Chapter 1

Local Volatility and Dupire's Equation

Local volatility model was invented around 1994 in [Dupire (1994)] for the continuous case and [Derman and Kani (1994a)] for the discrete case in response to the following problem.

In the celebrated Black-Scholes model, see e.g. [Hull (1997)], the dynamics of the stock price is modeled as a Geometric Brownian motion process with constant volatility parameter σ

$$dS_t = \mu S_t + \sigma S_t dW_t, \qquad S_t\Big|_{t=0} = S_0. \tag{1.1}$$

Here, S_t is the stock price at the time t, μ is the drift, W_t is the standard Brownian motion. It can be shown that under the risk-neutral measure $\mathbb{Q}$ the drift becomes $\mu = r - q$, where r, q are the constant interest rate and continuous dividends functions.

Next, we introduce a notion of the Black-Scholes implied volatility. In financial mathematics, the *implied volatility* σ_{BS} of an option contract is that value of the volatility of the underlying instrument which, when input in an Black-Scholes option pricing model will return a theoretical value equal to the current market price of the option. The implied volatility shows what the market implies about the underlying stock volatility in the future. For instance, the implied volatility is one of six inputs used in a simple option pricing (Black-Scholes) model, but is the only one that is not directly observable in the market. The standard way to determine it by knowing the market price of the contract and the other five parameters, is solving for the implied volatility by equating the model and market prices of the option contract. There exist various reasons why traders prefer considering option positions in term of the implied volatility, rather than the option price itself, see e.g., [Natenberg (1994)].

The Black-Scholes implied volatility is a useful measure, as it is a market practice instead of quoting the option premium in the relevant currency, the options are quoted in terms of the Black-Scholes implied volatility. Over the years, option traders have developed an intuition in this quantity. However, it can be further generalized by using a similar concept, but replacing the Black-Scholes framework with another one. For instance, in [Corcuera *et al.* (2009)] this is done under a Lévy framework, and, therefore, based on distributions that match more closely historical returns. Here we don't consider these generalizations, and are concentrated only on the Black-Scholes implied volatility.

Assume we are given an underlying and the continuous dividends function for this underlying. Also assume that the market constant interest rate is somehow known. Finally, assume that we are given a snapshot of market prices of, say European Call and Put options written on this underlying with the same expiration date T. Then one can compute the Black-Scholes implied volatilities for these options which will be a function of strikes K, and plot them against the strike price. Thus obtained line is called a *volatility smile* if it slopes upward on either end, or *volatility skew* if it slopes upward only on the left. The former behavior is typical for the stock options, while the latter — for the index options.

The important observation is that the volatility smiles should never occur based on standard Black-Scholes option theory, which normally requires a completely flat volatility curve. However, the first notable volatility smile was apparently seen back to 1987 following the stock market crash. Since that time the topic attracted a lot of attention in the financial industry. As mentioned in [Derman *et al.* (2016)], "After the crash, and ever since, equity index option markets have displayed a volatility smile, an anomaly in blatant disagreement with the Black-Scholes-Merton model. Since then, quants around the world have labored to extend the model to accommodate this anomaly". There exist various books on local and implied volatility with main focus on modeling, description, understanding, etc. We mentioned just few of them based on our own preferences, [Derman *et al.* (2016); Gatheral (2006); Natenberg (1994); Bossu (2014)], but the reader can also find numerous references therein.

The existence of the smile forced the quants to move from the simple Black-Scholes model to more sophisticated ones that would be able to describe this pattern. The idea of the *local volatility* model as proposed in [Dupire (1994); Derman and Kani (1994a)] was as follows. Assuming that only minimal changes should be applied to the Black-Scholes model, they proposed to replace the constant volatility σ with that which is a deterministic function of S_t and t. In other words, in this model instead of Eq.(1.31) we have

$$dS_t = \mu S_t + \sigma(S_t, t) S_t dW_t, \qquad S_t\Big|_{t=0} = S_0. \tag{1.2}$$

Thus, in this model the local volatility is a function of the stock level S_t and time t (rather than the constant value) which might be sufficient to build a smile. With that, several natural questions become subject of an immediately concern [Derman *et al.* (2016)]:

(1) Is there exist a unique local volatility function or surface $\sigma(S,t)$ to match the observed implied volatility surface $\sigma_{BS}(S,t,K,T,r,q)$?
(2) If yes, that means that we can explain the observed smile by means of a local volatility process for the stock. Is the explanation meaningful? Does the stock actually evolve according to an observable local volatility function? There are many different models that can match the implied volatility surface, but achieving a match doesn't mean that model is "correct."
(3) What does the local volatility model tell us about the hedge ratios of vanilla options and the values of exotic options? How do the results differ from those of the classic Black-Scholes model?

The first question was positively answered by [Derman and Kani (1994a)] who constructed a binomial-tree model with volatilities at every model being calibrated to some stock and options market data. This modification of the binomial tree is called now the implied tree, [Hull (1997); Derman *et al.* (2016)]. In turn, [Dupire (1994)] considered the stock's local volatility function as $\sigma(K,T)$, i.e. this is the local volatility $\sigma(S,t)$ when the future stock price is K at time T. He derived a forward PDE (known now as the Dupire equation) which describes the dynamics of $\sigma(K,T)$ in the continuous case, which means the derivatives of $\sigma(K,T)$ represent the market prices of infinitesimal strike spreads, calendar spreads, and butterfly spreads. We discuss this derivation in the next Section.

1.1 Dupire's equation

The derivation of the Dupire equation is provided in many textbooks and papers, e.g., [Dupire (1994); Gatheral (2004); Derman *et al.* (2016); Rouah (2001)]. Here, we consider two ways to obtain it.

1.1.1 *Derivation using the Fokker-Planck equation*

Below suppose that the interest rate is deterministic, $r = r(t)$ and denote the discount factor $D(t, T)$ as

$$D(t,T) = \exp\left(-\int_t^T r(k)dk\right). \tag{1.3}$$

By definition, a European option Call $C(S, t, T)$ and Put $P(S, t, T)$ prices can be defined as, [Hull (1997)]

$$\begin{aligned} C(S,K,T-t) &= D(t,T)\mathbb{E}_{\mathbb{Q}}[(S_T-K)^+], \\ P(S,K,T-t) &= D(t,T)\mathbb{E}_{\mathbb{Q}}[(K-S_T)^+], \end{aligned} \tag{1.4}$$

where for simplicity we dropped the dependence of the option prices on r, q, σ, $x^+ = \max(x, 0)$, and the expectation $\mathbb{E}_{\mathbb{Q}}$ is taken under the risk-neutral measure $\mathbb{Q}$. Therefore, it reads

$$\mathbb{E}_{\mathbb{Q}}[x] = \int xp(x, x_T, T-t)dx_T,$$

where $p(x, x_T, T - t)$ is the transition probability density from the state (x, t) into the state (x_T, T).

The function $p(S, S_T, T - t)$ satisfies the forward Kolmogorov (Fokker-Planck) equation (see, e.g. [Risken and Haken (1989); Soize (1994); Van Kampen (2007)] and references therein). With the simplified notation $p(S, t)$ instead of $p(S, S_T, T - t)$, it reads

$$\frac{\partial}{\partial t}p(S,t) = -\frac{\partial}{\partial S}[\mu S p(S,t)] + \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}[\sigma^2 S^2 p(S,t)]. \tag{1.5}$$

This equation should be solved subject to the initial condition $p(S, 0) = \delta(S - S_0)$, where $\delta(x)$ is the Dirac delta-function.

The next step is to find an explicit expression for the option Theta: $\Theta = \frac{\partial C}{\partial T}$ by using the definition of C in Eq.(1.4). Using the chain rule we get

$$\begin{aligned} \frac{\partial C}{\partial T} &= \frac{\partial D(t,T)}{\partial T}\int_K^\infty (S_T-K)p(S,S_T,T-t)dS_T \\ &\quad + D(t,T)\int_K^\infty (S_T-K)\frac{\partial p(S,S_T,T-t)}{\partial T}dS_T \\ &= -r(T)C + D(t,T)\int_K^\infty (S_T-K)\frac{\partial p(S,S_T,T-t)}{\partial T}dS_T. \end{aligned} \tag{1.6}$$

Now the derivative under the integral in Eq.(1.7) can be substituted with the right hands side of Eq.(1.5) taken at $t = T$ which yields

$$\begin{aligned} \frac{\partial C}{\partial T} + r(T)C &= D(t,T)\int_K^\infty (S_T - K) \\ &\cdot \left\{ -\frac{\partial}{\partial S_T}[\mu(T)S_T p(S, S_T, T-t)] + \frac{1}{2}\frac{\partial^2}{\partial \sigma^2}[\sigma^2 S_T^2 p(S, S_T, T-t)] \right\} dS_T \\ &= D(t,T)\left(-\mu(T)I_1 + \frac{1}{2}I_2\right), \end{aligned} \tag{1.7}$$

where the short notation I_1, I_2 is introduced for the integrals in the second line of Eq.(1.7). To evaluate these integrals we need two identities.

1.1.1.1 *First identity*

From the definition of the Call option price in Eq.(1.4) we have

$$\frac{C}{D(t,T)} = \int_K^\infty S_T p(S, S_T, T-t) dS_T - K\int_K^\infty p(S, S_T, T-t) dS_T. \tag{1.8}$$

On the other hand, from Eq.(1.4)

$$\frac{\partial C}{\partial K} = -D(t,T)\int_K^\infty p(S, S_T, T-t) dS_T. \tag{1.9}$$

Therefore,

$$\int_K^\infty S_T p(S, S_T, T-t) dS_T = \frac{1}{D(t,T)}\left(C - K\frac{\partial C}{\partial K}\right). \tag{1.10}$$

1.1.1.2 *The Breeden-Litzenberger identity*

Differentiating both sides of Eq.(1.9) on K and using the fundamental theorem of calculus, we obtain

$$\frac{\partial^2 C}{\partial K^2} = D(t,T)p(S, K, T-t). \tag{1.11}$$

This identity is known as the Breeden-Litzenberger formula, [Breeden and Litzenberger (1978)], which states that the risk-neutral probability of making a transition from S at time t to K at time T is proportional to the second partial derivative of the call price with respect to strike. □

With these identities we can proceed with evaluating the integrals I_1, I_2. For I_1 we get

$$I_1 = \int_K^\infty (S_T - K)\frac{\partial}{\partial S_T}[S_T p(S, S_T, T-t)] dS_T \tag{1.12}$$

$$
\begin{aligned}
&= (S_T - K)\Big[S_T p(S, S_T, T-t)\Big]_K^\infty - \int_K^\infty S_T p(S, S_T, T-t) dS_T \\
&= -\int_K^\infty S_T p(S, S_T, T-t) dS_T = \frac{1}{D(t,T)}\left(K\frac{\partial C}{\partial K} - C\right), \qquad (1.13)
\end{aligned}
$$

where in the last line the result in Eq.(1.10) is used, and also it is assumed that

$$
\lim_{S_T\to\infty}(S_T - K)S_T p(S, S_T, T-t) = 0.
$$

In other words, the first and the second moment of the density $p(S, S_T, T-t)$ are finite.

For I_2 we obtain

$$
\begin{aligned}
I_2 &= \int_K^\infty (S_T - K)\frac{\partial^2}{\partial S_T^2}[\sigma^2 S_T^2 p(S, S_T, T-t)] dS_T \qquad (1.14) \\
&= (S_T - K)\frac{\partial}{\partial S_T}\Big[\sigma^2 S_T^2 p(S, S_T, T-t)\Big]_K^\infty \\
&\quad - \int_K^\infty \frac{\partial}{\partial S_T}[S_T p(S, S_T, T-t)] dS_T \\
&= -\Big[\sigma^2 S_T^2 p(S, S_T, T-t)\Big]_K^\infty = \sigma^2 K^2 p(S, K, T-t).
\end{aligned}
$$

Here $\sigma^2 = \sigma(K,T)^2$, and it is assumed that

$$
\lim_{S_T\to\infty}(S_T - K)\frac{\partial}{\partial S_T}[S_T p(S, S_T, T-t)] = 0.
$$

Using the Breeden-Litzenberger we finally obtain

$$
I_2 = \frac{\sigma^2}{D(t,T)} K^2 \frac{\partial^2 C}{\partial K^2}. \qquad (1.15)
$$

1.1.1.3 *The final step*

Substituting Eq.(1.12) and Eq.(1.15) into Eq.(1.7) we obtain

$$
\frac{\partial C}{\partial T} + r(T)C = \mu(T)C - \mu(T)K\frac{\partial C}{\partial K} + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}. \qquad (1.16)
$$

Taking into account that under the risk-neutral measure, the drift reads $\mu(T) = r(T) - q(T)$, [Brigo and Mercurio (2006)], we finally get the Dupire equation

$$
\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2 K^2 - [(r(T) - q(T)]K\frac{\partial C}{\partial K} - q(T)C. \qquad (1.17)
$$

This equation can also be solved for the local variance $\sigma^2(K,T)$ to obtain

$$\sigma^2(K,T) = \frac{\frac{\partial C}{\partial T} + [(r(T) - q(T)]K\frac{\partial C}{\partial K} + q(T)C}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \tag{1.18}$$

1.1.2 *A probabilistic approach*

Here we derive the Dupire equation by using a probabilistic argument proposed in [Derman and Kani (1998)]. Let us define the stochastic variable

$$f(S_T, t, T) = D(t,T)(S_T - K)^+, \tag{1.19}$$

where S_t is the Geometric Brownian motion introduced in Eq.(1.31). Obviously, based on Eq.(1.4),

$$\mathbb{E}_{\mathbb{Q}}[f(S_T, t, T)|S_t = S] = C(S, K, T-t). \tag{1.20}$$

Using Itö's lemma at time $t = T$, one can find that $f(S_T, t, T)$ follows the process

$$df = \left(\frac{\partial f}{\partial T} + \mu(T)S_T\frac{\partial f}{\partial S_T} + \frac{1}{2}\sigma(T)^2 S_T^2\frac{\partial^2 f}{\partial S_t^2}\right)dT + \sigma(T)S_T\frac{\partial f}{\partial S_T}dW_T. \tag{1.21}$$

The partial derivatives in this expression could be easily found using the definition of f in Eq.(1.19):

$$\begin{aligned}
\frac{\partial f}{\partial T} &= -r(T)D(t,T)(S_T - K)^+, \qquad (1.22)\\
\frac{\partial f}{\partial S_T} &= D(t,T)\mathbf{1}_{S_T > K}, \qquad \mathbf{1}_x = \begin{cases} 1, & x \ge 0 \\ 0, & x < 0 \end{cases}\\
\frac{\partial^2 f}{\partial S_T^2} &= D(t,T)\delta(S_T - K).
\end{aligned}$$

Substituting this into Eq.(1.21) yields

$$\begin{aligned}
df = D(t,T)\Big[\Big(&- r(T)(S_T - K)^+ + \mu(T)S_T\mathbf{1}_{S_T > K} \qquad (1.23)\\
&+ \frac{1}{2}\sigma(T)^2 S_T^2\delta(S_T - K)\Big)dT + \sigma(T)S_T\mathbf{1}_{S_T > K}dW_T\Big].
\end{aligned}$$

The first two terms in parentheses can be re-written as follows

$$\begin{aligned}
-r(T)(S_T - K)^+ + \mu(T)S_T\mathbf{1}_{S_T > K} &= \mathbf{1}_{S_T > K}[-r(T)(S_T - K) + \mu(T)S_T]\\
&= r(T)K\mathbf{1}_{S_T > K} - q(T)S_T\mathbf{1}_{S_T > K}. \qquad (1.24)
\end{aligned}$$

Since W_t is a martingale, taking the risk-neutral expectation of Eq.(1.23) and using Eq.(1.20) yields

$$\begin{aligned} dC = D(t,T)\mathbb{E}_{\mathbb{Q}}\Big[r(T)K\mathbf{1}_{S_T>K} - q(T)S_T\mathbf{1}_{S_T>K} \\ + \frac{1}{2}\sigma(T)^2 S_T^2\delta(S_T-K)\Big]dT. \end{aligned} \tag{1.25}$$

Since

$$D(t,T)\mathbb{E}_{\mathbb{Q}}[S_T\mathbf{1}_{S_T>K}] = C + KD(t,T)\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S_T>K}],$$

and

$$\frac{\partial C}{\partial K} = -D(t,T)\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S_T>K}]. \tag{1.26}$$

Eq.(1.25) can also be represented as

$$\begin{aligned} \frac{\partial C}{\partial T} &= D(t,T)K[r(T)-q(T)]\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_{S_T>K}] - q(T)C \\ &\quad + \frac{1}{2}D(t,T)\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)S_T^2\delta(S_T-K)] \\ &= -K[r(T)-q(T)]\frac{\partial C}{\partial K} - q(T)C + \frac{1}{2}D(t,T)\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)S_T^2\delta(S_T-K)]. \end{aligned} \tag{1.27}$$

The last term in this equation can be simplified by using the sifting property of the Dirac delta function

$$\begin{aligned} \frac{1}{2}D(t,T)\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)S_T^2\delta(S_T-K)] &= \frac{1}{2}D(t,T)\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)S_T^2|S_T=K] \\ \cdot\, \mathbb{E}_{\mathbb{Q}}[\delta(S_T-K)] &= \frac{1}{2}D(t,T)\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)|S_T=K]K^2\mathbb{E}_{\mathbb{Q}}[\delta(S_T-K)] \\ &= \frac{1}{2}\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)|S_T=K]K^2\frac{\partial^2 C}{\partial K^2}, \end{aligned} \tag{1.28}$$

since it follows from Eq.(1.26) that

$$\frac{\partial^2 C}{\partial K^2} = D(t,T)\mathbb{E}_{\mathbb{Q}}[\delta(S_t-K)].$$

Thus, from Eq.(1.27) and Eq.(1.28) we finally obtain

$$\frac{\partial C}{\partial T} = -K[r(T)-q(T)]\frac{\partial C}{\partial K} - q(T)C + \frac{1}{2}\mathbb{E}_{\mathbb{Q}}[\sigma^2(T)|S_T=K]K^2\frac{\partial^2 C}{\partial K^2}. \tag{1.29}$$

Comparing this with the Dupire equation Eq.(1.17) we see that

$$\sigma^2(K,T) = \mathbb{E}_{\mathbb{Q}}[\sigma^2(S_T,T)|S_T=K].$$

This means that the local variance is the risk-neutral expectation of the instantaneous variance conditional on the final stock price S_T being equal to the strike price K.

1.2 Local volatility via the implied volatility

In this Section we derive the identity that connects the local and Black-Scholes implied variances. The identity was introduced in [Lipton (2002); Gatheral (2006)] and reads

$$\sigma^2(T,K) = \frac{\partial_T w}{\left(1 - \frac{y\partial_y w}{2w}\right)^2 - \frac{(\partial_y w)^2}{4}\left(\frac{1}{w} + \frac{1}{4}\right) + \frac{\partial_y^2 w}{2}}, \tag{1.30}$$

where $y = \log K/F$, $F = Se^{(r-q)T}$ is the stock forward price, and $w = \sigma_{BS}^2 T$ is the total implied variance. In terms of these variables the Black-Scholes formula for the future value of the Call option price becomes, [Gatheral (2006)]

$$\begin{aligned} C_{BS}(F_T, y, w) &= D(t,T)F_T[N(d_1) - e^y N(d_2)] \qquad (1.31)\\ &= D(t,T)F_T\left[N\left(-\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right) - e^y N\left(-\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\right], \end{aligned}$$

where $N(x)$ is the normal CDF, and

$$d_1 = \frac{\ln\frac{F_T}{K} + \frac{\sigma_{BS}^2}{2}T}{\sigma_{BS}\sqrt{T}}, \qquad d_2 = d_1 - \frac{1}{2}\sigma_{BS}\sqrt{T}. \tag{1.32}$$

The Dupire equation Eq.(1.17) can also be re-written in terms of the new variables. It is easy to check that this yields

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2\left[\frac{\partial^2 C}{\partial y^2} - \frac{\partial C}{\partial y}\right] - qC. \tag{1.33}$$

Computing derivatives of the Black-Scholes formula in Eq.(1.31), we obtain

$$\begin{aligned} \frac{\partial^2 C_{BS}}{\partial w^2} &= \left(-\frac{1}{8} - \frac{1}{2w} + -\frac{y^2}{2w^2}\right)\frac{\partial C_{BS}}{\partial w}, \qquad (1.34)\\ \frac{\partial^2 C_{BS}}{\partial w \partial y} &= \left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial C_{BS}}{\partial w},\\ \frac{\partial^2 C_{BS}}{\partial y^2} - \frac{\partial C_{BS}}{\partial y} &= 2\frac{\partial C_{BS}}{\partial w}. \end{aligned}$$

Taking into account that $w = w(y,T)$, the Dupire equation in Eq.(1.33) can be also transformed by using the identities

$$\frac{\partial C}{\partial y} = \frac{\partial C_{BS}}{\partial y} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial y}, \tag{1.35}$$

$$\frac{\partial^2 C}{\partial y^2} = \frac{\partial^2 C_{BS}}{\partial y^2} + 2\frac{\partial^2 C_{BS}}{\partial w \partial y}\frac{\partial w}{\partial y} + \frac{\partial^2 C_{BS}}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial C_{BS}}{\partial w}\frac{\partial^2 w}{\partial y^2},$$

$$\frac{\partial C}{\partial T} = \frac{\partial C_{BS}}{\partial T} + \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} = \frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} - qC_{BS}.$$

Substituting these expressions into Eq.(1.33) yields

$$\frac{\partial C_{BS}}{\partial w}\frac{\partial w}{\partial T} = \frac{1}{2}\sigma^2 \frac{\partial C_{BS}}{\partial w} A, \tag{1.36}$$

$$A = 2 - \frac{\partial w}{\partial y} + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2}$$
$$+ 2\left(\frac{1}{2} - \frac{\partial y}{\partial w}\right)\frac{\partial w}{\partial y}.$$

Taking out a factor $\frac{\partial C_{BS}}{\partial w}$ and simplifying, we finally obtain Eq.(1.30).

An interesting particular case is when $\frac{\partial w}{\partial y} = 0$. This implies $\frac{\partial \sigma_{BS}(S,K,T)}{\partial K} = 0$, i.e., there is no skew and the implied volatility is flat. Then it follows from Eq.(1.30) that $\sigma^2 = \frac{\partial w}{\partial T}$. In other words, in this case the local variance reduces to the forward Black-Scholes implied variance, and

$$w(T) = \int_0^T \sigma^2(k)dk. \tag{1.37}$$

Also, it is possible to show that the implied volatility $\sigma_{BS}(S,K,T)$ of an option is approximately the average of the local volatilities $\sigma(S,t)$ encountered over the life of the option between the current underlying price and the strike. A detailed discussion of this rule of thumb is provided in [Derman *et al.* (2016)].