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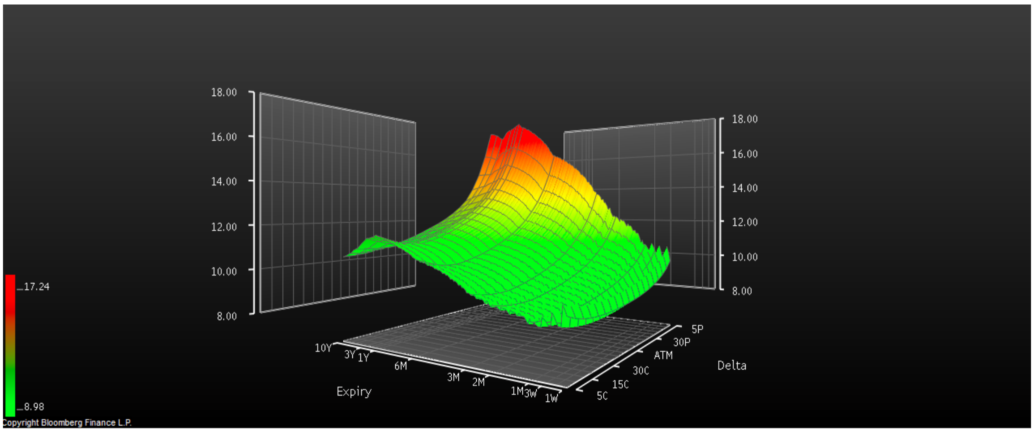
A Comparative Study of Various Versions of the SABR Model Adapted to Negative Interest Rates

IRINA SHADRINA



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Author:	Irina Shadrina
Student nr:	11088397
Email:	`ishadri@gmail.com`
Date:	July 23, 2017
Supervisor:	Prof. dr. L. J. van Gastel
Second reader:	Prof. dr. ir. M. H. Vellekoop

Abstract

In this thesis we compare different extensions of the SABR model adapted to negative interest rates. Namely, we introduce and discuss analytical properties of the Shifted SABR, the Normal SABR, the Free Boundary SABR and the Mixture SABR models.
To compare numerical outcomes of the models we compute approximation formulae for the implied Normal volatility in terms of the models parameters and compare the quality of the volatility smile approximation using data on swaptions as a benchmark.
Computational results show that the approximation quality of the Displaced SABR model outperforms the other models. Moreover, in some special cases of the market data the approximation quality is only sufficient for the Displaced SABR model. We also conclude that the Hagan approach can not be used for calibration of the parameters of the Free Boundary SABR and Mixture SABR models.
Finally, we conclude, using our quality estimation criteria, that the Displaced SABR model is the most suitable model for modelling the volatility smile in the presence of negative rates.

Keywords negative interest rates, Black's model, Bachelier's model, SABR model, Displaced SABR model, Normal SABR model, Free Boundary SABR model, Mixture SABR model, swaptions

Contents

Preface

Statement of Originality: This document is written by Irina Shadrina who declares to take full responsibility for the contents of this document. I declare that the text and the work presented in this document is original and that no sources other than those mentioned in the text and its references have been used in creating it. The Faculty of Economics and Business is responsible solely for supervision of completion of the work, not for the contents.
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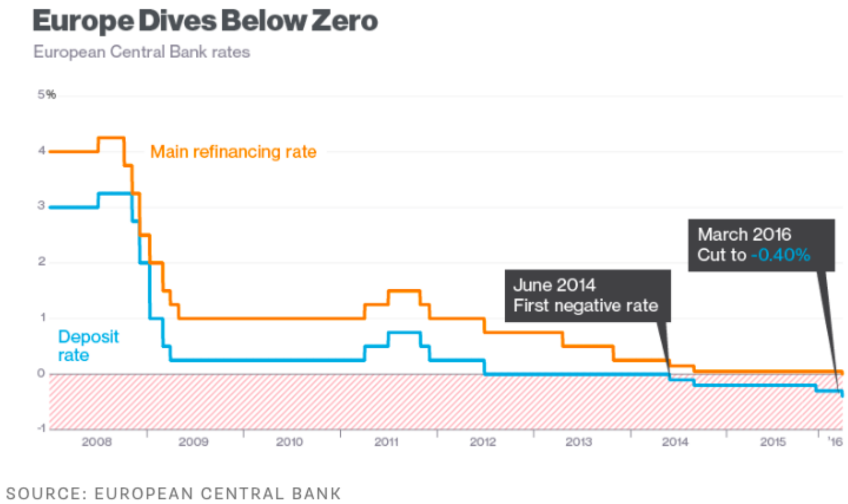
Chapter 1

Introduction

1.1 Economic background

In July 2012 The Central Bank of Denmark reduced the interest rate to -0.02%. Although highly unusual and innovative, negative interest rates did not damage the financial system in any essential way. Moreover, from the Denmark experience it was suggested that "negative rates may deserve to move from taboo to the standard monetary policy toolbox" [1]. In 2014, the European Central Bank introduced negative interest rates. Other countries that were not a part of the Eurozone, such as Japan, Sweden and Switzerland, followed as well.

We provide here an illustration from the European Central Bank.



The reasons for introduction negative interest rates

The main reason for introduction negative interest rates was the slow economy with the low inflation and the low level of investments. This needed to be speeded up, and was the basic reason to introduce negative interest rates.

In simple words, the idea was to motivate people to spend and invest, rather than save. In the situation of negative interest rates banks are encouraged to lend more and penalized for keeping cash. It stimulates investors to borrow money and invest into the economy and customers to spend more money. It increases the inflation and the economic growth rates. Banks and investors are encouraged not to invest in domestic currency. This will depreciate the currency value in the foreign markets and stimulates export.

[1] https://www.bloomberg.com/news/articles/2016-06-06/denmark-land-below-zero-where-negative-interest-rates-are-normal

Possible negative consequences

We would like to discuss possible negative implications if the negative interest rates will stay on the market for a long time.

One of the first concerns that comes to mind is whether the low rates will cause the bubbles on the housing market.

Also, banks are encouraged to move from safe assets, like government bonds, to more risky ones. This raises the financial stability risk and the risk of bank runs. Moreover, a possible result of that would by that the rules of banking capital requirements will imply the higher obligatory reserves.

Banks are not sure about the reaction of the depositors on the introduction of the negative interest rates for their deposits and are afraid to lose customers. This can make the banks less profitable. Lower profitability means lower earnings, and that will make it harder for banks to build up capital during the next credit cycle. A notable example is described in the article "In the era of negative interest rates, Japans second-largest bank finds itself having to run just to stand still." [1]

Negative interest rates put pressure profitability of insurers and pension companies. Negative interest rates reduce the discount on the cash flows from assets. The present value of liabilities of the insurance companies and pension funds is getting bigger, and so do the computed reserves. Moreover, to keep reserves on a bank account, companies would have to pay.

If negative interest rates stay for a substantially long period, financial institutions will be forced to change their behaviour. This will cause additional risks and unknown effects. Typical market behaviour could be changed as well.

If more countries will introduce negative interest rates, this will also diminish expected positive effects for the economy of the countries that did this first. For instance, the currency depreciation would probably be turned back.

Expected timeframe for the existence of the negative interest rates

It is difficult to predict how long these low interest rates will stay, but it seems possible that they will be low for quite some time. This is certainly the view of financial markets, where the return on government bonds is negative for a range of countries, even at long maturities. [2] For instance, most private-sector forecasters don not expect Denmarks central bank to return to the positive interest rates again before 2018, at the earliest. [3]

End-customer perspective on negative interest rates

It was always believed that the interest rates could not go below zero. Indeed, people can always withdraw cash and keep it outside the banking system (say, at home), earning the zero interest rate.

Nowadays, we realize that the customers are ready to pay for their bank account: it is convenient to use and can be cheaper, than keep the financial means in cash at home. The extra costs occur since cash needs to be stored, secured and insured. Thus, it is natural to suppose that the negative interest rates are limited from below, since a person or a company is able to choose to keep the money in cash, if it will be more profitable for them, taking into account all extra costs and risks mentioned above.

However, there are many reasons to pay for storing money on the bank account and so the idea of negative interest rate is something we have to get used to.

[1] https://www.bloomberg.com/news/articles/2017-05-31/japan-s-second-largest-bank-has-to-run-just-to-stand-still

[2] https://www.ecb.europa.eu/press/key/date/2016/html/sp160728.en.html

[3] https://www.bloomberg.com/news/articles/2016-06-06/denmark-land-below-zero-where-negative-interest-rates-are-normal

1.2 Problem statement

When most of the popular risk or pricing models in finance were developed, the general assumption that the interest rates can not go below zero was one of the basic required properties. Indeed, this reflected the market situation of that time and the expectations for the future interest rates behaviour.

For instance, the lognormal distribution was often used in stochastic modelling as a good reflection of the dynamics of the interest rates, financial instrument (like stocks) and some interest rate derivatives. The advantages of the lognormal distribution included that it was asymmetric (skewed to the right) and could not go below zero.

This kind of models is not able to reflect the negative interest rates. They are not defined in this case or give wrong results and predict the wrong dynamics. In order to cope with negative interest rates these models should be adapted or new models should be considered.

For instance, the Black-76 model, typically used to price options and compute the market implied volatilities, assumes that the underlying process follows lognormal distribution and assigns zero probability to non-positive interest rates.

The SABR model, developed by Hagan et. al. Hagan et al. [2002], is popular in financial industry because of the availability of an analytic asymptotic implied volatility formula. Practical applications of the SABR model include interpolation of volatility surfaces and the hedging of volatility risk. In its original formulation the model is not suitable for applications in the case of negative interest rates. In order to adapt the SABR model to the situation of negative interest rates, the following models were proposed in the literature (see Antonov et al. [2015a], Antonov et al. [2015b]): the Displaced SABR, the Free Boundary SABR, the Normal SABR, and the Mixture SABR.

The aim of this thesis is to compare these models in terms of the quality of the volatility smile interpolation. To our best knowledge there are no recent works where these models are calibrated to the nowadays market data and compared in this context.

The research question is the following.

Which extension of the SABR model is the most suitable for modelling the volatility smile in the presence of negative interest rates?

In order to answer this question we formulate a number of subquestions to set up comparison criteria. They are arranged below in two groups, one deals with the quality of the computational output, and the other one deals with the evaluation of the analytical properties of the models.

The quality of the computational output of the models

One of the most important criteria for evaluation of the quality of modelling the volatility smile is whether it fits the market data. We want to choose the model that provides the efficient interpolation of the volatility smile. Thus, our first subquestion is:

(1) Does the model has sufficient fit to the market data? In which circumstances / states of the market?

To model the volatility smile we implement the following procedure. First, we calibrate the model parameters using the market data. Second, we model the volatility smile using the estimated parameters.

There are two possible ways for the parameter calibration.

The first one is to use the approximation formula for the implied Normal volatility introduced in Hagan et al. [2002]. In this case we use the market data of the implied Normal volatility to calibrate the model parameters. Then we model the volatility smile with the formula mentioned above. In this thesis we refer to this method as Hagan's approach.

The second way is to use an analytical formula for option prices derived for the model. In this case we calibrate the model parameters using the market data of option prices (or compute the option prices from the market data of the implied volatility). Then we compute option prices using the analytical formula mentioned above. After that we model the volatility smile inserting the normal volatility values from option prices. In this thesis we refer to this method as Antonov's approach (we call it like that since it the only way to use their analytical formulas).

Since there is a one-to-one correspondence between the implied volatility and the option price, it is important that the computed option prices are consistent. Among other properties, they should not permit arbitrage. Thus, our second subquestion is:

(2) Does the model have arbitrage in option prices?

For the practical implementation of the model, the consistency of parameters is sufficient. We suppose that someone calibrates the parameters of the model to the current market data daily (or it could be the case that someone recalibrates just one parameter, keeping the rest fixed). In this case, we want to avoid jumps in the recalibrated model parameters. The reason to avoid jumps is that they can cause the corresponding jumps in the dependant risk metrics and/or option prices computed using the model. Thus, our third subquestion is:

(3) Does the model have consistent and stable parameters?

The other issue that we find important for practical implementation of the model is the complexity of its implementation. The model which permits an easy implementation and recalibration of parameters guarantees transparency and comparability of the results. We want that the volatility smile modelled by the model will not substantially depend on the algorithm we choose for implementation.

Therefore, our fourth subquestion is:

(4) Is the calibration of parameters stable and simple enough? Is the model easy to implement in a computer code? Are the computational costs relatively small?

Evaluation of the analytical properties of the models

The models that we use for the volatility smile modelling assume different behaviour of the underlying forward. It is important that the behaviour of the forward reflects the behaviour we observe on the market. This helps us to avoid the misspecification of the model. Unquestionably, in our analysis we have to take into account the change of measure, since we model the forward rate behaviour under the risk-neutral measure and the observed market behaviour corresponds to the real world measure. Still, there are properties conserved by the change of measure.

Our fifth subquestion is:

(5) Do the properties of the forward under the model reflect the expected market behaviour?

We find important to pay attention in our analysis to the following aspects. The model has to be not over-parametrized. All other things being equal, the model with less parameters should be preferred. The model has to be not too complex. Otherwise it can have difficulty with the analytical tractability.

Our next subquestion is:

(6) Which model is less parametrized? Which model is the least complex one?

One of the applications of the SABR model is the hedge of the volatility risk. It is important to pay attention to the possibility to implement the hedge in models we compare.

Our last subquestion is:

(7) Which model is the most consistent in terms of the hedge purposes? Is the hedge process for the model described in the literature?

1.3 Thesis outline

The thesis is organized as follows. In Chapter 2 we review some definitions related to the fixed income security market. In Chapter 3 we introduce the mathematical framework and notations. In Chapter 4 we introduce the concept of the implied volatility. First, we consider the fundamental models: the Black and the Bachelier ones. Further, we give a definition of the implied volatility and discuss its properties. The Chapter 5 is devoted to the SABR model. In Chapter 6 we introduce extensions of the SABR model adapted to negative interest rates: the Displaced SABR, the Free Boundary SABR, the Normal SABR, and the Mixture SABR. Chapter 7 contains computational results for all models. We investigate which extension of the SABR model is the most suitable for the modelling of the volatility smile. Chapter 8 is devoted to the conclusion.

Chapter 2

Interest rate derivatives

In this Chapter we introduce some definitions related to the fixed income security market. Our goal is to give a short recollection of the main concepts that we use throughout the thesis. For a detailed explanation we refer the reader to Veronesi [2010] and Hull [2011], which we follow below closely.

2.1 Market of interest rate derivatives

A derivative can be defined as a financial instrument whose value depends on the value of other, in general more basic, variables, usually assets [Hull, 2011, p.1].

In the case of an *interest rate derivative* [Hull, 2011, p.648] the underlying basic variable is an interest rate or a set of different interest rates. Here are some examples of interest rate derivatives: interest rate swaps, forward rate agreements, swaptions, interest rate caps and floors. Interest rate derivatives are commonly used to hedge against interest rate movements, or in order to reduce or increase the interest rate exposure.

The derivatives market is the financial market for derivatives. Interest rate derivatives are traded in Over-The-Counter (OTC) and exchange-traded markets. *Over-the-counter (OTC)* trading is done directly between two parties, as opposite to the *exchange* trading, which occurs via exchanges. Products traded on the exchange market must be well standardized. In the meanwhile, the OTC market does not have this limitation. The OTC derivatives market is significant in some asset classes, such as: interest rates, foreign exchange, stocks, and commodities. From our discussion above it is clear that the OTC derivatives are important for hedging purposes (to hedge against interest rate risk, for instance). They can also be used to increase or to refine holder's risk profile, for portfolio immunization or for speculation.

The interest rate derivatives play an important role in the current economy. The interest rates derivatives market is the largest in the world. The Bank for International Settlements estimates the notional amount of the OTC derivatives contracts outstanding in December 2016 by \$483 trillions BIS Monetary and Economic Department [2017]. Their gross market value (the cost of replacing all outstanding contracts at current market prices) \$15 trillion.Notional amount of the OTC interest rate derivatives was estimated by \$368 trillions at end of December 2016, op. cit. Interest rate swaps are the single largest segment in the OTC derivatives market. At the end of December 2016, they accounted for 57% of the notional amount of all outstanding OTC derivatives and 59% of the total gross market value, op. cit.

2.2 Interest rates

A *discount factor* [Veronesi, 2010, p.30] is the factor by which a future cash flow must be multiplied in order to obtain the present value. We determine the discount factor between two dates t and T as $Z(t,T)$.

Interest rates [Veronesi, 2010, p.32] are closely related to discount factors, but are more similar to the concept of return on an investment. The interest rate is the percentage on the amount of money that a borrower promises to pay the lender.

The compounding frequency of the interest accruals refers to the number of times within a year in which the interests are paid on the invested capital. For a given payoff, a higher compounding frequency results in a lower interest rate:

$$\text{Payoff} = P(1 + \frac{r_n(t,T)}{n})^n,$$

where P is the value of the principal, and $r_n(t,T)$ is the annualized n-times compounded interest rate, defined at time t for the future period T.

The continuously compounded interest rate is obtained by increasing the compounding frequency n to infinity.

$$\lim_{n\to\infty}(1 + \frac{r_n(t,T)}{n})^{n(T-t)} = e^{R(t,T)(T-t)}$$

The continuously compounded interest rate $R(t,T)$ is related with the n-times compounded interest rate $r_n(t,T)$ by the following formula:

$$R(t,T) = n\log(1 + \frac{r_n(t,T)}{n}).$$

People usually think of continuous compounding as daily compounding, since there is no substantial difference between the daily compounded interest and the rate obtained with higher frequency.

Let us recall some standard interest rates available on the market. The *London Interbank Offered Rate (LIBOR)* [Hull, 2011, p.76] is produced daily by the Association of British Banks, and it reflect the average of the interest rates estimated by the leading banks in London. It is an index that measures the cost of funds to large global banks operating in London financial markets. The *Euro Interbank Offered Rate (EURIBOR)* is a daily reference rate, published by the European Money Markets Institute, based on the averaged interest rates at which the banks in the Eurozone offer to lend unsecured funds to other banks.

The *forward rate* [Hull, 2011, p.84] is the rate of interest for the future period of time, based on the today's interest rates. To compute it from the given yield curve we can use the no-arbitrage argument. It is applied in the following way. Suppose we have two historical continuously compounded interest rates $r_1 = r(t,T_1)$ and $r_2 = r(t,T_2)$, defined at the moment t, where $t < T_1 < T_2$ are the parameters for the starting and ending moments for the action of these interest rates. To compute the value of the continuously compounded forward rate $f := f(t,T_1,T_2)$ we use the equation

$$e^{-r_2\cdot(T_2-t)} = e^{-r_1\cdot(T_1-t)} \cdot e^{-f\cdot(T_2-T_1)}.$$

Thus, for $t=0$ we obtain

$$f(0,T_1,T_2) = \frac{r_2T_2 - r_1T_1}{T_2 - T_1}.$$

2.3 Interest rate derivatives

The *Forward Rate Agreement (FRA)* [Veronesi, 2010, p.162] is a contract in which one party pays the forward rate $f(0,T_1,T_2)$ during a given future period from T_1 to T_2. The other pays future market floating rate $r_n(T_1,T_2)$.

A *forward contract* [Veronesi, 2010, p.168] is a contract in which one party agrees to buy and the other agrees to sell a given security at a given future time and at a given price, called the forward price.

A *plain vanilla fixed-for-floating interest rate swap contract* [Veronesi, 2010, p.172] is an agreement in which:

- One party makes n fixed payments per year at an (annualized) rate s (the swap rate) on a notional N up to date T (tenor);
- The other party makes payments linked to a floating rate index $r_n(t)$ (typically LIBOR).

Denote by $T_1, T_2, \ldots, T_M$ the payment days, $\Delta = T_i - T_{i-1}$ with $\Delta = \frac{1}{n}$. The net payment at each of these days is computed in the following way:

$$\text{Net payment at } T_i = N \times \Delta \times [r_n(0, T_{i-1}) - s].$$

Cash flows correspond to long position in floating rate bond, and short position in coupon bond. No-arbitrage swap rate could be computed by the formula:

$$s = n \times \frac{1 - Z(0,T)}{\sum_{j=1}^{M} Z(0,T_i)},$$

here T is the tenor of swap (duration of swap), T_i - payments dates, n - number of payments per year. A fixed-for-floating swap does not cost anything to enter, but can change the duration of a portfolio.

The *swap curve* [Veronesi, 2010, p.177] at time t is the set of swap rates for all maturities.

Futures and forward contracts [Veronesi, 2010, p.25] are contracts where two parties agree to exchange a security (or cash, or a commodity) at a fixed time in the future for a price agreed today. Futures are traded on regulated exchanges. Forwards are traded on the over-the-counter (OTC) market. A *forward swap contract* [Veronesi, 2010, p.179] means that two parties agree to enter into a fixed swap contract at a fixed future date and for a fixed forward swap rate.

A *call or put option* [Veronesi, 2010, p.25] is a contract which gives the holder the right, but not the obligation, to buy or sell an underlying asset at a fixed strike price on a fixed date (maturity). Intuitively, an option is the financial equivalent of an insurance contract.

A *swaption* [Veronesi, 2010, p.211] is the option on a swap rate. A swaption is *At-the-money* (ATM) if the forward swap rate is (approximately) equal to the strike.

An interest rate *cap* [Veronesi, 2010, p.665] is a contract that pays the difference between LIBOR and a fixed rate, if this difference is positive, otherwise it pays zero. It looks like series of call options on the LIBOR rate. The cap insures a borrower with a floating-rate loan against increases in the LIBOR rate. A *floor* is exactly the reverse: a series of put options on the LIBOR rate.

Chapter 3

Mathematical framework

In this Chapter we introduce some definitions related to the stochastic calculus. The basic reference for this Chapter is Etheridge [2002].

3.1 Assumptions

Before we introduce the main definitions and theorems important for our further work, we would like to mention the assumptions under which this theory was build up.

- The market is complete.
- No riskless arbitrage opportunity exists.
- Risk-free interest rate exists and is non-negative.
- It is possible to borrow and lend money at the same risk free interest rate.
- It is possible to sell (or buy) any quantity of any asset at any time.
- All securities are infinitely divisible.
- Short selling is allowed.
- There are no transaction costs.
- Underlying asset pays no dividends.
- You can buy and sell at the same time.
- Trading is continuous in time.

3.2 Ito calculus

The *arbitrage* [Etheridge, 2002, p.5] is the opportunity to earn a risk-free profit. In simple words, there exists a trading strategy as follows: either we receive a positive amount today and a non-negative tomorrow (with probability one) or we pay a zero amount today and receive non-negative amount tomorrow (with a positive probability for a positive profit).

A *complete market* [Etheridge, 2002, p.16] is the market where every contingent claim can be replicated.

Statement (Law of One Price). *[Veronesi, 2010, p.9] In the absence of arbitrage two assets with the same payoff must have the same market price.*

The *risk free rate* is the rate of interest that can be earned in the absence of any risks (for instance, no risk of default, no reinvestment risk). The risk-free rate plays an important role in the mathematical theory of asset pricing. In the past, the LIBOR rates were used as the risk-free rates. However, under stressed market conditions LIBOR turned out to be a weak proxy for the risk free rate. Therefore, nowadays the market practice is to discount with the overnight index swap (OIS) rates Hull and White [2013]. In the negative interest rate environment the suitable proxy for the risk-free rate is still a subject for discussion. We can use the negative value of OIS, set the risk-free interest rate equal to zero (since we can keep cash and earn approximately the zero risk free rate), take the average of the appropriate interest rates from the past, or choose another currency where the interest rates are positive (USD, for instance). As we can see, if we assume the negative value for the risk-free interest rate, we can create the arbitrage opportunity just by keeping money in cash, which is not consistent with our theory. Thus the fact that negative risk free rates are possible could only be explained with properties not included in our model, such as the cost of saving cash or the fact that there is no risk-free asset in the real market.

We consider the *probability space* Haugh [2010] $(\Omega, \mathcal{F}, P)$. Here P is the real world probability measure, Ω is the universe of possible outcomes, the σ-algebra $\mathcal{F}$ represents the set of possible events where an event is a subset of Ω. A *filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ reflects the evolution of the information throughout time. For example, if at the moment t we already know whether the event A happened, we assume that $A \in \{\mathcal{F}_t\}$.

The *Brownian motion* Haugh [2010] W_t, $t \geq 0$ is a stochastic process, such that

- $W_0 = 0$,
- W_t is almost surely continuous,
- W_t has independent increments,
- $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s \leq t$.

Note that W_t is almost surely nowhere differentiable.

The filtration $\{\mathcal{F}_t\}$ considered in this thesis is generated by the Brownian motion that is specified in the model description.

A stochastic process $X_t, t \geq 0$ is called *martingale* [Etheridge, 2002, p.33] with respect to the filtration $\mathcal{F}_t$ and probability measure P if

$$E^P[|X_t|] < \infty \text{ for all } t \geq 0 \text{ and } E^P[X_{t+s}|\mathcal{F}_t] = X_t \text{ for all } t, s \geq 0$$

A *stochastic differential equation* [Etheridge, 2002, p.91] is an equation of the following form:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

Here $\mu(t, X_t)$ is predictable and (Lebesgue) integrable function, and $\sigma(t, X_t)$ is Ito-integrable function. They satisfy $\int_0^t (\sigma_s^2 + |\mu_s^2|)ds < \infty$.

An *Ito-process* Haugh [2010] is a solution of a stochastic differential equation. It can be expressed as the sum of an integral with respect to the Brownian motion and an integral with respect to time:

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s.$$

We proof the property of the sum of two uncorrelated Brownian motions with the zero-drift. This property we will use in the Chapter 9 of this thesis.

Lemma 1 (Ito's Lemma). *Haugh [2010] Suppose $f(t,x)$ is a twice continuously differentiable function on $\mathbb{R}$. Consider the Ito process $dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t$. Then the process $Z_t := f(t,X_t)$ satisfies the following stochastic equation:*

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)(dX_t)^2 \\ &= (\frac{\partial f}{\partial t}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\mu(t,X_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_t)\sigma^2(t,X_t))dt + \frac{\partial f}{\partial x}(t,X_t)\sigma(t,X_t)dW_t. \end{aligned}$$

Theorem 1 (Fokker-Planck (the forward Kolmogorov) equation). *[Gardiner, 2004, p.138] The probability density function $p(t,x)$ of a stochastic Ito-process $dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t$ satisfies the following equation:*

$$\frac{\partial}{\partial t}p(t,x) = -\frac{\partial}{\partial x}[\mu(t,x)p(t,x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(t,x)p(t,x)].$$

Theorem 2 (Martingale representation theorem). *[Etheridge, 2002, p.100] Let W_t be the standard Brownian motion with a filtration $\mathcal{F}_t$. If M_t is a martingale, adapted to $\mathcal{F}_t$ and $E[M_t^2] < \infty$, then there exists a unique $\mathcal{F}_t$-predictable stochastic process C_t such that*

$$dM_t = C_t dW_t.$$

The coefficient C_t may be deterministic or random, and may depend on any information that is available at time t.

3.3 Pricing under the risk neutral measure

A *Risk-neutral probability measure* (or an equivalent martingale measure) is a probability measure Q such that the discounted price of the derivative process is martingale.

As we can see, under the risk-neutral measure the derivative earns risk-free interest rate. Under this measure the derivative price (with no intermediate payments) can be computed as the discounted value of its expected payoff at the payment day.

Theorem 3. *Haugh [2005] The market is free of arbitrage and complete if and only if there exists a risk-neutral probability measure Q equivalent to the real world measure P such that the discounted price process S_t/B_t is a martingale. Here we denote by S_t the price process of a risky asset, and by B_t the price process of a risk-free asset.*

We can price financial derivatives under the risk-neutral measure using the following formula:

$$V(0,T) = e^{-rT} \times (\text{expected payoff at T, computed under } Q).$$

Risk-neutral density-based tests

Here we present a no-arbitrage test and some tests for the existence of the risk-neutral density, discussed in the literature Breeden and Litzenberger [1978]. These tests are used to check whether the model is consistent.

- Put-Call parity must be satisfied.
- The implied risk-neutral probability density function is integrated to one.
- The implied risk-neutral probability density function is positive.
- The risk-neutral density function produces call prices decreasing monotonically with respect to the strike.

The *risk-neutral market implied probability density function* can be computed as the second order derivative of the call option price with respect to the strike Breeden and Litzenberger [1978]. The value at a price x of the discounted risk neutral density $f_{S_T}(x)$ is the second partial derivative of the call price with respect to the strike, evaluated at x:

$$e^{-rT} f_{S_T}(x) = \frac{\partial^2}{\partial K^2} C(K,T)|_{K=x}.$$

Chapter 4

Implied volatility

In this Chapter we introduce the conception of implied volatility. First, the fundamental models: the Black and the Bachelier models are examined. These models assume that the underlying forward rate follows the lognormal and the normal process, respectively, with a constant volatility and a constant risk-free rate. Further, we give the definition of the implied volatility and discuss their properties.

4.1 Black's model

The market standard for quoting option prices is Black's model, developed by Fischer Black in 1976. This variant of the BlackScholes option pricing model is usually used to price options on interest rate caps and floors or swaptions. The Black's model assumes that a forward rate F_t (for instance the LIBOR forward or a forward swap rate) follows a driftless lognormal process.

Namely, in Black's model the forward rate is modelled under the risk-neutral measure by the following dynamics:

$$dF_t = \sigma_B F_t dW_t,$$

$$F(0) = F_0.$$

The parameter σ_B is called the Black volatility. This stochastic equation has the following solution: $F = F_0 \times exp(-\frac{1}{2}\sigma^2 t + \sigma W_t)$. As we can see from this solution, the values of the forward rate F are always positive for $F_0 > 0$.

The price of the call option with the maturity T and the strike K under this model is given by:

$$V_{call}^B = e^{-rT}[F_0 N(d_1) - KN(d_2)],$$

where

$$d_1 = \frac{\log(F_0/K) + (\sigma_B^2/2)T}{\sigma\sqrt{T}},$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

$N(\cdot)$ is the cumulative normal distribution and r is the risk-free rate.

4.2 Bachelier's model

European option prices are often quoted by using the Normal (Bachelier) model. In this model the forward asset price F_t follows the following dynamics:

$$dF_t = \sigma_N dW_t,$$

where $F(0) = F_0$. The parameter σ_N is called the normal volatility. This stochastic equation has the following solution: $F(t) = F_0 + \sigma_N W(t)$. The price of the option under this model is:

$$V_{call}^N = e^{-rT}[(F_0 - K)N(d_1) + \frac{\sigma_N\sqrt{T}}{\sqrt{2\pi}}e^{-d_1^2/2}],$$

where $d_1 = \frac{F_0 - K}{\sigma_N\sqrt{T}}$.

Note that the formula for options prices contains only the difference between the strike and the initial forward value. Thus, if we shift a both on the same shift parameter s we obtain the same option price under this model (the price does not depends on the shift value).

4.3 Implied volatility

To find the implied volatility we solve the equation for σ, using the market data for the option price:

$$Price(r, T, \sigma, F_0) = \text{Market price}.$$

The option value under the Black or Bachelier model is the increasing function of σ. Thus we can find the unique solution of the equation:

$$\sigma_{impl} = f(r, T, F_0, \text{Market price}),$$

where σ_{imp} is the *implied volatility.*

There is no analytical solution for this equation but we can use numerical methods to find the approximate value of σ_{imp}.

European option prices are often quoted by using the Normal (Bachelier) or the Black model. The values of the implied volatility are given as annualized percentages:

$$\text{volatility} = 100 \times \sigma_{imp} \times \sqrt{250}\%$$

4.4 Comparison of Normal and Black volatilities

Implied lognormal volatility is measured by the relative changes of the forward rate. On the contrary, the Normal model measures the implied normal volatility in absolute changes of the forward rate. Thus it makes more sense to compare implied normal volatilities with historical moves of the underlying forward rate.

The Black model does not permit the process to go below zero, thus the implied Black volatility is determined only for positive values of the forward rate. It is possible to compute the shifted lognormal implied volatility in the case of negative or zero forwards. In this case, the value of the shifted lognormal volatility depends on the value of the shift parameter. Under the Normal model there is non-zero probability for forward rate to be negative. One of the disadvantages of the Normal model is that extreme negative values are possible (though with small probability).

The normal model seems to capture the rates dynamics better than the lognormal model Hohmann et al. [2015].

Since both volatilities are determined from the formulas

$$Price_{Black}(r, T, \sigma_B, F_0) = Price_{Normal}(r, T, \sigma_N, F_0) = \text{Market price},$$

there is a one-to-one relation between the implied Black and the implied Normal volatilities. Usually, we can approximately express one through the other. For instance, in the case of swaption pricing we have the following relationship between the volatilities for the ATM swaptions Hohmann et al. [2015]:

$$\sigma_B \approx \frac{\sigma_N}{dF}.$$

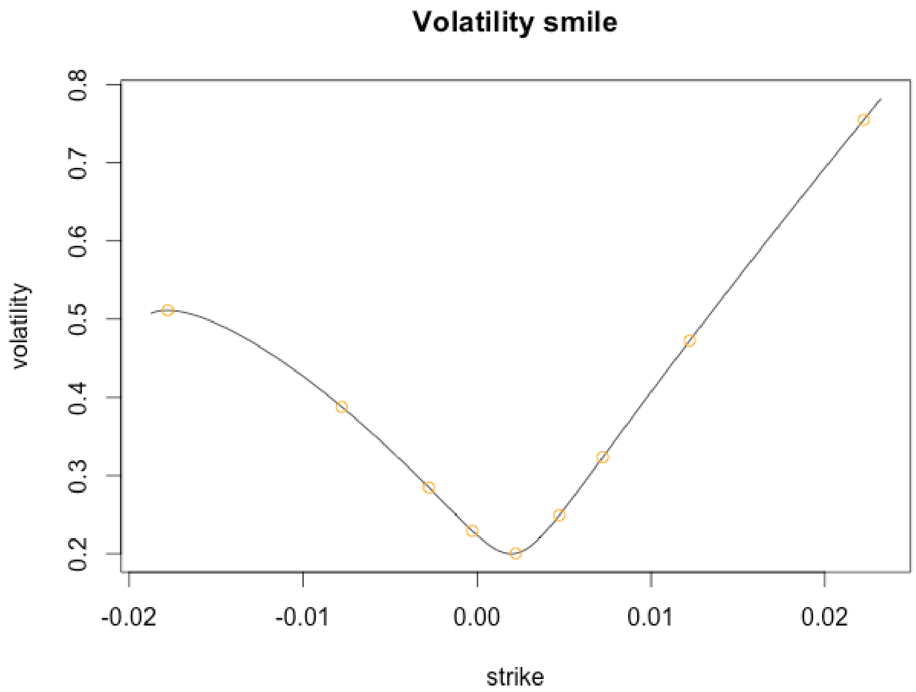
This approximation is less accurate in the case of low interest rates, when F is close to zero.

In the case of risk-neutral modelling the Normal volatility has advantages compared to the Black volatility Hohmann et al. [2015].

4.5 Volatility smile

Both Black and Bachelier models are not able to correctly price options with different strikes K and times-to-expiry T with the constant value of volatility. In practice, the value of the implied volatility depends on the maturity and on the strike of an option. If we make a graph of the estimated implied volatilities against strikes, we see that the graph usually looks like a smile. This is the reason why this graph is called the Volatility smile in the literature.

The value of the implied volatility approximately achieves its minimum for the At-The-Money strike (with the strike approximately equal to the current forward value) and is higher for lower and higher strikes. When the forward value increases, the smile should moves to the right, and when the forward value decreases, to the left.



4.6 Local volatility models

To price derivatives under the risk-neutral measure we suppose that underlying process F_t (forward rate in our case) is a martingale. It follows from the martingale representation theorem that there exits a unique stochastic process C_t such that under the risk-neutral measure

$$dF_t = C_t dW_t.$$

It is assumed that $C_t = F_t$ in the case of the Black model and $C_t = 1$ in the case of the Normal model.

Since the assumption about a constant volatility for all strikes was not confirmed by the market, more advanced model were developed. The goal of these models was to reflect the volatility smile and to interpolate it between the points, estimated from the market data.

One of extensions of the Black model is a class of models called local volatility models. Local volatility models are specified by assuming that C_t is a function of t and F_t. The stochastic process F_t follows the following stochastic equation:

$$dF_t = C(t, F(t))dW_t,$$

where $C(t, F)$ is an instantaneous volatility.

Local volatility models give correct option prices but predict wrong dynamics. Namely, when the forward $F(0) = f$ increases, the volatility curve shifts to left Hagan et al. [2002], which does not reflect the typical market behaviour. This property causes problems by hedging.

Constant Elasticity of Variance (CEV) model

The CEV model is one of the Local volatility models. The forward rate has the following dynamics:

$$dF_t = \sigma F_t^{\beta} dW_t,$$

where σ is a positive constant and $\beta \in [0, 1]$.

There is an analytical solution for the risk-neutral probability density function. Analytic pricing formulas for the CEV model in terms of Bessel function series have been derived for any value of β, but these formulas are computationally intensive. There are also various analytical approximations for the transition density function.

The transition probability density function can be found as a solution of the backward Kolmogorov equation with absorbing (for all values of β) or reflecting (for $\beta \in [\frac{1}{2}, 1)$) boundary conditions. The CEV process with the absorbing condition is a martingale Antonov and Spector [2012].

Chapter 5

SABR model

In this Chapter we introduce the SABR (stochastic alpha, beta, rho) model, developed by Hagan et. al. and introduced in Hagan et al. [2002]. The SABR model is widely used in the financial industry, especially in the interest rate derivative markets. The most attractive properties of the model is that it captures the volatility smile and reflects the correct dynamics of the volatility smile. Moreover, there is a closed-form approximation formula for the Black and Normal implied volatilities, as a function of the SABR model parameters.

5.1 Model definition

By comparison with the CEV model, the SABR model adds a new stochastic factor to the dynamics of the forward rate. Namely, it assumes that the volatility parameter itself follows a lognormal stochastic process.

The SABR model is specified under the risk-neutral probability measure by the following system of stochastic differential equations:

$$\begin{aligned}
&dF_t = \alpha_t C(F_t) dW_t, \\
&d\alpha_t = \nu \alpha_t dZ_t, \\
&dW_t dZ_t = \rho dt, \\
&F(0) = f, \ \alpha(0) = \alpha \text{ are the initial conditions.}
\end{aligned}$$

Here

- $C(x)$ is the local volatility function, defined for $x > 0$ and assumed to be positive, smooth, and integrable around zero Hagan et al. [2015, first draft in 2001],

$$\int_0^K \frac{du}{C(u)} < \infty;$$

- $F(0) = f > 0$ is the currently observed forward rate;
- $\nu > 0$ is the volatility-of-volatility (volvol) of the forward rate;
- $\alpha(0) = \alpha > 0$ is the initial value for α_t;
- dW_t and dZ_t are two $\rho-$correlated Brownian motions;
- $\rho \in (-1, 1)$.

The processes α_t and F_t are correlated. If F_t changes, then α_t changes as well. This change is proportional to the correlation coefficient ρ between the Brownian motions.

Two Brownian motions dW_t and dZ_t are correlated. We can express the second process dZ_t from the first one using the Choleski decomposition: $dZ_t = \rho dW_t + \sqrt{(1-\rho^2)} dW_t^{(1)}$, where $dW_t^{(1)}$ and dW_t are two uncorrelated Brownian motions. Therefore,

$$\begin{aligned} F_t &= F_0 + \alpha_0 \int_0^t C(F_s) dW_s + \nu\rho \int_0^t \alpha_k \int_0^s C(F_s) dW_s dW_k \\ &+ \nu\sqrt{1-\rho^2} \int_0^t C(F_s) \int_0^s \alpha_k dW_k^{(1)} dW_s. \end{aligned}$$

In order to determine the model we have to estimate 4 parameters (f, ν, α, ρ) and choose the function $C(F)$. For the standard SABR model we set $C(F) = F^\beta$, where $\beta \in [0, 1)$. In this case the model is defined by 5 parameters $(\alpha, \beta, \rho, \nu, f)$.

5.2 Risk-neutral probability density function

To apply the risk-neutral pricing theory described in the Chapter 3 we suppose that there exists a risk-neutral measure, under which the stochastic process F_t is a martingale. Our goal is to find the risk-neutral probability density function corresponding to the stochastic process F_t.

Let us define the risk-neutral probability density function implied by the two-dimensional process F_t as in Hagan et al. [2002]:

$$\begin{aligned} & p(t, f, \alpha, T, F, A) dF dA \\ & = P(F' < F(T) < F' + dF, A < \alpha(T) < A + dA) | F(t) = f, \alpha(t) = \alpha). \end{aligned}$$

Here we suppose that the economy of the market is in state $F(t) = f$, $\alpha(t) = \alpha$.

If we find the probability density function, then the value of the European call option at the maturity T is the expected payoff of the option:

$$\begin{aligned} & V(T, T) = (\text{expected payoff op the option at T}) \\ & = E_Q[|F(T) - K|^+ | F(t) = f, \alpha(t) = \alpha] \\ & = \int_{-\infty}^{\infty} \int_K^{\infty} (F - K) p(t, f, \alpha, T, F, A) dF dA \end{aligned}$$

The function $p(\cdot)$ satisfies the two-dimensional Fokker-Planck (Kolmogorov forward) partial differential equation Hagan et al. [2002]:

$$p_T = \frac{1}{2} A^2 [C^2(F) p]_{FF} + \rho\nu [A^2 C(F) p]_{FA} + \frac{1}{2} \nu^2 [A^2 p]_{AA},$$

for $T > t$ with

$$p = \delta(F - f)\delta(A - \alpha) \text{ at } T = t.$$

There is no analytical solution of this equation in the general case. When the SABR model was introduced, the approximation formula for the implied Normal and Black volatilities were derived in Hagan et al. [2002]. The explicit solution in the particular case of $\beta = 0$ (Normal SABR) was given in Antonov et al. [2015b]. The explicit solution in the zero-correlated case was given in Antonov et al. [2013] together with the mapping technique to the general case.

Once we find the approximate (or explicit) solution we check whether it is satisfactory, in the following sense. We have to check, whether the process F_t defined by the density function is martingale. It is also important that no-arbitrage conditions are preserved: the put-call parity is true, the implied risk neutral density function is integrated to one, and it is positive.

5.3 Results of Hagan et. al.

The SABR model was introduced in 2002 by Hagan et. al. in Hagan et al. [2015, first draft in 2001]. In Hagan et al. [2002] the approximation formula for the implied Normal and Black volatilities in terms of the SABR parameters was computed.

Hagan et. al. find the approximated solution of the Kolmogorov forward equation mentioned above. They use the singular perturbation technique and the known solution of the Kernel Heat equation. For the simplicity they expressed the implied Normal and Black volatilities in terms of the SABR parameters.

They suppose that $\alpha^2 T << 1$, $\nu^2 T << 1$. They also suppose that the values of f and K are close to each other. The authors suggest that under the typical market conditions this values are small. Thus the approximation formula is the most efficient in the area around the ATM values with relatively small T and volatilities.

The following approximation formula for the Normal implied volatility in the SABR setting is proposed by Hagan et. al. in Hagan et al. [2014] as the most robust one:

$$\sigma_N(K) \approx \frac{\alpha(f-K)}{\int_K^f \frac{dx}{C(x)}} \cdot \frac{\zeta}{\xi(\zeta)}$$
$$\left[1 + \left[g\alpha^2 + \frac{1}{4}\rho\nu\alpha\frac{C(f)-C(K)}{f-K} + \nu^2\frac{2-3\rho^2}{24}\right]\tau_{ex}\right],$$

where τ_{ex} is the time to exercise, and

$$\zeta := \frac{\nu}{\alpha}\int_K^f \frac{dx}{C(x)},$$
$$\xi(\zeta) := \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right),$$
$$g := \frac{\log\left(\frac{\sqrt{C(f)C(K)}\int_K^f \frac{dx}{C(x)}}{f-K}\right)}{\left(\int_K^f \frac{dx}{C(x)}\right)^2}$$

The typical choice for the function $C(F)$ is $C(F) = F^\beta$. In this case we obtain the following equations:

$$\sigma_N(K) \approx \frac{\alpha(1-\beta)(f-K)}{f^{1-\beta}-K^{1-\beta}} \cdot \frac{\zeta}{\xi(\zeta)}$$
$$\left[1 + \left[g\alpha^2 + \frac{1}{4}\rho\nu\alpha\frac{f^\beta-K^\beta}{f-K} + \nu^2\frac{2-3\rho^2}{24}\right]\tau_{ex}\right],$$

where τ_{ex} is the time to exercise,

$$\zeta = \frac{\nu}{\alpha}\frac{f^{1-\beta}-K^{1-\beta}}{1-\beta},$$
$$\xi(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}-\rho+\zeta}{1-\rho}\right),$$
$$g = \frac{(1-\beta)^2}{(f^{1-\beta}-K^{1-\beta})^2}\log\left((fK)^{\frac{\beta}{2}}\frac{f^{1-\beta}-K^{1-\beta}}{(1-\beta)(f-K)}\right),$$

and

$$\sigma_{ATM}(K) \approx \alpha f^\beta(1 + \left[\frac{-\beta(2-\beta)}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right]\tau_{ex})$$

In their recent article Hagan et al. [2016] written in 2016, the authors improved the accuracy of their approach. They introduce the boundary correction formula, that can be used for the Displaced SABR, for instance. If we assume that we have a barrier such that for the forward holds $F_t \geq F_{min}$, then we can correct the option price formula by adding an extra term:

$$V_{call}(\tau_{ex}, K) = V_{call}^N(\tau_{ex}, f, K, \sigma_N) - V_{call}^N(\tau_{ex}, f, -K + 2F_{min}, \sigma_N)$$

They also provide the new method to estimate the option price values by numerically solving the effective forward equation.

5.3.1 Impact of parameters

The parameter β represents the expectation of traders about the distribution of the forward rate. Setting β close to zero we get a model close to the stochastic Normal model. If we take β close to one we a model closer to the stochastic Lognormal model. The volatility smile shifts upwards if one decreases β and downwards if one increases β. Also β influences the curvature of the volatility smile. The higher is β, the flatter is the skew.

Note that $\beta = 0$ determines the stochastic Normal model, $\beta = \frac{1}{2}$ gives the stochastic CIR model, and $\beta = 1$ corresponds to the stochastic Lognormal model.

The parameter ρ controls the rotation of the curve. If we increase the value of ρ the curve turns counter clockwise around the ATM value.

The parameter ν (volvol) is the volatility of volatility. It has impact on the smile curvature. The growth of ν increases the smile effect and vice versa.

The parameter α governs the vertical location of the smile. The increase of α leads to an upward shift of the entire smile, while the decrease results in a downward shift. As we can see, the influence of the parameters α and β partly overlaps. Thus if we fix the value of β and estimate the value of α further, the parameter α can compensate the difference.

The Backbone is the curve that is traced out from the ATM volatility when the initial forward value f changes. It depends almost entirely on the value of β. If we take $\beta = 0$ the curve move down exponentially by increasing the initial forward value. If we take $\beta = 1$ it moves right through the line.

5.3.2 Problems with Hagan et. al. formula

The original SABR approximation formula (and its improved version as well) leads to negative probability densities for low strikes. Thus, the resulting pricing is not arbitrage free. However, the exact probability density obtained by numerically solving the effective forward equation proposed in Hagan et al. [2016] is always positive.

The approximation quality declines with time. For instance, for maturities larger than 1 year the approximation error can be higher than 1% or even more for the ATM values Antonov and Spector [2012].

In 2014 Hagan et. al. proposed in Hagan et al. [2014] the no-arbitrage technique to reconstruct the no-arbitrage probability density function. They suppose that possible forward values are limited on $[F_{min}, F_{max}]$ and put absorbing boundary conditions on the bounds. The probability density function, which is guaranteed to be positive (and satisfies no-arbitrage conditions), can be found by solving the effective forward equation. As soon as we solve it numerically, we can compute the option prices and find the corresponding Normal implied volatility which is guaranteed to be arbitrage free.

5.4 Results of Antonov et. al.

Antonov et. al. in Antonov and Spector [2012] find the explicit solution for the zero-correlation case ($\rho = 0$). Using the mapping technique (similarly to Hagan et. al.), they propose the mimicking approximation for the general case.

They set the absorbing boundary condition at zero, which guarantees the process F_t to be martingale.

5.5 Parameters calibration

The general idea is that we try to find the optimal values of parameters by minimizing the differences between the market values and the values estimated by the model. The objective function is

$$\min_{\nu,\alpha,\rho,\beta} \sum (\sigma_{market} - \sigma_{Hagan})^2.$$

Here σ_{market} are the market implied normal volatilities and σ_{Hagan} are the values computed in the model.

It is possible to use in the formula above a weighted sum and put more weight for the values around ATM. Indeed, these values are the most important in the case of swaption pricing.

5.5.1 Hagan's formula

The most convenient way to calibrate the SABR model to the market data is to use the Hagan's approximation formula. The model's parameters have the following restrictions: $\rho \in [-1, 1]$, $\beta \in [-1, 1)$, $\nu \geq 0$, $\alpha > 0$.

In Hagan et al. [2002] the authors advise to fix te value of β before other parameters calibration. Usually the value of β can be chosen from prior believes of the appropriate model. For instance, $\beta = 1$ determines the stochastic lognormal model (typically chosen for FX option markets), $\beta = \frac{1}{2}$ defines the stochastic CIR model (used for interest rate markets), and $\beta = 0$ means the stochastic normal model (used in the case of zero or negative forwards) . Other way to estimate it, is to run the liner regression between natural logarithm of the observed ATM volatility and natural logarithm of the forward. The following relationship is approximately true:

$$\log \sigma_{ATM} \approx \log \alpha - (1 - \beta) \log f.$$

Thus the value of β can be estimated from the value of the slope plus one. It follows from the parameters properties (see above), that the choice of β does not substantially affect the shape of the smile curve.

Since the ATM values should be fitted perfectly the value of α could be find as the solution of the cubic equation for α. The smallest positive real root of the equation below, solved for α should be taken:

$$\sigma_{ATM}(K) \approx \alpha f^{\beta}(1 + [\frac{-\beta(2-\beta)}{24} \frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4} \frac{\rho\nu\alpha\beta}{f^{1-\beta}} + \frac{2 - 3\rho^2}{24} \nu^2] \tau_{ex})$$

The objective function in this case is:

$$\min_{\nu,\rho} \sum (\sigma_{market} - \sigma_{Hagan}(\alpha(\beta, \rho, \nu), \beta, \rho, \nu))^2.$$

The local Levenberg-Marquardt algorithm was recommended in Floc'h and Kennedy [2014] as a fast and effective local minimizer for parameters calibration. Since the local minimum is possible and different sets of parameters could lead to almost the same volatility smile, the efficient initial guess is important. In Gauthier and Rivaille [2009] and Floc'h and Kennedy [2014] are suggested methods for computing the parameters.

5.5.2 Antonov's formula

It is possible to estimate the values of parameters using the explicit formula for the zero correlation case, introduced in Antonov and Spector [2012]. In this case the differences between the options prices, computed by Antonov's formula, and the market values of option prices should be minimized. It is computationally more time consuming, includes the computation of two dimensional integral (or the use of its approximation formula) and the results can depends on the grids, used by discretezation.

5.5.3 Fit to the market

Then the SABR model was introduced in 2002 by Hagan and all. Hagan et al. [2002] it was the following market situation: interest rates were significantly positive, the volatilities were on "normal levels", quotes were in log-normal volatility or premium. In this circumstances the model gave a reasonably good fit to the market observed volatility structure. In its original formulation the SABR model is not suitable for applications in the case of negative/zero interest rates. Therefore, several modifications of this model were proposed in the literature: Displaced SABR, Normal SABR, Free Boundary SABR, Mixture SABR.

Chapter 6

SABR extensions for negative interest rates

As we mentioned in the previous chapter, the SABR model is defined for the positive interest rate environment. In order to adapt the SABR model to the situation of negative interest rates, the following models were proposed in the literature: the Displaced SABR, the Free Boundary SABR, the Normal SABR, and the Mixture SABR. In this Chapter we introduce this models and discuss their analytical properties.

6.1 Displaced SABR

It is now commonly accepted that interest rates do not need to be strictly positive. As we discussed in the Introduction part, it seems to be natural to suppose that negative interest rates are limited from below on the negative side. Let us suppose that the general qualitative properties of the forward rate distribution did not change, when the negative interest rates were introduced in the market. Moreover, suppose we are able to estimate the minimal possible negative value of the forward. In this case, the most natural solution would be if we just shift the dynamics in a way that the boundary moves from zero to the minimum possible forward determined by the market.

The Displaced SABR model is similar to the classic SABR model apart from the fact that a shift parameter s is introduced in the stochastic process of the forward rate. We introduce the Displaced model as a change of variable $F' = F + s$ for $s > 0$ in the original SABR model. As we can see $dF' = dF$. That gives us the following system of equation:

$$
\begin{aligned}
&dF_t = \alpha_t (F_t + s)^\beta dW_t,\\
&d\alpha_t = \nu \alpha_t dZ_t,\\
&dW_t dZ_t = \rho dt,\\
&F(0) = f,\ \alpha(0) = \alpha
\end{aligned}
$$

This is equivalent to the assumption the F' follows the usual SABR model, that is, that F' satisfies the following system of stochastic equations:

$$
\begin{aligned}
&dF'_t = \alpha_t {F'}_t^\beta dW_t,\\
&d\alpha_t = \nu \alpha_t dZ_t,\\
&dW_t dZ_t = \rho dt,\\
&F'(0) = f + s > 0,\ \alpha(0) = \alpha
\end{aligned}
$$

Thus we are able to use the ordinary SABR results for F'. We use the original formulas given in the previous chapter, but apply the changes $f \to f + s$ and $K \to K + s$ in our computations. The function $C(F)$ is equal to $(F + s)^\beta$ in this case. Note that the Bachelier option pricing formula depends only on the difference between the initial forward f and the strike K. So the value of the shift is not relevant in the option prices computations. However, it has influence on the values of the implied normal volatility values that we compute from the Displaced SABR model.

6.1.1 Advantages of the Displaced SABR model

This set-up of the model reflects the current beliefs about existing of the minimum negative interest rate (the boundary from below on the negative side).

It is easy to transform the ordinary SABR model to the Displaced SABR model. In the Displaced SABR model, the arbitrage problem with the implied probability density function are shifted from zero to $-s$ (we typically see the arbitrage around zero for the ordinary SABR). Thus we have no problems with the probability density function around zero.

6.1.2 Disadvantages of the Displaced SABR model

The main drawback of this model is that in the case when the interest rates go below s (the floor for the rate value), a recalibration of the model parameters is necessary and can cause jumps in greeks and option values. Also, sometimes the Displaced SABR model does not attain certain market prices Antonov et al. [2015b].

6.2 Normal SABR

In order to set up the Normal SABR model we take $\beta = 0$ or, equivalently, the function $C \equiv 1$ in the SABR model. Thus under the Normal SABR model the forward rate follows the following stochastic process:

$$\begin{aligned} dF_t &= \alpha_t dW_t, \\ d\alpha_t &= \nu \alpha_t dZ_t, \\ dW_t dZ_t &= \rho dt. \end{aligned}$$

As we can see from the equations above, in the Normal SABR model the increments of the forward rate do not depend on its current value. Typically this does not reflect the beliefs about the forward rates behaviour. Another disadvantage is that the "real" risk-neutral distribution is probably not symmetric. Indeed, it is very improbable that interest rates ever become below -10%. Hence, this approach makes sense only for short time horizons.

6.2.1 Hagan's approximation formula

As it is advised in Hagan et al. [2002], in order to obtain the approximation formula for the implied Normal volatility we can set $C(f) = 1$ and assume the free boundary. The Hagan's approximation formula then simplifies to:

$$\sigma_N(K) \approx \alpha \frac{\zeta}{\xi(\zeta)} \left[1 + \nu^2 \frac{2 - 3\rho^2}{24} \tau_{ex} \right],$$

where τ_{ex} is the time to exercise, and

$$\begin{aligned} \zeta &= \frac{\nu}{\alpha}(f - K), \\ \xi(\zeta) &= \log \left(\frac{\sqrt{1 - 2\rho\zeta + \zeta^2} - \rho + \zeta}{1 - \rho} \right). \end{aligned}$$

In this case the At-The-Money volatility is given by

$$\sigma_{ATM}(K) \approx \alpha \left(1 + \frac{2 - 3\rho^2}{24} \nu^2 \tau_{ex}\right).$$

6.2.2 Antonov's explicit solution

In Antonov et al. [2015b] the explicit for the Normal SABR with the free boundary is given.

$$\begin{aligned} V_{call}(T,K) &= E_Q[F_T - K] \\ &= |f-K|^+ + \frac{V_0}{\pi}\int_{s_0}^{\infty}\frac{G(\gamma^2 T, s)}{\sinh s}\sqrt{\sinh^2 s-(k-\rho\cosh s)^2}ds, \end{aligned}$$

where

$$\begin{aligned} \cosh s_0 &= \frac{-\rho k+\sqrt{k^2+\tilde{\rho}^2}}{\tilde{\rho}^2} \\ k &= \frac{K-f}{V_0}+\rho \\ V_0 &= \frac{\alpha}{\nu} \\ \tilde{\rho} &= \sqrt{1-\rho^2} \\ G(t,s) &= 2\sqrt{2}\frac{e^{-\frac{t}{8}}}{t\sqrt{2\pi t}}\int_s^{\infty} ue^{-\frac{u^2}{2t}}\sqrt{\cosh u \cosh s}du \end{aligned}$$

Antonov et. al. suggest that this solution guarantees the positivity of the implied probability density function and that it is integrated to one.

The function $G(t,s)$ is a one dimensional integral. In Antonov et al. [2013] an approximation for it was given by a closed formula.

$$G(t,s) = \sqrt{\frac{\sinh s}{s}}e^{-\frac{s^2}{2t}-\frac{t}{8}}(R(t,s)+\delta R(t,s)),$$

where:

$$\begin{aligned} R(t,s) &= 1+\frac{3tg(s)}{8s^2}-\frac{5t^2(-8s^2+3g(s)^2+24g(s))}{128s^4} \\ &\quad +\frac{35t^3(-40s^2+3g(s)^3+24g(s)^2+120g(s))}{1024s^6} \\ g(s) &= s\coth s - 1 \\ \delta R(t,s) &= e^{\frac{t}{8}}-\frac{3072+384t+24t^2+t^3}{3072} \end{aligned}$$

6.2.3 Practical computations with Antonov's formula

We face with the problem with the uncertainties of the type $0/0$ and ∞/∞ in the implementation of Antonov's formula in the program code. To avoid it, we do the following steps.

First, we rewrite

$$\begin{aligned} V_{call}(T,K) &= E_Q[F_T - K] \\ &= |f-K|^+ + \frac{V_0}{\pi}\int_{s_0}^{\infty} G(\gamma^2 T, s)\sqrt{1-\frac{(k-\rho\cosh s)^2}{\sinh^2 s}}ds. \end{aligned}$$

Secondly, following advise of the authors given in Antonov et al. [2013] we replace $R(t,s)$ by its fourth-order expansion for small s as a square root expression in $G(t,s)$ expression. We obtain the following expression for the *small* s case (which determines the around at-the-money area).

$$R(t,s) \approx 1 + (1/8)t + (1/128)t^2 + (1/3072)t^3 +$$
$$(-(1/120)t - (1/4032)t^2 - (1/15360)t^3)s^2 +$$
$$((1/1260)t + (1/56320)t^3 - (1/40320)t^2)s^4$$

$$G(t,s) \approx \sqrt{1 + s^2/6 + s^4/120}\, e^{-s^2/(2t)}(R(t,s) + \delta R(t,s)).$$

6.2.4 Advantages of the Normal SABR model

This model gives a relatively good approximation to the volatility smile with a small number of parameters. It is arbitrage free, since an explicit solution exists Antonov et al. [2015b].

6.2.5 Disadvantages of the Normal SABR model

As we have already mentioned above, the main disadvantage of this model is that the dynamics of the forward is not really consistent with the observable market data.

6.3 Free Boundary SABR

In Antonov et al. [2015a] Antonov et. al. propose an extension of the SABR model called the Free Boundary SABR. Under the Free Boundary SABR model the forward rate follows the following system of stochastic equations:

$$\begin{aligned} dF_t &= \alpha_t |F_t|^\beta dW_t, \\ d\alpha_t &= \nu \alpha_t dZ_t, \\ dW_t dZ_t &= \rho dt, \\ F(0) &= f,\ \alpha(0) = \alpha \end{aligned} \tag{6.1}$$

with $\beta \in [0, 1/2)$. As we can see, we set $C(F) = |F|^\beta$ in this case. The non-negativeness of the function $C(F)$ guarantees that the increments of the stochastic process F_t are always well defined. This model does not assume boundary conditions for F_t.

6.3.1 Antonov's solution

In the case of $\rho = 0$, the explicit solution was computed in Antonov et al. [2015a]. This solution involves the computation of a two-dimensional integral. In Antonov et al. [2013] an asymptotic expansion of the function $G(t,s)$ is derived, which reduces the overall computation to a one-dimensional integral. In the general case the authors apply the Markovian projection to approximate this model by means of a projection onto the zero correlation case.

In Antonov et al. [2015a] authors determine the probability density function for the Free Boundary SABR model in terms of the Free Boundary CEV model. As a solution for the Fokker-Planck equation, the CEV probability density consists of two terms: the absorbing at zero solution and the reflecting at zero solution. Namely, $p(t,f) = \frac{1}{2}(p_R(t,|f|) + sign(f)p_A(t,|f|)$. Since one of these two densities is singular at zero, we observe the singularity around zero in the full density. In particular, the probability density function of the Free Boundary SABR model has two maxima. One of these maxima occurs exactly at $F = 0$. This singularity at $F = 0$ means a high probability that the forward attains this value. This stickiness at zero has the consequence that zero acts as an attractor of Monte-Carlo paths.

6.3.2 Hagan's approximation formula

For our computations we use the expression of the normal implied volatility obtained from the Hagan's approximation formula Hagan et al. [2014] by setting $C(F) = |F|^\beta$, as it was advised in Kienitz [2015].

$$\sigma_N(K) \approx \frac{\alpha(f-K)}{\int_K^f \frac{dx}{|x|^\beta}} \cdot \frac{\zeta}{\xi(\zeta)} \cdot \left[1 + \left[\frac{-\beta(2-\beta)}{24}|fK|^{\beta-1} sgn(fK)\alpha^2 + \frac{1}{4}\rho\nu\alpha\frac{|f|^\beta - |K|^\beta}{f-K} + \nu^2\frac{2-3\rho^2}{24}\right]\tau_{ex}\right],$$

where τ_{ex} is the time to exercise, and

$$\zeta = \frac{\nu}{\alpha}\int_K^f \frac{dx}{|x|^\beta},$$
$$\xi(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2} - \rho + \zeta}{1-\rho}\right),$$

and

$$\int_K^f = \frac{dx}{|x|^\beta} = \frac{\frac{f}{|f|^\beta} - \frac{K}{|K|^\beta}}{1-\beta}$$

Let us mention that in Hagan et al. [2002] it was assumed that $C(x)$ is smooth, symmetric and integrable around zero function. The function $|x|^\beta$ is not differentiable at zero. Although we can approximate is with a sequence of functions that satisfy the conditions of Hagan et al. [2002] that differ from $|x|^\beta$ only on a small punctured interval around zero, whose length tends to zero.

The quality of the approximation formula for the case $fK < 0$ is very questionable from the theoretical point of view. In this case f and K are lying on different sides of 0. The function $|F|^\beta$ has singularity at zero. Therefore, it is not correct to use the Tailor expansion to obtain the approximation formula for this case. Indeed, this formula is based on the approximation of the analytically derived function

$$g(K) := \log\left(\frac{\int_K^f \frac{\sqrt{C(f)C(K)}}{C(x)}dx}{f-K}\right) \cdot \left(\int_K^f \frac{dx}{C(x)}\right)^{-2}$$

by the first term of its Taylor series, in the case $C(x) := |x|^\beta$. But in this case (when f and K are of different sign) the corresponding Taylor series does not converge, so its first order approximation has nothing to do with the actual value of g at the point K.

6.3.3 The approximation formula for the At-The-Money volatility

For our computations we derive the At-The-Money volatility approximation separately from the general case to avoid the $0/0$ and ∞/∞ uncertainties. In our derivation we follow the steps proposed in Hagan et al. [2002] for the ordinary SABR model. The ATM volatility is computed for the case $K = f$. If we try to substitute $K = f$ in our formula, we obtain $0/0$ expressions. Thus we have to compute this case separately. If $f > 0$, $K > 0$, we can use the usual Hagan's approximation formula, see Chapter 4. So, here we assume that $f < 0$, $K < 0$.

Let us express f and K in the following way:

$$f = -e^{\log(-f)}, \quad K = -e^{\log(-K)}.$$

We obtain then the following expression:

$$f - K = -(e^{\log(-f)} - e^{\log(-K)}) =$$
$$= -e^{\frac{1}{2}\log(-f)+\frac{1}{2}\log(-K)}(e^{\frac{1}{2}\log(-f)-\frac{1}{2}\log(-K)} - e^{-(\frac{1}{2}\log(-f)-\frac{1}{2}\log(-K))}) =$$
$$= -\sqrt{fK}(e^{\frac{1}{2}\log\frac{f}{K}} - e^{-\frac{1}{2}\log\frac{f}{K}}).$$

Then we use the expansion of $\sinh x = \frac{e^x - e^{-x}}{2}$ and obtain the following Taylor series expansion for $x = \frac{1}{2}\log\frac{f}{K}$:

$$f - K = -2\sqrt{fK}(\frac{1}{2}\log\frac{f}{K} + \frac{1}{6}(\frac{1}{2}\log\frac{f}{K})^3 + \frac{1}{120}(\frac{1}{2}\log\frac{f}{K})^5 + \dots)$$
$$= -\sqrt{fK}\log\frac{f}{K}(1 + \frac{1}{24}(\log\frac{f}{K})^2 + \frac{1}{1920}(\log\frac{f}{K})^4 + \dots)$$

Following the same steps and taking the Tailor series expression for $x = \frac{1}{2}\beta\log\frac{f}{K}$:

$$(-f)^\beta - (-K)^\beta = \beta(fK)^{\frac{\beta}{2}}\log\frac{f}{K}(1 + \frac{1}{24}(\beta\log\frac{f}{K})^2 + \frac{1}{1920}(\beta\log\frac{f}{K})^4 + \dots)$$

Thus we obtain the following expressions:

$$(-f)^{1-\beta} - (-K)^{1-\beta} = (1-\beta)(fK)^{\frac{1-\beta}{2}}\log\frac{f}{K}(1 + \frac{1}{24}(1-\beta)^2(\log\frac{f}{K})^2 +$$
$$\frac{1}{1920}(1-\beta)^4(\log\frac{f}{K})^4 + \dots).$$

and

$$\frac{(-f)^\beta - (-K)^\beta}{f-K} = \frac{\beta(fK)^{\frac{\beta}{2}}\log\frac{f}{K}(1 + \frac{1}{24}(\beta\log\frac{f}{K})^2 + \frac{1}{1920}(\beta\log\frac{f}{K})^4 + \dots)}{-\sqrt{fK}\log\frac{f}{K}(1 + \frac{1}{24}(\log\frac{f}{K})^2 + \frac{1}{1920}(\log\frac{f}{K})^4 + \dots)}.$$

Since we consider the case $K = f$ we have $\log\frac{f}{K} = 0$. So, for $f < 0$, $K < 0$, the following approximation is valid:

$$\frac{(-f)^\beta - (-K)^\beta}{f-K} \approx -\beta|f|^{\beta-1}.$$

In the case $K = f$ the value of ζ is small and

$$\chi(\zeta) = \zeta + o(\zeta^2).$$

Therefore, we have:

$$\frac{\zeta}{\chi(\zeta)} = \frac{\zeta}{\zeta + o(\zeta^2)} \approx 1.$$

We substitute all expressions derived above in the main approximation formula for the implied volatility and obtain the following final formula:

$$\sigma_{ATM} \approx \alpha|f|^\beta(1 + [\frac{-\beta(2-\beta)}{24}|f|^{2(\beta-1)}\alpha^2 - \frac{1}{4}\rho\nu\alpha\beta|f|^{\beta-1} + \frac{2-3\rho^2}{24}\nu^2]\tau_{ex})$$

As it was mentioned above, for the case $f > 0$, $K > 0$ we obtain the ordinary SABR formula for ATM volatility.

$$\sigma_{ATM}(K) \approx \alpha f^\beta(1 + \left[\frac{-\beta(2-\beta)}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{1}{4}\frac{\rho\nu\alpha\beta}{f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right]\tau_{ex})$$

Thus we obtain the following final formula for the At-The-Money volatility:

$$\sigma_{ATM} \approx \alpha|f|^\beta(1 + [\frac{-\beta(2-\beta)}{24}|f|^{2(\beta-1)}\alpha^2 + \frac{1}{4}sgn(f)\rho\nu\alpha\beta|f|^{\beta-1} + \frac{2-3\rho^2}{24}\nu^2]\tau_{ex})$$

6.3.4 Advantages of the Free Boundary SABR model

Negative forward rates are allowed. No extra parameter is needed by comparison with the SABR model. The explicit solution exists in the zero correlation case.

6.3.5 Disadvantages of the Free Boundary SABR model

The model restricts the possible values for $\beta \in [0, \frac{1}{2})$. In the original SABR model we allow $\beta \in [0, 1)$. This limitation on β parameter means that models with properties similar to the stochastic CIR model and stochastic lognormal model are not included in the Free Boundary model. Although, in some circumstances this kind of models can reflect the market behaviour better.

Taking into account the stickiness of the risk-neutral density function at zero we can suppose that the initial sign of the stochastic forward rate has influence on the interest rate curve. Thus, in the case of implementing Monte Carlo simulations, the forward process with negative starting value predominantly stays on the negative side and vice versa. This property could affect the hedging parameters. Probably, if we take two starting values with small differences but different sign, the options prices we obtain will differ substantially. This can cause problems by hedging and affect the stability of the calibrated parameters.

6.4 Mixture SABR

The Mixture SABR model was introduced in Antonov et al. [2015b]. It is claimed to be the more efficient in comparison with the previous models introduced above, for the following reasons. This model can be solved exactly. It is arbitrage free. It has effect of "stickiness" near the zero forward rate (which reflects the observed market behaviour at that time).

In Antonov et al. [2015b] some analytical and computational research was done, though no explicit analytical formulas for the implied normal volatility, the probability density function and the price of the call option under this model were provided. In this chapter we derive explicit formulas for the implied volatility and option prices under the Mixture model. Furthermore, we discuss some properties of the model that could be useful to make a decision whether this model is appropriate for the volatility smile interpolation and option pricing.

6.4.1 Definition of Mixture SABR

The Mixture SABR was proposed first in Antonov et al. [2015b]. The goal was to neglect some drawbacks of the previous models, proposed and discussed in Antonov et al. [2015a].

Assume that the forward rate F_t can be written as

$$F_t = \chi F_t^{(1)} + (1-\chi) F_t^{(2)},$$

where

- $F^{(1)}$ follows a zero-correlation Free Boundary SABR model with the parameters $(\alpha_1, \beta_1, 0, \nu_1)$:

$$\begin{aligned}
&dF_t^{(1)} = \alpha_t^{(1)} |F_t^{(1)}|^{\beta_1} dW_t^{(1)}, \\
&d\alpha_t^{(1)} = \nu_1 \alpha_t^{(1)} dZ_t^{(1)}, \\
&dW_t^{(1)} dZ_t^{(1)} = 0 \\
&F^{(1)}(0) = f, \;\; \alpha^{(1)}(0) = \alpha_1
\end{aligned}$$

- $F^{(2)}$ follows a Normal SABR model with the parameters $(\alpha_2, 0, \rho_2, \nu_2)$,

$$
\begin{aligned}
&dF_t^{(2)} = \alpha_t^{(2)} dW_t^{(2)},\\
&d\alpha_t^{(2)} = \nu_2 \alpha_t^{(2)} dZ_t^{(2)},\\
&dW_t^{(2)} dZ_t^{(2)} = \rho_2\\
&F^{(1)}(0) = f,\ \alpha^{(2)}(0) = \alpha_2
\end{aligned}
$$

- χ is a random variable taking the value 1 with the probability p and 0 with the probability $1-p$.

We suppose that all Wiener processes are uncorrelated with each other and χ is independent from $F_t^{(1)}$, $F_t^{(2)}$, $\alpha_t^{(1)}$, $\alpha_t^{(2)}$.
This model is arbitrage free and permits negative interest rates Antonov et al. [2015b]. It has closed-form solution for option prices and gives extra degrees of freedom because of new parameters. The model has 7 parameters $(\alpha_1, \alpha_2, \beta_1, \rho_2, \nu_1, \nu_2, p)$ to be calibrated to market data.

Let us comment now on the parameter choices for the Mixture SABR. It is always useful to keep the same ATM volatility for both models Antonov et al. [2015b], leading to the following relation of the initial stochastic volatility:

$$\sigma_0 = \alpha_1 F_0^{\beta_1} = \alpha_2.$$

The value of p can be parametrized by an auxiliary parameter s via the following formula given in Antonov et al. [2015b]:

$$p = \frac{\sigma_0 \beta_1 e^s}{\sigma_0 \beta_1 e^s + |\nu_2 \rho_2|}$$

.

This guarantees that the Mixture SABR model is reduced to either the zero-correlated Free Boundary model or the Normal SABR model when $\rho_2 = 0$ or $\beta_1 = 0$, respectively. The probability parameter can be also used to control the singularity at zero. Namely, excluding it from the calibration parameters and setting it to small value reduces the singularity arising from the zero-correlated Free Boundary model Antonov et al. [2015b].

In Antonov et al. [2015b] the following reduced parametrization set was proposed.

$$
\begin{aligned}
\nu_2 &= \frac{\nu_1}{1-\beta_1}\\
p &= \frac{\sigma_0 \beta_1}{\sigma_0 \beta_1 + |\nu_2 \rho_2|.}
\end{aligned}
$$

The reduced parametrization, proposed by the authors, increases the number of parameters need to be calibrated to 4 parameters, like the original SABR model.

We would like to comment on the choice for p proposed by the authors.

We can rewrite the equation for p as:

$$p = \frac{1}{1 + \frac{|\rho_2 \nu_2|}{\beta_1 \sigma_0}}, \text{ for } \sigma_0 \beta_1 \neq 0.$$

For the case $\sigma_0 << 1$ and $0 < \beta_1 < 1/2$ we obtain close to extreme choices for p: $p \approx 0$ or $p = 1$.

6.4.2 Analytical formulas for the Mixture SABR model

Probability density function, option prices

First, let us compute the distribution function of F_t. We use the independence of the underlying stochastic processes and the random variable χ in our computation.

$$\begin{aligned} P(F_t \le x) &= P(F_t \le x, \chi \in \{0,1\}) \\ &= P(\chi = 1)P(F_t^{(1)} \le x) + P(\chi = 0)P(F_t^{(2)} \le x) \\ &= pP(F_t^{(1)} \le x) + (1-p)P(F_t^{(2)} \le x). \end{aligned}$$

Thus, the cumulative distribution function of the Mixture model process is the weighted sum of the Normal model distribution functions and the Free Boundary distribution function.

We introduce the following notation:

- f^1 is the probability density function of the zero-correlated Free SABR model;
- f^2 is the probability density function of the Normal SABR model.

Then the probability density function of the Mixture model can be computed by the following expression:

$$f_t = (P(F_t \le x))'_t = pf_t^{(1)} + (1-p)f_t^{(2)}.$$

Using these properties we can compute the call option price for the Mixture SABR model:

$$V^{call}_{Mixture} = pV^{call}_{\text{Zero Free Boundary}} + (1-p)V^{call}_{Normal}.$$

The explicit formulas for $V_{\text{Zero Free Boundary}}$ and V_{Normal} are given in Antonov et al. [2015b].

Martingale properties, no-arbitrage conditions

Below we show that the process F_t satisfies the no-arbitrage conditions. We assume that the underlying processes $F_t^{(1)}$ and $F_t^{(2)}$ satisfy the no-arbitrage conditions, as it was mentioned in Antonov et al. [2015b].

We check the following properties of the probability density function of the stochastic process F_t:

1. Two underlying models (the zero-correlated Free Boundary SABR and the Normal SABR) have the same domain of definition, with the same σ-algebras, which allows to consider their weighted sum.
2. Normalization: $\int_{-\infty}^{+\infty} f_t = p\int_{-\infty}^{+\infty} f_t^{(1)} + (1-p)\int_{-\infty}^{+\infty} f_t^{(2)} = 1$.
3. Martingale property: $E[F_T] = F_0$, since the processes $F_t^{(1)}$ and $F_t^{(2)}$ are martingales.
4. $f \ge 0$ as a sum of two non-negative functions with positive weights: $f_t = pf_t^{(1)} + (1-p)f_t^{(2)}$.

Implied normal volatility for the Mixture model

The value of the implied normal volatility under the Mixture model can be computed by the formula:

$$\sigma^N_{Mixture} = \sqrt{p^2\sigma^2_{\text{Zero Free Boundary}} + (1-p)^2\sigma^2_{Normal}}$$

Using the derivation of the approximation formulas for the Normal implied volatility Hagan et al. [2002] we find the implied normal volatilities for the underlying models that satisfy the following equations:

$$dF^{(1)}_t \approx \sigma^{(1)}_N dW^{(1)}_t,\ F^{(1)}(0) = f,$$
$$dF^{(2)}_t \approx \sigma^{(2)}_N dW^{(2)}_t,\ F^{(2)}(0) = f.$$

Our goal is to find an approximation for σ_N such that

$$dF_t \approx \sigma_N dW_t.$$

Let us define the process $F^{'}_t = pF^{(1)}_t + (1-p)F^{(2)}_t$. This process $F^{'}_t$ differs from F_t but it has the same distribution function and density function as F_t. The option prices computed for these two processes are the same. Thus, we can conclude that $\sigma_N = \sigma^{'}_N$ (this follows from the definition of the implied normal volatility). Here $\sigma^{'}_N$ is determined by the equation:

$$dF^{'}_t \approx \sigma^{'}_N dW^{'}_t,\ F^{'}(0) = f.$$

Since $F^{'}$ is the sum of two uncorrelated Ito's processes, the variance of $F^{'}$ and hence the σ_N is equal to $\sigma_N = \sqrt{p^2\sigma^{(1)2}_N + (1-p)^2\sigma^{(2)2}_N}$.

The other way to determine the value of σ_N is to solve the equation Hagan et al. [2002]

$$V(\tau_{ex}, f) = p\frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{(f-K)^2}{2\sigma^{(1)2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq + (1-p)\frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{(f-K)^2}{2\sigma^{(2)2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq = \frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{(f-K)^2}{2\sigma^{2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq$$

Then we solve the equation:

$$p\int_{\frac{(f-K)^2}{2\sigma^{(1)2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq + (1-p)\int_{\frac{(f-K)^2}{2\sigma^{(2)2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq = \int_{\frac{(f-K)^2}{2\sigma^{2}_N\tau_{ex}}}^{\infty}\frac{e^{-q}}{q^{\frac{3}{2}}}dq$$

This is equivalent to the following equation: Let us suppose that $\sigma^{(1)} \leq \sigma^{(2)}$ We can see from the formula, that $\sigma_N \in [\sigma^{(1)}, \sigma^{(2)}]$. Then we obtain the following formula

$$(1-p)\int_{\frac{(f-K)^2}{2\sigma^{(2)2}_N\tau_{ex}}}^{\frac{(f-K)^2}{2\sigma^{2}_N\tau_{ex}}}\frac{e^{-q}}{q^{\frac{3}{2}}}dq = p\int_{\frac{(f-K)^2}{2\sigma^{2}_N\tau_{ex}}}^{\frac{(f-K)^2}{2\sigma^{(1)2}_N\tau_{ex}}}\frac{e^{-q}}{q^{\frac{3}{2}}}dq$$

Thus we see that σ_N can be characterized by the following property. The value $(f-K)^2/2\sigma^2_N\tau_{ex}$ divides the segment $[(f-K)^2/2\sigma^{(2)2}_N\tau_{ex}, (f-K)^2/2\sigma^{(2)2}_N\tau_{ex}]$ in such a way that the square under the curve $e^q/q^{3/2}$ is divided in two parts with the proportion $p/(1-p)$.

6.4.3 Advantages of the Mixture SABR model

The model has a closed-form solution and gives an extra degree of freedom because of the new parameter p.

6.4.4 Disadvantages of the Mixture SABR model

The Mixture model can be seen as over-parametrized in comparison with the other models described above. It is difficult to calibrate 6 parameters of the model simultaneously. The use of the restriction authors advise for the parameters, makes the model less understandable from the practical point of view. There is no answer on the question "Why we mix the Normal and the Free Boundary models, except to the fact, that analytical solutions exist for both models?".

The structure of the model can cause jumps in risk metrics and in Monte Carlo simulated paths because of the mixture structure.

Some general drawbacks of mixture-type models are discussed in Piterbarg [2003]. We can apply the same arguments to the Mixture SABR model. Namely, it is not possible to price path dependent derivatives using the mixture model, and there is also explained a problem that occurs once one tries to apply a mixture models for hedging.

Chapter 7

Comparison of the models

In this Chapter we compare the models introduced in the previous Chapter. We assume that F_t is the stochastic process of the swap forward rate. Let us recall that we investigate which extension of the SABR model is the most suitable for the modelling of the volatility smile in the presence of negative interest rates. In order to answer this question we formulate comparison criteria.

7.1 Comparison criteria

We aim to evaluate the models by the criteria, introduced in the Chapter 1. Let us briefly recall them.

Computational: the quality of the fit (cf. the Subquestion (1) from the Introduction section), the absence of arbitrage in the computed option prices (cf. the Subquestion (2)), the parameters consistency (cf. the Subquestion (3)), easiness of implementation and computational costs (cf. the Subquestion (4)).

Analytical: reflection of the analytical properties of the model in real market (cf. the Subquestion (5)); number of parameters of the model; existence intuitive explanation for models parameters; the complexity of the model (cf. the Subquestion (6)); hedging under the model (cf. the Subquestion (7)).

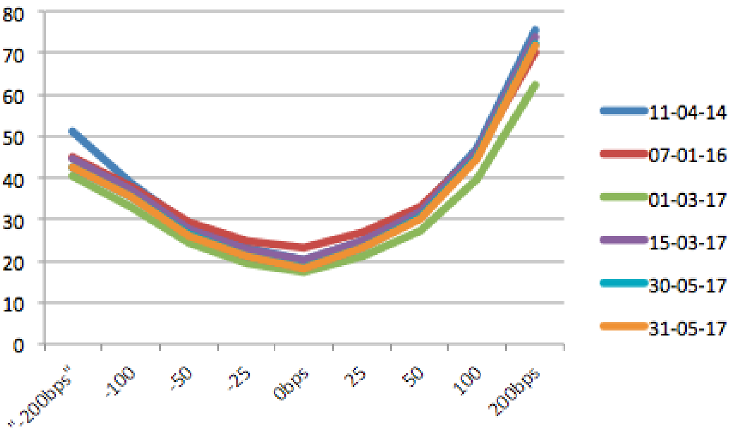
All other things being equal, the model with less parameters should be preferred. Ideally, we select the model that optimizes all our criteria with respect to the other models.

7.2 Market data

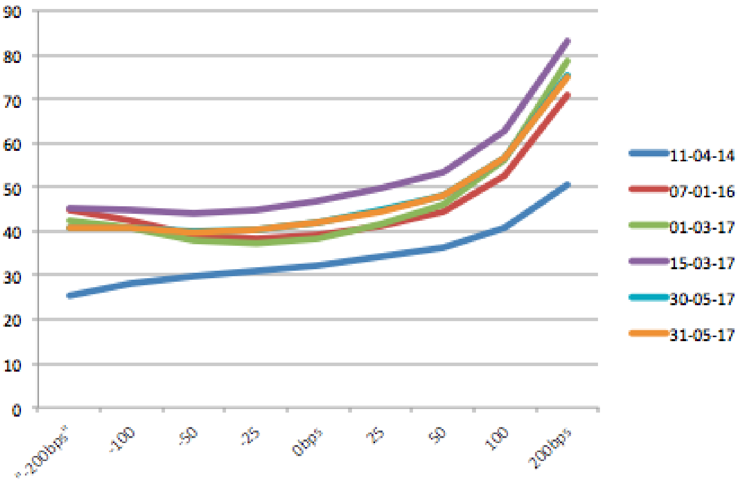
7.2.1 Implied normal volatility

For our analysis we use the data of the implied normal volatilities for the 6-months EURIBOR swaptions ranging from 1M X 2Y to 15Y x 30Y. We chose the following testing days: 04.11.2014, 01.07.2016, 03.01.2017, 15.03.2017, 30.05.2017, 31.05.2017. We choose a day with quite hight ATM (At-The-Money) value (04.11.2014), days with negative ATM values and two days in a row to check the stability of parameters.

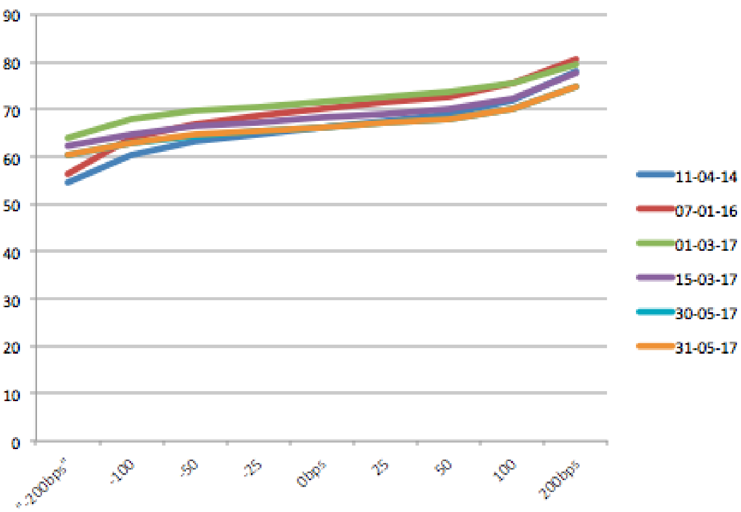
We observe different behaviour for the implied volatilities of swaptions with small expiry and maturity and swaption with long expiry and maturity. The volatility smile looks more like a smile in the case of small expiry and maturity (less than 2 years). Thus, we observe the lower values for the implied volatilities around At-The-Money values. If the distances between strikes and the At-The-Money value increase, then the values of the implied volatilities increase as well.



The graph of the implied volatility with middle maturity and expiry (between 2 and 5 years) has lower curvature. The smile is less expressed for small maturities. The ATM values are positive.



The graph of the implied volatilities of swaptions with high expiry and maturity (more than 5Y X 5Y) looks more like a straight line and not like a smile. We do not observe any substantial growth of the implied volatility values on the left side of the graph. The ATM values for swaptions with long maturity and tenor are positive.



7.3 Methodology

In order to calibrate model parameters we use extensions of Hagan's approximation formula. The other methods (calibration using the approach proposed by Antonov et. al.) could probably give other qualitative behaviour of the fit and lead to other conclusions. We choose Hagan's approximation formula because it is easy to implement and following the article of Kienitz [2015] we can suspect the same kind of errors as in the method of Antonov et. al. Moreover, Antonov's approach does not work for all possible combinations of parameters and can produce non arbitrage-free prices for some sets of parameters, especially in the area around ATM Youmbi [2017]. This property makes the calibration procedure difficult to implement.

We fit the models to different volatility smiles and measure the estimation errors. We measure the differences between the market data and the models results. The measure of errors is Root Mean Squared Error (RMSE):

$$RMSE = \sqrt{\sum (\sigma^N_{market} - \sigma^N_{Estimatred})^2 / n}.$$

We use the Levenberg-Marquardt algorithm for the calibration of parameters.

It is important to define the order of the approximation error that we suppose insufficient. Since the implied volatility values are quite small (if we measure them in the absolute values, then they are less than one), the error that seems to be small in the absolute value could potentially mean a bad approximation. We suppose that the error higher than 0.1% means insufficient approximation. Thus, we evaluate the order of the $RMSE$ higher than 0.001 (or less than 0.01 if we choose 1%) as insufficient.

We check the presence of arbitrage for each model. In order to do that we compute the second derivatives of the option prices produced by the models. If the implied risk-neutral probability density function has negative values, then we conclude that the model allows arbitrage.

7.4 Computational results

7.4.1 Displaced SABR

This model seems to be one of the most easiest and intuitive choices. The construction of the model reflects the belief that negative interest rates could not go below some level.

We fix the value of the shift before calibration. The most intuitive guess is to set the value of the shift equal to the absolute value of the minimal possible negative interest rate, which is supposed to exist. Nowadays, it is usually taken between -2% and $-2,5\%$.

We conclude from our computational results that the quality of the fit depends on the value of the shift chosen priorly. The best way is to choose the value of the shift close to the absolute value of the minimal negative strike used in calibration. If we choose a higher value of the shift, then the approximation error increases. This observation can be explained by the analytical properties of the Displaced SABR model. Suppose we choose a relatively big value of the shift (by comparison with the initial forward value). As we can see from the formula

$$dF_t = \alpha_t (F_t + s)^\beta dW_t,$$

if s is big, then it is the most influential term on the right hand side of this formula. The value of the shift dominates and the model approximately looks like

$$dF_t = \alpha_t (s)^\beta dW_t.$$

Thus we obtain approximately the Normal SABR model, which gives a worse fit to the market data then the Displaced SABR model with a small s.

We observe arbitrage near the boundary prescribed by the shift for some kinds of volatility smiles. It can be corrected by the usage of no-arbitrage techniques described in Hagan et al. [2014]. Moreover, we can conclude from our computational results that the values of the estimated parameters depend on the shift. The change of the shift by $0,1\%$ can cause jumps in the values of estimated parameters, especially in the values of α and ν.

The main disadvantages of the model are the hedge problems. If interest rates go lower than expected, a necessary change of the shift can cause jumps in calibration parameters and risk metrics.

7.4.2 Normal SABR and Free Boundary SABR

The Free Boundary SABR model and the Normal SABR model give worse fit to the data then the Displaced SABR model.

The RMSE of the Displaced SABR model has the order about 10^{-3} and the RMSE of Free Boundary SABR and the Normal SABR models are about 10^{-2} for some shapes of the volatility smile.

The Free Boundary model shows slightly better approximation then the Normal model, but the order of approximation is the same. The estimation of the parameter β is usually close to zero, so we obtain the model which is close to the Normal SABR model. The appropriate choice of the initial guess for the Free Boundary SABR model is essential. The Free Boundary SABR can not approximate the volatility smile in the case of very small initial forward values.

Moreover, the Free Boundary SABR model has the stickiness around zero, which is not a desirable property in our opinion. The Free Boundary model limits the values of $\beta \in (0, 1/2)$. That means that we are going in the direction of the stochastic normal distribution in our modelling. This makes the Free Boundary model less flexible then the Displaced SABR model. The other potential problem of the Free Boundary SABR model is the behaviour around zero of the modelled forward rate. Indeed, we use in this model function $C(F) = |F|^{\beta}$ which is not differentiable at zero.

Thus, we conclude that the Normal model satisfies our criteria better than the Free Boundary SABR model. Indeed, the Normal model has less parameters and gives the same order of the fit to the market data. The analytical properties of the Free Boundary SABR model do not reflect the market situation.

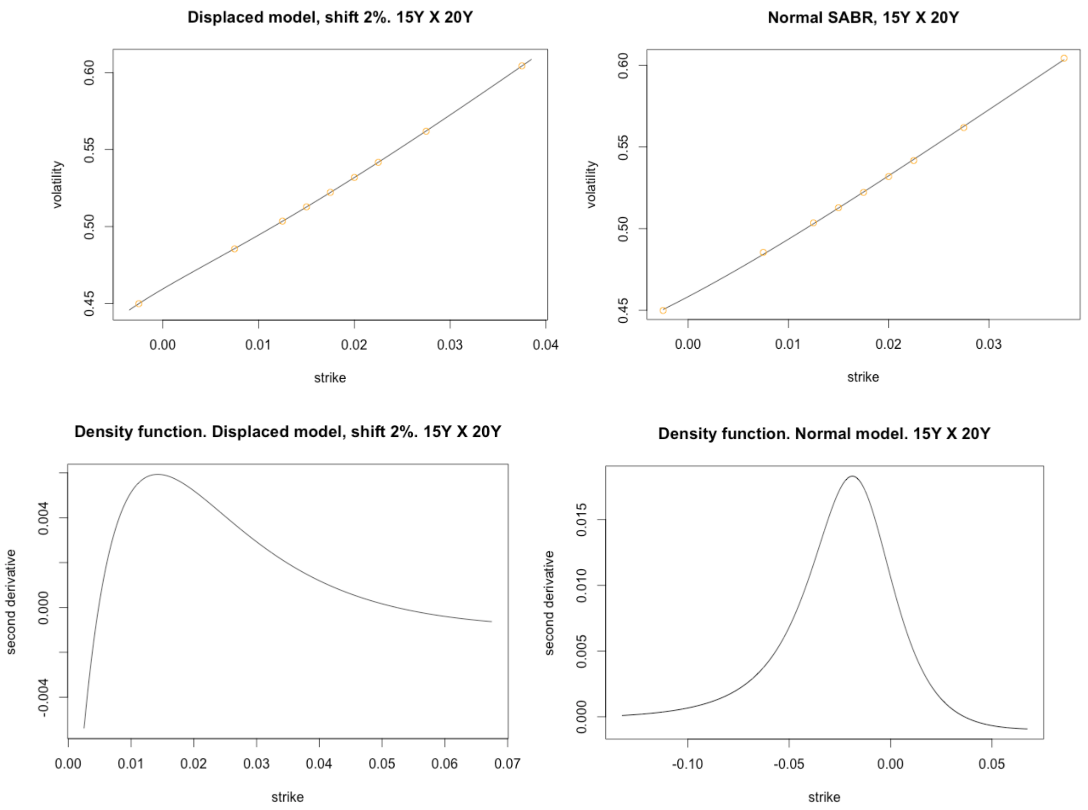
7.4.3 Approximation of different kind of smiles

Below we describe the approximation quality of the models for different types of smiles. As we mentioned above, in our data we have different kinds of the volatility smiles, depending on the maturity and expiry of the swaption.

Volatility smile for swaption with high expiry and tenor

The volatility smiles for swaptions with high expiry and maturity (higher than 5Y X 5Y) look like a straight line. We have positive and quite high ATM values for this kind of swaptions.

The approximation quality for this type of volatility smiles is sufficiently high for all models. It is not surprising, since this kind of curves is easier to approximate. For all available input data we observe the arbitrage near the boundary for the Displaced SABR model.

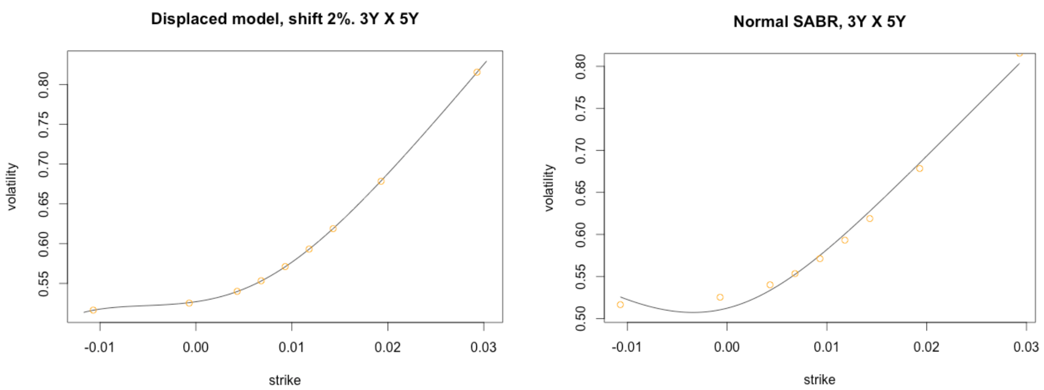


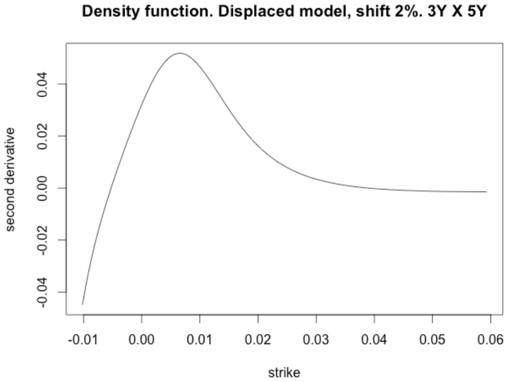
Volatility smile for the swaption with middle expiry and tenor

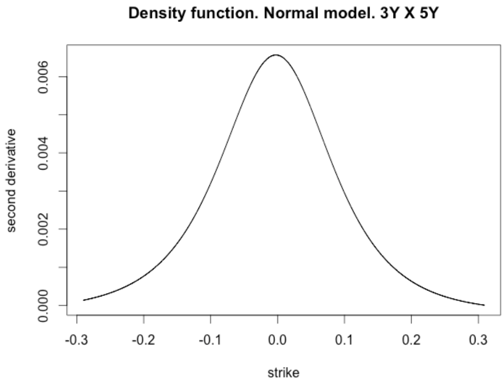
In this Section we consider swaptions with maturity and expiry between 2 and 5 years. As we can see from the graphs these smiles have higher curvature and are declining for small strikes. The ATM values are positive.

This type of smiles is much better approximated by the Displaced SABR model. The order of approximation error is 10^{-8} for the Displaced SABR model and 10^{-3} for the Free Boundary and Normal models.

However, the implied density function of the Displaced SABR model is negative near the boundary.



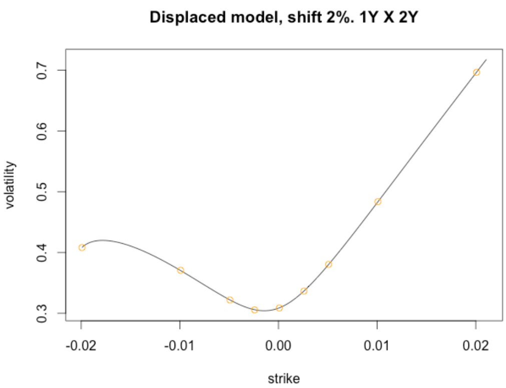


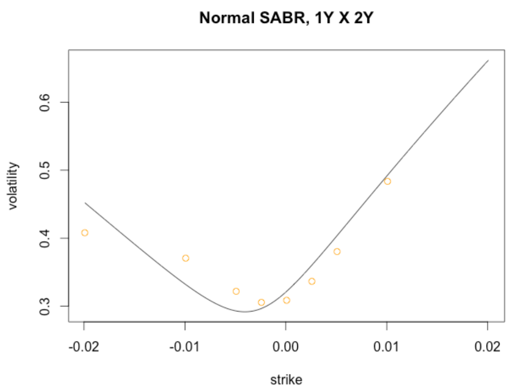


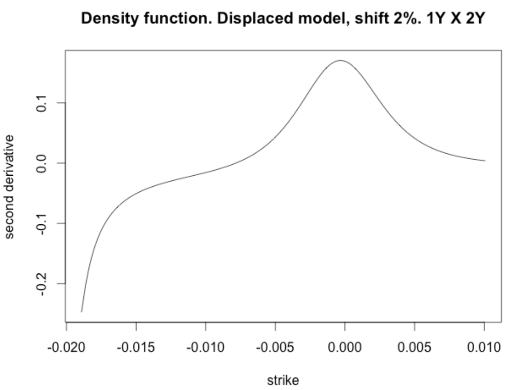
Volatility smile for the swaption with small expiry and tenor

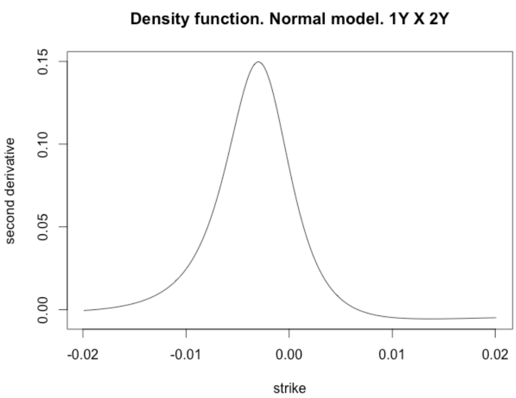
In this section we consider swaptions with maturity and expiry smaller than 2 years. We can see from the graph below that the implied normal volatilities of the 6M X 2Y swaption has a "smile" effect.The ATM values are negative.

In this case the we got insufficient fit applying the Free Boundary and Normal models with the error about 10^{-2}. The Displaced SABR model has the error about 10^{-3}. The value of β calibrated for the Displaced SABR model is close to one. This can be a reason why the Normal SABR model (with $\beta = 0$) and the Free Boundary SABR model (with $\beta \in (0, 1/2)$) are not able to give sufficiently close estimation for this type of volatility smiles. The implied density function of the Displaced SABR model is negative near the boundary.









7.5 Mixture model

We show in the previous Chapter that the value of the approximated normal volatility under the Mixture model can be computed as

$$\sigma^N_{Mixture} = \sqrt{p^2\sigma^2_{\text{Zero Free Boundary}} + (1-p)^2\sigma^2_{Normal}}.$$

We try to use this approach in our computations, but it gives unsatisfactory results. First, we calibrate the Normal SABR and the zero-correlated Free Boundary SABR separately. Second, we estimate the value of p. Unfortunately, it turns out, that it is impossible to calibrate the uncorrelated Free Boundary SABR model using Hagan's approach. We conclude that the implementation of the Mixture SABR model in terms of Hagan's approximation formula is not working.

The use of the Antonov method of calibration for the Mixture SABR model can give other results. In Youmbi [2017] this model was calibrated to swaption prices, using Antonov's approach, and the authors suggest that "it works quite well".

Chapter 8

Conclusions

8.1 Summary

The SABR model, developed by Hagan et. al. Hagan et al. [2002], is widely used for the interpolation of volatility surfaces and hedging of the volatility risk. The SABR model is not adapted to the negative forward rates.

In this thesis we introduce and compare different extensions of the SABR model, adapted to negative interest rates. Namely, we consider the following models: the Displaced SABR, the Free Boundary SABR, the Normal SABR, and the Mixture SABR.

We use the Hagan approximation formula for the Displaced SABR model and derive the same kind of approximation formulas for other extensions of the SABR model.

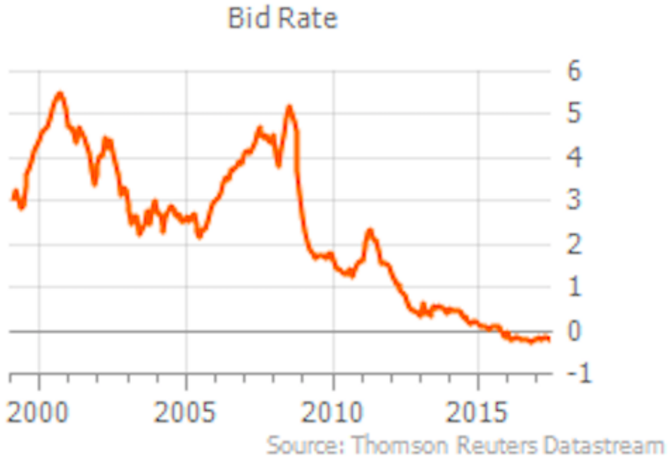
In particular, in this thesis we derive the Hagan approximation formula for the extensions of the SABR model and compute the expressions for the ATM volatilities. We describe the properties of the Mixture model and derive the Hagan approximation for its normal implied volatility in terms of underlying models volatilities. We calibrate models to the current market data. We discuss analytical properties of the models, and analyse the intrinsic limitations of the models.

8.2 Discussion

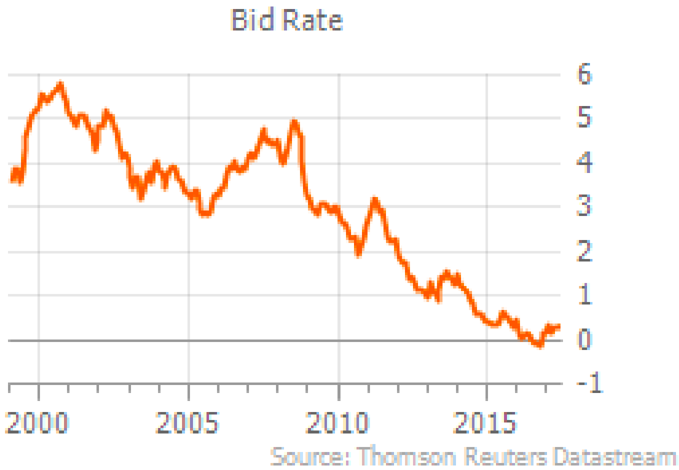
8.2.1 Stickiness at zero

The forward under the Free Boundary SABR model has stickiness around zero. We want to check, whether the stickiness at zero is indeed the case in the forward rate market behaviour. We inspect the behaviour of EURIBOR, LIBOR and Swiss Franc interest rates. This kind stickiness around zero was observed only for Swiss Franc.

Below we recall the graph of "Euribor 6 Months Interest Rate Swaps 2 Years". We conclude from this graph, that there is no stickiness at zero there, but we can see risk aversion of the rate.



Below we recall the graph of "ICAP Euro VS Euribor 6 Months Interest Rate Swaps 6 Years". We conclude from this graph, that there is no stickiness at zero there.



We conclude that although the stickiness could be observed for some currency, we should not assume that the stickiness at zero is something typical for swaption forward rates. The stickiness around zero, observed for the Swiss Franc, could be caused by the monetary policy. Thus it could be not random in its nature. So it is questionable whether it possible to demand this property in the stochastic process modelling.

Thus, we conclude that stickiness around zero is not the desirable analytical property we want to include in our model (cf. the Subquestion (5) from the Introduction section).

8.2.2 Shortcomings of Antonov's approach

Here we discuss some shortcomings of the Antonov approach. They give us more ground to choose the Displaced model over the others.

As it is mentioned in Youmbi [2017] the Antonov approach does not work for all possible combinations of parameteres and can produce non arbitrage-free prices for some sets of parameters Youmbi [2017], especially in the area around the ATM. This property make the calibration procedure very difficult to implement.

Because of the complexity of the Antonov approach the results coming from the implementation are very sensitive to particular implementations of the computational schemes. It really matters how the double integral is computed, which approximation is chosen, etc. This makes the model unstable and difficult to implement. Moreover, computational costs are much higher then in Hagan's approach.

The hedge procedure in the Antonov approach is not clarified in the literature. We can take the derivative of the option prices computed by the Antonov formula, but the double integration remains in the formula for the derivative and its convergence properties are unclear (cf. the Subquestion (7) from the Introduction section).

8.3 Conclusion

It turns out that the approximation quality of the Displaced SABR model outperforms the other models. Moreover, in some special cases of the market data the approximation quality is only sufficient for the Displaced SABR model. From our analysis one can conclude that the Displaced SABR model is, in many cases, the most efficient one. There are the following reasons to prefer this model to the other extensions of the SABR model.

- The model has the best approximation accuracy for all types of the volatility smile (cf. the Subquestion (1) from the Introduction section)

- The model is intuitively reasonable and reflects the current believes about the interest rate behaviour (cf. the Subquestion (5)).
- This model is easy to implement. Especially if the ordinary SABR model was already used in computations (cf. the Subquestion (4)).
- The calibration of parameters is computationally efficient and the model is not over-parametrized (cf. the Subquestions (4) and (6)).

There are the following attention points for this model:

- This model creates arbitrage in option prices near the boundary. This problem can be corrected using the no-arbitrage techniques Hagan et al. [2014] (cf. the Subquestion (2)).
- The proper and consistent initial guess for the minimization algorithm is essential (see Floc'h and Kennedy [2014]). For calibration, it is useful to take as the initial guess the values of the parameters estimated on the previous day. This help for the stability of the estimated parameters (cf. the Subquestion (3)).
- The choice of the value of the shift influences the calibrated parameters and approximation errors. Taking the shift as close as possible to the smallest strike gives the best approximation quality. Still if we set the shift equal to $2,5\%$ we obtain good accuracy for all data that we used (cf. the Subquestion (1)).

We would like to emphasize that our results are based on Hagan's approximation approach, which we used for computations. Other calibration methods and pricing formulas (Antonov's approach, for instance) could give other results and lead to other conclusions.

To conclude this thesis, we would like to mention topics, which are not covered in this thesis, but which seem to be very interesting to us. We almost do not discuss the hedge issues in this thesis, although it is a very important part of the applications of the SABR model. The questions related to this area could be the following ones: How does the shift parameter in the Displaced SABR model influence the risk metrics of the model? Would the influence be the same if we implement the new improved formula for the volatility smile approximation, recently presented at Hagan et al. [2016]?

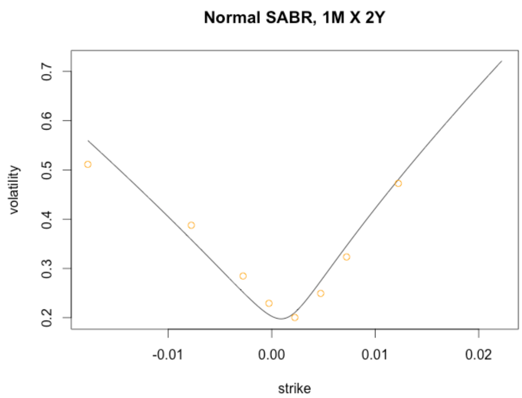
How the hedging under the Mixture SABR model should be organized? Compute the risk metrics for the model. For the analysis of the hedging under the Mixture model we recommend to keep in mind the issues raised up in the similar set-up with mixture models in Piterbarg [2003]. Namely, it is an open question, whether there is a way to hedge consistently under the mixture models. Moreover, it is suggested in op. cit. that it is not possible to price path dependent derivatives using mixture models.

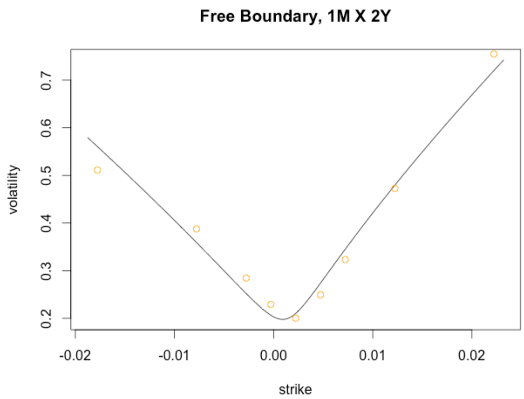
The other topic, which to our best knowledge is not completely covered in the literature, is the efficient and relatively fast calibration of the parameters of the model in the case of Antonov's approach. Although the computational results of the calibration are presented in the literature, it is not fully clarified how the calibration of parameters was performed. This raises the questions, whether the obtained results depend on the method chosen for calibration, whether this method is stable, and whether the calibrated parameters are consistent.

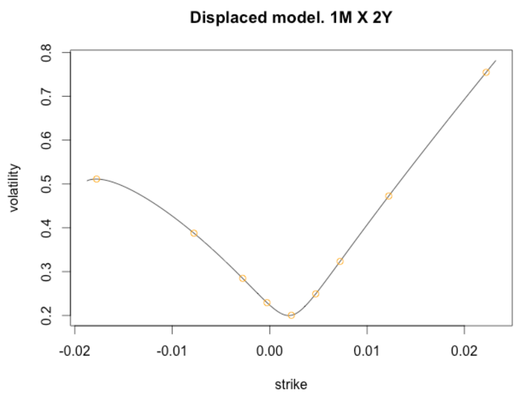
Appendix 1: Graphs

Swaptions with small maturity/expiry

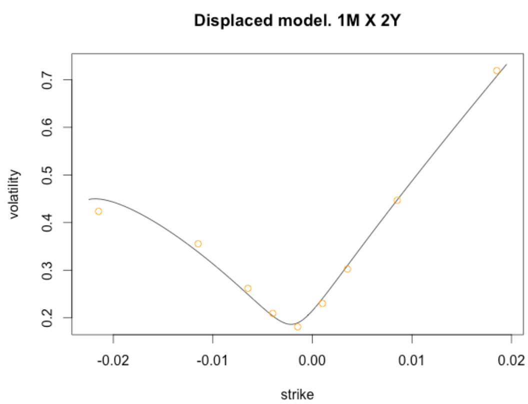
The volatility smile. The Normal SABR, the Free Boundary SABR model and the Displaced SABR model (shift $2,5\%$). 1M X 2Y swaption. 11.04.2014.

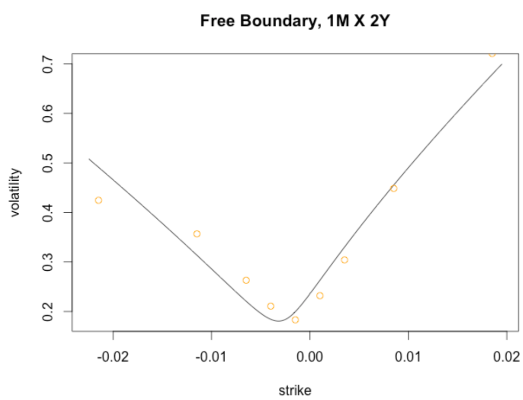




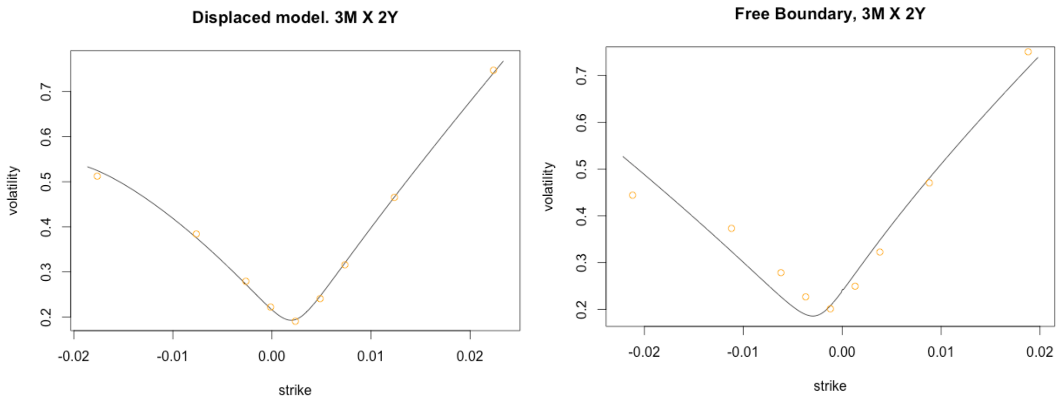


The volatility smile. The Displaced SABR model (shift $2,4\%$), the Normal SABR and the Free Boundary SABR model. 1M X 2Y swaption. 30.05.2017.

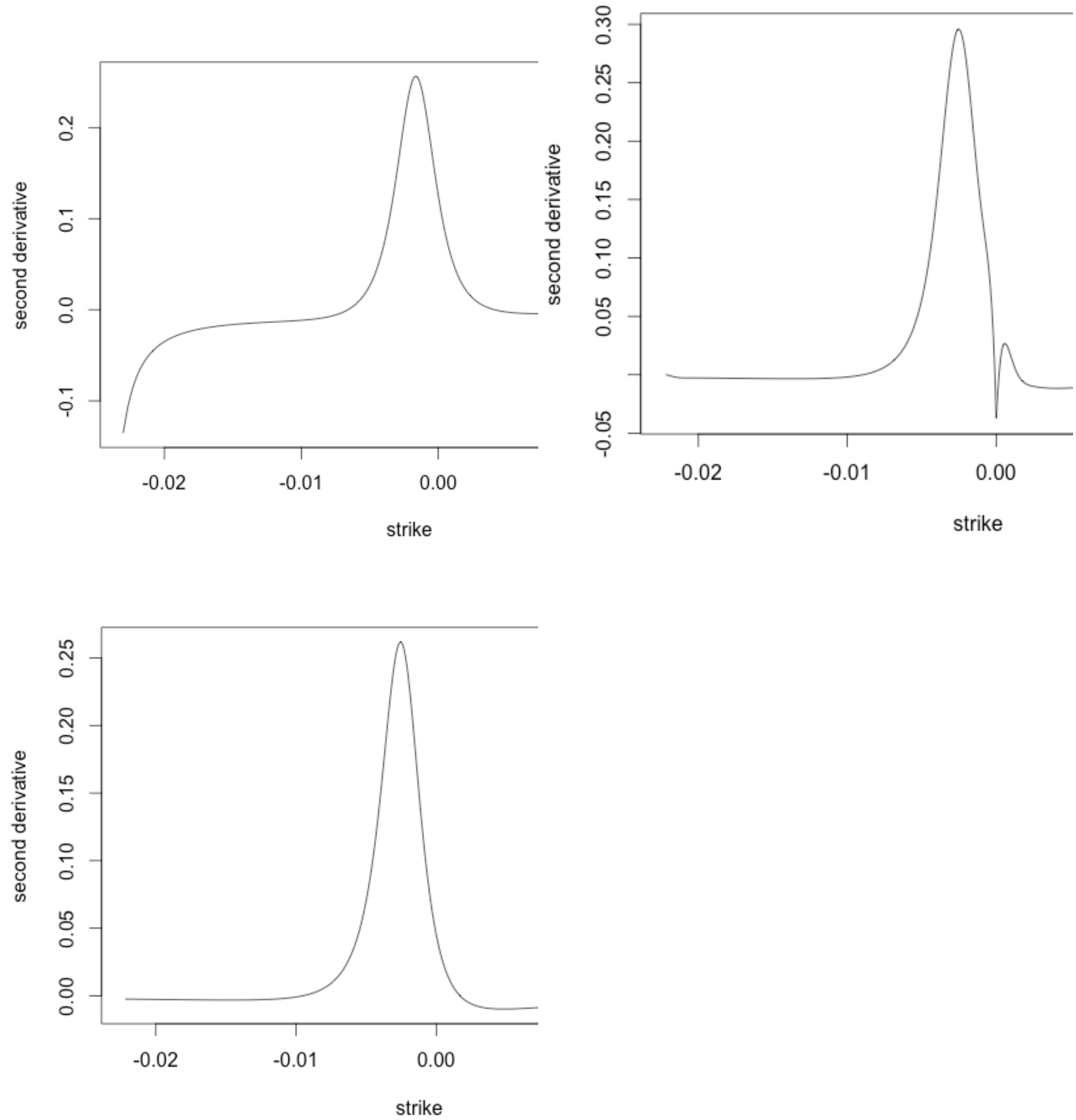




The volatility smile. The Displaced SABR model (shift $2,3\%$)and the Free Boundary SABR model. 3M X 2Y swaption. 30.05.2017.

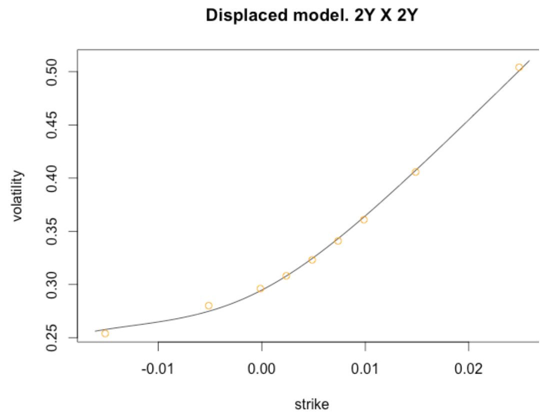


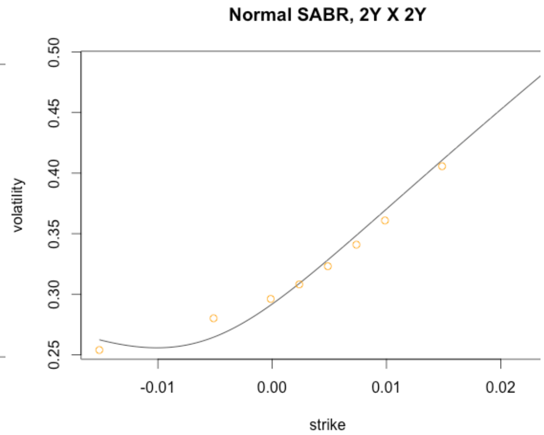
Probability density function. The Displaced SABR model (shift $2,5\%$),the Free Boundary SABR model and the Normal SABR. 3M X 2Y swaption. 31.05.2017.



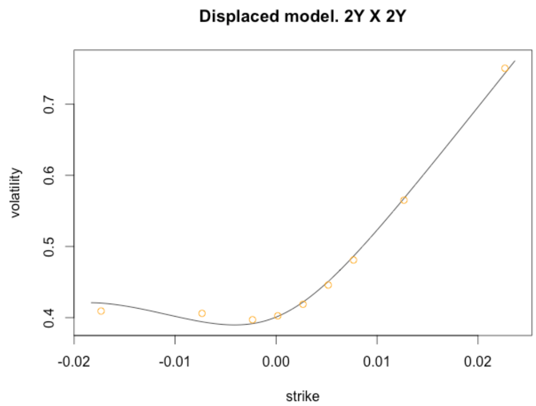
Swaptions with middle maturity/expiry

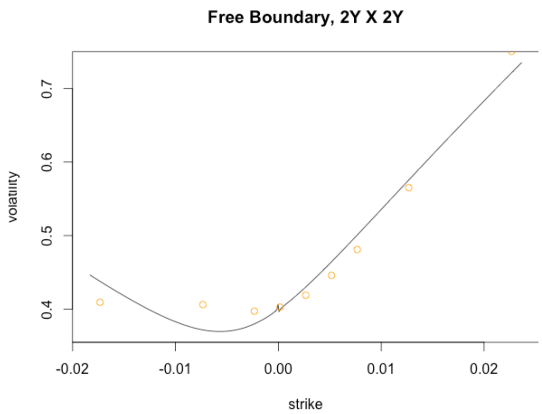
The volatility smile. The Displaced SABR model (shift 2, 5%) and the Normal SABR model. 2Y X 2Y swaption. 11.04.2014.



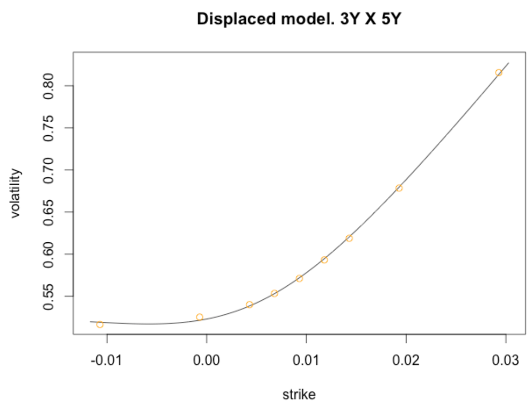


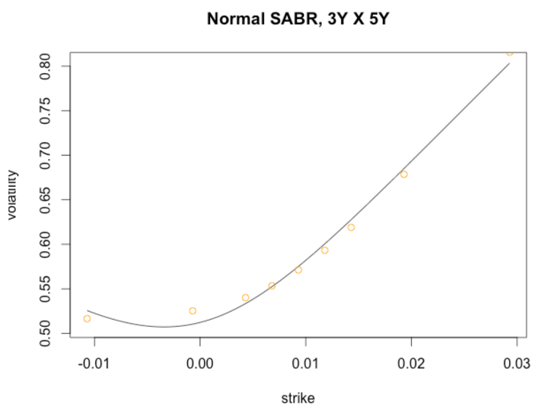
The volatility smile. The Displaced SABR model (shift 2, 5%) and the Free Boundary SABR model. 2Y X 2Y swaption. 31.05.2017.





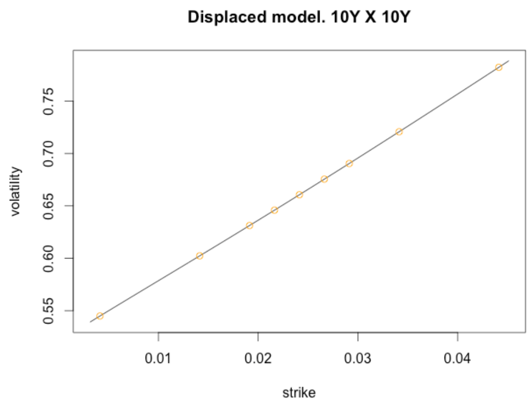
The volatility smile. The Displaced SABR model (shift 2, 5%) and the Normal SABR model. 3Y X 5Y swaption. 31.05.2017.

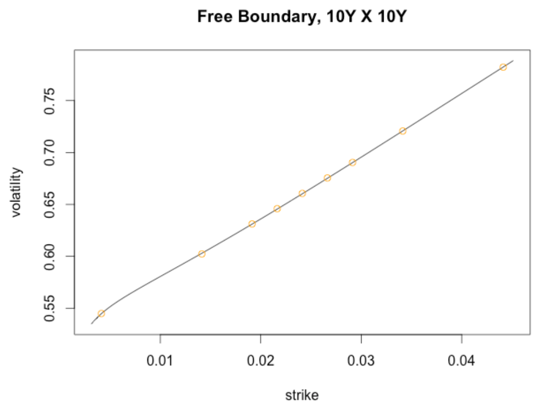




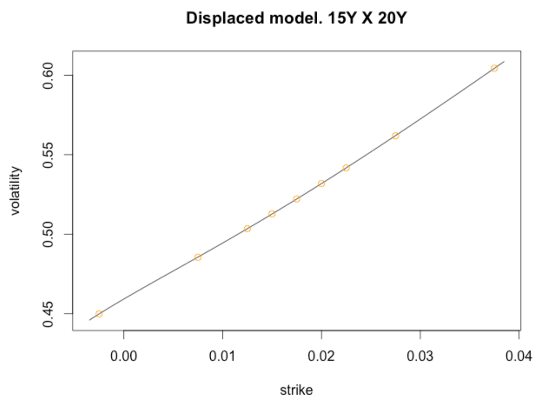
Swaptions with large maturity/expiry

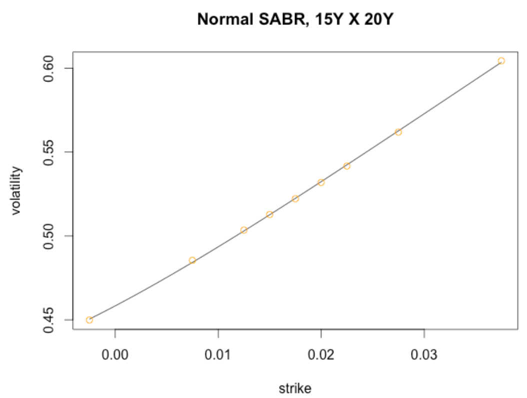
The volatility smile. The Displaced SABR model (shift $2,5\%$) and the Free Boundary SABR model. 10Y X 10Y swaption. 11.04.2014.





The volatility smile. The Displaced SABR model (shift $2,5\%$) and the Normal SABR model. 15Y X 20Y swaption. 31.05.2017.





Appendix 2: RMSE for small maturity/expiry

				Parameters estimations
type of swaption	Date	Model	RMSE	alpha, beta, rho, nu; for Normal model: alpha, rho, nu
1M X 2Y	11-04-14	Displaced, shift 2%	0,0000	parameter estimates: 2.00228248488612, 0.999010104940871, 0.000133237339927612, 22.4478972075505
1M X 2Y	11-04-14	Normal	0,0284	parameter estimates: 0.0532029164180086, 0.395448128374141, 23.5899491618086
1M X 2Y	11-04-14	Free boundary	0,0267	parameter estimates: 0.057162499712888, 0.0211714015112637, 0.398220455713508, 23.7544374471225
1M X 2Y	07-01-16	Displaced, shift 2,5%	0,0063	parameter estimates: 2.66139455546255, 0.9999, -0.0135333025012825, 20.6058167474078
1M X 2Y	07-01-16	Normal	0,0312	parameter estimates: 0.071401824421011, 0.398673612001967, 21.5234157462859
1M X 2Y	07-01-16	Free boundary	0,0314	parameter estimates: 0.0705832156613939, 1e-05, 0.418012992070098, 21.8215261110011
1M X 2Y	01-03-17	Displaced, shift 2,5%	0,0079	parameter estimates: 1.953800615923, 0.9999, 0.03584383170234, 20.2678864741856
1M X 2Y	01-03-17	Normal	0,0281	parameter estimates: 0.0535707764616658, 0.40654299545016, 21.5810011461827
1M X 2Y	01-03-17	Free boundary	0,0281	parameter estimates: 0.05358150436793, 1e-05, 0.398582303898223, 21.468611709626
1M X 2Y	15-03-17	Displaced, shift 2,5%	0,0141	parameter estimates: 2.09262903536116, 0.9999, 0.119550416770518, 21.1810785383067
1M X 2Y	15-03-17	Displaced, shift 2,3%	0,0092	parameter estimates: 2.21261387221433, 0.9999, 0.0699162157182418, 21.4343298020792
1M X 2Y	15-03-17	Normal	0,0361	parameter estimates: 0.0601880233456823, 0.468568248547471, 23.6037283471109
1M X 2Y	15-03-17	Free boundary	0,0362	parameter estimates: 0.0589959262385485, 1e-05, 0.482840784884815, 23.9956144732563

1M X 2Y	30-05-17	Displaced, shift 2,5%	0,0154	parameter estimates: 1.98225727029631, 0.9999, 0.138659321352081, 21.1708821531287
1M X 2Y	30-05-17	Displaced, shift 2,3%	0,0096	parameter estimates: 2.08259263889136, 0.9999, 0.0822830907560802, 21.4515136063057
1M X 2Y	30-05-17	Normal	0,0374	parameter estimates: 0.0564393626433933, 0.492395844546903, 23.9510663163909
1M X 2Y	30-05-17	Free boundary	0,0374	parameter estimates: 0.0563406609002188, 1e-05, 0.487415429665813, 23.8555193484575
1M X 2Y	31-05-17	Displaced, shift 2,5%	0,0154	parameter estimates: 1.96114578114848, 0.9999, 0.139509844443524, 21.1832983404022
1M X 2Y	31-05-17	Displaced, shift 2,3%	0,0096	parameter estimates: 2.06031191177528, 0.9999, 0.0832716954090121, 21.4598814456719
1M X 2Y	31-05-17	Normal	0,0373	parameter estimates: 0.0558229947067525, 0.492483240678179, 23.9683352203632
1M X 2Y	31-05-17	Free boundary	0,0373	parameter estimates: 0.0551578394429671, 1e-05, 0.504677336947944, 24.2986099398895
3M X 2Y	11-04-14	Displaced, shift 2,5%	0,0090	parameter estimates: 1.15636757232262, 0.9999, 0.093939601799998, 15.7236696696808
3M X 2Y	11-04-14	Displaced, shift 2,3%	0,0064	parameter estimates: 1.2290540191623, 0.9999, 0.0638024333841593, 15.7962194315174
3M X 2Y	11-04-14	Normal	0,0275	parameter estimates: 0.0356540921549453, 0.3888318058732, 16.9234515808985
3M X 2Y	11-04-14	Free boundary	0,0259	parameter estimates: 0.036094051120012, 0.0117093107157506, 0.400749594577115, 17.161413595945
3M X 2Y	31-05-17	Displaced, shift 2,5%	0,0154	parameter estimates: 1.51412018437409, 0.9999, 0.132299546103797, 15.1802401036762
3M X 2Y	31-05-17	Displaced, shift 2,3%	0,0099	parameter estimates: 1.5956672501433, 0.9999, 0.0793934487478871, 15.3612903666787
3M X 2Y	31-05-17	Normal	0,0380	parameter estimates: 0.0437331846355711, 0.484930561960288, 17.1967434349394
3M X 2Y	31-05-17	Free boundary	0,0386	parameter estimates: 0.0415098992190043, 1e-05, 0.512339571289439, 17.8132281815762
6M X 2Y	15-03-17	Displaced, shift 2,5%	0,0128	parameter estimates: 1.68098606878568, 0.9999, 0.0950349612084388, 11.873041589016
6M X 2Y	15-03-17	Displaced, shift 2,3%	0,0083	parameter estimates: 1.78786830012101, 0.9999, 0.0501545051694347, 12.0267629415467
6M X 2Y	15-03-17	Normal	0,0363	parameter estimates: 0.0504080706420939, 0.461179196666376, 13.2750009930065
6M X 2Y	15-03-17	Free boundary	0,1481	parameter estimates: 0.00184746135548561, 1e-05, 0.64692977901219, 24.2197516052072
1Y X 2Y	31-05-17	Displaced, shift 2,5%	0,0083	parameter estimates: 1.79292925157917, 0.9999, 0.0590402400695274, 8.41524743604011
1Y X 2Y	31-05-17	Displaced, shift 2,3%	0,0046	parameter estimates: 1.91348137620864, 0.9999, 0.00974512190497718, 8.59788365303436
1Y X 2Y	31-05-17	Normal	0,0277	parameter estimates: 0.0547476710749302, 0.482912871674516, 9.48775262227097
1Y X 2Y	31-05-17	Free boundary	0,0272	parameter estimates: 0.0564546082814246, 1e-05, 0.48618274760044, 9.43980585580006

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