# CS109: Gaussian RVs

# Normal Random Variable

def An **Normal** random variable $X$ is defined as follows:

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Support: $(-\infty, \infty)$

| | |
|---|---|
| PDF | $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$ |
| Expectation | $E[X] = \mu$ |
| Variance | $\text{Var}(X) = \sigma^2$ |

Other names: **Gaussian** random variable

mean

variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

# Carl Friedrich Gauss

Carl Friedrich Gauss (1777-1855) was a remarkably influential German mathematician.

**Johann Carl Friedrich Gauss** (/gaʊs/; German: *Gauß* [gaʊs] (listen); Latin: *Carolus Fridericus Gauss*; 30 April 1777 – 23 February 1855) was a German mathematician and physicist who made significant contributions to many fields, including algebra, analysis, astronomy, differential geometry, electrostatics, geodesy, geophysics, magnetic fields, matrix theory, mechanics, number theory, optics and statistics.

Sometimes referred to as the *Princeps mathematicorum*[1] (Latin for "the foremost of mathematicians") and "the greatest mathematician since antiquity", Gauss had an exceptional influence in many fields of mathematics and science, and is ranked among history's most influential mathematicians.[2]

Did not invent Normal distribution but rather popularized it

# Why the Normal?

- Common for natural phenomena: height, weight, etc.
- Most noise in the world is Normal
- Often results from the sum of many random variables
- Sample means are distributed normally

That's what they want you to believe…

# Why the Normal?

- Common for natural phenomena: height, weight, etc.

  Actually log-normal

- Most noise in the world is Normal

  Just an assumption

- Often results from the sum of many random variables

  Only if equally weighted

- Sample means are distributed normally

  (okay this one is true, we'll see this in 3 weeks)

# Okay, so why the Normal?

Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: model real life situations with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.

Fits perfectly!
But what about in another classroom?

# Okay, so why the Normal?

Part of CS109 learning goals:

- Translate a problem statement into a random variable

In other words: model real life situations with probability distributions

How do you model student heights?

- Suppose you have data from one classroom.

- Same mean/var
- Generalizes well

Occam's Razor:

*"Non sunt multiplicanda entia sine necessitate."*

Entities should not be multiplied without necessity.

A Gaussian maximizes entropy for a given mean and variance.

# Why the Normal?

- Common for natural phenomena: height, weight, etc.

  Actually log-normal

- Most noise in the world is Normal

  ...mption

- Often results from ... random va...

  Only if equally weighted

- Sample means are distributed normally

  (okay this one is true, we'll see this in 3 weeks)

because it's easy to use

I encourage you to stay critical of how to model real-world phenomena.

# Anatomy of a beautiful equation

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

The PDF of $X$ is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

normalizing constant

exponential tail

symmetric around $\mu$

variance $\sigma^2$ manages spread

# Campus bikes

You spend some minutes, $X$, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes$^2$

Suppose $X$ is normally distributed. What is the probability you spend $\geq 6$ minutes traveling?

$$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$$

$$P(X \geq 6) = \int_6^\infty f(x)dx = \int_6^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

(call me if you analytically solve this)

Loving, not scary
...except this time

# Computing probabilities with Normal RVs

For a Normal RV $X \sim \mathcal{N}(\mu, \sigma^2)$, its CDF has no closed form.

$$P(X \leq x) = F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

Cannot be solved analytically

However, we can solve for probabilities numerically using a function $\Phi$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

To get here, we'll first need to know some properties of Normal RVs.

CDF of $X \sim \mathcal{N}(\mu, \sigma^2)$

A function that has been solved for numerically

# Properties of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

1. Linear transformations of Normal RVs are also Normal RVs.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2).$$

2. The PDF of a Normal RV is symmetric about the mean $\mu$.

$$F(\mu - x) = 1 - F(\mu + x)$$

# 1. Linear transformations of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

Linear transformations of X are also Normal.

$$\text{If } Y = aX + b, \text{ then } Y \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

Proof:

- $E[Y] = E[aX + b] = aE[X] + b = a\mu + b$ Linearity of Expectation
- $\text{Var}(Y) = \text{Var}(aX + b) = a^2\text{Var}(X) = a^2\sigma^2$ $\text{Var}(aX + b) = a^2\text{Var}(X)$
- $Y$ is also Normal

Proof in Ross,
10th ed (Section 5.4)

# 2. Symmetry of Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $P(X \leq x) = F(x)$.

The PDF of a Normal RV is symmetric about the mean $\mu$.

$$F(\mu - x) = 1 - F(\mu + x)$$

# Using symmetry of the Normal RV

$F(\mu - x) = 1 - F(\mu + x)$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

How do we compute the following probabilities?

1. $P(Z \leq z) \quad = F(z)$
2. $P(Z < z)$
3. $P(Z \geq z)$
4. $P(Z \leq -z)$
5. $P(Z \geq -z)$
6. $P(y < Z < z)$

A. $F(z)$
B. $1 - F(z)$
C. $F(z) - F(y)$

# Using symmetry of the Normal RV

$F(\mu - x) = 1 - F(\mu + x)$

Let $Z \sim \mathcal{N}(0,1)$ with CDF $P(Z \leq z) = F(z)$.

Suppose we only knew numeric values for $F(z)$ and $F(y)$, for some $z, y \geq 0$.

How do we compute the following probabilities?

1. $P(Z \leq z) = F(z)$
2. $P(Z < z) = F(z)$
3. $P(Z \geq z) = 1 - F(z)$
4. $P(Z \leq -z) = 1 - F(z)$
5. $P(Z \geq -z) = F(z)$
6. $P(y < Z < z) = F(z) - F(y)$

A. $F(z)$
B. $1 - F(z)$
C. $F(z) - F(y)$

Symmetry is particularly useful when computing probabilities of zero-mean Normal RVs.

# Computing probabilities with Normal RVs

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

To compute the CDF, $P(X \leq x) = F(x)$:

- We cannot analytically solve the integral, since it has no closed form
- …but we ***can*** solve numerically using a function $\Phi$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

CDF of the Standard Normal, $Z$

# Standard Normal RV, $Z$

The **Standard Normal** random variable $Z$ is defined as follows:

$$Z \sim \mathcal{N}(0, 1)$$

Expectation $E[Z] = \mu = 0$

Variance $\text{Var}(Z) = \sigma^2 = 1$

Note: not a new distribution; just a special case of the Normal

Other names: **Unit Normal**

CDF of $Z$ defined as: $P(Z \leq z) = \Phi(z)$

# $\Phi$ has been numerically computed

**Standard Normal Table**

An entry in the table is the area under the curve to the left of $z$, $P(Z \leq z) = \Phi(z)$.

| $Z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0 | | |
| **0.1** | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0 | | |
| **0.2** | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0 | | |
| **0.3** | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0 | | |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | | |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | | |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | | |
| **0.7** | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7793 | | |
| **0.8** | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| **0.9** | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| **1.0** | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| **1.1** | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| **1.2** | 0.8849 | 0.8869 | 0.8888 | 0.8906 | 0.8925 | 0.8943 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| **1.3** | 0.9032 | $\boxed{0.9049}$ | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| **1.4** | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| **1.5** | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |

$$P(Z \leq 1.31) = \Phi(1.31)$$

Standard Normal Table only has probabilities $\Phi(z)$ for $z \geq 0$.

# History fact: Standard Normal Table

TABLES

SERVANT

AU CALCUL DES RÉFRACTIONS

APPROCHANTES DE L'HORIZON.

TABLE PREMIÈRE.

*Intégrales de $e^{-tt}\,dt$, depuis une valeur quelconque de $t$ jusqu'à $t$ infinie.*

| $t$ | Intégrale. | Diff. prem. | Diff. II. | Diff. III. |
|---|---|---|---|---|
| 0,00 | 0,88622692 | 999968 | 201 | 199 |
| 0,01 | 0,87622724 | 999767 | 400 | 199 |
| 0,02 | 0,86622957 | 999367 | 599 | 200 |
| 0,03 | 0,85623590 | 998768 | 799 | 199 |
| 0,04 | 0,84624822 | 997969 | 998 | 197 |
| 0,05 | 0,83626853 | 996971 | 1195 | 199 |
| 0,06 | 0,82629882 | 995776 | 1394 | 196 |

The first Standard Normal Table was computed by Christian Kramp, French astronomer (1760–1826), in *Analyse des Réfractions Astronomiques et Terrestres*, 1799

Used a Taylor series expansion to the third power

# Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use $\Phi$, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \leq x)$$ Definition of CDF

$$= P(X - \mu \leq x - \mu) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right)$$ Algebra + $\sigma > 0$

$$= P\left(Z \leq \frac{x-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of $X$.
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0,1)$
- In other words, $\frac{X-\mu}{\sigma} = Z \sim \mathcal{N}(0,1)$ with CDF $\Phi$.

# Probabilities for a general Normal RV

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. To compute the CDF $P(X \leq x) = F(x)$, we use $\Phi$, the CDF for the Standard Normal $Z \sim \mathcal{N}(0, 1)$:

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Proof:

$$F(x) = P(X \leq x)$$ Definition of CDF

$$= P(X - \mu \leq x - \mu) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right)$$ Algebra + $\sigma > 0$

$$= P\left(Z \leq \frac{x - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x - \mu}{\sigma}\right)$$

- $\frac{X-\mu}{\sigma} = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$ is a linear transform of $X$.
- This is distributed as $\mathcal{N}\left(\frac{1}{\sigma}\mu - \frac{1}{\sigma}\mu, \frac{1}{\sigma^2}\sigma^2\right) = \mathcal{N}(0, 1)$
- [illegible]

1. Compute $z = (x - \mu)/\sigma$.
2. Look up $\Phi(z)$ in Standard Normal table.

# Campus bikes

You spend some minutes, $X$, traveling between classes.

- Average time spent: $\mu = 4$ minutes
- Variance of time spent: $\sigma^2 = 2$ minutes$^2$

Suppose $X$ is normally distributed. What is the probability you spend $\geq 6$ minutes traveling?

$X \sim \mathcal{N}(\mu = 4, \sigma^2 = 2)$

✕ $P(X \geq 6) = \int_6^\infty f(x)dx$ (no analytic solution)

1. Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X \geq 6) = 1 - F_x(6)$$
$$= 1 - \Phi\left(\frac{6-4}{\sqrt{2}}\right)$$
$$\approx 1 - \Phi(1.41)$$

2. Look up $\Phi(z)$ in table

$$1 - \Phi(1.41)$$
$$\approx 1 - 0.9207$$
$$= 0.0793$$

# Is there an easier way? (yes)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

- Use Python

```
from scipy import stats
X = stats.norm(mu, std)
X.cdf(x)
```

SciPy reference:
https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html

- Use website tool

Website tool:
https://web.stanford.edu/class/cs109/handouts/normalCDF.html

# Mull over Gaussian

Let's take a ninety-second boba tea break.

Slide 29 presents four overlaid graphs, all Gaussian distributions to focus on as get your boba in.

Once you've converged some answers, stage your answer in chat as ACDB, but only send when I give the go-ahead.

# Normal Random Variable

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$
2. $\mathcal{N}(-2, 0.5)$
3. $\mathcal{N}(0, 5)$
4. $\mathcal{N}(0, 0.2)$

# Normal Random Variable

mean variance

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

Match PDF to distribution:

1. $\mathcal{N}(0, 1)$
2. $\mathcal{N}(-2, 0.5)$
3. $\mathcal{N}(0, 5)$
4. $\mathcal{N}(0, 0.2)$

# Computing probabilities with Normal RVs: Old school

**Standard Normal Table** *

Note: An entry in the table is the area under the curve to the left of $z$, $P(Z \le z) = \Phi(z)$

$\Phi(z)$ for non-negative $z$

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| **0.1** | 0.[illegible] | | | | | | | [illegible] | 0.5714 | 0.5753 |
| **0.2** | 0.[illegible] | | | | | | | [illegible] | 0.6103 | 0.6141 |
| **0.3** | 0.[illegible] | | | | | | | [illegible] | 0.6480 | 0.6517 |
| **0.4** | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| **0.5** | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| **0.6** | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |

*particularly useful when we have closed book exams with no calculator**

**we have open book exams with calculators this quarter

Knowing how to use a Standard Normal Table will still be useful in our understanding of Normal RVs.

# Computing probabilities with Normal RVs

Review

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. What is $P(X \leq x) = F(x)$?

1. Rewrite in terms of standard normal CDF $\Phi$ by computing $z = \frac{(x-\mu)}{\sigma}$.
   Linear transforms of Normals are Normal:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \qquad Z = \frac{(X-\mu)}{\sigma}, \text{ where } Z \sim \mathcal{N}(0,1)$$

2. Then, look up in a Standard Normal Table, where $z \geq 0$.
   Symmetry of Normal PDFs implies:

$$\Phi(-z) = 1 - \Phi(z)$$

**Standard Normal Table**

Note: An entry in the table is the area under the curve to the left of

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0 |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0. |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0. |

# Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ with CDF $F$.

$$P(|X-\mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$

$$= F(\mu+\sigma) - F(\mu-\sigma)$$

$$= \Phi\left(\frac{(\mu+\sigma)-\mu}{\sigma}\right) - \Phi\left(\frac{(\mu-\sigma)-\mu}{\sigma}\right)$$

$$= \Phi(1) - \Phi(-1) = \Phi(1) - \big(1 - \Phi(1)\big)$$

$$= 2\Phi(1) - 1 \quad \approx 2(0.8413) - 1 = 0.6826$$

# Why Be Normal? 68% rule

You may have heard the statement:

"68% of the class will fall within 1 standard deviation of the exam average."

In general, this is only true of normal distributions:

Counterexample: Let $X \sim \text{Uni}(\alpha, \beta)$.

$$\mu = E[X] = \frac{\alpha + \beta}{2}$$

$$\text{Var}(X) = \frac{(\beta - \alpha)^2}{12} \Rightarrow \sigma = \text{SD}(X) = \frac{\beta - \alpha}{\sqrt{12}}$$

$$P(|X - \mu| < \sigma) = P(\mu - \sigma < X < \mu + \sigma)$$

$$= \frac{1}{\beta - \alpha} \cdot [(\mu + \sigma) - (\mu - \sigma)]$$

$$= \frac{1}{\beta - \alpha}[2\sigma] = \frac{1}{\beta - \alpha} \cdot \left[2 \cdot \frac{\beta - \alpha}{\sqrt{12}}\right]$$

$$= 2/\sqrt{12} \approx 0.58$$

# Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

# Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$.
Note standard deviation $\sigma = 4$.

How would you write each of the below probabilities as a function of the standard normal CDF, $\Phi$?

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

1. $P(X > 0)$ (we just did this)
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

# Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-z) = 1 - \Phi(z)$

# Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

Look up $\Phi(z)$ in table

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \big(1 - F(9)\big)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

# Get your Gaussian On

Let $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 16)$. Std deviation $\sigma = 4$.

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 6)$

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
- Symmetry of the PDF of Normal RV implies $\Phi(-x) = 1 - \Phi(x)$

Compute $z = \frac{(x-\mu)}{\sigma}$

$$P(X < -3) + P(X > 9)$$

$$= F(-3) + \big(1 - F(9)\big)$$

$$= \Phi\left(\frac{-3-3}{4}\right) + \left(1 - \Phi\left(\frac{9-3}{4}\right)\right)$$

Look up $\Phi(z)$ in table

$$= \Phi\left(-\frac{3}{2}\right) + \left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$= 2\left(1 - \Phi\left(\frac{3}{2}\right)\right)$$

$$\approx 0.1337$$

# Puzzle Over Voltages

Let's take a ninety-second hike.

Slide 39 presents a problem about digital circuits, how 0 and 1 are encoded, and our confidence they'll be properly interpreted.

Puzzle over the problem while you take in some nature, and we'll discuss once we're all back.

# Noisy Wires

Send a voltage of 2 V or $-2$ V on wire (to denote 1 and 0, respectively).

- $X$ = voltage sent (2 or $-2$)
- $Y$ = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.

1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?
2. What is P(decoding error | original bit is 0)?

These probabilities are unequal. Why might this be useful?

# Noisy Wires

Send a voltage of 2 V or $-2$ V on wire (to denote 1 and 0, respectively).

- $X$ = voltage sent (2 or $-2$)
- $Y$ = noise, $Y \sim \mathcal{N}(0, 1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.

1. What is P(decoding error | original bit is 1)?
   i.e., we sent 1, but we decoded as 0?

$$P(R < 0.5 | X = 2) = P(2 + Y < 0.5) = P(Y < -1.5)$$ Y is Standard Normal

$$= \Phi(-1.5) = 1 - \Phi(1.5) \approx 0.0668$$

# Noisy Wires

Send a voltage of 2 V or $-2$ V on wire (to denote 1 and 0, respectively).

- $X$ = voltage sent (2 or $-2$)
- $Y$ = noise, $Y \sim \mathcal{N}(0,1)$
- $R = X + Y$ voltage received.

Decode: 1 if $R \geq 0.5$
0 otherwise.

1. What is P(decoding error | original bit is 1)? i.e., we sent 1, but we decoded as 0?

$0.0668$

2. What is P(decoding error | original bit is 0)?

$$P(R \geq 0.5 | X = -2) = P(-2 + Y \geq 0.5) = P(Y \geq 2.5) \approx 0.0062$$

Asymmetric decoding probability: We would like to avoid mistaking a 0 for 1. Errors the other way are tolerable.

# Gaussian Sampling and ELO ratings

What is the probability that the Warriors win?
How do you model zero-sum games?

# Gaussian Sampling and ELO ratings

Each team has an ELO score $S$, calculated based on its past performance.

- Each game, a team has ability $A \sim \mathcal{N}(S, 200^2)$.
- The team with the higher sampled ability wins.

What is the probability that Warriors win this game?

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

Arpad Elo

Warriors' $A_W \sim \mathcal{N}(S = 1657, 200^2)$

Opponent's $A_O \sim \mathcal{N}(S = 1470, 200^2)$

# Gaussian Sampling and ELO ratings

Want: $P(\text{Warriors win}) = P(A_W > A_O)$

```
from scipy import stats
WARRIORS_ELO = 1657
OPPONENT_ELO = 1470
STDEV = 200
NTRIALS = 10000

nSuccess = 0
for i in range(NTRIALS):
  w = stats.norm.rvs(WARRIORS_ELO, STDEV)
  o = stats.norm.rvs(OPPONENT_ELO, STDEV)
  if w > o:
    nSuccess += 1
print("Warriors sampled win fraction: ",
      float(nSuccess) / NTRIALS)
```

$\approx 0.7488$, calculated by sampling

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$

Opponents $A_O \sim \mathcal{N}(S = 1470, 200^2)$

# Is there a better way?

$$P(A_W > A_O)$$

- This is a probability of an event involving ***two continuous*** random variables!
- We'll solve this problem analytically in two weeks' time.

Big goal for next time: Events involving two ***discrete*** random variables.
Stay tuned!