# An empirical comparison for modern volatility models on commodity options using QuantLib and python.

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***Abstract*-** The Quantitative finance world has developed fast in the last decades and moved drastically for its century long beginnings. Since the beginning with Black’s formula in 1973, financial mathematicians, engineers, academics and practitioners have contributed to building models that more realistically identify the stochastic behavior of prices in the financial markets. Such behavior has changed drastically over decades and since the times where volatility was assumed to be constant over time. The presence of the equity skew and volatility smiles, moving from a flatter, simpler kind, suggested the utilization of a new way of thinking, with volatility not being fixed anymore, but following a stochastic process.

This research paper wants not only to study, but also to compare the current trends in identifying forward looking implied volatility, starting with the Dupire model – fixing volatility over time and treating as a constant; the Heston model – volatility following a stochastic process with a mean reverting structure; the more modern SABR models – stochastic alpha, beta and rho with complex structures that have many variants according to specific situations. Follows an in-depth analysis and plotting of such models that will serve our understanding of their accuracy and intricacy with respect to calculating market implied volatility.

1. INTRODUCTION

“Il secolo breve”, or the ‘short century’ is often the expression to which Italians appoint to the 20th century to indicate the fast advancements and developing that the world experienced over the period. And for finance and mathematics, as with many scientific fields, this was also an incredibly flourishing time. In fact, 1900 is the year where Louis Bachelier produced his major work “The theory of speculation”, where a fist application of Brownian motion was attributed to the movements of stock prices and the first steps towards the field of mathematical finance were done. A few decades and market crashes later, the development of the Black model was produced in 1973, which still to this day is used and thought across universities and academia around the world. Soon enough, as the world of business and finance changed drastically over the years, more mathematical theories and models were produced.

The main goal of this paper is to investigate how modern asset pricing models have developed and improved over the years and are trustful with respect to real world scenarios. In particular, the interests of this paper are to identify reliability of such models to explain implied volatility on commodities using call options market data. Moreover, in the analysis we shall go trough how much does implied volatility fit market black implied volatility. Models analyzed are the local volatility models, the Heston model and the more modern SABR models, which then comprehend popular variants such as the shifted SABR, the free boundary SABR and the mixture SABR models.

Our paper is divided in different chapters to better describe the research. This chapter described general topics covered in the paper and the overall structure. The second chapter covers mathematical definitions over which models are based. The third chapter goes trough a theoretical and historical explanation of the development of models across time together with a literature review of subjects. Chapter four goes first trough data, sources and technology used, then progresses into the analysis and results of models. Finally, the fifth and last chapter sums up, reviews results and concludes on findings.

1. MATHEMATICAL DEFINITIONS

In this chapter we will be going through the main mathematical definitions over which topics will work on, starting with general efficient markets theory assumptions, Brownian motion processes, Ito’s calculus and pricing under a risk-neutral framework.

## General Assumptions

For the purpose of the research study and for the models used to work, we assume the following:

* Markets are ideal. No transaction fees, taxes, inflation and no restrictions to short sell.
* Markets are complete.
* The risk-free rate exists and is non-negative.
* Markets are efficient, there are no material or non-material non-public information. All information is available to investors.
* It is possible to buy and sell securities at the same time and at any time.
* Securities are infinitesimally divisible.
* Securities do not pay dividends.

## Ito’s Calculus

Ito’s calculus is at the core foundation of the entire mathematical finance field. Without it the entire field would not be able to sustain calculations of any degree of accuracy. Thanks to the contributions given by Kyoshi Ito during the 1950s and the solution to integrating Brownian motion processes the field of mathematical finance has stepped up further and moved to develop exponentially in the last decades.

Let us start with the core process with which stochastic processes rely heavily on, Brownian motion. A Brownian motion is a stochastic process such that:

* is continuous almost surely and not differentiable

Given the probability measure and the filtered probability space which denotes the information available at time , we define a martingale , with , to be a stochastic process such that . That is the conditional expectation of any given next value of the process at time is equivalent to the present value of the same process at time .

An SDE, stochastic differential equation, is presented below in its general form:

where is the drift component, with being an integrable function, and being the random, or stochastic, component. More specifically, is a random process, differentiable using Ito’s rules, and that is why Ito’s discoveries are remarkably important to the field.

An Ito process is an adapted stochastic process that can be expressed as the sum of two integrals, with respect to a Brownian motion and with respect to time. Generally, the solution to a stochastic differential equation is an Ito process, which is shown below:

Moreover, Ito’s lemma states that for a given univariate drift-diffusion stochastic process, and an at least twice differentiable function , for being a stochastic process and , we have that the infinitesimal increment in is given by:

As an example, we consider the process , then the process satisfies the following stochastic equation:

Hence, we can use the relationship to find a solution to the stochastic process on , with initial condition :

## Pricing under the risk-neutral measure

If a probability measure is said to be risk-neutral, then the discounted price of the derivative process is a martingale. Under such probability measure, with constant interest rates , we have that the value of an option is equivalent to the expected value of its time discounted payoff at a given maturity conditional on the information set available today under the risk-free probability measure :

where is the information set we have at the current time, and are the risk-free rate and the conditional expectation under the probability measure .

It is important to distinguish the probability measure , which is a measure of convenience with respect to using the risk-neutral pricing, from the probability measure , which is the true probability measure in the real world. In our research we are assuming probability under the risk free measure for all circumstances.

1. THEORY AND LITERATURE REVIEW

## The Black-Scholes Option Pricing Model

### The Normal Model

The Normal model for pricing options was originally developed in 1900 by Louis Bachelier in his paper “The theory of speculation” [Bachelier, 1900]. Used for many decades, it served as a skeleton for what later became the most utilized model, the Black-Scholes [Black and Scholes, 1973]. In the normal model, the price of the stock does not pay dividends, it is assumed to be normally distributed and follows an Ornstein-Uhlenbeck process:

Here, is the constant risk-free rate, is the stock price, is the time to maturity, is the constant normal volatility and is a standard Brownian motion.

We now consider a European call option with payoff . The closed-formula for the option price using the Bachelier model is derived under probability measure :

The model produces nice closed-formulas for pricing vanilla options. Also, it is safe to note that pricing with the above model works the same if we apply a shift on both the strike and asset price such that and . We will see later in the research how this can be useful for pricing options with non-positive rates in SABR models.

### The Black Model

In 1973, Black developed the original theories by Bachelier and came up with a new, revolutionary model. As with the Normal model, Black and Scholes assumed an ideal market, a riskless rate , stock prices following geometric Brownian motion, no dividends paid. Most importantly, whereas the Normal model assumed a normal distribution for asset prices, Black-Scholes assumes the underlying to be log-normally distributed. Also, positions can be hedged using the put-call parity relationship for replicating a portfolio of both options and the underlying. Out of all the important features of the model, this was the most attractive element for risk managers, institutions and banks.

At the core of the model is the Black-Scholes partial differential equation, with and treated as constants.

From this, the premium for a call option with maturity evaluated at time is given by the following relationship.

The model assumes that changes in the underlying asset prices satisfy the following stochastic process.

Hence, the value of the option priced under probability measure yields the following.

## Implied Volatility

### Implied volatility under the Black-Scholes model

As we have seen, under the Black-Scholes model are all constants. Therefore, pricing options under Black-Scholes is a relatively simple process and allowed practitioners to calculate implied volatility remarkably well for more than a decade. Following the assumptions in the Black model, asset price returns - meaning - follow a lognormal distribution with no fat nor heavy tails. Holding everything equal, we can retrieve the value of the standard deviation of asset returns by simply reversing the equation and solving for .

Historically, credit to the model was mainly given to its simplicity and how market volatilities were moving during that time. The most profound flaw with respect to identifying volatility is treating as a constant. Where, for options on the same underlying and with a given maturity, should be a constant no matter how in or out of the money the price is trading at, this is not true when dealing with the idea that stock returns following fat tailed distributions as in real world scenarios. In essence, the risk of the underlying moving more than average is much higher than what the Black-Scholes model expected the market to do and Black implied volatility smiles resulted to be imprecise. As we will see, different theories developed to fix the issue.

### Volatility Smiles and Skews

The put-call parity formula implies that the volatility of calls and puts does not differ when the Black-Scholes option price is matched to the market, that is when . Because we generally are used to see out-of-money and in-the-money options trading substantially above at-the-money options, implied volatility for such options is higher. It is then fair to say that volatility is directly proportional to moneyness, hence the further the distance between ATM strike and option strike, the higher the volatility. This behavior started occurring after the famous Black Monday market crash in the October of 1987.

In recent times, OTM (out-the-money) and ITM (in-the-money) options tended to have unequal levels of implied volatilities, with OTM fluctuating more than ITM option prices. This feature is commonly referred to as “implied volatility skew” and is what commonly we see in markets today, especially with equities. Supposedly, the reason behind this being the increasing number of portfolio managers purchasing more OTM options than ITM for hedging purposes, thus resulting in raising prices and volumes of these options.

### Volatility Surfaces

Together with volatility smiles, the term structure of implied volatility is also incredibly useful for assessing overall stability of the option. Whereas volatility smiles are showing the variability in volatility with respect to different strikes on options with same tenor, term structures assess the future expectations of with respect to different maturities.

A volatility surface puts both these two sets of information together – volatility term structures and smiles - in a tridimensional chart where the x-axis is the moneyness of the option, the y-axis is the maturity, and the z-axis is implied volatility.

For a given underlying asset, we can define a volatility matrix , where T is the number of tenors and K the total number of strikes for each expiry.

where is the time to time to expiry and is referring to the option strike price. We then have that is the implied volatility for the option with strike expiring at time . We will see how this will be useful when dealing with for the use of matrices in volatility surfaces.

## Dupire’s Local Volatility Model

In 1994 Dupire brought in the field a real first step forward towards the analysis of modern volatility patterns [Dupire, 1994]. Assuming only minimal changes to the original model, he proposed a replacement of the constant with a deterministic function of time and stock price , hence now solving the Black-Scholes formula we have that the stock price follows a diffusion process.

is the instantaneous risk-free rate at time and is the time dependent continuous dividend yield, which in our case we can set up to zero. Hence and since our initial assumptions imply stocks not paying dividends. We then have that the forward price of a call option is given by the following.

In this case satisfies the Fokker-Planck equation. The mathematics above are more thoroughly explained in Dupire’s original paper. From the above relationships, we find Dupire’s equation.

This is a major step forward for the field as we move from calculating implied volatility straight and only from option prices to a traceable, calibrated expression for local variance. From the Dupire equation we solve for :

Now, the first real concern with the original Dupire resolution was to identify if there exists a single equation that matches implied Black volatility at each smile. The answer was found by Derman and Kani shortly after the release of the first paper by Dupire [Derman and Kani, 1994]. The solution consisted in using an implied binomial tree, where the local volatility is calculated at each node and calibrated across strikes and expirations along with market data. Furthermore, the local volatility surface can be extrapolated along with respective gradients w.r.t. prices and strikes. The results are then used to calibrate on the Dupire equation as we have seen above.

The most important benefit we have when pricing using Dupire’s formula is greater precision in matching implied skews for all strikes in a market with no smiles. With this the ease of calculation makes it a fast resolution for many circumstances.

On the other hand, issues with this model arise with implied variance for longer tenors. Whereas for short term calibration, smiles are registered fine by the model, with longer maturities the effect continues, which is highly unrealistic in a real-world situation. Longer maturities tend to have a flatter skew, which goes against what the model outputs. As stated by Hagan a few years later in the famous 2002 paper “Managing Smile Risk”, due to this contradiction between model and market, Delta and Vega hedges derived from the local volatility model can be unstable and may perform worse than naive Black-Scholes’ hedges [Hagan, 2002].

## Stochastic Volatility and the Heston Model

With the rising concern of proving a good statistical match with volatility surfaces in option prices, in 1993 Heston contributed to the field with a model that for the first time treated volatility as a stochastic process, in contrast with the previous deterministic (Dupire) and constant (Black Scholes) predecessors. Still being heavily used to this day for pricing options, the Heston stochastic volatility model is an extended version of Black-Scholes, with volatility following a CIR (Cox–Ingersoll–Ross) process [Rasmusson, 2008]. The asset price now obeys a diffusion process.

where is the instantaneous expected rate of return, is the volatility of variance , is the long term mean of the variance, is a positive constant indicating speed of mean reversion (how fast is variance approaching its mean value) and and are Brownian motions with being their correlation, typically negative, which is often called ‘leverage effect’. In addition, since volatility follows a CIR process, we need the Feller condition [Cox–Ingersoll–Ross 1985, Feller 1951]. This is a necessary constraint for having a strictly positive variance .

The reason why the Heston model gained so much popularity among professionals is the existence of a closed-form solution that quickly obtains prices with any given parameter set . The closed-form solution is presented.

and are probability-related quantities that are obtained using Fourier transforms as shown below. In the Heston model calibration parameters are which in order tare the initial value of the option, speed of the reversion process, the long-run mean, volatility of variance (which measures steepness of smiles), correlation between and .

The full derivation to this closed-form is shown in Heston’s paper “A closed-form solution for options with stochastic volatility with applications to bond and currency options” where interest rate is a stochastic process [Heston, 1993]. For the purpose of this paper, we shall keep interest rates constant. Again, the main motivation for using the Heston closed-form solution is to construct consistent smiles and skews that fit well with market data. For that, the calibration process under the Heston model proceeds to minimize the difference between market prices and model prediction prices. We have a total of five parameters to estimate and we want to choose the best fit that consistently replicates market prices.

The calibration process is somewhat cumbersome as the objective function presents multiple local minimas, hence the entire process is highly dependent on the choice of the initial parameters [Bin, 2007]. Different calibration methods can be used for the purpose. MrázekJan and Pospíšil have run calibrations using different procedures and concluded that even though the initial guess is critical, it might not be converging to a good fit [Mrázek and Pospíšil, 2017]. This complicates the issue even further. In order to work through the problem, a possible solution is to run the calibration multiple times with different initial guesses, until the best parameter set is found. This works but under the drawbacks of higher computational times and sometimes the best fit does not output a satisfiable result.

There is an alternative way to minimize times using global optimization methods. Nevertheless, this results in an imprecise calibration which in most cases is not ideal and convergence is also not always guaranteed [Goel et al. 2009].

## SABR Models

### Classic SABR Models

#### SABR, stochastic alpha, beta, rho

The SABR model was first introduced by Hagan in his work “Managing smile risk” [Hagan, 2002] as an extension of the CEV (constant elasticity of variance) and the Black model. It is a stochastic volatility model but, unlike the Heston model, it does not produce option prices directly. The main purpose behind SABR was to provide a solution for matching volatility smiles over longer maturities and take into account of changes in interest rates.

where is the forward price under the risk neutral measure at time t, and are Brownian motions with correlation , , which explains volatility skew curvature, and , which is the volatility of volatility, follow conditions .

The SABR model was originally intended to work with forward rates , whereas the Heston and previous models work with asset prices by definition. For equity options, we recall the relationship , where is the compounded present value of the asset under stochastic interest rates and no dividends are paid.

As it can be seen, the drift component is now missing from the forward stochastic equation. The variance of forward prices is now stochastic and does not have a mean reverting process. Since variance is now a function of both strike and time to maturity, we are expecting to have a more realistic fit to market data with regards to smiles with long-term maturities.

Since we are dealing with vanilla options, we can construct the SABR variant based on asset prices [Vlaming, 2011]. The model will now show a drift term:

Again, unlike in the Dupire or Black model, volatility is now a function of time. This means, just as with the Heston model, that SABR will not only capture the smiles behavior across moneyness, but also changes in volatility across longer maturities with greater accuracy. Hagan’s formula for , originally shown in his paper is the following.

With f being the current forward price and K being the strike price of the option.

We then find the optimal values for the parameters – being arbitrarily chosen – and plug the volatility back into the Black-Scholes formula to get the price.

By collecting volatility for each strike for options with same tenor and different strike, we end up with the volatility smile. We can then calculate smiles for each tenor and obtain the SABR volatility surface which can be then plugged in the Black-Scholes model for pricing call options. Overall, the SABR model successfully produces a result that explains the behavior of real world assets as the smile is generally pronounced for short-dated options and flattened for longer maturities.

#### Backbone in SABR models

When introducing the SABR model, a backbone was identified in the effects of changing the forward price in the calibration and resulting smiles. The inner behavior of the backbone seems to be solely dependent on the value of . With , as the value of decreases, volatility smiles move along the backbone curve to the upper left side of the plot. Instead, with and the price of decreasing, the backbone line seems to flatten as smiles move to the left and along the x-axis. See below for a graphical reconstruction of this response.

FIGURE (BACKBONE Beta=0)

FIGURE (BACKBONE BETA=1)

The most important outcome of this is that whenever the price of the forward changes, the implied volatility curve shifts in the same direction. This is extremely useful and is one of the major advantages of the SABR model.

#### SABR calibration

There are two main ways to calibrate the SABR parameters , , and . As a first step, is set arbitrarily to match with the condition . Now, it is common practice to set either or depending on the asset class and general matching priorities.

When , the movement of becomes independent of the price itself. Given that a Brownian motion follows a normal distribution , then the forward price will be normally distributed with stochastic variance. In this case, the model takes the name of Normal SABR, as the assumptions match with the Bachelier Normal model we have seen previously.

When , follows a lognormal distribution, which is more in line with the CEV model. In our case we will be looking also at , which produces a CIR model. In this case, the current level of the price is under a square root, and this will prevent the forward price to be negative. After having chosen a value for , the following calibration methods are used:

* First method: estimate , and directly when minimizing the sum of squared errors with market volatilities.
* Second method: calibrate and directly, and then find from , and ATM volatility .

For the purpose of this research, we will be using the first method only, as the differences in outputs are very small and not noticeable [Ivarsson, 2020]. After having specified arbitrarily, we want to minimize the sum of squared errors between the market volatilities and the SABR calibrated where and are the forward price and the strike price for a given option at a specified maturity. Therefore, we need to minimize the sum of squared errors between market and SABR implied volatility for each smile.

where is the volatility in the market for strike . The formula above works for a given smile. In order to construct a volatility surface, we would need to do the same for each maturity and return parameters for each smile. Finally, we use the generated parameters to obtain , which we then plug into the Black-Scholes formula to get the option price.

As for the second method, the procedure for minimizing the objective function is the same, with the only difference that , which is now a function of and , is found in using ATM volatility after calibrating and . The following third order polynomial is solved with respect to , the solution takes the minimum value of the roots:

Following, the same objective function of the first method is minimized, with the only difference that now is a function of and .

#### Pitfalls in SABR Models

Some implementations were added to the classical SABR model, including the ones by Paulot [Paulot, 2009] and Obloj [Obloj, 2008]. Most importantly, Obloj provided with a refinement to the original model that captured the flaws of incorrectly pricing options with low strikes and long maturities. In modern times, this is widely applied to SABR models and is known as the Obloj’s refinement which was tested to be an effective improvement to the model.

At a wider level, the main issue with Hagan’s proposed SABR model is generally reduced to its utilization in modern negative rates environments. Wu performed an in-depth analysis of SABR models and Obloj’s refinement using interest rate caps and found that resulting smiles were fitting market implied volatility well enough [Wu, 2008]. Nevertheless, this study did not include the Normal model, the shifted SABR and used only positive rates.

### SABR Extensions for Negative Rates

#### Normal SABR

For the Normal SABR model, we set , which simplifies the equations as follows:

If we solve the above process above, we get which means that the model follows a normal distribution and can be used for modelling options in a negative rates environment.

As seen, the infinitesimally difference in forward prices in the first equation does not depend anymore on increments, which follows that the distribution is not symmetric anymore. In this case, the original Hagan’s formula can be approximated further and so is the formula for ATM volatility. We can proceed the same way we would in the standard SABR model and minimize the sum of squared errors with the approximated formula for volatility.

#### Shifted SABR

Working with the SABR model allowed to work with more realistic volatility curves, which better resembles real world scenarios. In modern times, the need of pricing under negative interest rates started to be a relevant issue. The Shifted SABR answers this needs effectively by shifting values of the forward and strike prices by so to have enough range in the outputs to return a positive distribution of prices. The model is shown below:

We now have that the original forward price is shifted by with . The value of can be either calibrated as an additional parameter in the SABR model or fixed prior to calibration. Since the model only depends on the initial difference of the strike with the forward price, and because this does not affect the formulas from the original SABR model whatsoever, the shift will not be changing option prices.

The only drawback is that when pricing with the shift, this does lightly influence the value of implied volatility. Hence, the model needs to be handled with care and shifts cannot be extreme. The objective function remains the same, with the only difference of using the new shifted parameters:

Benefits of the shifted variant of the SABR model are that the computation is rather quick, and the model follows the same exact structure as with the classic SABR model. The results are excellent at fitting the data since it inherits the structure from the classic SABR model. The ending computation is the same, with the only difference of adding the shift to prices and strikes. Therefore, because of the inner structure of the Normal model and the shift characteristic that does not affect pricing, the shifted SABR model can be used effectively with negative rates environments.

#### Free Boundary SABR

In 2015 Antonov proposed a new solution to the SABR stochastic differential equation [Antonov, 2015], which is based on the discretization of the probability density function of forward rates which are assumed to follow a free-CEV [Brecher and Lindsay, 2010]. s

Although being very similar to the original solution, this version is arbitrage-free by construction and allows pricing with low and negative interest rates. In fact, allows is defined with negative rates whereas in the original model is not. The original solution consisted in reducing the bidimensional SABR model to one dimension, for an easier and faster computation [Antonov, 2015]. The Fokker-Planck PDE is used under probability density :

The innovative feature of Free Boundary version lies in the boundary condition and an implication on the formulas related to where we transform the value of to its absolute vale to keep signs positive:

Following, the process of discretization of the density function is done using Crank-Nicholson, which can lead to undesirable oscillations in the option prices.

In 2015, in order to make up for these issues, Le Floch et al. published a paper [Le Floch et al. 2015] where they acknowledged major flaws around the discretization process of the PDF function for both the arbitrage-free and free-boundary variants of the SABR model. Instead, they proposed a set of mathematical methods of which TR-BDF2 and Lawson-Swayne stand out in terms of both speed and stability [Lawson and Swayne, 1976].

The key point is the choice of which, especially when dealing with long term contracts, results in an inaccurate discretization and therefore is inefficient. Therefore, a change of variable for is proposed, while still preserving moments of the distribution:

Moreover, the partial differential equation for the free-boundary model is proposed and tested successfully with negative interest rates and boundary conditions. The new probability density function with the change of variable is given:

and are probability masses at and respectively. Then, the PDE for is given with boundary conditions:

Both Floch et al. and Hagan et al. claimed the new pdf with respect to to be normally distributed. Both the use of TR-BDF2 and the Lawson-Swayne solutions proposed by Le Floch et al. give very similar results [Le Floch et al. 2015].

After having the model setup, we can proceed in calculating relative implied volatility. For this purpose, we can go through the approximation formula advised by Hagan et al. in their 2015 paper:

#### Mixture SABR

Another quite popular approach in modern times specifies the forward rate as the weighted sum of a free-boundary SABR with a normal SABR. The formula was first introduced by Antonov et al. [Antonov et al. 2015], with the goals of approximating a solution between the two models.

:

* is a zero-correlation free boundary SABR model with parameters
* is a nonzero correlation Normal SABR model with parameters
* is a random variable, independent of and , taking value of with probability and with probability .
* is the relationship with stock prices

Due to the linear combination of the free-boundary zero-correlation SABR and the nonzero correlation Normal SABR, the structure of the Mixture SABR allows for negative rates and has closed-form solution, just as much as the two independent models alone. Seven parameters are estimated and calibrated, . Correlation in the FB model is set to and , which is an inner characteristic for both cases.

When dealing with at-the-money volatilities we want to use the following relationship on initial stochastic volatility.

Antonov et al. suggested that the probability parameter , which really defines the balance between the two models, can either be arbitrarily set or implied by the following relationship with an auxiliary parameter .

When , model reduces to a free-boundary SABR, else when it reduces to a Normal SABR. An important proposition shown in the same publication was to set parameters to small values in order to reduce singularity in the free-boundary model, namely, omitting the previously defined parameter , restrictions are as follow.

The value of the approximated normal volatility under the Mixture SABR model is given by the following.

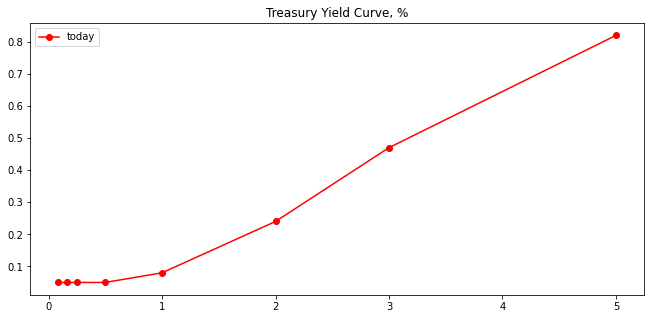
Consequently, due to the restrictions fit, we successfully reduced the number of parameters for the calibration procedure . For that purpose, we have the same setup as with the standard SABR model, minimizing the sum of squared errors from the market data. We set arbitrarily, as we proceeded before:

Overall, it can be said that the model successfully takes into consideration of previous models flaws and mitigate the issues with some new calibration processes. It also provides for a closed-form solution and a valid method for combining two models. Having said that, there are some major drawbacks mainly due to its weighting approach. The Mixture SABR is ultimately a very complex model, and practically difficult to grasp in its essence as it contains a weighted probabilistic solution and most general practitioners might not comprehend how it works. Also, despite a formula for the probability being proposed by Antonov, there is no real answer to how to calibrate it and why a mixture model would be more beneficial than individual models. Due to the weighted approach, it might cause jumps when simulating stock prices with Monte Carlo. A few more general issues with regards to mixture models are discussed in Piterbarg [2003].

1. ANALYSIS AND COMPARISON

## Data and Methodology

The following chapters will go through the analysis of models explained in the first part of the paper with a focus on how well they perform with respect to market implied volatility. Analyzed models are the Heston, the classic and normal SABR, the Shifted SABR, The Free-Boundary SABR and the mixture SABR. We shall be going through the analysis of each group independently and at the end of the chapter we shall compare models together to discuss which works best in explaining market implied volatility.

Spot rates used for the discounting and forwarding of the models are the US yield term structures recorded on the 30th of August 2021 provided by the Department of The Treasury of the United States. The full term structure can be seen below:



Call options chains data has been collected from barchart.com on the 30th of August 2021 on Gold, Silver and Coffee. Data consists of options on 17 different strike prices and 12 tenors for a total of 204 options on Gold, 18 strikes and 10 tenors for a total of 180 options on Silver, and finally 20 strikes, 7 tenors for options on Coffee for a total of 140. The full option chain data for the commodities options, along with market volatility rounded to is shown in the Appendix.

Moreover, upon collection, data was checked for stability in the following days and assessed with stated ATM prices. The full dataset comprehends a wider range of strike prices, but for the purpose of graphs and ease of calibration (mainly so to have valid inputs for calculations on ) only options on strikes with market volatility data available on each tenor were selected.

All models have been calibrated using Python and QuantLib, a library used for quantitative financial modelling. The full notebook can be viewed and downloaded on GitHub at <https://github.com/lucasomigli/SABR-volatility-models>. More regarding the QuantLib port for python and its full documentation can be found on the official developers’ GitHub page or on the website <https://www.quantlib.org/>. All volatility smiles and surfaces in are calculated using bicubic interpolation. SABR models are parametrized using the module and charts are plotted using the library. The code for each computation is available in the Appendix section. The machine on which models are run is a Ryzen 5 1600 6-Core Processor with 16 Gb of RAM.

## Analysis

### Black Volatility Surface

We start by introducing the implied volatility smiles for a range of tenors. These smiles are directly built from market data. Then, these smiles are used together in order to construct the volatility matrix. In this is done using . Moreover, the resulting matrix is plugged into the method by setting the calculation date for the time when data was collected and the day count convention to . This method constructs a surface using linear interpolation (although other ways are possible, such as or and more). The resulting Black implied volatility smile and surface can be seen below along with at-the-money market data.

FIGURE(Volatility Smiles for Black model)

FIGURE(Volatility Surface for implied Black model)

### Local Volatility Model

As explained previously, the Local Volatility Model builds by solving Dupire’s equation using implied volatility and market data. In , this is done by using the related process and plugging as arguments the yield term structure, the dividend term structure and the spot rate. As for our case, we assume no dividends are paid and hence an empty dividend term structure object was passed.

While we know that the Local Volatility Surface holds problems as with being a stochastic process in volatility and necessarily will not get the right prices at each strike, there are a few problems with the Dupire calibration function inherently in . In fact, the library currently does not support a constant calibration of parameters and the process requires the second derivative w.r.t the strike , which is a problem to compute with discrete data. So, in order to get as accurate as possible, we would need to use cubic spline interpolation and obtain valid enough output, but by doing so the calibration error increases and the algorithm returns an error as it does not find values allow for a smooth calibration. To cope with this, I have used the alternative method which allows less rigidity and gives more space for errors with the cost of a non-linear interpolation, which most likely results in an unrealistic surface. We will not use this model for our final comparison, but the results are yet important to analyze and discuss.

FIGURE(Local Volatility Surface)

Surprisingly, the model does its best job at fitting the data and in most cases, it performs fine, but this comes at a cost. Spikes are present on the entire fit and the ‘smile’ output is completely unrealistic, leaving us with a model that cannot be used for any serious setting. The area requires in-depth research that could shed a light on how to use a more efficient smoothing technique or to construct a more functional algorithm in QuantLib.

### Heston Model

We setup the initial conditions and run the calibration using and use the Levenberg-Marquardt algorithm for minimizing the objective function. Since this is generally advised by professionals for faster and, in most cases, accurate calculations [see Cui, Rollin, Germano, 2016], QuantLib uses this method by default in its module. The calibration and plotting took 3.4 seconds. As stated in the previous chapter, the process consists in minimizing the difference between model price and market price. In order to assess how the algorithm works with the underlying data, we have run through two cases with different initial conditions. This is to see how initial parameters result in different outputs. Parameters for the two runs are shown here:

FIGURE(Heston model initial conditions)

We can see below output tables and charts referring to relative error between actual market price and model calibrated prices as well as final parameters assigned through the calibration.

FIGURE (Individual models output Heston)

Clearly, in most cases the first model is more accurate as the relative error behaves more accurately with respect to the second model. In particular, the first model fits best especially with longer tenors. With Coffee, we have tried calibrating with different parameters multiple times, with the output only changing slightly and not enough for a notice, which means there are fewer solutions to the calibration process and any initial condition will lead to the same result. The relative errors plots show that where Model 1 is bound near zero, Model 2 presents more errors with change in moneyness. Remembering that the Heston model is calibrated on option prices - and not on the volatility RMSE as it is with SABR - we can see that model values and market values for Model 1 behave similarly whilst they do not quite match in Model 2. This is best explained by the average error which totals and for Silver, and for Gold and and for Coffee. As expected, the difference is minimal with Coffee data.

Of course, the selection process for the initial conditions can be automated by running an additional optimization process using . We run a series of Heston calibrations starting with different initial conditions (and appropriate constrains, as explained previously) and find the ones returning the lowest error . Computational time is then increasing depending on the quality and quantity of data.

The final parameters output for both models and datasets are shown below.

FIGURE(parameters output Heston)

We conclude that the Heston proves to be generally an accurate model, especially for pricing options with mid to long tenors. Nevertheless, tests confirm that sufficiently successful results are very much dependent on the initial parameters set and time for calibration can be excessively long depending on data. Using the newly obtained parameters , we can compute smiles for each period and add results together for building volatility surfaces. Volatility smiles, surfaces and charts for an empirical comparison between relative errors are shown below. Since the first Heston model has noticeably performed better, we shall use it for final comparisons at the end of the chapter.

FIGURE (Surfaces, smiles and Relative error comparison)

### SABR Model

We use to calibrate the root mean squared error function. For each smile, we use the fitting parameters into Hagan’s formula and solve using . After having parametrized all the smiles accordingly, we compute the volatility matrix and plug it into to construct the SABR volatility surface.

Again, as the model becomes a normal SABR and asset prices are assumed to follow a Gaussian distribution. The results are shown in the table below. A graph also shows how parameters change with different tenors and changing the value of .

FIGURE (SABR tables)

FIGURE (alpha, rho, nu comparison between models with errors)

As we can see, it is interesting how generally parameters behave similarly over time across datasets with same . In fact, is generally a flat looking function when plotted across time, seems to be increasing with time on average, is a decreasing function of time. The biggest exception comes with Coffee data, where parameters behave in a drastically different way than with the other models, and somehow result in a constant underperformance across the entire set of tenors. An important consideration to note is that when , tends to stay close to zero with small to no variability, whereas with the calibration results in higher values and variability with the parameter hitting more than 250 in the case of Gold. The parameters calibration on for the Normal SABR on Coffee would imply a sharper smile on each tenor and a higher slope on each smile, which in essence results in a bad model. Similarly, spikes on for Gold would indicate sharp smiles on some (not all) tenors. Just by looking at the errors plot, it seems that the model which underperforms the most is with , especially on Coffee and Gold, whereas Normal model and seem to perform significantly better.

The volatility surfaces are then plotted using , together with volatility smiles for a basket of tenors.

FIGURE(SABR volatility surfaces)

FIGURE(SABR volatility smiles comparison)

As expected, SABR models with are more accurate with a very light difference with respect to slope and curvature of smiles. The sharpness of smiles is the effect to why the Normal model performs bad on both Coffee and Gold. In particular, the surface on Coffee returns with a V looking shape across the entire plot which underperforms the most around ATM prices.

### Shifted SABR Model

For the shifted SABR model we run the calibration on the same data again choosing a value for our shift on both the strike and the forward value. In our case we used a shift of 50% of the original values. The calibration process runs the same way aside from this difference. The tables for the parameters on each case with different values are shown below together with charts.

FIGURE(charts for parameters over time)

For all models and datasets, we can see that all the parameters result in a shift to the output function over time, while keeping the original model behavior. The errors plot show that that there is no large difference between the two shifted and non-shifted SABR models. In particular, it can be seen that the largest differences in behavior of smiles are shown in the Normal SABR model, where the shift does change drastically, fitting well the data for some tenors, whilst still presenting the classic “V shape” for other tenors.

As expected, the volatility surfaces plot shows no large differences between models.

FIGURE(Volatility surfaces for Shifted sabr models)

FIGURE(Volatility smiles for Shifted sabr models)

### Free-Boundary SABR Model

The free-boundary conditions are set up automatically using the method which relies on the TR-BDF2 algorithm for calibration which has been presented in the previous chapter. We setup the model in a similar way to what was done for the classic SABR model, with the difference of minimizing the objective function using the Floch-Kennedy approach. Results are shown below with tables and graphs.

FIGURE(tables for parameters on floch-kennedy)

FIGURE(graphs for parameters on floch-kennedy)

We can see from the results that the normal Free-Boundary SABR presents generally more spikes and a higher range of the output for all parameters, especially with where on Gold data where it bounces from around 100 to 150 in value. The relative errors graph shows that the Normal model with underperforms in most cases and tenors.

We construct volatility surfaces for both models by building the volatility matrix for each in and aligning smiles with tenors, which we then plug into the black variance surface . Resulting plots are seen below.

FIGURE(Volatility smiles for floch-kennedy models)

As confirmed by the volatility smiles chart comparison, the normal case performs badly with short term tenors. The biggest difference relies in the small “bump” in the surface for the Normal Free-Boundary model with mid to long term maturities, when the calibration does not perform well. An exception is with data on Silver options, where the two model behave similarly and fit well for all tenors to implied volatility smiles.

FIGURE(Volatility surfaces for floch-kennedy models)

### Mixture SABR Model

In the previous chapter, we have shown the process under which the Mixture SABR model is calibrated, following the formula for the implied volatility of an option at a given expiry . We first calibrate the Free-Boundary SABR model, set initial conditions, then minimize the objective function for each maturity set under the constrains explained in the previous chapter and finally choose the best fitting curves. The final outputs for the results are shown below together with a comparison plot of the model volatility and market surfaces.

FIGURE(surfaces of mixture sabr and black variance)

The model output results in overstated volatilities, with smiles always plotting on the area above market curves. The reason behind volatilities not quite fitting the smiles is most likely due to balancing good fitting non-normal Free-Boundary smiles with the non-well performing zero-correlation Normal SABR model. As it can be seen by the error graphs over time, the Normal SABR has mostly an increasing RMSE whereas we see aa decreasing RMSE for the Free Boundary model. In such way, the final formula for the model turns out to increase the unbalanced error from the Normal SABR resulting in a bad fitting Mixture surface.

From this we can argue that the Mixture model works best when both models calibrated together show already good results independently. Generally, any type of mixture model tends not to perform well as errors present in each model are assimilated into the balancing process of the Mixture model, hence failing to calibrate well to the market. When one of the two models do not perform well by itself, as in our scenario, it is then advisable to use a single model only.

In the next section we will perform a comparison of all the best fitting models to finally draw our conclusions.

## Comparison

As a conclusion to our analysis, we shall inspect the root mean square error term for each model volatility with respect to the market and over time. Then, we shall plot volatility smiles for a range of tenors along with surfaces to analyze the fit. Finally, after we comment on the distribution of implied volatility across tenors and look at computational time for each model, we discuss results on our findings.

### Comparison of SABR, Heston and Black Volatility models

FIGURE(VOLATILIY SMILES FINAL COMPARISON)

By looking at the chart, we can clearly distinguish two groups that perform with results. These are formed by Gold and Coffee, where the Normal SABR and Mixture SABR are performing the worst on average, then Silver, where the Mixture and the Heston model have larger RMSE on average across most tenors. The other cases, classic non-normal and Free Boundary SABR models perform generally better across all datasets except for the Normal Free Boundary SABR that performs poorly on Coffee options data, especially for mid to long maturities. A table with average implied volatility RMSE on each commodity is shown below for further explaining overall performance.

FIGURE(MEAN performance of models for each commodity)

To further assess our results, we shall present the same errors chart showing top performing models. That is with the exclusion of Mixture SABR, Normal SABR and/or Heston model for each commodity. In addition, the shifted SABR is plotted for a wider comparison.

FIGURE(volatility smiles comparison for top performance)

FIGURE (MEAN performance of top performing models)

As stated before, whereas the Heston model is an outlier as it performs well mostly as time to maturity increases, most models behave similarly across all tenors. The shifted SABR does not perform too differently from the non-shifted counterparts. It is then a common trend for processes to perform best with longer maturities as shown in the graph. Moreover, it is important to note that there are outlier results in the calibration, the best example being SABR for Gold on options maturing on the 23rd November 2021. This might be due to computational error on modules parametrization process or even from the module . To understand better nonlinear results and outliers, the same data should be tested under different libraries and/or algorithms and compared.

Overall, by looking at the average error table we can see that Free Boundary models result in the smallest RMSE and hence show the best fit to market data.

### Implied Volatility Density

We shall now have a look at the probability density functions

### Computation time

Before commenting results, let us have a comparison of computational time across models, that is looking at the time it takes to the algorithm to calibrate and output results. A comparison table is shown below.

FIGURE(Computational time tables)

From a computation time standpoint, we clearly see that the all the SABR models do stand out more in comparison to the others. The Mixture model is the one taking longer to calibrate with approximately double the time of the Normal SABR or the Free Boundary SABR, and this is expected since it does involve calibrating the two processes together.   
On the one hand, the Heston model scores badly aside when used for Gold data, where it stands out with the fastest computation time. On the other hand, the calibration time for the Heston model varies, as said before, with respect to both initial conditions and quality or quantity of data, which means that computation time is generally understated in our case. And the real values are much larger than what we have shown here.

1. CONCLUSION

A conclusion section is not required. Although a conclusion may review the main points of the paper, do not replicate the abstract as the conclusion. A conclusion might elaborate on the importance of the work or suggest applications and extensions.

Appendix

Appendixes, if needed, appear before the acknowledgment.

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