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***Abstract*-** The Quantitative finance world has stepped up well since its last centuries beginnings. Since the beginning with Black’s formula in 1973, financial mathematicians, engineers, academics and practitioners have contributed to building models that more realistically identify the stochastic behavior of prices in the financial markets. Such behavior has changed drastically over the decades and since the times where volatility was assumed to be constant over time. The presence of the equity skew and volatility smiles, moving from a flatter, simpler kind, suggested the utilization of a new way of thinking, with volatility not being fixed anymore, but following a stochastic process.

This research paper wants not only to study, but also to compare the current trends in identifying forward looking implied volatility, starting with the Dupire model – fixing volatility over time and treating as a constant; the Heston model – volatility following a stochastic process with a mean reverting structure; the more modern SABR models – stochastic alpha, beta and rho with complex structures that have many variants according to specific situations. Follows an in-depth analysis and plotting of such models that will serve our understanding of their accuracy and intricacy with respect to calculating market implied volatility.

1. Introduction

T

his section is here to introduce the topics confronted by outlining the major discussions, divided by chronological order. Section 1 will do something, section 2 will follow with something else, etcetera.

Identify the constructs of a Journal – Essentially a journal consists of five major sections. The number of pages may vary depending upon the topic of research work but generally comprises up to 5 to 7 pages. These are:

1. Abstract
2. Introduction
3. Research Elaborations
4. Results or Finding
5. Conclusions

**In Introduction you can mention the introduction about your research.**

1. LITERAURE REVIEW

It's the foremost preliminary step for proceeding with any research work writing. While doing this go through a complete thought process of your Journal subject and research for it's viability by following means:

1. Read already published work in the same field.
2. Goggling on the topic of your research work.
3. Attend conferences, workshops and symposiums on the same fields or on related counterparts.
4. Understand the scientific terms and jargon related to your research work.
5. MATHEMATICAL DEFINITIONS

In this chapter we will be going through the main mathematical definitions over which topics will work on, starting with general efficient markets theory assumptions, Brownian motion processes, Ito’s calculus and pricing under a risk-neutral framework.

## General Assumptions

For the purpose of the research study and for the models used to work, we assume the following:

* Markets are ideal. No transaction fees, taxes, inflation and no restrictions to short sell.
* Markets are complete.
* The risk-free rate exists and is non-negative.
* Markets are efficient, there are no material or non-material non-public information. All information is available to investors.
* It is possible to buy and sell securities at the same time and at any time.
* Securities are infinitesimally divisible.
* Securities do not pay dividends.

## Ito’s Calculus

Ito’s calculus is at the core foundation of the entire mathematical finance field. Without it the entire field would not be able to sustain calculations of any degree of accuracy. Thanks to the contributions given by Kyoshi Ito during the 1950s and the solution to integrating Brownian motion processes the field of mathematical finance has stepped up further and moved to develop exponentially in the last decades.

Let us start with the core process with which stochastic processes rely heavily on, Brownian motion. A Brownian motion is a stochastic process such that:

* is continuous almost surely and not differentiable

Given the probability measure and the filtered probability space which denotes the information available at time , we define a martingale , with , to be a stochastic process such that . That is the conditional expectation of any given next value of the process at time is equivalent to the present value of the same process at time .

An SDE, stochastic differential equation, is presented below in its general form:

where is the drift component, with being an integrable function, and being the random, or stochastic, component. More specifically, is a random process, differentiable using Ito’s rules, and that is why Ito’s discoveries are remarkably important to the field.

An Ito process is an adapted stochastic process that can be expressed as the sum of two integrals, with respect to a Brownian motion and with respect to time. Generally, the solution to a stochastic differential equation is an Ito process, which is shown below:

Moreover, Ito’s lemma states that for a given univariate drift-diffusion stochastic process, and an at least twice differentiable function , for being a stochastic process and , we have that the infinitesimal increment in is given by:

As an example, we consider the process , then the process satisfies the following stochastic equation:

Hence, we can use the relationship to find a solution to the stochastic process on , with initial condition :

## Pricing under the risk-neutral measure

If a probability measure is said to be risk-neutral, then the discounted price of the derivative process is a martingale. Under such probability measure, with constant interest rates , we have that the value of an option is equivalent to the expected value of its time discounted payoff at a given maturity conditional on the information set available today under the risk-free probability measure :

It is important to distinguish the probability measure , which is measure of convenience with respect to using the risk-neutral pricing, from , the true probability measure. In this paper we are using only the risk neutral probability measure as calculations need implications which are satisfied by the choice.

1. THEORY: A HISTORY OF VOLATILITY, FROM CONSTANT TO STOCHASTIC

## The Normal Model

The Normal model for pricing options was originally developed and published in 1900 by Louis Bachelier in his paper “The theory of speculation”. Even though it has not been used too often after the seventies due to the discovery of the Black pricing model, the Bachelier model served as a skeleton for what became the more utilized model thanks to Black and Scholes in 1973. The Normal model considers an economy in which stock dividend payments do not exist. Hence, the price of the stock is assumed to be normally distributed and follows the given Ornstein-Uhlenbeck process with solution:

Here, is the constant risk-free rate, is the stock price, is the time to maturity, is the constant normal volatility and is a standard Brownian motion. We will keep using the same variables as stated for the same purposes until further notice.

We now consider a European call option based on the same underlying . The closed-formula for the option price is derived under probability measure :

where follows a standard Normal distribution, all the other constants were explained previously.

To note that pricing with the above model works the same if we apply a shift on both the strike and asset price and . We will see later in the research how this can be useful for pricing options with non-positive rates.

This model gives nice closed-formulas for pricing vanilla options and it is very suitable for negative interest rates too. We will see later its most important application, that is in SABR and in more modern pricing models.

## The Black Model

In 1973, Black came up with a new, revolutionary model, which was set to define how the entire industry would price options and implied volatility from that moment onwards. The model was based on the original Normal model by Bachelier but with a few important changes.

As with the Normal model, the Black and Scholes assumed an ideal market, a riskless rate , stock prices following geometric Brownian motion, no dividends paid. Most importantly, whereas the Normal model assumed a normal distribution for asset prices, the Black model assume the underlying to be log-normally distributed. In addition, in the original paper, it was shown that it is possible to hedge positions using the put-call parity relationship for replicating a portfolio using options and the underlying. This, out of all the important features of the model, was the most attractive element, capturing the interest of risk managers and banks.

The key behind the famous model is the Black-Scholes partial differential equation, which describes the option over time and the relationship with the underlying, risk-free rate and volatility:

Let us have a more in depth look at the underlying structure of the model with respect to options. The premium for a call option with maturity and evaluated at time is given by the following:

where is the asset price, is the strike price, is a constant annual risk free-rate and is the Gaussian cumulative distribution function. Now, the model assumes that changes in the underlying asset prices satisfy the following stochastic differential equation with solution below:

What follows is the option value priced under probability measure and substituting with the Black-Scholes formula for pricing options:

where are specified above, is the expected value under the risk-free probability measure and is time to maturity.

## Implied volatility under the Black-Scholes model

As we have seen, under the Black-Scholes model are all constants, and therefore pricing options is the most straight forward, easy to compute and reliable process, at least up to a certain extent. In fact, the requiring assumptions have led practitioners to calculate implied volatility remarkably well for more than a decade since the original release of the paper, with a good degree of accuracy. Following the assumptions, asset price returns - meaning - follow a lognormal distribution with no fat nor heavy tails. Holding everything equal, we can retrieve the value of standard deviation of asset prices by simply reversing the equation and solving for . As shown below, we have that now sigma is a linear function of defined constants:

These assumptions worked well because of a combination of both the way that the market was behaving up to a market crash in 1987 (Black Monday) and the reliability that professionals in the field were giving to the model. As a matter of fact, professionals were not even considering the idea of dealing really with fat tailed distributions instead of Gaussians when modelling derivatives. Historically, credit to the model was mainly given to how market volatilities were moving, and with a good reason.

## Volatility Smiles and Skews

The put-call parity formula implies that the volatility of calls and puts does not differ when the Black-Scholes option price is matched to the market, that is when . Because we generally are used to see out-of-money and in-the-money options trading substantially above at-the-money options, the implied volatility for such options is also higher than the ATM ones. It is then fair to say that volatility changes with respect to moneyness (how far in-the-money or out-of-money the option trades at). This phenomenon is commonly referred to implied volatility smile.

This behavior carried on up until Black Monday’s market crash in October of 1987, when these smiles started to have different shapes. The OTM and ITM options tended to have unequal levels of implied volatilities, most specifically with the OTM options trading at a higher risk than ITM ones. This feature is commonly referred to as “implied volatility skew” and is what commonly we see in markets today, especially in equity markets.

Supposedly, a reason behind this is the large number of portfolio managers purchasing more OTM options than ITM for hedging purposes, therefore raising prices and volumes of these options more than the opposite counterparts.

## Volatility Surfaces

Together with volatility smiles, the term structure of implied volatility is also incredibly useful for assessing overall stability of the option. Whereas volatility smiles are showcasing the variability in volatility with respect to different strikes on options with same tenor, and we have seen that at-the-money options have smaller , term structures assess the future expectations of with respect to different maturities.

A volatility surface puts both sets of information together – volatility term structures and smiles - in a tridimensional chart where the x-axis is the moneyness of the option, y-axis being the maturities and the z-axis being volatility. With respect to the Black-Scholes model implied volatility surface does not perform well for longer maturities. This is due to its nature of keeping a constant term which consequently keeps the unrealistic “smile” shape all along for longer terms.

## Constant Elasticity of Variance and Dupire’s Local Volatility

With the need of a more sophisticated model, in 1994 Dupire, alongside Derman and Kani, brought in the field a real first academic step forward to analyzing modern volatility patterns. Assuming only minimal changes to the original model, he proposed a replacement of the constant with a deterministic function of time and stock price . The stock price now follows the following diffusion process:

with and r bring a constant risk-free rate. With the first paper being released, a series of debates arose to identify the strength and durability of the local volatility model. First and foremost, doubts were around the idea that there can be a single equation to match the implied volatility surface and therefore fit well for each volatility smile.

Firstly, the solution was found by Derman and Kani shortly after the release of the first paper by Dupire. The solution was shown by constructing an implied binomial tree, where the local volatility is calculated at each node in time and calibrated across strikes and expirations along with market data. Furthermore, from this data we can extract implied volatility surface and respective gradients with respect to prices and strikes. The results are then used to calibrate on the Dupire equation.

This is a major step forward for the field. In fact, we move from calculating implied volatility straight and only from option prices to a traceable, calibrated expression for local variance. From the following Dupire equation, we solve for :

The derivation of the formula can be found either by using a probabilistic approach or by using Fokker-Plank equation. Again, the most important benefit we have when pricing using Dupire’s Local Volatility model with respect to Black-Scholes is a greater precision in matching implied skews for all strikes in a market with no smiles. Now, issues with this model arise with implied variance for longer tenors. Whereas for short term calibration, smiles are registered fine by the model, but with longer maturities the effect continues which is highly unrealistic in a real-world situation. Longer maturities tend to have a flatter skew, which goes against the model’s implications. The necessity of a process that would flatten the curve enough for longer maturities to behave in a more realistic manner was very much felt. As stated by Hagan a few years later in the famous 2002 paper “Managing Smile Risk”, due to this contradiction between model and market, delta and vega hedges derived from the local volatility model can be unstable and may perform worse than naive Black-Scholes’ hedges.

## Stochastic Volatility and the Heston Model

With the rising concern of proving a good statistical match with volatility surfaces in option prices, in 1993 Heston contributed to field with a model that for the first time treated volatility as a stochastic process, in contrast with the previous deterministic (Dupire) and constant (Black Scholes) predecessors. Still being heavily used to this day for pricing options, the Heston stochastic volatility model is an extended version of the Black-Scholes model, with volatility following a CIR-process (see Rasmusson, 2008). The asset price now obeys the following diffusion process:

where is the instantaneous expected rate of return, is the volatility of variance , is the long term mean of the variance, is a positive constant indicating speed of mean reversion (how fast is variance approaching its mean value) and and are Brownian motions with being their correlation, typically negative, which is often called ‘leverage effect’. The later imposed Feller classification implies the following condition: . This is a necessary constraint for having a strictly positive variance .

The reason why the Heston model gained so much popularity among professionals is the existence of a closed-form solution that quickly obtains prices with any given parameter set . The closed-form solution is presented:

Again, the main motivation for using the Heston closed-form solution is to construct consistent smiles and skews that fit well with market data.

## SABR, stochastic alpha, beta, rho

The SABR model was first introduced by Hagan in his 2002 paper “Managing smile risk”. The main idea behind the model was to provide a better fit and a more effective formula to compute volatility in order to match volatility skews over different maturities on forward rates. For equity options, we do need to remember the relationship with forwards , where is the compounded present value of the asset under risk-neutral assumptions. The model follows the following stochastic process:

where is the forward price under the risk neutral measure at time t, and are Brownian motions with correlation given by , , which explains volatility skew curvature, and , which is the volatility of volatility, follow conditions .

The SABR model works with forward rates , whereas the Heston and previous models work with asset prices . This means we need to consider of the drift when pricing options using SABR and the relationship with stochastic interest rates when the asset does not pay dividends. The drift component is now missing in the forward stochastic equation. The variance of forward prices is now stochastic and does not have a mean reverting process. Since variance is now a function of both strike and time to maturity, we are expecting to have a closer fit to market data with regards to smiles and long-term maturities.

Since we are dealing with vanilla options, we can construct the SABR variant based on asset prices [Vlaming, 2011]. The model will now show a drift term:

Again, unlike in the Dupire or Black model, volatility is now a function of time. This means, as with the Heston model, that SABR will not only capture the smiles behavior across moneyness, but also changes in volatility across longer maturities. The formula for was originally discovered by Hagan et al. (2002). All things equal we have:

With f being the current forward price and K being the strike price of the option.

We then find the optimal values for the parameters – being arbitrarily chosen – and plug the volatility back into the Black-Scholes formula to get the price:

By doing this for each strike set maturities we end up with the SABR volatility surface which can be then used for pricing call options under the Black-Scholes model. Ultimately the SABR model works well especially when dealing with mid to long term tenors. In fact, a known pitfall is when working with short term maturities, where the match does not capture volatility skews or smiles close enough. Depending on the objective and type of situation we have, we can choose a different way to calibrate our model and set the right values for the volatility surface.

## SABR calibration

There are two main ways to calibrate the SABR parameters , , and . As a first step, is set arbitrarily to match with the condition . Now, it is common practice to set either or depending on the asset class and general matching priorities.   
When , the movement of becomes independent of the price itself. Given that a Brownian motion follows a normal distribution , then the forward price will be normally distributed with stochastic variance. In this case, the model takes the name of Normal SABR, as the assumptions match with the Bachelier Normal model we have seen previously.   
When follows a lognormal distribution, which is more in line with the CEV model. In our case we will be looking also at , which produces the CIR (Cox–Ingersoll–Ross) model. Calibration methods are dealt with by using two methods:

* First method: estimate , and directly when minimizing the sum of squared errors with market volatilities.
* Second method: calibrate and directly, and then find from , and ATM volatility .

For the purpose of this research, we will be using the first method only, as the differences in outputs are very small and not noticeable (See paper \* negative interest rates with SABR). After having specified arbitrarily, we want to minimize the sum of squared errors between the market volatilities and the SABR calibrated where and are the forward price and the strike price for a given option. Therefore, the minimizing equation is what follows:

where is the volatility in the market for strike . Of course, this works for a volatility smile at a given maturity, in order to construct a volatility surface, we would need to do that for each maturity and will therefore return different parameters for each smile. Finally, we use the generated parameters and to obtain to then plug into the Black-Scholes formula and get the option price.

As for the second method, the procedure for minimizing the objective function is the same, with the only difference that , which is now a function of and , is found in using ATM volatility and previously calibrated and . The following third order polynomial is solved with respect to , the solution takes the minimum value of the roots:

Following, the same objective function of the first method is minimized, with the only difference that now is a function of and .

It is important to note that since the variables are found trough the ATM volatility, the second method might be more useful if priority is to fit values to the market ATM smiles or skews. As stated before, since the difference is very mild and the second method would take a bit longer to compute, for the purpose of the research we will be going with the first method only.

## Normal SABR

For the Normal SABR model, we set , which simplifies the equations as follows:

As seen, the infinitesimally difference in forward prices in the first equation does not depend anymore on increments, which follows that the distribution is not symmetric anymore. Of course, the original Hagan’s formula can now be approximated further:

In this case, the at-the-money volatility can be found by solving:

## Shifted SABR

The complexity of working with the SABR model allowed for the first time to work with more realistic volatility curves. In modern times, the need of pricing under negative interest rates started to make a way in the field. Along with the first release of the SABR model for pricing derivatives, soon enough, doubts came along especially with respect to how to deal with negative rates. The Shifted SABR answers the question effectively by shifting the values of the forward and strike prices so to have enough range in the outputs and returning a close enough approximation. The model is shown below:

We now have that the original forward price is shifted by with . If the value of the shift is minimal, then it will not be affecting the model too much. Ultimately, the only difference is that we get and . Since the Bachelier model only depends on the initial difference of the strike with the forward price, the shift will not be intrusive for calculating price. Nevertheless, when pricing with the shift, it will influence the value of implied volatility. Hence, the model needs to be handled with care and shift cannot be extreme.

Benefits of the shifted variant of the SABR model are that the computation is rather quick, and the model follows the same exact structure as with the classic SABR model. The ending computation is the same, with the difference of adding the shift to prices and strikes. Therefore, because of the inner structure of the Normal model and the shift characteristic that does not affect pricing, the shifted SABR model can be used effectively with negative rates environments.

## Free Boundary SABR

In 2014 Hagan et al. proposed a new solution to the SABR stochastic differential equation, which is based on the discretization of the probability density function. Although being very similar to the original solution, this version is arbitrage-free by construction and allows pricing with low and negative interest rates. The original solution provided by Hagan et al. consisted in reducing the bidimensional SABR model to one dimension, for an easier computation. The solution relies on using the Fokker-Planck PDE under probability density :

With the Free Boundary version lies in the boundary condition and an implication on the formulas related to where we transform the value of to its absolute vale to keep signs positive:

Following, the process of discretization of the density function is done by Hagan et al. using Crank-Nicholson, which can lead to undesirable oscillations in the option prices.

In 2015, in order to make up for these issues, Le Floch et al. published a paper where they acknowledged major flaws around the discretization process of the PDF function for both the arbitrage-free and free-boundary variants of the SABR model. Instead, they proposed a set of mathematical methods of which TR-BDF2 and Lawson-Swayne stand out in terms of both speed and stability.

The key point is the choice of which, especially when dealing with long term contracts, results in an inaccurate discretization and therefore is inefficient. Le Floch et al. then provided a change of variable for that works while still preserving moments of the distribution:

Moreover, the partial differential equation for the free-boundary model is proposed and tested successfully with negative interest rates and boundary conditions. The new probability density function with the change of variable is given:

and are probability masses at and respectively. Then, the PDE for is given with boundary conditions:

Both Floch et al. and Hagan et al. claimed the new pdf with respect to to be normally distributed. Thanks toe new bounds and the structure of this new model, pricing under negative interest rates is not an issue anymore. Both the use of TR-BDF2 and the Lawson-Swayne solutions proposed by Le Floch et al. give very similar results. For the purpose of this paper, we will be using the QuantLib method ‘ql.flochkennedy’ from the SABR modules for calibrating volatility surfaces, which uses the TR-BDF2 method for computations.

## Mixture SABR

Another quite popular approach in modern times specifies the forward rate as the weighted sum of a zero-correlation SABR with a normal SABR. The formula was first introduced by Antonov et al. in 2015, with the goals of minimizing common pitfalls of the two models:

:

* is a zero-correlation free boundary SABR model with parameters
* is a zero-correlation free boundary SABR model with parameters
* is a random variable, independent of and , taking value of with probability and with probability .
* is the relationship with stock prices

Due to the linear combination of the free-boundary zero-correlation SABR and the nonzero correlation Normal SABR, the structure of the Mixture SABR allows for negative rates and has closed-form solution, just as much as the two independent models. The total number of parameters to estimate and calibrate to market data is seven, . Correlation in the FB model is set to and , which is an inner characteristic for both cases.

When dealing with at-the-money volatilities we want to use the following relation on initial stochastic volatility:

Antonov et al. then suggested that the probability parameter , which really defines the balance between the two models, can either be arbitrarily set or implied by the following relationship with an auxiliary parameter :

which implies that when this reduces the model to the free-boundary SABR, else when it reduces to the Normal SABR. An important proposition given in the same paper was to set parameters to small values in order to reduce singularity in the free-boundary model, namely, omitting the previously defined parameter , restrictions are as follow:

Overall, it can be said that the model successfully takes into consideration of previous models flaws and mitigate the issues with some new calibration processes. It provides for a closed-form solution and a good method for combining two models. Having said that, there are some major drawbacks mainly due to its weighting approach. The Mixture SABR is ultimately a very complex model, and practically difficult to grasp in its essence as it contains a weighted probabilistic solution and most general practitioners might not comprehend how it works. Also, despite a formula for the probability being proposed by Anonov et al., there is no real answer to how to calibrate it and why a mixture model would be more beneficial than individual models. Due to the weighted approach, it might cause jumps when simulating stock prices with Monte Carlo. A few more general issues with regards to mixture models are discussed in Piterbarg [2003].

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1. CONCLUSION

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

Appendix

Appendixes, if needed, appear before the acknowledgment.

Acknowledgment

The preferred spelling of the word “acknowledgment” in American English is without an “e” after the “g.” Use the singular heading even if you have many acknowledgments.

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