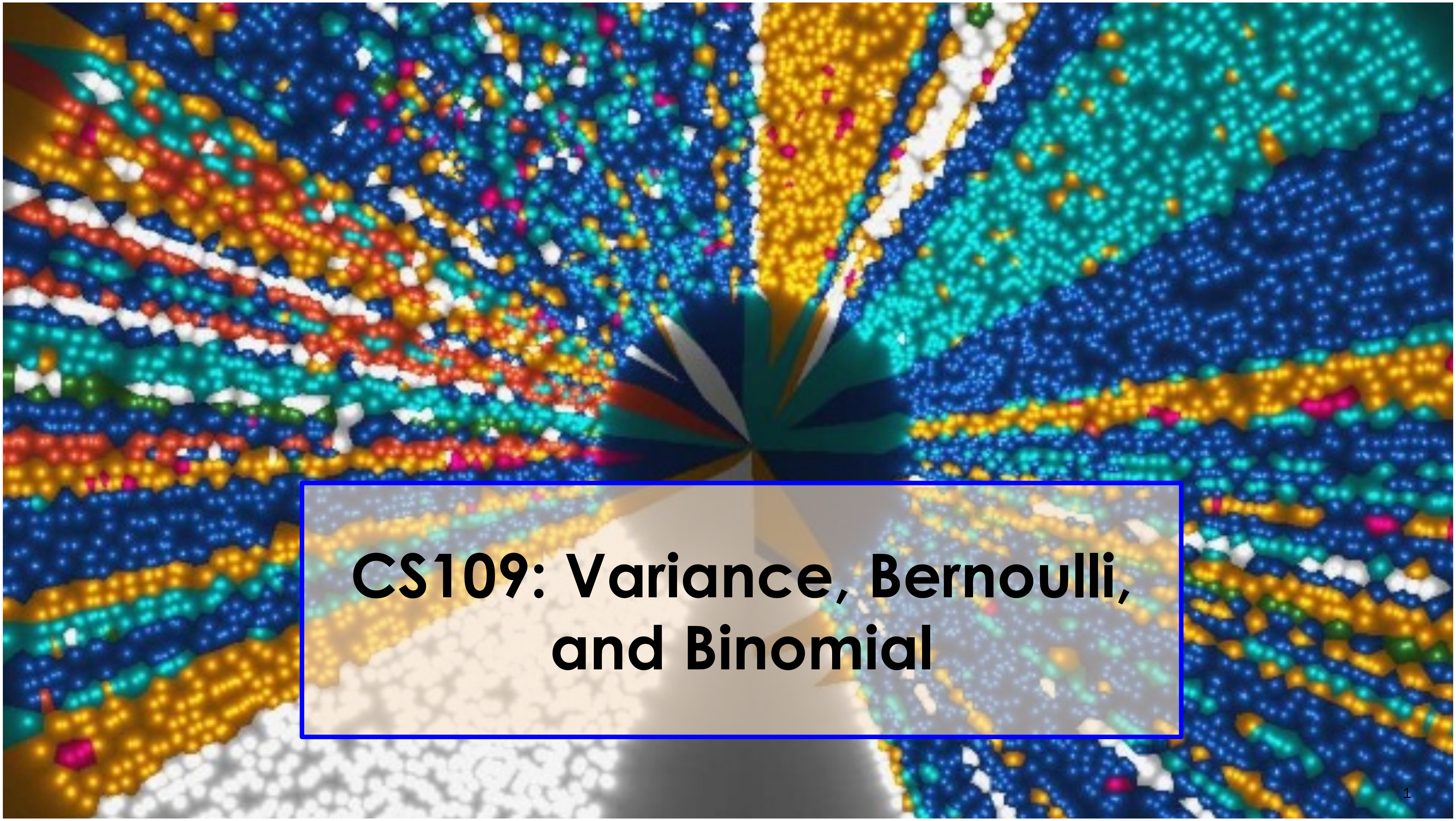


CS109: Variance, Bernoulli, and Binomial

Variance

Average annual weather

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$

Washington, DC

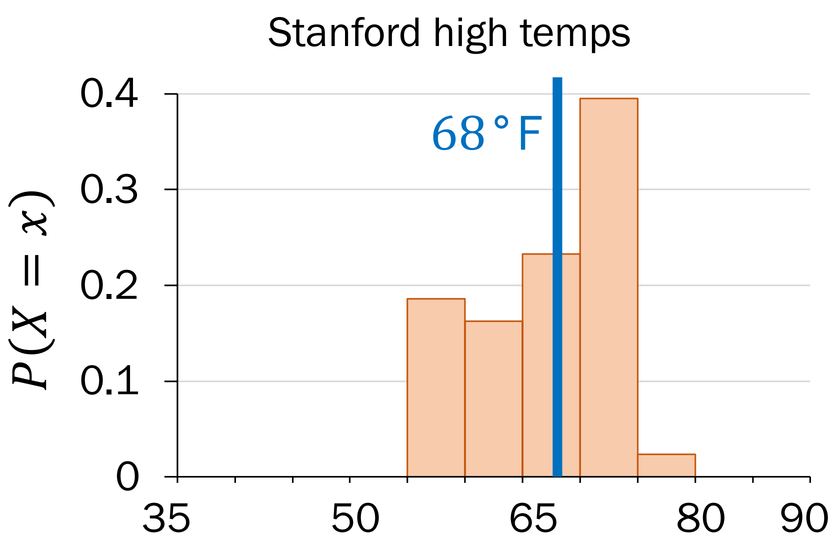
$E[\text{high}] = 67°\text{F}$

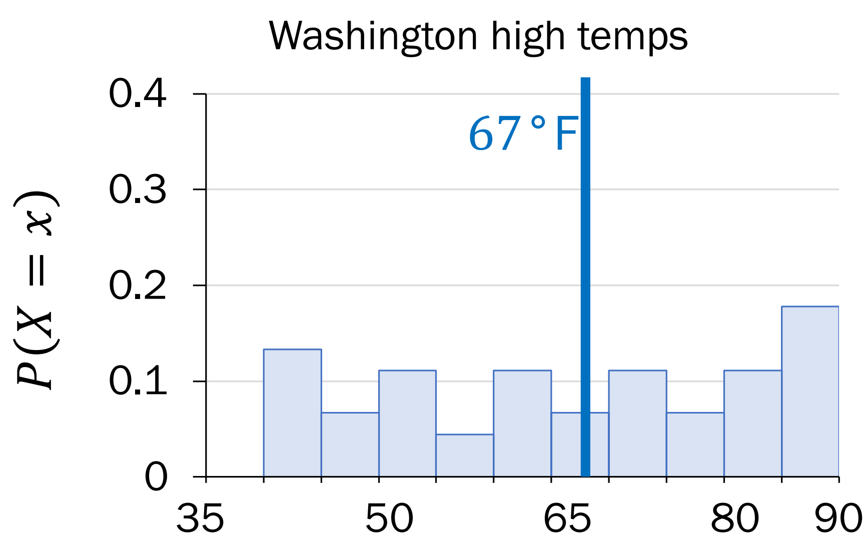
$E[\text{low}] = 51°\text{F}$

Is $E[X]$ enough?

Average annual weather

Stanford, CA	Washington, DC
$E[\text{high}] = 68°\text{F}$	$E[\text{high}] = 67°\text{F}$
$E[\text{low}] = 52°\text{F}$	$E[\text{low}] = 51°\text{F}$

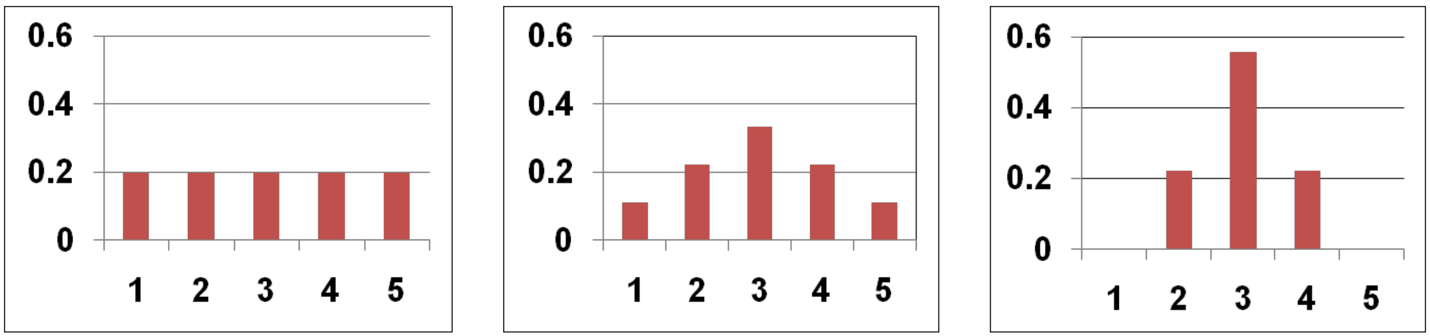




Normalized histograms are approximations of PMFs.

Variance = "spread"

Consider the following three distributions (PMFs):



- Expectation: $E[X] = 3$ for all distributions
- But the "spread" in the distributions is different!
- Variance, $\text{Var}(X)$: a formal quantification of “spread”

Variance

The variance of a random variable X with mean $E[X] = \mu$ is

$$\text{Var}(X) = E[(X - \mu)^2]$$

- Also written as: $E[(X - E[X])^2]$
- Note: Var(X) ≥ 0
- Other names: **2nd central moment**, or square of the standard deviation

	$\text{Var}(X)$	Units of X^2
def standard deviation	$\text{SD}(X) = \sqrt{\text{Var}(X)}$	Units of X

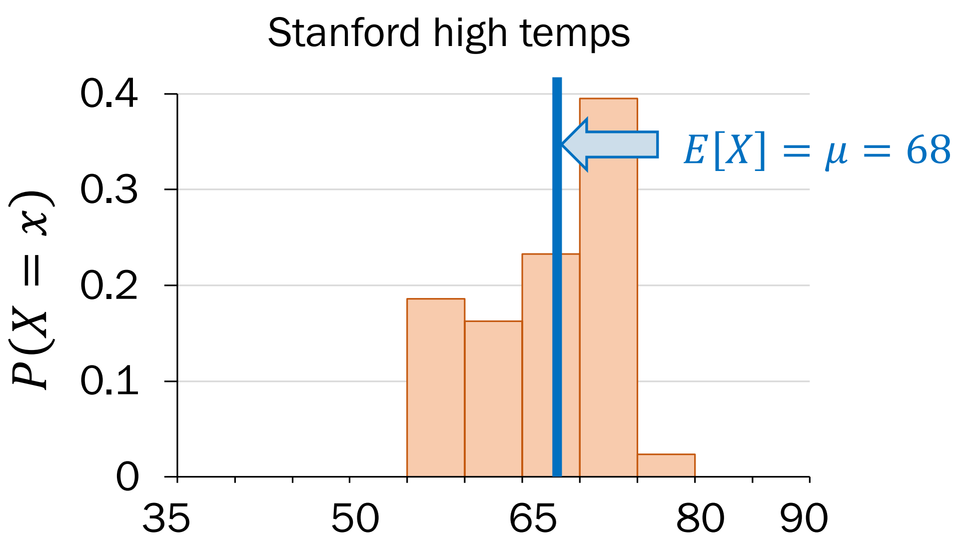
Variance of Stanford weather

$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA

$E[\text{high}] = 68°\text{F}$

$E[\text{low}] = 52°\text{F}$



X	$(X - \mu)^2$
57°F	121 (°F)^2
71°F	9 (°F)^2
75°F	49 (°F)^2
69°F	1 (°F)^2
…	…

Variance $E[(X - \mu)^2] = 39$ (°F)2

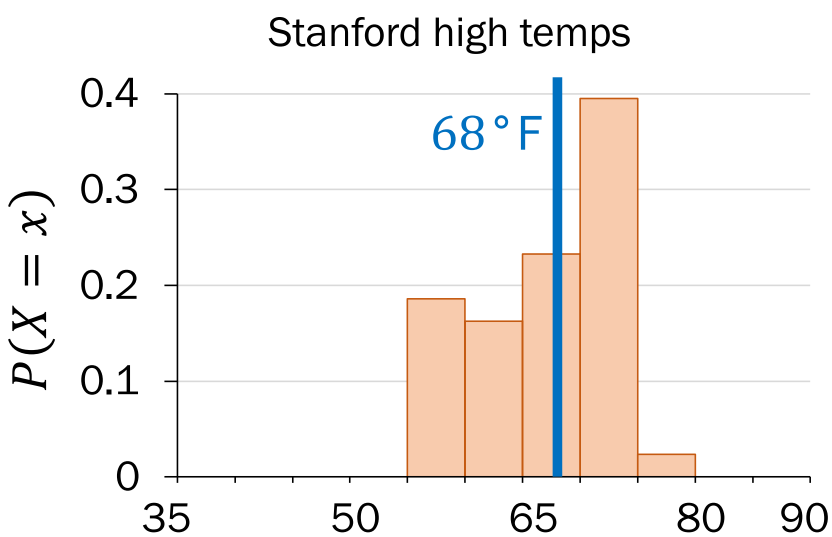
Standard deviation $= 6.2°\text{F}$

Comparing variance

$\text{Var}(X) = E[(X - E[X])^2]$ Variance of X

Stanford, CA

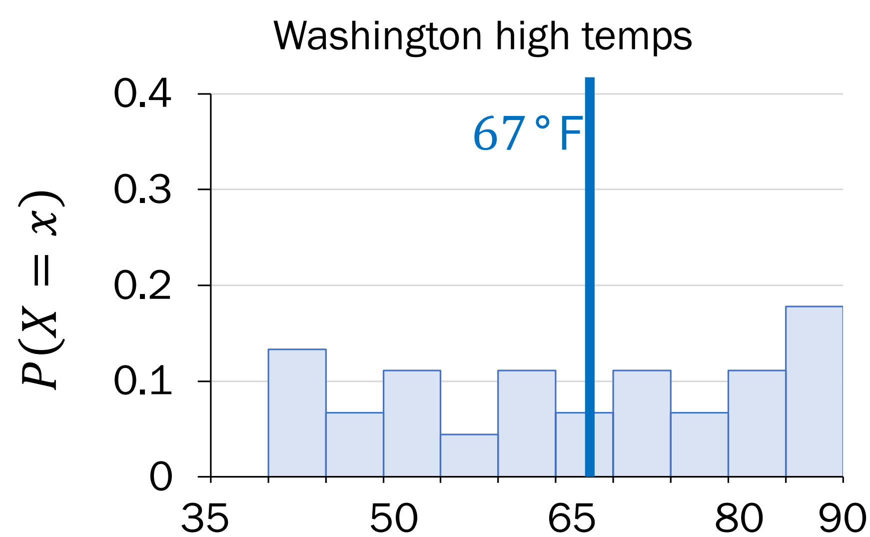
$E[\text{high}] = 68°\text{F}$



$\text{Var}(X) = 39\ (°\text{F})^2$

Washington, DC

$E[\text{high}] = 67°\text{F}$



$\text{Var}(X) = 248\ (°\text{F})^2$

Properties of Variance

Properties of variance

Definition $\quad \text{Var}(X) = E[(X - E[X])^2]$ $\quad$ Units of X^2

def standard deviation $\quad \text{SD}(X) = \sqrt{\text{Var}(X)}$ $\quad$ Units of X

Property 1 $\quad \text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\quad \text{Var}(aX + b) = a^2\text{Var}(X)$

- Property 1 is often easier to compute than the definition
- Unlike expectation, variance is not linear

Properties of variance

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Property 1 $\text{Var}(X) = E[X^2] - (E[X])^2$

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- Property 1 is often easier to compute than the definition
- Unlike expectation, variance is not linear

Computing variance, a proof

$\text{Var}(X) = E[(X - E[X])^2]$ Variance
$= E[X^2] - (E[X])^2$ of X

$$\text{Var}(X) = E[(X - E[X])^2] = E[(X-\mu)^2], \text{ where } E[X] = \mu$$

$$= \sum_x (x-\mu)^2 p(x)$$

$$= \sum_x (x^2 - 2\mu x + \mu^2) p(x)$$

$$= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x)$$

$$= E[X^2] - 2\mu E[X] + \mu^2 \cdot 1$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$= E[X^2] - \mu^2$$

$$= E[X^2] - (E[X])^2$$

Everyone, please welcome the second moment!

Variance of a 6-sided die

$\text{Var}(X) = E[(X - E[X])^2]$
$= E[X^2] - (E[X])^2$

Variance of X

Let Y = outcome of a single die roll. Recall $E[Y] = 7/2$.
Calculate the variance of Y.

1. Approach #1: Definition

$$\text{Var}(Y) = \frac{1}{6}\left(1-\frac{7}{2}\right)^2 + \frac{1}{6}\left(2-\frac{7}{2}\right)^2 + \frac{1}{6}\left(3-\frac{7}{2}\right)^2 + \frac{1}{6}\left(4-\frac{7}{2}\right)^2 + \frac{1}{6}\left(5-\frac{7}{2}\right)^2 + \frac{1}{6}\left(6-\frac{7}{2}\right)^2$$

$$= 35/12$$

2. Approach #2: A property

2nd moment

$$E[Y^2] = \frac{1}{6}[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2]$$
$$= 91/6$$

$$\text{Var}(Y) = 91/6 - (7/2)^2$$
$$= 35/12$$

Properties of variance

Definition $\text{Var}(X) = E[(X - E[X])^2]$ Units of X^2

def standard deviation $\text{SD}(X) = \sqrt{\text{Var}(X)}$ Units of X

Property 1 $\text{Var}(X) = E[X^2] - (E[X])^2$

Property 2 $\text{Var}(aX + b) = a^2\text{Var}(X)$

- Property 1 is often easier to compute than the definition
- Unlike expectation, variance is not linear

Property 2: A proof

Property 2 $\quad \text{Var}(aX + b) = a^2\text{Var}(X)$

Proof: $\text{Var}(aX + b)$

$$= E[(aX+b)^2] - (E[aX+b])^2 \qquad \text{Property 1}$$

$$= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2$$

$$= a^2E[X^2] + 2abE[X] + b^2 - (a^2(E[X])^2 + 2abE[X] + b^2)$$

Factoring/ Linearity of Expectation

$$= a^2E[X^2] - a^2(E[X])^2$$

$$= a^2(E[X^2] - (E[X])^2)$$

$$= a^2\text{Var}(X) \qquad \text{Property 1}$$

Bernoulli RV

Jacob Bernoulli

Jacob Bernoulli (1654-1705), also known as "James", was a Swiss mathematician

One of many mathematicians in Bernoulli family

The Bernoulli Random Variable is named for him

Bernoulli Random Variable

Consider an experiment with two outcomes: "success" and "failure".

def A **Bernoulli** random variable X maps "success" to 1 and "failure" to 0.
Other names: **indicator** random variable, Boolean random variable

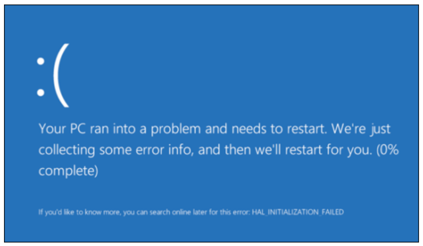
$X \sim \text{Ber}(p)$	PMF	$P(X=1) = p(1) = p$ $P(X=0) = p(0) = 1-p$
	Expectation	$E[X] = p$
Support: $\{0,1\}$	Variance	$\text{Var}(X) = p(1-p)$

Examples:
- Coin flip
- Random binary digit
- Whether you watched a movie

Remember this nice property of expectation. It will come back!

Defining Bernoulli RVs

$X \sim \text{Ber}(p)$ $\quad p_X(1) = p$
$E[X] = p$ $\quad p_X(0) = 1 - p$



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crash

$$X \sim \text{Ber}(p)$$

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

Serve an ad.

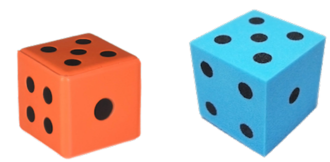
- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \text{Ber}(\_\_)$$

$$P(X = 1) = \_\_$$

$$P(X = 0) = \_\_$$



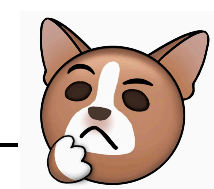
Roll two dice.

- Success: roll two 6's
- Failure: anything else

Let X : 1 if success

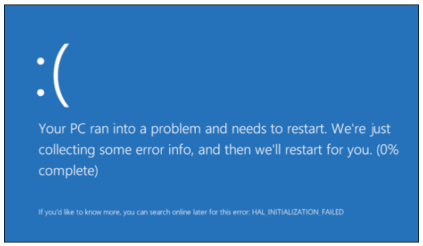
$$X \sim \text{Ber}(\_\_)$$

$$E[X] = \_\_$$



Defining Bernoulli RVs

$X \sim \text{Ber}(p)$ $\quad p_X(1) = p$
$E[X] = p$ $\quad p_X(0) = 1 - p$



Run a program

- Crashes w.p. p
- Works w.p. $1 - p$

Let X: 1 if crash

$$X \sim \text{Ber}(p)$$

$$P(X = 1) = p$$

$$P(X = 0) = 1 - p$$

Serve an ad.

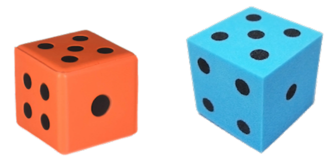
- User clicks w.p. 0.2
- Ignores otherwise

Let X: 1 if clicked

$$X \sim \text{Ber}(\_\_)$$

$$P(X = 1) = \_\_$$

$$P(X = 0) = \_\_$$



Roll two dice.

- Success: roll two 6's
- Failure: anything else

Let X : 1 if success

$$X \sim \text{Ber}(\_\_)$$

$$E[X] = \_\_$$

Binomial RV

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Support: $\{0,1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Reiterating notation

$$X \sim \text{Bin}(n, p)$$

1. The random variable
2. is distributed as a
3. Binomial
4. with parameters

The parameters of a Binomial random variable:

- n: number of independent trials
- p: probability of success on each trial

Reiterating notation

$$X \sim \text{Bin}(n, p)$$

If X is a binomial with parameters n and p, the PMF of X is

$$\underbrace{P(X = k)}_{\text{Probability that } X \text{ takes on the value } k} = \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{\textbf{Probability Mass Function} \text{ for a Binomial}}$$

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Three fair ("heads" with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$

Compute the following event probabilities:

$$P(X = 0)$$

$$P(X = 1)$$

$$P(X = 2)$$

$$P(X = 3)$$

$$P(X = 7)$$

P(event)

Three coin flips

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

Three fair ("heads" with $p = 0.5$) coins are flipped.

- X is number of heads
- $X \sim \text{Bin}(3, 0.5)$

Compute the following event probabilities:

$$P(X=0) = p(0) = \binom{3}{0} p^0 (1-p)^3 = \frac{1}{8}$$

$$P(X=1) = p(1) = \binom{3}{1} p^1 (1-p)^2 = \frac{3}{8}$$

$$P(X=2) = p(2) = \binom{3}{2} p^2 (1-p)^1 = \frac{3}{8}$$

$$P(X=3) = p(3) = \binom{3}{3} p^3 (1-p)^0 = \frac{1}{8}$$

$$P(X=7) = p(7) = 0$$

P(event) PMF

Extra math note:
By Binomial Theorem, we can prove $\sum_{k=0}^{n} P(X=k) = 1$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$

Range: $\{0, 1, \ldots, n\}$

PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Expectation	$E[X] = np$
Variance	$\text{Var}(X) = np(1-p)$

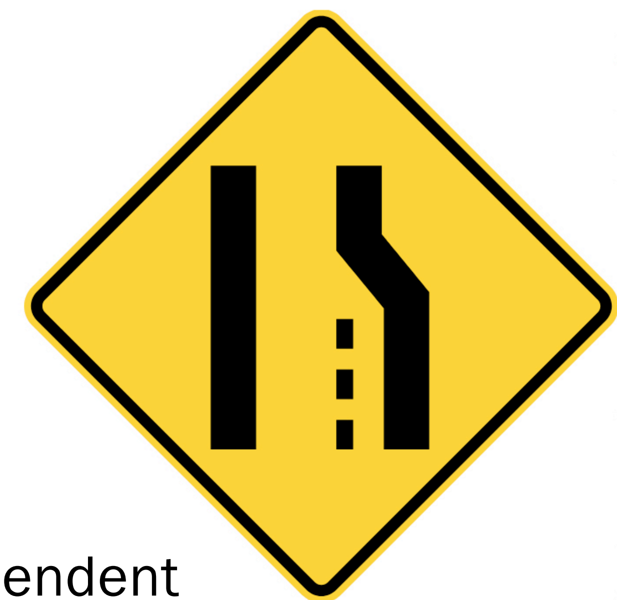
Examples:

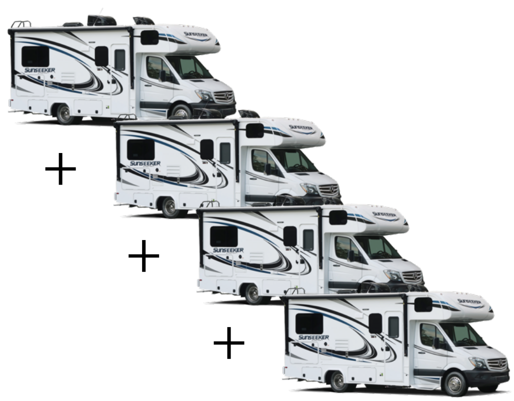
- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Binomial RV is sum of Bernoulli RVs

Bernoulli

- $X \sim \text{Ber}(p)$





Binomial

- $Y \sim \text{Bin}(n, p)$
- The sum of n independent Bernoulli RVs

$$Y = \sum_{i=1}^{n} X_i, \qquad X_i \sim \text{Ber}(p)$$

$$\boxed{\text{Ber}(p) = \text{Bin}(1, p)}$$

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$$X \sim \text{Bin}(n, p)$$

Range: $\{0, 1, \ldots, n\}$

PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
Expectation	$E[X] = np$
Variance	$\text{Var}(X) = np(1-p)$

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

Proof:

Binomial Random Variable

Consider an experiment: n independent trials of Ber(p) random variables.

def A **Binomial** random variable X is the number of successes in n trials.

$X \sim \text{Bin}(n, p)$	PMF	$k = 0, 1, \ldots, n$: $P(X = k) = p(k) = \binom{n}{k} p^k (1-p)^{n-k}$
	Expectation	$E[X] = np$
Range: $\{0, 1, \ldots, n\}$	Variance	$\text{Var}(X) = np(1-p)$

We'll prove this later in the course

Examples:

- # heads in n coin flips
- # of 1's in randomly generated length n bit string
- # of disk drives crashed in 1000 computer cluster (assuming disks crash independently)

No, give me the variance proof right now (way optional)

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

So:

$\mathrm{E}(X^2) = \sum_{k \geq 0}^{n} k^2 \binom{n}{k} p^k q^{n-k}$	Definition of Binomial Distribution: $p + q = 1$
$= \sum_{k=0}^{n} kn \binom{n-1}{k-1} p^k q^{n-k}$	Factors of Binomial Coefficient: $k\binom{n}{k} = n\binom{n-1}{k-1}$
$= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} q^{(n-1)-(k-1)}$	Change of limit: term is zero when $k - 1 = 0$
$= np \sum_{j=0}^{m} (j+1) \binom{m}{j} p^j q^{m-j}$	putting $j = k - 1, m = n - 1$
$= np\left(\sum_{j=0}^{m} j\binom{m}{j} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right)$	splitting sum up into two
$= np\left(\sum_{j=0}^{m} m\binom{m-1}{j-1} p^j q^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right)$	Factors of Binomial Coefficient: $j\binom{m}{j} = m\binom{m-1}{j-1}$
$= np\left((n-1)p \sum_{j=1}^{m} \binom{m-1}{j-1} p^{j-1} q^{(m-1)-(j-1)} + \sum_{j=0}^{m} \binom{m}{j} p^j q^{m-j}\right)$	Change of limit: term is zero when $j - 1 = 0$
$= np\left((n-1)p(p+q)^{m-1} + (p+q)^m\right)$	Binomial Theorem
$= np((n-1)p + 1)$	as $p + q = 1$
$= n^2p^2 + np(1-p)$	by algebra

proofwiki.org

Then:

$\mathrm{var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$	
$= np(1-p) + n^2p^2 - (np)^2$	Expectation of Binomial Distribution: $\mathrm{E}(X) = np$
$= np(1-p)$	

as required.

Ponder Bernoulli

Let’s take a two-minute stretch.

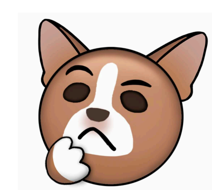
Slide 33 has three questions to think over by yourself while you stretch. We’ll go over it together afterwards.

Statistics: Expectation and variance

1. a. Let X = the outcome of a fair 4-sided die roll. What is $E[X]$?
   b. Let Y = the sum of three rolls of a fair 4-sided die. What is $E[Y]$?

2. Let Z = # of ***tails*** on 10 flips of a biased coin (w.p. 0.4 of heads). What is $E[Z]$?

3. Compare the variances of $B_1 \sim \text{Ber}(0.1)$ and $B_2 \sim \text{Ber}(0.5)$.



Statistics: Expectation and variance

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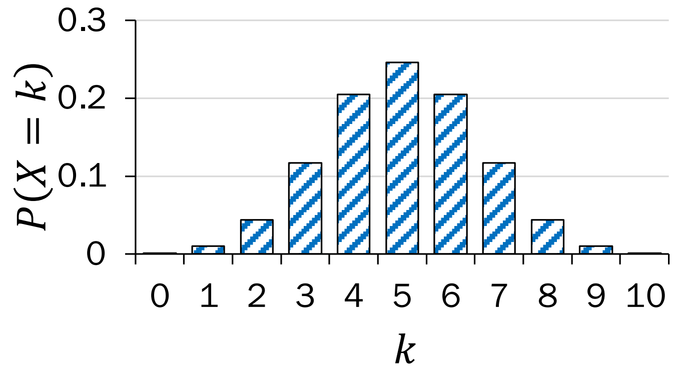
3. Compare the variances of $B_1 \sim \text{Ber}(0.1)$ and $B_2 \sim \text{Ber}(0.5)$.

If you can identify common RVs, just look up statistics instead of re-deriving from definitions.

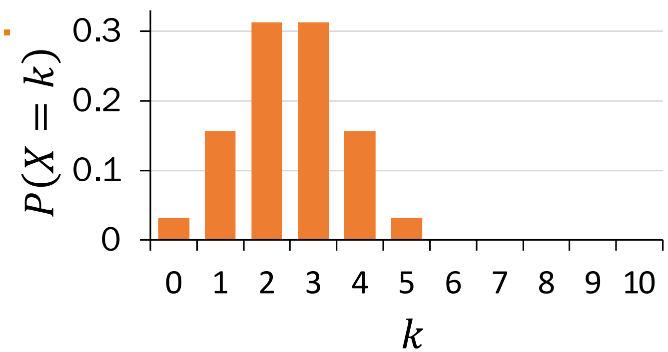
Visualizing Binomial PMFs

$X \sim \text{Bin}(n, p)$ $\quad E[X] = np \quad p(i) = \binom{n}{k} p^k (1-p)^{n-k}$

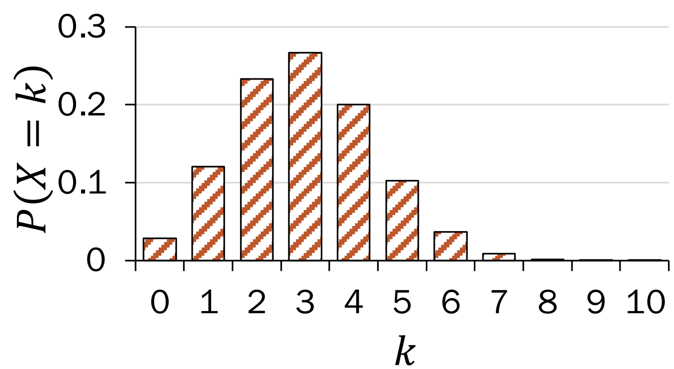
A.



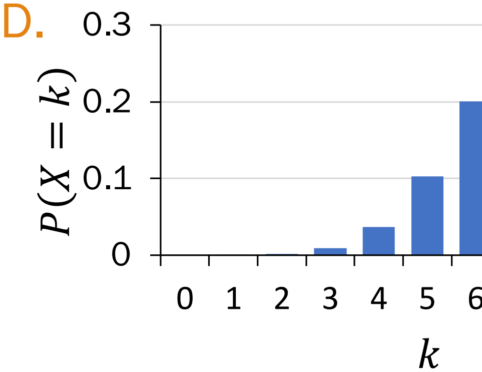
B.



C.



D.

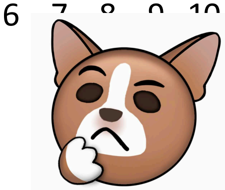


Match the distribution of X to the graph:

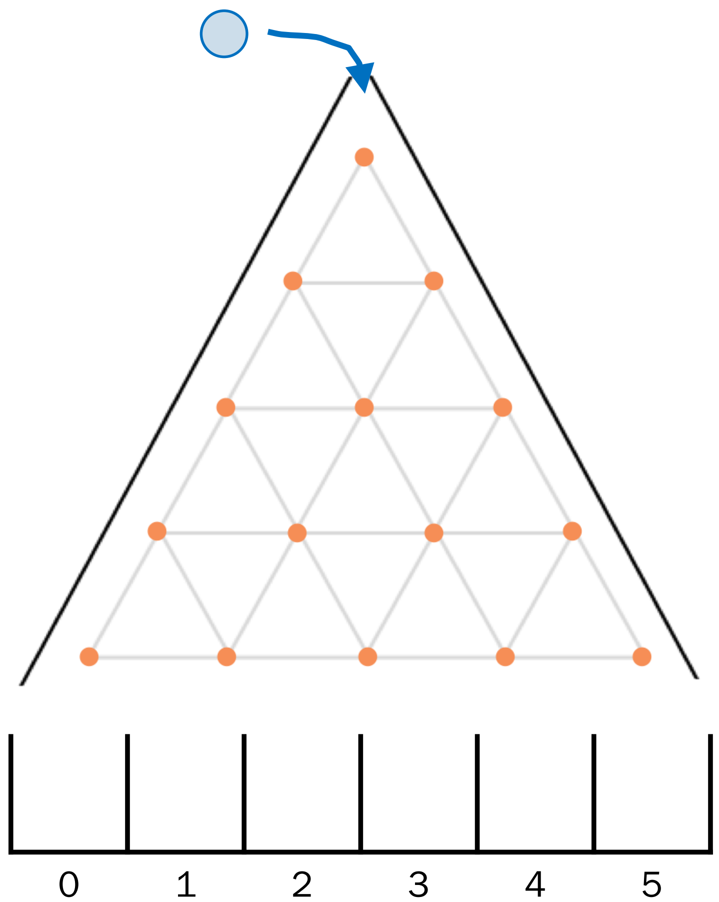
1. $\text{Bin}(10, 0.5)$
2. $\text{Bin}(10, 0.3)$
3. $\text{Bin}(10, 0.7)$
4. $\text{Bin}(5, 0.5)$

Type your answer in Zoom chat but don't press enter until time is up

Example: 1: A, 2: B, 3: C, 4: D



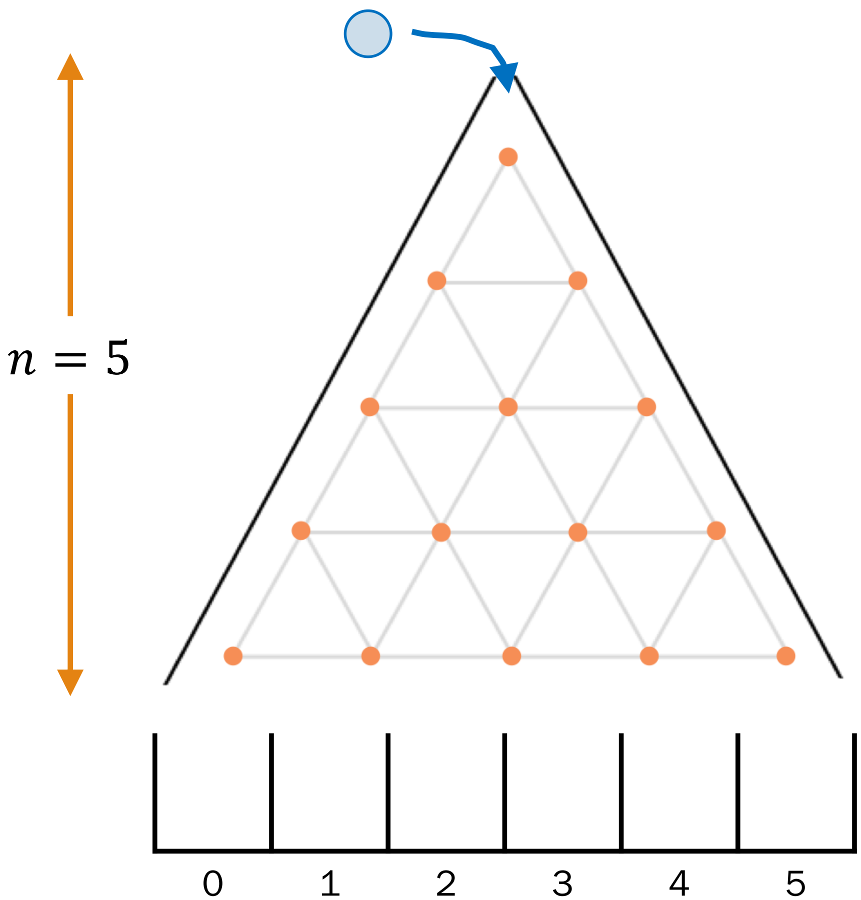
Galton Board



http://web.stanford.edu/class/cs109/demos/galton.html

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

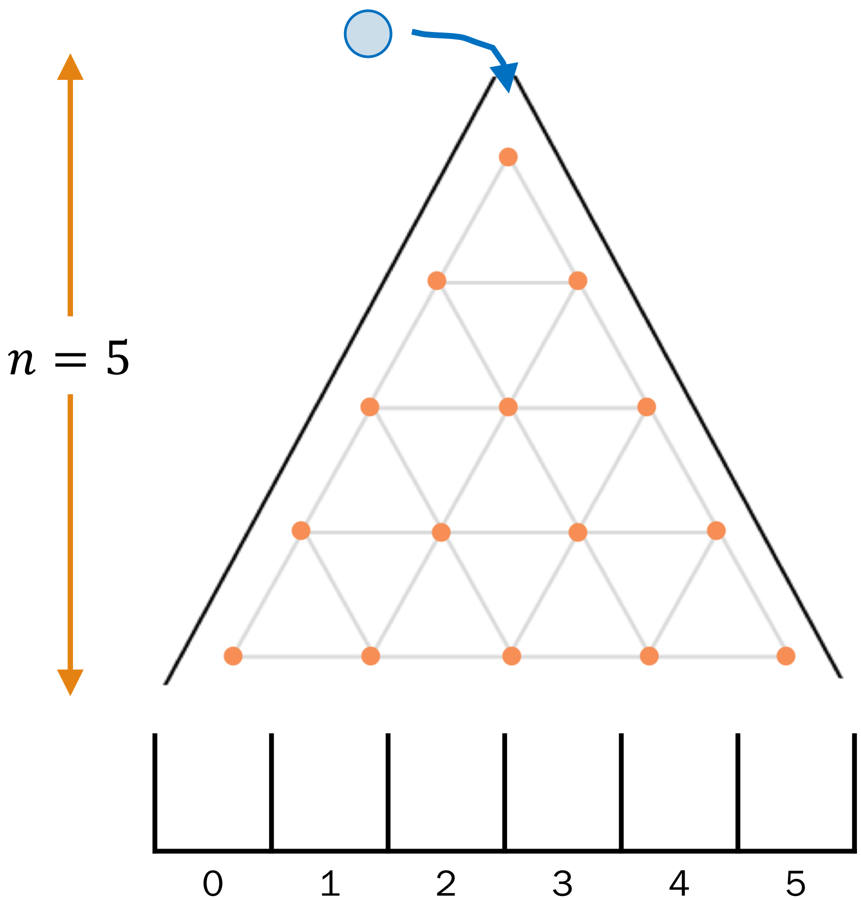


When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
What is the distribution of B?

(Interpret: If B is a common random variable, report it, otherwise report PMF)

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



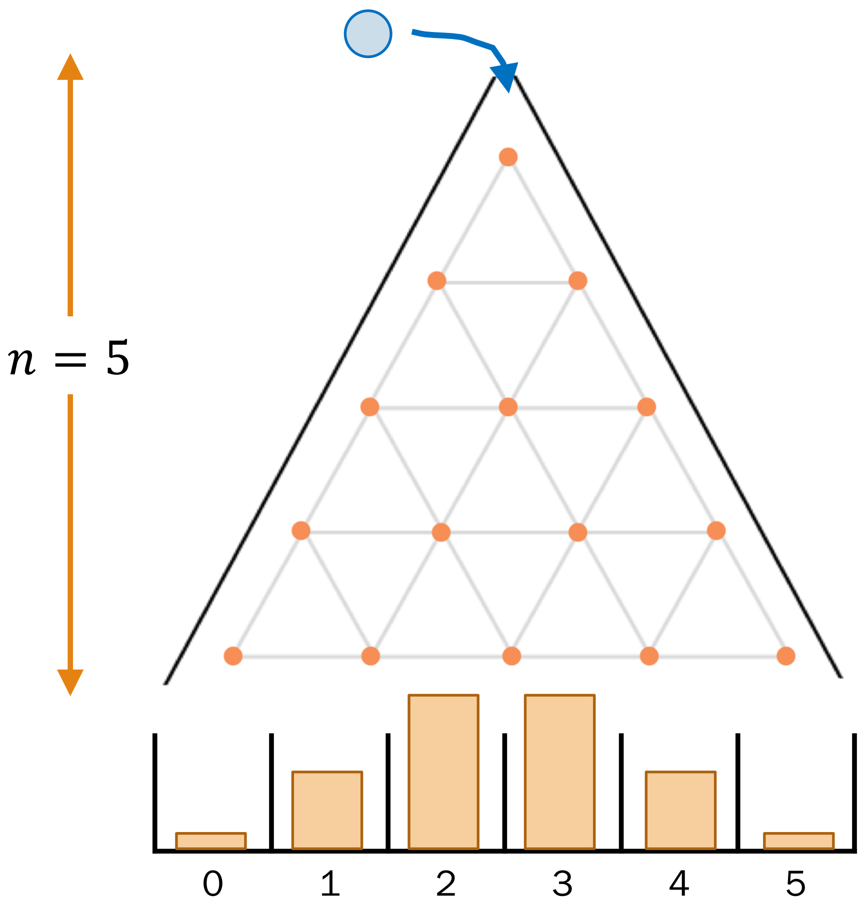
When a marble hits a pin, it has an equal chance of going left or right.
Let B = the bucket index a ball drops into.
What is the **distribution** of B?

- Each pin is an independent trial
- One decision made for level $i = 1, 2, .., 5$
- Consider a Bernoulli RV with success R_i if ball went right on level i
- Bucket index $B =$ # times ball went right

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Galton Board

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$



When a marble hits a pin, it has an equal chance of going left or right.

Let B = the bucket index a ball drops into.

B is distributed as a Binomial RV,

$$B \sim \text{Bin}(n = 5, p = 0.5)$$

Calculate the probability of a ball landing in bucket k.

$$P(B = 0) = \binom{5}{0} 0.5^5 \approx 0.03$$

$$P(B = 1) = \binom{5}{1} 0.5^5 \approx 0.16$$

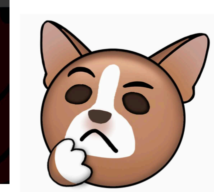
$$P(B = 2) = \binom{5}{2} 0.5^5 \approx 0.31$$

PMF of Binomial RV!

Ponder Genes & Basketball

Let’s take a thirty second water break.

Slide 41 has two questions to think over by yourself while you hydrate. We’ll go over it together afterwards.



Genetics and NBA Finals

$X \sim \text{Bin}(n, p) \quad p(k) = \binom{n}{k} p^k (1-p)^{n-k}$

1. Each person has 2 genes per trait (e.g., eye color).
   - Child receives 1 gene (equally likely) from each parent
   - Brown is dominant, blue is recessive:
     - Child has brown eyes if either or both genes are brown
     - Blue eyes only if both genes are blue.
   - Parents each have 1 brown and 1 blue gene.

A family has 4 children. What is P(exactly 3 children have brown eyes)?

2. Prediction: the LA Lakers are going to play the Miami Heat in a 7-game series during the 2021 NBA finals this coming October.
   - The Lakers have a probability of 58% of winning each game.
   - A team wins the series if they win at least 4 games (though they play all 7).

What is P(Lakers winning)?

Genetic inheritance

Each person has 2 genes per trait (e.g., eye color).

- Child receives 1 gene (equally likely) from each parent
- Brown is dominant, blue is recessive:
  - Child has brown eyes if either or both genes are brown
  - Blue eyes only if both genes are blue.
- Parents each have 1 brown and 1 blue gene.

- A family has 4 children. What is P(exactly 3 children have brown eyes)?

Big Q: Fixed parameter or random variable?

Parameters What is common among all outcomes of our experiment?

Random variable What differentiates our event from the rest of the sample space?

Genetic inheritance

Each person has 2 genes per trait (e.g., eye color).

- Child receives 1 gene (equally likely) from each parent
- Brown is dominant, blue is recessive:
  - Child has brown eyes if either or both genes are brown
  - Blue eyes only if both genes are blue.
- Parents each have 1 brown and 1 blue gene.

A family has 4 children. What is P(exactly 3 children with brown eyes)?

1. Define events/ RVs & state goal
2. Identify <u>known</u> probabilities
3. Solve

X: # brown-eyed children,
$X \sim \text{Bin}(4, p)$
p: $P(\text{brown}-\text{eyed child})$

Want: $P(X = 3)$

NBA Finals

Prediction: the LA Lakers are going to play the Miami Heat in a 7-game series during the 2021 NBA finals this coming October.

- The Lakers have a probability of 58% of winning each game.
- A team wins the series if they win at least 4 games (though they play all 7).

What is P(Lakers winning)?

1. Define events/ RVs & state goal

X: # games Lakers win

$X \sim \text{Bin}(7, 0.58)$

Want:

Big Q: Fixed parameter or random variable?

Parameters	# of total games prob. Lakers winning a game
Random variable	# of games Lakers win
Event based on RV	Lakers win 4 or more games

NBA Finals

Prediction: the LA Lakers are going to play the Miami Heat in a 7-game series during the 2021 NBA finals this coming October.

- The Lakers have a probability of 58% of winning each game.
- A team wins the series if they win at least 4 games (though they play all 7).

What is P(Lakers winning)?

1. Define events/ RVs & state goal
2. Solve

X: # games Lakers win
$X \sim \text{Bin}(7, 0.58)$

Want: $P(X \geq 4)$

$$P(X \geq 4) = \sum_{k=4}^{7} P(X = k) = \sum_{k=4}^{7} \binom{7}{k} 0.58^k (0.42)^{7-k}$$

Cool Algebra/Probability Fact: this is identical to the probability of winning if we define winning = first to win 4 games

See you next time