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Properties of the SABR model

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ABSTRACT

In the original Black-Scholes model, the volatility is assumed to be a constant. But with different strike prices, the options require different volatilities to match their market prices. In fact, when implied volatility is plotted against strike price, the resulting graph is typically downward sloping for equity markets called "volatility skew" , or valley-shaped for currency markets called "volatility smile". Modeling the volatility smile is an active area of research in quantitative finance. Local volatility model captures the static pattern of volatility smile. However, it predicts the wrong dynamics of the volatility smile. This causes unstable hedges. To explain the volatility smile better, several stochastic volatility models have been developed, in which models that the volatility is driven by a stochastic process. In the present paper, a review of the SABR (Stochastic Alpha, Beta, Rho) model is presented. The pricing function of European options is studied in detail. More importantly, we show that the SABR model predicts the correct dynamics of volatility smile. We also analyze how the parameters α, β, ρ influence the option price and the volatility smile and skew.

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1. INTRODUCTION

The original Black-Scholes model presumes that the volatility of the underlying asset as a constant parameter σ_B. To obtain this constant parameter σ_B, we observe the option prices from the market and match our theoretical value for the option under Black-Scholes model to it. Since all the other variables can be observed directly, there is a one-to-one relationship between the σ_B and the option price. We can use this relation to get the implied volatility. Unfortunately, options with different strikes K require different volatilities σ_B to match their market prices, this suggests that the volatility is in general not a constant but is dependent on the strike K. Indeed, if we depict the volatility as a function of strike, it forms the so called "volatility smile" or "volatility skew". See figure 1
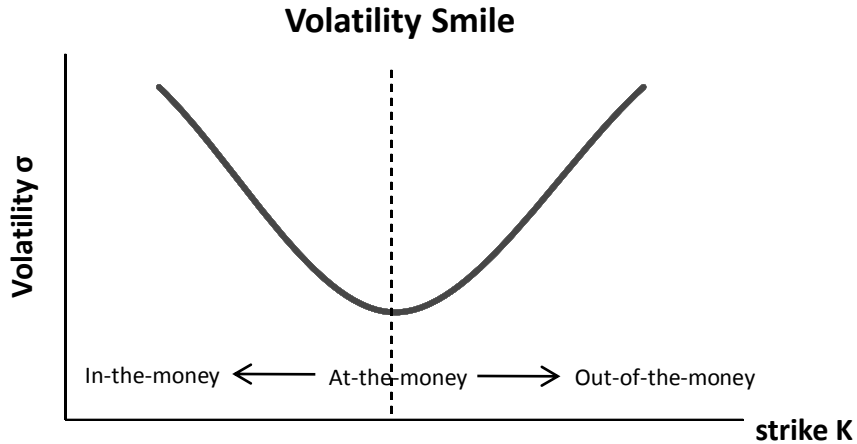

Figure 1. Volatility smile

To handle the smiles and skews, a local volatility model was developed by Dupire [2] and Derman-Kani [3]. In the local volatility model, the σ_B in the Black-Scholes model is replaced by the local volatility $\sigma_{loc}(t, F)$. The local volatility model is self-consistent, arbitrage-free and, can

be calibrated to match precisely the observed market smiles and skews. A problem with the local volatility model arises, that is the dynamic behavior of smiles and skews predicted by local volatility model is exactly opposite the behavior observed in the marketplace. This contradiction between the model and the marketplace tend to de-stabilize the delta risk $\frac{\partial V_{call}}{\partial f}$ and vega risk $\frac{\partial V_{call}}{\partial \sigma}$ hedges derived from local volatility model, and often these hedges perform worse than the original Black-Scholes' hedges. This will be discussed in detail later in section 2.2.

To resolve this problem, Hegan et al [4] developed the SABR model, a stochastic volatility model in which the asset price and volatility are correlated. Compared with other stochastic volatility models such as the famous Heston model, SABR model has a simpler form and allows the market price and the market risks, including vanna risk $\partial V/\partial \rho$ and volga risk $\partial V/\partial v$ (introduced by SABR model) to be obtained immediately from Black's formula. SABR model also provides good fits to the implied volatility curves observed in the marketplace. And, the formula shows that the SABR model captures the correct dynamics of the smile, and thus yields stable hedges.

The main purpose of this thesis is to study the SABR model in detail and to investigate the dependence on the parameters α, β, and ρ which altogether give the name of the model.

2. REVIEW OF BLACK-SHOLES MODEL AND LOCAL VOLATILITY MODEL

In this chapter, we take a little time to review the Black-Scholes model and the local volatility model based on the Martingale pricing theory.

2.1. Review of Black-Scholes model

Consider a European call option on an asset S with exercise date t_{ex}, and strike price K. If the holder exercises the option at t_{ex}, then on the settlement date t_{set} he receives the underlying asset S and pays the strike K. Under the Martingale pricing theory, the value of the option is

$$V_{call} = D(t_{set})E\{[F(t_{ex}) - K]^+ | \mathcal{F}_0\}, \tag{2.1}$$

here D(t) is the discount factor for date t, F(t) is the forward price of the forward contract that matures on the settlement date t_{set} that under the forward measure, i.e.

$$dF = C(t,*)dW, \tag{2.2}$$

$$F(0) = f. \tag{2.3}$$

By the call-put parity, the value of the corresponding European put option is

$$V_{put} = D(t_{set})E\{[K - F(t_{ex})]^+ | \mathcal{F}_0\} = V_{call} + D(t_{set})[K - f]. \tag{2.4}$$

The coefficient $C(t,*)$ may be deterministic or random; one cannot determine the coefficient $C(t,*)$ on purely theoretical grounds. That is, one must postulate a mathematical model for $C(t,*)$.

Black-Scholes model postulates the coefficient $C(t,*)$ as $\sigma_B F(t)$, where the volatility σ_B is a constant. Then the forward price is

$$dF = \sigma_B F(t)dW, \tag{2.5}$$

$$F(0) = f,$$

The value functions of call option and put option under this model yields

$$V_{call} = D(t_{set})E\{[F(t_{ex}) - K]^+ | \mathcal{F}_0\} = D(t_{set})\{f\mathcal{N}(d_1) - K\mathcal{N}(d_2)\}, \quad (2.6)$$

where

$$d_{1,2} = \frac{1}{\sigma_B\sqrt{t_{ex}}}\left\{\ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B{}^2 t_{ex}\right\}, \quad (2.7)$$

$$V_{put} = D(t_{set})E\{[K - F(t_{ex})]^+ | \mathcal{F}_0\} = V_{call} + D(t_{ex})[K - f] . \quad (2.8)$$

Differentiating the option value function, for say, a call option with respect to the volatility σ_B,

$$\frac{\partial V_{call}}{\partial \sigma_B} = \frac{\partial}{\partial \sigma_B}\{f\mathcal{N}(d_1) - K\mathcal{N}(d_2)\} = f\varphi(d_1)\frac{\partial d_1}{\partial \sigma_B} - K\varphi(d_2)\frac{\partial d_2}{\partial \sigma_B} \quad (2.9)$$

where

$$\mathcal{N}(d) = \int_{-\infty}^{d} \varphi(z)dz, \qquad \varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}, \quad (2.10)$$

We note that $d_2 = d_1 - \sigma_B\sqrt{t_{ex}}$. Substituting into $\varphi(d_2)$, we have

$$\varphi(d_2) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(d_1 - \sigma_B\sqrt{t_{ex}})^2}{2}} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1{}^2}{2}} \cdot e^{d_1\sigma_B\sqrt{t_{ex}} - \frac{\sigma_B{}^2 t_{ex}}{2}}$$

$$= \varphi(d_1) \cdot e^{d_1\sigma_B\sqrt{t_{ex}} - \frac{\sigma_B{}^2 t_{ex}}{2}} . \quad (2.11)$$

Therefore

$$\frac{\partial V_{call}}{\partial \sigma_B} = f\varphi(d_1)\frac{\partial d_1}{\partial \sigma_B} - K\varphi(d_1) \cdot e^{d_1\sigma_B\sqrt{t_{ex}} - \frac{\sigma_B{}^2 t_{ex}}{2}} \cdot \frac{\partial d_2}{\partial \sigma_B}$$

$$= f\varphi(d_1)\frac{\partial d_1}{\partial \sigma_B} - f\varphi(d_1)\frac{\partial d_2}{\partial \sigma_B} . \quad (2.12)$$

Substituting

$$\frac{\partial d_1}{\partial \sigma_B} = \frac{-1}{\sigma_B{}^2\sqrt{t_{ex}}}\ln\left(\frac{f}{K}\right) + \frac{\sqrt{t_{ex}}}{2}, \quad \frac{\partial d_2}{\partial \sigma_B} = \frac{-1}{\sigma_B{}^2\sqrt{t_{ex}}}\ln\left(\frac{f}{K}\right) - \frac{\sqrt{t_{ex}}}{2}, \quad (2.13)$$

into (2.12),

$$\frac{\partial V_{call}}{\partial \sigma_B} = f\varphi(d_1)\left\{\frac{-1}{\sigma_B{}^2\sqrt{t_{ex}}}\ln\left(\frac{f}{K}\right) + \frac{\sqrt{t_{ex}}}{2}\right\} - f\varphi(d_1)\left\{\frac{-1}{\sigma_B{}^2\sqrt{t_{ex}}}\ln\left(\frac{f}{K}\right) - \frac{\sqrt{t_{ex}}}{2}\right\}$$

$$= f\varphi(d_1)\sqrt{t_{ex}}\,. \tag{2.14}$$

Since $\frac{\partial V_{call}}{\partial \sigma_B} = f\varphi(d_1)\sqrt{t_{ex}} > 0$, we know that option prices under the B-S model are increasing

functions of σ_B, and the volatility implied by the market price of an option is unique. However,

with different strikes K, we need different volatilities to match the market data, which causes the

following problems:

1. When we price an exotic option, for instance, a down-and-out call option with strike K_1 and

   the barrier $K_2 > K_1$, it is not clear that whether we should use the implied volatility at the

   strike K_1, the implied volatility at the barrier K_2, or some combination of K_1 & K_2 to price

   the option.

2. Since different volatilities σ_B are used for different strikes K, it is not clear that the delta and

   vega risks calculated at one strike are consistent with the same risks calculated at other strikes

   or not.

3. Since the implied volatility σ_B depends on the strike K, it is likely to also depend on the

   current value f of the forward price: $\sigma_B = \sigma_B(f, K)$. In this case, some of the vega risks of

   Black's model would actually be due to changes in the price of the underlying asset, and

   would be hedged more properly as delta risks.

2.2. Review of the local volatility model

In the local volatility model, Dupire assumed the coefficient $C(t,*)$ has the form $\sigma_{loc}(t, F)F$, i.e. the forward price is given by

$$dF = \sigma_{loc}(t, F)FdW,\qquad\qquad (2.15)$$

$$F(0) = f$$

in the forward measure. The corresponding value function of European call and put options are

$$V_{call} = D(t_{set})E\{[F(t_{ex}) - K]^+ | F(0) = f\},\qquad\qquad (2.16)$$

$$V_{put} = V_{call} + D(t_{set})(K - f),\qquad\qquad (2.17)$$

Then according to the Dupire's formula we can obtain an explicit algebraic formula for the implied volatility of the local volatility model:

$$\sigma_{loc}(t_{ex}, K) = \sqrt{\frac{C_{t_{ex}}}{\frac{1}{2}K^2 C_{KK}}}\;;\qquad\qquad (2.18)$$

Once $\sigma_{loc}(t, F)$ has been obtained, the local volatility model becomes a single, self-consistent model which correctly reproduces the market prices of calls and puts for all strikes K and exercise dates t_{ex}. The problems of pricing exotics and hedging risks in the B-S model are now resolved by the local volatility model. Unfortunately, the local volatility model predicts the wrong dynamics of the implied volatility curve, which leads to inaccurate and often unstable hedges. To illustrate the problem, we use singular perturbation method (See Appendix) to find the equivalent Black volatility that is

$$\sigma_B(K, f) = \sigma_{loc}\left(\frac{1}{2}[f + K]\right)\left\{1 + \frac{1}{24}\frac{\sigma''_{loc}\left(\frac{1}{2}[f+K]\right)}{\sigma_{loc}\left(\frac{1}{2}[f+K]\right)}(f - K)^2 + \cdots\right\},\qquad (2.19)$$

The solution of this equation is dominated by the first term on the right hand side; Suppose that today the forward price is f_0 and the implied volatility curve seen in the market is $\sigma_B^0(K)$. Then we have

$$\sigma_B^0(K) = \sigma_{loc}\left(\frac{1}{2}[f_0 + K]\right)\{\cdots\}. \tag{2.20}$$

Calibrating the model to the market clearly requires choosing the local volatility to be

$$\sigma_{loc}(F) = \sigma_B^0(2F - f_0)\{1 + \cdots\}. \tag{2.21}$$

Now, suppose that the forward value changes from f_0 to f,

$$\sigma_B(K) = \sigma_{loc}\left(\frac{1}{2}[f + K]\right)\{1 + \cdots\}, \tag{2.22}$$

from (2.21) we see that the model predicts that the new implied volatility curve is

$$\sigma_B(K, f) = \sigma_B^0(K + f - f_0)\{1 + \cdots\}, \tag{2.23}$$

In particular, if the forward price f_0 increases to f, the implied volatility curve moves to the left; if f_0 decreases to f, the implied volatility curve moves to the right. This result is illustrated in figure 2 and figure 3.

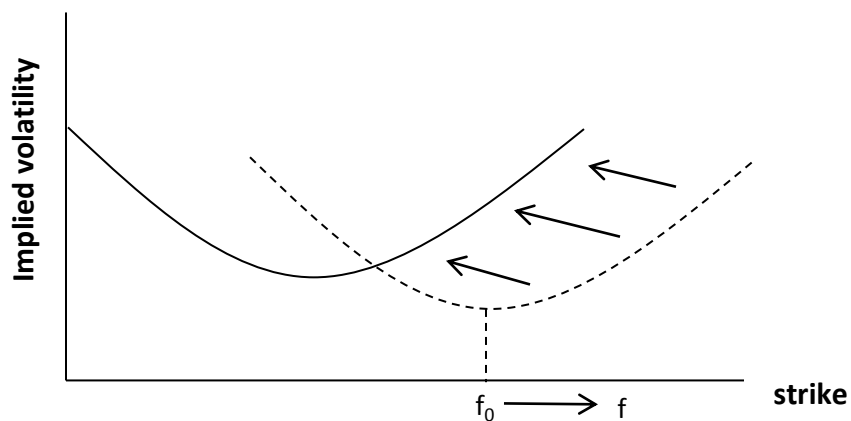Figure 2. Implied volatility if the forward price increases from f_0 to f
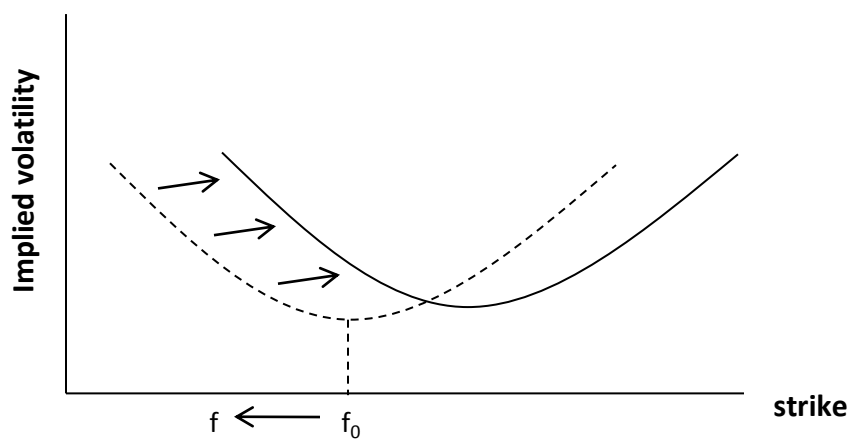

Figure 3. Implied volatility if the forward price decreases from f_0 to f

Local volatility model predicts that the market smile and skew moves in the opposite direction as the price of the underlying asset. This is in general opposite to market behavior, in which smiles and skews move in the same direction as the underlying.

Moreover, the delta risk of a call option predicted by the local volatility can be presented as

$$\Delta = \frac{\partial V_{call}}{\partial f} = \frac{\partial BS}{\partial f} + \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B}{\partial f}, \tag{2.24}$$

where

$$V_{call} = BS(f, K, \sigma_B(K, f), t_{ex}) = D(t_{set})[f\mathcal{N}(d_1) - K\mathcal{N}(d_2)], \tag{2.25}$$

$$d_1 = \frac{1}{\sigma_B\sqrt{t_{ex}}}\left\{\ln\left(\frac{f}{K}\right) + \frac{1}{2}\sigma_B^2 t_{ex}\right\}$$

is Black's formula for call option.

We note that the first term of the delta risk is just the delta risk calculated by Black's model using the implied volatility, and the second term is the correction term made by the local volatility model. Since the term $\frac{\partial \sigma_B}{\partial f}$ has the opposite sign to the real market behavior, this causes unstable hedges.

3. SABR MODEL

3.1. Introduction of SABR model

To demonstrate the correct dynamics of implied volatility and thus provide stable hedges, the SABR model (Hegan et al. 2002) was derived. SABR model assumes the volatility of the forward price is a stochastic variable. In the SABR model the forward price and the volatility are

$$dF = \alpha F^\beta dW_1, \quad F(0) = f \tag{3.1}$$

$$d\alpha = \nu\alpha dW_2, \quad \alpha(0) = \alpha \tag{3.2}$$

under the forward measure, where the two processes are correlated by

$$dW_1 dW_2 = \rho dt . \tag{3.3}$$

3.2. Option pricing functions under SABR model

Assuming both the overall volatility α and the volatility of volatility ("volvol") v are small. First, we re-write our model in general form

$$dF = \varepsilon \alpha C(F) dW_1 , \tag{3.4}$$

$$d\alpha = \varepsilon v \alpha dW_2 , \tag{3.5}$$

with

$$dW_1 dW_2 = \rho dt , \tag{3.3}$$

where $\varepsilon \ll 1$. We firstly consider the general case, $C(F)$, and then specialize the results to the power law F^β.

Suppose that $F(t) = f$, $\alpha(0) = \alpha$ at time t. Define the probability density $p(t, f, \alpha; T, F, A)$ by

$$p(t, f, \alpha; T, F, A) dF dA = prob\{F < F(T) < F + dF, A < \alpha(T) < A + dA \mid F(t) = f, \alpha(t) = \alpha\} . \tag{3.6}$$

As a function of the forward variables T, F,A, the probability density p satisfies the forward

Kolmogorov equation

$$p_T = \frac{1}{2}\varepsilon^2 A^2 [C^2(F)p]_{FF} + \varepsilon^2 \rho v [A^2 C(F)p]_{FA} + \frac{1}{2}\varepsilon^2 v^2 [A^2 p]_{AA} , \quad \text{for } T > t , \tag{3.7}$$

$$p = \delta(F - f)\delta(A - \alpha), \quad \text{at } T = t . \tag{3.8}$$

where

$$\delta(x - y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases} \tag{3.9}$$

Let $V(t, f, \alpha)$ be the value of a European call option at date t, we have

$$V(t, f, \alpha) = E\{[F(t_{ex}) - K]^+ \mid F(t) = f, \alpha(t) = \alpha\} \tag{3.10}$$

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$$= \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K) \cdot p(t, f, \alpha; \ t_{ex}, F, A) dF \, dA,$$

where we omit the discounting factor $D(t_{set})$ since it only affects the value at the last step.

From the forward Kolmogrov equation (3.7), (3.8), we have

$$p(t, f, \alpha; \ t_{ex}, F, A) = \delta(F - f)\delta(A - \alpha) + \int_{t}^{t_{ex}} p_T(t, f, \alpha; T, F, A) dT. \qquad (3.11)$$

We can re-write (3.10) as

$$V(t, f, \alpha) = [f - K]^+ + \int_{t}^{t_{ex}} \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K) \cdot p_T(t, f, \alpha; \ T, F, A) dF \, dA \, dT. \qquad (3.12)$$

We substitute (3.7) into the formula and then integrate over A. Since integrating both the terms $\varepsilon^2 \rho v [A^2 C(F)p]_{FA}$ and $\frac{1}{2}\varepsilon^2 v^2 [A^2 p]_{AA}$ overall A yields 0, our pricing function reduces to

$$V(t, f, \alpha) = [f - K]^+ + \int_{t}^{t_{ex}} \int_{K}^{\infty} \int_{-\infty}^{\infty} (F - K) \cdot \frac{1}{2}\varepsilon^2 A^2 [C^2(F)p]_{FF} dA \, dF dT \qquad (3.13)$$

$$= [f - K]^+ + \int_{t}^{t_{ex}} \int_{-\infty}^{\infty} \frac{1}{2}\varepsilon^2 A^2 \left\{ \int_{K}^{\infty} (F - K) \cdot [C^2(F)p]_{FF} dF \right\} dA dT$$

Integrating $\int_{K}^{\infty} (F - K) \cdot [C^2(F)p]_{FF} dF$ by parts twice w. r. t. F yields

$$V(t, f, \alpha) = [f - K]^+ + \int_{t}^{t_{ex}} \int_{-\infty}^{\infty} \frac{1}{2}\varepsilon^2 A^2 C^2(K)p(t, f, \alpha; T, K, A) dA dT \qquad (3.14)$$

$$= [f - K]^+ + \frac{1}{2}\varepsilon^2 C^2(K) \int_{t}^{t_{ex}} \int_{-\infty}^{\infty} A^2 p(t, f, \alpha; T, K, A) dA dT.$$

To simplify the problem, we define

$$P(t, f, \alpha; T, K) = \int_{-\infty}^{\infty} A^2 p(t, f, \alpha; T, K, A) dA, \qquad (3.15)$$

Then P satisfies the backward Kolmogorov equation

$$P_t + \frac{1}{2}\varepsilon^2 \alpha^2 C^2(f)P_{ff} + \varepsilon^2 \rho v \alpha^2 C(f)P_{f\alpha} + \frac{1}{2}\varepsilon^2 v^2 \alpha^2 P_{\alpha\alpha} = 0, \quad \text{for } t < T, \qquad (3.16)$$

$$P = \alpha^2 \delta(f - K), \quad \text{for } t = T. \tag{3.17}$$

Note that P depends on $T - t$, but not on t and T separately. We define

$$\tau = T - t, \quad \tau_{ex} = t_{ex} - t, \tag{3.18}$$

then our pricing function becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon^2 C^2(K) \int_0^{\tau_{ex}} P(\tau, f, \alpha; K) d\tau, \tag{3.19}$$

where $P(\tau, f, \alpha; K)$ satisfies

$$P_\tau = \frac{1}{2}\varepsilon^2\alpha^2 C^2(f) P_{ff} + \varepsilon^2 \rho\nu\alpha^2 C(f) P_{f\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha}, \quad \text{for } \tau > 0, \tag{3.20}$$

$$P = \alpha^2 \delta(f - K), \quad \text{for } \tau = 0. \tag{3.21}$$

Using perturbation expansion to this problem we would yield

$$P = \frac{\alpha}{\sqrt{2\pi\varepsilon^2 C^2(K)\tau}} e^{-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}} \{1 + \cdots\}. \tag{3.22}$$

We notice that since the expansion involves powers of $(f - K)/\varepsilon\alpha C(K)$, this expansion would become inaccurate as soon as $C(f)$ and $C(K)$ are significantly different. In other words, a small change in the exponent would be expanded by the power term in the probability density. Instead, a better approach is to re-cast the series as

$$P = \frac{\alpha}{\sqrt{2\pi\varepsilon^2 C^2(K)\tau}} e^{-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}\{1+\cdots\}}. \tag{3.23}$$

Now, Let us define a new variable

$$z = \frac{1}{\varepsilon\alpha} \int_K^f \frac{dg}{C(g)}, \tag{3.24}$$

So that the leading term of $e^{-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}\{1+\cdots\}}$ in (3.23) is simply $e^{-\frac{z^2}{2}}$. We also define

$$B(\varepsilon\alpha z) = C(f), \tag{3.25}$$

then

$$P_f = \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial f} = \frac{\partial P}{\partial z} \cdot \frac{1}{\varepsilon \alpha B(\varepsilon \alpha z)} \,, \qquad P_\alpha = \frac{\partial P}{\partial \alpha} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial \alpha} = \frac{\partial P}{\partial \alpha} - \frac{\partial P}{\partial z} \cdot \frac{z}{\alpha} \,, \quad (3.26)\&\ (3.27)$$

And

$$P_{ff} = \frac{1}{\varepsilon^2 \alpha^2 B^2(\varepsilon \alpha z)} \left\{ P_{zz} - \varepsilon \alpha \frac{B'(\varepsilon \alpha z)}{B(\varepsilon \alpha z)} \cdot P_z \right\}, \tag{3.28}$$

$$P_{f\alpha} = \frac{1}{\varepsilon \alpha B(\varepsilon \alpha z)} \left\{ P_{z\alpha} - \frac{z}{\alpha} P_{zz} - \frac{1}{\alpha} P_z \right\}, \tag{3.29}$$

$$P_{\alpha\alpha} = P_{\alpha\alpha} - \frac{2z}{\alpha} P_{z\alpha} + \frac{z^2}{\alpha^2} P_{zz} + \frac{2z}{\alpha^2} P_z \ ; \tag{3.30}$$

also,

$$\delta(f - K) = \delta\big(\varepsilon \alpha z \cdot C(K)\big) = \frac{1}{\varepsilon \alpha \cdot C(K)} \delta(z) \,. \tag{3.31}$$

Therefore, by substituting our new variable into the pricing function (3.19), (3.20) and (3.21), we have

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2} \varepsilon^2 C^2(K) \int_0^{\tau_{ex}} P(\tau, z, \alpha) d\tau \,, \tag{3.32}$$

where $P(\tau, z, \alpha)$ is the solution of

$$P_\tau = \frac{1}{2} (1 - 2\varepsilon \rho \nu z + \varepsilon^2 \nu^2 z^2) P_{zz} - \frac{1}{2} \varepsilon \alpha \frac{C'}{C} P_z + (\varepsilon \rho \nu - \varepsilon^2 \nu^2 z)(\alpha P_{z\alpha} - P_z) + \frac{1}{2} \varepsilon^2 \nu^2 \alpha^2 P_{\alpha\alpha} \,,$$

$$\text{for } \tau > 0 \,, \tag{3.33}$$

$$P = \frac{\alpha}{\varepsilon C(K)} \delta(z), \ \text{at } \tau = 0 \,. \tag{3.34}$$

Accordingly, it is quite natural to think that simplifying the problem by define $P(\tau, z, \alpha)$ by

$$\widehat{P}(\tau, z, \alpha) = \frac{\varepsilon}{\alpha} C(K) P(\tau, z, \alpha) \,, \tag{3.35}$$

In terms of $\widehat{P}$, we obtain

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2} \varepsilon \alpha C(K) \int_0^{\tau_{ex}} \widehat{P}(\tau, z, \alpha) d\tau \,, \tag{3.36}$$

where $\widehat{P}(\tau, z, \alpha)$ is the solution of

$$\hat{P}_\tau = \frac{1}{2}(1 - 2\varepsilon\rho vz + \varepsilon^2 v^2 z^2)\hat{P}_{zz} - \frac{1}{2}\varepsilon\alpha\frac{C'}{C}\hat{P}_z + (\varepsilon\rho v - \varepsilon^2 v^2 z)\alpha\hat{P}_{z\alpha} + \frac{1}{2}\varepsilon^2 v^2(\alpha^2\hat{P}_{\alpha\alpha} + 2\alpha\hat{P}_\alpha),$$

$$\text{for } \tau > 0, \tag{3.37}$$

$$\hat{P} = \delta(z), \text{ at } \tau = 0. \tag{3.38}$$

Re-arranging the terms according to the order of ε, $O(\varepsilon)$ and $O(\varepsilon^2)$, we have

$$\hat{P}_\tau = \frac{1}{2}\hat{P}_{zz} + \varepsilon\left(\rho v\alpha\hat{P}_{z\alpha} - \rho vz\hat{P}_{zz} - \frac{1}{2}\alpha\frac{C'}{C}\hat{P}_z\right) + \varepsilon^2\left(\frac{1}{2}v^2 z^2\hat{P}_{zz} - v^2 z\alpha\hat{P}_{z\alpha} + \frac{1}{2}v^2\alpha^2\hat{P}_{\alpha\alpha} + \alpha v^2\hat{P}_\alpha\right),$$

$$\text{for } \tau > 0,$$

with

$$\hat{P} = \delta(z), \qquad \text{at } \tau = 0.$$

Note that the variable α does not enter the problem for $\hat{P}$ until $O(\varepsilon)$, so

$$\hat{P}(\tau, z, \alpha) = \hat{P}_0(\tau, z) + \hat{P}_1(\tau, z, \alpha) + \cdots \tag{3.39}$$

Consequently, the derivatives $\hat{P}_{z\alpha}$, $\hat{P}_{\alpha\alpha}$, and $\hat{P}_\alpha$ are all $O(\varepsilon)$. So we can re-write our problem as

$$\hat{P}_\tau = \frac{1}{2}\hat{P}_{zz} + \varepsilon\left(\rho v\alpha\hat{P}_{z\alpha} - \rho vz\hat{P}_{zz} + \frac{1}{2}\alpha\frac{C'}{C}\hat{P}_z\right) + \varepsilon^2\left(\frac{1}{2}v^2 z^2\hat{P}_{zz}\right), \text{ for } \tau > 0, \tag{3.40}$$

$$\hat{P} = \delta(z), \qquad \text{at } \tau = 0.$$

Now we need to eliminate the $\frac{1}{2}\varepsilon\alpha\frac{C'}{C}\hat{P}_z$ term. Define $H(\tau, z, \alpha)$ by

$$\hat{P} = \sqrt{\frac{C(f)}{C(K)}} \cdot H, \tag{3.41}$$

then

$$\hat{P}_z = \sqrt{\frac{C(f)}{C(K)}} \cdot \left\{H_z + \frac{1}{2}\varepsilon\alpha\frac{C'}{C}H\right\}, \tag{3.42}$$

$$\hat{P}_{zz} = \sqrt{\frac{C(f)}{C(K)}} \cdot \left\{H_{zz} + \varepsilon\alpha\frac{C'}{C}H_z + \varepsilon^2\alpha^2\left[\frac{C''}{2C} - \frac{C'^2}{4C^2}\right]H\right\}, \tag{3.43}$$

$$\hat{P}_{z\alpha} = \sqrt{\frac{C(f)}{C(K)}} \cdot \left\{H_{z\alpha} + \frac{1}{2}\varepsilon z\frac{C'}{C}H_z + \frac{1}{2}\varepsilon\alpha\frac{C'}{C}H_\alpha + \frac{1}{2}\varepsilon\frac{C'}{C}H + O(\varepsilon^2)\right\}. \tag{3.44}$$

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The option price now becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon\alpha\sqrt{C(f)C(K)}\int_0^{\tau_{ex}} H(\tau, z, \alpha)d\tau,$$ (3.45)

where

$$H_\tau = \frac{1}{2}(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2)H_{zz} - \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{C'}{C}(zH_z - H) + \varepsilon^2\alpha^2\left(\frac{C''}{4C} - \frac{3C'^2}{8C^2}\right)H$$

$$+\varepsilon\rho\nu\alpha\left(H_{z\alpha} + \frac{1}{2}\varepsilon\alpha\frac{C'}{C}H_\alpha\right), \qquad \tau > 0,$$ (3.46)

$$H = \delta(z), \ \tau = 0.$$ (3.47)

As above, re-arranging terms as

$$H_\tau = \frac{1}{2}H_{zz} + \varepsilon(\rho\nu\alpha H_{z\alpha} - \rho\nu z H_{zz})$$

$$+ \varepsilon^2\left(\frac{1}{2}\nu^2 z^2 H_{zz} - \frac{1}{2}\rho\nu\alpha z\frac{C'}{C}H_z + \frac{1}{2}\rho\nu\alpha\frac{C'}{C}H + \frac{C''}{4C}\alpha^2 H - \frac{3C'^2}{8C^2}H\alpha^2\right.$$

$$\left.+ \frac{1}{2}\rho\nu\alpha^2\frac{C'}{C}H_\alpha\right), \qquad \tau > 0.$$

Since the leading term is independent of α, as before, we can conclude that the α derivatives $H_{z\alpha}$ and H_α are $O(\varepsilon)$. So we can re-write the problem as

$$H_\tau = \frac{1}{2}H_{zz} - \varepsilon\rho\nu z H_{zz} + \varepsilon^2\left(\frac{1}{2}\nu^2 z^2 H_{zz} - \frac{1}{2}\rho\nu\alpha z\frac{C'}{C}H_z + \frac{1}{2}\rho\nu\alpha\frac{C'}{C}H + \frac{C''}{4C}\alpha^2 H - \frac{3C'^2}{8C^2}H\alpha^2\right)$$

$$+\varepsilon\rho\nu\alpha H_{z\alpha}, \ \tau > 0.$$ (3.48)

For the same reason, H is independent of α until $O(\varepsilon^2)$ and therefore $H_{z\alpha}$ is actually $O(\varepsilon^2)$. Thus,

$$H_\tau = \frac{1}{2}H_{zz} - \varepsilon\rho\nu z H_{zz} + \varepsilon^2\left(\frac{1}{2}\nu^2 z^2 H_{zz} - \frac{1}{2}\rho\nu\alpha z\frac{C'}{C}H_z + \frac{1}{2}\rho\nu\alpha\frac{C'}{C}H + \frac{C''}{4C}\alpha^2 H - \frac{3C'^2}{8C^2}H\alpha^2\right),$$

$$\tau > 0,$$ (3.49)

$$H = \delta(z), \qquad \tau = 0.$$

At this step, there are no longer any α derivatives, so we can treat α as a parameter instead of an

independent variable. We have succeeded in reducing the problem to one dimension.

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Let us now remove the H_z term through $O(\varepsilon^2)$. To leading order, $\frac{C'}{C}$ and $\frac{C''}{C}$ are constants. We can replace these ratios by

$$b_1 = B'(\varepsilon\alpha z_0)/B(\varepsilon\alpha z_0), \tag{3.50}$$

$$b_2 = B''(\varepsilon\alpha z_0)/B(\varepsilon\alpha z_0), \tag{3.51}$$

committing an $O(\varepsilon)$ error, where the constant z_0 will be chosen later. We now define $\widehat{H}$ by

$$H = e^{\frac{\varepsilon^2 \rho v \alpha b_1 z^2}{4}} \cdot \widehat{H}, \tag{3.52}$$

then our option price becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)} \cdot e^{\frac{\varepsilon^2 \rho v \alpha b_1 z^2}{4}} \int_0^{\tau_{ex}} \widehat{H}(\tau, z)d\tau, \tag{3.53}$$

where $\widehat{H}$ is the solution of

$$\widehat{H}_\tau = \frac{1}{2}(1 - 2\varepsilon\rho v z + \varepsilon^2 v^2 z^2)\widehat{H}_{zz} + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\widehat{H} + \frac{3}{4}\varepsilon^2\rho v\alpha b_1\widehat{H}, \text{ for } \tau > 0 \tag{3.54}$$

$$\widehat{H} = \delta(z), \qquad \tau = 0. \tag{3.55}$$

We now define

$$x = \frac{1}{\varepsilon v}\int_0^{\varepsilon v z} \frac{d\zeta}{\sqrt{1 - 2\rho\zeta + \zeta^2}} = \frac{1}{\varepsilon v}\ln\left(\frac{\sqrt{1 - 2\varepsilon\rho v z + \varepsilon^2 v^2 z^2} - \rho + \varepsilon v z}{1 - \rho}\right). \tag{3.56}$$

In terms of x, our problem becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)} \cdot e^{\frac{\varepsilon^2 \rho v \alpha b_1 z^2}{4}} \int_0^{\tau_{ex}} \widehat{H}(\tau, x)d\tau, \tag{3.57}$$

with

$$\widehat{H}_\tau = \frac{1}{2}\widehat{H}_{xx} - \frac{1}{2}\varepsilon v I'(\varepsilon v z)\widehat{H}_x + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\widehat{H} + \frac{3}{4}\varepsilon^2\rho v\alpha b_1\widehat{H}, \text{ for } \tau > 0, \tag{3.60}$$

$$\widehat{H} = \delta(x), \qquad \tau = 0.$$

here

$$I(\zeta) = \sqrt{1 - 2\rho\zeta + \zeta^2}. \tag{3.61}$$

Now the final step is to define Q by

$$\hat{H} = I^{\frac{1}{2}}\big(\varepsilon v z(x)\big)Q = (1 - 2\varepsilon\rho v z + \varepsilon^2 v^2 z^2)^{\frac{1}{4}}Q, \tag{3.62}$$

then

$$\hat{H}_x = I^{\frac{1}{2}}(\varepsilon v z)\left[Q_x + \frac{1}{2}\varepsilon v I'(\varepsilon v z)Q\right], \tag{3.63}$$

$$\hat{H}_{xx} = I^{\frac{1}{2}}(\varepsilon v z)\left[Q_{xx} + \varepsilon v I'Q_x + \varepsilon^2 v^2\left(\frac{1}{2}I''I + \frac{1}{4}I'I'\right)Q\right] \tag{3.64}$$

Hence

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)} \cdot I^{\frac{1}{2}}(\varepsilon v z)e^{\frac{\varepsilon^2\rho v\alpha b_1 z^2}{4}}\int_0^{\tau_{ex}} Q(\tau, x)d\tau, \tag{3.65}$$

where Q is the solution of

$$Q_\tau = \frac{1}{2}Q_{xx} + \varepsilon^2 v^2\left(\frac{1}{4}I''I - \frac{1}{8}I'I'\right)Q + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1{}^2\right)Q + \frac{3}{4}\varepsilon^2\rho v\alpha b_1 Q, \text{ for } \tau > 0, \tag{3.66}$$

with

$$Q = \delta(x), \qquad \tau = 0. \tag{3.67}$$

As above, we can replace $I(\varepsilon v z)$, $I'(\varepsilon v z)$, $I''(\varepsilon v z)$ by the constants $I(\varepsilon v z_0)$, $I'(\varepsilon v z_0)$, $I''(\varepsilon v z_0)$,

and commit only $O(\varepsilon)$ errors. Define the constant κ by

$$\kappa = v^2\left(\frac{1}{4}I''(\varepsilon v z_0)I(\varepsilon v z_0) - \frac{1}{8}[I'(\varepsilon v z_0)]^2\right) + \alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1{}^2\right) + \frac{3}{4}\rho v\alpha b_1, \tag{3.68}$$

where z_0 will be chosen later. Then we can simplify our equation to

$$Q_\tau = \frac{1}{2}Q_{xx} + \varepsilon^2\kappa Q, \text{ for } \tau > 0, \tag{3.69}$$

$$Q = \delta(x), \ \tau = 0. \tag{3.70}$$

Now the solution of this problem is clearly

$$Q = \frac{1}{\sqrt{2\pi\tau}}e^{-\frac{x^2}{2\tau}}e^{\varepsilon^2\kappa\tau}. \tag{3.71}$$

Our option price is accordingly becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(0)B(\varepsilon\alpha z)} \cdot I^{\frac{1}{2}}(\varepsilon v z)e^{\frac{\varepsilon^2 \rho v \alpha b_1 z^2}{4}} \int_0^{\tau_{ex}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}e^{\varepsilon^2 \kappa \tau}d\tau, \quad (3.72)$$

which can be written as

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2} \cdot \frac{f - K}{x} \int_0^{\tau_{ex}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}e^{\varepsilon^2 \kappa \tau}e^{\varepsilon^2 \theta}d\tau, \quad (3.73)$$

where

$$\varepsilon^2 \theta = \ln\left(\frac{\varepsilon\alpha z}{f - K}\sqrt{B(0)B(\varepsilon\alpha z)}\right) + \ln\left(\frac{xI^{\frac{1}{2}}(\varepsilon v z)}{z}\right) + \frac{1}{4}\varepsilon^2 \rho v \alpha b_1 z^2. \quad (3.74)$$

Moreover, quite amazingly,

$$e^{\varepsilon^2 \kappa \tau} = \frac{1}{\left(1 - \frac{2}{3}\kappa\varepsilon^2\tau\right)^{3/2}} = \frac{1}{\left(1 - 2\varepsilon^2\tau\frac{\theta}{x^2}\right)^{3/2}} + O(\varepsilon^4) \quad (3.75)$$

through $O(\varepsilon^2)$. Therefore our option price is

$$V(t, f, \alpha) = [f - K]^+ + \frac{1}{2} \cdot \frac{f - K}{x} \int_0^{\tau_{ex}} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\tau}}e^{\varepsilon^2 \theta} \frac{d\tau}{\left(1 - \frac{2\tau}{x^2}\varepsilon^2\theta\right)^{3/2}}. \quad (3.76)$$

Define

$$q = \frac{x^2}{2\tau}, \quad (3.77)$$

option price becomes

$$V(t, f, \alpha) = [f - K]^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{ex}}}^{\infty} \frac{e^{-q + \varepsilon^2 \theta}}{(q - \varepsilon^2 \theta)^{3/2}} dq. \quad (3.78)$$

That is, the value of a European call option is given by

$$V(t, f, \alpha) = [f - K]^+ + \frac{|f - K|}{4\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{ex}} - \varepsilon^2 \theta}^{\infty} \frac{e^{-q}}{q^{3/2}} dq, \quad (3.79)$$

with

$$\varepsilon^2 \theta = \ln\left(\frac{\varepsilon\alpha z}{f - K}\sqrt{B(0)B(\varepsilon\alpha z)}\right) + \ln\left(\frac{xI^{\frac{1}{2}}(\varepsilon v z)}{z}\right) + \frac{1}{4}\varepsilon^2 \rho v \alpha b_1 z^2 \quad (3.74)$$

through $O(\varepsilon^2)$.

We have derived European call option pricing function under general SABR model. To utilize the pricing function more conveniently, we derive the implied normal volatility and the implied Black volatility now.

First, we consider the normal model

$$dF = \sigma_N dW, \qquad F(0) = f, \qquad (3.75)$$

where the normal volatility σ_N is a constant. Setting $C(F) = 1$, $\varepsilon\alpha = \sigma_N$, and $\nu = 0$ in our general SABR model (3.3) – (3.5) and repeat the preceding analysis we would find that the option price for the normal model is

$$V(t, f, \alpha) = [f - K]^+ + \frac{|f-K|}{4\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_N^2\tau_{ex}}}^{\infty} \frac{e^{-q}}{q^{3/2}} dq. \qquad (3.76)$$

Equating this to the option value for the SABR model implies that

$$\sigma_N{}^2 = \frac{(f-K)^2}{x^2 - 2\varepsilon^2\theta\tau_{ex}}. \qquad (3.77)$$

Taking the square root now shows that the option's implied normal volatility is given by

$$\sigma_N = \frac{f-K}{x}\left\{1 + \varepsilon^2 \frac{\theta}{x^2}\tau_{ex} + \cdots\right\} \qquad (3.78)$$

through $O(\varepsilon^2)$. Since $x = z[1 + O(\varepsilon)]$, we can re-write the answer as

$$\sigma_N = \left(\frac{f-K}{z}\right)\left(\frac{z}{x(z)}\right)\{1 + \varepsilon^2(\phi_1 + \phi_2 + \phi_3)\tau_{ex} + \cdots\}, \qquad (3.79)$$

where

$$\varepsilon^2\phi_1 = \frac{1}{z^2}\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{C(f)C(K)}\right) = \frac{2\gamma_2 - \gamma_1{}^2}{24}\varepsilon^2\alpha^2 C^2(f_{av}) + \cdots, \qquad (3.80)$$

$$\varepsilon^2\phi_2 = \frac{1}{z^2}\ln\left(\frac{x}{z}[1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2]^{\frac{1}{4}}\right) = \frac{2 - 3\rho^2}{24}\varepsilon^2\nu^2 + \cdots, \qquad (3.81)$$

$$\varepsilon^2\phi_3 = \frac{1}{4}\varepsilon^2\rho\alpha\nu\frac{B'(\varepsilon\nu z_0)}{B(\varepsilon\nu z_0)} = \frac{1}{4}\varepsilon^2\rho\alpha\nu\gamma_1 C(f_{av}) + \cdots, \qquad (3.82)$$

$$\gamma_1 = \frac{C'(f_{av})}{C(f_{av})}, \qquad \gamma_2 = \frac{C''(f_{av})}{C(f_{av})}. \qquad (3.83)$$

The first factor

$$\frac{f-K}{z} = \frac{\varepsilon\alpha(f-K)}{\int_K^f \frac{df'}{C(f')}} = \left(\frac{1}{f-K}\int_K^f \frac{df'}{\varepsilon\alpha C(f')}\right)^{-1} \tag{3.84}$$

represents the average difficulty in diffusing from today's forward f to the strike K. And the second factor

$$\frac{z}{x(z)} = \frac{\zeta}{\ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right)}, \tag{3.85}$$

where

$$\zeta = \varepsilon\nu z = \frac{\nu}{\alpha}\int_K^f \frac{df'}{C(f')} = \frac{\nu}{\alpha}\frac{f-K}{C(f_{av})}\{1 + O(\varepsilon^2)\} \tag{3.86}$$

represents the main effect of the stochastic volatility.

Now, let us consider the Black's model

$$dF = \varepsilon\sigma_B FdW, \qquad F(0) = f. \tag{3.87}$$

For Black's model, the value of a European call with strike K and exercise date τ_{ex} is

$$V_{call} = D(t_{set}) \cdot \{f\mathcal{N}(d_1) - K\mathcal{N}(d_2)\}, \tag{3.88}$$

$$V_{put} = V_{call} + D(t_{set})[K - f], $$

with

$$d_{1,2} = \frac{1}{\varepsilon\sigma_B\sqrt{\tau_{ex}}}\left\{\ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\varepsilon^2\sigma_B^2\tau_{ex}\right\}. \tag{3.89}$$

As in the case of normal volatility model, we repeat the analysis for the SABR model with $C(F) = F$ and $\nu = 0$ shows that

$$\sigma_N(K) = \frac{\varepsilon\sigma_B(f-K)}{\ln\left(\frac{f}{K}\right)}\left\{1 - \frac{1}{24}\varepsilon^2\sigma_B^2\tau_{ex} + \cdots\right\} \tag{3.90}$$

through $O(\varepsilon^2)$. We can find the implied Black volatility for the SABR model by setting (3.90) equal to the implied normal volatility for the SABR model. This yields

$$\sigma_B(K) = \frac{\alpha \ln\left(\frac{f}{K}\right)}{\int_K^f \frac{df'}{C(f')}} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \cdot \left\{1 + \left[\frac{2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2}}{24} \alpha^2 C^2(f_{av}) + \frac{1}{4}\rho\nu\alpha\gamma_1 C(f_{av}) + \frac{2-3\rho^2}{24}\nu^2\right]\varepsilon^2\tau_{ex} + \cdots\right\}.$$

(3.91)

Now, let us go back to the special case of SABR model (3.1) – (3.3). The according implied normal volatility for this model is

$$\sigma_N(K) = \frac{\varepsilon\alpha(1-\beta)(f-K)}{f^{1-\beta} - K^{1-\beta}} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right) \cdot \left\{1 + \left[\frac{-\beta(2-\beta)\alpha^2}{24 f_{av}^{2-2\beta}} + \frac{\rho\alpha\nu\beta}{4 f_{av}^{1-\beta}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\varepsilon^2\tau_{ex} + \cdots\right\} \quad (3.92)$$

through $O(\varepsilon^2)$, where $f_{av} = \sqrt{fK}$ and

$$\zeta = \frac{\nu}{\alpha} \cdot \frac{f-K}{f_{av}^{\beta}}, \qquad \hat{x}(\zeta) = \ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \qquad (3.93) \text{ \& } (3.94)$$

We can simplify this formula by expanding

$$f - K = \sqrt{fK} \ln\left(\frac{f}{K}\right)\left\{1 + \frac{\ln^2\left(\frac{f}{K}\right)}{24} + \frac{\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots\right\} \qquad (3.95)$$

$$f^{1-\beta} - K^{1-\beta} = (1-\beta)(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right)\left\{1 + \frac{(1-\beta)^2\ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots\right\} \qquad (3.96)$$

This expansion reduces the implied normal volatility to

$$\sigma_N(K) = \varepsilon\alpha(fK)^{\frac{\beta}{2}} \cdot \frac{1 + \dfrac{\ln^2\left(\frac{f}{K}\right)}{24} + \dfrac{\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots}{1 + \dfrac{(1-\beta)^2\ln^2\left(\frac{f}{K}\right)}{24} + \dfrac{(1-\beta)^4\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$

$$\cdot \left\{1 + \left[\frac{-\beta(2-\beta)\alpha^2}{24 f_{av}^{2-2\beta}} + \frac{\rho\alpha\nu\beta}{4 f_{av}^{1-\beta}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\varepsilon^2\tau_{ex} + \cdots\right\} \qquad (3.97)$$

where

$$\zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right), \qquad \hat{x}(\zeta) = \ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right). \qquad (3.98) \text{\& } (3.99)$$

Again, equating the implied normal volatility $\sigma_N(K)$ for the SABR model to the implied normal volatility for Black's model yields the implied Black volatility for the SABR model

$$\sigma_B(K) = \frac{\varepsilon\alpha}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1-\beta)^2\ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$

$$\cdot \left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\alpha\nu\beta}{4(fK)^{\frac{1-\beta}{2}}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\varepsilon^2\tau_{ex} + \cdots\right\} \qquad (3.100)$$

through $O(\varepsilon^2)$.

Up to now, we have derived the implied Black volatility for SABR model, which provides us great convenience in pricing options. We can simply use the Black model (3.88), (3.89) to keep our pricing function structural simple. Here the SABR model and the implied Black volatility are given by the formulas (3.1) – (3.3) and (3.100).

3.3. Hedging smile risks

As we have stated before, for local volatility model, $\frac{\partial\sigma_B}{\partial f}$ has the opposite sign to the real market behavior, and this causes unstable hedges. Now, we want show that the SABR model has succeeded in overcoming the problem and produce stable hedges.

Under the SABR model, the implied Black volatility is given by (3.100)

$$\sigma_B(f, K) = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1-\beta)^2\ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4\ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$

$$\cdot \left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\alpha\nu\beta}{4(fK)^{\frac{1-\beta}{2}}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\tau_{ex} + \cdots\right\}$$

where ζ and $\hat{x}(\zeta)$ are given by (3.98) and (3.99)

$$\zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right), \qquad \hat{x}(\zeta) = \ln\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right).$$

For the special case of at-the-money options, the underlying price is the same as the strike price, this formula reduces to

$$\sigma_{ATM} = \sigma_B(f,f) = \frac{\alpha}{f^{(1-\beta)}} \cdot \left\{1 + \left[\frac{(1-\beta)^2\alpha^2}{24f^{(2-2\beta)}} + \frac{\rho\alpha\nu\beta}{4f^{(1-\beta)}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\tau_{ex} + \cdots\right\}. \qquad (3.101)$$

Suppose that the strike price K is not far from the current forward price f, we can approximate the above formula as

$$\sigma_B(f,K) = \frac{\alpha}{f^{(1-\beta)}} \cdot \left\{1 - \frac{1}{2}(1-\beta-\rho\lambda)\ln\left(\frac{f}{K}\right) + \frac{1}{12}[(1-\beta)^2 + (2-3\rho^2)\lambda^2]\ln^2\left(\frac{K}{f}\right)\cdots\right\}$$

$$(3.102)$$

where

$$\lambda = \frac{\nu}{\alpha}f^{(1-\beta)}. \qquad (3.103)$$

This equation cannot be used to price deals, but it is accurate enough to expose the qualitative

behavior of SABR model.

As f varies, the curve that the at-the-money volatility $\sigma_B(f,f)$ traces is known as the "backbone",

while considering the implied volatility smile $\sigma_B(f,K)$ as a function of strike K for a fixed f. The

first term $\frac{\alpha}{f^{(1-\beta)}}$ in (3.102) is the implied volatility for at-the-money options. Therefore, the

backbone traversed by at-the-money option is essentially $\sigma_B(f,f) = \frac{\alpha}{f^{(1-\beta)}}$ for SABR model. The

backbone is almost entirely determined by the exponent β, with the exponent $\beta = 0$ (a stochastic

Gaussian model) giving a steeply downward sloping backbone, $\frac{\partial\sigma_B}{\partial f} < 0$; and the exponent $\beta = 0$

(stochastic log normal model) giving a nearly flat backbone, $\frac{\partial\sigma_B}{\partial f} \sim 0$. See figure 4 and 5:
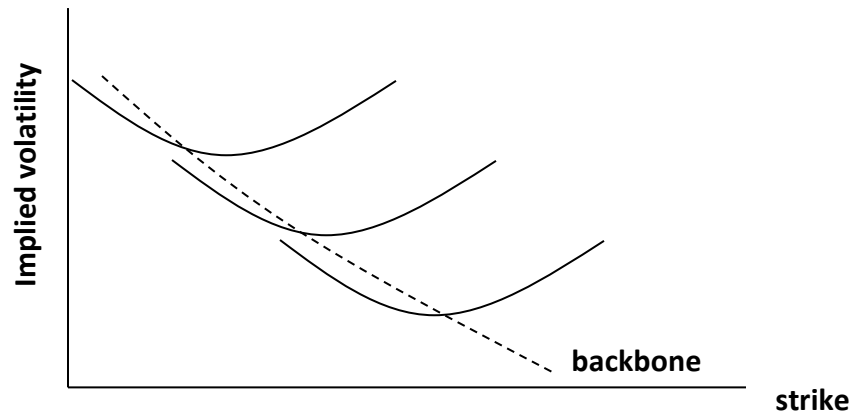
Figure 4. Backbone and volatility for β=0.

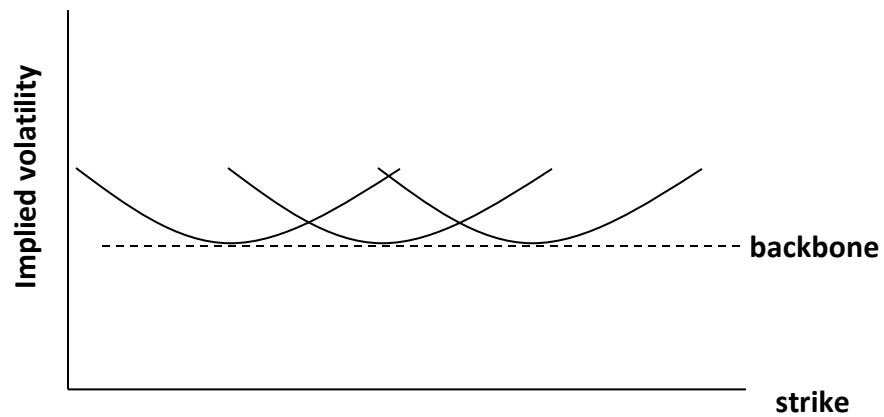Figure 5. Backbone and volatility for β=1.

Now we see, once the value of parameters ν, β, ρ and α have been chosen properly for the specific market, the SABR model (3.1) – (3.3) is a single, self-consistent model that fits the option prices for all strikes K without "adjustment", so we can use this model to price exotic options without ambiguity. Moreover, because of the downward sloping backbone, the SABR model reveals that

whenever the forward price f changes, the implied volatility curve shifts in the same direction. This is one of the most important advantages of the SABR model.

Now, Let us consider the smile risks. First of all, we note that since the SABR model is a single, self-consistent model for all strikes K, so the risks calculated at one strike are consistent with the risks calculated at other strikes. Therefore, the risks of all the options on the same asset can be added together, and only the residual risk needs to be hedged.

Let $BS(f, K, \sigma_B, t_{ex})$ be Black's formula

$$V_{call} = D(t_{set})\{f\mathcal{N}(d_1) - K\mathcal{N}(d_2)\}$$

$$V_{put} = V_{call} + D(t_{set})[K - f]$$

$$d_{1,2} = \frac{\ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B{}^2 t_{ex}}{\sigma_B\sqrt{t_{ex}}}$$

for, say, a call option. According to the SABR model, the value of a call is

$$V_{call} = BS(f, K, \sigma_B(K, f), t_{ex}) \tag{3.104}$$

where the volatility $\sigma_B(K, f) = \sigma_B(K, f; \alpha, \beta, \rho, v)$ is given by equations $(3.98) - (3.100)$.

Differentiating with respect to α yields the vega risk, the risk to overall changes in volatility:

$$vega = \frac{\partial V_{call}}{\partial \alpha} = \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K, f; \alpha, \beta, \rho, v)}{\partial \alpha}. \tag{3.105}$$

Note that to leading order, $\frac{\partial \sigma_B}{\partial \alpha} \sim \frac{\sigma_B}{\alpha}$, so the vega risk is roughly given by

$$vega \sim \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\sigma_B}{\alpha}. \tag{3.106}$$

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Since for the SABR model, we also have two parameters ρ and ν that are stochastic, therefore the SABR model has risks to those two parameters. We name the risk to ρ changing as vanna and the risk to ν changing as volga:

$$\text{vanna} = \frac{\partial V_{call}}{\partial \rho} = \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \rho}, \qquad (3.107)$$

$$\text{volga} = \frac{\partial V_{call}}{\partial \nu} = \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \nu}. \qquad (3.108)$$

Vanna basically expresses the risk to the skew increasing, and volga expresses the risk to the smile becoming more pronounced.

The delta risk expressed by the SABR model is calculated by differentiating the option price with respect to f,

$$\Delta = \frac{\partial V_{call}}{\partial f} = \frac{\partial BS}{\partial f} + \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial f}. \qquad (3.109)$$

The first term is the ordinary delta risk one would calculate from Black's model; the second term is SABR model's correction to the delta risk. It consists of the Black vega times the predicted change in the forward f. As discussed above, the predicted change consists of a sideways movement of the volatility curve in the same direction as the change in the forward price f.

However, we may notice that the delta risk calculated above has a slight problem. In the above expression of delta risk, we actually regarded α as a constant. But once α and f are correlated, i.e. $\rho \neq 0$, whenever f changes, α, at least on average, changes as well. This suggests that the delta risk could be calculated more accurately. To solve this problem, B. Bartlett [5] rewrites the dynamics of the SABR model as

$$dF = \alpha F^{\beta} dW_1,$$

$$d\alpha = \nu \alpha dW_2 = \nu \alpha \left(\rho dW_1 + \sqrt{1 - \rho^2} dY \right), \qquad (3.110)$$

where Y_t is another Brownian motion which is independent of W_t

$$dW_1 dY = 0. \qquad (3.111)$$

Clearly, this modification of the SABR model implies that

$$d\alpha = \frac{\rho \nu}{F^{\beta}} dF + \nu \alpha \sqrt{1 - \rho^2} dY. \qquad (3.112)$$

In other words, the dynamics of α can be decomposed into two independent components: one due to the change of F, and the other one due to the idiosyncratic change in α. In terms of this modified SABR model, the new delta risk is then given by

$$\Delta = \frac{\partial V_{call}}{\partial f} = \frac{\partial BS}{\partial f} + \frac{\partial BS}{\partial \sigma_B} \cdot \left\{ \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial f} + \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \alpha} \cdot \frac{\rho \nu}{f^{\beta}} \right\}. \qquad (3.113)$$

Hedging this risk should be more effective than hedging the original SABR delta risk.

We note that the new term in the delta risk

$$\frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \alpha} \cdot \frac{\rho \nu}{f^{\beta}} \qquad (3.114)$$

is just $\frac{\rho \nu}{f^{\beta}}$ times the original SABR vega risk. In a vega-hedged portfolio this term is zero, so if

the original vega and delta risk are both hedged, so are the new delta risk.

Under the new SABR model, it is obvious that the new vega risk is given by

$$vega = \frac{\partial V_{call}}{\partial \alpha} = \frac{\partial BS}{\partial \sigma_B} \cdot \left\{ \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \alpha} + \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial f} \cdot \frac{\rho f^{\beta}}{\nu} \right\} + \frac{\partial BS}{\partial f} \cdot \frac{\rho f^{\beta}}{\nu}. \qquad (3.115)$$

4. ANALYSIS OF THE DEPENDENCE OF PARAMETERS α, β, ρ

Now, it would be interesting to consider how the parameters α, β, ρ affects the option price and volatility smile.

First, let us consider how an option price is affected by the overall volatility parameter α. As before, let $BS(f, K, \sigma_B, t_{ex})$ be Black's formula (3.88) – (3.89),

$$V_{call} = D(t_{set})\{fN(d_1) - KN(d_2)\}$$

$$V_{put} = V_{call} + D(t_{set})[K - f]$$

$$d_{1,2} = \frac{\ln\left(\frac{f}{K}\right) \pm \frac{1}{2}\sigma_B^2 t_{ex}}{\sigma_B\sqrt{t_{ex}}}$$

According to the SABR model, the value of a call option is given by (3.104)

$$V_{call} = BS(f, K, \sigma_B(K, f), t_{ex})$$

where the volatility $\sigma_B(K, f) = \sigma_B(K, f; \alpha, \beta, \rho, \nu)$ is given by equations (3.98) – (3.100)

$$\sigma_B(f, K) = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1-\beta)^2 \ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4 \ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right)$$

$$\cdot \left\{1 + \left[\frac{(1-\beta)^2 \alpha^2}{24(fK)^{1-\beta}} + \frac{\rho\alpha\nu\beta}{4(fK)^{\frac{1-\beta}{2}}} + \frac{(2-3\rho^2)\nu^2}{24}\right]\tau_{ex} + \cdots\right\}$$

where

$$\zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}}\ln\left(\frac{f}{K}\right), \qquad \hat{x}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right).$$

Differentiating the option price V_{call} with respect to α, we note that this is the vega risk for SABR model. Using our result of the vega risk directly from the previous section, we see that

$$\frac{\partial V_{call}}{\partial \alpha} = \text{vega} = \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \alpha} . \tag{4.1}$$

From our study of the Black-Scholes model, we know that an option's value is a monotonic increasing function of implied volatility, provided all other parameters being equal, so the first factor $\frac{\partial BS}{\partial \sigma_B} > 0$. We only need to consider the monotonicity of the implied Black volatility $\sigma_B(K,f)$ with respect to α. To leading order, we see

$$\frac{\partial \sigma_B}{\partial \alpha} \sim \frac{\sigma_B}{\alpha} = \frac{1}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1-\beta)^2 \ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4 \ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \left(\frac{\zeta}{\hat{x}(\zeta)}\right), \tag{4.2}$$

with

$$\zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right), \qquad \hat{x}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right).$$

Since

$$\frac{1}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1 - \beta)^2 \ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1 - \beta)^4 \ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} > 0,$$

and

$$\hat{x}(\zeta) = \ln\left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho}\right) > 0,$$

We only need to consider

$$\zeta = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{K}\right).$$

Now, unfortunately, because the 'volvol' ν is a stochastic variable which need to be fitted for the specific market, we cannot say anything more in general about the sign of this term. Therefore we conclude that the monotonicity of option price V_{call} as a function of volatility α depends on the specific ν for the specific market.

Next, we consider the exponent β. As we discussed in the previous section, β mainly affects the slope of the backbone of volatility smile. When β=0, the slope of backbone is steep, and when β=1, the backbone is nearly flat. Moreover, like what is stated in the Hegan et al. 2002, market smiles can be fit equally well with any specific value of β, so selecting β is usually depending on the shape of the backbone that is believed for specific market.

At last, to see how the correlation parameter ρ affects a call option price V_{call} and the volatility smile, we consider the vanna risk

$$\text{vanna} = \frac{\partial V_{call}}{\partial \rho} = \frac{\partial BS}{\partial \sigma_B} \cdot \frac{\partial \sigma_B(K, f; \alpha, \beta, \rho, \nu)}{\partial \rho}.$$

Since we know $\frac{\partial BS}{\partial \sigma_B} > 0$, we only consider the term $\frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \rho}$. To leading order,

$$\frac{\partial \sigma_B(K, f; \alpha, \beta, \rho, \nu)}{\partial \rho} = \frac{\alpha}{(fK)^{\frac{1-\beta}{2}}} \cdot \frac{1}{1 + \frac{(1-\beta)^2 \ln^2\left(\frac{f}{K}\right)}{24} + \frac{(1-\beta)^4 \ln^4\left(\frac{f}{K}\right)}{1920} + \cdots} \cdot \zeta$$

$$\cdot \left\{ \frac{1-\rho}{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta} \left[\frac{-1}{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta} + \frac{\rho-1}{\left(\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta\right)^2} \cdot \left(\frac{-\zeta}{\sqrt{1-2\rho\zeta+\zeta^2}} - 1 \right) \right] \right\}. \quad (4.3)$$

When the forward price and the volatility are positively correlated, ρ=1, $\frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \rho} = 0$, this causes a nearly flat skew in $\sigma_B(K, f)$ as a function of K varies; When the forward price and the volatility are uncorrelated or negatively correlated, ρ=0 or ρ=-1, $\frac{\partial \sigma_B(K,f;\alpha,\beta,\rho,\nu)}{\partial \rho} < 0$, so the slope of the skew is downward. Therefore, we see that the correlation parameter ρ actually affects the volatility smile in the similar way as the exponent parameter β, which decides the slope of the volatility smile and skew.

5. CONCLUSION AND DISCUSSION

The dynamics of the volatility smile which cannot be explained by the local volatility model can be successfully predicted by SABR model. The changing of parameters β and ρ affects the slope of the "backbone" of the volatility smile. The monotonicity of an option price as a function of volatility α is not so clear; it also depends on the "volvol" factor ν.

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APPPENDIX: SINGULAR PERTURBATION METHOD

Consider a European call with exercise date t_{ex}, settlement date t_{set}, and strike price K. Let F(t) be the forward price at date t. We assume that

$$dF = \alpha A(F)dW \tag{a1}$$

under the forward measure. The corresponding value function of the option $V\big(t, F(t)\big)$ under this measure is given by

$$V(t, f) = D(t, t_{set}) \cdot E\{[F(t_{ex}) - K]^+ | F(t) = f\}, \tag{a2}$$

here $D(t, t_{set})$ is the discount factor to the settlement date t_{set} at date t.

Define

$$Q(t, f) = E\{[F(t_{ex}) - K]^+ | F(t) = f\}, \tag{a3}$$

Then we have

$$V(t, f) = D(t, t_{set}) \cdot Q(t, f). \tag{a4}$$

From Martingale pricing theory, we know under the forward measure Q(t, f) satisfies the backward Kolmogorov equation:

$$Q_t + \frac{1}{2}\alpha^2(t)A^2(F)Q_{ff} = 0, \quad t < t_{ex} \tag{a5}$$

with the final condition

$$Q = [f - K]^+, \quad t = t_{ex}. \tag{a6}$$

Now, define

$$\varepsilon \equiv A(K) \ll 1 \tag{a7}$$

and the new variables

$$\tau(t) = \int_t^{t_{ex}} \alpha^2(s)\, ds, \qquad x = \frac{1}{\varepsilon}(f - K), \qquad \tilde{Q}(\tau, x) = \frac{1}{\varepsilon}Q(t, f). $$

$$\text{(a8), (a9) \& (a10)}$$

In terms of the new variables,

$$Q_t = \varepsilon\tilde{Q}_\tau \cdot \tau_t = -\varepsilon\alpha^2(t)\tilde{Q}_\tau, \qquad Q_f = \varepsilon\tilde{Q}_x x_f, \qquad Q_{ff} = \varepsilon\big[\tilde{Q}_{xx}(x_f)^2 + \tilde{Q}_x x_{ff}\big] = \frac{1}{\varepsilon}\tilde{Q}_{xx}$$

$$\text{(a11), (a12) \& (a13)}$$

We scale the original pricing PDE (a5) as

$$\tilde{Q}_\tau - \frac{1}{2} \cdot \frac{A^2(F)}{\varepsilon^2} \tilde{Q}_{xx} = 0. \tag{a14}$$

Since $\varepsilon = A(K) \ll 1$, we have

$$\tilde{Q}_\tau - \frac{1}{2} \cdot \frac{A^2(K + \varepsilon x)}{A^2(K)} \tilde{Q}_{xx} = 0, \qquad \text{for } \tau > 0 \tag{a15}$$

with

$$\tilde{Q} = x^+, \qquad \text{at } \tau = 0. \tag{a16}$$

After solving this scaled pricing PDE for $\tilde{Q}(\tau, x)$, the option value will be given by

$$V(t, f) = D(t, t_{set}) A(K) \tilde{Q}\left(\tau(t), \frac{f - K}{A(K)}\right) \tag{a17}$$

$\left(\text{in the unscaled case before,} \qquad V(t, f) = D(t, t_{ex}) Q(t, f), \qquad Q(t, f) = \varepsilon \tilde{Q}(\tau, x)\right)$,

Expanding $A(K + \varepsilon x)$,

$$A(K + \varepsilon x) = A(K) + A'(K)\varepsilon x + \frac{A''(K)}{2}(\varepsilon x)^2 + \cdots$$

$$= A(K)\left\{1 + v_1 \varepsilon x + \frac{1}{2} v_2 (\varepsilon x)^2 + \cdots\right\}, \tag{a18}$$

where

$$v_1 = \frac{A'(K)}{A(K)} \qquad v_2 = \frac{A''(K)}{A(K)}, \qquad \cdots \tag{a19}$$

The pricing PDE becomes

$$\tilde{Q}_\tau - \frac{1}{2}\tilde{Q}_{xx} = \left\{v_1 \varepsilon x + \frac{1}{2}\varepsilon^2 x^2 (v_2 + v_1^2) + \cdots\right\}\tilde{Q}_{xx}, \qquad \text{for } \tau > 0 \tag{a20}$$

with

$$\tilde{Q} = x^+, \qquad \text{at } \tau = 0.$$

We can solve this problem by expanding $\tilde{Q}$ as

$$\tilde{Q}(\tau, x) = Q^0 + \varepsilon Q^1(\tau, x) + \varepsilon^2 Q^2(\tau, x) + \cdots \tag{a21}$$

Substituting this expansion into (a20) and equating the same powers of ε yields:

At leading order, we obtain

$$Q^0{}_\tau - \frac{1}{2}Q^0{}_{xx} = 0, \qquad \text{for } \tau > 0 \tag{a22}$$

with

$$Q^0 = x^+, \qquad \text{at } \tau = 0 \tag{a23}$$

We solve this problem yields

$$Q^0(\tau, x) = G(\tau, x) = xN\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\frac{\tau}{2\pi}}\, e^{-\frac{x^2}{2\tau}}. \tag{a24}$$

For later convenience, we also calculate

$$G_\tau = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{x^2}{2\tau}}, \qquad G_x = N\left(\frac{x}{\sqrt{\tau}}\right), \tag{a25),(a26}$$

$$G_{\tau\tau} = \frac{x^2-\tau}{4\tau^2} \cdot \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}}, \qquad G_{x\tau} = \frac{-x}{2\tau} \cdot \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}}, \qquad G_{xx} = \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \, (= 2G_\tau),$$

$$\tag{a27)-(a29}$$

$$G_{\tau\tau\tau} = \frac{x^4-6x^2\tau+3\tau^2}{8\tau^4} \cdot \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}} \, ; \tag{a30}$$

At order ε, we obtain

$$Q^1{}_\tau - \frac{1}{2}Q^1{}_{xx} = v_1 x Q^0{}_{xx}, \qquad \text{for } \tau > 0 \tag{a31}$$

$$Q^1 = 0, \qquad \text{at } \tau = 0 \tag{a32}$$

Substituting the (a24) into (a31) and using the relation $G_{xx} = \frac{-2\tau}{x} G_{x\tau}$, we have

$$Q^1{}_\tau - \frac{1}{2}Q^1{}_{xx} = v_1 x G_{xx} = -2v_1\tau G_{x\tau}, \qquad \text{for } \tau > 0 \tag{a33}$$

$$Q^1 = 0, \qquad \text{at } \tau = 0$$

The solution of this problem is

$$Q^1 = -v_1\tau^2 G_{x\tau} = v_1\tau x G_\tau \, ; \tag{a34}$$

At order ε^2,

$$Q^2{}_\tau - \frac{1}{2}Q^2{}_{xx} = v_1 x Q^1{}_{xx} + \frac{1}{2}(v_2 + v_1)^2 x^2 Q^0{}_{xx}, \qquad \text{for } \tau > 0 \tag{a35}$$

$$Q^2 = 0, \qquad \text{at } \tau = 0 \tag{a36}$$

We substitute (a24) and (a34) into the problem, this yields

$$Q^2{}_\tau - \frac{1}{2}Q^2{}_{xx} = \left(v_1{}^2 \cdot \frac{x^4-2\tau x^2}{2\tau} + \frac{1}{2}v_2 x^2\right)\frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau}}, \qquad \text{for } \tau > 0 \tag{a37}$$

$$Q^2 = 0, \qquad \text{at } \tau = 0$$

Solving this problem we obtain

$$Q^2 = v_1^2 \left[\tau^4 G_{\tau\tau\tau} + \frac{8}{3}\tau^3 G_{\tau\tau} \frac{1}{2}\tau^2 G_\tau \right] + v_2 \left[\frac{2}{3}\tau^3 G_{\tau\tau} + \frac{1}{2}\tau^2 G_\tau \right]. \tag{a38}$$

Re-writing this solution as

$$Q^2 = \frac{1}{2}v_1^2 \tau^2 x^2 G_{\tau\tau} + \frac{1}{12}v_1^2 (x^2 - \tau)\tau G_\tau + \frac{1}{6}v_2(2x^2 + \tau)\tau G_\tau. \tag{a39}$$

Now, substituting (a24), (a34), and (a39) back into (a21) we get

$$\widetilde{Q}(\tau,x) = G + \varepsilon v_1 \tau x G_\tau + \frac{1}{2}\varepsilon^2 v_1^2 \tau^2 x^2 G_{\tau\tau} + \varepsilon^2 \left(\frac{4v_1 + v_1^2}{12} x^2 + \frac{2v_2 - v_1^2}{12} \tau \right) \tau G_\tau + \cdots \tag{a40}$$

Recall that $G(\tau,x) = xN\left(\frac{x}{\sqrt{\tau}}\right) + \sqrt{\frac{\tau}{2\pi}} e^{-\frac{x^2}{2\tau}}$, this can be re-write as

$$\widetilde{Q}(\tau,x) = G(\tilde{\tau},x) \tag{a41}$$

where

$$\tilde{\tau} = \tau \left[1 + \varepsilon v_1 x + \varepsilon^2 \left(\frac{4v_1 + v_1^2}{12} x^2 + \frac{2v_2 - v_1^2}{12} \tau \right) + \cdots \right]. \tag{a42}$$

To obtain the option value, recall that

$$V(t,f) = D(t,t_{set})A(K)\widetilde{Q}\left(\tau(t),\frac{f-K}{A(K)}\right) = D(t,t_{set})\varepsilon G(\tilde{\tau},x), \tag{a43}$$

we note that

$$\varepsilon G(\tilde{\tau},x) = G(\varepsilon^2 \tilde{\tau}, \varepsilon x) \tag{a44}$$

$$= G[A^2(K)\tilde{\tau}, f - K].$$

Thus the value of the option is

$$V(t,f) = D(t,t_{set})G(\tau^*, f - K), \tag{a45}$$

with

$$\tau^* = A^2(K)\tilde{\tau} = A^2(K)\tau \left[1 + v_1(f - K) + \frac{4v_2 + v_1^2}{12}(f - K)^2 + \frac{2v_2 - v_1^2}{12}A^2(K)\tau + \cdots \right] \tag{a46}$$

Now, we obtain the asymptotic solution for the option's price. It is not very convenient to use, so, instead, we compute the equivalent Black volatility implied by this price.

To simplify the calculation, we take the square root of τ^* and obtain

$$\sqrt{\tau^*} = A(K)\sqrt{\tau}\left[1 + \frac{1}{2}v_1(f - K) + \frac{2v_2 - v_1^2}{12}(f - K)^2 + \frac{2v_2 - v_1^2}{24}A^2(K)\tau + \cdots\right] \qquad (a47)$$

Since

$$v_1 = \frac{A'(K)}{A(K)} \qquad v_2 = \frac{A''(K)}{A(K)}, \qquad \cdots \qquad (a19)$$

The first two terms on the right hand side are

$$A(K)\sqrt{\tau} + \frac{1}{2}A'(K)(f - K)\sqrt{\tau} \qquad (a48)$$

This suggests expanding A around the average

$$f_{av} = \frac{1}{2}(f + K) \qquad (a49)$$

instead of K. Therefore, we define

$$\gamma_1 = \frac{A'(f_{av})}{A(f_{av})} \qquad \gamma_2 = \frac{A''(f_{av})}{A(f_{av})}, \qquad \cdots \qquad (a50)$$

and re-write (a46) in terms of γ_1 and γ_2. This then shows that the option price is

$$V(t, f) = D(t, t_{set})G(\tau^*, f - K), \qquad (a51)$$

with

$$\sqrt{\tau^*} = A(f_{av})\sqrt{\tau}\left[1 + \frac{\gamma_2 - 2\gamma_1^2}{24}(f - K)^2 + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{av})\tau + \cdots\right] \qquad (a52)$$

Now we repeat the preceding analysis for the special case, Black's model, where $\alpha(t) = \sigma_B$, A(F) = F, i.e.

$$dF = \sigma_B F(t)dW \qquad (a53)$$

Our analysis indicates that the option price is

$$V(t, f) = D(t, t_{set})G(\tau_B, f - K), \qquad (a54)$$

with

$$\sqrt{\tau_B} = \sigma_B f_{av}\sqrt{t_{ex} - t}\left[1 - \frac{(f - K)^2}{12f_{av}^2} - \frac{\sigma_B^2(t_{ex} - t)}{24} + \cdots\right] \qquad (a55)$$

Since it is easy to see that $G(\tau_B, f - K)$ is an increasing function of τ_B, the Black price matches the correct price if and only if

$$\sqrt{\tau_B} = \sqrt{\tau^*} \tag{a56}$$

Equating yields the implied volatility

$$\sigma_B = a \frac{A(f_{av})}{f_{av}} \left\{ 1 + \left(\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{av}^2} \right) \frac{(f-K)^2}{24} + \left(2\gamma_2 - \gamma_1^2 + \frac{1}{f_{av}^2} \right) \frac{a^2 A^2(f_{av})(t_{ex}-t)}{24} + \cdots \right\} \tag{a57}$$

where

$$f_{av} = \frac{1}{2}(f + K), \qquad \gamma_1 = \frac{A'(f_{av})}{A(f_{av})}, \qquad \gamma_2 = \frac{A''(f_{av})}{A(f_{av})},$$

and

$$a = \frac{1}{t_{ex}-t} \int_t^{t_{ex}} \alpha^2(s)ds . \tag{a58}$$

This gives us the implied volatility for the European call option for the model

$$dF(t) = \alpha(t)A(F)dW(t).$$

A similar analysis shows that the implied volatility for an European put option is given by the same formula.

In particular, for local volatility model

$$dF = \sigma_{loc}(t, F)FdW , \tag{a59}$$

$$F(0) = f ,$$

Substituting $\alpha(t) = 1$, $A(F) = \sigma_{loc}(t, F)F$ into our equation for the implied volatility yields

$$\sigma_B(K, f) = \sigma_{loc}\left(\frac{1}{2}[f + K]\right)\left\{ 1 + \frac{1}{24} \frac{\sigma_{loc}''\left(\frac{1}{2}[f+K]\right)}{\sigma_{loc}\left(\frac{1}{2}[f+K]\right)}(f - K)^2 + \cdots \right\}. \tag{a60}$$