweede is vir stadig-wisselende fluktuasies of ’n vinniger tyd-skaal. Die voormalige is ’n Ergodiese-Markov-proses en die laasgenoemde is ’n sterk oplossing vir ’n Lipschitz stogastiese differensiaalvergelyking. Hierdie werk behels hoofsaaklik modellering, analise en skattingstegnieke, wat die konsep van terugkeer to die gemiddelde van die wisseling gebruik. Die benadering wat gebruik word is rubuust in die sin dat dit nie ’n aanname van ’n spesifieke wisselvalligheid model maak nie. Deur singulêre en reëlmatige steuringstegnieke te gebruik op die PDV kan ’n eerste-orde pryskorreksie aan die Black-Scholes opsie-waardasiemodel afgelei word. Belangrike groeperings van mark parameters is geïdentifiseer en hul geskatte waardes van mark data is uiters doeltreffend en stabiel. Die geïmpliseerde onbestendigheid word uitgedruk as ’n lineêre (affiene) funksie van die log-geldkarakter-tot-verval verhouding, en kan maklik gekalibreer word deur gegroepeerde mark parameters te beraam van die waargenome geïmpliseerde wisselvalligheids vlak. Wat belangrik is, is dat dieselfde gegroepeerde parameters gebruik kan word om ander komplekse afgeleide instrumente buite die Europese en Amerikaanse opsies te prys, dié sluit in Barrier, Asiatiese, Basket en Stuur opsies. Hierdie semi-analitiese steurings benadering is effektief vir langer termyne en onstabiel wanneer pryse naby aan die vervaldatum beraam word. As gevolg hiervan is ’n meer akkurate tegniek, die ontbinding prys benadering wat eksplisiete analitiese eerste- en tweede-orde pryse en geïmpliseerde wisselvalligheid formules gee as een van die huidige alternatiewe bespreek. Hier word slegs die metode vir Europese opsies gebruik, maar ’n uitbreiding na ander opsies kan’n idee vir verdere navorsing wees. Die enigste vereistes vir hierdie metode is integreerbaarheid en reëlmatigheid van die stogastiese wisselvalligheid proses. Korreksies tot [3] se noemenswaardige werk word ook hier bespreek.

# Dedication

*To my lovely girlfriend, Estelle Piedt.*

*"Problems can not be solved at the same level of awareness that created them"*

Albert Einstein.

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# Chapter 1

# Introduction

## 1.1 Background

The use of stochastic volatility models in studying financial markets has for the past twenty-five years showed significant developments in financial modelling. These models arose and momentarily gained popularity after the realization of the existence of a non-flat implied volatility surface. Their origin is traced way back in the early 1980's and gradually became pronounced especially after the 1987 market crash. The models serve a considerable improvement to the classical Black-Scholes approach which assumes constant volatility pricing for options with different strikes written on the same underlying. However, it is worth mentioning that Black-Scholes model still remains the benchmark for most of the current research developments in financial modelling, due to its inevitable and desirable features that actually led to its popularity and longevity.

The focus of current research is more on derivative pricing and parameter estimation for a class of models where volatility is mean-reverting and bursty or persistent in nature. These models are good at capturing most of the observed market features viz. volatility smiles and skews, the leverage effect, jumps in asset returns and volatility time-scales. This has made them attractive to both practitioners and academicians in the financial industry, for market analysis and modelling.

However, modelling with stochastic volatility is a non-trivial problem. The models perform poorly in regard to analyticity and tractability features, it is not easy to obtain closed form solutions for prices. It is for this reason that numerical schemes prove useful, where parameters can be estimated from observed data for calibration. Nevertheless, the volatility process is not directly observed which makes it difficult to calibrate these models in regard to stability of parameter estimation.

One theory that has been adopted in pricing and hedging various financial instruments exploits a semi-analytic tool through asymptotic expansions. The analysis yields pricing and implied volatility models that are easy to calibrate. The strength of the approach lies in the fact that, it reduces the number of model parameters needed for estimation to a few global market parameters. Moreover, these parameters are stable within periods where the underlying volatility is close to being stationary. Interestingly, the implied volatility can be expressed as an affine function of log-moneyness-to-maturity ratio composed of these parameters. Finally, the same grouped parameters can be used to price other complex derivatives beyond the European and American options.

## 1.2 Literature Review

A great deal of research has been published in regard to employing stochastic volatility models in pricing market instruments such as options, bonds and credit derivatives. Perturbation and asymptotic methods have greatly contributed to the reliability and effectiveness of stochastic volatility models. Different authors have employed perturbation techniques to the corresponding PDE with respect to a specific model parameter like, mean-reversion, see [40, 41, 43], volatility, [55] or correlation, [7]. All these methods restrict the region of validity of results to either short or long maturities.

It has been shown that perturbation techniques can generate corrections of different orders to Black-Scholes price. The approach has been used on a short, long and both short and long volatility time scales. The derivation of the first-order approximation associated with a short-time scale $\varepsilon$ (singular perturbation), or fast mean-reversion, with a smooth payoff function appears in [40]. The case for non-smooth European call option is presented in [39]. Perturbation on a long-time scale associated with a small parameter $\delta$ (regular perturbation) or slow factor for that matter, has been considered in [74] and [89]. Related literature on regular perturbation appear in [59] and [75].

Asymptotic methods have been widely applied in pricing various market instruments ranging from commodity to option markets, see for instance [47] where the techniques have been employed under fast mean-reversion, to determine prices of oil and gas. In [20], authors use similar techniques to value currency options, they show the effectiveness and efficiency of their asymptotic formula over the common Monte Carlo approach. A case for asymptotic approximations based on large strike price limits has been discussed in [8]. The authors, [32] study stochastic volatility models in regimes where the maturity is small but large compared to the mean-reversion time of the stochastic volatility factor. They derive a large deviation principle and deduce asymptotic prices for Out-of-The-Money call and put options, and their corresponding implied volatilities. Extending the idea to the bond market, consider the works of [43, 44] and [117]. Similar techniques have been employed in coming up with strategic investment decisions under fast mean-reverting stochastic volatility, see [104]. Using singular perturbations, Ma and Li [79] designed a uniform asymptotic expansion for stochastic volatility model in pricing multi-asset European options, see also [27]. These techniques continue to find wide applications in option pricing including complex derivatives. In his recent research findings, Siyanko [101] used asymptotics in form of Taylor series expansion to derive analytic prices for both fixed-strike and floating-strike Asian options. The author represents the price as an analytical expression constructed from a cumulative normal distribution function, an exponential function and finite sums.

Significant improvements by different authors on the work by [40] have proved the effectiveness and reliability of their approach. For instance, Sovan [105] builds on their work to construct a more accurate option pricing model with a very small relative error. Alòs [1, 2, 3] derives a decomposition formula from which first- and second-order price approximation formulae, that are also valid for options near maturity, are deduced.

This work is mainly involved with modelling, analysis and estimation techniques, exploiting the concept of mean-reversion of volatility. It identifies vital groupings of market parameters where their estimation from market data is extremely efficient and stable. The approach used is robust in the sense that it does not assume a specific volatility model. Lastly, a review of the decomposition formula is discussed for pricing near-maturity European options.

The next section explains the main mathematical tool employed here. Both regular and singular perturbation techniques are explicitly discussed with relevant examples, to motivate their applicability in obtaining the main result of this work.

## 1.3 Perturbation Theory

This section introduces regular and singular perturbation methods through simple examples of ordinary differential equations to motivate the theory's applicability in option pricing. Perturbation theory is a vital topic in mathematics and its applications to the natural and engineering sciences. Perturbation methods were first used by astronomers to predict the effects of small disturbances on the nominal methods of celestial bodies, see [95]. Today, perturbation methods are employed in solving problems involving differential equations (with particular conditions) whose exact solutions are difficult to derive.

A problem inclines to perturbation analysis if it is in the neighbourhood of a much simpler problem that can be solved exactly. This 'neighbourhood' or closeness is measured by the occurrence of a small dimensionless parameter (e.g. $0 < \varepsilon \ll 1$) in the governing system (this consists of differential equations and boundary conditions) such that when $\varepsilon = 0$, the resulting system becomes solvable exactly. The mathematical tool employed is asymptotic analysis with respect to a suitable asymptotic sequence of functions of $\varepsilon$. Perturbation methods fall into two categories; regular and singular depending on the nature of the problem.

### 1.3.1 Asymptotic Sequences and Expansions

This section explains the general implications of asymptotic sequences and expansion.

#### Big $\mathcal{O}$ and small $o$ Notation

Firstly, define the commonly used order symbols in asymptotic analysis, i.e. $\mathcal{O}$ and $o$. Given two functions $f(\varepsilon)$ and $g(\varepsilon)$, then $f = \mathcal{O}(g)$ as $\varepsilon \to 0$ if $|f(\varepsilon)/g(\varepsilon)|$ is bounded as $\varepsilon \to 0$, and $f = o(g)$ as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon) \to 0$ as $\varepsilon \to 0$.

#### Sequence and Expansion

Let $Q = \{\phi_n(\varepsilon)\}$, $n = 1, 2, 3, \cdots$ be an arbitrary sequence, $Q$ is an asymptotic sequence if

$$\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0, \tag{1.1}$$

for each $n = 1, 2, 3, \cdots$. Equation (1.1) implies that $|\phi_{n+1}(\varepsilon)|$ becomes small compared to $|\phi_n(\varepsilon)|$ as $\varepsilon \to 0$.

If $u(x;\varepsilon)$ is taken to be some arbitrary function dependent on $x$ and a small parameter $\varepsilon$, such that $u(x;\varepsilon)$ is in some domain $\mathcal{D}$ of $x$ and in the neighbourhood of $\varepsilon = 0$, then, the series

$$v(x;\varepsilon) = \sum_{n=1}^{N} \phi_n(\varepsilon) u_n(x) \quad \text{as} \quad \varepsilon \to 0, \tag{1.2}$$

is referred to as an asymptotic expansion of $u(x;\varepsilon)$ to the $N$-*th* term with respect to the asymptotic sequence $\{\phi_n(\varepsilon)\}$ if

$$u(x;\varepsilon) - \sum_{n=1}^{M} \phi_n(\varepsilon) u_n(x) = o\left(\phi_M(\varepsilon)\right) \quad \text{as} \quad \varepsilon \to 0,$$

for each $M = 1, 2, 3, \cdots, N$. If $N = \infty$, $u(x;\varepsilon)$ is said to be asymptotically equal to $v(x;\varepsilon)$:

$$u(x;\varepsilon) \sim \sum_{n=1}^{\infty} \phi_n(\varepsilon) u_n(x) \quad \text{as} \quad \varepsilon \to 0. \tag{1.3}$$

### 1.3.2 Regular and Singular Perturbations

In a regular perturbation problem, a straight forward procedure leads to a system of differential equations and boundary conditions for each term in the asymptotic expansion. This system can be solved recursively, the accuracy of the result improves as $\varepsilon$ becomes smaller for all values of the independent variables throughout a particular domain of interest. However, this approach is not always valid especially under certain circumstances such as, trying to find a solution under an infinite domain containing small terms with a cumulative effect. In this case, another approach can be used referred to as, singular perturbation technique.

In singular perturbation or layer-type problem, there is one or more thin layers at the boundary or in the interior of the domain where the regular technique fails. The regular perturbation technique usually fails when the small parameter $\varepsilon$ multiplies the highest derivative in the differential equation, setting the leading approximation to follow a lower-order equation. This creates 'chaos' in the sense that the resulting solution does not satisfy the whole set of given boundary conditions.

Further, consider a boundary value problem $P^{\varepsilon}$ depending on a small parameter $\varepsilon$ under specific conditions. A solution $u(x;\varepsilon)$ of $P^{\varepsilon}$ can be constructed by perturbation methods as a power series in $\varepsilon$ with the first term $u_0(x)$ being the solution of the problem $P^0$. If this series expansion converges uniformly in the entire domain $\mathcal{D}$ of $x$ as $\varepsilon \to 0$, then it's a regular perturbation problem. However, if $u(x;\varepsilon)$ does not have a uniform limit in $\mathcal{D}$ as $\varepsilon \to 0$, the regular perturbation method fails and the problem is said to be singularly perturbed.

### 1.3.3 Outer and Inner expansions

Using asymptotic expansions to approximate the solution of a differential equation given some boundary conditions (over some defined domain $\mathcal{D}$ of the independent variable $x$) may result into the asymptotic expansion, nicely approximating the exact solution

(i) over all $\mathcal{D}$,

(ii) only when one is far from a particular boundary point in $\mathcal{D}$, say $x = 0$, or,

(iii) when close to that same ($x = 0$) boundary point.

Case (i) is always the desired scenario. Cases (ii) and (iii) respectively, yield to *outer* and *inner* expansions and provide outer and inner solutions of the general approximation to the exact solution. This general approximation is obtained through *asymptotic matching* of the two solutions. The last two cases commonly occur in singularly perturbed problems.

### 1.3.4 Matched Asymptotics

In singular perturbation problems, the expansion in equation (1.3) can not be valid uniformly in domain $\mathcal{D}$ of $x$, it fails to satisfy all the boundary conditions. Suppose

$$u^o(x;\varepsilon) = \sum_{n=0}^{\infty} a_n(\varepsilon)u_n(x) \quad \text{as} \quad \varepsilon \to 0$$

is an outer solution, where $\{a_n(\varepsilon)\}$ is an asymptotic sequence, then this expansion satisfies the outer region away from (part of) the boundary of $\mathcal{D}$. In order to investigate regions of non-uniform convergence, one can introduce some stretching transformations

$$\xi = \psi(x;\varepsilon)$$

which ''blows up'' a region of non-uniformity. For instance, if $\xi := x/\varepsilon$, one observes that if $\xi$ is fixed and $\varepsilon \to 0$, $x \to 0$, while for fixed $x > 0$ and $\varepsilon \to 0$, $\xi \to \infty$. Suppose in terms of the stretched variable $\xi$ the asymptotic solution becomes

$$u^{\mathrm{i}}(\xi;\varepsilon) = \sum_{n=0}^{\infty} b_n(\varepsilon)u_n(\xi) \quad \text{as} \quad \varepsilon \to 0$$

and is valid for values of $\xi$ in some inner region, where $\{b_n(\varepsilon)\}$ is an asymptotic sequence, then the expansion $u^{\mathrm{i}}$ is referred to as an inner solution[1].

In most cases, it is impossible to determine both the outer and inner expansions $u^{\mathrm{o}}$ and $u^{\mathrm{i}}$ completely by straight forward expansion procedures. However, both expansions should represent the solution of the original problem asymptotically in different regions. Thus, there is need for matching the two expansions, i.e; relating the outer expansion in the inner region $(u^{\mathrm{o}})^{\mathrm{i}}$ and the inner expansion in the outer region $(u^{\mathrm{i}})^{\mathrm{o}}$ by using the stretching transformation $\xi = \psi(x;\varepsilon)$. After successful matching, the asymptotic solution to a well-posed problem becomes completely known in both the inner and outer regions. It is always convenient to obtain a composite expansion $u^{\mathrm{c}}$ uniformly valid in $\mathcal{D}$ where

$$u^{\mathrm{c}} = u^{\mathrm{o}} + u^{\mathrm{i}} - (u^{\mathrm{i}})^{\mathrm{o}}$$

and making appropriate modifications if several regions of non-uniform convergence (e.g several inner regions) are necessary.

### 1.3.5 Simple Cases

The following example is obtained from [67].

[1] The inner expansion accounts for boundary conditions neglected by the outer expansion and vice versa.

**Regular Problem**

Consider the regular problem (1.4) where $\varepsilon$ is a small perturbation parameter;

$$u' + 2xu - \varepsilon u^2 = 0 \quad \text{as} \quad \varepsilon \to 0, \tag{1.4}$$

with $0 \leq x < \infty$ and $u(0) = 1$. Here, $u' = \frac{du}{dx}$. An unperturbed ($\varepsilon = 0$) version of equation (1.4) takes the form,

$$u_0{}' + 2xu_0 = 0 \quad \text{with} \quad u_0(0) = 1, \tag{1.5}$$

The solution of equation (1.5) is $u_0(x) = e^{-x^2}$. Suppose the general solution to (1.4) is

$$u(x;\varepsilon) = u_0(x) + \sum_{n=1}^{\infty} \phi_n(\varepsilon) u_n(x). \tag{1.6}$$

Substituting equation (1.6) in equation (1.4), yields the following equation,

$$\begin{aligned}\left[u_0'(x) + \phi_1(\varepsilon)u_1'(x) + \ o(\phi(\varepsilon))\right] + 2x\left[u_0(x) + \phi_1(\varepsilon)u_1(x) + \ o(\phi_1(\varepsilon))\right] \\ -\varepsilon\left[u_0(x) + \phi_1(\varepsilon)u_1(x) + \ o(\phi_1(\varepsilon))\right]^2 = 0.\end{aligned}$$

This reduces to

$$\phi_1(\varepsilon)[u_1'(x) + 2xu_1(x)] - \varepsilon u_0^2(x) = \ o(\phi_1(\varepsilon)), \tag{1.7}$$

where $0 < \varepsilon \ll 1$. It remains to determine the kind of function that $\phi_1(\varepsilon)$ should take. Consider the following two cases:

- Case I: If $\varepsilon \ll \phi_1$, then $u_1$ satisfies the homogeneous equation

  $$u_1' + 2xu_1 = 0, \quad u_1(0) = 0.$$

  However, this gives a trivial solution $u_1(x) = 0$.
- Case II: If $\phi_1 \ll \varepsilon$, it implies that $u_0^2(x) = 0$ which is an inconsistent condition.

Therefore, a non-trivial solution $u_1(x)$ would only be obtained if $\phi_1 = \mathcal{O}(\varepsilon)$. For simplicity, take $\phi_1 = \varepsilon$ which reduces the problem to

$$u_1' + 2xu_1 = u_0^2, \quad u_1(0) = 0.$$

Substituting $u_0 = e^{-x^2}$ and solving the inhomogeneous ODE gives the solution $u_1$ as

$$u_1(x) = e^{-x^2} \int_0^x e^{-s^2}\, ds.$$

Thus,

$$u(x;\varepsilon) = e^{-x^2} + \varepsilon e^{-x^2} \int_0^x e^{-s^2}\, ds + \; o(\varepsilon). \tag{1.8}$$

Observe that $u(x;\varepsilon)$ is uniformly valid in $\mathcal{D} : 0 \leq x < \infty$, since $e^{-x^2} < 1$ and

$$\int_0^x e^{-s^2}\, ds < \int_0^\infty e^{-s^2}\, ds = \frac{1}{2}\sqrt{\pi} \text{ on } \mathcal{D}.$$

Hence, $0 \leq u_1(x) < \sqrt{\pi}/2$ on $\mathcal{D}$.

**Singular Problem**

It is not always the case that problems can be represented using uniformly valid expansions. Most equations exhibit singular behaviour which leads to asymptotic expansions that eventually break down, resulting in the need to rescale and probably invoke the matching principle. This is illustrated in the following example obtained from [62].

Consider the singular problem

$$\frac{dy}{dx} + \left[1 + \frac{\varepsilon^2}{x^2+\varepsilon^2}\right] y + \varepsilon y^2 = 0\,; \quad 0 \leq x \leq 1, \quad \text{with} \quad y(1;\varepsilon) = 1, \tag{1.9}$$

as $\varepsilon \to 0$. Using asymptotic expansion, the solution is assumed to take the form

$$y(x;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n y_n(x). \tag{1.10}$$

Substituting the above expansion up to the term of order $\mathcal{O}(\varepsilon^2)$ and comparing $\mathcal{O}(1)$, $\mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ order terms yields the following problems

$$y_0' + y_0 = 0; \quad y_1' + y_1 + y_0^2 = 0; \quad y_2' + y_2 + \frac{y_0}{x^2} + 2y_0y_1 = 0. \tag{1.11}$$

The boundary condition requires that

$$y_0(1) = 1, \quad y_n(1) = 0 \quad \text{for} \quad n \geq 1. \tag{1.12}$$

The evaluation of the expansion is valid on $x = 1$ and becomes difficult as $x \to 0$. The solutions $y_0(x)$ and $y_1(x)$ are easily obtained as

$$y_0(x) = e^{1-x}; \quad y_1(x) = e^{2-2x} - e^{1-x}. \tag{1.13}$$

Thus, the asymptotic expansion up to the second term, i.e. $y = y_0 + \varepsilon y_1$, is uniformly valid throughout the domain of $x$. Next, is to find the next term, $y_2$ in the expansion that satisfies the last problem in equation (1.11), i.e.

$$y_2' + y_2 + \frac{e^{1-x}}{x^2} + 2e^{1-x}\left[e^{2-2x} - e^{1-x}\right] = 0 \quad \text{with} \quad y_2(1) = 0. \tag{1.14}$$

Using the integrating factor, $e^x$, gives

$$y_2(x) = \frac{e^{1-x}}{x} + e^{1-x}\left[e^{2-2x} - 2e^{1-x}\right]. \tag{1.15}$$

Note the singularity in $y_2(x)$ at $x = 0$, this is what leads to a breakdown in the expansion given up to order $\mathcal{O}(\varepsilon^2)$

$$y(x;\varepsilon) \sim e^{1-x} + \varepsilon[e^{2-2x} - e^{1-x}] + \varepsilon^2\left[\frac{e^{1-x}}{x} + e^{3-3x} - 2e^{2-2x}\right], \tag{1.16}$$

as $x \to 0$ for $x = \mathcal{O}(1)$. Observe that as $x \to 0$, i.e. $x = \mathcal{O}(\varepsilon)$, the second and third terms in (1.16) become of the same size. This is a breakdown, which moreover, occurs for larger size of $x$ than that between the first and third terms. The remedy is to reformulate the problem to consider $x = \mathcal{O}(\varepsilon)$.

The problem for $x = \mathcal{O}(\varepsilon)$ is formulated as

$$x = \varepsilon X \quad \text{and} \quad y(\varepsilon X;\varepsilon) \equiv Y(X;\varepsilon). \tag{1.17}$$

The original problem (1.9), expressed in terms of $X$ and $Y$ requires that

$$\frac{dy}{dx} = \frac{dY}{dx} = \frac{d}{dx}Y(x/\varepsilon;\varepsilon) = \varepsilon^{-1}\frac{dY}{dX}, \tag{1.18}$$

to get

$$\frac{dY}{dX} + \varepsilon\left[1 + \frac{1}{1+X^2}\right]Y + \varepsilon^2 Y^2 = 0. \tag{1.19}$$

The boundary condition to (1.19) is unavailable because it is specified where $x = \mathcal{O}(1)$. The solution takes the form

$$Y(X;\varepsilon) \sim \sum_{n=0}^{N} \varepsilon^n Y_n(X), \tag{1.20}$$

from which the following problems are obtained

$$Y_0' = 0; \quad Y_1' + \left[1 + \frac{1}{1+X^2}\right]Y_0 = 0. \tag{1.21}$$

The first equation implies that $Y_0 = A_0$, an arbitrary constant which leads to a general solution to the second equation, given as

$$Y_1(X) = -A_0[X + \tan^{-1} X] + A_1, \tag{1.22}$$

where $A_1$ is another arbitrary constant. Thus, the expansion up to the second term is

$$Y(X;\varepsilon) \sim A_0 + \varepsilon[A_1 - A_0[X + \tan^{-1} X]], \qquad X = \mathcal{O}(1). \tag{1.23}$$

The constants $A_0$ and $A_1$ can be determined by using the matching principle. Solution $y(x;\varepsilon)$ in (1.16) has to match with $Y(X;\varepsilon)$ in (1.23).

The matching procedure is as follows: express $y(x;\varepsilon)$ as a function of $X$ as $\varepsilon \to 0$ and retain terms of $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ which appear in (1.23); Conversely, express $Y(X;\varepsilon)$ as a function of $x$, expand and retain terms $\mathcal{O}(1), \mathcal{O}(\varepsilon)$ and $\mathcal{O}(\varepsilon^2)$ which appear in (1.16). This yields[2]

$$\begin{aligned} &e^{1-\varepsilon X} + \varepsilon\left[e^{2-2\varepsilon X} - e^{1-\varepsilon X}\right] + \varepsilon^2\left[\frac{e^{1-\varepsilon X}}{\varepsilon X} + e^{3-3\varepsilon X} - 2e^{2-2\varepsilon X}\right] \\ &\qquad \sim e + \varepsilon\left[e^2 - e\right] \quad \text{for} \quad X = \mathcal{O}(1); \end{aligned} \tag{1.24}$$

and

$$\begin{aligned} &A_0 + \varepsilon\left[A_1 - A_0\left[\frac{x}{\varepsilon} + \tan^{-1}\frac{x}{\varepsilon}\right]\right] \\ &\qquad \sim A_0 - A_0 X + \varepsilon\left[A_1 - A_0\frac{\pi}{2}\right] + \frac{\varepsilon^2 A_0}{x} \quad \text{for} \quad x = \mathcal{O}(1). \end{aligned} \tag{1.25}$$

The two expansions match with the choices $A_0 = e$ and $A_1 = e^2 - e + e\pi/2$, yielding the asymptotic expansion for $X = \mathcal{O}(1)$ as

$$Y(X;\varepsilon) \sim e + \varepsilon\left[\frac{e\pi}{2} + e^2 - e - e[X + \tan^{-1} X]\right]. \tag{1.26}$$

Note that

$$y(0;\varepsilon) \sim Y(0;\varepsilon) \sim e + \varepsilon[e^2 - e] \quad \text{as} \quad \varepsilon \to 0. \tag{1.27}$$

The following section introduces the basic ideas and methods of Black-Scholes theory of European option pricing, motivating the need for random volatility.

[2] Note that, $\tan^{-1} X \approx \frac{\pi}{2} - \frac{1}{X} + \frac{1}{3X^3}$ as $X \to +\infty$.

## 1.4 Black Scholes Model

The section explains the dynamics of Black-Scholes model for pricing European options[3]. Throughout this work, $[0, T]$ denotes the trading interval. The uncertainity under the real world physical probability measure $\mathbb{P}$ shall be completely specified by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\Omega$ denotes the complete set of all possible outcomes $\omega \in \Omega$. All the available information in the economy up to time $t$ shall be contained within the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The level of uncertainty shall be resolved over $[0, T]$ with respect to the information filtration[5]. According to Girsanov's Theorem [87], there exists an equivalent martingale measure (EMM) under which all discounted tradable assets are martingales[6].

The Black-Scholes economy is a complete market framework[7] that assumes under a unique risk-neutral or EMM $\mathbb{Q}$ that the stock price $X_t$ satisfies the following stochastic differential equation with initial condition

$$dX_t = X_t\, dN_t\,; \quad X_0 = x_0, \tag{1.28}$$

where $N_t = rt + \sigma W_t$, $r$ is the risk-free rate of return, $\sigma$ is constant volatility and $W$ is a $\mathbb{Q}$-Brownian motion. According to [33], the solution $X_t$ to equation (1.28) takes the form

$$X_t = x_0 \exp\left\{N_t - \frac{1}{2}\langle N\rangle_t\right\}, \tag{1.29}$$

that is,

$$X_t = x_0 \exp\left\{rt + \sigma W_t - \frac{1}{2}\sigma^2 t\right\}. \tag{1.30}$$

Observe that the log-returns are normally distributed, i.e.

$$\log[X_t/X_0] \sim \mathcal{N}\left([r - \sigma^2/2]t, \sigma^2 t\right). \tag{1.31}$$

[3]Bachelier [9] developed the first model[4] to study stock price dynamics using *arithmetic Brownian motion*. His result was further developed by [88] and [92] who proposed a *Geometric Brownian motion model*. Based on the latter, [14] derived an analytic formula for a European call option price which won them a noble prize in 1997.

[5]A concrete discussion about filtered probability space is presented in [60].

[6]The EMM is unique in complete markets and is denoted by $\mathbb{Q}$ unlike in incomplete markets where they are numerous, usually denoted by $\mathbb{P}^*$.

[7]Completeness implies existence of a unique equivalent risk-neutral pricing measure with which options are priced and perfect hedging is possible.

### 1.4.1 Black-Scholes Pricing PDE

It is documented [90] that under measure $\mathbb{Q}$, the no-arbitrage price $P(t,x)$ of a European option is the present value of the conditional expected payoff given by

$$P(t,x) = \mathbb{E}^{\mathbb{Q}}\left\{e^{-r(T-t)}h(X_T)|\mathcal{F}_t\right\}, \tag{1.32}$$

where $T$ is the maturity date and $h(X_T)$ is the payoff function. Note that this expectation takes the form of the Feynman-Kac formula defined in Appendix A, Section A.5, where $R = r$ and $f = h(X_T)$. From the general equation (A.17) one obtains the PDE with terminal condition

$$P_t = rxP_x + \frac{1}{2}\sigma^2 x^2 P_{xx} - rP\,; \quad P(T,x) = h(x). \tag{1.33}$$

Equation (1.33) is known as Black-Scholes pricing partial differential equation. This equation has *variable coefficients* and can easily be solved by transforming it into the *constant coefficient* heat equation. The motivation is that the heat equation has a well known analytic solution.

### 1.4.2 Diffusion and Heat Equations

The pricing PDE in equation (1.33) can be reduced to the heat equation by re-scaling the independent variables $X_t$ and $t$. Introducing new dimensionless parameters $s$ and $\tau$ such that $X_t = Ke^s$ and $t = T - 2\tau/\sigma^2$, transforms the PDE into

$$P_\tau + P_s - P_{ss} - \frac{2r}{\sigma^2}\left[P_s + P\right] = 0. \tag{1.34}$$

Assuming $P(t,X_t) = K\exp\left\{\alpha s + \beta\tau\right\}u(\tau,s)$, substituting it in equation (1.34) yields

$$[\beta u + u_\tau] + [\alpha u + u_s] - [\alpha^2 u + 2\alpha u_s + u_{ss}] - \frac{2r}{\sigma^2}[\alpha u + u_s - u] = 0,$$

which is a *constant coefficient* PDE.

Observe that eliminating the terms $u_s$ and $u$ leads to the heat equation. Therefore,

$$[\alpha + \beta] - \alpha^2 - \tfrac{2r}{\sigma^2}\left[\alpha - 1\right] = 0 \quad \text{and} \quad 1 - 2[\alpha + \tfrac{r}{\sigma^2}] = 0, \tag{1.35}$$

from which formulae for $\alpha$ and $\beta$ in terms of $r$ and $\sigma^2$, are derived. If $k_1 := \frac{2r}{\sigma^2}$, then $\alpha = -\frac{1}{2}[k_1 - 1]$ and $\beta = \frac{1}{4}[k_1 + 1]^2$ and the Black-Scholes PDE is reduced to the heat equation

$$u_\tau = u_{ss}\,; \quad -\infty < s < \infty, \tag{1.36}$$

Next is to obtain the initial condition to equation (1.36). Consider a European call option with payoff $\max\{X_T - K, 0\}$, and recall that $u(\tau, s)$ is related to $P(t, X_t)$ as

$$u(\tau, s) = \frac{1}{K} P(t, X_t) \exp\{-\alpha s - \beta\tau\} \tag{1.37}$$

and that $\tau = 0$ corresponds to $t = T$, then

$$\begin{aligned} u(0, s) &= \tfrac{1}{K} P(T, X_T) \exp\{-\alpha s\} = \tfrac{1}{K} \max\{X_T - K, 0\} \exp\{-\alpha s\}. \\ &= \tfrac{1}{K} \max\{K \exp\{s\} - K, 0\} \exp\{-\alpha s\} = \max\{\exp\{s\} - 1, 0\} \exp\{-\alpha s\}. \\ &= \max\left\{\exp\left\{\tfrac{1}{2}[k_1 + 1]s\right\} - \exp\left\{\tfrac{1}{2}[k_1 - 1]s\right\}, 0\right\} := u_0(s). \end{aligned}$$

### 1.4.3 Solution of the Black-Scholes PDE

The *fundamental solution* to the dimensionless heat equation (1.36) is given by, [112]

$$G(\tau, s) = \frac{1}{\sqrt{4\pi\tau}} \exp\left\{-s^2/4\tau\right\}, \tag{1.38}$$

for all $\tau > 0$ and $s \in \mathbb{R}$. This can be checked by direct substitution into the equation. Note, $G(\tau, s) = \phi_{0,\sqrt{2\tau}}(s)$, the probability density function of the normal distribution with zero mean and variance $2\tau$. For a given initial condition $u_0(s)$ the solution of the heat equation can be written as a convolution integral of $G$ and $u_0$, that is

$$u(\tau, s) = \int_{-\infty}^{\infty} G(\tau, s - \xi) u_0(\xi)\, d\xi, \tag{1.39}$$

for $\tau > 0$. Observe from Appendix A.4 that with this representation, the function $G(\tau, s - \xi)$ is also the *Green's function* for the diffusion equation. It is not difficult to show that the convolution integral is indeed a solution to the heat equation and satisfies

$$\lim_{\tau \to 0^+} u(\tau, s) = u_0(s).$$

Therefore, the solution of the heat equation which satisfies the initial condition $u_0(s)$ can be written from equation (1.39) as

$$u(\tau, s) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} \exp\left\{-[s - \xi]^2/4\tau\right\} u_0(\xi)\, d\xi. \tag{1.40}$$

Evaluating this integral[8] and transforming the solution back to the original variables gives Black-Scholes analytic formula for a European call option:

$$C_{BS} = xN(\mathrm{d}_+) - Ke^{-r[T-t]} N(\mathrm{d}_+ - \sigma\sqrt{T - t}), \tag{1.41}$$

[8] The computation for this integral is found in [112].

where $N(\cdot)$ is defined as the standard normal cumulative distribution function for some variable $\zeta$,

$$N(\cdot) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\cdot} e^{-\frac{\zeta^2}{2}} \mathrm{d}\zeta. \tag{1.42}$$

and $\mathrm{d}_+$ is given by

$$\mathrm{d}_+ = \frac{\log\left[x/K\right] + [r + \frac{\sigma^2}{2}][T-t]}{\sigma\sqrt{T-t}}. \tag{1.43}$$

The corresponding pricing formula, $P_{BS}$ for a European put option can be deduced from the *put-call parity principle*

$$x + P_{BS} - C_{BS} = Ke^{-r[T-t]}, \tag{1.44}$$

where $x$ denotes the current stock price and $K$ is the strike price.

### 1.4.4 The Greeks

In this section, analytic formulae for the *Greeks* from result (1.41), are derived. Greeks are simply derivatives of this solution with respect to the model parameters and variables.

#### The Delta

This is defined as the derivative of the function $C_{BS}$ with respect to the stock price $x$. To derive the delta function, follow the simple approach by [17]: The Black-Scholes pricing formula for a call option can be rewritten as

$$C_{BS} = xN(\mathrm{d}_0 + \sigma\sqrt{T-t}) - Ke^{-r[T-t]}N(\mathrm{d}_0),$$

where

$$\mathrm{d}_0 = \frac{\log[xe^{r[T-t]}/K]}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}.$$

If $\mathrm{d}_0$ is considered variant and all other parameters fixed, $C_{BS}$ remains a function of only $\mathrm{d}_0$:

$$C_{BS} = C_{BS}(\mathrm{d}_0). \tag{1.45}$$

Note that $\mathrm{d}_0$ generates the maximum value of the function $C_{BS}(\mathrm{d})$, that is

$$C_{BS} = C_{BS}(\mathrm{d}_0) = \sup_{\mathrm{d}\in\mathbb{R}} \left\{C_{BS}(\mathrm{d})\right\}.$$

Thus, the delta $\Delta$ of $C_{BS}(\mathrm{d}_0)$ is easily computed as

$$\Delta = \frac{\partial C_{BS}}{\partial x} + \frac{\partial C_{BS}}{\partial \mathrm{d}} \cdot \frac{\partial \mathrm{d}_0}{\partial x} = N(\mathrm{d}_0 + \sigma\sqrt{T-t}). \tag{1.46}$$

Note that $\mathrm{d}_+ = \mathrm{d}_0 + \sigma\sqrt{T-t}$, so $\Delta$ is given as

$$\frac{\partial C_{BS}}{\partial x} = N(\mathrm{d}_+). \tag{1.47}$$

### The Gamma

The *Gamma* is the derivative of the *Delta* with respect to the stock price

$$\frac{\partial}{\partial x}\left[\frac{\partial C_{BS}}{\partial x}\right] = \frac{\partial N(\mathrm{d}_+)}{\partial \mathrm{d}_+} \cdot \frac{\partial \mathrm{d}_+}{\partial x} = \frac{1}{x\sigma\sqrt{2\pi[T-t]}} e^{-\frac{\mathrm{d}_+^2}{2}}. \tag{1.48}$$

### The Vega

The derivative of (1.41) with respect to $\sigma$ gives the *Vega*. Let $\omega := \mathrm{d}_+ - \sigma\sqrt{T-t}$, then

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial \sigma} &= x\frac{\partial N(\mathrm{d}_+)}{\partial \sigma} - Ke^{-r(T-t)}\frac{\partial N(\omega)}{\partial \sigma} \\
&= x\frac{\partial N(\mathrm{d}_+)}{\partial \mathrm{d}_+} \cdot \frac{\partial \mathrm{d}_+}{\partial \sigma} - Ke^{-r(T-t)}\frac{\partial N(\omega)}{\partial \omega} \cdot \frac{\partial \omega}{\partial \sigma}.
\end{aligned}$$

From (1.42) and (1.43) follows

$$\begin{aligned}
\frac{\partial C_{BS}}{\partial \sigma} = &\frac{x}{\sqrt{2\pi}} e^{-\frac{\mathrm{d}_+^2}{2}} \left[\left[\frac{3}{2}\sigma^2 + r\right][T-t]^{\frac{3}{2}} - \frac{\log[\frac{x}{K}]}{\sigma^2\sqrt{T-t}}\right]. \\
&- \frac{K}{\sqrt{2\pi}} e^{-r[T-t]-\omega^2/2} \left[\left[\frac{3}{2}\sigma^2 + r\right][T-t]^{\frac{3}{2}} - \frac{\log[\frac{x}{K}]}{\sigma^2\sqrt{T-t}} - [T-t]^{\frac{1}{2}}\right].
\end{aligned}$$

Substituting for $\omega$ and factorizing gives

$$\begin{aligned}
&\frac{\partial C_{BS}}{\partial \sigma} = \frac{x[T-t]^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{\mathrm{d}_+^2}{2}} \left[\left[\frac{3}{2}\sigma^2 + r\right][T-t] - \frac{\log[\frac{x}{K}]}{\sigma^2[T-t]}\right] - \\
&\frac{x[T-t]^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{\mathrm{d}_+^2}{2}} \frac{K}{x} e^{\sigma \mathrm{d}_+\sqrt{T-t}-[r+\sigma^2/2][T-t]} \left[\left[\frac{3}{2}\sigma^2 + r\right][T-t] - \frac{\log[\frac{x}{K}]}{\sigma^2[T-t]} - 1\right].
\end{aligned}$$

From (1.43) one obtains the ratio $K/x$ as

$$\frac{K}{x} = e^{-\sigma \mathrm{d}_+\sqrt{T-t}+[r+\sigma^2/2][T-t]}. \tag{1.49}$$

Substituting for the ratio $K/x$ gives the explicit formula of the *Vega* as

$$\frac{\partial C_{BS}}{\partial \sigma} = \frac{x[T-t]^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{d_+^2}{2}}. \tag{1.50}$$

It is well known and documented that Black-Scholes model is considered the biggest success in financial theory both in terms of approach and applicability. However, it has counter-factual assumptions that are well explained in [109].

## 1.5 Thesis Structure

The rest of the thesis is organised as follows. Chapter 2 introduces and describes the immediate effects and adjustments in equity market modelling after the crash of 1987. Important market features such as volatility smiles, skews and term structure are discussed. Local and stochastic volatility models are briefly explained, and the general result by [45] presented.

The main ideas of volatility mean-reversion and clustering are discussed in Chapter 3, using the Ornstein-Uhlenbeck model as an example of a mean-reverting process. The convergence of the Hull-White model to Black-Scholes model is set as an example of the effects of these facets. Further, the Heston model is discussed as an example of a square-root mean-reverting model that yields a closed-form solution.

Chapter 4 which is also the main part of this work, introduces the methodology of asymptotic pricing that exploits the concepts of volatility mean-reversion and clustering. The singular perturbation method is explained in details to solve a perturbed pricing problem. Applications are given in form of pricing a perpetual American put option and delta hedging derivatives under stochastic volatility. An improved model, see [41] and [43] that uses a multi-scale volatility is discussed.

Chapter 5 reviews the decomposition pricing approach of Alòs [3]. The method addresses the difficult challenge of pricing derivatives near maturity faced in Chapter 4. Using the classical Itô's formula, one can construct a decomposition formula from which easy-to-compute first- and second-order approximation pricing formulae can be deduced. The chapter ends with a conclusion of the entire document.

# Chapter 2

# Beyond Black-Scholes Model

The purpose of this chapter is to explain the state of the financial industry after the market crash[1] of October 19$^{\text{th}}$, 1987, commonly referred to as Black Monday. The focus is on the post-crash pricing models from the point of view of equity-based option derivatives.

Prior to the stock market crash of 1987, options written on equity where basically priced using the classical Black-Scholes model [14]. The model assumes that the implied volatility of an option is independent of the strike price and expiration date. There is no smile and the implied volatility surface is relatively flat. The smile first appeared after the crash triggering the need for adjustment of the model. In fact, it is reported [24] that the volatility smile surfaced in almost all option markets about 15 years after the crash, forcing traders and quants to design new pricing models. The Black-Scholes model was no-longer reliable, there was a discrepancy between the classical Black-Scholes stock returns and the observed stock market returns. Moreover, it is observed that the underlying asset's log-returns do not exhibit Gaussian distribution, instead their distribution displays large tails and high peaks compared to normal distribution. Observed data suggested randomness in volatility.

Figure 2.1 shows a plot of implied volatilities against different strike levels for equity options depicting the structure of the volatility skew[2] before and after the crash of 1987. The relatively horizontal line shows the nature of the smile before the crash, implying that in the perfect Black-Scholes model, volatility is constant for all options on the same underlying asset and independent of strike level. The curved line indicates the nature of the smile after the crash, observe a totally different shape from Black-Scholes assumption. Figure 2.3 shows two different-maturity skews on recent data of the S&P 500 index.

[1] see [80] for the probable reasons for the crash.

[2] In equity markets, the smile is generally referred to as volatility skew.

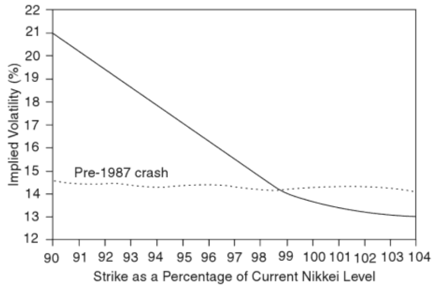


Figure 2.1. The volatility skew before and after the market crash of 1987. *Source: My life as a Quant by Emmanuel Derman, John Wiley & Sons, 2004, p.227.*

Figure 2.2 shows daily log-returns on the SPX index from December 31, 1984 to December 31, 2004. Observe an abnormal log-return of $-22.9\%$ on October 19$^{\text{th}}$, 1987.

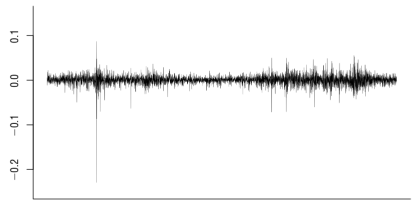

Figure 2.2. Daily log-returns on the SPX index from December 31, 1984 to December 31, 2004. *Source: The volatility surface: A practitioner's guide.*

In pursuit of correcting the constant volatility model, different pertinent models have been proposed. These models are discussed later in Sections 2.2 and 2.3, respectively.

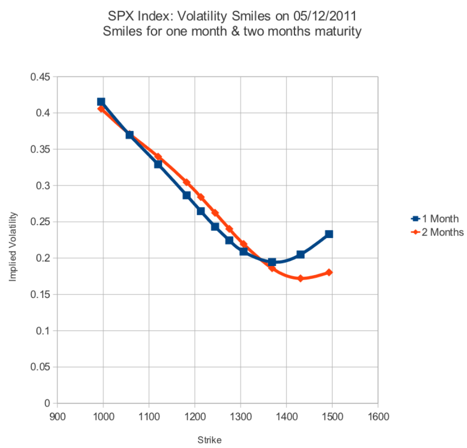


Figure 2.3. Volatility skews of S&P 500 index on 05/12/2011 for one month and two months maturities with stock price at 1244.28.

## 2.1 The Implied Volatility

Traders have turned Black-Scholes loop-hole of quoting European option prices in terms of their dollar value into a useful feature. The prices are instead expressed in terms of their equivalent implied volatilities. It is a common activity on trading floors to both quote and observe prices in this way. The advantage of expressing prices in such dimensionless units allows easy comparison between products with different characteristics. Implied volatility, $I$, is the value of $\sigma$ which must be plugged into the Black-Scholes formula to reproduce the market price of that particular option. If the market price of some call option is $C^{obs}$, then the implied volatility[3] $I$ is uniquely defined through the relationship

$$C_{\mathrm{BS}}(t, X; K, T; I) = C^{obs}(K, T), \tag{2.1}$$

where $C_{BS}$ is the Black-Scholes price of the option. The *put-call parity* indicates that puts and calls with same strike price and maturity have the same implied volatility. If at any instance, the Black-Scholes price $C_{\mathrm{BS}}(t, X, K, T, \sigma)$ equals the market price, then $\sigma = I$, where $\sigma$ is the historical volatility[4].

### 2.1.1 The Smile

It is observed in the market that the implied volatility of OTM option strike prices may trade substantially above that of the ATM options. This feature is referred to as 'implied volatility smile' because of the smile-like structure of the graph of implied volatility against strike prices. Traders utilize this pricing technique to correct for fat tails, the observed tendency of unlikely events happening more frequently than Black-Scholes option pricing would predict. The smile, see figure 2.4, is common in currency option markets.

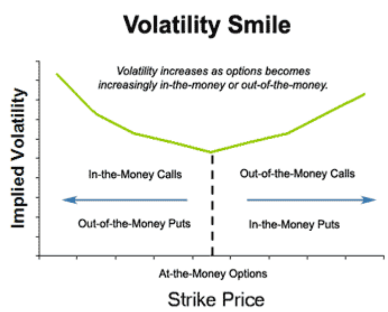


Figure 2.4. The volatility smile commonly observed in currency markets. *Source: The Options & Futures Guide: Volatility Smiles & Smirks Explained.*

[3] Implied volatility is expressed as a function of asset price $X$, strike $K$ and maturity $T$.
[4] Historical volatility is obtained from historical market data over a particular period of time.

### 2.1.2 The Skew

There is a tendency of the implied volatility for OTM put or ITM call options struck below the current ATM option price to trade at different levels as compared to similar OTM call or ITM put options struck above the same ATM price. This feature is referred to as 'implied volatility skew'. It is common in equity markets and it explains issues of supply and demand observed, when a trend develops in the underlying exchange rate that favours the direction of the strike prices with the higher implied volatility levels. It is a result of equity portfolio risk managers purchasing OTM puts to protect their equity holdings and selling off covered OTM calls against their equity positions to cap their profits. This hedging practice results in a supply and demand effect that raises the implied volatility of the OTM put over that seen for OTM calls. An example of 'implied volatility skew' is given in figure 2.3.

### 2.1.3 The Surface

The term structure of implied volatility is a common fact. A plot of implied volatility against a set of strikes and their corresponding maturities produces the surface, figure 2.5.

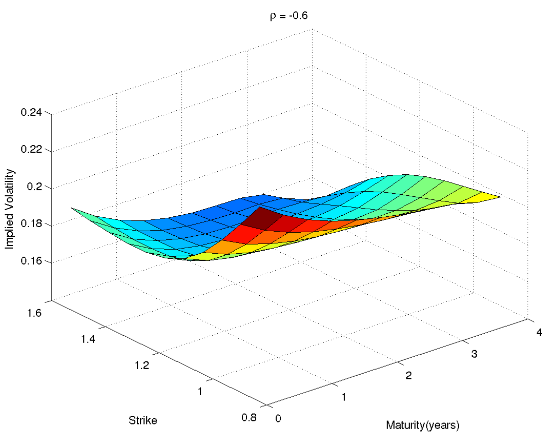


Figure 2.5. A simulation of the implied volatility surface from Heston model with parameters: $\rho = -0.6$, $\alpha = 1.0$, $m = 0.04$, $\beta = 0.3$, $\mathbf{v}_0 = 0.01$.

The following sections explain the different volatility models that have been used in pursuit of capturing the term structure of implied volatility that Black-Scholes model fails to address.

## 2.2 Local Volatility Models

Constant volatility models fail to explain both the leverage effect and the smile[5]. To capture these features, local volatility models consider volatility as a function of both time $t$ and $X_t$.

### 2.2.1 Time-dependent Volatility Models

Time-dependent volatility models are a special case of local volatility where the parameter $\sigma$ varies with time. In a deterministic time-dependent volatility, $\sigma(t)$, the stock price satisfies the stochastic differential equation[6]

$$dS_t = rS_t\,dt + \sigma(t)S_t\,dW_t^*.$$

Through logarithm transformation, $X_t = \log S_t$, and using Itô's formula gives

$$X_T = X_t \exp\left\{ r[T-t] - \frac{1}{2}\int_t^T \sigma^2(s)ds + \int_t^T \sigma(s)dW_s^* \right\}.$$

Which implies

$$\log[X_T/X_t] \sim \mathcal{N}\left( [r - \frac{1}{2}\overline{\sigma^2}][T-t], \overline{\sigma^2}[T-t] \right),$$

where

$$\overline{\sigma^2} := \frac{1}{[T-t]}\int_t^T \sigma^2(s)\,ds.$$

Computing the following expectation for a European call option with a payoff function $h(X_T)$ under a risk-neutral measure $P^*$:

$$C(t,x) = \mathbb{E}^*\left\{ e^{-r[T-t]}h(X_T)|\mathcal{F}_t \right\}, \tag{2.2}$$

gives Black-Scholes European call option price with volatility level $\sqrt{\overline{\sigma^2}}$. Time-dependent volatility models account for the observed *term structure of implied volatility* but not the *smile*. To obtain the smile, volatility has to depend on $X_t$ as well or, modelled as a process on its own. The following subsection exploits this idea.

[5]The term "smile" shall be used in general to refer also to "skews".

[6]The *Asterisk*-$*$ shall be used throughout the document to emphasize modeling under a risk-neutral equivalent measure $\mathbb{P}^*$.

### 2.2.2 Dupire Equation

Generally, in local volatility models, the dynamics of the stock price returns with dividends is given as

$$dX_t = [r - D]X_t\,dt + \sigma(t, X_t)X_t\,dW_t\,. \tag{2.3}$$

Note that there is only one source of randomness generated by a tradable asset, which makes the market complete. Completeness is very important because it guarantees unique prices. It is documented, [25] and [28] that local volatility can be extracted from prices of traded call options and local volatility surfaces from the implied volatility surface. Based on [46], [74] and [105], a summary of Dupire's local volatility model is discussed here. Dupire [28] showed that, given a distribution of terminal stock prices $X_T$, conditioned by the current stock price, $x_0$ for a fixed maturity time $T$, there exists a unique risk-neutral diffusion process that generates this distribution with dynamics described in (2.3).

Let $\phi(T, x)$ be the risk-neutral probability density function[7] of the underlying asset price at maturity, from the no-arbitrage arguments the price of a European call option is given as

$$\begin{aligned} C &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left\{[X_T - K]^+|\mathcal{F}_0\right\}. \\ &= e^{-rT}\int_K^\infty [x - K]\phi(T, x)\,dx\,. \end{aligned} \tag{2.4}$$

To obtain the formula for local volatility, one has to differentiate equation (2.4) with respect to $K$ and $T$:

$$\frac{\partial C}{\partial K} = -e^{-rT}\int_K^\infty \phi(T, x)dx. \tag{2.5}$$

The integral gives the cumulative density function. Consequently, a second derivative with respect to $K$ leads to the risk-neutral probability density function

$$\frac{\partial^2 C}{\partial K^2}e^{rT} = \phi(T, K). \tag{2.6}$$

Intuitively speaking, (2.6) suggests that the risk-neutral probability density $\phi$ can be extracted from option data. The idea is that $\phi$ gives information about the current view of the future outcome of the stock price. Since $\phi$ is a density function of time and space, it satisfies the forward Kolmogorov (Fokker-Planck) equation, see [107],

$$\frac{\partial}{\partial t}\phi(t, x) + [r - D]\frac{\partial}{\partial x}\left[x\phi(t, x)\right] - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\sigma^2(t, x)x^2\phi(t, x)\right] = 0. \tag{2.7}$$

[7] Note that $\phi(T, x)$ is actually $\phi(T, x; t, x_0)$, the transitional density function from $(t, x_0)$ to $(T, x)$ where $(t, x_0)$ are known constants.

Differentiating equation (2.4) with respect to time $T$, yields

$$\frac{\partial C}{\partial T} = -rC + e^{-rT}\int_K^{\infty}[x-K]\frac{\partial}{\partial T}\phi(T,x)\,dx\,.$$

Using the general equation (2.7) leads to

$$\frac{\partial C}{\partial T} = -rC + e^{-rT}\int_K^{\infty}[x-K]\left[\frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2 x^2\phi] - [r-D]\frac{\partial}{\partial x}[x\phi]\right]dx\,.$$

Compute the integral using *integration by parts*:

$$\begin{aligned}\frac{\partial C}{\partial T} + rC =& \frac{1}{2}e^{-rT}\left[[x-K]\frac{\partial}{\partial x}[\sigma^2 x^2\phi]|_{x=K}^{x=\infty} - \int_K^{\infty}\frac{\partial}{\partial x}[\sigma^2 x^2\phi]dx\right] \\ & - e^{-rT}[r-D]\left[[x-K]x\phi|_{x=K}^{x=\infty} - \int_K^{\infty}x\phi dx\right].\end{aligned}$$

Note that $\sigma$ and $\phi$ are functions[8] of $x$ and $T$. Thus, the above equation reduces to

$$\begin{aligned}\frac{\partial C}{\partial T} &= -\frac{1}{2}e^{-rT}\sigma^2(T,x)x^2\phi(T,x)|_{x=K}^{x=\infty} - rC + [r-D]e^{-rT}\int_K^{\infty}x\phi(T,x)dx. \\ &= \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\phi(T,K) - rC + [r-D]\left[C + Ke^{-rT}\int_K^{\infty}\phi(T,x)dx\right].\end{aligned}$$

Substituting equations (2.5) and (2.6) gives

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 C}{\partial K^2} - [r-D]K\frac{\partial C}{\partial K} - DC,$$

from which the local volatility is deduced as

$$\sigma(T,K) = \sqrt{\frac{\frac{\partial C}{\partial T} + [r-D]K\frac{\partial C}{\partial K} + DC}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}}. \tag{2.8}$$

For no dividends,

$$\sigma(T,K) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K}}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}}. \tag{2.9}$$

Equation (2.8) is referred to as Dupire equation, where volatility is a deterministic function[9]. To compute local volatility, partial derivatives of the option price $C$ with respect to $K$ and $T$ are required. This necessitates the need for a continuous set of options data for all $K$ and $T$. Common examples of local volatility models include Constant Elasticity of Variance (CEV), [11] and the Sigma-Alpha-Beta-Rho (SABR) model, [56].

[8]The arguments of these functions are relaxed for simplicity.

[9]Since values of call options with different strikes and times to maturity can be observed in the market at any point in time.

### 2.2.3 Local Volatility as Conditional Expectation

This section discusses a different approach for deriving local volatility without using the forward Kolmogorov equation. An interesting property of local volatility is demonstrated.

The no-arbitrage call option pricing formula (2.4) can be rewritten as

$$C = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left\{[X_T - K]\mathbb{I}_{\{X_T>K\}}|\mathcal{F}_0\right\}. \tag{2.10}$$

where $\mathbb{I}$ denotes the indicator function with properties:

$$\begin{aligned}
\mathbb{I}_{\{x>K\}} &= \begin{cases} 1 & \text{if} \quad x > K, \\ 0 & \text{if} \quad x \leq K. \end{cases} \\
\frac{\partial}{\partial x}\mathbb{I}_{\{x>K\}} &= \delta(x-K). \\
\frac{\partial}{\partial K}\mathbb{I}_{\{x>K\}} &= \frac{\partial}{\partial K}[1-\mathbb{I}_{\{K\geq x\}}] = -\delta(x-K).
\end{aligned}$$

where $\delta(\cdot)$ denotes the Dirac-delta function.

Assuming normal integrability and interchange of derivative and expectation operators are justified, then

$$\frac{\partial C}{\partial K} = \frac{\partial}{\partial K}\mathbb{E}^{\mathbb{Q}}\left\{e^{-rT}[X_T-K]\mathbb{I}_{\{X_T>K\}}|\mathcal{F}_0\right\} = -\mathbb{E}^{\mathbb{Q}}\left\{e^{-rT}\mathbb{I}_{\{X_T>K\}}|\mathcal{F}_0\right\}. \tag{2.11}$$

$$\frac{\partial^2 C}{\partial K^2} = -\mathbb{E}^{\mathbb{Q}}\left\{e^{-rT}\frac{\partial}{\partial K}\mathbb{I}_{\{X_T>K\}}|\mathcal{F}_0\right\} = \mathbb{E}^{\mathbb{Q}}\left\{e^{-rT}\delta(X_T-K)|\mathcal{F}_0\right\}. \tag{2.12}$$

With reference to (2.6), observe that the probability density function for the stock price at maturity is the expected value of the Dirac-delta function

$$\phi(T,K) = \mathbb{E}^{\mathbb{Q}}\left\{\delta(X_T-K)|\mathcal{F}_0\right\}. \tag{2.13}$$

Note that $C = C(T, X_T)$ thus, applying Itô's formula to equation (2.10) leads to

$$\begin{aligned}
dC = \mathbb{E}^{\mathbb{Q}}\Bigg\{&\frac{\partial}{\partial T}[e^{-rT}[x-K]\mathbb{I}_{\{x>K\}}]dT + e^{-rT}\frac{\partial}{\partial x}\left[[x-K]\mathbb{I}_{\{x>K\}}\right]dX_T \\
&+\frac{1}{2}e^{-rT}\frac{\partial^2}{\partial x^2}[[x-K]\mathbb{I}_{\{x>K\}}]d\langle X\rangle_T|\mathcal{F}_0\Bigg\}.
\end{aligned}$$

Substitute the following identities in the above derivative of the call price:

$$\frac{\partial}{\partial T}[e^{-rT}] = -re^{-rT},$$
$$\frac{\partial}{\partial x}\left[[x-K]\mathbb{I}_{\{x>K\}}\right] = \mathbb{I}_{\{x>K\}} + [x-K]\delta(x-K),$$
$$\frac{\partial^2}{\partial x^2}\left[[x-K]\mathbb{I}_{\{x>K\}}\right] = \frac{\partial}{\partial x}\mathbb{I}_{\{x>K\}} = \delta(x-K),$$

to obtain

$$\begin{aligned}dC &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left\{-r[x-K]\mathbb{I}_{\{x>K\}}dT + x\mathbb{I}_{\{x>K\}}[[r-D]dT + \sigma(T,x)dW_T]\right.\\ &\quad \left.+\frac{1}{2}\delta(x-K)x^2\sigma^2(T,x)dT|\mathcal{F}_0\right\}.\\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left\{rK\mathbb{I}_{\{x>K\}} - Dx\mathbb{I}_{\{x>K\}} + \frac{1}{2}\delta(x-K)K^2\sigma^2(T,x)|\mathcal{F}_0\right\}dT.\end{aligned}$$

from which

$$\begin{aligned}\frac{\partial C}{\partial T} &= re^{-rT}K\mathbb{E}^{\mathbb{Q}}\left\{\mathbb{I}_{\{x>K\}}|\mathcal{F}_0\right\} - D[C + e^{-rT}K\mathbb{E}^{\mathbb{Q}}\left\{\mathbb{I}_{\{x>K\}}|\mathcal{F}_0\right\}]\\ &\quad + \frac{1}{2}e^{-rT}K^2\mathbb{E}\left\{\delta(x-K)\sigma^2(T,x)|\mathcal{F}_0\right\}.\\ &= [r-D]e^{-rT}K\mathbb{E}^{\mathbb{Q}}\left\{\mathbb{I}_{\{x>K\}}|\mathcal{F}_0\right\} - DC\\ &\quad + \frac{1}{2}e^{-rT}K^2\mathbb{E}^{\mathbb{Q}}\left\{\delta(x-K)\sigma^2(T,x)|\mathcal{F}_0\right\}.\end{aligned}$$

The expectation of the last term can be expressed as

$$\mathbb{E}^{\mathbb{Q}}\left\{\delta(x-K)\sigma^2(T,x)|\mathcal{F}_0\right\} = \mathbb{E}^{\mathbb{Q}}\left\{\sigma^2(T,x)|x=K|\mathcal{F}_0\right\}\mathbb{E}^{\mathbb{Q}}\left\{\delta(x-K)|\mathcal{F}_0\right\}.$$

Applying (2.11) and (2.12) gives

$$\frac{\partial C}{\partial T} = -[r-D]\frac{\partial C}{\partial K} - DC + \frac{1}{2}K^2\mathbb{E}^{\mathbb{Q}}\left\{\sigma^2(T,x)|x=K|\mathcal{F}_0\right\}\frac{\partial^2 C}{\partial K^2},$$

from which local volatility is deduced in terms of conditional expectation

$$\mathbb{E}^{\mathbb{Q}}\left\{\sigma^2(T,x)|x=K|\mathcal{F}_0\right\} = \frac{\frac{\partial C}{\partial T} + [r-D]\frac{\partial C}{\partial K} + DC}{\frac{K^2}{2}\frac{\partial^2 C}{\partial K^2}}. \tag{2.14}$$

Equations (2.8) and (2.14) show that local volatility can be observed as the expected volatility at maturity given that, at maturity the stock price is equal to the strike price. Research shows that this result is analogous to interest rates. The local volatility surface is comparable to the yield curve[10]. It is the expectation of future instantaneous volatilities (future spot

[10] In the interest rates market, the long-term rates are given as average values of the expected future short-term rates, see [98].

rates). It is not guaranteed that this expectation will be realised. However, it is reasonable in current times to consider it by trading different financial instruments. For instance, in interest rates market one would consider buying and selling bonds with different maturities. Similarly, it would mean buying and selling options with different strikes and maturities, [26]. With reference to [65], the implied volatility is the constant value for the volatility which is consistent with option prices in the market, just as the yield is the constant value for the interest rate consistent with bond prices in the market.

Compared to the Black-Scholes model, local volatility models are seen as an improvement in financial market modelling. They account for empirical observations and theoretical arguments on volatility. They can be calibrated to perfectly fit the observed surface of implied volatilities [29]. There is no additional or untradable source of randomness is introduced in the model which makes the market complete. Thus, theoretically, perfect hedging of any contingent claim is possible. However, they also have weakness, see for instance [28]. Option maturities correspond to the end of a particular fixed period which means the number of different maturities is always limited, the same applies to the strikes. Therefore, extracting the local volatility surface from the option price given as a function of strike and maturity, is not a well-posed problem[11].

## 2.3 Stochastic Volatility

Stochastic volatility models assume realistic dynamics for the underlying asset where its volatility is modelled as a stochastic process[12]. They explain in a self-consistent way why options with different strikes and expirations have different implied volatilities. Stochastic volatility models are characterized by more than one source of risk which may or may not be correlated. At least one of the sources is not observable and thus, not tradable, which makes the market incomplete, see examples in [10], [58], [59], [96] and [106].

Volatility is not directly observed from the market but it can be estimated from stock price returns[13]. In fact, the size of fluctuations in returns is volatility. Figure 2.6 shows daily returns on S&P 500 stock index for the year 2010. Notice the high volatility during the months of May and June.

[11] This could lead to an unstable and not unique, solution.

[12] Stochastic volatility models can be seen as continuous time versions of ARCH-type models introduced by R. Engle, a 2003-noble prize winner with C. Granger, see [43] pg. 62.

[13] This can be achieved through the Maximum Likelihood Estimator for instance, [99].

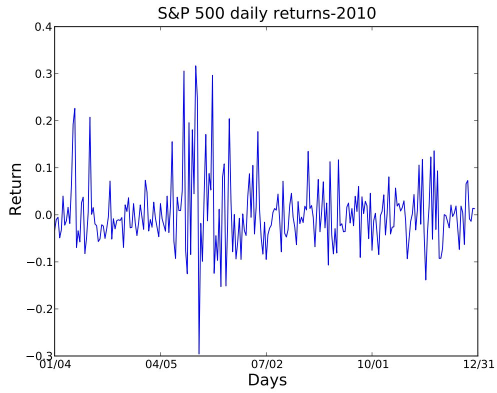


Figure 2.6. S&P 500 2010 Returns

Stochastic volatility thickens tails of returns distributions with respect to normal distribution. This enables modelling of more extreme stock price movements. The correlation effect is captured through a constant parameter $\rho \in [-1, 1]$, the correlation coefficient[14].

### 2.3.1 Generalized Garman Equation

The purpose of this subsection is to derive a general partial differential equation for pricing stock or equity derivatives under stochastic volatility, proposed by [45]. It is interesting to mention that most of the common stochastic volatility models mentioned above are derived from this general model. For instance, under particular conditions, this generalization leads to the standard Garman's equation for Heston's model or Black-Scholes classes of equations, see [102, 103].

#### The General Model

In a stochastic volatility model, the stock price $X_t$, satisfies the stochastic differential equation

$$\begin{cases} dX_t = A(t, X_t, \mathbf{v}_t)\, dt + B(t, X_t, \mathbf{v}_t)\, dW_t^{(1)} , \\ d\mathbf{v}_t = C(t, \mathbf{v}_t)\, dt + D(t, \mathbf{v}_t)\, dW_t^{(2)} , \\ d\langle W^{(1)}, W^{(2)} \rangle_t = \rho dt , \end{cases} \tag{2.15}$$

[14]This parameter determines the heaviness of the tails. Intuitively speaking, a positive correlation implies that an increase in volatility leads to an increase in the asset price returns and a negative correlation is the converse. The latter is a common fact in equity markets and is usually referred to as the leverage effect. Positive correlation generates a fat right-tailed distribution of asset price returns whereas negative correlation produces a fat left-tailed distribution. Also, $\rho$ has an indirect impact on the shape of the implied volatility surface. Altering the skew changes the shape of the surface.

where $\rho$ is the correlation between the two standard Brownian motions, $W^{(1)}$ and $W^{(2)}$; $A, B, C$ and $D$ are functions of $X_t$ and $\mathbf{v}_t$.

Suppose $f(t, X_t, \mathbf{v}_t)$ is a twice differentiable time-dependent function of $X_t$ and $\mathbf{v}_t$ then Itô's formula gives

$$df = f_t\,dt + f_x\,dX_t + f_{\mathbf{v}}\,d\mathbf{v}_t + \frac{1}{2}f_{xx}\,d\langle X\rangle_t + \frac{1}{2}f_{\mathbf{vv}}\,d\langle \mathbf{v}\rangle_t + f_{x\mathbf{v}}\,d\langle \mathbf{v}, X\rangle_t.$$

Substituting for $dX_t$ and $d\mathbf{v}_t$ from equation (2.15) yields

$$df = \left[f_t + Af_x + Cf_{\mathbf{v}} + \frac{1}{2}B^2 f_{xx} + \frac{1}{2}D^2 f_{\mathbf{vv}} + BDf_{x\mathbf{v}}\right]dt + Bf_x\,dW_t^{(1)} + Df_{\mathbf{v}}\,dW_t^{(2)}. \tag{2.16}$$

**Derivation of Garman's PDE**

The general model (2.15) contains two sources of randomness from the Brownian motion processes. Thus, the Black-Scholes approach of hedging with only the underlying asset and a risk-less bond is not applicable. Hedging requires a portfolio $\Pi(t)$ of $a$ shares, $c$ by weight of a derivative $\psi_2$ with known price $P^{(2)}(t, X_t, \mathbf{v}_t)$ and maturity $T_2$ and the target derivative $\psi_1$ with (unknown) price $P^{(1)}(t, X_t, \mathbf{v}_t)$ and maturity $T_1$ such that $t \leq T_1 < T_2$, $\psi_1$ and $\psi_2$ are assumed to have the same payoff. The value of this portfolio is given by

$$\Pi(t) = P^{(1)}(t, X_t, \mathbf{v}_t) + aX_t + cP^{(2)}(t, X_t, \mathbf{v}_t). \tag{2.17}$$

Its return is given by (relax the arguments for simplicity)

$$d\Pi(t) = dP^{(1)} + adX_t + cdP^{(2)} \tag{2.18}$$

where

$$dX_t = A\,dt + B\,dW_t^{(1)}. \tag{2.19}$$

Using equation (2.16), deduce the expressions for the derivatives $dP^{(i)}$, $i = 1, 2$,

$$\begin{cases} dP^{(i)} & = u_i dt + v_i dW_t^{(1)} + w_i dW_t^{(2)} \\ w_i & = DP_{\mathbf{v}}^{(i)} \\ v_i & = BP_x^{(i)} \\ u_i & = P_t^{(i)} + AP_x^{(i)} + CP_{\mathbf{v}}^{(i)} + \frac{1}{2}B^2P_{xx}^{(i)} + \frac{1}{2}D^2P_{\mathbf{vv}}^{(i)} + \rho BDP_{x\mathbf{v}}^{(i)}. \end{cases} \tag{2.20}$$

Substituting (2.20) in (2.18) yields

$$d\Pi(t) = \left[u_1 dt + v_1 dW_t^{(1)} + w_1 dW_t^{(2)}\right] + a\left[Adt + BdW_t^{(1)}\right] \\ + c\left[u_2 dt + v_2 dW_t^{(1)} dt + w_2 dW_t^{(2)}\right]. \tag{2.21}$$

After rearranging,

$$d\Pi(t) = [u_1 + aA + u_2 c]\, dt + [v_1 + aB + v_2 c]\, dW_t^{(1)} + [w_1 + w_2 c]\, dW_t^{(2)}.$$

Recall that the aim is to hedge away the collective risk resulting from the two Brownian motions, therefore it suffices to set their coefficients to zero,

$$v_1 + aB + v_2 c = 0 \quad \text{and} \quad w_1 + w_2 c = 0. \tag{2.22}$$

This leads to a risk-free return on the portfolio

$$d\Pi(t) = [u_1 + aA + u_2 c]\, dt. \tag{2.23}$$

To eliminate any arbitrage opportunities, this return must be equal to the risk-free rate of return

$$d\Pi(t) = r\Pi dt. \tag{2.24}$$

Consequently,

$$r\Pi(t) = u_1 + aA + u_2 c. \tag{2.25}$$

From equation (2.22), deduce

$$c = -w_1/w_2 \quad \text{and} \quad a = [-w_2 v_1 + w_1 v_2]/[w_2 B]. \tag{2.26}$$

Thus, substituting for $a$ and $c$ in equation (2.25) yields

$$r\Pi(t) = u_1 + u_2[-w_1/w_2] + [-w_2 v_1 + w_1 v_2]A/[w_2 B]. \tag{2.27}$$

Substituting for $\Pi(t)$ from equation (2.17) yields

$$r\left[P^{(1)} + [-w_2 v_1 + w_1 v_2]X_t/[w_2 B] + P^{(2)}[-w_1/w_2]\right] = \\ u_1 + u_2[-w_1/w_2] + [-w_2 v_1 + w_1 v_2]A/[w_2 B]. \tag{2.28}$$

Multiplying throughout by $w_2$ generates

$$rP^{(1)}w_2 - w_2v_1[rX_t/B] + w_1v_2[rX_t/B] - rP^{(2)}w_1 =$$
$$u_1w_2 - u_2w_1 - w_2v_1[A/B] + w_1v_2[A/B].$$

Finally, multiplying throughout by $w_1w_2$ and rearranging leads to

$$rP^{(1)}/w_1 - [v_1/w_1][rX_t/B] - [u_1/w_1] + [v_1/w_1][A/B]$$
$$= rP^{(2)}/w_2 - [v_2/w_2][rX_t/B] - [u_2/w_2] + [v_2/w_2][A/B].$$

Note that the *l.h.s* contains terms that depend only on $T_1$ and those on the *r.h.s* only on $T_2$. Thus, either side of this equation must be equal to a function say $\wedge(t, x, \mathbf{v})$, independent of maturity date. Therefore,

$$rP/w - [v/w][rX_t/B] - [u/w] + [v/w][A/B] = \wedge. \tag{2.29}$$

By substituting $w = DP_{\mathbf{v}}$, $v = BP_x$ and

$$u = P_t + AP_x + CP_{\mathbf{v}} + \frac{1}{2}B^2P_{xx} + \frac{1}{2}D^2P_{\mathbf{vv}} + \frac{1}{2} + \rho BDP_{x\mathbf{v}}$$

in equation (2.29) gives

$$\begin{cases} P_t + \frac{1}{2}B^2P_{xx} + r[xP_x - P] + [C - D\wedge]P_{\mathbf{v}} + \frac{1}{2}D^2P_{\mathbf{vv}} + \rho BDP_{x\mathbf{v}} = 0 \\ P(T, x, \mathbf{v}) = h(x). \end{cases} \tag{2.30}$$

Equation (2.30) is a boundary-value problem known as the generalized Garman equation. Under certain conditions, the function $\wedge$, known as the *risk premium*, can be expressed as

$$\wedge(t, x, \mathbf{v}) = \left[\rho\frac{[A - r]}{B} + \gamma(t, x, \mathbf{v})\sqrt{1 - \rho^2}\right], \tag{2.31}$$

where $\gamma(t, x, \mathbf{v})$ denotes the market price of volatility.

Chapter 3 will focus on mean-reverting stochastic volatility models, using Garman's general framework to deduce the corresponding pricing PDE easy to solve using perturbation techniques. The next chapter introduces the notion of mean-reverting volatility-driver processes. A common example is the Ornstein-Uhlenbeck (OU) process, see [111]. The motivation is that mean-reversion is an observed characteristic of volatility.

# Chapter 3

# Mean Reverting Stochastic Volatility Processes

Mean reversion refers to the characteristic time it takes a diffusion process to get back to the mean level of its invariant or long-run distribution. In derivative pricing, mean reversion may surface as a pull-back in the drift of the volatility process or in the drift of an underlying process of the volatility function. A stochastic price series is considered to be reverting towards its long-run mean $m$ if the price shows a downward trend when greater than $m$ and an upward trend when less than $m$.

The concept of mean-reversion is a realistic trend in investment markets, see [31]. It cuts across market commodities such as oil, [77] and [91] to foreign exchange markets, [49]. It also finds useful application in the study of profitability of market making[1] strategies [19].

Mean reverting stochastic processes are studied as a major class of pricing models in contrast to stochastic processes with directional drift, or with no drift like Brownian motion. The commonest and widely studied mean reverting stochastic processes are, the *Ornstein-Uhlenbeck*, [111] and the *Cox-Ingersoll-Ross* [22] processes. The former is pivotal to this work.

## 3.1 The Ornstein-Uhlenbeck Process

**Definition 3.1.1.** *The Ornstein-Uhlenbeck process is a diffusion process that satisfies the following stochastic differential equation*

$$dY_t = \alpha[m - Y_t]\,dt + \beta\,dW_t, \tag{3.1}$$

[1]Market making refers broadly to trading strategies that seek to profit by providing liquidity to other traders, while avoiding accumulating a large net position in stock.

*where $W$ is standard Brownian motion. The constant parameters are defined as:*

- *$\alpha > 0$ is the rate of mean reversion.*
- *$m$ is the long-run mean of the process.*
- *$\beta > 0$ is the volatility or average magnitude, per square root time, of the random fluctuations that is modelled as Brownian motion.*

### 3.1.1 Distribution of the OU Process

The OU process $(Y_t)_{t\geq 0}$ is a Gaussian-Markov process as verified in the following. Using variation of constants, the solution to equation (3.1) is given by

$$Y_t = m + [y - m]e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)}\, dW_s\,, \tag{3.2}$$

where $Y_0 = y$ is the initial state process. The expectation and variance of $Y_t$ are given as

$$\mathbb{E}\left\{Y_t\right\} = m + [y-m]e^{-\alpha t} \quad \text{and} \quad \mathrm{Var}\left\{Y_t\right\} = \tfrac{\beta^2}{2\alpha}[1 - e^{-2\alpha t}].$$

Consequently

$$Y_t \sim \mathcal{N}\left(m + [y-m]e^{-\alpha t}, \tfrac{\beta^2}{2\alpha}[1 - e^{-2\alpha t}]\right).$$

As $\alpha \to \infty$ or $t \to \infty$, $Y_t \sim \mathcal{N}\left(m, \beta^2/2\alpha\right)$. This implies that for high rates of mean-reversion or after a long period of time the process becomes independent of its initial state, $y$.

To verify the Markov property of the process, it is enough to show that its future expectation conditioned on its current value is independent of its past, [90]. From equation (3.1) deduce

$$Y_{t+1} = Y_t + dY_t = Y_t + \alpha[m - Y_t]dt + \beta dW_t.$$

The expectation conditioned on all past information up to the nearest value is given as

$$\begin{aligned}\mathbb{E}_{t+1}\left\{Y_{t+1}|Y_t, Y_{t-1}, \cdots, Y_0\right\} = \mathbb{E}_{t+1}&\left\{Y_t|Y_t, Y_{t-1}, \cdots, Y_0\right\} \\ &+ \mathbb{E}_{t+1}\left\{\alpha[m - Y_t]dt|Y_t, Y_{t-1}, \cdots, Y_0\right\} \\ &+ \mathbb{E}_{t+1}\left\{\beta dW_t|Y_t, Y_{t-1}, \cdots, Y_0\right\}.\end{aligned}$$

Thus,

$$\mathbb{E}_{t+1}\left\{Y_{t+1}|Y_t, Y_{t-1}, \cdots, Y_0\right\} = Y_t + \alpha[m - Y_t]dt + \beta\mathbb{E}_{t+1}\left\{dW_t\right\}.$$

This follows from the fact that $\mathbb{E}\left\{X_1|X_1, X_2, \cdots\right\} = X_1$ and that the increment $dW_t$ is independent of the past such that $\mathbb{E}_{t+1}\left\{dW_t|Y_t, Y_{t-1}, \cdots, Y_0\right\} = \mathbb{E}_{t+1}\left\{dW_t\right\}$. The OU process is ergodic and exhibits a unique invariant distribution, [63]. This follows in the next subsection.

### 3.1.2 Invariant Distribution of the OU process

A distribution of a diffusion process $(Y_t)_{t\geq 0}$ is said to be stationary or invariant if it is the same for all $t \geq 0$, [68]. This is a significant property with diverse applications in the financial industry, see for instance the work by [47] in the study of oil and natural gas commodity prices in the energy market and [70] in pricing electricity derivatives.

**Proposition 3.1.2.** *Given a Markov process $(Y_t)_{t\geq 0}$ whose semi-group is a family $(P_t)_{t\geq 0}$ then for any bounded measurable function $f$, $P_t f = \mathbb{E}\{f(Y_t)\}$.*

**Definition 3.1.3.** *(i) A measure $\mu$ is said to be invariant for the process $(Y_t)_{t\geq 0}$ if and only if*

$$\int \mu(dy) P_t f(y) = \int \mu(dy) f(y),$$

*for any bounded function $f$. (ii) $\mu$ is invariant for $(Y_t)_{t\geq 0}$ if and only if $\mu P_t = \mu$. Equivalently, the law of $(Y_{t+\tau})_{\tau\geq 0}$ is independent of $t$, starting at $t = 0$ with measure $\mu$.*

**Definition 3.1.4.** *Let $(Y_t)_{t\geq 0}$ be a Markov process then this process is said to exhibit mean reversion if and only if it admits a finite invariant measure.*

**Proposition 3.1.5.** *The existence of an invariant measure implies that the process $(Y_t)_{t\geq 0}$ is stationary. If this process admits a limit law independent of its initial state, then this limit law is an invariant measure.*

**Proposition 3.1.6.** *The Ornstein-Uhlenbeck process admits a finite invariant measure, and this measure is Gaussian.*

The proofs of Propositions 3.1.2 and 3.1.5 are presented in [47].

The infinitesimal generator of the OU process is given by, see Appendix A

$$\mathcal{L} = \alpha[m - y]\frac{\partial}{\partial y} \cdot + \frac{1}{2}\beta^2 \frac{\partial^2}{\partial y^2} \cdot . \tag{3.3}$$

Taking $Y_0$ as the initial value of the OU process $(Y_t)_{t\geq 0}$ under invariant distribution, $Y_0 \overset{\mathrm{d}}{=} Y_t$ for all $t > 0$ such that

$$\frac{d}{dt}\mathbb{E}\left\{g(Y_t)\right\}|_{t=0} = 0 \quad \text{or} \quad \mathbb{E}\{\mathcal{L}g(Y_0)\} = 0, \tag{3.4}$$

where $g$ is a smooth bounded function. This follows from equation (A.10) in Appendix A.3. In other words, if $\phi(y)$ is the density function of the invariant distribution of $(Y_t)_{t\geq 0}$, then the invariant distribution requires

$$\int_{-\infty}^{\infty} \phi(y)\mathcal{L}g(y)\, dy \ = 0, \tag{3.5}$$

where $\phi(y)$ and $\frac{\partial}{\partial y}\phi(y)$ tend to zero as $y$ tends to positive or negative infinity.

Using integration by parts and the product rule techniques, equation (3.5) can be written as

$$\int_{-\infty}^{\infty} g(y)\mathcal{L}^*\phi(y)\, dy \; = 0. \tag{3.6}$$

where the operator $\mathcal{L}^*$, known as the *adjoint* of $\mathcal{L}$ is defined as

$$\mathcal{L}^* := -\alpha\frac{\partial}{\partial y}[[m-y]\cdot] + \frac{\beta^2}{2}\frac{\partial^2}{\partial y^2}\cdot . \tag{3.7}$$

Thus, from equation (3.6), it is deduced that

$$-\alpha\frac{\partial}{\partial y}[[m-y]\phi(y)] + \frac{\beta^2}{2}\frac{\partial^2}{\partial y^2}\phi(y) = 0, \tag{3.8}$$

since $g$ is a smooth and bounded function. Note that equation (3.8) is the stationary version of Kolmogorov forward equation for the process $(Y_t)_{t\geq 0}$ with density function $\phi$, see [107]. Integrating this equation once leads to

$$\alpha[y-m]\phi + \frac{\beta^2}{2}\frac{\partial\phi}{\partial y} = 0. \tag{3.9}$$

Further integration and setting $\beta^2/2\alpha = \nu^2$ yields

$$\log\left\{\frac{\phi}{\phi_0}\right\} = -\frac{1}{2\nu^2}[y^2 - 2my]. \tag{3.10}$$

Completing squares on the *r.h.s* and rearranging gives

$$\phi(y) = \phi_0(y)e^{m^2}\cdot\exp\left\{-\frac{[y-m]^2}{2\nu^2}\right\}. \tag{3.11}$$

Comparing equation (3.11) with a normally distributed random variable $\eta$, mean $m$ and variance $\nu^2$ such that

$$\frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{\infty}\exp\left\{-\frac{[\eta-m]^2}{2\nu^2}\right\} d\eta \; = 1, \tag{3.12}$$

implies that

$$\phi_0 e^{m^2} = \frac{1}{\sqrt{2\pi\nu^2}}. \tag{3.13}$$

Therefore, the density function of the invariant distribution of the OU process is

$$\phi(y) = \frac{1}{\sqrt{2\pi\nu^2}}\exp\left\{-\frac{[y-m]^2}{2\nu^2}\right\}, \tag{3.14}$$

which is a normal distribution with mean $m$ and variance $\nu^2$. Notice that this is in agreement with the long-run distribution of the process as discussed earlier.

### 3.1.3 Autocorrelation

There is a significant relationship between the rate of mean reversion and correlation of the OU process under the invariant distribution.

Define the rate of mean reversion $\alpha := 1/\varepsilon$, where $0 < \varepsilon \ll 1$ is a time scale of the OU process. Under invariant distribution, the autocorrelation of $(Y_t)_{t\geq 0}$ follows from computing the expectation $\mathbb{E}\left\{[Y_\tau - m][Y_s - m]\right\}$ where $m$ is the long-run mean of the process. Equation (3.2) yields

$$\mathbb{E}\left\{[Y_\tau - m][Y_s - m]\right\} = \mathbb{E}\left\{Y_\tau Y_s\right\} - m^2, \tag{3.15}$$

where

$$\mathbb{E}\left\{Y_\tau Y_s\right\} = m^2 + \mathbb{E}\left\{\beta \int_0^\tau e^{-\alpha[\tau-u]}\, dW_u \cdot \beta \int_0^s e^{-\alpha[s-u]}\, dW_u \right\}.$$

Applying Itô's isometry gives

$$\mathbb{E}\left\{Y_\tau Y_s\right\} = m^2 + \beta^2 \int_0^{\tau\wedge s} e^{-\alpha[\tau+s-2u]}\, du = m^2 + \frac{\beta^2}{2\alpha} \exp\left\{-\alpha|\tau - s|\right\},$$

where $\tau \wedge s := \min\{s, \tau\}$. Thus, equation (3.15) becomes

$$\mathbb{E}\left\{[Y_\tau - m][Y_s - m]\right\} = \nu^2 \exp\left\{-\frac{|\tau - s|}{\varepsilon}\right\}, \tag{3.16}$$

where $\nu^2$ is the variance of the stationary process $(Y_t)_{t\geq 0}$. Observe that on the time scale $0 < \varepsilon \ll 1$, the process decorrelates exponentially fast.

**Remark 3.1.7.** *The variance $\nu^2$ of the invariant distribution represents the size of the fluctuations of the process.*

**Remark 3.1.8.** *The process $(Y_t)_{t\geq 0}$ is normally distributed and $Y_t|_{t\to\infty} \stackrel{d}{=} Y_t|_{\alpha\to\infty}$.*

**Remark 3.1.9.** *The time scale $\varepsilon$ can be observed as the autocorrelation time. If it is small then any two values of the process $(Y_t)_{t\geq 0}$ observed at different times, become less correlated, however close the event times could be. Conversely, for a big $\varepsilon$-value the two process values will be highly correlated.*

## 3.2 Volatility-driver Processes

In stochastic volatility models, the variance of returns on the underlying is usually modelled as a bounded continuous function of a stochastic process. The OU process discussed in the previous section is one of such processes commonly used to capture both the mean-reversion and clustering features of volatility. Other volatility-driving processes include, [6] and [76]

- Log-normal (LN): $dY_t = c_1 Y_t\, dt + c_2 Y_t\, dW_t\,,$
- Cox-Ingersoll-Ross (CIR): $dY_t = \alpha_2[m_2 - Y_t]\, dt + \beta\sqrt{Y_t}\, dW_t\,,$

the coefficients $c_1$ and $c_2$ are positive constants. Observe that the log-normal process is not mean-reverting. Some common examples of choices for volatility functions are given in [76].

Figure 3.1 shows a simulation of the stock price process under stochastic volatility driven by a Gaussian-Markovian mean-reverting diffusion process.

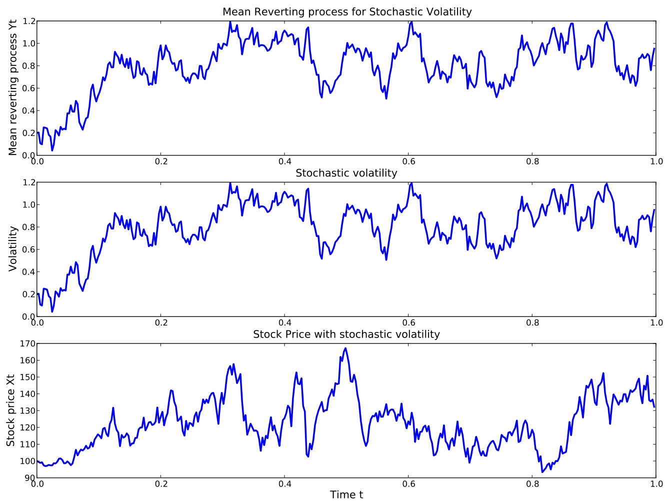


Figure 3.1. Simulated mean reverting volatility, Ornstein-Uhlenbeck process, $(Y_t)_{t\geq 0}$ and the stock price, $X_t$. $f(Y_t) = |Y_t|$, $\alpha = 1.0$, $\beta = \sqrt{2}$, long-run average volatility $\bar{\sigma} = 0.1$, the correlation between the two Brownian motions $\rho = -0.2$ and the mean growth rate of the stock is $\mu = 0.15$.

### 3.2.1 Volatility Clustering

Varying the rate of mean reversion has a significant impact on volatility. There is a tendency of volatility clustering in 'packets' of low and high values for almost similar time intervals as the rate is increased, see figure 3.2. Volatility fluctuates rapidly about its long-run mean in clusters of low and high values, a behaviour commonly known as burstiness. Increasing the rate of mean reversion increases the rate at which volatility goes back to its mean value. The existence of this behaviour in the market has been confirmed, [40] through empirical data analysis. Intuitively speaking, volatility clustering suggests that large price fluctuations are more likely to be followed by large price fluctuations and vice versa.

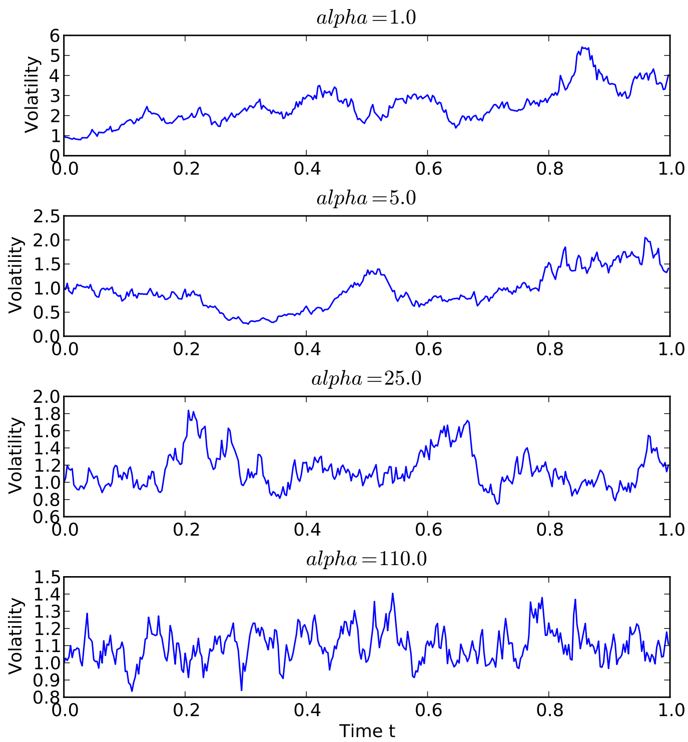


Figure 3.2. The effect of rate of mean-reversion on volatility. In the first two panels, $\alpha = 1$ and $\alpha = 5$, observe that volatility generally keeps at low values for almost 7 months and then goes up later in the year. However, as the rate of mean reversion is increased, notice that volatility fluctuates rapidly about its average value; panels: 3 and 4.

## 3.3 Convergence of Hull-White Model under Mean-Reversion

This section explains convergence of the Hull-White model to the Black-Scholes model.

### 3.3.1 Time and Statistical averages

It is clear from simulations in figure 3.2 that when the rate of mean reversion is high, the volatility process $(Y_t)_{t\geq 0}$ frequently goes back to its mean value.

**Theorem 3.3.1** (Ergodic Theorem, [40])**.** *Let $(Y_t)_{t\geq 0}$ be an ergodic process*[2] *and let $g(Y_t)$ be a square-integrable function in time, then the long-run time average of $g(Y_t)$ is close to its statistical average or its expected value with respect to the invariant distribution of $Y_t$,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t g(Y_s)\,ds \;\approx \langle g\rangle,$$

*where $\langle g\rangle := \int_{-\infty}^{\infty} g(y)\phi(y)\,dy$ is the average with respect to the invariant distribution $\phi(y)$ of the process $(Y_t)_{t\geq 0}$.*

Recall that the long-run distribution (*i.e.* $t \to \infty$) of the OU process is similar to that when the rate of mean reversion becomes large (*i.e.* $\alpha \to \infty$). Thus, under fast mean-reversion,

$$\frac{1}{t}\int_0^t g(Y_s)\,ds \;\approx \langle g\rangle.$$

Generally, for a fast mean-reverting process the mean-square time-averaged volatility is approximately constant

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T f^2(Y_s)\,ds \;\approx \langle f^2\rangle = \bar{\sigma}^2. \tag{3.17}$$

Note the difference between $\overline{\sigma^2}$ and $\bar{\sigma}^2$. The former is a random process whereas the latter is constant expected square volatility under the invariant distribution of $y$.

### 3.3.2 Hull-White Model

The Hull-White model [59] assumes the following stochastic processes $X_t$ for the asset price and its return volatility $\mathbf{v}_t = \sigma^2(t)$:

$$\begin{cases} dX_t = \mu X_t\,dt\, + \sigma(t) X_t\,dB_t^x\,, \\ d\mathbf{v}_t = k\mathbf{v}_t\,dt\, + \xi\mathbf{v}_t\,dB_t^\sigma\,, \\ \langle B^x, B^\sigma\rangle_t = 0, \end{cases} \tag{3.18}$$

where $B_t^x$ and $B_t^\sigma$ are two standard Brownian motions and parameters, $\mu, k, \sigma(t)$ and $\xi$ are independent of $X_t$.

[2] A stochastic process is said to be ergodic if its statistical properties (i.e. mean and variance) can be estimated from a single, sufficiently long sample of the process.

### 3.3.3 Convergence

By using iterated expectations[3], the price of a European call in equation (2.2) can be obtained by conditioning on the path of the volatility process,

$$C(t,x,y) = \mathbb{E}^* \left\{ \mathbb{E}^* \left\{ e^{-r(T-t)} h(X_T) | \mathcal{F}_t, \sigma(s); 0 \le t < s \le T \right\} | \mathcal{F}_t \right\}. \tag{3.19}$$

The ‘inner’ expectation is Black-Scholes call option price with mean-square time-averaged volatility. Precisely, the Hull-White model gives the price of a European call option as

$$C(t,x,y) = \mathbb{E}^* \left\{ C_{BS}(\sqrt{\overline{\sigma^2}}) | Y_t = y \right\},$$

where $\sqrt{\overline{\sigma^2}}$ is the root-mean square time-averaged volatility over the remaining trajectory of each realization. Observe from equation (3.17) that under fast mean-reversion, $\overline{\sigma^2} \approx \bar{\sigma}^2$.

## 3.4 The Heston Model

The Heston model [58] is one of the most popular models in pricing derivatives due to its robustness and tractability. Its structure is based largely on financial, economical and mathematical considerations i.e. volatility is stochastic, positive and bounded in a range.

Let $X_t$ be the price of a stock and denote by $r$, the risk-free rate of return under a pricing risk-neutral probability measure $\mathbb{P}^*$ then, Heston model takes the form

$$\begin{cases} dX_t = rX_t \, dt + \sigma_t X_t \, dW_t^{*(1)} \\ d\sigma_t = -\xi_t \sigma_t \, dt + \epsilon_t \, dW_t^{*(2)} \\ d\langle W^{*(1)}, W^{*(2)} \rangle_t = \rho dt \end{cases} \tag{3.20}$$

where $W^{(*1)}$ and $W^{*(2)}$ are standard Brownian motions. Most authors use variance $\mathbf{v}_t = \sigma_t^2$ instead of $\sigma_t$ where, according to Itô's formula

$$d\mathbf{v}_t = [\epsilon_t^2 - 2\xi_t \mathbf{v}_t] \, dt + 2\epsilon_t \sqrt{\mathbf{v}_t} \, dW_t^{*(2)}, \tag{3.21}$$

and then rewrite equations (3.20) and (3.21) as

$$\begin{cases} dX_t = rX_t \, dt + \sqrt{\mathbf{v}_t} X_t \, dW_t^{*(1)} \\ d\mathbf{v}_t = \alpha[m - \mathbf{v}_t] \, dt + \beta \sqrt{\mathbf{v}_t} \, dW_t^{*(2)} \end{cases} \tag{3.22}$$

where $\alpha$ and $\beta$ are respectively the rate of mean reversion and the volatility of variance.

[3]This is also referred to as the smoothing property of conditional expectation, see [68].

European option prices are efficiently priced through the method of characteristic functions. The solution is derived from the general Garman equation (2.30) and is desired to take the form corresponding to the Black-Scholes model,

$$P(X_t, K, \mathbf{v}_t, t, T) = X_t q_1 - K \exp\{-r[T-t]\} q_2, \tag{3.23}$$

where $q_1$ denotes the delta of the European call option and $q_2$ is the conditional risk neutral probability that the asset price will be greater than $K$ at maturity. Both $q_1$ and $q_2$ satisfy the general PDE (2.30). Given the characteristic functions[4] $\psi_1$ and $\psi_2$, the terms $q_1$ and $q_2$ are defined under the inverse Fourier transformation as, [58] and [61]

$$q_j(S, \log K, \mathbf{v}_t, t, T) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left\{\frac{\exp\{-i\phi\log K\}\,\psi_j(S, \mathbf{v}_t, t, T, \phi)}{i\phi}\right\} d\phi$$

where $j = 1, 2$ and $S = \log X_t$. The characteristic functions $\psi_1$ and $\psi_2$ assume the form

$$\psi_j(S, \mathbf{v}_t, t, T; \phi) = \exp\{C_j(\tau;\phi) + D_j(\tau;\phi)\mathbf{v}_t + i\phi S\}; \quad \tau = T - t, \tag{3.24}$$

By substituting $\psi_1$, $\psi_2$ in the general Garman equation[5] (2.30) gives the following ordinary differential equations for the unknown functions $C_j(\tau, \phi)$ and $D_j(\tau, \phi)$:

$$\frac{dC_j(\tau,\phi)}{d\tau} - \alpha m D_j(\tau;\phi) - r\phi i = 0, \tag{3.25}$$

$$\frac{dD_j(\tau,\phi)}{d\tau} - \frac{\sigma^2 D_j^2(\tau;\phi)}{2} + [b_j - \rho\sigma\phi i]D_j(\tau;\phi) - u_j\phi i + \frac{\phi^2}{2} = 0 \tag{3.26}$$

with zero initial (i.e. $t = T$) conditions

$$C_j(0;\phi) = D_j(0;\phi) = 0. \tag{3.27}$$

The solution set to the system of equations (3.25)-(3.27) is given as, see [82]

$$C_j(\tau;\phi) = r\phi i\tau + \frac{\alpha m}{\sigma^2}\left\{[b_j - \rho\sigma\phi i + d]\tau - 2\log\left[\frac{1 - ge^{d\tau}}{1-g}\right]\right\}, \tag{3.28}$$

$$D_j(\tau;\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}\right] \tag{3.29}$$

[4] The characteristic function $\psi_j$, $j = 1, 2$ is also given by: $\psi_j = \mathbb{E}^{\mathbb{P}_j}\{\exp\{i\phi S(T)\}\}$, where $\mathbb{P}_j$ is EMM corresponding to numeraire $N_j$. See Radon-Nikodym derivative $\frac{d\mathbb{P}_j}{d\mathbb{P}}$ in [61].

[5] Note that the functions $B, C$ and $D$ in equation (2.30) are defined as $B = \sqrt{\mathbf{v}_t}$, $C = \alpha[m - \mathbf{v}_t]$ and $D = \beta\sqrt{\mathbf{v}_t}$ in the case of Heston model.

where

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \quad ; \quad d = \sqrt{[\rho\sigma\phi i - b_j]^2 - \sigma^2[2u_j\phi - \phi^2]}.$$

$$u_1 = 0.5,\ \ u_2 = -0.5,\ \ a = \alpha m,\ \ b_1 = \alpha + \wedge - \rho\sigma,\ \ b_2 = \alpha + \wedge. \tag{3.30}$$

There are quite a number of excellent sources where one can read about the Heston model in details. The purpose here is to demonstrate an example under which mean reverting volatility models would produce closed form solutions. Chapter 5 will discuss an analytic technique of improving Heston pricing fromula using the Decomposition pricing approach, see [3]. The following chapter gives a detailed study on the general pricing of option derivatives under stochastic volatility framework using asymptotic expansion methods.

# Chapter 4

# Asymptotic Pricing

This chapter focuses on pricing and hedging derivatives under mean-reverting volatility. Single (fast mean-reversion)- and multi-scale volatility[1], [43] are discussed. The idea can be extended to Fractional Brownian motion, [86], and jump diffusion processes, [78]. Theorem 4.10 of [43] gives the necessary conditions for the pricing model discussed in the following.

## 4.1 Model Setup

The stock price is modelled as a Geometric Brownian motion and volatility as a positive function of a OU process.

### 4.1.1 Under Physical Measure $\mathbb{P}$

From the generalized Garman equation derived in Section 2.3.1, the model takes the form

$$\begin{cases} dX_t \; = \mu X_t \, dt \; + \sigma_t X_t \, dW_t^{(1)}, \\ \sigma_t = f(Y_t), \\ dY_t \; = \alpha[m - Y_t] \, dt \; + \beta \, dW_t^{(2)}, \\ W_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(3)}, \\ \langle W^{(1)}, W^{(2)} \rangle_t = \rho t, \end{cases} \tag{4.1}$$

[1]A case where a slow mean-reverting process is considered is discussed in [54]. A multi-scale volatility model comprising both the slow and fast mean-reverting processes driving volatility is studied in [76] in determining oil prices in the energy market, see also [41].

where $\alpha$, $m$ are respectively, rate of mean-reversion and long-run mean of $(Y_t)_{t\geq 0}$, $W^{(1)}$ and $W^{(3)}$ are independent[2] Brownian motions under $\mathbb{P}$ and $f$ is a real, positive and bounded.

### 4.1.2 Under Risk-neutral Measure $\mathbb{P}^*$

Stochastic volatility leads to an incomplete market with infinite equivalent martingale measures $\mathbb{P}^*$ for pricing. Under $\mathbb{P}^*$, discounted prices of all tradable instruments are martingales, e.g. the discounted stock price $\tilde{X}_t = e^{-r[T-t]}X_t$, where $r$ is the risk-free rate of return. To construct such a measure, one uses Girsanov Theorem, [87], where

$$W_t^{*(1)} := W_t^{(1)} + \int_0^t \frac{[\mu - r]}{f(Y_s)}\, ds\,. \tag{4.2}$$

Since $W_t^{(1)}$ and $W_t^{(3)}$ are independent, any transformation of $W_t^{(3)}$ under measure $\mathbb{P}^*$ has no effect on the discounted stock price, thus

$$W_t^{*(3)} := W_t^{(3)} + \int_0^t \gamma_s\, ds\,, \tag{4.3}$$

where the parameter $\gamma_t$ is determined by the market[3]. The change of measure[4] leads to

$$\begin{cases} dX_t \; = rX_t\, dt \; + \sigma_t X_t\, dW_t^{*(1)}\,, \\ \sigma_t = f(Y_t), \\ dY_t \; = \alpha[[m - Y_t] - \wedge(t, X_t, Y_t)]\, dt \; + \beta\, W_t^{*(2)}\,, \\ W_t^{*(2)} = \rho W_t^{*(1)} + \sqrt{1-\rho^2} W_t^{*(3)}, \\ \langle W^{*(1)}, W^{*(2)} \rangle_t = \rho t, \end{cases} \tag{4.4}$$

where $\wedge$ is defined as

$$\wedge(t, x, y) = \rho \frac{[\mu - r]}{f(y)} + \gamma(t, x, y)\sqrt{1-\rho^2}, \tag{4.5}$$

Note, $[\mu - r]/f(y)$ and $\gamma_t$ are risk premia due to the source of randomness $W^{*(1)}$ and $W^{*(2)}$.

[2]The parameter $\rho$ enables the capturing of the skew effect (asymmetry in returns distribution). If $\rho < 0$, an increase in volatility yields a decrease in stock price and vice versa. Thus, at low volatilities, holders of far in-the-money call options are most likely to exercise at maturity, unlike in high volatility markets especially when the option is too close at-the-money or out-of-money.

[3]This is what makes the market incomplete. There is no unique equivalent measure for pricing since $\gamma(t)$ depends on $\mathbb{P}^*$.

[4]Note that volatility is not affected by the choice of measure. Intuitively speaking, all traders encounter the same volatility effects irrespective of the choice of measure. In addition, there is a change in the drift term of the stock price dynamics from $\mu$ to $r$, this guarantees a risk free model.

Hence under the new measure, the price of the option is given as

$$P(t,x,y) = \mathbb{E}^{*(\gamma)}\left\{e^{-r(T-t)}h(X_T)|\mathcal{F}_t\right\}. \tag{4.6}$$

This expectation can be computed in two ways: by using Feynman-Kac fromula, see Appendix A.5 or through the replicating portfolio strategy. The following section employs the latter approach[5] which is summarised in Chapter 2, Section 2.3.1.

## 4.2 Pricing Derivatives

Black-Scholes approach of pricing through hedging with only the underlying asset is not applicable in this case where the model has two risks from the asset and volatility shocks. Both risks need to be balanced for risk-neutral pricing. The derivative price $P$, equation (4.6) must satisfy the generalized Garman equation (2.30),

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{1}{2}f^2(y)x^2\frac{\partial^2 P}{\partial x^2} + r\left[x\frac{\partial P}{\partial x} - P\right] + \rho\beta x f(y)\frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2}&\\ +[\alpha[m-y] - \beta \wedge (t,x,y)]\frac{\partial P}{\partial y} = 0,&\end{aligned} \tag{4.7}$$

with terminal condition $P(T,x,y) = h(x)$. Using Itô's formula and equation (4.7), the return on the target derivative is given as

$$\begin{aligned} dP(t,x,y) &= \left[\frac{[\mu - r]}{f(y)}\left[xf(y)\frac{\partial P}{\partial x} + \rho\beta\frac{\partial P}{\partial y}\right] + rP + \gamma\beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}\right] dt \\ &\quad + \left[xf(y)\frac{\partial P}{\partial x} + \rho\beta\frac{\partial P}{\partial y}\right] dW_t^{(1)} + \beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}\, dW_t^{(3)}. \end{aligned} \tag{4.8}$$

**Remark 4.2.1.** *Note from equation (4.7) that the pricing differential equation comprises of the Black-Scholes differential operator, $\mathcal{L}_{BS}(f(y))$ with volatility level, $f(y)$, the infinitesimal generator, $\mathcal{L}_{OU}$ of the OU process, the term due to correlation and a term due to market price of volatility.*

**Remark 4.2.2.** *Note from equation (4.8) that an infinitesimal fractional increase $\Delta\beta$ in volatility risk $\beta$, increases the infinitesimal rate of return on the option by $\gamma\Delta\beta$ plus an increase in the excess return-to-risk ratio, $[\mu - r]/f(y)$. Thus, high risk corresponds to the possibility of big returns.*

## 4.3 Asymptotic Approach

A derivation of a first-order correction to Black-Scholes solution is the aim of this section. However, it is difficult to solve the pricing partial differential equation (4.7) analytically.

[5]The former approach shall be employed under the multi-scale volatility framework later in the chapter.

Here, the course of action is to find an approximate solution in the neighbourhood of Black-Scholes price[6]. This is achieved through introducing a small parameter $\varepsilon$ in the pricing PDE. The solution is assumed to be a power series with respect to, preferably $\sqrt{\varepsilon}$. Define $\varepsilon$ as the inverse of the rate of mean-reversion[7] of $Y_t$, i.e. $\varepsilon = 1/\alpha$ and parameter$\beta = \nu\sqrt{2}/\sqrt{\epsilon}$.

Assume the market price of risk $\wedge(y)$ depends only on the current value of volatility $y$ and denote the price of the target derivative by $P^{\varepsilon}(t, x, y)$, indicating its dependence on the small parameter $\varepsilon$. Then, the perturbed pricing PDE follows in the next section.

**Remark 4.3.1.** *It is important to set $\alpha$ to a large value preferably $\alpha \gg \frac{1}{T-t}$, to enable fast mean-reversion. Consequently, too-near-to-maturity derivatives will not welcome this pricing approach for low rates since volatility will not have enough time to perform sufficient fluctuations.*

### 4.3.1 The Perturbed Pricing PDE

A perturbed version of the pricing PDE in equation (4.7) is given as

$$\begin{aligned}\frac{\partial P^{\varepsilon}}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2 P^{\varepsilon}}{\partial x^2} + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \rho x f(y) \frac{\partial^2 P^{\varepsilon}}{\partial x \partial y} + \frac{\nu^2}{\varepsilon} \frac{\partial^2 P^{\varepsilon}}{\partial y^2} \\ + r\left[x\frac{\partial P^{\varepsilon}}{\partial x} - P^{\varepsilon}\right] + \left[\frac{1}{\varepsilon}[m-y] - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \wedge (y)\right] \frac{\partial P^{\varepsilon}}{\partial y} = 0,\end{aligned} \tag{4.9}$$

for all $t < T$ with a terminal condition $P^{\varepsilon}(T, x, y) = h(x)$. Rearranging terms in orders of $1/\varepsilon$, $1/\sqrt{\varepsilon}$ and 1, and revisiting Remark 4.2.1, Equation (4.9) can be written as[8]

$$\begin{cases} \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right] P^{\varepsilon} = 0 \\ P^{\varepsilon}(T, x, y) = h(x) \end{cases}, \tag{4.10}$$

where

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + [m-y]\frac{\partial}{\partial y}, \tag{4.11}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu x f(y) \frac{\partial^2}{\partial x \partial y} - \sqrt{2}\nu \wedge (y) \frac{\partial}{\partial y}, \text{ and} \tag{4.12}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2} + r\left[x\frac{\partial}{\partial x} - \cdot\right]. \tag{4.13}$$

[6]Recall that asymptotic methods rely on a problem with known exact solution. This makes the Black-Scholes model a better choice.

[7]Recall from Section 3.1.2 that under its invariant distribution, the volatility driving process $Y_t \sim \mathcal{N}(m, \nu^2)$, where $\nu^2 = \beta^2/2\alpha$.

[8]Observe that $\mathcal{L}_2 = \mathcal{L}_{BS}(f(y))$, $\frac{1}{\varepsilon}\mathcal{L}_0 = \mathcal{L}_{OU}$ and $\mathcal{L}_1$ comprises of the mixed partial derivatives arising from the correlation between $W_t^{(1)}$ and $W_t^{(2)}$ plus the term due to the market price of volatility risk.

Equation (4.10) is a well-posed singularly perturbed boundary value problem with diverging terms and an order-1 term, $\mathcal{L}_2$, containing the partial derivative with respect to time. The next section explains the procedure followed to compute the solution $P^\varepsilon$ to this problem.

### 4.3.2 Asymptotic Expansion

This section explains a step-by-step description of the asymptotic expansion pricing method.

#### Outer Expansion

The outer expansion as discussed in Section 1.3 occurs away from the boundary[9]. Asymptotic expansion assumes that the solution to equation (4.10) is given in terms of power series of $\sqrt{\varepsilon}$ and converges as $\varepsilon$ goes to zero:

$$P^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^{\frac{n}{2}} P_n, \tag{4.14}$$

where the $P_n$'s depend on $t, x$ and $y$ for all $n$. Substituting equation (4.14) in (4.10) gives

$$\left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right]\left[P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2 + \cdots\right] = 0. \tag{4.15}$$

Collecting terms with similar order in $\varepsilon$ leads to,

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}\left[\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right] + \left[\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right] \\ &+ \sqrt{\varepsilon}\left[\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right] + \varepsilon\left[\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2\right] + \cdots = 0. \end{aligned} \tag{4.16}$$

Comparing terms on both sides of equation (4.16) with same order in $\varepsilon$ gives,

$$\begin{cases} \mathcal{L}_0 P_0 = 0 & \text{with} \quad P_0(T,x,y) = h(x) \\ \mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0 & \text{with} \quad P_1(T,x,y) = 0 \\ \mathcal{L}_0 P_n + \mathcal{L}_1 P_{n-1} + \mathcal{L}_2 P_{n-2} = 0 & \text{with} \quad P_n(T,x,y) = 0,\ \forall n \geq 2 \end{cases} \tag{4.17}$$

This method is efficient when pricing is far from maturity, so it is important to set suitable boundary conditions on the $P_i$ terms near maturity. This requires an inner expansion, [53], it is shown that the boundary conditions should behave in such a way as indicated in equation (4.17). The $P_i$ components are determined through iteration by a step-by-step analysis on the terms of order $1/\varepsilon$, $1/\sqrt{\varepsilon}$ and 1, as explained in the following.

[9]In this region, a stable and reliable solution is obtained. An inner expansion would only help in determining all the necessary boundary conditions.

**Collection of $\mathcal{O}(1/\varepsilon)$ terms**

The terms diverge as $\varepsilon \to 0$, an indication of a singularly-perturbed problem. Grouping them yields

$$\mathcal{L}_0 P_0 = 0, \tag{4.18}$$

where the differential operator $\mathcal{L}_0$ is defined in equation (4.11). Note that $\mathcal{L}_0$ contains derivatives with respect to only $y$, thus, equation (4.18) implies $P_0$ is independent of $y$, and so, $P_0 = P_0(t, x)$.

**Collection of $\mathcal{O}(1/\sqrt{\varepsilon})$ terms**

Similarly, these terms diverge as $\varepsilon \to 0$ but slowly compared to those of $\mathcal{O}(1/\varepsilon)$:

$$\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0, \tag{4.19}$$

where $\mathcal{L}_1$ is defined by equation (4.12). The fact that $\mathcal{L}_1$ contains derivatives with respect to only $y$ implies that $\mathcal{L}_1 P_0 = 0$, since $P_0$ does not depend on $y$. Thus,

$$\mathcal{L}_0 P_1 = 0. \tag{4.20}$$

This implies, $P_1$ is independent of $y$, that is, $P_1 = P_1(t, x)$. Equations (4.18) and (4.20) suggest that the first two terms of the expansion (4.14), $P_0 + \sqrt{\varepsilon} P_1$, do not depend on $y$, the current volatility value. This is interesting because volatility is not directly observable.

**Collection of $\mathcal{O}(1)$ terms**

These terms lead to

$$\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0, \tag{4.21}$$

where $\mathcal{L}_2$ is given by equation (4.13) and $\mathcal{L}_1 P_1 = 0$ for reasons already explained above. Thus,

$$\mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0. \tag{4.22}$$

By keeping $x$ fixed, one observes that even though $P_0$ is independent of $y$, $\mathcal{L}_2 P_0$ is a function of $y$ since $\mathcal{L}_2$ contains $y$ in form of $f^2(y)$.

**Poisson Equation**

Equation (4.22) is Poisson's equation for $P_2(y)$ with respect to the operator $\mathcal{L}_0$ in the variable $y$ provided $\mathcal{L}_2 P_0$ is known. This is a singular linear problem solvable if and only if $\mathcal{L}_2 P_0$ is in the orthogonal complement of the null space of $\mathcal{L}_0^*$, the adjoint of $\mathcal{L}_0$, see [119]. Alternatively, (4.22) admits a solution if the average of $\mathcal{L}_2 P_0$ with respect to the invariant distribution of $(Y_t)_{t\geq 0}$ is zero, a property referred to as *centering*. The verification is given in Appendix C,

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0, \tag{4.23}$$

since $P_0$ is independent of $y$. Combining equations (4.13) and (4.23), gives

$$\langle \mathcal{L}_2 \rangle P_0 = \frac{\partial P_0}{\partial t} + \frac{1}{2} \langle f^2(y) \rangle x^2 \frac{\partial^2 P_0}{\partial x^2} + r \left[ x \frac{\partial P_0}{\partial x} - P_0 \right] = 0. \tag{4.24}$$

Equation (4.24) is the Black-Scholes pricing PDE with volatility level $\langle f(y) \rangle$ with a terminal condition $P_0(T, x) = h(x)$. Recall that the mean-square time-averaged volatility,

$$\overline{\sigma^2} = \frac{1}{T-t} \int_t^T f^2(Y_s)\, ds \approx \langle f^2 \rangle = \bar{\sigma}^2, \tag{4.25}$$

when the rate of mean reversion is fast, where $f$ is a positive bounded function of an ergodic process, $(Y_t)_{t\geq 0}$ and $\bar{\sigma}^2$ is the expected square volatility under the invariant distribution. This follows from the Ergodic Theorem 3.3.1. Since $\langle \mathcal{L}_2 P_0 \rangle = 0$, one can express $\mathcal{L}_2 P_0$ as $\mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle$ such that

$$\begin{aligned} \mathcal{L}_2 P_0 &= \frac{1}{2} f^2(y) x^2 \frac{\partial^2 P_0}{\partial x^2} - \frac{1}{2} \langle f^2(y) \rangle x^2 \frac{\partial^2 P_0}{\partial x^2}. \\ &= \frac{1}{2} \left[ f^2(y) - \bar{\sigma}^2 \right] x^2 \frac{\partial^2 P_0}{\partial x^2}. \end{aligned} \tag{4.26}$$

Then equation (4.22) becomes

$$\mathcal{L}_0 P_2 + \frac{1}{2} \left[ f^2(y) - \bar{\sigma}^2 \right] x^2 \frac{\partial^2 P_0}{\partial x^2} = 0, \tag{4.27}$$

from which the second order term $P_2$ is obtained as a function of $t, x$ and $y$:

$$P_2(t, x, y) = -\frac{1}{2} \mathcal{L}_0^{-1} \left[ f^2(y) - \bar{\sigma}^2 \right] x^2 \frac{\partial^2 P_0}{\partial x^2}. \tag{4.28}$$

The inverse $\mathcal{L}_0^{-1}$ is an integral operator thus, if $F(y) := f^2(y) - \bar{\sigma}^2$, then

$$\mathcal{L}_0^{-1} F(y) = \mathcal{G}(y) + k(t, x), \tag{4.29}$$

where $k(t,x)$ (given explicitly in Appendix B.1) is a constant dependent on $t$ and $x$ only. Therefore, equation (4.28) can be written as

$$P_2(t,x,y) = -\frac{1}{2}\mathcal{G}(y)x^2\frac{\partial^2 P_0}{\partial x^2} + k(t,x). \tag{4.30}$$

From equation (4.29) it follows that $\mathcal{G}(y)$ satisfies

$$\mathcal{L}_0\mathcal{G}(y) = f^2(y) - \bar{\sigma}^2. \tag{4.31}$$

Recall that $\mathcal{L}_0$ is the OU differential operator defined in equation (4.11) and that $\bar{\sigma}^2 = \langle f^2(y)\rangle$, so equation (4.31) is of the form

$$\nu^2\mathcal{G}''(y) + [m-y]\mathcal{G}'(y) = f^2(y) - \left\langle f^2(y)\right\rangle. \tag{4.32}$$

Using the density function $\phi(y)$ of the invariant distribution of $(Y_t)_{t\geq 0}$ (i.e. $Y_t \sim \mathcal{N}(m,\nu^2)$) given in equation (3.14), one can obtain the function $\mathcal{G}(y)$ in terms of $\phi(y)$, $f(y)$ and $\langle f^2(y)\rangle$:

$$\begin{aligned}
\left[\mathcal{G}'(y)\phi(y)\right]' &= \mathcal{G}'(y)\phi'(y) + \phi(y)\mathcal{G}''(y).\\
&= -\mathcal{G}'(y)\frac{[y-m]}{\nu^2}\cdot\frac{1}{\sqrt{2\pi\nu^2}}e^{-\frac{[y-m]}{2\nu^2}} + \frac{1}{\sqrt{2\pi\nu^2}}e^{-\frac{[y-m]}{2\nu^2}}\mathcal{G}''(y).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\left[\mathcal{G}'(y)\phi(y)\right]' &= \left[-\mathcal{G}'(y)\frac{[y-m]}{\nu^2} + \mathcal{G}''(y)\right]\phi(y).\\
&= \frac{1}{\nu^2}\left[\nu^2\mathcal{G}''(y) + [m-y]\mathcal{G}'(y)\right]\phi(y).
\end{aligned} \tag{4.33}$$

Comparing equation (4.33) with equation (4.32), observe that

$$\left[\mathcal{G}'(y)\phi(y)\right]' = \frac{1}{\nu^2}\left[f^2(y) - \langle f^2(y)\rangle\right]\phi(y). \tag{4.34}$$

Integrating equation (4.34) once, leads to

$$\mathcal{G}'(y) = \frac{1}{\phi(y)\nu^2}\int_{-\infty}^{y}\left[f^2(w) - \left\langle f^2(w)\right\rangle\right]\phi(w)\,dw\,. \tag{4.35}$$

The upper limit is the current level of $(Y_t)_{t\geq 0}$. Observe that if $f^2$ is bounded then $\mathcal{G}(y)$ is bounded by a linear function[10] in $|y|$.

[10]This follows from the polynomial growth condition $|f(y)| \leq C|1+|y|^n|$, where $C$ is an arbitrary constant and $n$ is an integer.

## 4.4 First-order Correction to BS Model

In this section, a derivation of the first correction $p_1 = \sqrt{\varepsilon}P_1$ of equation (4.15) is given. The second correction is presented in Appendix B.1.

### Collection of $\mathcal{O}(\sqrt{\varepsilon})$ terms

From equation (4.16), collect all terms of $\mathcal{O}(\sqrt{\varepsilon})$ from which it is deduced:

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0, \tag{4.36}$$

a Poisson equation of $P_3$ with respect to $\mathcal{L}_0$. For equation (4.36) to admit a solution,

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0. \tag{4.37}$$

Recall that $P_1$ is independent of $y$ from equation (4.20), so from equation (4.37) it follows that

$$\langle \mathcal{L}_2 \rangle P_1 = -\langle \mathcal{L}_1 P_2 \rangle, \tag{4.38}$$

where $\langle \mathcal{L}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma})$ gives the Black-Scholes partial differential operator with volatility level $\langle f(y) \rangle = \bar{\sigma}$ as explained above. Substituting equation (4.30) in equation (4.38), yields

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma})P_1 &= \langle \mathcal{L}_1 \left[ \frac{1}{2} \left[ \mathcal{G}(y) + k(t,x) \right] x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \rangle. \\
&= \frac{1}{2} \langle \mathcal{L}_1(\mathcal{G}(y)) + \mathcal{L}_1(k(t,x)) \rangle x^2 \frac{\partial^2 P_0}{\partial x^2}. \\
&= \frac{1}{2} \langle \mathcal{L}_1(\mathcal{G}(y)) x^2 \frac{\partial^2 P_0}{\partial x^2} \rangle,
\end{aligned} \tag{4.39}$$

where $\mathcal{L}_1(k(t,x)) = 0$ since $k(t,x)$ is independent of $y$. Observe from equation (4.12) that

$$\langle \mathcal{L}_1 \mathcal{G}(y) \cdot \rangle = \sqrt{2}\rho \nu x \langle f(y) \frac{\partial \mathcal{G}(y)}{\partial y} \rangle \frac{\partial}{\partial x} \cdot - \sqrt{2}\nu \langle \wedge(y) \frac{\partial \mathcal{G}(y)}{\partial y} \rangle \cdot, \tag{4.40}$$

thus, equation (4.39) can be written as

$$\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma})P_1 = & \frac{\sqrt{2}}{2} \rho \nu x \langle f(y) \frac{\partial \mathcal{G}(y)}{\partial y} \rangle \frac{\partial}{\partial x} \left[ x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \\
& - \frac{\sqrt{2}}{2} \nu \langle \wedge(y) \frac{\partial \mathcal{G}(y)}{\partial y} \rangle \left[ x^2 \frac{\partial^2 P_0}{\partial x^2} \right].
\end{aligned}$$

On further expansion, it follows

$$\mathcal{L}_{BS}(\bar{\sigma})P_1 = \frac{\sqrt{2}}{2}\rho\nu x\langle f(y)\frac{\partial \mathcal{G}(y)}{\partial y}\rangle\left[2x\frac{\partial^2 P_0}{\partial x^2} + x^2\frac{\partial^3 P_0}{\partial x^3}\right] - \frac{\sqrt{2}}{2}\nu\langle \wedge(y)\frac{\partial \mathcal{G}(y)}{\partial y}\rangle\left[x^2\frac{\partial^2 P_0}{\partial x^2}\right]. \tag{4.41}$$

Equation (4.41) can finally be simplified to

$$\mathcal{L}_{BS}(\bar{\sigma})P_1 = \frac{\sqrt{2}}{2}\rho\nu\langle f(y)\mathcal{G}'(y)\rangle x^3\frac{\partial^3 P_0}{\partial x^3} + \left[\sqrt{2}\rho\nu\langle f(y)\mathcal{G}'(y)\rangle - \frac{\sqrt{2}}{2}\langle \wedge(y)\mathcal{G}'(y)\rangle\right]x^2\frac{\partial^2 P_0}{\partial x^2}. \tag{4.42}$$

To obtain the first correction $p_1 = \sqrt{\varepsilon}P_1(t,x)$, to the classical Black-Scholes pricing PDE, multiply equation (4.42) by $\sqrt{\varepsilon}$ to obtain

$$\mathcal{L}_{BS}(\bar{\sigma})p_1 = \frac{\sqrt{2\varepsilon}}{2}\rho\nu\langle f(y)\mathcal{G}'(y)\rangle x^3\frac{\partial^3 P_0}{\partial x^3} + \frac{\sqrt{2\varepsilon}}{2}\left[2\rho\nu\langle f(y)\mathcal{G}'(y)\rangle - \langle \wedge(y)\mathcal{G}'(y)\rangle\right]x^2\frac{\partial^2 P_0}{\partial x^2}. \tag{4.43}$$

Now, recall that the small parameter $\varepsilon = 1/\alpha$. Thus, equation (4.43) can be expressed as

$$\mathcal{L}_{BS}(\bar{\sigma})p_1 = V_2x^2\frac{\partial^2 P_0}{\partial x^2} + V_3x^3\frac{\partial^3 P_0}{\partial x^3}, \tag{4.44}$$

where $p_1(T,X_T) = 0$ and coefficients $V_2$ and $V_3$ are defined as,

$$V_2 = \frac{1}{\sqrt{2\alpha}}\rho\nu\langle f(y)\mathcal{G}'(y)\rangle \quad \text{and} \quad V_3 = \frac{1}{\sqrt{2\alpha}}\left[2\rho\nu\langle f(y)\mathcal{G}'(y)\rangle - \langle \wedge(y)\mathcal{G}'(y)\rangle\right].$$

It remains to solve equation (4.44) to obtain the first correction term to Black-Scholes PDE. The following lemmas are useful in determining $p_1$.

**Lemma 4.4.1.** *Let $k \in \mathbb{R}_+$, the operator $D_k$ be defined as*

$$D_k = V_kx^k\frac{\partial^k}{\partial x^k}, \tag{4.45}$$

*where $V_k$ is constant with respect to $x$, and $\mathcal{L}_{BS}(\bar{\sigma})$ denote Black-Scholes operator with volatility level $\bar{\sigma}$. Then for any smooth and bounded function $P_0(t,x)$ dependent on time, $t$ and a space variable $x$ the following equation holds:*

$$\mathcal{L}_{BS}(\bar{\sigma})(D_kP_0) = D_k\mathcal{L}_{BS}(\bar{\sigma})P_0. \tag{4.46}$$

*Proof.* By induction, if equation (4.46) holds for $k = 1$, $k = 2$ and for some positive numbers $n$ and $n+1$ then it holds for all positive numbers, $k$.

For $k = 1$,

$$\mathcal{L}_{BS}(\bar{\sigma})\left[Vx\frac{\partial P_0}{\partial x}\right] = Vx\frac{\partial}{\partial x}\frac{\partial P_0}{\partial t} + \frac{1}{2}V\bar{\sigma}^2x^2\left[2\frac{\partial^2 P_0}{\partial x^2} + Vx\frac{\partial^3 P_0}{\partial x^3}\right] + rVx^2\frac{\partial P_0}{\partial x}.$$

This last equation is exactly the same as the expansion of $D_1\mathcal{L}_{BS}(\bar{\sigma})P_0$, thus,

$$\mathcal{L}_{BS}(\bar{\sigma})(D_1P_0) = D_1\mathcal{L}_{BS}(\bar{\sigma})P_0.$$

Similarly, for $k = 2$,

$$\mathcal{L}_{BS}(\bar{\sigma})(D_2P_0) = V_2x^2\frac{\partial^2}{\partial x^2}\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2V_2x^2\left[2\frac{\partial^2 P_0}{\partial x^2} + 4\frac{\partial^3 P_0}{\partial x^3} + x^2\frac{\partial^4 P_0}{\partial x^4}\right]$$
$$+ rV_2x^2\frac{\partial^2 P_0}{\partial x^2} + rV_2x^3\frac{\partial^3 P_0}{\partial x^3}.$$

This is the same result one would get by expanding $D_2\mathcal{L}_{BS}(\bar{\sigma})P_0$. Therefore,

$$\mathcal{L}_{BS}(\bar{\sigma})(D_2P_0) = D_2\mathcal{L}_{BS}(\bar{\sigma})P_0.$$

Suppose that for some positive number $n$ the following is true,

$$\mathcal{L}_{BS}(\bar{\sigma})(D_nP_0) = D_n\mathcal{L}_{BS}(\bar{\sigma})P_0,$$

Next, is to show that it is also true for $n + 1$:

$$\begin{aligned}\mathcal{L}_{BS}(\bar{\sigma})(D_{n+1}P_0) &= \mathcal{L}_{BS}(\bar{\sigma})(D_1D_nP_0).\\ &= D_1\mathcal{L}_{BS}(\bar{\sigma})(D_nP_0).\\ &= D_1D_n\mathcal{L}_{BS}(\bar{\sigma})(P_0).\\ &= D_{n+1}\mathcal{L}_{BS}(\bar{\sigma})(P_0).\end{aligned}$$

Letting $k = n + 1$ concludes the proof. □

**Lemma 4.4.2.** *Given a differential equation of the form:*

$$\mathcal{L}_{BS}(\bar{\sigma})u_{k,l} = -[T-t]^lD_kP_0(t, x; \bar{\sigma}), \tag{4.47}$$

*where $P_0(t, x; \bar{\sigma})$ is the solution to Black-Scholes PDE with volatility level $\bar{\sigma}$ and $D_k$ is defined as:*

$$D_k = V_kx^k\frac{\partial^k}{\partial x^k}. \tag{4.48}$$

*where $V_k$ is constant with respect to $x$. Then the solution to equation (4.47) is given as,*

$$u_{k,l} = \frac{[T-t]^{l+1}}{l+1}D_kP_0(t, x; \bar{\sigma}). \tag{4.49}$$

*Proof.* By definition of the Black-Scholes operator $\mathcal{L}_{BS}(\bar{\sigma})$ with volatility level $\bar{\sigma}$, it follows that,

$$\mathcal{L}_{BS}(\bar{\sigma})\left(\frac{[T-t]^{l+1}}{l+1}D_kP_0\right) = \frac{\partial}{\partial t}\left[\frac{[T-t]^{l+1}}{l+1}D_kP_0\right] + \frac{1}{2}\bar{\sigma}^2x^2\frac{\partial^2}{\partial x^2}\left[\frac{[T-t]^{l+1}}{l+1}D_kP_0\right] + rx\frac{\partial}{\partial x}\left[\frac{[T-t]^{l+1}}{l+1}D_kP_0\right] - rx\left[\frac{[T-t]^{l+1}}{l+1}D_kP_0\right]. \tag{4.50}$$

With the knowledge from Lemma 4.4.1, one can reduce the *r.h.s* of equation (4.50) to,

$$r.h.s = -[T-t]^lD_kP_0 + \frac{[T-t]^{l+1}}{l+1}D_k\left[\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2x^2\frac{\partial^2 P_0}{\partial x^2} + r\left[x\frac{\partial P_0}{\partial x} - P_0\right]\right].$$

The last term is zero since $P_0 = P_0(t,x;\bar{\sigma})$ satisfies Black-Scholes PDE, $\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0$. Hence,

$$\mathcal{L}_{BS}(\bar{\sigma})u_{k,l} = -[T-t]^lD_kP_0(t,x;\bar{\sigma}),$$

where $u_{k,l}$ is defined as

$$u_{k,l} = \frac{[T-t]^{l+1}}{l+1}D_kP_0(t,x;\bar{\sigma}).$$

This concludes the proof. □

From Lemma 4.4.2, if $l=0$ in equation (4.47) then, $k=2,3$ gives

$$\mathcal{L}_{BS}(\bar{\sigma})u_2 = -D_2P_0(t,x;\bar{\sigma}). \tag{4.51}$$
$$\mathcal{L}_{BS}(\bar{\sigma})u_3 = -D_3P_0(t,x;\bar{\sigma}). \tag{4.52}$$

Since Black-Scholes operator is linear, then equations (4.51) and (4.52) give

$$\mathcal{L}_{BS}(\bar{\sigma})(u_2+u_3) = -[D_2+D_3]P_0(t,x;\bar{\sigma}). \tag{4.53}$$

Deduce $u_2$ and $u_3$ from (4.49) with $l=0$ and substitute them in equation (4.53), to obtain

$$\mathcal{L}_{BS}(\bar{\sigma})([T-t][D_2+D_3]P_0(t,x;\bar{\sigma})) = -[D_2+D_3]P_0(t,x;\bar{\sigma}).$$

Let $\mathcal{A} = [D_2 + D_3]$ and rewrite the above equation[11] as

$$\mathcal{L}_{BS}(\bar{\sigma})(-[T-t]\mathcal{A}P_0(t,x;\bar{\sigma})) = \mathcal{A}P_0(t,x;\bar{\sigma}), \tag{4.54}$$

where

$$\mathcal{A} = V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3}, \tag{4.55}$$

Comparing equations (4.44) and (4.54) implies the correction,

$$p_1 = -[T-t]\left[V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right]. \tag{4.56}$$

Hence, the corrected Black-Scholes derivative price up to the first leading term in the asymptotic expansion is given by

$$P(t,x) = P_0 - [T-t]\left[V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}\right]. \tag{4.57}$$

**Lemma 4.4.3.** *Define the operator $\mathcal{D}_k^*$ as*

$$\mathcal{D}_k^* = x^k \frac{\partial^k}{\partial x^k},$$

*then for all $k > 0$, the following expansion is valid:*

$$\mathcal{D}_1^* \mathcal{D}_k^* = k\mathcal{D}_k^* + \mathcal{D}_{k+1}^*.$$

The proof of Lemma 4.4.3 can be obtained by induction. For $k = 2$, $\mathcal{D}_1^*\mathcal{D}_2^* = 2\mathcal{D}_2^* + \mathcal{D}_3^*$. Using this decomposition, the corrected price given by equation (4.57) can be written as

$$\boxed{P(t,x) = P_0 - [T-t]\left[[V_2 - 2V_3] + V_3\mathcal{D}_1^*\right]\mathcal{D}_2^* P_0} \tag{4.58}$$

**Remark 4.4.4.** *The corrected price in equation (4.57) does not depend on the current value of volatility, $y$.*

**Remark 4.4.5.** *The first corrected Black-Scholes price is a composition of the gamma and the delta-gamma, see equation (4.58).*

[11] Note that $\mathcal{A}$ can also be expressed as

$$\mathcal{A} = V_3\mathcal{D}_1^*\mathcal{D}_2^* + V_2\mathcal{D}_2^*$$

### Explicit form of the $\mathcal{A}P_0$-term

Recall from equation (4.55) that

$$\mathcal{A}P_0 = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3},$$

with the small parameters $V_2$ and $V_3$ defined as

$$V_2 = \frac{\nu}{\sqrt{2\alpha}} \left[ 2\rho \langle f(y) \mathcal{G}'(y) \rangle - \langle \wedge(y) \mathcal{G}'(y) \rangle \right]. \tag{4.59}$$

$$V_3 = \frac{\rho\nu}{\sqrt{2\alpha}} \langle f(y) \mathcal{G}'(y) \rangle. \tag{4.60}$$

Using the definition of the function $\mathcal{G}'(y)$ in (4.35), $V_2$ and $V_3$ can be obtained explicitly.

Firstly, computing the averaged terms with respect to the invariant distribution of $Y_t$ yields

$$\begin{aligned} \langle f(y)\mathcal{G}'(y) \rangle &= \frac{1}{\nu^2} \langle \frac{f(y)}{\phi(y)} \int_{-\infty}^{y} \left[ f^2(w) - \langle f^2(w) \rangle \right] \phi(w)\, dw \, \rangle. \\ &= \frac{1}{\nu^2} \int_{-\infty}^{\infty} \left[ f(y) \int_{-\infty}^{y} \left[ f^2(w) - \langle f^2(w) \rangle \right] \phi(w)\, dw \right] dy \,. \end{aligned} \tag{4.61}$$

To evaluate equation (4.61), use the method of integration by parts:

Let,

$$\mathbf{u} = \int_{-\infty}^{y} \left[ f^2(w) - \langle f^2(w) \rangle \right] \phi(w)\, dw \quad \text{then} \quad \frac{d\mathbf{u}}{dy} = \left[ f^2(y) - \langle f^2(y) \rangle \right] \phi(y),$$

and

$$\frac{d\mathbf{F}(y)}{dy} = f(y) \quad \text{such that} \quad \mathbf{F}(y) = \int f(w)\, dw.$$

Thus, equation (4.61) becomes

$$\begin{aligned} \langle f(y)\mathcal{G}'(y) \rangle &= \frac{1}{\nu^2} \mathbf{F}(y) \int_{-\infty}^{y} [f^2(w) - \langle f^2(w) \rangle] \phi(w)\, dw \,|_{-\infty}^{\infty} \\ &\quad - \frac{1}{\nu^2} \int_{-\infty}^{\infty} \mathbf{F}(w) \left[ f^2(w) - \langle f^2(w) \rangle \right] \phi(w)\, dw \,. \\ &= -\frac{1}{\nu^2} \langle \mathbf{F}(y) \left[ f^2(y) - \langle f^2(y) \rangle \right] \rangle. \end{aligned} \tag{4.62}$$

Similarly,

$$\langle \wedge(y)\mathcal{G}'(y) \rangle =$$
$$\langle \left[ \rho \frac{[\mu - r]}{f(y)} + \sqrt{1-\rho^2}\gamma(y) \right] \cdot \left[ \frac{1}{\nu^2 \phi(y)} \int_{-\infty}^{y} \left[ f^2(w) - \langle f^2(w) \rangle \right] \phi(w)\, dw \right] \rangle.$$

Expansion of this, yields

$$\langle f(y)\mathcal{G}'(y)\rangle = \rho\frac{[\mu - r]}{\nu^2}\langle\frac{1}{f(y)\phi(y)}\int_{-\infty}^{y}\left[f^2(w) - \langle f^2(w)\rangle\right]\phi(w)\,dw\,\rangle$$
$$+\frac{\sqrt{1-\rho^2}}{\nu^2}\langle\frac{\gamma(y)}{\phi(y)}\int_{-\infty}^{y}\left[f^2(w) - \langle f^2(w)\rangle\right]\phi(w)\,dw\,\rangle \tag{4.63}$$

Using the same approach of integrating by parts the two terms on the *r.h.s* of (4.63) give

$$\langle f(y)\mathcal{G}'(y)\rangle = -\rho\frac{[\mu - r]}{\nu^2}\langle\tilde{\mathbf{F}}(y)\left[f^2(y) - \langle f^2(y)\rangle\right]\rangle$$
$$-\frac{\sqrt{1-\rho^2}}{\nu^2}\langle\mathbf{\Gamma}(y)\left[f^2(y) - \langle f^2(y)\rangle\right]\rangle, \tag{4.64}$$

where the functions $\tilde{\mathbf{F}}(y)$ and $\mathbf{\Gamma}(y)$ are defined as,

$$\tilde{\mathbf{F}}(y) = \int\frac{1}{f(w)}\,dw \quad \text{and} \quad \mathbf{\Gamma}(y) = \int\gamma(w)\,dw\,.$$

Hence, equations (4.59) and (4.60) are given as

$$V_2 = \frac{1}{\nu\sqrt{2\alpha}}\langle\left[-2\rho\mathbf{F}(y) + \rho[\mu - r]\tilde{\mathbf{F}}(y) + \sqrt{1-\rho^2}\mathbf{\Gamma}(y)\right][f^2(y) - \langle f^2(y)\rangle]\rangle.$$
$$V_3 = -\frac{\rho}{\nu\sqrt{2\alpha}}\langle\mathbf{F}(y)[f^2(y) - \langle f^2(y)\rangle]\rangle.$$

It is interesting to notice the existence of all model parameters $\mu, m, \nu, \rho$ and $\alpha$ in the expressions of $V_2$ and $V_3$. To this end, the specific choice of the model is not relevant. All that is required is $f(y)$ to able to obtain explicit formulae for $V_2$ and $V_3$. For instance, [6] showed that if $f(y) = e^y$, where the invariant distribution of $y$ is given by equation (3.14), then $V_2$ and $V_3$ can be expressed as

$$V_2 = -\frac{2\rho}{\nu\sqrt{2\alpha}}\left[e^{9\nu^2/2+3m} - e^{5\nu^2/2+3m}\right] - \frac{\rho}{\nu\sqrt{2\alpha}}[\mu - r]\left[e^{\nu^2/2+m} - e^{5\nu^2/2+m}\right]$$
$$+\frac{2}{\sqrt{2}}\sqrt{1-\rho^2}\gamma\bar{\sigma}^2\nu. \tag{4.65}$$
$$V_3 = -\frac{\rho}{\nu\sqrt{2\alpha}}\left[e^{9\nu^2/2+3m} - e^{5\nu^2/2+3m}\right]. \tag{4.66}$$

**Remark 4.4.6.** *Since $V_2$ and $V_3$ contain all the model parameters, they are referred to as universal market group parameters and thus, only these parameters together with $\bar{\sigma}$, are enough to calibrate the model to market data.*

**Remark 4.4.7.** *The corrected price in equation (4.57) applies to any stochastic volatility model where volatility is driven by an ergodic process such that the Poisson equations admit well-behaved solutions.*

Next, is a discussion on the significance of the derivatives of the Black-Scholes solution $P_0$, the leading term in the asymptotic expansion. One can refer to the formulae of the Greeks derived in Section 1.4.4. Note that

$$P_0 = xN(\mathrm{d}_+) - Ke^{-r[T-t]}N(\mathrm{d}_+ - \sigma\sqrt{[T-t]}), \tag{4.67}$$

with constant volatility as $\bar{\sigma}$. It follows that the $\mathcal{A}P_0 - term$ is composed of the Gamma, (see also equation (1.48))

$$\frac{\partial^2 P_0}{\partial x^2} = \frac{1}{x\bar{\sigma}\sqrt{2\pi[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}}. \tag{4.68}$$

Consequently, the third derivative of $P_0$ with respect to $x$ is

$$\begin{aligned}\frac{\partial^3 P_0}{\partial x^3} &= -\frac{1}{x^2\bar{\sigma}\sqrt{2\pi[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}} + \frac{1}{x\bar{\sigma}\sqrt{2\pi[T-t]}}\left[\frac{-\mathrm{d}_+}{x\bar{\sigma}\sqrt{[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}}\right].\\ &= -\frac{1}{x^2\bar{\sigma}\sqrt{2\pi[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}}\left[1 + \frac{\mathrm{d}_+}{\bar{\sigma}\sqrt{[T-t]}}\right].\end{aligned} \tag{4.69}$$

This third derivative is sometimes referred to as the *Epsilon*. Substituting equations (4.68) and (4.69) in equation (4.55) gives

$$\mathcal{A}P_0(t,x;\bar{\sigma}) = \frac{x}{\bar{\sigma}\sqrt{2\pi[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}}\left[V_2 - V_3 - \frac{V_3\mathrm{d}_+}{\bar{\sigma}}\sqrt{[T-t]}\right]. \tag{4.70}$$

Consequently, the correction in equation (4.56) becomes

$$\begin{aligned}p_1 &= -[T-t]\left[\frac{x}{\bar{\sigma}\sqrt{2\pi[T-t]}}e^{-\frac{\mathrm{d}_+^2}{2}}\left[V_2 - V_3 - \frac{V_3\mathrm{d}_+}{\bar{\sigma}}\sqrt{[T-t]}\right]\right].\\ &= \frac{x}{\bar{\sigma}\sqrt{2\pi}}e^{-\frac{\mathrm{d}_+^2}{2}}\left[\frac{V_3\mathrm{d}_+}{\bar{\sigma}} + [V_3 - V_2]\sqrt{[T-t]}\right].\end{aligned} \tag{4.71}$$

## 4.5 Volatility Correction and Skewness

This section explains the origin of the skew observed in the implied volatility surface as a result of correcting the BS-pricing PDE. Since $V_2$ contains the market price of volatility risk[12], the corrected effective volatility $\tilde{\sigma}$ for pricing can be obtained by performing a little shift to Black-Scholes constant volatility $\bar{\sigma}$ by the small parameter $V_2$ as

$$\tilde{\sigma}^2 = \bar{\sigma}^2 - 2V_2. \tag{4.72}$$

[12]This is clear from equations (4.59) and (4.5).

Under this effective corrected volatility, the first-order correction $P_0 + p_1$, satisfies

$$\mathcal{L}_{BS}(\tilde{\sigma})(P_0 + p_1) = \mathcal{L}_{BS}(\tilde{\sigma})P_0 + \mathcal{L}_{BS}(\tilde{\sigma})p_1. \tag{4.73}$$

By definition of the operator $\mathcal{L}_{BS}(\tilde{\sigma})$, the *r.h.s* of equation (4.73) is written as

$$r.h.s = \frac{\partial P_0}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + r\left[x\frac{\partial P_0}{\partial x} - P_0\right] \tag{4.74}$$

$$+ \frac{\partial p_1}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 x^2 \frac{\partial^2 p_1}{\partial x^2} + r\left[x\frac{\partial p_1}{\partial x} - p_1\right]. \tag{4.75}$$

Substituting equation (4.72) in equation (4.74) gives

$$\begin{aligned} r.h.s &= \frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2 P_0}{\partial x^2} + r\left[x\frac{\partial P_0}{\partial x} - P_0\right] \\ &+ \frac{\partial p_1}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2 \frac{\partial^2 p_1}{\partial x^2} + r\left[x\frac{\partial p_1}{\partial x} - p_1\right] - V_2 x^2\left[\frac{\partial^2 P_0}{\partial x^2} + \frac{\partial^2 p_1}{\partial x^2}\right]. \\ &= \mathcal{L}_{BS}(\bar{\sigma})(P_0 + p_1) - V_2 x^2\left[\frac{\partial^2 P_0}{\partial x^2} + \frac{\partial^2 p_1}{\partial x^2}\right]. \end{aligned} \tag{4.76}$$

From equation (4.44) and the fact that $P_0$ is a solution to the classical Black-Scholes equation $\mathcal{L}_{BS}(\bar{\sigma})P_0 = 0$, it follows that

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0 + p_1) = V_2 x^2 \frac{\partial^2 P_0}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}, \tag{4.77}$$

substituting this in equation (4.76), implies that equation (4.73) can be written as

$$\mathcal{L}_{BS}(\tilde{\sigma})(P_0 + p_1) = -V_2 x^2 \frac{\partial^2 p_1}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}. \tag{4.78}$$

Note that the first term on the *r.h.s* of equation (4.78) is of $\mathcal{O}(\varepsilon)$ because $V_2 = \mathcal{O}(\sqrt{\varepsilon})$ and $p_1 = \sqrt{\varepsilon}P_1 = \mathcal{O}(\sqrt{\varepsilon})$. Now, $V_3 = \mathcal{O}(\sqrt{\varepsilon})$ meaning that the first term on the *r.h.s* is negligible compared to the $V_3$ term. This shows then, that the order of the corrected price $P_0 + p_1$ is the same as that of a function $p$ that would satisfy

$$\mathcal{L}_{BS}(\tilde{\sigma})p = V_3 x^3 \frac{\partial^3 P_0}{\partial x^3}. \tag{4.79}$$

Observe from (4.60), the $V_3$ term entirely relies on the correlation between stock price and volatility processes. This term vanishes completely if the two processes are uncorrelated. For no correlation, equation (4.79) is purely Black-Scholes model with volatility level $\tilde{\sigma}$, the effective corrected volatility and with a payoff, $p(T, x)$. In a classical sense, if $\eta$ is a random variable , its skewness is defined as its third standardized moment, that is, $\mathbb{E}\left\{[\eta - \mu_0]^3/\sigma^3\right\}$, with $\mu_0$ as its mean and $\sigma$ as its standard deviation. This suggests that the $V_3$ term which contains $x^3$ accounts for the skewness of the distribution of the stock price returns.

**Remark 4.5.1.** *In the corrected derivative price, the $V_2$ term is for adjusting the volatility level to an effective value as a result of the market price of volatility and the $V_3$ term accounts for the skewness of the distribution of the stock price returns.*

## 4.6 First-order Correction to Implied Volatility

The first-order correction of the implied volatility using asymptotic expansion is discussed here. A derivation of the second-order correction is given in Appendix B.2. From equation (2.1), the implied volatility $I$ satisfies the equation,

$$C_{\mathrm{BS}}(t, X, K, T, I) = C^{obs}(K, T), \tag{4.80}$$

where all parameters and variables carry their usual meaning. Expand $I$ as

$$I = \sum_{n=0}^{\infty} \varepsilon^{\frac{n}{2}} I_n, \tag{4.81}$$

Expressing the *l.h.s* of equation (4.80) as a function of $I$ keeping other factors constant, the expansion of $C_{\mathrm{BS}}(I)$ using the Taylor series about $I_0$ yields

$$C_{\mathrm{BS}}(I) = C_{\mathrm{BS}}\left(\sum_{n=0}^{\infty} \varepsilon^{\frac{n}{2}} I_n\right) = \sum_{k=0}^{\infty} \frac{1}{k!} C_{\mathrm{BS}}(I_0)^k \left[\sum_{n=0}^{\infty} \varepsilon^{\frac{n}{2}} I_n - I_0\right]^k, \tag{4.82}$$

where $C_{\mathrm{BS}}(I_0)^k = \frac{\partial^k}{\partial \sigma^k} C_{\mathrm{BS}}(I_0)$. Expanding up to the first order term in $\varepsilon$ gives

$$C_{\mathrm{BS}}(I) = C_{\mathrm{BS}}(I_0) + \sqrt{\varepsilon} I_1 \frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(I_0) + \cdots . \tag{4.83}$$

Thus, substituting equation (4.83) in equation (4.80) with an assumption that the corrected option price equals to the observed market price, leads to

$$C_{\mathrm{BS}}(t, X, K, T, I_0) + \sqrt{\varepsilon} I_1 \frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(t, X, K, T, I_0) + \cdots = P_0(\bar{\sigma}) + p_1(\bar{\sigma}) + \cdots$$

By comparison, first, note that $P_0(t, x, \bar{\sigma}) = C_{\mathrm{BS}}(t, X, K, T, I_0)$ from which it can be concluded that $I_0 = \bar{\sigma}$ and secondly, that the second terms on both sides lead to

$$\sqrt{\varepsilon} I_1 = p_1(t, x) \left[\frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(t, X, K, T, \bar{\sigma})\right]^{-1}. \tag{4.84}$$

Thus, equation (4.81) can be written as

$$I = \bar{\sigma} + p_1(t, x) \left[\frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(t, X, K, T, \bar{\sigma})\right]^{-1} + \sum_{n=2}^{\infty} \varepsilon^{\frac{n}{2}} I_n. \tag{4.85}$$

Revisiting (1.50) and (4.71) and taking $\varepsilon = 1/\alpha$, equation (4.85) can further be written as

$$I = \bar{\sigma} + \frac{V_3 \mathrm{d}_+}{\bar{\sigma}\sqrt{[T-t]}} + \frac{V_3 - V_2}{\bar{\sigma}} + \mathcal{O}(1/\alpha). \tag{4.86}$$

Substituting for $\mathrm{d}_+$ defined by equation (1.43) (observe that volatility$= \bar{\sigma}$), (4.86) becomes,

$$I = \bar{\sigma} + \frac{V_3}{\bar{\sigma}\sqrt{[T-t]}} \left[ \frac{\log\left[x/K\right] + [r + \frac{\bar{\sigma}^2}{2}][T-t]}{\bar{\sigma}^2\sqrt{[T-t]}} \right] + \frac{V_3 - V_2}{\bar{\sigma}} + \mathcal{O}(1/\alpha).$$

This can be simplified to

$$I = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left[r + \frac{3}{2}\bar{\sigma}^2\right] + \frac{V_3}{\bar{\sigma}^3}\frac{\log\left[x/K\right]}{[T-t]} - \frac{V_2}{\bar{\sigma}} + \mathcal{O}(1/\alpha). \tag{4.87}$$

For convenience, $\log[x/K]$ can be expressed as a *logarithm of moneyness*, $\log[K/x]$. Thus,

$$I = -\frac{V_3}{\bar{\sigma}^3}\frac{\log\left[K/x\right]}{[T-t]} + \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left[r + \frac{3}{2}\bar{\sigma}^2\right] - \frac{V_2}{\bar{\sigma}} + \mathcal{O}(1/\alpha). \tag{4.88}$$

Observe that equation (4.88) is of the form: $\Psi(\eta) = L(\eta) + b$, where $L$ is a linear function and $b$ is a vector.

**Definition 4.6.1.** *A function $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ is regarded affine if there exists a linear function $L$ and $b \in \mathbb{R}^n$ such that*

$$\Psi(\eta) = L(\eta) + b,$$

*for all $\eta$ in $\mathbb{R}^m$. Note that $L$ could be an $m \times n$ matrix and $b$ an $n$-vector. If $m = 1 = n$, there exist real numbers, $a$ and $b$ such that*

$$\Psi(\eta) = a\eta + b,$$

*for all real numbers $\eta$.*

Hence, equation (4.88) can equivalently be written as

$$\boxed{I(K, T-t) = a \cdot \kappa + b + \mathcal{O}(1/\alpha)} \tag{4.89}$$

which is an *affine function* of the *log-moneyness-to-maturity ratio* (LMMR) up to order $\mathcal{O}(1/\alpha)$, with

$$\kappa = \frac{\log\left[K/x\right]}{T-t}, \tag{4.90}$$

$$a = -\frac{V_3}{\bar{\sigma}^3}, \tag{4.91}$$

$$b = \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}\left[r + \frac{3}{2}\bar{\sigma}^2\right] - \frac{V_2}{\bar{\sigma}}. \tag{4.92}$$

Solving equations (4.91) and (4.92) simultaneously for $V_2$ and $V_3$ yields

$$V_2 = \bar{\sigma}\left[[\bar{\sigma} - b] - a\left[r + \frac{3}{2}\bar{\sigma}^2\right]\right], \tag{4.93}$$

$$V_3 = -a\bar{\sigma}^3, \tag{4.94}$$

Therefore, one can obtain estimates for $V_2$ and $V_3$ by calibrating parameters $a$ and $b$ to the implied volatility surface and estimating $\bar{\sigma}$ from historical data.

**Remark 4.6.2.** *The formula for implied volatility given in equation (4.89) is only valid when far from the expiry date, that is, $T \gg t$. This is because sufficient time is required for enormous fluctuations about the long-run mean of $Y_t$.*

## 4.7 Calibration

The procedure for calibrating the model in (4.89) to market data is stipulated in [40].

### 4.7.1 Procedure

The step-by-step procedure for calibration is as follows:

- Estimate the effective historical volatility, $\bar{\sigma}$ from stock price returns.
- Confirm fast mean-reversion of volatility by performing a variogram analysis[13] of historical stock price returns.
- Fit the implied volatility model, (4.89), to the implied volatility surface across strikes and maturities for liquid options to obtain estimates of the slope $a$ and the intercept $b$.
- From the estimated slope $a$, the intercept $b$, and the effective volatility $\bar{\sigma}$, compute the Global parameters $V_2$ and $V_3$ using equations (4.93) and (4.94). Take the instantaneous rate $r$ to be constant.

[13]Here, variogram analysis involves the study of the empirical structure function of the log absolute value $L_n$ of normalized fluctuations of the historical data:

$$V_i^N = \frac{1}{N}\sum_{n=1}^{N}[L_{n+i} - L_n]^2\,; \quad \text{where } \; L_n = \left|\frac{2[X_n - X_{n-1}]}{\sqrt{\Delta t}[X_n + X_{n-1}]}\right|,$$

$i$ is the lag and $N$ is the total number of data points. $V_i^N$ estimates the empirical structure function (variogram), i.e. $V_i^N \approx 2k^2 + 2\nu_f^2[1 - e^{-\alpha|i|}]$, where $k^2 = \text{var}\{\log|\varepsilon|\}$ and $2\nu_f^2$ denotes the variance of $\log(f(Y_n))$. Full details of the variogram analysis in regard to estimation of the rate of fast mean-reversion, are well presented in [40].

- The estimated parameters $\bar{\sigma}$, $V_2$ and $V_2$ are substituted in equation (4.57) to compute the corrected price of the corresponding derivative, see for instance the next section that discusses one application by pricing an Asian average-strike option. More applications to exotic derivatives are presented in [43].

### 4.7.2 Estimating $V_2$ and $V_3$

Estimates for $V_2$ and $V_3$ are obtained from values of $a$ and $b$. The latter are obtained using the method of least squares as follows:

$$a = \frac{\frac{1}{n}\left[\sum_{i=1}^{n} I_i \sum_{i=1}^{n} \kappa_i\right] - \sum_{i=1}^{n} I_i \kappa_i}{\frac{1}{n}\left[\sum_{i=1}^{n} I_i\right]^2 - \sum_{i=1}^{n} \kappa_i^2}, \tag{4.95}$$

where $I$ is the observed implied volatility and $\kappa_i$ is defined as

$$\kappa_i = \frac{\log[K_i/x]}{T-t}.$$

Consequently, the parameter $b$ is given as

$$b = \frac{1}{n}\left[\sum_{i=1}^{n} I_i - a\sum_{i=1}^{n} \kappa_i\right]. \tag{4.96}$$

## 4.8 Application to Asian Options

This section explains the pricing of an Asian average-strike option using the singular perturbation technique discussed above. Details of Asian options can be found in most mathematical finance book on exotic derivatives, see for instance Zhang [118] page 113.

The price of the Asian average-strike option considered is a function $P(t, X_t, Y_t, I_t)$ of time, $t$, the underlying stock price $X_t$, the volatility driving process $Y_t$ and a process $I_t$ defined as

$$I_t = \int_0^t X_s \, ds. \tag{4.97}$$

Under an equivalent martingale measure $\mathbb{P}^*$, the price $P(t, x, y, I)$ at time $0 \leq t < T$, satisfies the expectation, see [43]

$$P(t, x, y, I) = \mathbb{E}^*\left\{ e^{-r[T-t]} \left[X_T - \frac{I_T}{T}\right]^+ | X_t = x, Y_t = y, I_t = I \right\}. \tag{4.98}$$

According to Feynman-Kac formula, Appendix A.5, Section A.5.1, the expectation (4.98) solves the problem with terminal condition

$$\begin{cases} \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2\right] P = 0, \\ P(T,x,y,I) = \left[x - \frac{I}{T}\right]^+, \end{cases} \tag{4.99}$$

where $\mathcal{L}_0$ , $\mathcal{L}_1$ , $\mathcal{L}_2$ are defined in (4.11), (4.12) and (4.13) respectively. The operator $\hat{\mathcal{L}}_2$ due to process $I_t$, is defined as

$$\hat{\mathcal{L}}_2 = \mathcal{L}_2 + x\frac{\partial}{\partial I}. \tag{4.100}$$

Thus, by following the singular pricing technique discussed above, where $\mathcal{L}_2$ is replaced by $\hat{\mathcal{L}}_2$, the corrected price of the Asian average-strike option is given by

$$\tilde{P}(t,x,I) = P_0(t,x,I) + \tilde{P}_1(t,x,I), \tag{4.101}$$

where $P_0$ solves the problem with terminal condition

$$\begin{cases} \langle\hat{\mathcal{L}}_2\rangle P_0 = 0, \\ P_0(T,x,I) = \left[x - \frac{I}{T}\right]^+. \end{cases} \tag{4.102}$$

Note that $P_0$ is the Black-Scholes Asian price[14] with constant volatility $\bar{\sigma}$. The correction $\tilde{P}_1(t,x,I)$ solves the problem with terminal condition equal to zero

$$\begin{cases} \langle\hat{\mathcal{L}}_2\rangle \tilde{P}_1 = \hat{\mathcal{A}}P_0, \\ \tilde{P}_1(T,x,I) = 0, \end{cases} \tag{4.103}$$

where the source term is expressed as

$$\hat{\mathcal{A}}P_0 = \sqrt{\varepsilon}\langle\mathcal{L}_1\mathcal{L}_0^{-1}\left[\hat{\mathcal{L}}_2 - \langle\hat{\mathcal{L}}_2\rangle\right]\rangle P_0. \tag{4.104}$$

Note that the additive term in $\hat{\mathcal{L}}_2$ is independent of $y$, the current level of volatility which implies

$$\hat{\mathcal{L}}_2 - \langle\hat{\mathcal{L}}_2\rangle = \mathcal{L}_2 - \langle\mathcal{L}_2\rangle \tag{4.105}$$

and thus, $\hat{\mathcal{A}} = \mathcal{A}$. Therefore $\tilde{P}_1(t,x,I)$ solves the problem

$$\begin{cases} \left[\mathcal{L}_{BS}(\bar{\sigma}) + x\frac{\partial}{\partial I}\right] \tilde{P}_1(t,x,I) = \mathcal{A}P_0(t,x,I), \\ \tilde{P}_1(T,x,I) = 0, \end{cases} \tag{4.106}$$

where $\mathcal{A}$ is defined in (4.55). Equation (4.106) can be solved using numerical schemes, [43].

[14] This price is usually solved numerically, see [118].

## 4.9 Accuracy of Approximation

The purpose here is to determine the order of the error arising due to approximating the solution up to the first two leading terms in the asymptotic expansion, that is, $P_0 + \sqrt{\varepsilon}P_1$.

### 4.9.1 Regularization of the Payoff function

The payoff $h(X_T)$ defined as

$$h(X_T) := \max\{X_T - K, 0\},$$

is a continuous function. If $X := \log S$, such that

$$h(S_T) = \max\left\{e^{S_T} - K, 0\right\},$$

then the payoff function has a discontinuous derivative with respect to stock price $S$ at maturity when $S \approx \log K$ (that is $X \approx K-$ *at-the-money*). It requires a smooth and bounded payoff function to analyse the error.

Consider a small time to maturity of order of a very small parameter, $\zeta$, and denote the regularized price from equation (4.14), by $P^{\varepsilon,\zeta}$ and its regularized first-order correction from equation (4.57), by $P^{\zeta}$. Then, the regularized problem takes the form

$$\mathcal{L}^{\varepsilon} P^{\varepsilon,\zeta} = 0, \tag{4.107}$$

where the operator $\mathcal{L}^{\varepsilon}$ is defined as

$$\mathcal{L}^{\varepsilon} := \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

with the regularized payoff, $h^{\zeta}(X_T)$ as the terminal condition. In particular, $h^{\zeta}(X_T)$ is taken to be the Black-Scholes payoff at time $T - \zeta$

$$h^{\zeta}(X_T) = C_{BS}(T - \zeta, x; K, T; \bar{\sigma}). \tag{4.108}$$

Thus, unlike $h(X_T)$, $h^{\zeta}(X_T)$ is a smooth function of $\mathcal{C}^{\infty}$-class, for $0 < \zeta \ll 1$. Consequently, the regularized first-order correction to problem (4.107) is given as,

$$P^{\zeta} = P_0^{\zeta} + p_1^{\zeta}, \tag{4.109}$$

where $P_0^\zeta$ and $p_1^\zeta$ are defined as

$$P_0^\zeta = C_{BS}(t-\zeta, x; K, T; \bar{\sigma}) \quad \text{and,} \tag{4.110}$$

$$p_1^\zeta = -[T-t]\left[V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3}\right] P_0^\zeta. \tag{4.111}$$

### 4.9.2 Accuracy of the Approximation

To determine the magnitude of the error, $|P^\varepsilon - P|$, it requires to compute first, $|P^\varepsilon - P^{\varepsilon,\zeta}|$, $|P^{\varepsilon,\zeta} - P^\zeta|$ and $|P^\zeta - P|$ such that one can easily obtain $|P^\varepsilon - P|$ from the inequality

$$|P^\varepsilon - P| \leq |P^\varepsilon - P^{\varepsilon,\zeta}| + |P^{\varepsilon,\zeta} - P^\zeta| + |P^\zeta - P|, \tag{4.112}$$

with a requirement that $P^\varepsilon \approx P^{\varepsilon,\zeta}$, $P^{\varepsilon,\zeta} \approx P^\zeta$ and $P^\zeta \approx P$.

**Lemma 4.9.1.** *Suppose a fixed point $(t, x, y)$ with $t \leq T$, then there exist small parameters $\bar{\zeta}_1 > 0$, $\bar{\varepsilon}_1 > 0$ and a constant $c_1^* > 0$ which might depend on $t, T, x$ and $y$ such that,*

$$|P^\varepsilon(t, x, y) - P^{\varepsilon,\zeta}(t, x, y)| \leq c_1^* \zeta, \tag{4.113}$$

*for all $0 < \zeta < \bar{\zeta}_1$ and $0 < \varepsilon < \bar{\varepsilon}_1$.*

The proof of equation (4.113) can easily be developed from the concept of risk neutral valuation, see Appendix C, Section C.2.

**Lemma 4.9.2.** *Suppose a fixed point $(t, x, y)$ with $t \leq T$, then there exist small parameters $\bar{\zeta}_2 > 0$, $\bar{\varepsilon}_2 > 0$ and a constant $c_2^* > 0$ which might depend on $t, T, x$ and $y$ such that,*

$$|P(t, x) - P^\zeta(t, x)| \leq c_2^* \zeta. \tag{4.114}$$

The proof is given in Appendix C, Section C.3

**Lemma 4.9.3.** *Suppose a fixed point $(t, x, y)$ with $t \leq T$, then there exist small parameters $\bar{\zeta}_3 > 0$, $\bar{\varepsilon}_3 > 0$ and a constant $c_3^* > 0$ which might depend on $t, T, x$ and $y$ such that,*

$$|P^{\varepsilon,\zeta}(t, x) - P^\zeta(t, x)| \leq c_3^* \left[\varepsilon |\log \zeta| + \varepsilon \sqrt{\frac{\varepsilon}{\zeta}} + \varepsilon\right].$$

The proof can be found in Appendix C, Section C.4. Choosing small parameters $\zeta$ and $\varepsilon$

defined[15] as $\zeta = \min\{\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3\}$ and $\varepsilon = \min\{\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3\}$, then from equation (4.112),

$$\begin{aligned}|P^{\varepsilon} - P| &\leq |P^{\varepsilon} - P^{\varepsilon,\zeta}| + |P^{\varepsilon,\zeta} - P^{\zeta}| + |P^{\zeta} - P|. \\ &\leq [c_1^* + c_2^*]\zeta + c_3^*\left[\varepsilon|\log\zeta| + \varepsilon\sqrt{\frac{\varepsilon}{\zeta}} + \varepsilon\right]. \\ &\leq 2\max\{c_1^*, c_2^*\}\,\zeta + c_3^*\left[\varepsilon|\log\zeta| + \varepsilon\sqrt{\frac{\varepsilon}{\zeta}} + \varepsilon\right].\end{aligned}$$

Now, if $\zeta \approx \varepsilon$, it can be deduced that

$$|P^{\varepsilon} - P| \leq c_4^*\left[\varepsilon + \varepsilon|\log\varepsilon|\right],$$

for some constant $c_4^* > 0$. Hence, at a fixed point $t < T$ and $x, y \in \mathbb{R}$, the accuracy of the approximation of call prices is given by

$$\lim_{\varepsilon\downarrow 0}\frac{|P^{\varepsilon}(t,x,y) - P(t,x)|}{\varepsilon|\log\varepsilon|^{1+l}} = 0, \tag{4.115}$$

for any $l > 0$. Therefore, the error generated due to first-order approximation is of order $\varepsilon$.

## 4.10 Applications of Asymptotic Pricing

In this section, some applications of the asymptotic expansion technique are discussed.

### 4.10.1 Pricing a Perpetual American Put option

Standard financial options are expressed in terms of pre-determined maturity. Their life time ranges from a few days to several years. They are only exercised at a pre-determined date. On the other hand, perpetual options have no fixed period for exercise, the investor can exercise at any time. This section derives the price of an American perpetual put under a stochastic volatility framework. The key idea is to find an optimal underlying price that would suggest an optimal value of the option.

#### Pricing Under Constant Volatility

The stock price dynamics under measure $\mathbb{P}^*$ is considered to follow the SDE

$$dX_t = rX_t\,dt + \sigma X_t\,dW_t\ ;\quad X_0 = x, \tag{4.116}$$

[15] Refer to the proofs in Appendix C for the significance of $(\bar{\zeta}_1, \bar{\zeta}_2, \bar{\zeta}_3)$ and $(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_3)$.

where $\sigma$ is a constant parameter and $r$ is the risk-free rate of return. Then the price $P_A$ of an American put option is given as [16]

$$P_A = \sup_{\tau} \mathbb{E}^* \left\{ e^{-r\tau} \left[K - X_\tau\right]^+ \right\}, \tag{4.117}$$

an optimal stopping problem with the supremum taken over all finite stopping times.

There exists an optimal underlying asset price $\hat{x} < K$ upon which exercising the option gives its maximum value. Thus, the domain of $x$ can be divided into two parts; the continuation sub-domain where $x > \hat{x}$ and the exercise sub-domain, $x \leq \hat{x}$. In the latter the option owner can exercise any time most probably at the lowest value of $x \leq \hat{x}$ and for the former, one waits until the asset price hits $\hat{x}$. If $\hat{x}$ is never hit, then the option would never be exercised.

At this juncture, note that the dependence on time in determining the value of the option does not really matter. Thus, if $V(x)$ is considered as the solution to the optimal problem (4.117), then in the domain $x > \hat{x}$, $V(x)$ satisfies the time-independent Black-Scholes PDE[17]

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2V}{dx^2} + r\left[x\frac{dV}{dx} - V\right] = 0. \tag{4.118}$$

However, in the exercise region (i.e. $x \leq \hat{x}$), the value of the option is given by,

$$V(x) = [K - x]^+. \tag{4.119}$$

In summary,

$$\begin{aligned} V(x) &\geq [K-x]^+; \text{ for all } x \geq 0. \\ \frac{1}{2}\sigma^2 x^2 V'' + r[xV' - V] &= 0; \text{ for } x \in (\hat{x}, \infty). \\ V(x) &= [K-x]^+; \text{ for } x \in (0, \hat{x}]. \\ V' &= -1\,; \text{ for } x = \hat{x}. \end{aligned}$$

In addition, $V(x) \in \mathcal{C}^2((0,\infty)\backslash\{\hat{x}\})$ and $\mathcal{C}^1$ everywhere. According to [84], if $V(x)$ satisfies all the above conditions, then it is equal to the price, $P_A$, of the perpetual American put option. Applying the continuity and smooth pasting conditions to the ODE (4.118), at $\hat{x}$:

$$V(\hat{x}) = K - \hat{x} \quad \text{and} \quad V'(x) = -1, \tag{4.120}$$

[16]Since immediate exercise (i.e. at $\tau = 0$) in this case is possible, the price of a perpetual American put is atleast equal to its current payoff.

[17]Note in this case that this PDE reduces to an ODE.

gives the solution $V(x)$ and hence, the value $P_A$ under constant volatility as

$$P_A(x) = \begin{cases} [K-\hat{x}]\,[\hat{x}/x]^{2\gamma} & \text{for } x > \hat{x}, \\ K - x & \text{for } x \le \hat{x} \end{cases} \tag{4.121}$$

where $\gamma = r/\sigma^2$ and $\hat{x} = 2K\gamma/[1+2\gamma]$. The optimal stopping time $\tau_{\hat{x}}$ or the time when $x$ first hits $\hat{x}$ is given by

$$\tau_{\hat{x}} = \inf\{\tau \ge 0; X_\tau \le \hat{x}\}. \tag{4.122}$$

**Pricing Under Stochastic Volatility**

Consider the OU model under the equivalent martingale measure $\mathbb{P}^*$:

$$\begin{cases} dX_t = rX_t\,dt + f(Y_t)X_t\,dW_t^{*(1)}, \\ dY_t = [\alpha[m-Y_t] - \beta\wedge(Y_t)]\,dt + \beta\,dW_t^{*(2)}, \\ W_t^{*(2)} = \rho W_t^{*(1)} + \sqrt{1-\rho^2}W_t^{*(3)}, \\ \langle W^{*(1)}, W^{*(2)}\rangle_t = \rho t, \end{cases}$$

then the price of a perpetual American option is given by the optimal problem

$$P_A(x,y) = \sup_\tau \mathbb{E}^*\left\{e^{-r\tau}[K-X_\tau]^+ | X_t = x, Y_t = y\right\}, \tag{4.123}$$

where the supremum is taken over all stopping times. The solution $P_A(x,y)$ satisfies the free-boundary problem:

$$\begin{cases} \mathcal{L}P_A(x,y) = 0 & \text{for } x > \hat{x}(y). \\ P_A(x,y) = K - x & \text{for } x \le \hat{x}(y), \end{cases}$$

with the following conditions:

$$\begin{aligned} &P_A(\hat{x}(y),y) = K - \hat{x} \ \text{: continuity at } \hat{x}. \\ &\frac{\partial}{\partial x}P_A(\hat{x}(y),y) = -1. \\ &\frac{\partial}{\partial y}P_A(\hat{x}(y),y) = 0. \end{aligned} \tag{4.124}$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L} = \frac{1}{2}f^2(y)x^2\frac{\partial}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2}{\partial x\partial y} + \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + r\left[x\frac{\partial}{\partial x} - \right] + [\alpha[m-y] - \beta\wedge(y)]\frac{\partial}{\partial y}.$$

The derivatives in equations (4.124) emphasis the smooth pasting condition (Continuity at $\hat{x}$) in the case where $x \leq \hat{x}$. The problem is now well-defined, using the method of asymptotics, one can obtain an approximation to the exact solution. To introduce perturbation in the problem, let $\alpha = 1/\varepsilon$ and $\beta = \sqrt{2}\nu/\sqrt{\varepsilon}$, then

$$\mathcal{L}^{\varepsilon} P_A^{\varepsilon}(x,y) = \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right] P_A^{\varepsilon}(x,y) = 0, \tag{4.125}$$

where the differential operators; $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined as

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} \cdot + [m - y] \frac{\partial}{\partial y} \cdot, \tag{4.126}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu x f(y) \frac{\partial^2}{\partial x \partial y} \cdot - \sqrt{2}\nu \wedge (y) \frac{\partial}{\partial y} \cdot, \tag{4.127}$$

$$\mathcal{L}_2 = \frac{1}{2} f^2(y) x^2 \frac{\partial^2}{\partial x^2} \cdot + r\left[x \frac{\partial}{\partial x} \cdot - \cdot\right]. \tag{4.128}$$

Note, $\mathcal{L}_2$ denotes time-independent Black-Scholes PDE with constant volatility level, $f(y)$.

### Asymptotic Expansions

For simplicity, the price $P_A$ of the perpetual American put option shall be denoted by $P$ and the unknown boundary by $\hat{x}$. Then, from asymptotic analysis, $P$ and $\hat{x}$ are expanded as

$$P^{\varepsilon}(x,y) = P_0(x,y) + \sqrt{\varepsilon} P_1(x,y) + \varepsilon P_2(x,y) + \cdots \tag{4.129}$$

$$\hat{x}^{\varepsilon}(y) = \hat{x}_0(y) + \sqrt{\varepsilon}\hat{x}_1(y) + \varepsilon \hat{x}_2(y) + \cdots \tag{4.130}$$

Substituting these equations in (4.125) and grouping similar order terms in $\varepsilon$, gives

$$\begin{aligned} &\frac{1}{\varepsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\varepsilon}}\left[\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0\right] + \left[\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0\right] + \\ &\sqrt{\varepsilon}\left[\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1\right] + \cdots = 0. \end{aligned} \tag{4.131}$$

with the following conditions[18]:

$$
\begin{aligned}
&P_0(x_0(y),y)+\sqrt{\varepsilon}\left[\hat{x}_1(y)\frac{\partial}{\partial x}P_0(\hat{x}_0(y),y)+P_1(\hat{x}_0(y),y)\right]+\cdots=\\
&K-\hat{x}_0(y)-\sqrt{\varepsilon}\hat{x}_1(y)-\cdots,\\
&\frac{\partial}{\partial x}P_0(\hat{x}_0(y),y)+\sqrt{\varepsilon}\left[\hat{x}_1(y)\frac{\partial^2}{\partial x^2}P_0(\hat{x}_0(y),y)+\frac{\partial}{\partial x}P_1(\hat{x}_0(y),y)\right]+\cdots=-1,\\
&\frac{\partial}{\partial y}P_0(\hat{x}_0(y),y)+\sqrt{\varepsilon}\left[\hat{x}_1(y)\frac{\partial^2}{\partial x\partial y}P_0(\hat{x}_0(y),y)+\frac{\partial}{\partial y}P_1(\hat{x}_0(y),y)\right]+\cdots=0.
\end{aligned}
$$

The $P_n$'s, $n=0,1,2,3,\cdots$, from equation (4.131) can be obtained through iterations:

**Collecting terms of $\mathcal{O}(1/\varepsilon)$**

The problem that corresponds to this order satisfies

$$
\begin{aligned}
&\mathcal{L}_0P_0(x,y)=0\,; \quad \text{for } x>\hat{x}_0(y),\\
&P_0(x,y)=[K-x]^+\,; \quad \text{for } x<\hat{x}_0(y),\\
&P_0(\hat{x}_0(y),y)=[K-\hat{x}_0(y)]^+,\\
&\frac{\partial}{\partial x}P_0(\hat{x}_0(y),y)=-1.
\end{aligned}
\tag{4.132}
$$

Recall, $\mathcal{L}_0$ contains only derivatives with respect to $y$, so $P_0$ is independent of $y$ in the continuation region. Moreover, in the exercise region $P_0$ does not depend on $y$. Therefore, $P_0$ and $\hat{x}_0$ are independent of $y$ everywhere in the domain of $x$ and as a result, $\hat{x}_0=\hat{x}$ and $P_0=P_0(x)$.

**Collecting terms of $\mathcal{O}(1/\sqrt{\varepsilon})$**

The resulting problem takes the form

$$
\begin{aligned}
\mathcal{L}_0P_1(x,y)+\mathcal{L}_1P_0(x)&=0\,; \quad \text{for} \quad x>\hat{x},\\
P_1(x,y)&=0\,; \quad \text{for} \quad x<\hat{x},\\
P_1(\hat{x},y)&=0,\\
x_1(y)\frac{\partial^2}{\partial x^2}P_0(\hat{x})+\frac{\partial}{\partial x}P_1(\hat{x},y)&=0.
\end{aligned}
\tag{4.133}
$$

Equation (4.133) shows in the continuation region, $\mathcal{L}_0P_1(x,y)=0$, since $\mathcal{L}_1P_0(x)=0$. Using similar arguments for $P_0$, $P_1$ is also independent of $y$, so $P_1=P_1(x)$. In the exercise region, it is required that the contribution to the payoff by the terms $P_n$ ($n=1,2,3,\cdots$), be zero.

[18]These follow from expanding $P_0(\hat{x}_0(y)+\sqrt{\varepsilon}\hat{x}_1(y),y)$ at $\hat{x}_0$ using Taylor's expansion, with $y$ fixed.

**Collecting terms of $\mathcal{O}(1)$**

Grouping terms of $\mathcal{O}(1)$ gives

$$\begin{aligned} \mathcal{L}_0 P_2(x,y) + \mathcal{L}_2 P_0(x) = 0\,; &\quad \text{for} \quad x > \hat{x}, \\ P_2(x,y) = 0\,; &\quad \text{for} \quad x \leq \hat{x}, \end{aligned} \tag{4.134}$$

where $\mathcal{L}_1 P_1 = 0$. Equation (4.134) is a Poisson equation in $P_2$ given that $\mathcal{L}_2 P_0$ is known. This equation admits a reasonably growing solution at infinity if

$$\langle \mathcal{L}_2 P_0 \rangle = 0. \tag{4.135}$$

By definition of $\mathcal{L}_2$ and the fact that $P_0$ is independent of $y$, one can write

$$\langle \mathcal{L}_2 P_0 \rangle = \langle \mathcal{L}_2 \rangle P_0 = 0,$$

where $\langle \mathcal{L}_2 \rangle$ is simply,

$$\langle \mathcal{L}_2 \rangle = \frac{1}{2} \langle f^2(y) \rangle x^2 \frac{\partial^2}{\partial x^2} + r \left[ x \frac{\partial}{\partial x} - \cdot \right], \tag{4.136}$$

and $\langle f^2(y) \rangle = \bar{\sigma}^2$ is a constant volatility.

Define $\langle \mathcal{L}_2 \rangle := \mathcal{L}_C(\bar{\sigma})$ as Black-Scholes time-independent derivative operator with a constant volatility level, $\bar{\sigma}$. Then, one can determine the value of $P_0$ in the same way as pricing under constant volatility given that the function $f(y)$ is known. In this case, one is also able to determine the value of the optimal boundary level, $\hat{x}$ for exercising the option.

Thus, $P_0$ satisfies

$$\begin{aligned} &P_0 = [K - x]^+\,; \quad \text{for} \quad x \leq \hat{x}, \\ &\mathcal{L}_C(\bar{\sigma}) P_0 = 0\,; \quad \text{for} \quad x > \hat{x}, \\ &P_0(\hat{x}) = [K - \hat{x}]^+, \\ &\frac{\partial}{\partial x} P_0(\hat{x}) = -1, \end{aligned}$$

where the solution is given as

$$P_0(x) = \begin{cases} [K - \hat{x}] \left[ \frac{\hat{x}}{x} \right]^{2\gamma} & \text{for } x > \hat{x} \\ K - x & \text{for } x \leq \hat{x} \end{cases}, \tag{4.137}$$

It is interesting to note that $P_0$ corresponds to Black-Scholes price as it was shown in the main result. Next, is a collection of terms with order $\mathcal{O}(\sqrt{\varepsilon})$.

### Collecting terms of $\mathcal{O}(\sqrt{\varepsilon})$

This gives forth the following problem

$$\begin{aligned} \mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0\,; &\quad \text{for} \quad x > \hat{x}, \\ P_3 = 0\,; &\quad \text{for} \quad x \leq \hat{x}. \end{aligned} \tag{4.138}$$

Equation (4.138) is a Poisson equation in $P_3$ given that $[\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1]$ is known. Thus, its admittance of a solution necessitates the condition

$$\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0. \tag{4.139}$$

From equation (4.134),

$$P_2 = -\mathcal{L}_0^{-1} \mathcal{L}_2 P_0. \tag{4.140}$$

Since $\langle \mathcal{L}_2 P_0 \rangle = 0$, then equation (4.140) can also be written as

$$P_2 = -\mathcal{L}_0^{-1} \left( \mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \right) P_0. \tag{4.141}$$

Substituting for $P_2$ in equation (4.139) leads to

$$\langle \mathcal{L}_2 P_1 - \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0 \rangle = 0. \tag{4.142}$$

Recall, that $P_1$ is independent of $y$ so,

$$\langle \mathcal{L}_2 P_1 \rangle = \langle \mathcal{L}_2 \rangle P_1 = \mathcal{L}_C(\bar{\sigma}) P_1. \tag{4.143}$$

Thus, equation (4.142) can be expressed as

$$\mathcal{L}_C(\bar{\sigma}) P_1 = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0 \rangle. \tag{4.144}$$

### The First Corrected Price

In this section the price of the perpetual American put is derived as a first-order correction to the Black-Scholes.

Let $p_1 = \sqrt{\varepsilon} P_1$ and multiply $\sqrt{\varepsilon}$ through equation (4.144) to obtain,

$$\mathcal{L}_C(\bar{\sigma}) p_1 = \sqrt{\varepsilon} \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0 \rangle. \tag{4.145}$$

Now, using the definition of $\mathcal{L}_2$ from equation (4.128) one gets

$$\mathcal{L}_{\mathrm{C}}(\bar{\sigma})p_1 = \frac{\sqrt{\varepsilon}}{2}\langle \mathcal{L}_1\mathcal{L}_0^{-1}\left[f^2(y) - \bar{\sigma}^2\right]\rangle x\frac{\partial^2 P_0}{\partial x^2}. \tag{4.146}$$

Let $\mathcal{G}(y)$ be a function of $y$ that satisfies

$$\mathcal{L}_0\mathcal{G}(y) = f^2(y) - \bar{\sigma}^2, \tag{4.147}$$

then equation (4.146) can be written as,

$$\mathcal{L}_{\mathrm{C}}(\bar{\sigma})p_1 = \frac{\sqrt{\varepsilon}}{2}\mathcal{L}_1[\mathcal{G}(y) + k(t,x)]x\frac{\partial^2 P_0}{\partial x^2}, \tag{4.148}$$

with $k(t,x)$ as a constant independent of $y$, thus, $\mathcal{L}_1 k(t,x) = 0$.

Following the same procedure as that from equations (4.39) – (4.44), one notices that equation (4.148) yields to

$$\mathcal{L}_{\mathrm{C}}(\bar{\sigma})p_1 = v_3 x^3\frac{\partial^3 P_0}{\partial x^3} + v_2 x^2\frac{\partial^2 P_0}{\partial x^2}, \tag{4.149}$$

where $v_2$ and $v_3$ are defined as

$$v_2 = \frac{\nu}{\sqrt{2\alpha}}[2\rho\langle f(y)\mathcal{G}'(y)\rangle - \langle\wedge(y)\mathcal{G}'(y)\rangle].$$
$$v_3 = \frac{\rho\nu}{\sqrt{2\alpha}}\langle f(y)\mathcal{G}'(y)\rangle.$$

The parameters $v_2$ and $v_3$ can be easily calibrated to market data, where $\mathcal{G}'(y)$ is given as

$$\mathcal{G}'(y) = \frac{1}{\nu^2\phi(y)}\int_{-\infty}^{y}[f^2(w) - \langle f^2(w)\rangle]\phi(w)\,dw, \tag{4.150}$$

with $\phi(y)$ as the probability density distribution of the invariant distribution of $y$.

$P_0$ is the price to a perpetual American put option under constant volatility and thus, satisfies

$$P_0(x) = \begin{cases} [K-\hat{x}]\left[\frac{\hat{x}}{x}\right]^{2\gamma}\ ; & \text{for} \quad x > \hat{x}. \\ K - x\,; & \text{for} \quad x \le \hat{x}. \end{cases} \tag{4.151}$$

In the continuation region, $P_0(x)$ can also be rewritten as

$$P_0(x) = \left[\frac{1}{x}\right]^{2\gamma}[K-\hat{x}]\hat{x}^{2\gamma}, \tag{4.152}$$

from which the first, second and third derivatives with respect to $x$ are derived:

$$\frac{\partial P_0(x)}{\partial x} = -\frac{2\gamma}{x^{2\gamma+1}}[K-\hat{x}]\hat{x}^{2\gamma}. \tag{4.153}$$

$$\frac{\partial^2 P_0(x)}{\partial x^2} = \frac{2\gamma[2\gamma+1]}{x^{2\gamma+2}}[K-\hat{x}]\hat{x}^{2\gamma}. \tag{4.154}$$

$$\frac{\partial^3 P_0(x)}{\partial x^3} = -\frac{2\gamma[2\gamma+1][2\gamma+2]}{x^{2\gamma+3}}[K-\hat{x}]\hat{x}^{2\gamma}. \tag{4.155}$$

Substituting these derivatives in equation (4.149) gives

$$\mathcal{L}_C(\bar{\sigma})p_1 = \mathcal{V}x^{-2\gamma}. \tag{4.156}$$

where $\mathcal{V}$ is defined as

$$\mathcal{V} = [v_3[-2\gamma][-2\gamma-1][-2\gamma-2] + v_2[-2\gamma][-2\gamma-1]][K-\hat{x}]\hat{x}^{2\gamma}.$$

Therefore, the first corrected price of a perpetual American put option in this case is

$$P_A = P_0(x) + p_1(x), \tag{4.157}$$

where $p_1(x)$ satisfies:

$$\mathcal{L}_C(\bar{\sigma})p_1(x) = \mathcal{V}x^{-2\gamma}\;;\text{for}\quad x > \hat{x}. \tag{4.158}$$

$$p_1(x) = 0 \qquad ;\text{for}\quad x \leq \hat{x}. \tag{4.159}$$

### 4.10.2 Hedging under Stochastic Volatility

This section explains how traders can hedge themselves by constructing a portfolio of shares and a bank account, to replicate a particular derivative of their interest. Due to incompleteness of the market under stochastic volatility, it is almost impossible to establish a perfect hedging portfolio. There is a cost involved as discussed in the following. Consider the following hedging strategy, where $a_t$ and $b_t$ are respectively, the number of units of shares and bond required to construct the hedge portfolio:

$$\begin{cases} a_t(t,X_t) = \frac{\partial}{\partial x}P_0(t,X_t) \\ b_t(t,X_t) = e^{-rt}\left[P_0(t,X_t) - X_t\frac{\partial}{\partial x}P_0(t,X_t)\right] \end{cases}$$

where $P_0(t,X_t)$ denotes the solution to Black-Scholes PDE with volatility level $\langle f(y)\rangle$. The value of this portfolio is equal to $P_0(t,X_t)$ for all $t \leq T$. Holding this portfolio has a cost as mentioned before.

**Proposition 4.10.1.** *The value of the costs accumulated in hedging the target derivative is given by*

$$C_t^1 = \sqrt{\varepsilon}[B_t + M_t] + \mathcal{O}(\varepsilon),$$

*where $M_t$ is a martingale and*

$$B_t = -\frac{\rho\nu}{\sqrt{2}} \int_0^t \mathcal{G}' f(Y_s) \left[ 2X_s \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3} \right] ds, \tag{4.160}$$

*with $\mathbb{E}\{M_t\} = 0$ and $\mathbb{E}\{B_t\} \neq 0$.*

**Definition 4.10.2.** *The infinitesimal cost $dC_t$ of a hedging strategy is the difference between the variation of the derivative value and the variation of the value of the portfolio due to market variations:*

$$dC_t = dP_0(t, X_t) - [a_t(t, X_t)\, dX_t + rb_t(t, X_t)\, dt].$$

*The cumulative cost $C_t$ is the sum of the infinitesimal costs, $C_t^i$, $i = 1, 2, 3, \cdots$, over a period of time:*

$$C_t = \int_0^t C_s^i \, ds.$$

*Note that in order to maintain the variation $dC_t \approx 0$ at any time $t$, the investor has to add (remove) more money to (from) this portfolio.*

According to Itô, the variation in the option price is given as

$$dP_0(t, X_t) = \frac{\partial}{\partial x} P_0(t, x)\, dX_t + \left[ \frac{\partial}{\partial t} P_0(t, X_t) + \frac{1}{2} f^2(Y_t) X_t^2 \frac{\partial^2}{\partial x^2} P_0(t, X_t) \right] dt.$$

Assuming that the first cost of hedging is $C_t^1$ and using Definition 4.10.2, Then,

$$\begin{aligned} dC_t^1 &= dP_0(t, X_t) - [a_t(t, X_t)\, dX_t + rb_t(t, X_t)\, dt]. \\ &= \frac{\partial}{\partial x} P_0(t, x)\, dX_t + \left[ \frac{\partial}{\partial t} P_0(t, X_t) + \frac{1}{2} f^2(Y_t) X_t^2 \frac{\partial^2}{\partial x^2} P_0(t, X_t) \right] dt \\ &- \left[ \frac{\partial}{\partial x} P_0(t, X_t)\, dX_t + r \left[ P_0(t, X_t) - X_t \frac{\partial}{\partial x} P_0(t, X_t) \right] dt \right]. \end{aligned}$$

The last bracket can be rewritten using Black-Scholes PDE with constant volatility $\bar{\sigma}$ as,

$$\frac{\partial}{\partial x} P_0(t, X_t)\, dX_t + \left[ \frac{\partial}{\partial t} P_0(t, X_t) + \frac{1}{2} \bar{\sigma}^2 X_t^2 \frac{\partial^2}{\partial x^2} P_0(t, X_t) \right] dt.$$

Consequently,

$$dC_t^1 = \frac{1}{2} \left[ f^2(Y_t) - \bar{\sigma}^2 \right] X_t^2 \frac{\partial^2}{\partial x^2} P_0(t, X_t)\, dt.$$

### The hedging Process

Suppose in writing a derivative at time $t = 0$ the trader receives an amount $P^{\varepsilon} = P_0 + \sqrt{\varepsilon}P_1 + \varepsilon P_2$, where $P_1$ and $P_2$ can either be positive or negative. Then, invests $P_0 = a_0 + b_0$ in the portfolio in proportions say, $a_0$ in the risky asset and $b_0$ in bonds and borrows (lends) a sum $\sqrt{\varepsilon}P_1 + \varepsilon P_2$ from (to) the bank. To maintain this portfolio with $a_t$ in the risk asset and $b_t$ in bonds between, $t = 0$ and a time $t$, the trader would have to spend or receive an amount

$$C_t^1 = \frac{1}{2}\int_0^t \left[f^2(Y_s) - \bar{\sigma}^2\right] X_s^2 \frac{\partial^2}{\partial x^2} P_0(s, X_s)\, ds, \tag{4.161}$$

that is, he spends $C_t$ for $C_t > 0$ or receives the same amount if $C_t < 0$.

**Remark 4.10.3.** *Equation (4.161) shows that the cost of maintaining a hedging portfolio will be as small as possible if the actual volatility $f(Y_t)$ happens to be so close to the average volatility $\bar{\sigma}$. Moreover, if the difference, $[f^2(Y_t) - \bar{\sigma}^2]$ is negligible for all times $t$, then the portfolio is self-financing since $C_t^1 \approx 0$, i.e. no money is added or removed from this portfolio at any time $t$.*

**Remark 4.10.4.** *The cost of maintaining a hedging portfolio will be smaller as the convexity or positiveness of the second derivative of $P_0(t, X_t)$ is closer to zero. This is certainly clear from equation (4.161).*

The next section gives the analysis of the hedging strategy under fast mean-reversion.

### The Averaging Effect

This section is devoted to the analysis of the hedging strategy when the volatility process $f(Y_t)$ fluctuates rapidly about its long-run mean value. Under fast mean-reversion, $\langle f(Y_t)\rangle \approx \bar{\sigma}$. This implies that

$$\frac{1}{2}\int_0^t \langle f(Y_s)\rangle X_s^2 \frac{\partial^2}{\partial x^2} P_0(s, X_s)\, ds \approx \frac{1}{2}\bar{\sigma}^2 \int_0^t X_s^2 \frac{\partial^2}{\partial x^2} P_0(s, X_s)\, ds.$$

Comparing this with equation (4.161) one can notice that under fast mean-reversion the cumulative costs of the hedging portfolio get reduced.

### Second Order Hedging Costs

To have a better understanding of the hedging cost, $C_t$, consider its second order dynamics. Recall from (4.31) where the function $\mathcal{G}(Y_t)$ is assumed to satisfy the Poisson equation

$$\mathcal{L}_0 \mathcal{G}(Y_t) = f^2(Y_t) - \bar{\sigma}^2, \tag{4.162}$$

where $\mathcal{L}_0$ is the infinitesimal generator of the OU-process. Using Itô's formula,

$$d\mathcal{G}(Y_t) = \mathcal{G}'(Y_t)\,dY_t + \frac{1}{2}\mathcal{G}''(Y_t)d\langle Y\rangle_t.$$

According to the dynamics of the process $(Y_t)_{t\geq 0}$, it follows that,

$$d\mathcal{G}(Y_t) = \mathcal{G}'(Y_t)\left[\frac{1}{\varepsilon}[m - Y_t]\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}dW_t^{(2)}\right] + \frac{1}{2}\left[\frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\right]^2 \mathcal{G}''(Y_t)\,dt.$$

Applying the definition of $\mathcal{L}_0$, leads to

$$d\mathcal{G}(Y_t) = \frac{1}{\varepsilon}\mathcal{L}_0\mathcal{G}(Y_t)\,dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}'(Y_t)\,dW_t^{(2)}. \tag{4.163}$$

Alternatively,

$$\mathcal{L}_0\mathcal{G}(Y_t)\,dt = \varepsilon\left[d\mathcal{G}(Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}'(Y_t)\,dW_t^{(2)}\right]. \tag{4.164}$$

Comparing equations (4.162) and (4.164) implies that the cost $C_t^1$ from equation (4.161) is

$$\begin{aligned} C_t^1 &= \frac{\varepsilon}{2}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\left[d\mathcal{G} - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}'(Y_s)\,dW_s^{(2)}\right]. \\ &= \frac{\varepsilon}{2}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\,d\mathcal{G} - \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\mathcal{G}'\,dW_s^{(2)}. \\ &= \frac{\varepsilon}{2}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\,d\mathcal{G} + \sqrt{\varepsilon}M_t, \end{aligned} \tag{4.165}$$

where $M_t$ is a martingale with respect to the invariant distribution of $Y_t$, defined as

$$M_t = -\frac{\nu}{\sqrt{2}}\int_0^t X_s^2\frac{\partial^2 P_0}{\partial x^2}\mathcal{G}'\,dW_s^{(2)}. \tag{4.166}$$

It is clear that $\mathbb{E}\{M_t\} = 0$ with $M_0 = 0$ since $dW^{(2)}$ is a standard Brownian motion. It is also vital to mention that the expectation of $M_t$ being zero does not in any way make $C_t^1$ biased. To carry on with asymptotic expansion of the cost $C_t^1$, there is need to modify the first term on the *r.h.s* of equation (4.165), because $d\mathcal{G}$ is of $\mathcal{O}(\sqrt{1/\varepsilon})$. Using integration by parts,

$$d\left[X_t^2\frac{\partial^2 P_0}{\partial x^2}\mathcal{G}\right] = X_t^2\frac{\partial^2 P_0}{\partial x^2}\,d\mathcal{G} + \mathcal{G}\,d\left[X_t^2\frac{\partial^2 P_0}{\partial x^2}\right] + d\langle X^2\frac{\partial^2 P_0}{\partial x^2}, \mathcal{G}\rangle_t, \tag{4.167}$$

and applying Itô's formula gives

$$d\left[X_t^2\frac{\partial^2 P_0}{\partial x^2}\right] = \mathbb{A}_t + \left[2X_t^2\frac{\partial^2 P_0}{\partial x^2} + X_t^3\frac{\partial^3 P_0}{\partial x^3}\right]f(Y_t)\,dW_t^{(1)}. \tag{4.168}$$

where $\mathbb{A}$ has finite variation. The focus is on the martingale part, since $\mathbb{A}$ vanishes on computing the bracket of equation (4.167) so, there is no need of expanding $\mathbb{A}_t$.

Using equations (4.163) and (4.168) gives the bracket of equation (4.167) as,

$$d\langle X^2 \frac{\partial^2 P_0}{\partial x^2}, \mathcal{G}\rangle_t = \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}' \left[2X_t^2 \frac{\partial^2 P_0}{\partial x^2} + X_t^3 \frac{\partial^3 P_0}{\partial x^3}\right] f\,dt \tag{4.169}$$

Now, one can substitute equation (4.167) in equation (4.165) to obtain the hedging cost as,

$$\begin{aligned} C_t^1 = \frac{\varepsilon}{2}\int_0^t \Bigg[ d\left[X_s^2 \frac{\partial^2 P_0}{\partial x^2}\mathcal{G}\right] - \mathcal{G}\, d\left[X_s^2 \frac{\partial^2 P_0}{\partial x^2}\right] \\ - \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}' \left[2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right] f\,ds \Bigg] + \sqrt{\varepsilon} M_t. \end{aligned}$$

Alternatively,

$$\begin{aligned} C_t^1 =& \frac{\varepsilon}{2}\left[X_t^2 \frac{\partial^2 P_0}{\partial x^2}\mathcal{G}(Y_t) - X_0^2 \frac{\partial^2 P_0}{\partial x^2}\mathcal{G}(Y_0)\right] \\ &- \frac{\varepsilon}{2}\int_0^t \mathcal{G}\, d\left[X_s^2 \frac{\partial^2 P_0}{\partial x^2}\right] \\ &- \frac{\varepsilon}{2}\int_0^t \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}\mathcal{G}' f \left[2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right] ds \\ &+ \sqrt{\varepsilon} M_t. \end{aligned} \tag{4.170}$$

Note that the terms $X_t^2 \frac{\partial^2 P_0}{\partial x^2}$ and $X_t^3 \frac{\partial^3 P_0}{\partial x^3}$ are bounded and of order $\mathcal{O}(1)$ with respect to $\varepsilon$. Moreover, if $\langle \mathcal{G}' f\rangle \neq 0$, then the first and second terms are of order $\mathcal{O}(\varepsilon)$ and the third and fourth terms are of order $\mathcal{O}(\sqrt{\varepsilon})$. Thus, $C_t^1$ is of order $\mathcal{O}(\sqrt{\varepsilon})$ and can be expressed as,

$$C_t^1 = \sqrt{\varepsilon}[B_t + M_t] + \mathcal{O}(\varepsilon), \tag{4.171}$$

where $M_t$ is given by equation (4.166) and

$$B_t = -\frac{\rho\nu}{\sqrt{2}}\int_0^t \mathcal{G}' f \left[2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + X_s^3 \frac{\partial^3 P_0}{\partial x^3}\right] ds. \tag{4.172}$$

Observe that $\mathbb{E}(B_t) \neq 0$, but it can take on both negative and positive values which means that the self-financing strategy can not hold. The investor will always have to add (remove) money to (from) the portfolio in an attempt of hedging himself. To achieve a self-financing strategy (hedging at zero cost), the $B_t$-term has to vanish. For no correlation between stock and volatility shocks, $B_t$ is zero. Thus, if the stock price and its volatility are uncorrelated, then hedging under stochastic volatility is similar to that in Black-Scholes for order $\mathcal{O}(\varepsilon)$. The next section exploits one approach by which one can eliminate the $B_t$-term in equation (4.171), by shifting the bias to the next order.

**A Mean Self-financing Strategy**

According to [53], $B_t$ is eliminated by introducing on the *r.h.s* of equation (4.171), a quantity,

$$+\frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\int_0^t \langle \mathcal{G}'f\rangle\left[2X_s^2\frac{\partial^2 P_0}{\partial x^2}+X_s^3\frac{\partial^3 P_0}{\partial x^3}\right]ds. \tag{4.173}$$

This gives the value of the cumulative costs as,

$$C_t^{1^*}=\sqrt{\varepsilon}[B_t^*+M_t]+\mathcal{O}(\varepsilon), \tag{4.174}$$

where

$$B_t^*=\frac{\rho\nu}{\sqrt{2}}\int_0^t\left[\langle \mathcal{G}'f\rangle-\mathcal{G}'f\right]\left[2X_s^2\frac{\partial^2 P_0}{\partial x^2}+X_s^3\frac{\partial^3 P_0}{\partial x^3}\right]ds. \tag{4.175}$$

Taking expectation of the cumulative costs given by equation (4.174) generates a term of order[19] $\varepsilon = 1/\alpha$. Intuitively speaking, this implies that in a very fast mean-reverting economy, the cumulative costs are minimal. Consider a function $\tilde{p}_1(t,x)$ that satisfies the problem

$$\begin{cases}\mathcal{L}_{\mathrm{BS}}(\bar{\sigma})\tilde{p}_1=\frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\langle \mathcal{G}'f\rangle\left[2X_t^2\frac{\partial^2 P_0}{\partial x^2}+X_t^3\frac{\partial^3 P_0}{\partial x^3}\right],\\ \tilde{p}(T,X_T)=h(X_T)=0.\end{cases} \tag{4.176}$$

Applying Lemma 4.4.2 gives the solution, $\tilde{p}_1$ to equation (4.176) as,

$$\tilde{p}_1=-[T-t]\frac{\rho\nu}{\sqrt{2}}\hbar\left[2X_t\frac{\partial^2 P_0}{\partial x^2}+X_t^3\frac{\partial^3 P_0}{\partial x^3}\right]. \tag{4.177}$$

where $\hbar := \frac{1}{\sqrt{\varepsilon}}\langle \mathcal{G}'f\rangle$. If the price of the target derivative is assumed to be $(P_0+\tilde{p}_1)$, then the hedging strategy takes the form

$$\begin{cases}a_t(t,X_t)=\frac{\partial}{\partial x}[P_0+\tilde{p}_1],\\ b_t(t,X_t)=e^{-rt}\left[[P_0+\tilde{p}_1]-X_t\frac{\partial}{\partial x}[P_0+\tilde{p}_1]\right].\end{cases} \tag{4.178}$$

The value of this portfolio is equal to that of the target derivative at all times up to maturity, i.e. $a_tX_t+b_te^{rt}=[P_0+\tilde{p}_1]$ and thus, pays off a value $h(X_T)$ at maturity. However, the self-financing strategy does not hold which incur costs of maintaining the portfolio. Applying Itô's formula to $d[P_0+\tilde{p}_1]$, the infinitesimal return on the derivative, gives

$$\begin{aligned}d[P_0+\tilde{p}_1]=&\left[\frac{\partial}{\partial t}[P_0+\tilde{p}_1]+\mu X_t\frac{\partial}{\partial x}[P_0+\tilde{p}_1]+\frac{1}{2}f^2(Y_t)X_t^2\frac{\partial^2}{\partial x^2}[P_0+\tilde{p}_1]\right]dt\\ &+f(Y_t)X_t\frac{\partial}{\partial x}[P_0+\tilde{p}_1]\,dW_t.\end{aligned} \tag{4.179}$$

[19] Recall that $\alpha$ is the rate of mean reversion.

The infinitesimal return on the portfolio is given as,

$$a_t\,dX_t + rb_te^{rt}\,dt = [\mu X_t\,dt + f(Y_t)X_t\,dW_t]\frac{\partial}{\partial x}[P_0+\tilde{p}_1] \\ + r\left[[P_0+\tilde{p}_1] - X_t\frac{\partial}{\partial x}[P_0+\tilde{p}_1]\right]\,dt. \tag{4.180}$$

The infinitesimal cost is computed by taking the difference between (4.179) and (4.180):

$$\begin{aligned} dC_t^2 &= d[P_0+\tilde{p}_1] - [a_t\,dX_t + rb_te^{rt}\,dt]. \\ &= \left[\left[\frac{\partial}{\partial t} + \frac{1}{2}f^2(Y_t)X_t^2\frac{\partial^2}{\partial x^2} + r\left[X_t\frac{\partial}{\partial x} - \cdot\right]\right][P_0+\tilde{p}_1]\right]\,dt. \\ &= \mathcal{L}_{\mathrm{BS}}(\bar{\sigma})[P_0+\tilde{p}_1] + \frac{1}{2}[f^2(Y_t)-\bar{\sigma}^2]X_t^2\frac{\partial^2}{\partial x^2}[P_0+\tilde{p}_1]\,dt. \end{aligned}$$

Recall that Black-Scholes operator, $\mathcal{L}_{\mathrm{BS}}$ is linear and $P_0(t,X_t)$ satisfies $\mathcal{L}_{BS}(\bar{\sigma})(P_0)=0$, thus,

$$dC_t^2 = \mathcal{L}_{\mathrm{BS}}(\tilde{p}_1) + \frac{1}{2}[f^2(Y_t)-\bar{\sigma}^2]X_t^2\frac{\partial^2}{\partial x^2}[P_0+\tilde{p}_1]\,dt.$$

Using equation (4.178), it can be deduced that

$$dC_t^2 = \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\langle\mathcal{G}'f\rangle\left[2X_t^2\frac{\partial^2 P_0}{\partial x^2} + X_t^3\frac{\partial^3 P_0}{\partial x^3}\right] + \frac{1}{2}[f^2(Y_t)-\bar{\sigma}^2]X_t^2\frac{\partial^2}{\partial x^2}[P_0+\tilde{p}_1]\,dt.$$

Integrating this equation yields the cumulative costs, $C_t^2$ given as,

$$\begin{aligned} C_t^2 =& \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\langle\mathcal{G}'f\rangle\int_0^t\left[2X_s^2\frac{\partial^2 P_0}{\partial x^2} + X_s^3\frac{\partial^3 P_0}{\partial x^3}\right]\,ds + \frac{1}{2}\int_0^t\frac{1}{2}[f^2(Y_s)-\bar{\sigma}^2]X_s^2\frac{\partial^2 P_0}{\partial x^2}\,ds \\ & - \int_0^t[f^2(Y_s)-\bar{\sigma}^2]X_s^2\frac{\partial^2\tilde{p}_1}{\partial x^2}\,ds. \end{aligned}$$

Note that the second term is an integral that gives $C_t^1$ defined in equation (4.171). Therefore, substituting $C_t$ in the last equation yields,

$$\begin{aligned} C_t^2 =& \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}}\int_0^t[\langle\mathcal{G}'f\rangle - \mathcal{G}'f]\left[2X_s^2\frac{\partial^2 P_0}{\partial x^2} + X_s^3\frac{\partial^3 P_0}{\partial x^3}\right]\,ds + \frac{\varepsilon}{2}\left[X_t^2\frac{\partial P_0}{\partial x^2}\mathcal{G}(Y_t) - X_0^2\frac{\partial P_0}{\partial x^2}\mathcal{G}(Y_0)\right] \\ & - \frac{\varepsilon}{2}\int_0^t\mathcal{G}\,d\left[X_s^2\frac{\partial^2 P_0}{\partial x^2}\right]\,ds + \sqrt{\varepsilon}M_t - \frac{1}{2}\int_0^t[f^2(Y_s)-\bar{\sigma}^2]X_s^2\frac{\partial^2\tilde{p}_1}{\partial x}\,ds + \mathcal{O}(\varepsilon). \end{aligned}$$

Using the averaging effect, observe that $C_t^2$ is of $\mathcal{O}(\sqrt{\varepsilon})$ and the $B_t$-term has been eliminated:

$$C_t^2 = \sqrt{\varepsilon}M_t + \mathcal{O}(\varepsilon). \tag{4.181}$$

This result ensures a self-financing portfolio. Now the question is, can the bias be extended further, from order $\mathcal{O}(\sqrt{\varepsilon})$ to order $\mathcal{O}(\varepsilon)$, of course yes, by introducing another term say, $\tilde{p}_2$ in the replicating portfolio of (4.176). However, the challenge will be, how to eliminate:

(a) all $\mathcal{O}(\varepsilon)$ terms in resulting $C_t^3$, i.e. the second and third terms on the r.h.s of (4.181).

(b) the terms:

$$\int_0^t \frac{1}{2}[f^2(Y_s) - \bar{\sigma}^2]X_s^2 \frac{\partial^2 \tilde{p}_1}{\partial x^2}\, ds \quad \text{and} \quad \int_0^t \frac{1}{2}[f^2(Y_s) - \bar{\sigma}^2]X_s^2 \frac{\partial^2 \tilde{p}_2}{\partial x^2}\, ds,$$

depending on the order of $\tilde{p}_1$.

(c) the remaining terms as a result of trying to get rid of the $B_t$-term by introducing:

$$\frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}} \int_0^t \left[ [\langle \mathcal{G}' f \rangle - \mathcal{G}' f] \left[ 2X_s^2 \frac{\partial^2 P_0}{\partial x^2} + \frac{\partial^3 P_0}{\partial x^3} \right] \right] ds.$$

The above methods are efficient for only fast-mean reverting volatility models and prove not to work for the slow case. Using the method of *Monte Carlo simulations with antithetic variates*, [115] analysed results by [40] and [54] for fast-mean and non-fast mean reverting volatilities respectively, together with the classical Black-Scholes price. By comparing *difference rates*[20] for different *times-to-maturity* of both, *at-the-money* (ATM) and *out-of-the-money* (OTM) options, through numerical experiments, they showed that:

- difference rates for non-fast mean reverting volatility are much higher than those of the fast mean-reverting volatility.
- first order approximation is not reliable for non-fast mean reverting volatility but the converse is true.
- for a given time to maturity, the difference rates for the first order approximation prices increase with depth of OTM for a given time to maturity.
- first order approximation prices decrease in time to maturity.

For some particular maturities and OTM options, the first order approximation reflected relatively large errors. However, [115] improved the accuracy of the prices by approximating the price up to the $\varepsilon P_2(t,x,y)$-term in the asymptotic expansion. An explicit form of $P_2$ is derived in Appendix B.1. In the fast-mean reverting setting, the difference between first and second order approximations for near ATM options and options with longer maturities is almost negligible. The second order approximation improves the accuracy for long-maturity options or ATM options in the case of non-fast mean reverting volatility.

More applications of the perturbation pricing approach are given in details, in [43].These include pricing interest rates, bonds, Asian options and credit derivatives. The authors also employ the same technique to correct the Heston volatility model.

[20] The ratio of *the difference between analytic value and Monte Carlo value to Monte Carlo value*.

Market data analysis, see [41], shows that pricing with a single-factor volatility model is not accurate enough. There is need to introduce a slowly varying factor in the volatility model. It is observed through empirical results that modelling with multi-scale stochastic volatility generates a better fit to the observed implied volatility. In this case, both regular and singular perturbation techniques apply in the analysis of the model. The next section covers this.

## 4.11 Pricing with Multi-Scale Volatility

This section includes a new stochastic process[21] introduced by [43], that contributes to the dynamics of volatility in addition to the fast mean-reverting process discussed above.

In this case, the volatility is modelled as a positive and bounded function of two processes; $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$. The model of the stock price returns described in (4.4) under an equivalent martingale measure $\mathbb{P}^*$, takes the following form:

$$\begin{aligned}
dX_t &= rX_t\,dt + \sigma_t X_t\,dW_t^{*(1)}\\
\sigma_t &= f(Y_t, Z_t)\\
dY_t &= \left[\frac{1}{\varepsilon}[m - Y_t] - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\wedge(Y_t, Z_t)\right]dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\,dW_t^{*(2)}\\
dZ_t &= \left[\delta c(Z_t) - \sqrt{\delta}g(Z_t)\Gamma(Y_t, Z_t)\right]dt + \sqrt{\delta}g(Z_t)\,dW_t^{*(4)}
\end{aligned}$$

$$\begin{aligned}
d\langle W^{*(1)}, W^{*(2)}\rangle_t &= \rho_1\,dt\\
d\langle W^{*(1)}, W^{*(4)}\rangle_t &= \rho_2\,dt\\
d\langle W^{*(2)}, W^{*(4)}\rangle_t &= \rho_{24}\,dt
\end{aligned}$$

where $\varepsilon, \delta$ are small positive constants and the functions $c$ and $g$ satisfy Lipschitz continuity and growth conditions see [69], so that the diffusion process $(Z_t)_{t\geq 0}$ has a strong unique solution[22]. The parameters $\wedge$ and $\Gamma$ are the combined market prices of volatility risk. The processes $Y_t$ and $Z_t$ respectively, represent fast and slow fluctuation features of volatility[23].

The corresponding risk-neutral price of a European option is given by

$$P^{\varepsilon,\delta}(t,x,y,z) = \mathbb{E}^*\left\{e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y, Z_t = z\right\}.$$

where the dependence on the two small parameters $\varepsilon$ and $\delta$, is emphasized.

[21] This process is a persistent factor (slow) in mean-reversion.

[22] The correlations coefficients are such that $|\rho_1| < 1, \rho_2 < 1, \rho_{24} < 1$ and $1 + 2\rho_1\rho_2\rho_{24} - \rho_1^2 - \rho_2^2 - \rho_{24}^2 > 0$ in order to ensure positive definiteness of the covariance matrix of the three Brownian motion, see [43].

[23] Observe that $\varepsilon^{-1}$ and $\varepsilon^{-\frac{1}{2}}$ increase the rate of mean-reversion of $Y_t$ whereas $c$ and $g$ reduce that of $Z_t$.

### 4.11.1 The Pricing Equation

The Feyman-Kac formula described in Appendix A, Section A.5 gives the characterization of $P^{\varepsilon,\delta}$ as a solution of the following parabolic PDE with a final condition

$$\begin{cases} \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right] P^{\varepsilon,\delta} = 0 \\ P^{\varepsilon,\delta}(T,x,y,z) = h(x) \end{cases} \tag{4.182}$$

where

$$\begin{aligned} \mathcal{L}_0 &= [m-y]\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, & \mathcal{M}_1 &= -g\Gamma\frac{\partial}{\partial z} + \rho_2 g f x\frac{\partial^2}{\partial x \partial z}, \\ \mathcal{L}_1 &= \nu\sqrt{2}\left[\rho_1 f x\frac{\partial^2}{\partial x \partial y} - \wedge\frac{\partial}{\partial y}\right], & \mathcal{M}_2 &= c\frac{\partial}{\partial z} + \frac{1}{2}g\frac{\partial^2}{\partial z^2}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f^2x^2\frac{\partial^2}{\partial x^2} + r\left[x\frac{\partial}{\partial x} - \cdot\right], & \mathcal{M}_3 &= \nu\sqrt{2}\rho_{24}g\frac{\partial^2}{\partial y \partial z}. \end{aligned} \tag{4.183}$$

Note that $\mathcal{L}_2$ is the Black-Scholes operator, corresponding to volatility level $f(y,z)$, denoted by $\mathcal{L}_{BS}(f(y,z))$. Operator $\mathcal{M}_2$ is the infinitesimal generator of the slow volatility factor, $Z_t$.

### 4.11.2 Asymptotics

In this subsection, a formal derivation of the price approximation is given in the regime of small parameters, $\varepsilon$ and $\delta$. Using asymptotic expansion, the price, $P^{\varepsilon,\delta}$ can be expressed as

$$P^{\varepsilon,\delta} = \sum_{i\geq 0}\sum_{j\geq 0}\varepsilon^{\frac{i}{2}}\delta^{\frac{j}{2}}P_{i,j}. \tag{4.184}$$

It is convenient to expand $P^{\varepsilon,\delta}$ first with respect to $\delta$ and subsequently with respect to $\varepsilon$ though the converse yields the same result.

#### Expansion in the Slow-Scale

This subsection considers an expansion of the price in powers of $\sqrt{\delta}$. This results in a regularly perturbed problem

$$P^{\varepsilon,\delta} = P_0^{\varepsilon} + \sqrt{\delta}P_1^{\varepsilon} + \delta P_2^{\varepsilon} + \cdots . \tag{4.185}$$

Substituting equation (4.185) in (4.182) and collecting terms of order $\mathcal{O}(1)$ with respect to $\sqrt{\delta}$, leads to the following PDE with terminal condition

$$\begin{cases} \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right] P_0^\varepsilon = 0, \\ P_0^\varepsilon(T,x,y) = h(x). \end{cases} \tag{4.186}$$

Subsequently, collecting terms of order $\mathcal{O}(\sqrt{\delta})$ yields

$$\begin{cases} \left[\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right] P_1^\varepsilon = -\left[\mathcal{M}_1 + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3\right] P_0^\varepsilon, \\ P_1^\varepsilon(T,x,y) = 0. \end{cases} \tag{4.187}$$

The next step is to expand both $P_0^\varepsilon$ and $P_1^\varepsilon$ in powers of $\varepsilon$ to obtain an approximation for the price $P^{\varepsilon,\delta}$. Note that only first-order expansions of $P_0^\varepsilon$ and $P_1^\varepsilon$ will be considered here.

**Expansion in the Fast-Scale**

First, consider the expansion of $P_0^\varepsilon$ in powers of $\sqrt{\varepsilon}$:

$$P_0^\varepsilon = P_0 + \sqrt{\varepsilon}P_{1,0} + \varepsilon P_{2,0} + \varepsilon\sqrt{\varepsilon}P_{3,0} + \cdots . \tag{4.188}$$

For interests in first-order expansion, only explicit expressions for $P_0$ and $P_{1,0}$ will be derived. Substituting (4.188) in (4.186), and collecting order $\mathcal{O}(1/\varepsilon)$ and $\mathcal{O}(1/\sqrt{\varepsilon})$- terms, yields the following ODE's associated with the first two leading terms:

$$\mathcal{L}_0 P_0 = 0. \tag{4.189}$$

$$\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0. \tag{4.190}$$

According to the nature[24] of $\mathcal{L}_0$ and $\mathcal{L}_1$, it is deduced that $P_0 = P_0(t,x,z)$ and $P_{1,0} = P_{1,0}(t,x,z)$. Note the independence on the current value of volatility, $y$. Subsequently, collecting the $\mathcal{O}(1)$ gives the following Poisson equation in $P_{2,0}$:

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0, \tag{4.191}$$

where $\mathcal{L}_1 P_{1,0} = 0$. As discussed before, this problem admits a solution if $\langle \mathcal{L}_2 P_0 \rangle = 0$. This argument gives

$$P_{2,0} = -\mathcal{L}_0^{-1}[\mathcal{L}_2 - \langle L_2 \rangle]P_0. \tag{4.192}$$

[24] These operators contain derivatives with respect to only $y$.

Define the following problem to be satisfied by $P_0$:

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_0 = 0, \\ P_0(T,x,z) = h(x), \end{cases} \tag{4.193}$$

where

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\langle f(\cdot,z)\rangle x^2 \frac{\partial^2}{\partial x^2} + r\left[x\frac{\partial}{\partial x} - \cdot\right], \tag{4.194}$$

is the Black-Scholes operator with volatility level $\langle f(\cdot,z)\rangle = \bar{\sigma}^2(z)$ which depends on the slow volatility factor $z$,

**Remark 4.11.1.** *$P_0(t,x,z) = P_0(t,x;\bar{\sigma})$ is the Black-Scholes price at the volatility level $\bar{\sigma}(z)$.*

Next, collecting terms of order $\mathcal{O}(\varepsilon)$ gives the following Poisson equation with respect to $P_{3,0}$,

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0. \tag{4.195}$$

which admits a reasonable solution if

$$\langle \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} \rangle = 0. \tag{4.196}$$

This argument leads to a problem that defines $P_{1,0}(t,x,z)$.

Define the following inhomogeneous problem to be satisfied by $P_{1,0}$:

$$\begin{cases} \langle \mathcal{L}_2 \rangle P_{1,0} = \mathcal{A} P_0, \\ P_{1,0}(T,x,z) = 0, \end{cases} \tag{4.197}$$

where $\mathcal{A}$ is given as

$$\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1}[\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle]\rangle. \tag{4.198}$$

Recall that the solution takes the form

$$P_{1,0} = -[T-t]\mathcal{A}P_0, \tag{4.199}$$

and that if $\mathcal{G}(y,z)$ is assumed to satisfy the following Poisson equation:
$\mathcal{L}_0 \mathcal{G}(y,z) = f^2(y,z) - \bar{\sigma}(z)$, then,

$$\mathcal{L}_0^{-1}[\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle] = \frac{1}{2}\mathcal{G}(y,z)x^2\frac{\partial^2}{\partial x^2}, \tag{4.200}$$

and thus, the operator $\mathcal{A}$ is explicitly given as

$$\mathcal{A} = \frac{\nu\rho_1}{\sqrt{2}}\langle f\frac{\partial \mathcal{G}}{\partial y}\rangle x\frac{\partial}{\partial x}\left[x^2\frac{\partial^2}{\partial x^2}\right] - \frac{\nu}{\sqrt{2}}\langle \wedge\frac{\partial \mathcal{G}}{\partial y}\rangle x^2\frac{\partial^2}{\partial x^2}. \tag{4.201}$$

Next, is the expansion of $P_1^\varepsilon$ in equation (4.185).

#### Expansion of $P_1^\varepsilon$

The expansion in terms of powers of $\sqrt{\varepsilon}$, takes the form

$$P_1^\varepsilon = P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1} + \varepsilon\sqrt{\varepsilon}P_{3,1} + \cdots . \tag{4.202}$$

The interest is only, in the explicit form of $P_{0,1}$ for first-order expansion. Substituting (4.202) in (4.187) and collecting terms of order $\mathcal{O}(1/\varepsilon)$, gives

$$\mathcal{L}_0 P_{0,1} = 0, \tag{4.203}$$

which implies, $P_{0,1}$ is independent of the current value of $y$, $P_{0,1} = P_{0,1}(t,x,z)$. Subsequently, collecting terms of order $\mathcal{O}(1/\sqrt{\varepsilon})$ gives

$$\mathcal{L}_0 P_{1,1} = 0, \tag{4.204}$$

$\mathcal{M}_3 P_0 = 0$ and $\mathcal{L}_1 P_{0,1} = 0$, since $\mathcal{M}_3$ and $\mathcal{L}_1$ contain derivatives with respect to only $y$.

Collecting terms of order $\mathcal{O}(1)$ and using $\mathcal{M}_3 P_{1,0} = \mathcal{L}_1 P_{1,1} = 0$, yields

$$\mathcal{L}_0 P_{2,1} + \mathcal{L}_2 P_{0,1} = -\mathcal{M}_1 P_0, \tag{4.205}$$

which is a Poisson equation in $y$ for $P_{2,1}$ with a condition $\langle \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0\rangle = 0$, to admit a solution. This yields to a problem that defines $P_{0,1}$. Define $P_{0,1}(t,x,z)$ to be the unique solution to the following problem:

$$\begin{cases} \langle \mathcal{L}_2\rangle P_{0,1} = -\langle \mathcal{M}_1\rangle P_0, \\ P_{0,1}(T,x,z) = 0. \end{cases} \tag{4.206}$$

**Proposition 4.11.2.** *The solution $P_{0,1}$ to equation (4.206) is given explicitly in terms of derivatives with respect to $x$ and $z$ of $P_0$:*

$$P_{0,1} = \frac{[T-t]}{2}\langle \mathcal{M}_1\rangle P_0. \tag{4.207}$$

Proposition 4.11.2 can be verified in the following way:

*Verification.* Observe first the relationship between the vega and the gamma:

$$\frac{\partial C_{BS}}{\partial \sigma} = [T-t]\sigma x^2 \frac{\partial^2 C_{BS}}{\partial x^2}, \tag{4.208}$$

where $C_{BS} = C_{BS}(t,x;\sigma)$ denotes the Black-Scholes price with respect to volatility $\sigma$. Thus, it can be deduced[25] that

$$\frac{\partial P_0}{\partial z} = [T-t]\bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2 P_0}{\partial x^2}. \tag{4.209}$$

where $\bar{\sigma}' = d\bar{\sigma}/dz$. Now, introducing the operator

$$\langle \mathcal{M}_1 \rangle = \left[ -g\langle \Gamma \rangle + \rho_2 g \langle f \rangle x \frac{\partial}{\partial x} \right] \frac{\partial}{\partial z} := M_1 \frac{\partial}{\partial z}, \tag{4.210}$$

it can easily be checked that $P_{0,1}$ given in equation (4.207) satisfies

$$\langle \mathcal{L}_2 \rangle P_{0,1} = \langle \mathcal{L}_2 \rangle \left[ \frac{[T-t]}{2} \left[ M_1 \frac{\partial P_0}{\partial z} \right] \right]. \tag{4.211}$$

Using the fact that the operator $\langle \mathcal{L}_2 \rangle$ commutes with $x^k \partial^k / \partial x^k$ and that $P_0$, satisfies $\langle \mathcal{L}_2 \rangle P_0 = 0$, then

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_{0,1} &= \langle \mathcal{L}_2 \rangle \left[ \frac{[T-t]^2}{2} M_1 \left[ \bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \right]. \\ &= -[T-t]M_1 \left[ \bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2 P_0}{\partial x^2} \right] \\ &\quad + \frac{[T-t]^2}{2} M_1 \bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2}{\partial x^2} \langle \mathcal{L}_2 \rangle P_0. \\ &= -[T-t]M_1 \left[ \bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2 P_0}{\partial x^2} \right]. \\ &= -\langle \mathcal{M}_1 \rangle P_0. \end{aligned} \tag{4.212}$$

This ends the verification. □

[25] This is obtained by using the chain rule:

$$\frac{\partial P_0}{\partial z} = \frac{\partial \bar{\sigma}(z)}{\partial z} \cdot \frac{\partial P_0}{\partial \bar{\sigma}(z)}.$$

### 4.11.3 First-Order Price Approximation

The first-order price approximation can be deduced from the expansions of $P^{\varepsilon,\delta}$, $P_0^{\varepsilon}$ and $P_1^{\varepsilon}$ given respectively, in (4.185), (4.188) and (4.202), as

$$\begin{aligned} P^{\varepsilon,\delta} \approx \tilde{P^{\varepsilon,\delta}} &:= P_0 + \sqrt{\varepsilon}P_{1,0} + \sqrt{\delta}P_{0,1}. \\ &= P_0 + [T-t]\left[-\sqrt{\varepsilon}\mathcal{A} + \frac{\sqrt{\delta}}{2}\langle\mathcal{M}_1\rangle\right]P_0, \end{aligned} \tag{4.213}$$

where $\mathcal{M}_1$ and $\mathcal{A}$ are respectively, defined in (4.183) and (4.201).

In this case, the group market parameters (which also depend on $z$) are given as

$$V_0^{\delta} = -\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle\wedge_s\rangle\bar{\sigma}', \qquad V_2^{\varepsilon} = \frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\langle\wedge_f\frac{\partial\mathcal{G}}{\partial y}\rangle,$$

$$V_1^{\delta} = \rho_2\frac{\nu_s\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\bar{\sigma}', \qquad V_3^{\varepsilon} = -\rho_1\frac{\nu_f\sqrt{\varepsilon}}{\sqrt{2}}\langle f\frac{\partial\mathcal{G}}{\partial y}\rangle. \tag{4.214}$$

where parameters $\wedge_s$ and $\wedge_f$ respectively, denote the market prices of volatility risk corresponding to the slow and the fast volatility factors[26]. Therefore, the components in the price approximation given in equation (4.213) take the form

$$-\frac{\sqrt{\delta}}{2}\langle\mathcal{M}\rangle P_0(t,x,z) = \frac{1}{\bar{\sigma}}\left[V_0^{\delta}\frac{\partial}{\partial\sigma} + V_1^{\delta}x\frac{\partial^2}{\partial x\partial\sigma}\right]P_0(t,x,z), \tag{4.215}$$

$$\sqrt{\varepsilon}\mathcal{A}P_0(t,x,z) = \left[V_2^{\varepsilon}x^2\frac{\partial^2}{\partial x^2} + V_3^{\varepsilon}x\frac{\partial}{\partial x}\left[x^2\frac{\partial^2}{\partial x^2}\right]\right]P_0(t,x,z). \tag{4.216}$$

Recall from Lemma 4.11.1 that $P_0 = P_0(t,x;\bar{\sigma}(z))$, the Black-Scholes price with volatility level, $\bar{\sigma}(z)$. Therefore, the first-order price approximation takes the form

$$\tilde{P^{\varepsilon,\delta}} = P_0 - [T-t]\left[\frac{1}{\bar{\sigma}}\left[V_0^{\delta}\frac{\partial}{\partial\sigma} + V_1^{\delta}x\frac{\partial^2}{\partial x\partial\sigma}\right] + \left[V_2^{\varepsilon}x^2\frac{\partial^2}{\partial x^2} + V_3^{\varepsilon}x\frac{\partial}{\partial x}\left[x^2\frac{\partial^2}{\partial x^2}\right]\right]\right]P_0.$$

Thus, the first-order correction depends on the parameters[27] $(\bar{\sigma}(z), V_0^{\delta}(z), V_1^{\delta}(z), V_2^{\varepsilon}(z), V_3^{\varepsilon}(z))$.

[26] Observe the relationship between group parameters discussed in Section 4.4 and those given in (4.214), i.e;

$$V_3^{\varepsilon} = V_3, \quad V_2^{\varepsilon} = V_2 - 2V_3.$$

[27] These parameters can be reduced to only four in total by defining $\sigma^*(z) := \sqrt{\bar{\sigma}^2(z) - 2V_2^{\varepsilon}}$, see [43] for details.

### 4.11.4 Accuracy of the Approximation

This subsection derives the error generated in approximating the price $P^{\varepsilon,\delta}$ with $\tilde{P^{\varepsilon,\delta}}$ when pricing with multi-scale volatility[28]. This error is summarised in the lemma below, by [41].

**Lemma 4.11.3.** *Given a smooth payoff $h(x)$, fix (t,x,y,z), then for any $\varepsilon \leq 1, \delta \leq 1$, there exists a constant $C > 0$ such that*

$$|P^{\varepsilon,\delta} - \tilde{P^{\varepsilon,\delta}}| \leq C[\varepsilon + \delta + \sqrt{\varepsilon\delta}]. \tag{4.217}$$

*In the case of call and put options, where the payoff is continuous but only piecewise smooth, the accuracy is given by*

$$|P^{\varepsilon,\delta} - \tilde{P^{\varepsilon,\delta}}| \leq C[\varepsilon|\log\varepsilon| + \delta + \sqrt{\varepsilon\delta}]. \tag{4.218}$$

The proof of Lemma 4.11.3 is given in Appendix C, Section C.5.

### 4.11.5 Implied Volatility

The first-order approximation for the implied volatility under multi-scale volatility, is derived in a similar way as in Section 4.6. By definition,

$$C_{BS}(t,x;T,K,I) = \tilde{P^{\varepsilon,\delta}}(t,x,z). \tag{4.219}$$

where $\tilde{P^{\varepsilon,\delta}}(t,x,z)$ is the model price for a call option and $C_{BS}$ is the Black-Scholes call option price with volatility $I$. The implied volatility $I$ can be expanded as

$$I = I_0 + I_1^\varepsilon + I_1^\delta + \cdots. \tag{4.220}$$

Using Taylor's expansion of $C_{BS}$ about $I_0$ and rewriting $\tilde{P^{\varepsilon,\delta}}(t,x,z)$ as

$$\tilde{P^{\varepsilon,\delta}}(t,x,z) = P_0 - \frac{1}{\bar{\sigma}}\left[\left[V_2^\varepsilon + V_3^\varepsilon x\frac{\partial}{\partial x}\right] + \tau\left[V_0^\delta + V_1^\delta x\frac{\partial}{\partial x}\right]\right]\frac{\partial}{\partial\sigma}P_0, \tag{4.221}$$

where $\tau = T - t$, gives

$$I_0 = \bar{\sigma}(z). \tag{4.222}$$

$$I_1^\varepsilon\frac{\partial}{\partial\sigma}C_{BS} = -\frac{1}{\bar{\sigma}}\left[V_2^\varepsilon + V_3^\varepsilon x\frac{\partial}{\partial x}\right]P_0. \tag{4.223}$$

$$I_1^\delta\frac{\partial}{\partial\sigma}C_{BS} = -\frac{\tau}{\bar{\sigma}}\left[V_0^\delta + V_1^\delta x\frac{\partial}{\partial x}\right]P_0. \tag{4.224}$$

[28] A smooth payoff is considered here. The case of a nonsmooth payoff can be found in [41]. The latter is more important since in particular, the European payoff is not smooth at the strike.

Using the substitution

$$\left[x\frac{\partial}{\partial x}\right]\frac{\partial}{\partial\sigma}C_{BS}=\left[1-\frac{\mathrm{d}_+}{\sigma\sqrt{\tau}}\right]\frac{\partial}{\partial\sigma}C_{BS}, \tag{4.225}$$

where $\mathrm{d}_+$ is defined in equation (1.43), gives

$$I_1^{\varepsilon}=-\frac{1}{\bar{\sigma}}\left[V_2^{\varepsilon}+V_3^{\varepsilon}\left[1-\frac{\mathrm{d}_+}{\bar{\sigma}\sqrt{\tau}}\right]\right]. \tag{4.226}$$

$$I_1^{\delta}=-\frac{\tau}{\bar{\sigma}}\left[V_0^{\delta}+V_1^{\delta}\left[1-\frac{\mathrm{d}_+}{\bar{\sigma}\sqrt{\tau}}\right]\right]. \tag{4.227}$$

Therefore, the $z$-dependent first-order approximation of the term structure of implied volatility is given by

$$I_0+I_1^{\varepsilon}+I_1^{\delta}=\bar{\sigma}+b^{\varepsilon}+a^{\varepsilon}\,\frac{\log[K/x]}{T-t}+a^{\delta}\,\log[K/x]+b^{\delta}\,[T-t], \tag{4.228}$$

where the parameters $\bar{\sigma},a^{\varepsilon},a^{\varepsilon},b^{\varepsilon}$ and $b^{\delta}$ depend on $z$ and are related to the group market parameters $(V_0^{\delta},V_1^{\delta},V_2^{\varepsilon},V_3^{\varepsilon})$ by

$$a^{\varepsilon}=-\frac{V_3^{\varepsilon}}{\bar{\sigma}^3},\qquad b^{\varepsilon}=-\frac{V_2^{\varepsilon}}{\bar{\sigma}}+\frac{V_3^{\varepsilon}}{\bar{\sigma}^3}[r-\bar{\sigma}^2/2],$$

$$a^{\delta}=-\frac{V_1^{\delta}}{\bar{\sigma}^3},\qquad b^{\delta}=-\frac{V_0^{\delta}}{\bar{\sigma}}+\frac{V_1^{\delta}}{\bar{\sigma}^3}[r-\bar{\sigma}^2/2]. \tag{4.229}$$

Equation (4.228) can be rewritten as a time-varying *log-moneyness-to-maturity-ratio* parametrization:

$$I\approx\bar{\sigma}+[a^{\varepsilon}+a^{\delta}\,[T-t]]\,\frac{\log[K/x]}{[T-t]}+[b^{\varepsilon}+b^{\delta}\,[T-t]]. \tag{4.230}$$

Estimation of the above parameters, (4.229), is done in a similar way to that explained in Subsection 4.7.1, through calibration of (4.228) to the observed term structure of implied volatility. Once the parameters are determined, then for pricing and hedging of derivatives, one only needs a set of parameters $(V_0^{\delta},V_1^{\delta},V_2^{\varepsilon},V_3^{\varepsilon})$ given as

$$V_0^{\delta}=-\bar{\sigma}\left[b^{\delta}+a^{\delta}\,[r-\bar{\sigma}^2/2]\right],\qquad V_1^{\delta}/\bar{\sigma}=-a^{\delta}\bar{\sigma}^2,$$

$$V_2^{\varepsilon}=-\bar{\sigma}\left[b^{\varepsilon}+a^{\varepsilon}\,[r-\bar{\sigma}^2/2]\right],\qquad V_3^{\varepsilon}=-a^{\varepsilon}\bar{\sigma}^3.$$

It has been confirmed empirically, that the two-scale volatility model with additional parameters performs better than either of the single-scale models. This can be checked by comparing results from [40, 41, 42]. The strength of the asymptotic approach is that the same set of parameters can be used to price path dependent contracts.

Asymptotic methods work well when pricing is done far from the expiry date. Pricing with a very short time to maturity may not permit sufficient time for enormous fluctuations about long-term mean. This is because the method involves averaging effects of the rapidly mean-reverting volatility-driving process. In addition, for options far OTM (i.e. $\log[K/x]$-large), the formula (4.89) is not reliable. In these situations, the value of *vega* (see equation (1.50)) becomes small yielding to a large correction value which is supposed to be small.

It has been shown [3] that by using classical Itô's calculus one can construct a decomposition formula that allows to establish first- and second-order option pricing approximation formulae that fit well the implied volatility, extremely easy to compute and permit easy accuracy analysis. These pricing formulae are suitable for options that are near to maturity as opposed to asymptotic methods. The following chapter exploits this approach in details.

# Chapter 5

# The Decomposition Pricing Approach

This chapter is devoted to deriving a decomposition pricing formula for option prices under fast mean-reverting volatility. The derivative price is expressed as a sum of the classical Black-Scholes formula with a root-mean-square future average volatility, plus a term due to correlation, and a term due to volatility of volatility. The approach hinges on two supports namely; integrability and regularity conditions on the volatility process. The method is valid for any stochastic volatility model that satisfies these conditions. The key idea is that the decomposition allows the derivation of the first- and second-order approximation formulae for option prices and implied volatilities. Moreover, the approach works well even for options near expiration as opposed to asymptotic methods of [40, 41].

Alòs [1, 2], employs Malliavin calculus to derive a decomposition formula for pricing option derivatives based on [59] or [10] models for uncorrelated volatility. The approach discussed here is based on [3], which does not require rigorous mathematical techniques from Malliavin calculus and, assumes a correlation between volatility and the stock price dynamics. This work corrects and improves some proofs presented in [3]. The results remain valid.

The Heston model discussed in Section 3.4 is a particular case under study. This model has a closed-form solution, but does not allow in general, for a fast calibration of parameters.

## 5.1 Mathematical Background

Throughout this chapter, $W^{*(2)}$ and $W^{*(3)}$, will be independent Brownian motions[1] defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with an EMM $\mathbb{P}^*$, and $\mathcal{F}_t := \mathcal{F}_t^{W^{*(2)}} \vee \mathcal{F}_t^{W^{*(3)}}$, where $\mathcal{F}_t^{W^{*(2)}}$ and $\mathcal{F}_t^{W^{*(3)}}$ respectively, correspond to the filtrations generated by $W^{*(2)}$ and $W^{*(3)}$.

Referring to the model given in Section 3.4, for less cumbersome computations, asset log-price dynamics will be considered, i.e. $S_t = \log X_t$ for $t \in [0, T]$. By Itô's formula

$$dS_t = [r - \sigma_t^2/2]dt + \sigma_t[\rho dW_t^{*(2)} + \sqrt{1-\rho^2}dW_t^{*(3)}], \tag{5.1}$$

where $r$ is the instantaneous interest rate[2] and

$$d\sigma_t^2 = \alpha[m - \sigma_t^2]\, dt + \beta\sigma_t\, dW_t^{*(2)}. \tag{5.2}$$

It is known that the value $P_t$ of a derivative with a payoff function $h(S_T)$ is given by

$$P_t = \mathbb{E}^* \left\{ e^{-r[T-t]} h(S_T) | \mathcal{F}_t \right\}. \tag{5.3}$$

Recall that if $\sigma_t = \sigma = constant$, $\rho = 0$, (5.3) leads to Black-Scholes call option pricing formula

$$C_{BS}(t, s; \sigma) = e^s N(\mathrm{d}_+) - K e^{-r[T-t]} N(\mathrm{d}_-), \tag{5.4}$$

where $N(\cdot)$ denotes the cumulative standard normal distribution and $\mathrm{d}_\pm$ defined as

$$\mathrm{d}_\pm = \frac{s - s^*}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t}. \tag{5.5}$$

Note that $s$ denotes the current log-stock price and $s^*$ is defined as

$$s^* := \log K - r[T-t]. \tag{5.6}$$

Black-Scholes differential operator in the log-variable with volatility level $\sigma$ is given as[3]

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial s^2} + \left[ r - \frac{1}{2}\sigma^2 \right] \frac{\partial}{\partial s} - r\cdot, \tag{5.7}$$

where $C_{BS}$ satisfies the equation $\mathcal{L}_{BS}(\sigma) C_{BS}(\cdot, \cdot; \sigma) = 0$.

[1]Note that $W^{*(1)}$ in Section 3.4 is related to $W^{*(2)}$ and $W^{*(3)}$ in equation (5.1) as

$$W_t^{*(1)} = \rho W_t^{*(2)} + \sqrt{1-\rho^2} W_t^{*(3)}.$$

[2]The instantaneous interest rate is considered constant in this case.

[3]This operator derives from the Delta hedging strategy of using a portfolio comprised of a contingent claim and delta shares.

In the case of stochastic volatility, the root mean-square of the time-averaged volatility, $v_t$ is

$$v_t := \left[\frac{1}{T-t}\int_t^T \mathbb{E}^*\left\{\sigma_\tau^2|\mathcal{F}_t\right\}d\tau\right]^{\frac{1}{2}}. \tag{5.8}$$

Also, define

$$L_t := \int_0^T \mathbb{E}^*\left\{\sigma_\tau^2|\mathcal{F}_t\right\}d\tau, \tag{5.9}$$

such that

$$v_t^2 = \frac{1}{T-t}\left[L_t - \int_0^t \sigma_\tau^2 d\tau\right], \tag{5.10}$$

and $dL_t$ is given as

$$dL_t = \beta\sigma_t\left[\int_t^T e^{-\alpha[\tau-t]}d\tau\right]dW_t^{*(2)}, \tag{5.11}$$

Define the centred Gaussian kernel $p(s,\epsilon)$ with variance $\epsilon^2$ for any $\epsilon > 0$, as

$$p(s,\epsilon) := \frac{1}{\epsilon\sqrt{2\pi}}\exp\left\{-s^2/2\epsilon^2\right\}. \tag{5.12}$$

Define $G(t,S_t;\sigma_t)$ as

$$G(t,S_t;\sigma_t) := [\partial_{ss}-\partial_s]C_{BS}(t,S_t;\sigma_t).$$

Before deriving the decomposition pricing formula for derivative prices, consider the following lemma similar to Lemma 2 in [5]:

**Lemma 5.1.1.** *Let $0 \leq t \leq \tau \leq T$, define $\mathcal{C}_t := \mathcal{F}_t \vee \mathcal{F}_T^{W^{*(2)}}$ and $G(\tau,S_\tau;v_\tau) := (\partial_{ss} - \partial_s)C_{BS}(\tau,S_\tau;v_\tau)$, then for every $n \geq 0$ there exists a $C = C(n,\rho)$ such that*

$$|\mathbb{E}^*\left\{\partial_s^n G(\tau,S_\tau;v_\tau)|\mathcal{C}_t\right\}| \leq C\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-\frac{1}{2}[n+1]}. \tag{5.13}$$

*Proof.* From Black-Scholes formula given in equation (5.4), obtain $\partial_s$ and $\partial_{ss}$ as

$$\begin{aligned}\partial_s C_{BS}(\tau,S_\tau,v_\tau) &= e^s N(\mathrm{d}_+)\\ \partial_{ss}C_{BS}(\tau,S_\tau,v_\tau) &= e^s N(\mathrm{d}_+) + e^s\partial_s N(\mathrm{d}_+)\\ &= e^s N(\mathrm{d}_+) + Ke^{-r[T-\tau]}N(\mathrm{d}_-).\end{aligned}$$

Thus,

$$G(\tau,S_\tau;v_\tau) = Ke^{-r[T-\tau]}\partial_s N(\mathrm{d}_-). \tag{5.14}$$

Equation (5.5) yields

$$\partial_s N(\mathrm{d}_-) = \frac{\partial N}{\partial \mathrm{d}_-} \cdot \frac{\partial \mathrm{d}_-}{\partial s} = \frac{1}{\sigma\sqrt{2\pi[T-\tau]}} \exp\left\{-\mathrm{d}_-^2/2\right\}. \tag{5.15}$$

This implies that $G(\tau, S_\tau; v_\tau)$ is given as

$$\begin{aligned} G(\tau, S_\tau; v_\tau) &= Ke^{-r[T-\tau]} \frac{1}{v_\tau\sqrt{2\pi[T-\tau]}} \exp\left\{-\mathrm{d}_-^2/2\right\}. \\ &= Ke^{-r[T-\tau]} \frac{1}{v_\tau\sqrt{2\pi[T-\tau]}} \exp\left\{-\frac{1}{2}\left[\frac{s-s^*}{v_\tau\sqrt{T-\tau}} - \frac{v_\tau}{2}\sqrt{T-\tau}\right]^2\right\}. \end{aligned}$$

Using equation (5.6), substitute for $s^*$ and rewrite the above equation as

$$G(\tau, S_\tau; v_\tau) = Ke^{-r[T-\tau]} \frac{1}{v_\tau\sqrt{2\pi[T-\tau]}} \exp\left\{-\frac{1}{2}\frac{s - \log K - [r - \frac{v_\tau^2}{2}][T-\tau]}{v_\tau^2[T-\tau]}\right\}^2.$$

By definition of the centred Gaussian kernel in equation (5.12), it follows,

$$G(\tau, S_\tau; v_\tau) = Ke^{-r[T-\tau]} p\left(S_\tau - \mu, v_\tau\sqrt{T-\tau}\right), \tag{5.16}$$

where $\mu := \log K - [r - v_\tau^2/2][T-\tau]$. Differentiating equation (5.16) up to the $n^{th}$ derivative and taking conditional expectation on both sides of the equation yields

$$\mathbb{E}^*\left\{\partial_s^n G(\tau, S_\tau, v_\tau)|\mathcal{C}_t\right\} = (-1)^n Ke^{-r[T-\tau]}\partial_\mu^n \mathbb{E}^*\left\{p\left(S_\tau - \mu, v_\tau\sqrt{T-\tau}\right)|\mathcal{C}_t\right\}.$$

Notice that one can express the log-price dynamics in equation (5.1) as

$$S_\tau = S_t + \int_t^\tau [r - \sigma_\theta^2/2]d\theta + \int_t^\tau \sigma_\theta\left[\rho dW_\theta^{*(2)} + \sqrt{1-\rho^2}dW_\theta^{*(3)}\right].$$

Since $W_t^{*(2)}$ is $\mathcal{C}_t$-adapted, $S_\tau$ is normally distributed with mean $\varphi$ and variance $\vartheta$ given as

$$\varphi = S_t + \int_t^\tau [r - \sigma_\theta^2/2]d\theta + \rho\int_t^\tau \sigma_\theta dW_\theta^{*(2)} \quad \text{and} \quad \vartheta = [1-\rho^2]\int_t^\tau \sigma_\theta^2 d\theta.$$

Therefore, by definition,

$$\mathbb{E}^*\left\{p\left(S_\tau - \mu, v_\tau\sqrt{T-\tau}\right)|\mathcal{C}_t\right\} = \int_{\mathbb{R}} p(w-\mu, v_\tau\sqrt{T-\tau}) \cdot p(w-\varphi, \sqrt{\vartheta})dw.$$

Recall the expression for $v_\tau$ given by equation (5.8). Using the semi-group property[4] of the

[4] A family of density functions $f_{t,z}$ with respect to the measure $\mu$, is said to have the semi-group property in the parameter $t \in \Theta$, where $\Theta_1 = (0, \infty)$ or $\Theta_1 = \{1, 2, 3, \cdots\}$, if

$$f_{t_1,z} * f_{t_2,z} = f_{t_1+t_2,z} \quad t_1 \in \Theta_1, \quad t_2 \in \Theta_2$$

Gaussian density function [97], the above equation becomes

$$\mathbb{E}^* \left\{ p \left( S_\tau - \mu, v_\tau \sqrt{T-\tau} \right) | \mathcal{C}_t \right\}$$
$$= p \left( \varphi - \mu, \sqrt{\int_\tau^T \mathbb{E}^* \{\sigma_r^2 | \mathcal{F}_\tau\} \, dr + [1-\rho^2] \int_t^\tau \sigma_\theta^2 d\theta} \right). \tag{5.17}$$

Consequently,

$$\mathbb{E}^* \{ \partial_s^n G(\tau, S_\tau, v_\tau) | \mathcal{C}_t \}$$
$$= (-1)^n K e^{-r[T-\tau]} \partial_\mu^n p \left( \varphi - \mu, \sqrt{\int_\tau^T \mathbb{E}^* \{\sigma_r^2 | \mathcal{F}_\tau\} \, dr + [1-\rho^2] \int_t^\tau \sigma_\theta^2 d\theta} \right).$$

Now, let $\mathcal{X}$ be defined as

$$\mathcal{X} := \int_\tau^T \mathbb{E}^* \left\{ \sigma_r^2 | \mathcal{F}_\tau \right\} dr + [1-\rho^2] \int_t^\tau \sigma_\theta^2 d\theta,$$

then

$$p(\varphi - \mu, \sqrt{\mathcal{X}}) = \frac{1}{\sqrt{2\pi\mathcal{X}}} \exp\left\{ -\frac{1}{2} \frac{[\varphi-\mu]^2}{\mathcal{X}} \right\}. \tag{5.18}$$

After differentiating equation (5.18) up to the $n^{th}$ derivative with respect to $\mu$, one can observe that

$$|\partial_\mu^n p(\varphi - \mu, \sqrt{\mathcal{X}})| \leq C \mathcal{X}^{-\frac{1}{2}[2n+1]},$$

where $C$ is a positive non decreasing constant. Hence,

$$|\partial_\mu^n p \left[ \varphi - \mu, \sqrt{\int_\tau^T \mathbb{E}^* \{\sigma_r^2 | \mathcal{F}_\tau\} \, dr + [1-\rho^2] \int_t^\tau \sigma_\theta^2 d\theta} \right] |$$
$$\leq C \left[ \int_\tau^T \mathbb{E}^* \left\{ \sigma_r^2 | \mathcal{F}_\tau \right\} dr + [1-\rho^2] \int_t^\tau \sigma_\theta^2 d\theta \right]^{-\frac{1}{2}[n+1]}.$$
$$\leq C \left[ \int_\tau^T \mathbb{E}^* \left\{ \sigma_\theta^2 | \mathcal{F}_\tau \right\} d\theta \right]^{-\frac{1}{2}[n+1]}.$$

This concludes the proof. □

where

$$f_{t_1,z} * f_{t_2,z}(x) = \int_R f_{t_1,z}(v) f_{t_2,z}(x-v) \, d\mu(v).$$

## 5.2 The Decomposition Formula

A decomposition pricing formula for Heston model [58] based on Theorem 2 of [3] is presented.

**Theorem 5.2.1** (Decomposition formula, [3]). *Consider the model in equation (3.22) with the volatility process $\sigma = \{\sigma_\tau, \tau \in [0,T]\}$ satisfying Feller's condition; $2\alpha m > \beta^2$ for strict positivity of the process $\sigma_t^2$, then for all $t \in [0,T]$*

$$
\begin{aligned}
P_t &= C_{BS}(t, S_t; v_t) \\
&+ \frac{1}{2}\mathbb{E}^* \left\{ \int_t^T e^{-r[\tau-t]} H(\tau, S_\tau; v_\tau)\rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau | \mathcal{F}_t \right\} \\
&+ \frac{1}{8}\mathbb{E}^* \left\{ \int_t^T e^{-r[\tau-t]} K(\tau, S_\tau; v_\tau)\sigma_\tau d\langle L\rangle_\tau | \mathcal{F}_t \right\},
\end{aligned} \tag{5.19}
$$

*where*

$$
\begin{aligned}
H(t, S_t; v_t) &:= \left[ \frac{\partial^3}{\partial S^3} - \frac{\partial^2}{\partial S^2} \right] C_{BS}(t, S_t; v_t). \\
K(t, S_t; v_t) &:= \left[ \frac{\partial^4}{\partial S^4} - 2\frac{\partial^3}{\partial S^3} + \frac{\partial^2}{\partial S^2} \right] C_{BS}(t, S_t; v_t).
\end{aligned}
$$

*Proof.* The result assumes integrability and regularity conditions on volatility. The idea is to apply Itô's formula to the discounted regularized[5] B-S price, i.e. $e^{-rt}C_{BS}(t, S_t; v_t^\delta)$, where

$$
v_t^\delta := \left[ \frac{1}{T-t} \left[ \delta + L_t - \int_0^t \sigma_\tau^2 d\tau \right] \right]^{\frac{1}{2}}. \tag{5.20}
$$

First, using the integration by parts formula gives

$$
d\left[ e^{-rt} C_{BS}(t, S_t; v_t^\delta) \right] = e^{-rt} dC_{BS}(t, S_t; v_t^\delta) - re^{-rt} C_{BS}(t, S_t; v_t^\delta). \tag{5.21}
$$

Next, apply Itô's formula to $dC_{BS}(t, S_t; v_t^\delta)$. For simplicity, define $F := C_{BS}(t, S_t; v_t^\delta)$. Then

$$
dF = F_t dt + F_s dS_t + F_v dv_t^\delta + F_{sv} d\langle S, v^\delta\rangle_t + \frac{1}{2}F_{ss} d\langle S\rangle_t + \frac{1}{2}F_{vv} d\langle v^\delta\rangle_t.
$$

Using Itô's formula together with equations (5.20) and (5.11), compute

$$
\begin{aligned}
dv_t^\delta &= \frac{\partial v_t^\delta}{\partial t} dt + \frac{\partial v_t^\delta}{\partial L} dL_t + \frac{\partial v_t^\delta}{\partial N} dN_t \\
&+ \frac{\partial^2 v_t^\delta}{\partial L \partial N} d\langle L, N\rangle_t + \frac{1}{2}\frac{\partial^2 v_t^\delta}{\partial L^2} d\langle L\rangle_t + \frac{1}{2}\frac{\partial^2 v_t^\delta}{\partial N^2} d\langle N\rangle_t,
\end{aligned}
$$

[5] The derivatives of Black-Scholes price are not bounded, see [3]. Thus, the need for regularization.

where

$$N_t = -\int_0^t \sigma_\tau^2 d\tau.$$

Observe that $N_t$ has finite variation. Therefore the bracket $\langle N\rangle_t$ is equal zero. Hence,

$$\begin{aligned}
dv_t^\delta &= \frac{\partial v_t^\delta}{\partial t}dt + \frac{\partial v_t^\delta}{\partial L}dL_t + \frac{\partial v_t^\delta}{\partial N}dN_t + \frac{1}{2}\frac{\partial^2 v_t^\delta}{\partial L^2}d\langle L\rangle_t. \\
&= -\frac{[\sigma_t^2 - (v_t^\delta)^2]}{2v_t^\delta[T-t]}dt + \frac{\partial v_t^\delta}{\partial L}dL_t + \frac{1}{2}\frac{\partial^2 v_t^\delta}{\partial L^2}d\langle L\rangle_t. \\
&= -\frac{[\sigma_t^2 - (v_t^\delta)^2]}{2v_t^\delta[T-t]}dt + \frac{1}{2v_t^\delta[T-t]}dL_t - \frac{1}{8(v_t^\delta)^3[T-t]^2}d\langle L\rangle_t.
\end{aligned}$$

Now substitute $dS_t$ from equation (5.1) and $dv_t$ above in the expression for $dF$:

$$\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt + \frac{1}{2}\sigma_t^2\frac{\partial^2 F}{\partial S^2}dt + \left[r - \sigma_t^2/2\right]\frac{\partial F}{\partial S}dt \\
& + \frac{\partial F}{\partial S}\left[\sigma_t[\rho dW_t^{*(2)} + \sqrt{1-\rho^2}dW_t^{*(3)}]\right] \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial S \partial v_t^\delta}\left[\frac{\rho\sigma_t}{v_t^\delta[T-t]}d\langle L, W^{*(2)}\rangle_t\right] \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial v_t^{\delta 2}}\left[\frac{1}{4(v_t^\delta)^2[T-t]^2}d\langle L\rangle_t\right] \\
& + \frac{1}{2}\frac{\partial F}{\partial v_t^\delta}\left[-\frac{[\sigma_t^2 - (v_t^\delta)^2]}{v_t^\delta[T-t]}dt + \frac{1}{v_t^\delta[T-t]}dL_t - \frac{1}{4(v_t^\delta)^3[T-t]^2}d\langle L\rangle_t\right].
\end{aligned}$$

Alternatively,

$$\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt + \frac{1}{2}\sigma_t^2\frac{\partial^2 F}{\partial S^2}dt + \left[r - \sigma_t^2/2\right]\frac{\partial F}{\partial S}dt \\
& + \frac{\partial F}{\partial S}\left[\sigma_t[\rho dW_t^{*(2)} + \sqrt{1-\rho^2}dW_t^{*(3)}]\right] \\
& + \frac{1}{2v_t^\delta[T-t]}\frac{\partial F}{\partial v_t^\delta}dL_t \\
& + \frac{1}{2}\frac{\partial^2 F}{\partial S \partial v_t^\delta}\left[\frac{\rho\sigma_t}{v_t^\delta[T-t]}d\langle L, W^{*(2)}\rangle_t\right] \\
& + \frac{1}{8(v_t^\delta)^2[T-t]^2}\left[\frac{\partial^2 F}{\partial v_t^{\delta 2}} - \frac{1}{v_t^\delta}\frac{\partial F}{\partial v_t^\delta}\right]d\langle L\rangle_t \\
& - \frac{1}{2}\frac{\partial F}{\partial v_t^\delta}\frac{[\sigma_t^2 - (v_t^\delta)^2]}{v_t^\delta[T-t]}dt.
\end{aligned} \tag{5.22}$$

Consider the following relation between the Gamma, the Vega and the Delta:

$$\frac{\partial F}{\partial v_t^\delta} = v_t^\delta [T-t] \left[ \frac{\partial^2 F}{\partial S^2} - \frac{\partial F}{\partial S} \right], \tag{5.23}$$

substituting this in equation (5.22) gives

$$\begin{aligned} dF = &\frac{\partial F}{\partial t} dt + \frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S^2} dt + \left[ r - \sigma_t^2/2 \right] \frac{\partial F}{\partial S} dt \\ &+ \frac{\partial F}{\partial S} \left[ \sigma_t [\rho dW_t^{*(2)} + \sqrt{1-\rho^2} dW_t^{*(3)}] \right] \\ &+ \frac{1}{2} \left[ \frac{\partial^2 F}{\partial s^2} - \frac{\partial F}{\partial s} \right] dL_t \\ &+ \frac{\rho \sigma_t}{2} \left[ \frac{\partial^3 F}{\partial S^3} - \frac{\partial^2 F}{\partial S^2} \right] d\langle L, W^{*(2)} \rangle_t \\ &+ \frac{1}{8} \left[ \frac{\partial^4 F}{\partial S^4} - 2\frac{\partial^3 F}{\partial S^3} + \frac{\partial^2 F}{\partial S^2} \right] d\langle L \rangle_t \\ &- \frac{1}{2} [\sigma_t^2 - (v_t^\delta)^2] \left[ \frac{\partial^2 F}{\partial S^2} - \frac{\partial F}{\partial S} \right] dt. \end{aligned}$$

Substituting for $dF$ in equation (5.21) yields

$$\begin{aligned} &d\left[ e^{-rt} C_{BS}(t, S_t; v_t^\delta) \right] \\ &= e^{-rt} \left[ \mathcal{L}_{BS}(v_t^\delta) + \frac{1}{2} \left[ \sigma_t^2 - (v_t^\delta)^2 \right] \left[ \frac{\partial^2}{\partial S^2} - \frac{\partial}{\partial S} \right] \right] C_{BS}(t, S_t; v_t^\delta)\, dt \\ &+ e^{-rt} \frac{\partial}{\partial S} C_{BS}(t, S_t; v_t^\delta) \sigma_t \left[ \rho\, dW_t^{*(2)} + \sqrt{1-\rho^2}\, dW_t^{*(3)} \right] \\ &+ \frac{1}{2} e^{-rt} \left[ \frac{\partial^2}{\partial S^2} - \frac{\partial}{\partial S} \right] C_{BS}(t, S_t; v_t^\delta)\, dL_t \\ &+ \frac{\rho \sigma_t}{2} e^{-rt} \left[ \frac{\partial^3}{\partial S^3} - \frac{\partial^2}{\partial S^2} \right] C_{BS}(t, S_t; v_t^\delta)\, d\langle L, W^{*(2)} \rangle_t \\ &+ \frac{1}{8} e^{-rt} \left[ \frac{\partial^4}{\partial S^4} - 2\frac{\partial^3}{\partial S^3} + \frac{\partial^2}{\partial S^2} \right] C_{BS}(t, S_t; v_t^\delta)\, d\langle L \rangle_t \\ &- \frac{1}{2} e^{-rt} [\sigma_t^2 - (v_t^\delta)^2] \left[ \frac{\partial^2}{\partial S^2} - \frac{\partial}{\partial S} \right] C_{BS}(t, S_t; v_t^\delta)\, dt. \end{aligned} \tag{5.24}$$

Note, $\mathcal{L}_{BS}(v_t^\delta)C_{BS}(t,S_t,v_t^\delta)=0$. Simplifying and integrating the above equation (5.24) gives

$$\begin{aligned}
e^{-rT}C_{BS}(T,S_T;v_T^\delta) &= e^{-rt}C_{BS}(t,S_t;v_t^\delta)\\
&+\int_t^T e^{-r\tau}\frac{\partial}{\partial S}C_{BS}(\tau,S_\tau;v_\tau^\delta)\sigma_\tau\left[\rho\, dW_\tau^{*(2)}+\sqrt{1-\rho^2}\, dW_\tau^{*(3)}\right]\\
&+\frac{1}{2}\int_t^T e^{-r\tau}\left[\frac{\partial^2}{\partial S^2}-\frac{\partial}{\partial S}\right]C_{BS}(\tau,S_\tau;v_\tau^\delta)\, dL_\tau\\
&+\frac{\rho}{2}\int_t^T e^{-r\tau}\sigma_\tau\left[\frac{\partial^3}{\partial S^3}-\frac{\partial^2}{\partial S^2}\right]C_{BS}(\tau,S_\tau;v_\tau^\delta)\, d\langle L,W^{*(2)}\rangle_\tau\\
&+\frac{1}{8}\int_t^T e^{-r\tau}\left[\frac{\partial^4}{\partial S^4}-2\frac{\partial^3}{\partial S^3}+\frac{\partial^2}{\partial S^2}\right]C_{BS}(\tau,S_\tau;v_\tau^\delta)\, d\langle L\rangle_\tau.
\end{aligned}$$

Multiplying all through by $e^{rt}$ and taking conditional expectation gives

$$\begin{aligned}
&\mathbb{E}^*\left\{e^{-r[T-t]}C_{BS}(T,S_T;v_T^\delta)|\mathcal{F}_t\right\}\\
&=C_{BS}(t,S_t;v_t^\delta)\\
&+\frac{1}{2}\mathbb{E}^*\left[\int_t^T e^{-r[\tau-t]}\rho\sigma_\tau H(\tau,S_\tau;v_\tau^\delta)\, d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right]\\
&+\frac{1}{8}\mathbb{E}^*\left[\int_t^T e^{-r[\tau-t]}\sigma_\tau K(\tau,S_\tau;v_\tau^\delta)\, d\langle L\rangle_\tau|\mathcal{F}_t\right].
\end{aligned}\tag{5.25}$$

Let $\delta\to 0$ and recall from (5.3) that the expected discounted payoff gives the price $P_t$. Hence

$$\begin{aligned}
P_t &= C_{BS}(t,S_t;v_t)\\
&+\frac{1}{2}\mathbb{E}^*\left[\int_t^T e^{-r[\tau-t]}\rho\sigma_\tau H(\tau,S_\tau;v_\tau)\, d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right]\\
&+\frac{1}{8}\mathbb{E}^*\left[\int_t^T e^{-r[\tau-t]}\sigma_\tau K(\tau,S_\tau;v_\tau)\, d\langle L\rangle_\tau|\mathcal{F}_t\right].
\end{aligned}\tag{5.26}$$

This concludes the proof with the terms in the brackets, given as

$$\begin{aligned}
d\langle L,W^{*(2)}\rangle_\tau &= \beta\sigma_\tau\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]\, d\tau.\\
d\langle L\rangle_\tau &= \beta^2\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^2\, d\tau.
\end{aligned}$$

□

**Remark 5.2.2.** *The value of the derivative is given by Black-Scholes price with volatility level equal to the root-mean-square future volatility, plus a term due to correlation between the stock price and volatility and a term due to volatility of volatility.*

**Remark 5.2.3.** *From the decomposition formula, observe that if the stock price and volatility are not correlated (i.e. $\rho=0$) the second term on the r.h.s of equation (5.19) vanishes.*

The above decomposition formula leads to the construction of the first- and second-order option pricing approximation formulae. This is discussed explicitly in the following section.

## 5.3 Approximate Pricing Formula

This section theoretically states and explains the first- and second-order option pricing approximation formulae according to [3]. To begin, consider the following useful lemmas[6]:

**Lemma 5.3.1.** *Let* $\delta := \frac{4\alpha m}{\beta^2} \geq 4$ *and take* $n \leq \delta - 2$. *Then, for all* $t, \tau \in [0, T]$ *with* $t < \tau$

$$\mathbb{E}^* \left\{ \frac{1}{\sigma_\tau^n} | \mathcal{F}_t \right\} \leq C_n(T, \sigma_t), \tag{5.27}$$

*where* $C_n(T, \sigma_t)$ *is a positive constant non-decreasing as a function of* $T$.

**Lemma 5.3.2.** *Assume Feller's condition:* $2\alpha m > \beta^2$, *and let* $\delta := \frac{4\alpha m}{\beta^2} < 4$. *For all* $t, \tau \in [0, T]$ *with* $t < \tau$ *and for all* $p < \frac{2}{4-\delta}$

$$\mathbb{E}^* \left\{ \frac{1}{\sigma_\tau^2} | \mathcal{F}_t \right\} \leq \frac{C(T, \sigma_t)}{[[\tau - t]^2 \beta^2 \, [p[\delta/2 - 2] + 1]]^{\frac{1}{p}}}, \tag{5.28}$$

*where* $C(T, \sigma_t)$ *is a positive constant non-decreasing as a function of* $T$.

**Lemma 5.3.3.** *Assume* $\delta := \frac{4\alpha m}{\beta^2} \in (3, 4)$. *Then, for all* $t \in [0, T]$ *with* $t < \tau$ *and for all* $p < \frac{2}{5-\delta}$

$$\mathbb{E}^* \left\{ \frac{1}{\sigma_\tau^3} | \mathcal{F}_t \right\} \leq \frac{C(T, \sigma_t)}{\beta^{2[1-1/p]} \, \left[\frac{p}{2}[\delta - 5] + 1\right]^{\frac{1}{p}}}, \tag{5.29}$$

The first-order approximation formula for pricing derivative options follows.

**Theorem 5.3.4** (First-Order Approximation Formula, [3]). *Assume the model presented in equation* (3.22) *with volatility process* $\sigma = \{\sigma_\tau, \tau \in [0, T]\}$ *satisfying Feller's condition* $2\alpha m > \beta^2$. *If* $\delta \geq 4$, *for all* $t \in [0, T]$ *such that* $T - t < 1$, *then*

$$\begin{aligned} &|P_t - C_{BS}(t, S_t; v_t) - \frac{1}{2} H(t, S_t; v_t) \, \mathbb{E}^* \left\{ \int_t^T \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\} | \\ &\leq C(T, \sigma_t) \beta^2 [T - t]^{\frac{3}{2}} [\rho^2 + \rho + 1]. \end{aligned} \tag{5.30}$$

[6] The corresponding proofs to these lemmas can be found in [3], [15] and [4].

*and for $\delta < 4$*

$$
\begin{aligned}
&|P_t - C_{BS}(t, S_t; v_t) - \frac{1}{2} H(t, S_t; v_t)\, \mathbb{E}^* \left\{ \int_t^T \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\} | \\
&\leq C(T, \sigma_t) \beta^{2-2\sqrt{2-\delta/2}} \left[ \frac{1}{1 - \sqrt{2 - \delta/2}} \right]^{1+\sqrt{2-\delta/2}} \\
&\times \left[ \rho^2 [T-t]^{\frac{1}{2}[3-\sqrt{2-\delta/2}]} + \rho [T-t]^{2[1-\sqrt{2-\delta/2}]} + [T-t]^{\frac{1}{2}[3-\sqrt{2-\delta/2}]} \right], \qquad (5.31)
\end{aligned}
$$

*where $C(T, \sigma_t)$ is a positive constant non-decreasing as a function of $T$.*

*Proof.* The proof follows from [3]. Consider a process $e^{-rT} H(t, S_t; v_t) U_t$ where

$$
U_t := \mathbb{E}^* \left\{ \int_t^T \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\}. \qquad (5.32)
$$

Observe, $e^{-rT} H(T, S_T; v_T) U_T = 0$. A similar approach employed in Theorem 5.2.1, gives

$$
\begin{aligned}
0 &= H(t, S_t; v_t) U_t \\
&- \mathbb{E}^* \left\{ \int_t^T e^{-[\tau-t]} H(\tau, S_\tau; v_\tau) \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\} \\
&+ \frac{1}{2} \mathbb{E}^* \left\{ \int_t^T e^{-[\tau-t]} \left[ \frac{\partial^3}{\partial S^3} - \frac{\partial^2}{\partial S^2} \right] H(\tau, S_\tau; v_\tau^\delta) U_\tau \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\} \\
&+ \frac{1}{8} \mathbb{E}^* \left\{ \int_t^T e^{-[\tau-t]} \left[ \frac{\partial^4}{\partial S^4} - 2\frac{\partial^3}{\partial S^3} + \frac{\partial^2}{\partial S^2} \right] H(\tau, S_\tau; v_\tau^\delta) U_\tau \sigma_\tau d\langle L \rangle_\tau | \mathcal{F}_t \right\}.
\end{aligned}
$$

Using the above equation, substitute for the second term on the r.h.s into (5.26) to get

$$
\begin{aligned}
P_t &= C_{BS}(t, S_t; v_t) + \frac{1}{2} H(t, S_t; v_t) U_t \\
&+ \frac{1}{4} \mathbb{E}^* \left\{ \int_t^T e^{-[\tau-t]} \left[ \frac{\partial^3}{\partial S^3} - \frac{\partial^2}{\partial S^2} \right] H(\tau, S_\tau; v_\tau^\delta) U_\tau \rho \sigma_\tau d\langle L, W^{*(2)} \rangle_\tau | \mathcal{F}_t \right\} \\
&+ \frac{1}{16} \mathbb{E}^* \left\{ \int_t^T e^{-[\tau-t]} \left[ \frac{\partial^4}{\partial S^4} - 2\frac{\partial^3}{\partial S^3} + \frac{\partial^2}{\partial S^2} \right] H(\tau, S_\tau; v_\tau^\delta) U_\tau \sigma_\tau d\langle L \rangle_\tau | \mathcal{F}_t \right\} \\
&+ \frac{1}{8} \mathbb{E}^* \left\{ \int_t^T e^{-r[\tau-t]} \sigma_\tau K(\tau, X_\tau; v_\tau) d\langle L \rangle_\tau | \mathcal{F}_t \right\}. \\
&= C_{BS}(t, S_t; v_t) + \frac{1}{2} H(t, S_t; v_t) U_t + J_1 + J_2 + J_3.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|U_\tau| &\leq \rho \beta\, \mathbb{E}^* \left\{ \int_\tau^T \sigma_r^2 \left[ \int_r^T e^{-\alpha[u-r]}\, du \right] dr | \mathcal{F}_\tau \right\}. \\
&= \rho \beta \int_\tau^T \mathbb{E}^* \left\{ \sigma_r^2 | \mathcal{F}_\tau \right\} \left[ \int_r^T e^{-\alpha[u-r]}\, du \right] dr. \qquad (5.33)
\end{aligned}
$$

As some side work, take a look at the following expansion:

$$\frac{\partial^3}{\partial S^3}H = \frac{\partial^3}{\partial S^3}\left[\frac{\partial^3}{\partial S^3} - \frac{\partial^2}{\partial S^2}\right]C_{BS} = \frac{\partial^4}{\partial S^4}\left[\frac{\partial^2}{\partial S^2} - \frac{\partial}{\partial S}\right]C_{BS} = \frac{\partial^4}{\partial S^4}G.$$

Comparing this with Lemma 5.1.1, one can deduce that $n = 4$, this together with equation (5.33), implies that $J_1$ is bounded as

$$\begin{aligned}
J_1 \leq & \frac{\rho^2\beta^2}{4}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}\right]^{-\frac{5}{2}}\right. \\
& \times\left.\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_r^2|\mathcal{F}_\tau\right\}\left[\int_r^T e^{-\alpha[u-r]}\,du\right]\,dr\right]\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}\,du\right]\,d\tau|\mathcal{F}_t\right\}. \\
\leq & \frac{\rho^2\beta^2}{4}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-\frac{3}{2}}\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^2 d\tau|\mathcal{F}_t\right\}.
\end{aligned}$$

Using the fact

$$\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}\,d\theta \geq \sigma_\tau^2\int_\tau^T e^{-\alpha[r-\tau]}\,dr, \tag{5.34}$$

it follows that

$$\begin{aligned}
J_1 \leq & \frac{\rho^2\beta^2}{4}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\sigma_\tau^2\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{-\frac{3}{2}}\right. \\
& \left.\times\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^2 d\tau|\mathcal{F}_t\right\}. \\
\leq & \frac{\rho^2\beta^2}{4}\int_t^T e^{-r[\tau-t]}\mathbb{E}^*\left\{\sigma_\tau^{-1}|\mathcal{F}_t\right\}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{1}{2}}d\tau. \\
\leq & \frac{\rho^2\beta^2}{4}\int_t^T e^{-r[\tau-t]}\sqrt{\mathbb{E}^*\left\{\sigma_\tau^{-2}|\mathcal{F}_t\right\}}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{1}{2}}d\tau.
\end{aligned}$$

It can be deduced from Lemma 5.3.1 where $\delta \geq 4$, that

$$\begin{aligned}
J_1 \leq & C(T,\sigma_t)\rho^2\beta^2\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{1}{2}}d\tau. \\
\leq & C(T,\sigma_t)\rho^2\beta^2[T-t]^{\frac{3}{2}}.
\end{aligned}$$

The last inequality follows from the fact that

$$\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{1}{2}}d\tau \leq \left[\int_t^T e^{-\alpha[u-t]}\,du\right]^{\frac{3}{2}}.$$

and the approximation

$$e^{\alpha[u-t]} \approx 1 + \alpha[u-t].$$

Consequently, from Lemma 5.3.2, where $\delta < 4$, it can be shown that

$$J_1 \leq \frac{C(T,\sigma_t)\beta^2\rho^2}{\beta^{2/p}\left[p[\delta/2-2]+1\right]^{\frac{1}{2p}}} \int_t^T \frac{e^{-r[\tau-t]}}{[\tau-t]^{\frac{1}{2p}}} \left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{1}{2}} d\tau.$$

Using the same arguments as above leads to

$$J_1 \leq \frac{C(T,\sigma_t)\beta^2\rho^2}{\beta^{2/p}\left[p[\delta/2-2]+1\right]^{\frac{1}{2p}}}[T-t]^{\frac{3}{2}-1/2p}.$$

Next, define $p := 1/\sqrt{2-\delta/2}$, then

$$J_1 \leq \frac{C(T,\sigma_t)\rho^2\beta^{2-2\sqrt{2-\delta/2}}}{\left[1-\sqrt{2-\delta/2}\right]^{\frac{\sqrt{2-\delta/2}}{2}}}[T-t]^{\frac{1}{2}\left[3-\sqrt{2-\delta/2}\right]}. \tag{5.35}$$

Similarly, comparing $J_2$ with Lemma 5.1.1 one notices that $n=5$. Thus,

$$\begin{aligned}
J_2 &\leq \frac{\rho\beta^2}{16}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-3}\right.\\
&\times\left.\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_r^2|\mathcal{F}_\tau\right\}\left[\int_r^T e^{-\alpha[u-r]}\,du\right]\,dr\right]\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}\,du\right]^2 d\tau|\mathcal{F}_t\right\}.\\
&\leq \frac{\rho\beta^2}{16}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-2}\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^3 d\tau|\mathcal{F}_t\right\}.
\end{aligned}$$

Using the idea in equation (5.34),

$$\begin{aligned}
J_2 &\leq \frac{\rho\beta^2}{16}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\sigma_\tau^2\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{-2}\right.\\
&\times\left.\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^3 d\tau|\mathcal{F}_t\right\}.\\
&\leq \frac{\rho\beta^2}{16}\int_t^T e^{-r[\tau-t]}\mathbb{E}^*\left\{\sigma_\tau^{-2}|\mathcal{F}_t\right\}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]d\tau.
\end{aligned}$$

Thus, for $\delta \geq 4$ and using Hölder's inequality one can deduce

$$J_2 \leq C(T,\sigma_t)\rho\beta^2[T-t]^2.$$

Suppose $[T-t] < 1$, then one can as well write

$$J_2 \leq C(T,\sigma_t)\rho\beta^2[T-t]^{\frac{3}{2}}.$$

For $\delta < 4$:

$$\begin{aligned}
J_2 &\leq C(T,\sigma_t)\rho\beta^2 \int_t^T e^{r[\tau-t]}\mathbb{E}^*\left\{\sigma_\tau^{-2}|\mathcal{F}_t\right\}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]d\tau. \\
&\leq \frac{C(T,\sigma_t)\rho\beta^2}{\beta^{2/p}[p[\delta/2-2]+1]^{\frac{1}{p}}}\int_t^T \frac{e^{-r[\tau-t]}}{[\tau-t]^{1/p}}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]d\tau. \\
&\leq \frac{C(T,\sigma_t)p\beta^2\rho}{[p-1]\beta^{2/p}[p[\delta/2-2]+1]^{\frac{1}{p}}}[T-t]^{2-1/p}. \\
&= C(T,\sigma_t)\rho\beta^2[T-t]^2\left[\frac{p}{p-1}\right]\left[\frac{1}{\beta^2[T-t][p[\delta/2-2]+1]}\right]^{\frac{1}{p}}.
\end{aligned}$$

Next, define

$$p := \frac{1}{\sqrt{2-\delta/2}},$$

such that

$$J_2 \leq C(T,\sigma_t)\rho[\beta[T-t]]^{2-2\sqrt{2-\delta/2}}\left[\frac{1}{1-\sqrt{2-\delta/2}}\right]^{1+\sqrt{2-\delta/2}}.$$

The last term $J_3$ also follows the same arguments as above, that is

$$\begin{aligned}
J_3 \leq& \frac{\beta^2}{8}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-\frac{3}{2}}\right. \\
&\left.\times\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^2 d\tau|\mathcal{F}_t\right\}. \\
\leq& C(T,\sigma_t)\beta^2\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\left[\sigma_\tau^2\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{-\frac{3}{2}}\right. \\
&\left.\times\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^2 d\tau|\mathcal{F}_t\right\}. \\
\leq& C(T,\sigma_t)\beta^2\int_t^T \mathbb{E}^*\left\{\sigma_\tau^{-1}|\mathcal{F}_t\right\}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{\frac{1}{2}}d\tau.
\end{aligned}$$

Thus, for $\delta \geq 4$, it follows that

$$J_3 \leq\ C(T,\sigma_t)\beta^2[T-t]^{\frac{3}{2}},$$

and for $\delta < 4$

$$J_3 \leq\ C(T,\sigma_t)\frac{\beta^{2-2\sqrt{2-\delta/2}}}{[1-\sqrt{2-\delta/2}]^{\frac{\sqrt{2-\delta/2}}{2}}}[T-t]^{\frac{1}{2}[3-\sqrt{2-\delta/2}]}.$$

This concludes the proof. □

**Remark 5.3.5.** *Observe that equations (5.30) and (5.31) are equivalent when $\delta = 4$ since $T - t < 1$. Thus, for a fixed $\delta$, the accuracy of the first order approximation increases with decrease in values of volatility of volatility and/or time to maturity. However, the approximation in equation (5.31) becomes inaccurate as $\delta \to 2$.*

**Remark 5.3.6.** *Note that this first order approximation is more accurate for shorter maturities compared to the approach by [40] which works well for longer maturities.*

The first decomposition formula above is derived using only the first term on the *r.h.s* of equation (5.19), including the second term leads to the second decomposition pricing formula. To do this, one requires Lemma 5.3.3.

**Theorem 5.3.7** (Second-Order Approximation Formula, [3])**.** *Assume the model presented in equation (3.22) with volatility process $\sigma = \{\sigma_\tau, \tau \in [0,T]\}$ satisfying Feller's condition $2\alpha m > \beta^2$. Then for all $t \in [0,T]$ such that $T - t < 1$, the following three cases are valid:*

**Case I:**

*If $\delta \geq 5$,*

$$\begin{aligned}
&|P_t - C_{BS}(t, S_t; v_t) - \frac{1}{2}H(t, S_t, v_t)\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau | \mathcal{F}_t\right\} \\
&- \frac{1}{8}K(t, S_t, v_t)\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau | \mathcal{F}_t\right\}| \\
&\leq C(T, \sigma_t)[\rho^2\beta^2[T-t]^{3/2} + \beta^3\rho[T-t]^2 + \beta^4[T-t]^{5/2}].
\end{aligned} \tag{5.36}$$

**Case II:**

*If $\delta \in [4,5)$,*

$$\begin{aligned}
&|P_t - C_{BS}(t, S_t; v_t) - \frac{1}{2}H(t, S_t, v_t)\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau | \mathcal{F}_t\right\} \\
&- \frac{1}{8}K(t, S_t, v_t)\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau | \mathcal{F}_t\right\}| \\
&\leq C(T, \sigma_t)\left[\rho^2\beta^2[T-t]^{3/2} + \beta^3\rho[T-t]^2\right. \\
&\left. + \beta^{4-2\sqrt{5/2-\delta/2}}[T-t]^{5/2-2\sqrt{5/2-\delta/2}}\left[\frac{1}{1-\sqrt{5/2-\delta/2}}\right]^{1+\sqrt{5/2-\delta/2}}\right].
\end{aligned} \tag{5.37}$$

**Case III:**

*If* $\delta \in [3,4)$,

$$
\begin{aligned}
&|P_t - C_{BS}(t,S_t;v_t) - \frac{1}{2}H(t,S_t,v_t)\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \\
&- \frac{1}{8}K(t,S_t,v_t)\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau|\mathcal{F}_t\right\}| \\
&\leq C(T,\sigma_t)\left[\beta^{2-2\sqrt{2-\delta/2}}\left[\frac{1}{1-\sqrt{2-\delta/2}}\right]^{1+\sqrt{2-\delta/2}}\right. \\
&\times\left[\rho^2[T-t]^{\frac{1}{2}[3-\sqrt{2-\delta/2}]} + \rho[T-t]^{2[1-\sqrt{2-\delta/2}]}\right] \\
&\left. + \beta^{4-2\sqrt{5/2-\delta/2}}[T-t]^{5/2-2\sqrt{5/2-\delta/2}}\left[\frac{1}{1-\sqrt{5/2-\delta/2}}\right]^{1+\sqrt{5/2-\delta/2}}\right].
\end{aligned}
\tag{5.38}
$$

*where* $C(T,\sigma_t)$ *is a positive constant non-decreasing as a function of* $T$.

*Proof.* In a similar way to that in the derivation of the first-order approximation formula, one can consider the process $e^{-rt}K(t,S_t;v_t)R_t$, where $R_t := \mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau|\mathcal{F}_t\right\}$, observe that $e^{-rT}K(T,S_T;v_T)R_T = 0$. By using similar arguments applied to $e^{-rt}H(t,S_t;v_t)U_t$ above,

$$
\begin{aligned}
0 &= K(t,S_t,v_t)R_t \\
&- \mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}K(\tau,S_\tau,v_\tau)\sigma_\tau d\langle L,L\rangle_\tau|\mathcal{F}_t\right\} \\
&+ \frac{1}{2}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^3}{\partial S^3}-\frac{\partial^2}{\partial S^2}\right]K(\tau,S_\tau,v_\tau^\delta)R_\tau\rho\sigma_\tau d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \\
&+ \frac{1}{8}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^4}{\partial S^4}-2\frac{\partial^3}{\partial S^3}+\frac{\partial^2}{\partial S^2}\right]K(\tau,S_\tau,v_\tau^\delta)R_\tau\sigma_\tau d\langle L\rangle_\tau|\mathcal{F}_t\right\}.
\end{aligned}
$$

Using this equation, substitute for the second term on the r.h.s into equation (5.26) to obtain

$$
\begin{aligned}
P_t &= C_{BS}(t,S_t,v_t) + \frac{1}{2}H(t,S_t,v_t)U_t + \frac{1}{8}K(t,S_t,v_t)R_t \\
&+ \frac{1}{4}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^3}{\partial S^3}-\frac{\partial^2}{\partial S^2}\right]H(\tau,S_\tau,v_\tau^\delta)U_\tau\rho\sigma_\tau d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \\
&+ \frac{1}{16}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^4}{\partial S^4}-2\frac{\partial^3}{\partial S^3}+\frac{\partial^2}{\partial S^2}\right]H(\tau,S_\tau,v_\tau^\delta)U_\tau\sigma_\tau d\langle L\rangle_\tau|\mathcal{F}_t\right\} \\
&+ \frac{1}{16}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^3}{\partial S^3}-\frac{\partial^2}{\partial S^2}\right]K(\tau,S_\tau,v_\tau^\delta)R_\tau\rho\sigma_\tau d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \\
&+ \frac{1}{64}\mathbb{E}^*\left\{\int_t^T e^{-[\tau-t]}\left[\frac{\partial^4}{\partial S^4}-2\frac{\partial^3}{\partial S^3}+\frac{\partial^2}{\partial S^2}\right]K(\tau,S_\tau,v_\tau^\delta)R_\tau\sigma_\tau d\langle L\rangle_\tau|\mathcal{F}_t\right\}. \\
&= C_{BS}(t,S_t,v_t) + \frac{1}{2}H(t,S_t,v_t)U_t + \frac{1}{8}K(t,S_t,v_t)R_t + J_1 + J_2 + J_3 + J_4.
\end{aligned}
$$

From the proof of the first-order decomposition, observe that if $\delta \geq 4$ then

$$J_1 + J_2 \leq C(T,\sigma_t)\rho^2\beta^2[T-t]^{3/2},$$

and for $\delta < 4$,

$$J_1 + J_2 \leq C(T,\sigma_t)\rho\beta^{2-2\sqrt{2-\delta/2}}\left[\frac{1}{1-\sqrt{2-\delta/2}}\right]^{1+\sqrt{2-\delta/2}} \times \left[\rho[T-t]^{\frac{1}{2}[3-\sqrt{2-\delta/2}]} + [T-t]^{2-2\sqrt{2-\delta/2}}\right].$$

The $R_t$-term can be computed as

$$|R_\tau| \leq \beta^2\mathbb{E}^*\left\{\int_\tau^T \sigma_r^2\left[\int_r^T e^{-\alpha[u-r]}du\right]^2 dr|\mathcal{F}_t\right\}. = \beta^2\int_\tau^T \mathbb{E}^*\left\{\sigma_r^2|\mathcal{F}_\tau\right\}\left[\int_r^T e^{-\alpha[u-r]}du\right]^2 dr.$$

Using Lemma 5.1.1 gives

$$J_3 \leq \frac{\beta^3\rho}{16}\mathbb{E}^*\{\int_t^T e^{-\alpha[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-3} \times\left[\int_r^T \mathbb{E}^*\left\{\sigma_r^2|\mathcal{F}_\tau\right\}\left[\int_r^T e^{-\alpha[u-r]}du\right]^2 dr\right]\times\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]d\tau\}$$

From Lemma 5.3.1, if $\delta \geq 4$ then

$$J_3 \leq C(T,\sigma_t)\beta^3\rho[T-t]^2,$$

and from Lemma C.7, it can be deduced for $\delta < 4$ that

$$J_3 \leq C(T,\sigma_t)\rho\beta[\beta[T-t]]^{2-2\sqrt{2-\delta/2}}\left[\frac{1}{1-\sqrt{2-\delta/2}}\right]^{1+\sqrt{2-\delta/2}}.$$

Term $J_4$ is derived from Lemma 5.1.1, applying similar arguments as in the derivation of $J_3$,

$$\begin{aligned}J_4 &\leq \frac{\beta^4}{64}\mathbb{E}^*\{\int_t^T e^{-r[\tau-t]}\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_\theta^2|\mathcal{F}_\tau\right\}d\theta\right]^{-\frac{7}{2}} \\ &\quad\times\left[\int_\tau^T \mathbb{E}^*\left\{\sigma_r^2|\mathcal{F}_\tau\right\}\left[\int_r^T e^{-\alpha[u-r]}du\right]^2 dr\right]\times\sigma_\tau^2\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^2 d\tau|\mathcal{F}_t\}. \\ &\leq \frac{\beta^4}{64}\mathbb{E}^*\left\{\int_t^T e^{-r[\tau-t]}\sigma_\tau^{-3}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{\frac{3}{2}}d\tau|\mathcal{F}_t\right\}. \\ &\leq \frac{\beta^4}{64}\int_t^T e^{-r(\tau-t)}\mathbb{E}^*\left\{\sigma_\tau^{-3}|\mathcal{F}_t\right\}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{\frac{3}{2}}d\tau.\end{aligned}$$

Using Lemma 5.3.1, note that for $\delta \geq 5$

$$J_4 \leq C(T,\sigma_t)\beta^4[T-t]^{5/2},$$

and from Lemma 5.3.3 it can be deduced that if $\delta \in (3,5)$ then

$$\begin{aligned}
J_4 &\leq C(T,\sigma_t)\beta^4 \int_t^T e^{-r[\tau-t]}\mathbb{E}^*\left\{\sigma_\tau^{-3}\right\}\left[\int_\tau^T e^{-\alpha[r-\tau]}dr\right]^{\frac{3}{2}} d\tau. \\
&\leq C(T,\sigma_t)\frac{\beta^4}{\beta^{2/p}[p[\delta/2-5/2]+1]^{\frac{1}{p}}}\int_t^T \frac{e^{-r[\tau-t]}}{[\tau-t]^{\frac{1}{p}}}\left[\int_\tau^T e^{-\alpha[u-\tau]}du\right]^{\frac{3}{2}} d\tau. \\
&\leq C(T,\sigma_t)\frac{p\beta^4}{[p-1]\beta^{2/p}[p[\delta/2-5/2]+1]^{\frac{1}{p}}}[T-t]^{5/2-1/p}. \\
&= C(T,\sigma_t)\beta^4[T-t]^{5/2}\left[\frac{p}{p-1}\right]\left[\frac{1}{\beta^2[T-t][p[\delta/2-5/2]+1]}\right]^{\frac{1}{p}}.
\end{aligned}$$

Taking $p=\sqrt{\frac{2}{5-\delta}}$, it follows that

$$J_4 \leq C(T,\sigma_t)\beta^{4-2\sqrt{5/2-\delta/2}}[T-t]^{5/2-2\sqrt{5/2-\delta/2}}\left[\frac{1}{1-\sqrt{5/2-\delta/2}}\right]^{1+\sqrt{5/2-\delta/2}},$$

which completes the proof. □

**Proposition 5.3.8** (Remark , [3]). *Using the formula for a European call option given in equation (5.4) and the Greeks, it can be shown that*

$$\begin{aligned}
H(t,S;\sigma) &= \frac{e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\left\{-\mathrm{d}_+^2/2\right\}\left[1-\frac{\mathrm{d}_+}{\sigma\sqrt{T-t}}\right]. \\
K(t,S;\sigma) &= \frac{e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\left\{\mathrm{d}_+^2/2\right\}\left[\left[\frac{\mathrm{d}_+^2}{\sigma^2[T-t]}-\frac{\mathrm{d}_+}{\sigma\sqrt{T-t}}\right]-\frac{1}{\sigma^2[T-t]}\right].
\end{aligned}$$

*Moreover, it can be deduced from the Heston model, equation (5.2) that*

$$\begin{aligned}
&\mathbb{E}^*\left\{\int_t^T \sigma_\tau^2\, d\tau|\mathcal{F}_t\right\} = m[T-t]+\frac{1}{\alpha}\left[\sigma_t^2-m\right]\left[1-e^{-\alpha[T-t]}\right], \\
&\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau\, d\langle L, W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \\
&= \frac{\rho\beta}{\alpha^2}\left[\alpha m[T-t]-2m+\sigma_t^2+e^{-\alpha[T-t]}[2m-\sigma_t^2]-\alpha[T-t]e^{-\alpha[T-t]}[\sigma_t^2-m]\right]
\end{aligned}$$

*and*

$$\begin{aligned}\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau|\mathcal{F}_t\right\} = \frac{\beta^2}{\alpha^2}\Bigg[& m[T-t]+\frac{[\sigma_t^2-m]}{\alpha}\left[1-e^{-\alpha[T-t]}\right]\\ &-\frac{2m}{\alpha}\left[1-e^{-\alpha[T-t]}\right]-2[\sigma_t^2-m][T-t]e^{-\alpha[T-t]}\\ &\frac{m}{2\alpha}\left[1-e^{-2\alpha[T-t]}\right]+\frac{[\sigma_t^2-m]}{\alpha}\left[e^{-\alpha[T-t]}-e^{-2\alpha[T-t]}\right]\Bigg].\end{aligned}$$

*Thus, substituting these quantities in Theorems 5.3.4 and 5.3.8, one can easily obtain explicit first-order and second-order approximation pricing formulas.*

*Proof.* From the Black-Scholes formula (1.41), let $S=\log x$. Then

$$\begin{aligned}\frac{dC}{dS} &= e^S N(\mathrm{d}_+);\quad \frac{d^2C}{dS^2}=e^S N(\mathrm{d}_+)+\frac{e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\{-\mathrm{d}_+^2/2\};\\ \frac{d^3C}{dS^3} &= e^S N(\mathrm{d}_+)+\frac{2e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\{-\mathrm{d}_+^2/2\}-\frac{\mathrm{d}_+e^S}{\sigma^2\sqrt{2\pi[T-t]}}\exp\{-\mathrm{d}_+^2/2\};\\ \frac{d^4C}{dS^4} &= e^S N(\mathrm{d}_+)+\frac{3e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\{-\mathrm{d}_+^2/2\}-\frac{2\mathrm{d}_+e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\{-\mathrm{d}_+/2\}\\ &\quad -\frac{e^S}{\sigma^3\sqrt{2\pi}[T-t]}\exp\{-\mathrm{d}_+^2/2\}\left[1-\mathrm{d}_+^2\right].\end{aligned}$$

Thus, from the expressions of $H(t,S;\sigma)$ and $K(t,S;\sigma)$ given in Theorem 5.2 where $v_t=\sigma$,

$$\begin{aligned}H(t,S;\sigma) &= \frac{e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\left\{-\mathrm{d}_+^2/2\right\}\left[1-\frac{\mathrm{d}_+}{\sigma\sqrt{T-t}}\right].\\ K(t,S;\sigma) &= \frac{e^S}{\sigma\sqrt{2\pi[T-t]}}\exp\left\{\mathrm{d}_+^2/2\right\}\left[\left[\frac{\mathrm{d}_+^2}{\sigma^2[T-t]}-\frac{\mathrm{d}_+}{\sigma\sqrt{T-t}}\right]-\frac{1}{\sigma^2[T-t]}\right].\end{aligned}$$

Computing the expectation of (5.2) and integrating from $t$ to $\tau$ gives

$$\sigma_\tau^2=m+[\sigma_t^2-m]e^{-\alpha[\tau-t]}. \tag{5.39}$$

Next, integrate the expectation of (5.39) from $t$ to $T$ conditioned with a filtration to obtain

$$\int_t^T \mathbb{E}^*\left\{\sigma_\tau^2|\mathcal{F}_t\right\}d\tau=m[T-t]+\frac{1}{\alpha}[\sigma_t^2-m][1-e^{-\alpha[t-t]}]. \tag{5.40}$$

By using equation (5.39) and the expressions of $d\langle L,W^{*(2)}\rangle$ and $d\langle L\rangle$ it is easy to show that

$$\begin{aligned}&\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau\, d\langle L,W^{*(2)}\rangle_\tau|\mathcal{F}_t\right\} \qquad (5.41)\\ &=\frac{\rho\beta}{\alpha^2}\left[\alpha m[T-t]-2m+\sigma_t^2+e^{-\alpha[T-t]}[2m-\sigma_t^2]-\alpha[T-t]e^{-\alpha[T-t]}[\sigma_t^2-m]\right].\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau|\mathcal{F}_t\right\} = \frac{\beta^2}{\alpha^2}&\left[m[T-t]+\frac{[\sigma_t^2-m]}{\alpha}\left[1-e^{-\alpha[T-t]}\right]\right.\\
&-\frac{2m}{\alpha}\left[1-e^{-\alpha[T-t]}\right]-2[\sigma_t^2-m][T-t]e^{-\alpha[T-t]}\\
&\left.\frac{m}{2\alpha}\left[1-e^{-2\alpha[T-t]}\right]+\frac{[\sigma_t^2-m]}{\alpha}\left[e^{-\alpha[T-t]}-e^{-2\alpha[T-t]}\right]\right].
\end{aligned}
$$

This ends the proof. □

**Remark 5.3.9** (Approximations for the implied volatility, [3]). *It can be deduced from the expressions of Theorems 5.3.4 and 5.3.8, by using Taylor series expansions, that the first-order and second-order approximations for the implied volatility take the form*

$$
\begin{aligned}
I^{(1)} &= v_t + \frac{1}{2v_t[T-t]}\left[1-\frac{\mathrm{d}_+}{v_t\sqrt{T-t}}\right]\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau\right\}.\\
I^{(2)} &= v_t + \frac{1}{2v_t[T-t]}\left[1-\frac{\mathrm{d}_+}{v_t\sqrt{T-t}}\right]\mathbb{E}^*\left\{\int_t^T \rho\sigma_\tau d\langle L, W^{*(2)}\rangle_\tau\right\}\\
&\quad+\frac{1}{2v_tT}\left[\left[-\frac{\mathrm{d}_+}{v_t\sqrt{T-t}}+\frac{\mathrm{d}_+^2}{v_t^2[T-t]}\right]-\frac{1}{v_t^2[T-t]}\right]\mathbb{E}^*\left\{\int_t^T d\langle L\rangle_\tau\right\}.
\end{aligned}
$$

*where*

$$
\mathrm{d}_+ = \frac{s_t - s_t^*}{v_t\sqrt{T-t}}.
$$

*Observe that the first expression is linear in the initial log-stock price $s_t$, and the second one is quadratic in $s_t$. Thus, it can be deduced that the first-order approximation formula can be used to describe the skew effect and the second-order term, the smile.*

## 5.4 Conclusion

Based on the work by Fouque et al. [40, 41, 43], we have shown by using perturbation methods that the constant volatility Black-Scholes pricing model can be corrected up to the second order term. The first-order corrected price is a sum of Black-Scholes price of the corresponding option, i.e. European, American, Asian, Look-back, Barrier or Forward option, with a corrected constant volatility , plus a correction term comprising of small market grouped parameters that can be easily extracted from the observed implied volatility. The implied volatility model is expressed as a linear function of the log-moneyness-to-maturity ratio. In particular, the least squares method can be employed to estimate these grouped parameters from the observed implied volatility. We have been able to demonstrate the model-independence, by capturing all the effects of the model parameters in the grouped parameters i.e. the specifications of the stochastic volatility model are not necessary. In addition, the model is parsimonious in parameters, i.e. the number of parameters that any traditional stochastic volatility model would require for calibration, is reduced. The approach only requires volatility to be correlated and mean-reverting on short and long time scales, to effectively capture the market skews, smiles and the leverage. The approach can be extended to pricing interest rates derivatives, bonds and credit derivatives. In principle, the approach is unreliable when pricing is close to expiry. Instead, the maturity period should be small but large with respect to the rate of mean reversion.

According to [3], we have shown by using classical Itô's formula that a more accurate pricing model can be derived, that comprises of the Black-Scholes model with volatility level as the root-mean-square timed averaged volatility, plus a term due to correlation and a term due to volatility of volatility. A first- and second-order approximate pricing analytic formulae are generated. The approach only requires some general integrability and regularity conditions on the volatility process for its validity. In this case, it is also accurate when pricing is done close to expiry compared to the asymptotic methods. This technique gives a more accurate second-order approximation of the implied volatility model, by expressing it as a quadratic function of initial log-stock price.

Further research would seek employing malliavin calculus, the decomposition pricing technique and the Donsker delta function to derive general pricing and hedging formulae for Exotic options. The motivation is that there is sufficient literature on pricing exotics, but challenges arise from constructing their hedging strategies. So many exotics are listed on the exchange with their hedging unknown yet, investors interested in trading these option would certainly be concerned with hedging themselves against risk or be able to cover up their liabilities. Some related work appear in [13] where malliavin calculus has been applied to price Look-backs and Barrier options. The interest is to extend the ideas to all Exotics.

# Appendix A

# Itô Diffusion Processes

## A.1 Infinitesimal Generator of an Itô Diffusion Processes

Consider a time-homogeneous diffusion process $X$ that solves the stochastic differential equation;

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \tag{A.1}$$

where the coefficients $\mu$ and $\sigma$ are taken to be regular enough to satisfy Itô's formula. Suppose $f$ is a twice differentiable continuous function of $X$ then, from Itô's formula the derivative of $f$ is given by

$$\begin{aligned} df(X_t) &= \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\sigma^2(X_t)\frac{\partial^2 f}{\partial x^2}dt \\ &= \left[\mu(X_t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(X_t)\frac{\partial^2 f}{\partial x^2}\right]dt + \sigma(X_t)\frac{\partial f}{\partial x}dW_t \\ &= \mathcal{L}f(X_t)dt + \sigma(X_t)\frac{\partial f}{\partial x}dW_t, \end{aligned}$$

where

$$\mathcal{L}f(x) = \mu(x)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\,, \tag{A.2}$$

and $\mathcal{L}$ is known as the *infinitesimal generator* of the process $X_t$.

## A.2 Relevant Properties of Ergodic Markov Processes

Ergodic Markov processes such as the Ornstein-Ulhenbeck Process have some properties that serve relevant to this work, [40].

- The processes are characterised by an infinitesimal generator denoted by $\mathcal{L}$ that can be in form of a matrix, a differential operator or an integral.
- The invariant-distribution density, $\phi$ of the process satisfies the equation

  $$\mathcal{L}^*\phi = 0, \tag{A.3}$$

  where $\mathcal{L}^*$ denotes the adjoint of the operator $\mathcal{L}$.
- Given a function $\phi$, the homogeneous equation

  $$\mathcal{L}\phi = 0, \tag{A.4}$$

  has only constant solutions.
- The processes are associated with a characteristic mean reversion, holding, or correlation time $\alpha^{-1}$, whereby at infinite times or when $\alpha$ is large, the process tends to its invariant distribution.

## A.3 Expectation and the $\mathcal{L}$-operator

Consider a process $X_t$ that satisfies equation (A.1). If $g$ is a twice differentiable function of a random variable $x$ (independent of $t$) with bounded derivatives, then the operator $\mathcal{L}$ defined by equation (A.2) acts on $g$ as follows

$$\mathcal{L}g(X_t) = \mu(X_t)\frac{\partial}{\partial x}g(X_t) + \frac{1}{2}\sigma^2(X_t)\frac{\partial^2}{\partial x^2}g(X_t). \tag{A.5}$$

Moreover, the derivative of $g(X_t)$ can be obtained using Itô's formula as

$$\begin{aligned} dg(X_t) &= \left[\mu\frac{\partial}{\partial x}g(X_t) + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}g(X_t)\right]dt + \sigma\frac{\partial}{\partial x}g(X_t)dW_t. \\ &= \mathcal{L}g(X_t)dt + \sigma\frac{\partial}{\partial x}g(X_t)dW_t. \end{aligned} \tag{A.6}$$

Integrating equation (A.6) leads to

$$g(X_t) = \int_0^t \mathcal{L}g(X_s)ds + \left[g(X_0) + \int_0^t \sigma(s)\frac{\partial}{\partial s}g(X_s)dW_s\right]. \tag{A.7}$$

The integral term in the brackets is a martingale, where $g(X_0)$ is the value of the function evaluated at the initial value $X_0$, of the diffusion process. Taking expectation on both sides of equation (A.7) yields

$$\mathbb{E}\left\{g(X_t)\right\} - g(X_0) = \mathbb{E}\left\{\int_0^t \mathcal{L}g(X_s)ds\right\}. \tag{A.8}$$

From Lebesgue Convergence Theorem, it follows that

$$\frac{d}{dt}\mathbb{E}\left\{g(X_t)\right\}|_{t=0} = \lim_{t\downarrow 0}\frac{\mathbb{E}\left\{g(X_t)\right\} - g(X_0)}{t}. \tag{A.9}$$

Combining equations (A.8) and (A.9) leads to

$$\frac{d}{dt}\mathbb{E}\left\{g(X_t)\right\}|_{t=0} = \lim_{t\downarrow 0}\mathbb{E}\left\{\frac{1}{t}\int_0^t \mathcal{L}g(X_s)ds\right\} = \mathcal{L}g(X_0). \tag{A.10}$$

## A.4 Green's Function

A function $G(y,s)$ of a linear differential operator $\mathcal{L}$ is known as *Green's function* if

$$\mathcal{L}G(y,s) = \delta(y-s), \tag{A.11}$$

where $\delta$ is the *Dirac delta function*. This function is of significant importance in solving partial differential equations whose closed form solutions are hard to find, an example of such equations is the well-known *Poission equation*. Consider the following equation

$$\mathcal{L}u(y) = f(y), \tag{A.12}$$

where $f(y)$ is known and $\mathcal{L}$ is a linear differential operator, then the solution $u(y)$ to equation (A.12) can be obtained in terms of *Green's function* as:

$$u(y) = \int G(y,s)f(s)\mathrm{d}s. \tag{A.13}$$

*Proof.* The proof is simple. Multiply equation (A.11) by $f(s)$ and integrate with respect to $s$:

$$\int \mathcal{L}G(y,s)f(s)\mathrm{d}s = \int \delta(y-s)f(s)\mathrm{d}s = f(y). \tag{A.14}$$

Since the differential operator $\mathcal{L}$ is linear, it can be swapped with the integral operator as:

$$\mathcal{L}\int G(y,s)f(s)\mathrm{d}s = f(y) = \mathcal{L}u(y), \text{ from equation (A.12).}$$

From which one can conclude that

$$u(y) = \int G(y,s)f(s)\mathrm{d}s. \tag{A.15}$$

□

Notice that equation (A.15) is a *Fredholm Integral*.

## A.5 Feynman-Kac Formula

With reference to the definition of the infinitesimal generator described in A, the following theorem is relevant:

**Theorem A.5.1.** *Assume $\mu(x)$ and $\sigma(x)$ satisfy the global Lipschitz conditions and growth hypotheses. Let $f(x)$ and $R(x)$ be continuous functions such that $R \geq 0$ and $f(x) = \mathcal{O}(|x|)$ as $|x| \to \infty$. Then the function $u(t,x)$ defined by*

$$u(t,x) = \mathbb{E}^* \left\{ \exp\left\{ -\int_0^t R(X_s)ds \right\} f(X_t) \right\}, \tag{A.16}$$

*satisfies the diffusion equation*

$$\frac{\partial u}{\partial t} = \mathcal{L}u - Ru \tag{A.17}$$

*and the initial condition $u(0,x)$. Moreover, $u$ is the only solution to the Cauchy problem that is of at most polynomial growth in $x$.*

# Appendix B

# Second-order Approximations

## B.1 Second-order Approximations to BS price

Herewithin, a derivation of the explicit form of the third term in the analytic expansion for the derivative price, (i.e. the $P_2$ term in equation (4.15)) is presented. To obtain the explicit form of $P_2$, one needs to find out what the value of $k(t,x)$ in equation (4.30) is. From equation (4.37) and the definition of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ presented in (4.12) and (4.13), it follows

$$\begin{aligned}
\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 &= \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 - \langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle. \\
&= \sqrt{2}\rho\nu x \left[ f(y)\frac{\partial^2 P_2}{\partial x \partial y} - \langle f(y)\frac{\partial^2 P_2}{\partial x \partial y} \rangle \right] \\
&\quad - \sqrt{2}\nu \left[ \wedge(y)\frac{\partial P_2}{\partial y} - \langle \wedge(y)\frac{\partial P_2}{\partial y} \rangle \right] + \frac{1}{2}[f^2(y) - \langle f^2(y) \rangle] x^2 \frac{\partial^2 P_1}{\partial x^2}.
\end{aligned} \tag{B.1}$$

Substituting the general form of $P_2$ given by equation (4.30) simplifies equation (B.1) to:

$$\begin{aligned}
\mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 &= [f(y)\mathcal{G}'(y) - \langle f(y)\mathcal{G}'(y) \rangle] Q_1^x(t,x) \\
&\quad + [\wedge(y)\mathcal{G}'(y) - \langle \wedge(y)\mathcal{G}'(y) ] Q_2^x(t,x) \\
&\quad + [f^2(y) - \langle f^2(y) \rangle] Q_3^x(t,x),
\end{aligned} \tag{B.2}$$

where $Q_1^x(t,x), Q_2^x(t,x)$ and $Q_3^x(t,x)$ have been defined as

$$\begin{aligned}
Q_1^x(t,x) &= -\frac{\rho\nu}{\sqrt{2}} x \frac{\partial}{\partial x}\left[ x^2 \frac{\partial^2 P_0}{\partial x^2} \right], \\
Q_2^x(t,x) &= \frac{\nu}{\sqrt{2}} x^2 \frac{\partial^2 P_0}{\partial x^2}, \\
Q_3^x(t,x) &= \frac{1}{2} x^2 \frac{\partial^2 P_1}{\partial x^2}.
\end{aligned}$$

Substituting equation (B.2) in equation (4.36) leads to

$$\begin{aligned}\mathcal{L}_0 P_3 = &- [f(y)\mathcal{G}'(y) - \langle f(y)\mathcal{G}'(y)\rangle] Q_1^x(t,x) \\ &- [\wedge(y)\mathcal{G}'(y) - \langle \wedge(y)\mathcal{G}'(y)] Q_2^x(t,x) \\ &- [f^2(y) - \langle f^2(y)\rangle] Q_3^x(t,x).\end{aligned} \tag{B.3}$$

Following the same procedure presented through equations (4.28)–(4.35), one obtains the value of $P_3$ as:

$$P_3(t,x,y) = -Q_1^x(t,x)Q_1^y(y) - Q_2^x(t,x)Q_2^y(y) - Q_3^x(t,x)Q_3^y(y), \tag{B.4}$$

with $Q_1^y(y), Q_2^y(y)$ and $Q_1^y(y)$ defined as:

$$\begin{aligned}Q_1^y(y) &= \frac{1}{\phi(y)\nu^2}\int_\infty^y [f(w)\mathcal{G}'(w) - \langle f(w)\mathcal{G}'(w)\rangle]\phi(w)dw, \\ Q_2^y(y) &= \frac{1}{\phi(y)\nu^2}\int_\infty^y [\wedge(w)\mathcal{G}'(w) - \langle \wedge(w)\mathcal{G}'(w)\rangle]\phi(w)dw, \\ Q_3^y(y) &= \frac{1}{\phi(y)\nu^2}\int_\infty^y [f^2(w) - \langle f^2(w)\rangle]\phi(w)dw.\end{aligned}$$

Comparing the coefficients of $\varepsilon$ lead to

$$\mathcal{L}_0 P_4 + \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 = 0, \tag{B.5}$$

a *Poisson equation* with solution $P_4$. Moreover $P_4$ has reasonable growth at infinity if:

$$\langle \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 \rangle = 0. \tag{B.6}$$

From equation (4.12),

$$\begin{aligned}\langle \mathcal{L}_1 P_3 \rangle &= \sqrt{2}\rho\nu x \langle f(y)\frac{\partial^2 P_3}{\partial x \partial y}\rangle - \sqrt{2}\nu\langle \wedge(y)\frac{\partial P_3}{\partial y}\rangle. \\ &= -\sqrt{2}\rho\nu x \sum_{i=1}^{3}\langle f(y)Q_i^y(y)\rangle Q_{ix}^x(t,x) + \sqrt{2}\nu\sum_{i=1}^{3}\langle \wedge(y)Q_i^y(y)\rangle Q_i^x(t,x).\end{aligned}$$

Using equation (4.13) gives,

$$\begin{aligned}\langle \mathcal{L}_2 P_2 \rangle &= \langle \mathcal{L}_2 \left[-\frac{1}{2}\mathcal{G}(y)x^2\frac{\partial^2 P_0}{\partial x^2} + k(t,x)\right]\rangle. \\ &= -\frac{1}{2}\langle \tilde{\mathcal{L}}_2\rangle \left[x^2\frac{\partial^2 P_0}{\partial x^2}\right] + \langle \mathcal{L}_2\rangle k(t,x),\end{aligned}$$

where $\langle\tilde{\mathcal{L}}_2\rangle$ has been defined as

$$\langle\tilde{\mathcal{L}}_2\rangle = \langle\mathcal{G}(y)\rangle\frac{\partial}{\partial t} + \frac{\langle f^2(y)\mathcal{G}(y)\rangle}{2}x^2\frac{\partial^2}{\partial x^2} + \langle\mathcal{G}(y)\rangle r\left[x\frac{\partial}{\partial x} - \cdot\right].$$

Substituting these terms in equation (B.6), it can be deduced:

$$\begin{aligned}\langle\mathcal{L}_2\rangle k(t,x) = \frac{1}{2}\langle\tilde{\mathcal{L}}_2\rangle\left[x^2\frac{\partial^2 P_0}{\partial x^2}\right] + \sqrt{2}\rho\nu x\sum_{i=1}^{3}\langle f(y)Q_i^y(y)\rangle Q_{ix}^x(t,x)\\ - \sqrt{2}\nu\sum_{i=1}^{3}\langle\wedge(y)Q_i^y(y)\rangle Q_i^x(t,x).\end{aligned} \tag{B.7}$$

It now remains to solve the PDE (B.7) to obtain $k(t,x)$. Recall that $P_n(T,x,y) = 0$ at maturity in the asymptotic expansion, for all $n > 0$. Thus, from equation (4.30), one obtains the value of $k(T,x)$ at maturity as:

$$k(T,x) = \frac{1}{2}\mathcal{G}(y)x^2\check{\mathcal{L}}P_0(T,K,x), \tag{B.8}$$

where the operator $\check{\mathcal{L}}$ has been defined as $\frac{\partial^2}{\partial x^2}$. It is also clear that $P_0(T,K,x) = \max(x-K,0)$ for a call option. Thus, by definition of *Green's function* presented in A.4 it follows that:

$$k(T,x) = \begin{cases} 0\,, & \text{for } x \neq K \\ \frac{1}{2}\mathcal{G}(y)x^2\delta(x-K)\,, & \text{for } x = K. \end{cases} \tag{B.9}$$

This presents a well posed problem of determining $k(t,x)$. One can therefore conclude from Lemma 4.4.2 (*with* $l = 0$) that,

$$k(t,x) = -[T-t]\left[\frac{1}{2}\langle\tilde{\mathcal{L}}_2\rangle\left[x^2\frac{\partial^2 P_0}{\partial x^2}\right] + \sqrt{2}\rho\nu x\sum_{i=1}^{3}\langle f(y)Q_i^y(y)\rangle Q_{ix}^x(t,x)\right].$$

Hence, equation (4.30) can now be explicitly given as:

$$P_2(t,x,y) = -\frac{1}{2}\mathcal{G}(y)x^2\frac{\partial^2 P_0}{\partial x^2} - [T-t]\mathcal{H}, \tag{B.10}$$

where $\mathcal{H}$ is defined as:

$$\mathcal{H} = \frac{1}{2}\langle\tilde{\mathcal{L}}_2\rangle\left[x^2\frac{\partial^2 P_0}{\partial x^2}\right] + \sqrt{2}\rho\nu x\sum_{i=1}^{3}\langle f(y)Q_i^y(y)\rangle Q_{ix}^x(t,x).$$

The $P_2$-term is also of significant use when it comes to say obtaining the the second–order correction to the implied volatility, see Appendix B.2.

## B.2 Second-order Correction of the Implied Volatility Surface

This section can be regarded as a continuation of Section 4.6. A derivation of the second-order correction to the implied volatility is discussed. Some ideas shall be extracted from Appendix B.1. Recall that by definition,

$$C_{\mathrm{BS}}(t,x;K,T;I) = C^{\mathrm{obs}}(K,T). \tag{B.11}$$

Unlike in Section 4.6, consider an expansion of the implied volatility up to the $\mathcal{O}(\varepsilon)$ term:

$$I = I_0 + \sqrt{\varepsilon} I_1 + \varepsilon I_2. \tag{B.12}$$

From equation (4.82) it follows:

$$C_{\mathrm{BS}}(I) = C_{\mathrm{BS}}(I_0) + \sqrt{\varepsilon} I_1 \frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(I_0) + \varepsilon \left[ I_2 \frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(I_0) + \frac{1}{2} I_1^2 \frac{\partial^2}{\partial \sigma^2} C_{\mathrm{BS}}(I_0) \right].$$

Correspondingly, the second-order corrected price is as follows:

$$\tilde{P} = P_0(\bar{\sigma}) + \sqrt{\varepsilon} P_1(\bar{\sigma}) + \varepsilon P_2(t,x,y).$$

Comparing the *r.h.s* of $\tilde{P}$ and $C_{\mathrm{BS}}(I)$ one can conclude that $I_0 = \bar{\sigma}$ and also:

$$P_2(t,x,y) = I_2 \frac{\partial}{\partial \sigma} C_{\mathrm{BS}}(I_0) + \frac{1}{2} I_1^2 \frac{\partial^2}{\partial \sigma^2} C_{\mathrm{BS}}(I_0). \tag{B.13}$$

Therefore, from equation (B.10) it can be deduced:

$$-\frac{1}{2} \mathcal{G}(y) x^2 \frac{\partial^2 P_0}{\partial x^2} - [T-t]\mathcal{H} = I_2 \frac{\partial P_0}{\partial \sigma} + \frac{1}{2} I_1^2 \frac{\partial^2 P_0}{\partial \sigma^2}. \tag{B.14}$$

The partial derivatives are given as follows:

$$\begin{aligned}
\frac{\partial P_0}{\partial \sigma} &= \frac{x[T-t]^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{\mathrm{d}_+^2}{2}}. \\
\frac{\partial^2 P_0}{\partial \bar{\sigma}^2} &= \frac{x[T-t]\mathrm{d}_+}{\sqrt{2\pi\bar{\sigma}}} \left[ r + \frac{3}{2}\bar{\sigma}^2 \right] e^{-\frac{\mathrm{d}_+^2}{2}}.
\end{aligned}$$

Substituting these in equation (B.14) yields

$$I_2 = -\frac{1}{2} \frac{\mathcal{G}(y)\mathcal{D}_2 P_0 \sqrt{2\pi} e^{\frac{\mathrm{d}_+^2}{2}}}{x\sqrt{T-t}} - \frac{\sqrt{2\pi}[T-t]\mathcal{H} e^{\frac{\mathrm{d}_+^2}{2}}}{x} - \frac{1}{2} \frac{1}{\bar{\sigma} x^2 [T-t]} \left[ r + \frac{3}{2}\bar{\sigma}^2 \right] I_1^2.$$

# Appendix C

# Proofs

## C.1 Verification of the solution to Poisson equation

*Proof.* The verification for the solution to the general Poisson equation discussed in Chapter 4, Section 4.3.2 follows. Equation (4.22) is of the form

$$\mathcal{L}_0(y) + g(y) = 0. \tag{C.1}$$

which is Poisson equation for $\mathcal{X}(y)$ with respect to $\mathcal{L}_0$ given $g(y)$. Taking expectation with respect to the invariant distribution of $(Y_t)_{t\geq 0}$ gives

$$\langle g(y)\rangle = -\langle \mathcal{L}_0\mathcal{X}(y)\rangle = -\int_{-\infty}^{\infty}[\mathcal{L}_0\mathcal{X}(y)]\phi(y)\,dy = -\int_{-\infty}^{\infty}\mathcal{X}(y)[\mathcal{L}_0^*\phi(y)]\,dy = 0.$$

where $\phi(y)$ is the density of the invariant distribution of $(Y_t)_{t\geq 0}$, and the fact that $\mathcal{L}_0^*\phi(y) = 0$ has been applied. Recall that the invariant distribution of $(Y_t)_{t\geq 0}$ is the same as its long-run distribution,

$$\lim_{t\to+\infty}\mathbb{E}\{g(Y_t)|Y_0 = y\} = \langle g\rangle, \tag{C.2}$$

where this convergence is exponential. Assuming the centering condition $\langle g(y)\rangle = 0$, the solution to equation (C.1) becomes

$$\mathcal{X} = \int_0^{+\infty}\mathbb{E}\{g(Y_t)|Y_0 = y\}\,dt\,. \tag{C.3}$$

To verify equation (C.3), write

$$\mathcal{L}_0\int_0^{+\infty}\mathbb{E}\{g(Y_t)|Y_0 = y\}\,dt = \int_0^{+\infty}\mathcal{L}_0\mathbb{E}\{g(Y_t)|Y_0 = y\}\,dt\,, \tag{C.4}$$

with the assumption that the integral and differential operators are interchangeable. By using Itô's formula the derivative $dg(Y_t)$ is given as

$$g(Y_t) = g(y) + \int_0^t \mathcal{L}_0 g(Y_s)\, ds + \mathrm{M}, \tag{C.5}$$

where M denotes the martingale part. Taking conditional expectation on both sides of this equation yields:

$$\mathbb{E}\{g(Y_t)|Y_0 = y\} - g(y) = \int_0^t \mathcal{L}_0 \mathbb{E}\{g(Y_s)|Y_0 = y\}\, ds\,, \tag{C.6}$$

differentiating this with respect to $t$ yields

$$\frac{d}{dt}\mathbb{E}\{g(Y_t)|Y_0 = y\} = \mathcal{L}_0 \mathbb{E}\{g(Y_s)|Y_0 = y\}. \tag{C.7}$$

The above relation implies equation (C.4) can be written as:

$$\begin{aligned}\mathcal{L}_0 \int_0^{+\infty} \mathbb{E}\{g(Y_t)|Y_0 = y\}\, dt &= \int_0^{+\infty} \frac{d}{dt}\mathbb{E}\{g(Y_t)|Y_0 = y\}\, dt\,. \\ &= \mathbb{E}\{g(Y_t)|Y_0 = y\}|_{t=+\infty} - \mathbb{E}\{g(Y_t)|Y_0 = y\}|_{t=0}.\end{aligned}$$

Applying equation (C.2) and the centering condition $\langle g(Y_t)\rangle = 0$ leads to

$$\mathcal{L}_0 \int_0^{+\infty} \mathbb{E}\{g(Y_t)|Y_0 = y\}\, dt = -\mathbb{E}\{g(Y_t)|Y_0 = y\}|_{t=0} = -g(y).$$

This verifies the claim that provided $\langle g\rangle = 0$, the solution $\mathcal{X}$ has polynomial growth at infinity and is given by equation (C.3) . Solutions to the Poisson equation (C.1) can be obtained by adding constants to equation (C.3). □

## C.2 Proof of Lemma 4.9.1

*Proof.* Under a risk neutral measure, $\mathbb{P}^*$, recall that the model in equation (4.1), becomes[1]

$$\begin{aligned} dX_t &= \left[r - \frac{1}{2}f^2(Y_t)\right] dt + \sigma_t\, dW_t^{*(1)}\,, \\ \sigma_t &= f(Y_t)\,, \\ dY_t &= \left[\frac{1}{\varepsilon}[m - Y_t] - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \wedge (Y_t)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\, dW_t^{*(2)}\,, \end{aligned} \tag{C.8}$$

Consequently, the price of the derivative is given by

$$P^\varepsilon(t,x,y) = \mathbb{E}^*\left\{e^{-r[T-t]}h(X_T)|X_t = x\right\}, \tag{C.9}$$

[1]By using the logarithm transform: $X_t = \log \mathbf{X}_t$ and Itô's formula.

where $r$ denotes the risk free rate of return, and $\mathbb{E}^*\{\cdot\}$ denotes the expectation operator with respect to the risk neutral measure $\mathbb{P}^*$. Now, suppose that under regularization the stock price dynamics follows:

$$d\tilde{X}_t = \left[r - \frac{1}{2}\tilde{f}(t,Y_t)^2\right] dt + \tilde{f}(t,Y_t)[\sqrt{1-\rho^2}\, dW_t^{*(1)} + dW_t^{*(3)}], \tag{C.10}$$

where $(W_t^{*(1)})$ and $(W_t^{*(3)})$ are two independent Brownian motions and,

$$\tilde{f}(t,Y_t) = \begin{cases} f(y) & \text{for} \quad t \leqslant T \\ \bar{\sigma} & \text{for} \quad t > T \end{cases}$$

Then, the regularized and unregularized prices are respectively, given as

$$\begin{aligned} P^{\varepsilon,\zeta}(t,x,y) &= \mathbb{E}^*\left\{e^{-r(T-t+\zeta)}h(\tilde{X}_{T+\zeta})\right\}, \quad \text{and} \\ P^{\varepsilon}(t,x,y) &= \mathbb{E}^*\left\{e^{-r(T-t)}h(\tilde{X}_T)\right\}, \end{aligned}$$

from which, it follows that

$$P^{\varepsilon,\zeta}(t,x,y) - P^{\varepsilon}(t,x,y) = \mathbb{E}^*\left\{e^{-r[T-t+\zeta]}h(\tilde{X}_{T+\zeta})\right\} - \mathbb{E}^*\left\{e^{-r[T-t]}h(\tilde{X}_T)\right\}.$$

Given that the process $(W_s^{*(3)})$ is adapted for $t \leq s \leq T$ and that $\tilde{X}_s = x$, then, by using iterated expectation, one can rewrite the last equation as follows:

$$\begin{aligned} &P^{\varepsilon,\zeta}(t,x,y) - P^{\varepsilon}(t,x,y) = \\ &\mathbb{E}^*\left\{\mathbb{E}^*\left\{e^{-r[T-t+\zeta]}h(\tilde{X}_{T+\zeta}) - e^{-r[T-t]}h(\tilde{X}_T)\right\}|(W_s^{*(3)})_{t\leq s\leq T}\right\}. \end{aligned} \tag{C.11}$$

Taking $\tilde{f}(t,Y_t)$ to be $(W_s^{*(3)})$-adapted, the solution to equation (C.10) at time $T$, is given by

$$\tilde{X}_T = x + \int_t^T \tilde{f}(s,Y_s)\left[\sqrt{1-\rho^2}\, dW_s^{*(1)} + \rho\, dW_s^{*(3)}\right] + r[T-t] - \frac{1}{2}\int_t^T \tilde{f}(s,Y_s)^2\, ds.$$

From this equation, the variance of $\tilde{X}_T$ takes the form

$$\vartheta_1 = \mathrm{Var}\{\tilde{X}_T\} = [1-\rho^2]\int_t^T \tilde{f}(s,Y_s)^2\, ds.$$

and the mean is given as

$$\tilde{m}_1 = \mathrm{Mean}\{\tilde{X}_T\} = x + \rho\int_t^T \tilde{f}(s,Y_s)\, dW_s^{*(3)} + r[T-t] - \frac{1}{2}\int_t^T \tilde{f}(s,Y_s)^2\, ds.$$

One can also rewrite the variance, $\vartheta_1$ and mean, $\tilde{m}_1$ as

$$\vartheta_1 = \bar{\sigma}^2[T-t], \tag{C.12}$$

$$\tilde{m}_1 = x + \zeta_{t,T} + [r - \frac{1}{2}\bar{\sigma}^2][T-t], \tag{C.13}$$

where $\bar{\sigma}$ and $\zeta_{t,T}$ are defined as

$$\bar{\sigma}^2 = \frac{1-\rho^2}{T-t}\int_t^T \tilde{f}(s,Y_s)^2\,ds \quad \text{and,} \tag{C.14}$$

$$\zeta_{t,T} = \rho\int_t^T \tilde{f}(s,Y_s)\,dW_s^{*(3)} - \frac{1}{2}\rho^2\int_t^T \tilde{f}(s,Y_s)^2\,ds\,. \tag{C.15}$$

Thus, $\tilde{X}_T$ is normally distributed with mean $\tilde{m}_1$ and variance $\vartheta_1$. The regularized process $\tilde{X}_{T+\zeta}$ can also be written as

$$\begin{aligned}\tilde{X}_{T+\zeta} &= x + \sqrt{[1-\rho^2]}\int_t^{T+\zeta} \tilde{f}(\tau,Y_\tau)\,dW_\tau^{*(1)} + \rho\int_t^{T+\zeta} \tilde{f}(\tau,Y_\tau)\,dW_\tau^{*(3)} \\ &+ r[T-\zeta-t] - \frac{1}{2}\int_t^{T+\zeta} \hat{f}(\tau,Y_\tau)^2\,d\tau\,.\end{aligned} \tag{C.16}$$

Alternatively,

$$\begin{aligned}\tilde{X}_{T+\zeta} = &\sqrt{1-\rho^2}\int_t^T \tilde{f}(s,Y_s)\,dW_s^{*(1)} + \sqrt{1-\rho^2}\int_T^{T+\zeta} \bar{\sigma}\,dW_{\tau^*}^{*(3)} \\ &+ \rho\int_t^T \tilde{f}(s,Y_s)\,dW_s^{*(3)} + \rho\int_T^{T+\zeta} \bar{\sigma}\,dW_{\tau^*}^{*(3)} + r[T+\zeta-t] \\ &- \frac{1}{2}\int_t^T \tilde{f}(s,Y_s)^2\,ds - \frac{1}{2}\int_T^{T+\zeta} \bar{\sigma}\,d\tau^*\,.\end{aligned} \tag{C.17}$$

From equation (C.17), one can obtain the variance, $\vartheta_2$ of $X_{T+\zeta}$ as

$$\begin{aligned}\vartheta_2 &= [1-\rho^2]\int_t^T \tilde{f}(s,Y_s)^2\,ds + [1-\rho^2]\bar{\sigma}^2\zeta + \rho^2\bar{\sigma}^2\zeta. \\ &= [1-\rho^2]\int_t^T \tilde{f}(s,Y_s)^2\,ds + \bar{\sigma}^2\zeta,\end{aligned} \tag{C.18}$$

and the mean $\tilde{m}_2$ of $X_{T+\zeta}$ as

$$\tilde{m}_2 = \rho\int_t^T \tilde{f}(s,Y_s)\,dW_s^{*(3)} + r[T-t] + r\zeta - \frac{1}{2}\int_t^T \tilde{f}(s,Y_s)^2\,ds - \frac{1}{2}\bar{\sigma}\zeta. \tag{C.19}$$

Equations (C.18) and (C.19) can be simplified as:

$$\vartheta_2 = \tilde{\sigma}_\zeta^2[T-t], \tag{C.20}$$

$$\tilde{m}_2 = x + \zeta_{t,T} + r\zeta + [r - \frac{1}{2}\tilde{\sigma}_\zeta^2][T-t]. \tag{C.21}$$

where $\tilde{\sigma}_\zeta^2$ is defined as:

$$\tilde{\sigma}_\zeta^2 = \bar{\sigma}^2 + \frac{\bar{\sigma}^2\zeta}{T-t}, \tag{C.22}$$

and $\bar{\sigma}^2$ and $\zeta_{t,T}$ are given respectively, by equations (C.14) and (C.15). Thus, $X_{T+\zeta}$ is normally distributed with mean $\tilde{m}_2$ and variance $\vartheta_2$. Now, since the distributions of $X_T$ and $X_{T+\zeta}$ are known, one can compute the inner expectation in equation (C.11), to obtain the difference between the Black-Scholes prices with respect to the two distributions:

$$P^{\varepsilon,\zeta}(t,x,y) - P^{\varepsilon}(t,x,y) = \\ \mathbb{E}^* \left\{ P_0(t, x+\zeta_{t,T}+r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, x+\zeta_{t,T}; K, T; \bar{\sigma}) \right\}. \tag{C.23}$$

Note that $P_0$ satisfies Black-Scholes PDE, so if $z := \log x$, then $P_0$ can be given explicitly as:

$$P_0(t,z;K,T;\sigma) = e^z N(\mathrm{d}_+) - Ke^{-r\frac{\tau^2}{\sigma^2}} N(\mathrm{d}_+ - \tau), \tag{C.24}$$

where $\mathrm{d}_+$ from Section 1.4.4 has been simplified to

$$\mathrm{d}_+ = \frac{z - \log K}{\tau} + b\tau \quad \text{with,} \\ b = \frac{r^2}{\sigma^2} + \frac{1}{2}, \text{ and } \quad \tau = \sigma\sqrt{T-t}.$$

Applying this definition to equation (C.23) above, where $\tilde{\sigma}_\zeta$ is defined in equation (C.22), taking note that $\bar{\sigma}$ is bounded and that $0 \le N(\cdot) \le 1$, it follows that

$$P_0(t, z+\zeta_{t,T}+r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, z+\zeta_{t,T}; K, T; \bar{\sigma}) \\ = e^{z+\zeta_{t,T}} \left[ e^{r\zeta} N(\mathrm{d}_+^\zeta) - N(\mathrm{d}_+) \right] - K \left[ e^{-r\frac{\tilde{\tau}^2}{\tilde{\sigma}^2}} N(\mathrm{d}_+^\zeta - \tilde{\tau}) - e^{-r\frac{\bar{\tau}^2}{\bar{\sigma}^2}} N(\mathrm{d}_+ - \bar{\tau}) \right].$$

Taking the magnitude on both sides of the above equation yields

$$|P_0(t, z+\zeta_{t,T}+r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, z+\zeta_{t,T}; K, T; \bar{\sigma})| \\ \le c_1 \zeta e^{\zeta_{t,T}} |N(\mathrm{d}_+^\zeta) - N(\mathrm{d}_+)| + c_2, \tag{C.25}$$

where $e^z$ and $K$ are taken to be constants. Using the fact that $e^u \approx 1 + |u|$ for $0 < u \ll 1$, then

$$N(k_1) - N(k_2) = \int_{k_1}^{k_2} e^{-\frac{y^2}{2}}\, dy \le |k_2 - k_1|, \tag{C.26}$$

for some constants $k_1$ and $k_2$. Thus,

$$|N(\mathrm{d}_+^\zeta) - N(\mathrm{d}_+)| = |N \left[ \frac{z+\zeta_{t,T}+r\zeta}{\tilde{\tau}} + \tilde{b}\tilde{\tau} \right] - N \left[ \frac{z+\zeta_{t,T}}{\bar{\tau}} + \bar{b}\bar{\tau} \right] |,$$

taking $z$ as a constant and $\bar{\sigma}$ being bounded below and above, it is deduced that

$$|N(\mathrm{d}_+^\zeta) - N(\mathrm{d}_+)| = |c_3\zeta + c_4| - |c_5\zeta + c_6|. \\ \le c_7|\zeta| + c_8.$$

Therefore, equation (C.25) becomes

$$|P_0(t, z+\zeta_{t,T}+r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, z+\zeta_{t,T}; K, T; \bar{\sigma})| \le c_1 \zeta e^{\zeta_{t,T}} (c_7|\zeta| + c_8) + c_2.$$

Let $c = \max\{c_1, c_2, c_7, c_8\}$, thus:

$$|P_0(t, z + \zeta_{t,T} + r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, z + \zeta_{t,T}; K, T; \bar{\sigma})| \leq c\zeta(e^{\zeta_{t,T}}(|\zeta_{t,T}| + 1) + 1).$$

By definition of $\zeta_{t,T}$ from equation (C.15) and existence of its moments, it follows that

$$|P_0(t, z + \zeta_{t,T} + r\zeta; K, T; \tilde{\sigma}_\zeta) - P_0(t, z + \zeta_{t,T}; K, T; \bar{\sigma})| \leq c_1^*\zeta. \tag{C.27}$$

for some constant $c_1^* > 0$ and for a small $\zeta$. This concludes the proof. □

## C.3 Proof of Lemma 4.9.2

*Proof.* It follows that

$$P^\zeta(t, x) - P(t, x) = \left[1 - [T - t]\left[V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x^3 \frac{\partial^3}{\partial x^3}\right]\right][P_0^\zeta - P_0].$$

Recall that $V_2$ and $V_3$ are given as

$$V_2 = \frac{1}{\nu\sqrt{2\alpha}}\langle\left[-2\rho\mathbf{F}(y) + \rho[\mu - r]\tilde{\mathbf{F}}(y) + \sqrt{1-\rho^2}\mathbf{\Gamma}(y)\right][f^2(y) - \langle f^2(y)\rangle]\rangle.$$
$$V_3 = -\frac{\rho}{\nu\sqrt{2\alpha}}\langle\mathbf{F}(y)[f^2(y) - \langle f^2(y)\rangle]\rangle,$$

where $\varepsilon = 1/\alpha$. Note that $\mathbf{F}$, $\tilde{\mathbf{F}}$ and $\mathbf{\Gamma}$ are bounded since $f$ and $\gamma$ are bounded, that is; $0 < k_1 < |f(y)| < k_2 < \infty$ and $|\gamma(y)| < l < \infty, \forall\, y \in \mathcal{R}$ given some positive numbers $k_1$, $k_2$ and $l$. Thus,

$$\max\ \{|V_2|, |V_3|\} \leq c_1\sqrt{\varepsilon},$$

where $c_1$ is some positive constant greater than zero. Note that both $P_0 = C_{BS}(t, x; K, T; \bar{\sigma})$ and $P_0^\zeta = C_{BS}(t - \zeta, x; K, T; \bar{\sigma})$ together with their derivatives with respect to $x$ are differentiable for all $t$. If $t, x$ and $y$ are fixed, then the following is true:

$$|P(t, x) - P^\zeta(t, x)| \leq c_2^*\zeta,$$

for some constant $c_2^* > 0$ and $0 < \zeta << 1$. □

## C.4 Proof of Lemma 4.9.3

*Proof.* Suppose that the error created by approximating $P^\varepsilon$ with $P = P_0 + p_1$ is $R^\varepsilon$ and recall that some terms dependent on the current value of volatility, $y$ where left out which implies that $R^\varepsilon$ depends on $t, x$ and $y$. The regularized error, $R^{\varepsilon,\zeta}$ is thus, given by

$$\begin{aligned} R^{\varepsilon,\zeta}(t,x,y) &= P^{\varepsilon,\zeta} - [P_0^\zeta + \varepsilon P_1^\zeta] \\ &= \varepsilon P_1^\zeta + \varepsilon\sqrt{\varepsilon} P_2^\zeta + \cdots . \end{aligned}$$

This suggests that the error $R^{\varepsilon,\zeta}$ is of order $\mathcal{O}(\varepsilon)$ which converges to zero as $\varepsilon$ goes to zero. The verification follows immediately: Compute the error $R^{\varepsilon,\zeta}$ by obtaining the difference between the two prices, that is, the regularized exact price, $P^{\varepsilon,\zeta}$ and its regularized approximation, $[P_0^\zeta + p_1^\zeta]$:

$$\begin{aligned} R^{\varepsilon,\zeta}(t,x,y) = {} & P_0^\zeta(t,x) + \sqrt{\varepsilon} P_1^\zeta(t,x) + \varepsilon P_2^\zeta(t,x) \\ & + \varepsilon\sqrt{\varepsilon} P_3^\zeta(t,x,y) + \mathcal{O}(\varepsilon^2) - [P_0^\zeta(t,x) + p_1^\zeta(t,x)]. \end{aligned} \tag{C.28}$$

Recall that at maturity, $T$, the payoff $P_0^\zeta(T,x) = h^\zeta(x)$ and $p_1^\zeta(T,x) = 0$, where $h^\zeta(x)$ is a smooth bounded function for all $t \leq T$. Thus, the value of the error at maturity is

$$\begin{aligned} R^{\varepsilon,\zeta}(T,x,y) &= P_0^\zeta(T,x) + \sqrt{\varepsilon} P_1^\zeta(T,x) + \varepsilon P_2^\zeta(T,x,y) \\ &\quad + \varepsilon\sqrt{\varepsilon} P_3^\zeta(T,x,y) + \mathcal{O}(\varepsilon^2) - [P_0^\zeta(T,x) + p_1^\zeta(T,x)]. \\ &= \varepsilon \left[ P_2^\zeta(T,x,y) + \sqrt{\varepsilon} P_3^\zeta(T,x,y) + \mathcal{O}(\varepsilon) \right] := H^{\varepsilon,\zeta}(T,x,y). \end{aligned} \tag{C.29}$$

This shows that at maturity this error would be of order $\mathcal{O}(\varepsilon)$. Consider the expansion of the regularized price $P^{\varepsilon,\zeta}$:

$$P^{\varepsilon,\zeta} = P_0^\zeta(t,x) + \sqrt{\varepsilon} P_1^\zeta(t,x) + \varepsilon P_2^\zeta(t,x) + \varepsilon\sqrt{\varepsilon} P_3^\zeta(t,x,y) - R^{\varepsilon,\zeta}, \tag{C.30}$$

where $R^{\varepsilon,\zeta} = R^{\varepsilon,\zeta}(t,x,y)$ is the regularized error in approximating $P^\varepsilon$ with $P$. Applying the operator $\mathcal{L}^\varepsilon$ to equation (C.30) and revisiting equation (4.107) implies that

$$\begin{aligned} \mathcal{L}^\varepsilon R^{\varepsilon,\zeta}(t,x,y) = {} & \left[ \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right] P_0(t,x) \\ & \left[ \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right] \sqrt{\varepsilon} P_1^\zeta(t,x) \\ & \varepsilon \left[ \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right] P_2^\zeta(t,x,y) \\ & \varepsilon\sqrt{\varepsilon} \left[ \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 \right] P_3^\zeta(t,x,y). \end{aligned} \tag{C.31}$$

Rearranging equation (C.31) and relaxing the arguments in the meantime, leads to

$$\begin{aligned}\mathcal{L}^{\varepsilon}R^{\varepsilon,\zeta}(t,x,y) = &\left[\frac{1}{\varepsilon}\mathcal{L}_0P_0^{\zeta} + \frac{\sqrt{\varepsilon}}{\varepsilon}\mathcal{L}_0P_1^{\zeta} + \sqrt{\varepsilon}\mathcal{L}_0P_3^{\zeta}\right]\\ &\left[\frac{1}{\varepsilon}\mathcal{L}_1P_0^{\zeta} + \frac{\sqrt{\varepsilon}}{\varepsilon}\mathcal{L}_1P_1^{\zeta} + \sqrt{\varepsilon}\mathcal{L}_1P_3^{\zeta}\right]\\ &\left[\frac{1}{\varepsilon}\mathcal{L}_2P_0^{\zeta} + \frac{\sqrt{\varepsilon}}{\varepsilon}\mathcal{L}_2P_1^{\zeta} + \sqrt{\varepsilon}\mathcal{L}_2P_3^{\zeta}\right].\end{aligned} \tag{C.32}$$

Observe that $\mathcal{L}_0P_0^{\zeta} = \mathcal{L}_1P_0^{\zeta} = \mathcal{L}_1P_1^{\zeta} = 0$, since $P_0^{\zeta}$ and $P_1^{\zeta}$ are independent of $y$ so,

$$\begin{aligned}\mathcal{L}^{\varepsilon}R^{\varepsilon,\zeta}(t,x,y) = &\left[\mathcal{L}_0P_2^{\zeta} + \mathcal{L}_1P_1^{\zeta} + \mathcal{L}_2P_0^{\zeta}\right] + \sqrt{\varepsilon}\left[\mathcal{L}_0P_3^{\zeta} + \mathcal{L}_1P_2^{\zeta} + \mathcal{L}_2P_1^{\zeta}\right]\\ &+\varepsilon\left[\mathcal{L}_1P_3 + \mathcal{L}_2P_2^{\zeta}\right] + \varepsilon\sqrt{\varepsilon}\mathcal{L}_2P_3^{\zeta}.\end{aligned} \tag{C.33}$$

Revisiting equations (4.21) and (4.36) shows that

$$\mathcal{L}^{\varepsilon}R^{\varepsilon,\zeta}(t,x,y) = \varepsilon\left[\mathcal{L}_1P_3^{\zeta} + \mathcal{L}_2P_2^{\zeta}\right] + \varepsilon\sqrt{\varepsilon}\mathcal{L}_2P_3^{\zeta} := G^{\varepsilon,\zeta}(t,x,y). \tag{C.34}$$

Authors [39], showed that the terms $G(t,x,y)$ and $H(t,x)$ can be expressed as,

$$\begin{aligned}G^{\varepsilon,\zeta}(t,x,y) = &\,\varepsilon\left[\sum_{i=1}^{4} g_i^{(1)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta} + [T-t]\sum_{i=1}^{6} g_i^{(2)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta}\right]\\ &+\varepsilon\sqrt{\varepsilon}\left[\sum_{i=1}^{5} g_i^{(3)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta} + [T-t]\sum_{i=1}^{7} g_i^{(4)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta}\right].\end{aligned} \tag{C.35}$$

$$H^{\varepsilon,\zeta}(T,x,y) = \varepsilon\left[\sum_{i=1}^{2} h_i^{(1)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta}(T,x)\right] + \varepsilon\sqrt{\varepsilon}\left[\sum_{i=1}^{3} h_i^{(2)}(y)\frac{\partial^i}{\partial x^i}P_0^{\zeta}(T,x)\right], \tag{C.36}$$

which leads to the following lemmas:

**Lemma C.4.1.** *Let $\mathcal{G} = g_i^{(j)}$ or $\mathcal{G} = h_i^{(j)}$ where $g_i^{(j)}$ and $h_i^{(j)}$ are defined as in equations (C.35) and (C.36). Then there exists a constant $c > 0$ such that $\mathbb{E}^*\{|\mathcal{G}(Y_s)|Y_t = y\} \le c < \infty$ for $t \le s \le T$.*

**Lemma C.4.2.** *Assume $T - t > \zeta > 0$ and $\mathbb{E}^*\{|\mathcal{G}(Y_s)|Y_t = y\} \le c_1 < \infty$ for some constant $c_1$ then there exist constants $c_2 > 0$ and $\bar{\zeta}$ such that for $\zeta < \bar{\zeta}$ and $t \le s \le T$,*

$$|\mathbb{E}^*\left\{\sum_{i=1}^{n}\mathcal{G}(Y_s)\frac{\partial^i}{\partial x^i}P_0^{\zeta}(s,X_s)\right\}| \le c_2[T+\zeta-s]^{\min[0,1-n/2]}, \textit{ and as a result,}$$

$$\begin{aligned}&|\mathbb{E}^*\left\{\int_t^T (T-s)^P\sum_{i=1}^{n} e^{-r(s-t)}\mathcal{G}(Y_s)\frac{\partial^i}{\partial x^i}P_0^{\zeta}(s,X_s)\,ds\right\}|\\ &\le \begin{cases} c_2|\log(\zeta)| & \textit{for } n = 4+2p\\ c_2\zeta^{\min[0,p+(4-n)/2]} & \textit{else}\end{cases}.\end{aligned}$$

Thus, one can now use the probabilistic representation of equation (C.34), that is, $\mathcal{L}^{\varepsilon}R^{\varepsilon,\zeta} =$

$G^{\varepsilon,\zeta}$ with $H^{\varepsilon,\zeta}(T,x,y)$ as its terminal condition, to finalize the proof of Lemma 4.9.3:

$$R^{\varepsilon,\zeta}(t,x,y) = \mathbb{E}^* \left\{ e^{-r(T-t)} H^{\varepsilon,\zeta}(X_s,Y_s)\,ds - \int_t^T e^{-r(s-t)} G^{\varepsilon/\zeta}(s,X_s,Y_s)\,ds \right\}.$$

Lemma C.4.2 implies that there exists a constant $c > 0$ such that,

$$|\mathbb{E}^* \left\{ \int_t^T e^{-r(s-t)} G^{\varepsilon,\zeta}(X_s,Y_s)\,ds \right\}| \leq c \left\{ \varepsilon + \varepsilon|\log(\zeta)| + \varepsilon\sqrt{\varepsilon/\zeta} \right\},$$

$$|\mathbb{E}^* \left\{ H^{\varepsilon,\zeta}(X_T,Y_T) \right\}| \leq c \left\{ \varepsilon + \varepsilon\sqrt{\varepsilon/\zeta} \right\}.$$

Thus, for $t, x$ and $y$ fixed with $t < T$:

$$|P^{\varepsilon,\zeta} - P^{\zeta}| = |\varepsilon P_2^{\zeta} + \varepsilon\sqrt{\varepsilon} P_3^{\zeta} - R^{\varepsilon,\zeta}| \leq c_3^* \left\{ \varepsilon + \varepsilon|\log(\zeta)| + \varepsilon\sqrt{\varepsilon/\zeta} \right\}$$

Note that $P_2^{\zeta}$ is bounded since $\mathcal{G}(y)$ is bounded (see equation (4.30)) and as a consequence, $P_3^{\zeta}$ is also bounded because $P_1^{\zeta}$ is bounded, (see equation (4.36)). This concludes the proof. □

## C.5 Proof of Lemma 4.11.3

*Proof.* The following proof is given in [41]. In order to establish the accuracy of the approximation, the following higher order approximation is considered:

$$\begin{aligned} \widehat{P^{\varepsilon,\delta}} &= \tilde{P^{\varepsilon,\delta}} + \varepsilon[P_2,0 + \sqrt{\varepsilon}P_{3,0}] + \sqrt{\delta}[\sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1}] \\ &= P_0 + \sqrt{\varepsilon}P_{1,0} + \sqrt{\varepsilon}P_{2,0} + \varepsilon\sqrt{\varepsilon}P_{3,0} + \sqrt{\delta}[P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1}]. \end{aligned} \tag{C.37}$$

where $P_0$ and $P_{1,0}$ are respectively, defined in equations (4.193) and (4.197), $P_{2,0}$ and $P_{3,0}$ respectively, by (4.192) and (4.195). Moreover, $P_{0,1}$ is given in equation (4.206). Next, is to find $P_{1,1}$ and $P_{2,1}$. Define $P_{2,1}$ as

$$P_{2,1} = -\mathcal{L}_0^{-1}[[\mathcal{L}_2 - \langle\mathcal{L}_2\rangle]P_{0,1} + [\mathcal{M}_1 - \langle\mathcal{M}_1\rangle]P_0], \tag{C.38}$$

as a solution of the Poisson equation (4.205). Now, collecting terms of order $\sqrt{\varepsilon}$ in equation (4.187) gives the Poisson equation for $P_{3,1}$:

$$\mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} = -\mathcal{M}_1 P_{1,0} - \mathcal{M}_3 P_{2,0}, \tag{C.39}$$

which admits a reasonable solution if given that

$$\langle\mathcal{L}_2\rangle P_{1,1} = \mathcal{A}P_{0,1} + \mathcal{B}P_0 - \langle\mathcal{M}_1 P_{1,0} - \langle\mathcal{M}_3 P_{2,0}\rangle, \tag{C.40}$$

where $\mathcal{A}$ is defined in (4.198) and $\mathcal{B}$ is defined similarly, as

$$\mathcal{B} = \langle\mathcal{L}_1\mathcal{L}_0^{-1}[\mathcal{M}_1 - \langle\mathcal{M}_1\rangle]\rangle. \tag{C.41}$$

Next, is to introduce the residual

$$R^{\varepsilon,\delta} = \widehat{P^{\varepsilon,\delta}} - P^{\varepsilon,\delta} \tag{C.42}$$

which satisfies

$$\begin{aligned}
R^{\varepsilon,\delta} = & \frac{1}{\varepsilon}[\mathcal{L}_0 P_0] + \frac{1}{\sqrt{\varepsilon}}[\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0] + [\mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0] \\
& + \sqrt{\varepsilon}[\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}] \\
& + \sqrt{\delta}\left[\frac{1}{\varepsilon}[\mathcal{L}_0 P_{0,1}] + \frac{1}{\sqrt{\varepsilon}}[\mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0]\right] \\
& + \sqrt{\delta}[\mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 + \mathcal{M}_3 P_{1,0}] \\
& + \varepsilon R_1^{\varepsilon} + \sqrt{\varepsilon\delta} R_2^{\varepsilon} + \delta R_3^{\varepsilon},
\end{aligned} \tag{C.43}$$

where

$$R_1^{\varepsilon} = \mathcal{L}_2 P_{2,0} + \mathcal{L}_1 P_{3,0} + \sqrt{\varepsilon}\mathcal{L}_2 P_{3,0}, \tag{C.44}$$
$$R_2^{\varepsilon} = \mathcal{L}_2 P_{1,1} + \mathcal{L}_1 P_{2,1} + \mathcal{M}_1 P_{1,0} + \mathcal{M}_3 P_{2,0} \tag{C.45}$$
$$+ \sqrt{\varepsilon}[\mathcal{L}_2 P_{2,1} + \mathcal{M}_1 P_0 + \mathcal{M}_3 P_{1,1}] \tag{C.46}$$
$$R_3^{\varepsilon} = \mathcal{M}_1 P_{0,1} + \mathcal{M}_2 P_0 + \mathcal{M}_3 P_{1,1} \tag{C.47}$$
$$+ \sqrt{\varepsilon}[\mathcal{M}_1 P_{1,1} + \mathcal{M}_2 P_{1,0} + \mathcal{M}_3 P_{2,1}] + \varepsilon[\mathcal{M}_1 P_{2,1} + \mathcal{M}_2 P_{2,0}], \tag{C.48}$$

are smooth functions of $t, x, y$ and $z$ that are,for $\varepsilon \leq 1, \delta \leq 1$, bounded by smooth functions of $t, x, y, z$ independent of $\varepsilon$ and $\delta$, uniformly bounded in $t, x, z$ and at most linearly growing in $y$ through the solution of the Poisson equation,

$$\mathcal{L}_0 \mathcal{G}(y,z) = f^2(y,z) - \bar{\sigma}(z). \tag{C.49}$$

Now, by comparing and cancelling terms on both sides of equation (C.43), it is deduced:

$$\mathcal{L}^{\varepsilon,\delta} R^{\varepsilon,\delta} \varepsilon \mathcal{R}_1^{\varepsilon} + \sqrt{\varepsilon\delta} R_2^{\varepsilon} + \delta R_3^{\varepsilon}. \tag{C.50}$$

At maturity $T$, it follows that

$$\begin{aligned}
R^{\varepsilon,\delta}(T,x,y,z) = & \widehat{P^{\varepsilon,\delta}}(T,x,y,z). \\
& = \varepsilon[P_{2,0} + \sqrt{\varepsilon}P_{3,0}](T,x,y,z) + \sqrt{\varepsilon\delta}[P_{1,1} + \sqrt{\varepsilon}P_{2,1}](T,x,y,z). \\
& := \varepsilon G_1(x,y,z) + \sqrt{\varepsilon\delta} G_2(x,y,z),
\end{aligned} \tag{C.51}$$

where $G_1$ and $G_2$ are independent of $t$ and have the same properties as the $R$'s, above. Thus, from (C.50) and (C.51), it follows that

$$R^{\varepsilon,\delta} = \varepsilon\, \mathbb{E}^* \left\{ e^{-[T-t]} G_1(X_T, Y_T, Z_T) - \int_t^T e^{-[\tau-t]} R_1^{\varepsilon}(\tau, X_\tau, Y_\tau, Z_\tau,) d\tau | X_t, Y_t, Z_t \right\}$$
$$+ \sqrt{\varepsilon\delta}\, \mathbb{E}^* \left\{ e^{-[T-t]} G_2(X_T, Y_T, Z_T) - \int_t^T e^{-[\tau-t]} R_2^{\varepsilon}(\tau, X_\tau, Y_\tau, Z_\tau,) d\tau | X_t, Y_t, Z_t \right\}$$
$$+ \delta\, \mathbb{E}^* \left\{ - \int_t^T R_3^{\varepsilon}(\tau, X_\tau, Y_\tau, Z_\tau,) d\tau | X_t, Y_t, Z_t \right\}. \tag{C.52}$$

This accounts for the first part of Lemma 4.11.3, that is: $P^{\varepsilon,\delta} - \tilde{P^{\varepsilon,\delta}} = \mathcal{O}(\varepsilon, \delta, \sqrt{\varepsilon\delta})$. The second part can be verified through generalization of the regularization technique discussed in Section 4.9.1. A brief insight is given in [41]. □

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