# Title for paper submitted to International Journal of Scientific and Research Publications

##### **Luca Somigli**

Department of Economics, Mathematics and Statistics

Birkbeck, University of London

Dissertation submitted in partial fulfillment of the requirements for the degree of

**Master of Science in Mathematical Finance**

Student Number: 13197480

Words count: XXXXXXXX

***Abstract*-** The Quantitative finance world has stepped up well since its last centuries beginnings. Since the beginning with Black’s formula in 1973, financial mathematicians, engineers, academics and practitioners have contributed to building models that more realistically identify the stochastic behavior of prices in the financial markets. Such behavior has changed drastically over the decades and since the times where volatility was assumed to be constant over time. The presence of the equity skew and volatility smiles, moving from a flatter, simpler kind, suggested the utilization of a new way of thinking, with volatility not being fixed anymore, but following a stochastic process.

This research paper wants not only to study, but also to compare the current trends in identifying forward looking implied volatility, starting with the Dupire model – fixing volatility over time and treating as a constant; the Heston model – volatility following a stochastic process with a mean reverting structure; the more modern SABR models – stochastic alpha, beta and rho with complex structures that have many variants according to specific situations. Follows an in-depth analysis and plotting of such models that will serve our understanding of their accuracy and intricacy with respect to calculating market implied volatility.

1. Introduction

T

his section is here to introduce the topics confronted by outlining the major discussions, divided by chronological order. Section 1 will do something, section 2 will follow with something else, etcetera.

Identify the constructs of a Journal – Essentially a journal consists of five major sections. The number of pages may vary depending upon the topic of research work but generally comprises up to 5 to 7 pages. These are:

1. Abstract
2. Introduction
3. Research Elaborations
4. Results or Finding
5. Conclusions

**In Introduction you can mention the introduction about your research.**

1. LITERATURE REVIEW

It's the foremost preliminary step for proceeding with any research work writing. While doing this go through a complete thought process of your Journal subject and research for it's viability by following means:

1. Read already published work in the same field.
2. Goggling on the topic of your research work.
3. Attend conferences, workshops and symposiums on the same fields or on related counterparts.
4. Understand the scientific terms and jargon related to your research work.
5. MATHEMATICAL DEFINITIONS

In this chapter we will be going through the main mathematical definitions over which topics will work on, starting with general efficient markets theory assumptions, Brownian motion processes, Ito’s calculus and pricing under a risk-neutral framework.

## General Assumptions

For the purpose of the research study and for the models used to work, we assume the following:

* Markets are ideal. No transaction fees, taxes, inflation and no restrictions to short sell.
* Markets are complete.
* The risk-free rate exists and is non-negative.
* Markets are efficient, there are no material or non-material non-public information. All information is available to investors.
* It is possible to buy and sell securities at the same time and at any time.
* Securities are infinitesimally divisible.
* Securities do not pay dividends.

## Ito’s Calculus

Ito’s calculus is at the core foundation of the entire mathematical finance field. Without it the entire field would not be able to sustain calculations of any degree of accuracy. Thanks to the contributions given by Kyoshi Ito during the 1950s and the solution to integrating Brownian motion processes the field of mathematical finance has stepped up further and moved to develop exponentially in the last decades.

Let us start with the core process with which stochastic processes rely heavily on, Brownian motion. A Brownian motion is a stochastic process such that:

* is continuous almost surely and not differentiable

Given the probability measure and the filtered probability space which denotes the information available at time , we define a martingale , with , to be a stochastic process such that . That is the conditional expectation of any given next value of the process at time is equivalent to the present value of the same process at time .

An SDE, stochastic differential equation, is presented below in its general form:

where is the drift component, with being an integrable function, and being the random, or stochastic, component. More specifically, is a random process, differentiable using Ito’s rules, and that is why Ito’s discoveries are remarkably important to the field.

An Ito process is an adapted stochastic process that can be expressed as the sum of two integrals, with respect to a Brownian motion and with respect to time. Generally, the solution to a stochastic differential equation is an Ito process, which is shown below:

Moreover, Ito’s lemma states that for a given univariate drift-diffusion stochastic process, and an at least twice differentiable function , for being a stochastic process and , we have that the infinitesimal increment in is given by:

As an example, we consider the process , then the process satisfies the following stochastic equation:

Hence, we can use the relationship to find a solution to the stochastic process on , with initial condition :

## Pricing under the risk-neutral measure

If a probability measure is said to be risk-neutral, then the discounted price of the derivative process is a martingale. Under such probability measure, with constant interest rates , we have that the value of an option is equivalent to the expected value of its time discounted payoff at a given maturity conditional on the information set available today under the risk-free probability measure :

where is the information set we have at the current time, and are the risk-free rate and the conditional expectation under the probability measure .

It is important to distinguish the probability measure , which is a measure of convenience with respect to using the risk-neutral pricing, from the probability measure , which is the true probability measure in the real world. In our research we are assuming probability under the risk free measure for all circumstances.

1. THEORY: A HISTORY OF VOLATILITY, FROM CONSTANT TO STOCHASTIC

## The Normal Model

The Normal model for pricing options was originally developed and published in 1900 by Louis Bachelier in his paper “The theory of speculation”. Even though it has not been used too often after the seventies due to the discovery of the Black pricing model, the Bachelier model served as a skeleton for what became the more utilized model thanks to Black and Scholes with their paper “The Pricing of Options and Corporate Liabilities”, written in 1973. The Normal model considers an economy in which stock dividend payments do not exist. The price of the stock is assumed to be normally distributed and follows an Ornstein-Uhlenbeck process:

Here, is the constant risk-free rate, is the stock price, is the time to maturity, is the constant normal volatility and is a standard Brownian motion.

We now consider a European call option based on the same underlying, the formula is: . The closed-formula for the option price is derived under probability measure :

where follows a standard Normal distribution, is the standard normal cumulative distribution function and is the respective density function.

It is safe to note that pricing with the above model works the same if we apply a shift on both the strike and asset price such that and . We will see later in the research how this can be useful for pricing options with non-positive rates.

This model gives nice closed-formulas for pricing vanilla options, and it is very suitable for dealing with negative interest rates. We will see later its most important application, that is in SABR and in more modern option pricing models.

## The Black Model

In 1973, Black came up with a new, revolutionary model, which was set to define how the entire industry would price options and implied volatility from that moment onwards. The model was based on the original Normal model by Bachelier but with a few important changes.

As with the Normal model, Black and Scholes assumed an ideal market, a riskless rate , stock prices following geometric Brownian motion, no dividends paid. Most importantly, whereas the Normal model assumed a normal distribution for asset prices, the Black model assumes the underlying to be log-normally distributed. In addition, in the original paper, it was shown that it is possible to hedge positions using the put-call parity relationship for replicating a portfolio of both options and the underlying asset. This, out of all the important features of the model, was the most attractive element, which captured the interest of risk managers, institutions and banks.

The key behind the famous model is the Black-Scholes partial differential equation, which describes the option over time and the relationship with the underlying, risk-free rate and volatility:

Let us have a more in depth look at the underlying structure of the model with respect to options. The premium for a call option with maturity and evaluated at time is given by the following:

where is the asset price, is the strike price, is a constant annual risk free-rate and is the Gaussian cumulative distribution function. Now, the model assumes that changes in the underlying asset prices satisfy the following stochastic differential equation with solution below:

What follows is the option value priced under probability measure and substituting with the Black-Scholes formula for pricing options we have:

where are specified above.

## Implied volatility under the Black-Scholes model

As we have seen, under the Black-Scholes model are all constants, and therefore pricing options is the most straight forward, easy to compute and reliable process, at least up to a certain extent. In fact, the requiring assumptions have led practitioners to calculate implied volatility remarkably well for more than a decade since the original release of the paper. Following the assumptions in the Black model, asset price returns - meaning - follow a lognormal distribution with no fat nor heavy tails. Holding everything equal, we can retrieve the value of the standard deviation of asset prices by simply reversing the equation and solving for . As shown below, we have that sigma is a linear function of defined constants:

These assumptions worked well because of a combination of both the way that the market was behaving and the reliability that professionals in the field were giving to the model. As a matter of fact, professionals were not even considering the idea of dealing with fat tailed distributions instead of Gaussians when modelling derivatives. Historically, credit to the model was mainly given to how market volatilities were moving, and with a good reason. The most profound flaw in the model is treating as a constant. Where, for options on the same underlying and with a given maturity, should be a constant no matter how in or out of the money the price is trading at, this is not true when dealing with the idea that stock prices do not follow Gaussian distributions but follow fat tailed ones. In essence, the risk of the underlying moving more than average is much higher than what the Black-Scholes model expected the market to do. As we will see, most modern models will take this into account.

## Volatility Smiles and Skews

The put-call parity formula implies that the volatility of calls and puts does not differ when the Black-Scholes option price is matched to the market, that is when . Because we generally are used to see out-of-money and in-the-money options trading substantially above at-the-money options, the implied volatility for such options is also higher than the ATM ones. It is then fair to say that volatility changes with respect to moneyness (how far in-the-money or out-of-money the option trades at). This behavior started out after the famous Black Monday market crash in the October of 1987, when implied volatility charts started having different shapes.

In recent times, especially after the numerous market crises we had in the past few decades, the OTM (out-the-money) and ITM (in-the-money) options tended to have unequal levels of implied volatilities, with the OTM options trading at a higher risk than ITM ones. This feature, which slightly detaches from the previous smiles, is commonly referred to as “implied volatility skew” and is what commonly we see in markets today, especially with equities.

Supposedly, a reason behind this is the large number of portfolio managers purchasing more OTM options than ITM for hedging purposes, therefore raising prices and volumes of these options more than the opposite side.

## Volatility Surfaces

Together with volatility smiles, the term structure of implied volatility is also incredibly useful for assessing overall stability of the option. Whereas volatility smiles are showing the variability in volatility with respect to different strikes on options with same tenor, term structures assess the future expectations of with respect to different maturities.

A volatility surface puts both these two sets of information together – volatility term structures and smiles - in a tridimensional chart where the x-axis is the moneyness of the option, the y-axis is the maturities, and the z-axis is volatility. With respect to the Black-Scholes model implied volatility surface does not perform well for longer maturities. This is due to its nature of keeping a constant term which consequently keeps the unrealistic “smile” shape all along for longer terms.

For facility of calculations and visualization we can define a volatility matrix which contains the implied volatilities by strike prices and maturities:

where is the time to time to expiry and is referring to the option strike price. We then have that is the implied volatility for the option with strike and the maturity . We will see how this will be useful when dealing with Quantlib’s use of matrices .

## Dupire’s Local Volatility Model

With the need of a more sophisticated model, in 1994 Dupire, alongside Derman and Kani, brought in the field a real first academic step forward to analyzing modern volatility patterns. Assuming only minimal changes to the original model, he proposed a replacement of the constant with a deterministic function of time and stock price . The stock price now follows the following diffusion process:

with , where is the instantaneous risk-free rate at time and is the time dependent continuous dividend yield, which in our case we can set up to zero since our initial assumptions state that stocks do not pay dividends, . With the first paper being released, a series of debates arose to identify the strength and durability of the local volatility model. First and foremost, doubts were around the idea that there can be a single equation to match the implied volatility surface and therefore fit well for each volatility smile.

Firstly, the solution was found by Derman and Kani shortly after the release of the first paper by Dupire. The solution was shown by constructing an implied binomial tree, where the local volatility is calculated at each node in time and calibrated across strikes and expirations along with market data. Furthermore, from this data we can extract the implied volatility surface and respective gradients with respect to prices and strikes. The results are then used to calibrate on the Dupire equation.

This is a major step forward for the field. In fact, we move from calculating implied volatility straight and only from option prices to a traceable, calibrated expression for local variance. From the following Dupire equation, we solve for :

When constructing the smile for a specific tenor we then apply the formula above with respect to the strikes set:

where is the strike and is the market value of the call for the maturity and the strike . We therefore compute the entire smile for each option with different strike prices and the same expiry period. We then do that for each tenor and finally construct the local volatility surface.

The derivation of the formula can be found either by using a probabilistic approach or by using the Fokker-Plank equation. Again, the most important benefit we have when pricing using Dupire’s Local Volatility model with respect to Black-Scholes is greater precision in matching implied skews for all strikes in a market with no smiles. Now, issues with this model arise with implied variance for longer tenors. Whereas for short term calibration, smiles are registered fine by the model, with longer maturities the effect continues, which is highly unrealistic in a real-world situation. Longer maturities tend to have a flatter skew, which goes against the model’s outputs. The necessity of a process that would flatten the curve enough for longer maturities to behave in a more realistic manner was very much felt. As stated by Hagan a few years later in the famous 2002 paper “Managing Smile Risk”, due to this contradiction between model and market, delta and vega hedges derived from the local volatility model can be unstable and may perform worse than naive Black-Scholes’ hedges.

## Stochastic Volatility and the Heston Model

With the rising concern of proving a good statistical match with volatility surfaces in option prices, in 1993 Heston contributed to the field with a model that for the first time treated volatility as a stochastic process, in contrast with the previous deterministic (Dupire) and constant (Black Scholes) predecessors. Still being heavily used to this day for pricing options, the Heston stochastic volatility model is an extended version of the Black-Scholes model, with volatility following a CIR-process [Cox–Ingersoll–Ross , see Rasmusson, 2008]. The asset price now obeys the following diffusion process:

where is the instantaneous expected rate of return, is the volatility of variance , is the long term mean of the variance, is a positive constant indicating speed of mean reversion (how fast is variance approaching its mean value) and and are Brownian motions with being their correlation, typically negative, which is often called ‘leverage effect’. The later imposed Feller classification implies the following condition: . This is a necessary constraint for having a strictly positive variance .

The reason why the Heston model gained so much popularity among professionals is the existence of a closed-form solution that quickly obtains prices with any given parameter set . The closed-form solution is presented:

The full derivation to this closed-form is shown in 1993 Heston’s paper where interest rate is stochastic, for the purpose of this paper, we shall keep it constant. Again, the main motivation for using the Heston closed-form solution is to construct consistent smiles and skews that fit well with market data. The goal of calibration under the Heston model is to minimize the distance between model prediction prices and actual option market prices. We have a total of five parameters to estimate and we want to choose the best fit that consistently replicates market prices. A good optimization scheme is relative error between market prices and model prices as shown below:

The calibration process is somewhat cumbersome as the objective function presents multiple local minimas, hence the entire process is highly dependent on the choice of the initial parameters [Bin, 2007]. Also, sometimes, even though the initial guess is critical, it might not be converging to a good fit. This complicates the issue even further. In order to work through the problem, a possible solution is to run the calibration multiple times with different initial guesses, until the best parameter set is found. This works but under the drawbacks of higher running times.

In our case, we shall be using the Levenberg–Marquardt algorithm for least-squares curve fitting on the calibration. The option prices data is directly downloaded from a financial data provider (barchart.com).

A note to say that there is a way to minimize times using global optimization methods, the drawback is to have a less accurate calibration which in most cases is not ideal [Goel et al. 2009] and convergence is also not always guaranteed.

## SABR, stochastic alpha, beta, rho

The SABR model was first introduced by Hagan in his 2002 paper “Managing smile risk”. The main purpose behind SABR was to provide a solution for matching volatility smiles over longer maturities and take into account of changes in interest rates. For equity options, we do need to remember the relationship with forwards , where is the compounded present value of the asset under risk-neutral assumptions. The SABR model follows the following stochastic process:

where is the forward price under the risk neutral measure at time t, and are Brownian motions with correlation , , which explains volatility skew curvature, and , which is the volatility of volatility, follow conditions .

The SABR model was originally intended to work with forward rates , whereas the Heston and previous models work with asset prices by definition. This means we need to take into account of the drift when pricing options using SABR and the relationship , that is when interest rates follow a stochastic process and stocks does not pay dividends. As it can be seen, the drift component is now missing from the forward stochastic equation. The variance of forward prices is now stochastic and does not have a mean reverting process. Since variance is now a function of both strike and time to maturity, we are expecting to have a closer fit to market data with regards to smiles and long-term maturities.

Since we are dealing with vanilla options, we can construct the SABR variant based on asset prices [Vlaming, 2011]. The model will now show a drift term:

Again, unlike in the Dupire or Black model, volatility is now a function of time. This means, just as with the Heston model, that SABR will not only capture the smiles behavior across moneyness, but also changes in volatility across longer maturities. The formula for was originally discovered by Hagan et al. (2002). All things equal we have:

With f being the current forward price and K being the strike price of the option.

We then find the optimal values for the parameters – being arbitrarily chosen – and plug the volatility back into the Black-Scholes formula to get the price:

By collecting volatility for each strike for options with same tenor and different strike, we end up with the volatility smile for that basket of options. We can then calculate smiles for each tenor and end up with the SABR volatility surface which can be then plugged in the Black-Scholes model for pricing call options. Ultimately the SABR model works well especially when dealing with mid to long term tenors. In fact, a known pitfall for the model is when working with short term maturities, where the match does not capture volatility skews or smiles close enough. Depending on the objective and asset class we have, we can choose a different way to calibrate our model and set the right values for the volatility surface.

## Backbone in SABR models

Hagan et al. (2002), when introducing their SABR model, identified the so called backbone which is drawn by these skews or smiles when the price of the forward rate changes. The inner behavior of this backbone seems to be solely dependent on the value of . With , as the value of decreases, the volatility smiles start to move along the backbone curve to the upper left side of the plot. Instead, with and the price of decreasing, the backbone seems to flatten as the smiles are moving to the left and along the x-axis. See below for a graphical reconstruction of this response.

FIGURE (BACKBONE Beta=0)

FIGURE (BACKBONE BETA=1)

The most important outcome of this is that whenever the price of the forward changes, the implied volatility curve shifts in the same direction. This is extremely useful and is one of the major advantages of the SABR model.

## SABR calibration

There are two main ways to calibrate the SABR parameters , , and . As a first step, is set arbitrarily to match with the condition . Now, it is common practice to set either or depending on the asset class and general matching priorities. When , the movement of becomes independent of the price itself. Given that a Brownian motion follows a normal distribution , then the forward price will be normally distributed with stochastic variance. In this case, the model takes the name of Normal SABR, as the assumptions match with the Bachelier Normal model we have seen previously. When , follows a lognormal distribution, which is more in line with the CEV model. In our case we will be looking also at , which produces the CIR model. In this case, the current level of the price is under a square root, and this will prevent the forward price to be negative. After having chosen a value for , the following calibration methods are followed:

* First method: estimate , and directly when minimizing the sum of squared errors with market volatilities.
* Second method: calibrate and directly, and then find from , and ATM volatility .

For the purpose of this research, we will be using the first method only, as the differences in outputs are very small and not noticeable [See paper \* negative interest rates with SABR]. After having specified arbitrarily, we want to minimize the sum of squared errors between the market volatilities and the SABR calibrated where and are the forward price and the strike price for a given option at a given maturity. Therefore, we need to minimize the following equation:

where is the volatility in the market for strike . Of course, this works for a volatility smile at a given maturity, in order to construct a volatility surface, we would need to do that for each maturity and will return different parameters for each smile. Finally, we use the generated parameters and to obtain , which we then plug into the Black-Scholes formula to get the option price.

As for the second method, the procedure for minimizing the objective function is the same, with the only difference that , which is now a function of and , is found in using ATM volatility and previously calibrated and . The following third order polynomial is solved with respect to , the solution takes the minimum value of the roots:

Following, the same objective function of the first method is minimized, with the only difference that now is a function of and .

It is important to note that since the variables are found trough the ATM volatility, the second method might be more useful if priority is to fit values to the market ATM smiles or skews.

## Normal SABR

For the Normal SABR model, we set , which simplifies the equations as follows:

As seen, the infinitesimally difference in forward prices in the first equation does not depend anymore on increments, which follows that the distribution is not symmetric anymore. In this case, the original Hagan’s formula can now be approximated further:

It follows that the at-the-money volatility can be found by solving the approximated equation:

Of course, when capturing implied volatility in the model, we can just proceed the same way we would do in the standard SABR model and minimize the sum of squared errors with the approximated formula for volatility:

## Shifted SABR

The complexity of working with the SABR model allowed for the first time to work with more realistic volatility curves, which better resembled real world scenarios. In modern times, the need of pricing under negative interest rates started to be a relevant issue. For that matter, the Shifted SABR answers the question effectively by shifting the values of the forward and strike prices so to have enough range in the outputs and returning a close enough approximation. The model is shown below:

We now have that the original forward price is shifted by with . If the value of the shift is minimal, then it will not be affecting the model too much. Since the model only depends on the initial difference of the strike with the forward price, and because this does not affect the formulas from the original SABR model whatsoever, the shift will not be intrusive for calculating prices. The only drawback is that when pricing with the shift, this will influence the value of implied volatility. Hence, the model needs to be handled with care and shift cannot be extreme.

The objective function is the same, with the only difference of using the new shifted parameters:

Benefits of the shifted variant of the SABR model are that the computation is rather quick, and the model follows the same exact structure as with the classic SABR model. The results are excellent at fitting the data since it inherits the structure from the classic SABR model. The ending computation is the same, with the only difference of adding the shift to prices and strikes. Therefore, because of the inner structure of the Normal model and the shift characteristic that does not affect pricing, the shifted SABR model can be used effectively with negative rates environments and is commonly used.

## Free Boundary SABR

In 2014 Hagan et al. proposed a new solution to the SABR stochastic differential equation, which is based on the discretization of the probability density function. Although being very similar to the original solution, this version is arbitrage-free by construction and allows pricing with low and negative interest rates. The original solution provided by Hagan et al. consisted in reducing the bidimensional SABR model to one dimension, for an easier and faster computation. The solution relies on using the Fokker-Planck PDE under probability density :

The innovative feature of Free Boundary version lies in the boundary condition and an implication on the formulas related to where we transform the value of to its absolute vale to keep signs positive:

Following, the process of discretization of the density function is done by Hagan et al. using Crank-Nicholson, which can lead to undesirable oscillations in the option prices.

In 2015, in order to make up for these issues, Le Floch et al. published a paper where they acknowledged major flaws around the discretization process of the PDF function for both the arbitrage-free and free-boundary variants of the SABR model. Instead, they proposed a set of mathematical methods of which TR-BDF2 and Lawson-Swayne stand out in terms of both speed and stability.

The key point is the choice of which, especially when dealing with long term contracts, results in an inaccurate discretization and therefore is inefficient. Le Floch et al. then provided a change of variable for that works while still preserving moments of the distribution:

Moreover, the partial differential equation for the free-boundary model is proposed and tested successfully with negative interest rates and boundary conditions. The new probability density function with the change of variable is given:

and are probability masses at and respectively. Then, the PDE for is given with boundary conditions:

Both Floch et al. and Hagan et al. claimed the new pdf with respect to to be normally distributed. Thanks to new bounds and the structure of this new model, pricing under negative interest rates is not an issue anymore. Both the use of TR-BDF2 and the Lawson-Swayne solutions proposed by Le Floch et al. give very similar results. For the purpose of this paper, we will be using the QuantLib method from the SABR modules for calibrating volatility surfaces, which uses the TR-BDF2 method for computations.

After having the model setup, we can proceed in calculating relative implied volatility. For that purpose, we can go through the approximation formula advised by Hagan et al. in their 2015 paper:

To construct a volatility smile we go through each strike for all options with same expiry time and calculate the relative where stands for the different strike price. For the volatility surface we can get the smile for each maturity and obtain a volatility matrix.

## Mixture SABR

Another quite popular approach in modern times specifies the forward rate as the weighted sum of a free-boundary SABR with a normal SABR. The formula was first introduced by Antonov et al. in 2015, with the goals of minimizing common pitfalls of the two models:

:

* is a zero-correlation free boundary SABR model with parameters
* is a nonzero correlation Normal SABR model with parameters
* is a random variable, independent of and , taking value of with probability and with probability .
* is the relationship with stock prices

Due to the linear combination of the free-boundary zero-correlation SABR and the nonzero correlation Normal SABR, the structure of the Mixture SABR allows for negative rates and has closed-form solution, just as much as the two independent models alone. The total number of parameters to estimate and calibrate to market data is seven, . Correlation in the FB model is set to and , which is an inner characteristic for both cases.

When dealing with at-the-money volatilities we want to use the following relation on initial stochastic volatility:

Antonov et al. then suggested that the probability parameter , which really defines the balance between the two models, can either be arbitrarily set or implied by the following relationship with an auxiliary parameter :

which implies that when , it reduces the model to a free-boundary SABR, else when it reduces to a Normal SABR. An important proposition shown in the same publication was to set parameters to small values in order to reduce singularity in the free-boundary model, namely, omitting the previously defined parameter , restrictions are as follow:

The value of the approximated normal volatility under the Mixture SABR model is given by the following:

Consequently, due to the restrictions fit, we successfully reduced the number of parameters for the calibration procedure . For that purpose, we have the same setup as with the standard SABR model, minimizing the sum of squared errors from the market data. We set arbitrarily, as we proceeded before:

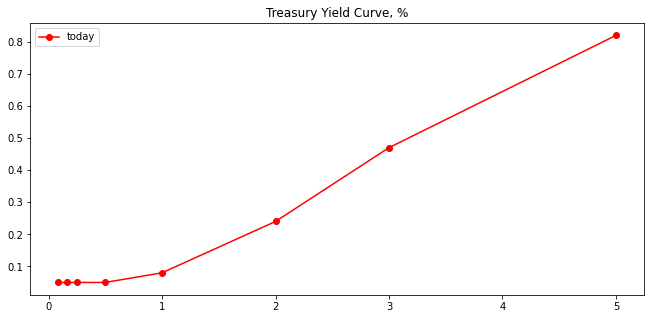
Overall, it can be said that the model successfully takes into consideration of previous models flaws and mitigate the issues with some new calibration processes. It provides for a closed-form solution and a good method for combining two models. Having said that, there are some major drawbacks mainly due to its weighting approach. The Mixture SABR is ultimately a very complex model, and practically difficult to grasp in its essence as it contains a weighted probabilistic solution and most general practitioners might not comprehend how it works. Also, despite a formula for the probability being proposed by Anonov et al., there is no real answer to how to calibrate it and why a mixture model would be more beneficial than individual models. Due to the weighted approach, it might cause jumps when simulating stock prices with Monte Carlo. A few more general issues with regards to mixture models are discussed in Piterbarg [2003].

1. ANALYSIS AND COMPARISON

## Data and Technology

The following chapters will go trough the analysis of models explained in the first part of the paper with a focus on how well they perform with respect to market data. Models which are going to be analyzed will be the Heston, the classic and normal SABR, the Shifted SABR, The Free-Boundary SABR and the mixture SABR. We shall be going through the analysis of each group independently and then compare ones fitting best with market values at the end of the chapter.

Spot rates used for the discounting and forwarding of the models in this paper are the US yield term structures recorded on the 10th of August 2021 provided by the Department of The Treasury of United States. The full term structure can be seen below:





Data has been collected from barchart.com on the 10th of August and 30th of August 2021 for call options on the SP500 index and the NASDAQ100. The data collects a range of 51 options with different strike prices for each tenor on the SP500 and a total of 32 call options for each of 12 tenors on the NASDAQ. The full data matrices for both indexes, along with market volatility rounded to is shown in the Appendix section.

Moreover, the data was checked for stability in the following days and assessed with stated ATM values. Entire dataset comprehends a wider range of strike prices. For the purpose of graphs and ease of calibration, only options with strike prices available and given implied volatility on each tenor were selected.

All models have been calibrated using Python and the QuantLib library for quantitative finance. The full notebook can be seen on the GitHub page <https://github.com/lucasomigli/SABR-volatility-models>. More regarding the python-QuantLib library and its documentation can be found on the official GitHub page or on the website <https://www.quantlib.org/>. All volatility smiles and surfaces in QuantLib are calculated using bicubic interpolation. SABR models are parametrized using the module and charts are plotted using the library. The code for each computation is available in the Appendix section. The machine on which models are run is a Ryzen 5 1600 6-Core Processor with 16 Gb of RAM.

## Black Volatility Surface

We start by introducing the implied volatility smiles for a range of tenors. These smiles are directly built from market data. Then, these smiles are used together in order to construct the volatility matrix. In Quantlib this is done using . Moreover, the resulting matrix is plugged into the method by setting the calculation date for the time when data was collected and the day count convention to . The method constructs a surface using linear interpolation. The resulting Black implied volatility smile and surface can be seen below along with market data.

FIGURE(Volatility Smiles for Black model)

FIGURE(Volatility Surface for implied Black model)

## Heston Model

We setup the initial conditions and run the calibration using and use the Levenberg-Marquardt algorithm for minimizing the objective function. This is generally advised by professionals for fast and effective calculations [see Cui, Rollin, Germano, 2016], and QuantLib uses this method by default in the Heston module. The calibration and plotting took 3.4 seconds. As stated in the previous chapter, the process consists in minimizing the difference between model price and market price. In order to assess how the algorithm works with the underlying data, we have run through two cases with different initial conditions over both datasets. This is to see how initial parameters result in different outputs. Parameters for the two runs are shown here:

FIGURE(heston model initial conditions)

We can see below output tables and charts referring to relative error between actual market price and model calibrated price as well as final parameters assigned through the calibration.

FIGURE (Individual models output Heston)

Clearly the first model is more accurate as the relative error behaves more accurately with respect to the second model. In both datasets, it seems that the first model fits better to market volatility especially with longer tenors and near ATM prices. Particularly, SPX presents spikes in Model 2 for deep out-of-money and in-the-money options with short tenors. The relative errors plot shows that where Model 1 is bound near zero, Model 2 presents more errors with change in moneyness. In fact, looking at the chart, we can see that model value and market value are very close functions for Model 1 and do not quite match well in Model 2. This is best explained by the average error which totals and for SPX, then and for NASDAQ. The final parameters output for both models and datasets are shown below.

FIGURE(parameters output Heston)

It is fair to say that the Heston model proves to be generally an accurate one. Nevertheless, tests confirm that sufficiently successful results are very much dependent on the initial parameters set.

Using the newly obtained parameters , we can compute smiles for each period and add the results together for building the respective volatility surface. Volatility smiles, surfaces and charts for an empirical comparison between relative errors for both cases, are shown below.

FIGURE (Surfaces, smiles and Relative error comparison)

Since the first Heston model has noticeably performed better, we shall use it for final comparisons at the end of the chapter.

## SABR Model

Using market data, we can compute the market implied volatility smile on each tenor, choose an arbitrary , apply the Hagan’s formula to retrieve volatility and parametrize the function to find the best fitting values for each smile on , as was shown in the previous chapter.

We use for calibration of the objective function. For each smile, we then use for inserting parameters into Hagan’s formula. After having parametrized all the smiles accordingly, we then use the volatility matrix onto to construct the SABR volatility surface.

It is important to note again that as the model becomes a normal SABR, as the asset prices follow a Bachelier Normal model. The results are shown in a table below for each different value of . A graph of values is also plotted to see how parameters values change with different tenors and different between values of .

FIGURE (SABR tables)

FIGURE (alpha, rho, nu comparison between models)

As we can see, values perform differently with different values of . One important thing to note is when , , and tend to stay close to zero with small to no variability. This implies a flat smile on each tenor and no curvature on the surface, which in essence results in a bad model. On the other hand, for the Normal SABR model and when values change more and do not seem to share similar behavior across datasets. This allows for more accurate surfaces. Just by looking at the errors plot, it seems that the model which underperforms the most is with , whereas the others seem to perform better, with being the closest fit on both SPX and NASDAQ. For this case, we won’t be comparing it to other models as clearly this is insufficient for our discovery of the most performing models to explain equity markets.

FIGURE(Error comparison across models on SABR)

To have a better visualization on each model it is necessary to show the respective volatility surface, which is constructed using , together with volatility smiles for a basket of tenors.

FIGURE(SABR volatility surfaces)

FIGURE(SABR volatility smiles comparison)

Results show that all models result in a similar surface, with the normal SABR having a spike on the global minimum point ATM prices, unlike the other two variants where the ATM volatility is smoothed out. As expected, volatility smiles are flat with and underperform across all maturities. On the other hand, The Normal model and SABR model with are more accurate but still generally divergent, the main difference being on ATM prices where one surface is smooth and the other is not. The slope of smiles, which tends to be higher with the Normal model, is another important difference with leads to the SABR performing considerably better with both datasets.

## Shifted SABR Model

For the shifted SABR model we run the calibration on the same data again choosing a value for our shift on both the strike and the forward value. In our case we used a shift of 50% of the original values. The calibration process runs the same way aside from this difference. The tables for the parameters on each case with are shown below together with charts.

FIGURE(charts for parameters over time)

For all models and datasets, we can see that all the parameters result in slightly more spikes in the calibration results but keeping the function behavior like what the original model presented. The errors plot show that that there is no real difference between the two shifted and non-shifted SABR models, which confirms its powerful characteristic when it comes to using it in case of negative prices. As expected, the volatility surfaces plot shows no large differences between models.

FIGURE(Volatility surfaces for Shifted sabr models)

Where surfaces look essentially the same as with the original non shifted-models, a closer look shows that a mild difference is visible in the slope of options when prices are distant from ATM value. Overall lines almost write onto each other, with the Normal model still not fitting well to data, whereas the model with fits the market better.

FIGURE(Volatility smiles for Shifted sabr models)

## Free-Boundary SABR Model

The free-boundary conditions are set up automatically using the method which relies on the TR-BDF2 algorithm for calibrating parameters which has been presented in the previous chapter. We calibrate the model in a similar way to what was done for the classic SABR model but minimizing the objective function using the Floch-Kennedy approach. Results are shown below with tables and graphs.

FIGURE(tables for parameters on floch-kennedy)

FIGURE(graphs for parameters on floch-kennedy)

We can see from the results that the normal Free-Boundary SABR presents generally more spikes and higher volatility for all parameters, especially with , which with the SPX option data reaches above 750 in value. The relative errors graph shows that the Normal model with underperforms with shorter expiries. Moreover, both models tend to the same results for mid to long term maturities.

We then construct volatility surfaces for both models by building the volatility matrix for each in and aligning smiles with tenors, which we then plug into the black variance surface . Resulting plots are seen below.

FIGURE(Volatility surfaces for floch-kennedy models)

As confirmed by the volatility smiles chart comparisons, the normal case performs badly with short term tenors. The biggest difference relies in the small “bump” in the surface for the Normal Free-Boundary model with mid maturities, when the calibration does not perform well. Nevertheless, models perform almost the same for longer tenors.

FIGURE(Volatility surfaces for floch-kennedy models)

## Mixture SABR Model

In the previous chapter, we have shown the process under which the Mixture SABR model is calibrated, following the formula for the implied volatility of an option at a given expiry . We first calibrate the Free-Boundary SABR model, set initial conditions, then minimize the objective function for each maturity set under the constrains explained in the previous chapter and finally choose the best fitting curves. The final outputs for the results are shown below together with a comparison plot of the model volatility and market surfaces.

FIGURE(surfaces of mixture sabr and black variance)

Unexpectedly, the model performs fine and better than the Normal SABR model, which we have seen underperforming, but volatilities are generally overstated, with smiles always plotting on the area above market curves. The reason behind volatilities not quite fitting the smiles is most likely due to balancing good fitting non-normal Free-Boundary smiles with the non-well performing zero-correlation Normal SABR model. From this we can argue that the Mixture model works best when both models calibrated together show already good results independently. Generally, any type of mixture model tends to not perform well as there is some space for errors due to the balancing process. When one of the two models do not perform well by itself, as in our scenario, it is then advisable to use a single model only.

In the next section we will perform a comparison of all the best fitting models to finally draw our conclusions.

## Comparison of SABR, Heston and Black Volatility models

As a conclusion for the analysis of the models we treated so far, we shall have a better look at the mean square error term for each model volatility with respect to the market volatility. Then, we shall plot the volatility smiles for a range of tenors and visualize how good of a fit each model is. Finally, we shall compare Black implied volatility and each model’s volatility surface.

FIGURE(VOLATILIY SMILES FINAL COMPARISON)

By looking at the models’ MSE along different tenors, we can clearly see that the classic SABR model is the finest, with the Free-Boundary SABR as a close second. The Normal Free-Boundary SABR does beat the non-normal variant on some tenors but fails to be accurate on a number of other maturities. On the other hand, the Normal SABR and the Mixture SABR share the worst performance out of the models analyzed.

It is important to note that the Mixture model does not present any values for the first maturities on SPX as otherwise the model would not have been able to calibrate. This is an important point with respect to this model, as it shows weaknesses that sometimes just cannot be resolved due to the nature of its calibration process.

The volatility surfaces for each model is plotted and shown in the Appendix.

1. CONCLUSION

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

Appendix

Appendixes, if needed, appear before the acknowledgment.

Acknowledgment

Firstly, I would like to thank my thesis supervisor, for helping me in the final format of the thesis and for responding with critical feedback when this was needed.

Secondly, I would also like to thank my mother and my brother who have helped indirectly by both being available and helping with their emotional support, especially during these unsure times of pandemic.

References

1. G. O. Young, “Synthetic structure of industrial plastics (Book style with paper title and editor),” in *Plastics*, 2nd ed. vol. 3, J. Peters, Ed. New York: McGraw-Hill, 1964, pp. 15–64.
2. W.-K. Chen, *Linear Networks and Systems* (Book style)*.* Belmont, CA: Wadsworth, 1993, pp. 123–135.
3. H. Poor, *An Introduction to Signal Detection and Estimation*. New York: Springer-Verlag, 1985, ch. 4.
4. B. Smith, “An approach to graphs of linear forms (Unpublished work style),” unpublished.
5. E. H. Miller, “A note on reflector arrays (Periodical style—Accepted for publication),” *IEEE Trans. Antennas Propagat.*, to be published.
6. J. Wang, “Fundamentals of erbium-doped fiber amplifiers arrays (Periodical style—Submitted for publication),” *IEEE J. Quantum Electron.*, submitted for publication.

Authors

**Author** – Luca Somigli, Birkbeck, University of London.

**Correspondence Author** – Author name, email address, alternate email address (if any), contact number.