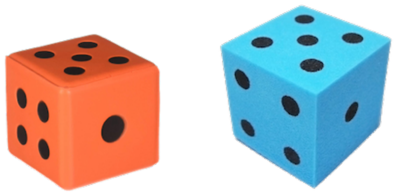
Conditional Probability and Bayes

Quick slide reference

Conditional Probability

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



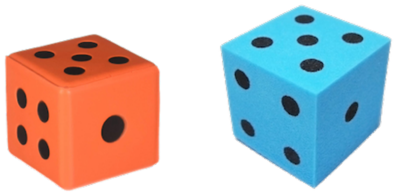
What is the best outcome for $P(D_1)$?

Your Choices:

- A. 1 and 3 tie for best
- B. 1, 2 and 3 tie for best
- C. 2 is the best
- D. Other/none/more than one

Dice, our misunderstood friends

Roll two 6-sided dice, yielding values D_1 and D_2.



Let E be event: $D_1 + D_2 = 4$.

What is $P(E)$?

$|S| = 36$

$E = \{(1,3), (2,2), (3,1)\}$

$P(E) = 3/36 = 1/12$

Let F be event: $D_1 = 2$.

What is $P(E$, given F already observed$)$?

$S = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)\}$

$E = \{(2,2)\}$

$P(E) = 1/6$

Conditional Probability

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Written as: $P(E|F)$

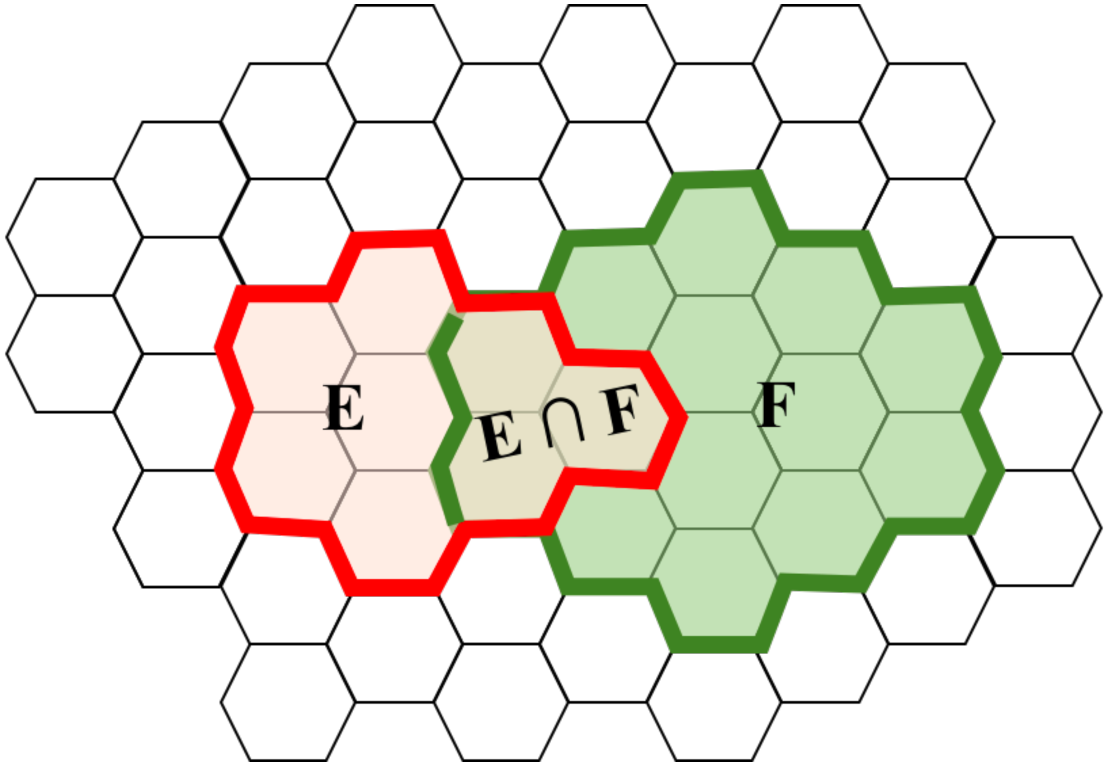
Means: "$P(E,$ given F already observed$)$"

Sample space → all possible outcomes consistent with F (i.e. $S \cap F$)

Event → all outcomes in E consistent with F (i.e. $E \cap F$)

Conditional Probability, visual intuition

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

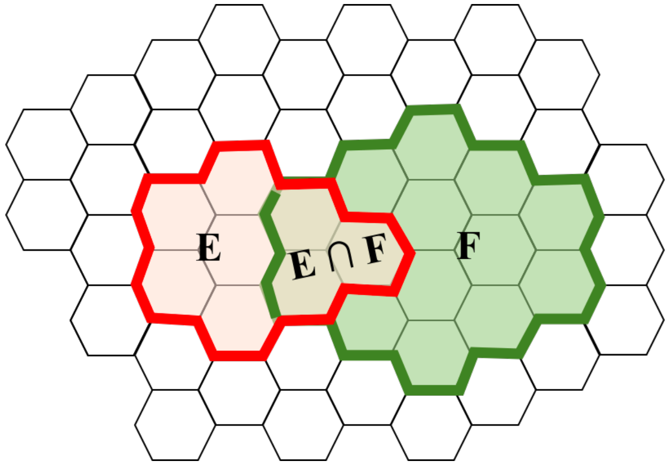
Conditional Probability, equally likely outcomes

The conditional probability of E given F is the probability that E occurs given that F has already occurred. This is known as conditioning on F.

Shorthand notation for set intersection (aka set "and")

With equally likely outcomes:

$$\Pr(E|F) = \frac{\text{\# of outcomes in } E \text{ consistent with } F}{\text{\# of outcomes in } S \text{ consistent with } F} = \frac{|EF|}{|SF|} = \frac{|EF|}{|F|}$$



$$P(E) = \frac{8}{50} \approx 0.16$$

$$P(E|F) = \frac{3}{14} \approx 0.21$$

Conditional probability in general

These properties hold even when outcomes are not equally likely.

General definition of conditional probability:

$$P(E|F) = \frac{P(EF)}{P(F)}$$

The Chain Rule (aka Product rule):

$$P(EF) = P(F)P(E|F)$$

What if P(F) = 0?

- P(E | F) undefined
- *Congratulations! Observed impossible*

Generalized Chain Rule

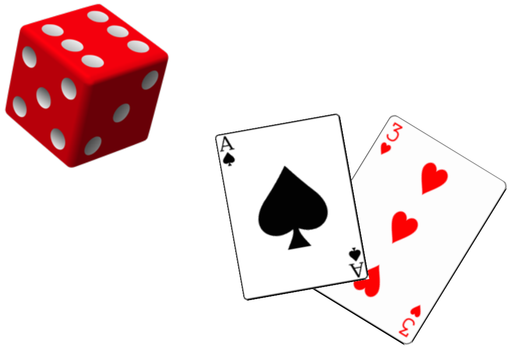
$$\Pr(E_1 \text{ and } E_2 \text{ and } E_3 \text{ and } \ldots E_n)$$

$$= \Pr(E_1) \cdot \Pr(E_1|E_2) \cdot \Pr(E_3|E_1, E_2) \cdots \Pr(E_n|E_1, E_2 \ldots E_{n_1})$$

This class going forward

<u>Last week</u>

Equally likely events

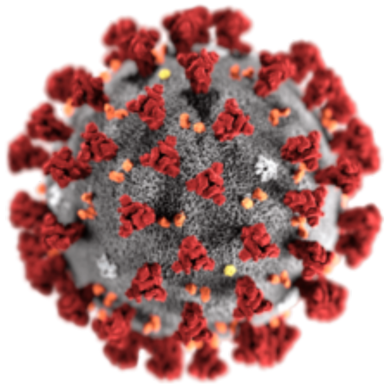


$P(E \cap F)$ $\quad$ $P(E \cup F)$

(counting, combinatorics)

<u>Today and for most of this course</u>

Not equally likely events



NETFLIX

and Learn

Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$

S = {Watch, Not Watch}

E = {Watch}

P(E) = ½ ?

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$$P(E)$$

Netflix and Learn

What is the probability that a user will watch Life is Beautiful?

$P(E)$

$$P(E) = \lim_{n \to \infty} \frac{n(E)}{n} \approx \frac{\#\text{people who watched movie}}{\#\text{people on Netflix}}$$

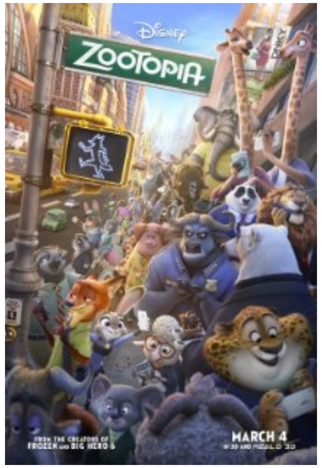
$P(E) = 10{,}234{,}231 \,/\, 50{,}923{,}123 = 0.20$

Netflix and Learn

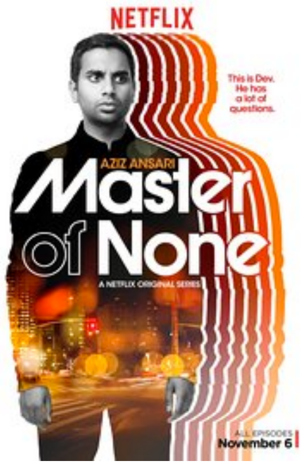
$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.

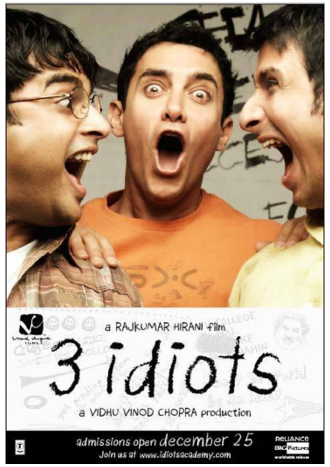
$P(E) = 0.19$



$P(E) = 0.32$



$P(E) = 0.20$



$P(E) = 0.09$

$P(E) = 0.20$

Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E = a user watches Life is Beautiful.

Let F = a user watches Amelie.

What is the probability that a user watches Life is Beautiful, given they watched Amelie?

$$P(E|F)$$

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{\text{\# people who have watched both}}{\text{\# people on Netflix}}}{\frac{\text{\# people who have watched Amelie}}{\text{\# people on Netflix}}}$$

$$= \frac{\text{\# people who have watched both}}{\text{\# people who have watched Amelie}}$$

$$\approx 0.42$$

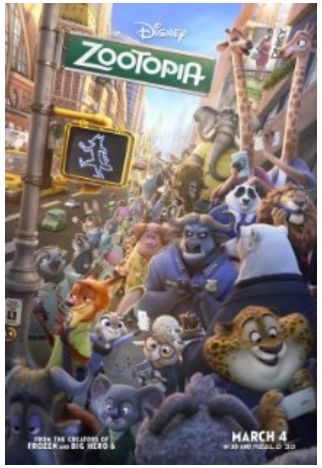
Netflix and Learn

$$P(E|F) = \frac{P(EF)}{P(F)}$$ Definition of Cond. Probability

Let E be the event that a user watches the given movie.
Let F be the event that the same user watches Amelie.

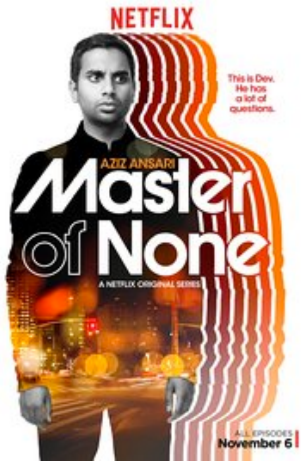
$P(E) = 0.19$

$P(E|F) = 0.14$



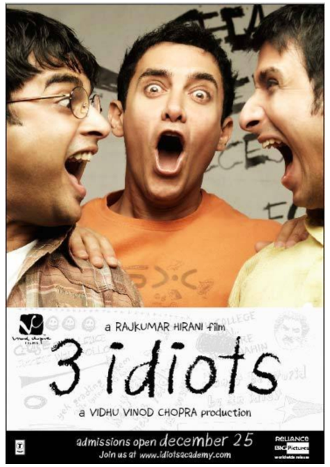
$P(E) = 0.32$

$P(E|F) = 0.35$



$P(E) = 0.20$

$P(E|F) = 0.20$



$P(E) = 0.09$

$P(E|F) = 0.72$

$P(E) = 0.20$

$P(E|F) = 0.42$

Machine Learning

Machine Learning is:
Probability + Data + Computers

Law of Total Probability

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule
(Product rule)

Definition of
conditional probability

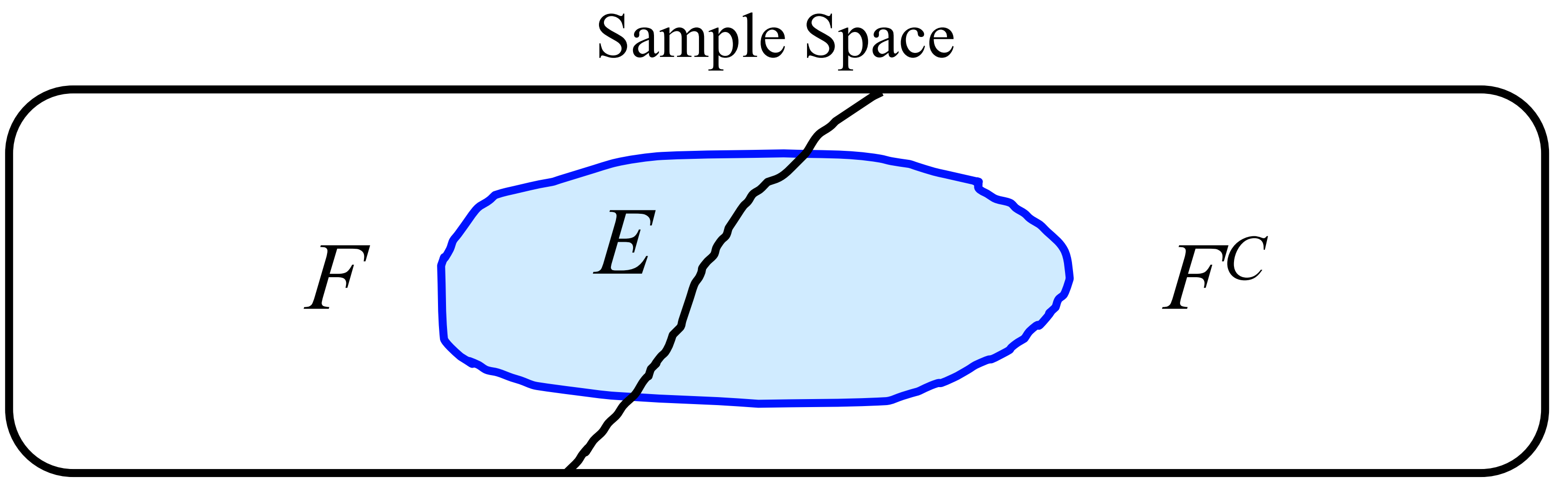
$$P(E|F)$$

Law of Total
Probability

$$P(E)$$

Law of Total Probability

Say E and F are events in S



$$E = EF \cup EF^c$$

$$P(E) = P(EF) + P(EF^c)?$$

Law of Total Probability

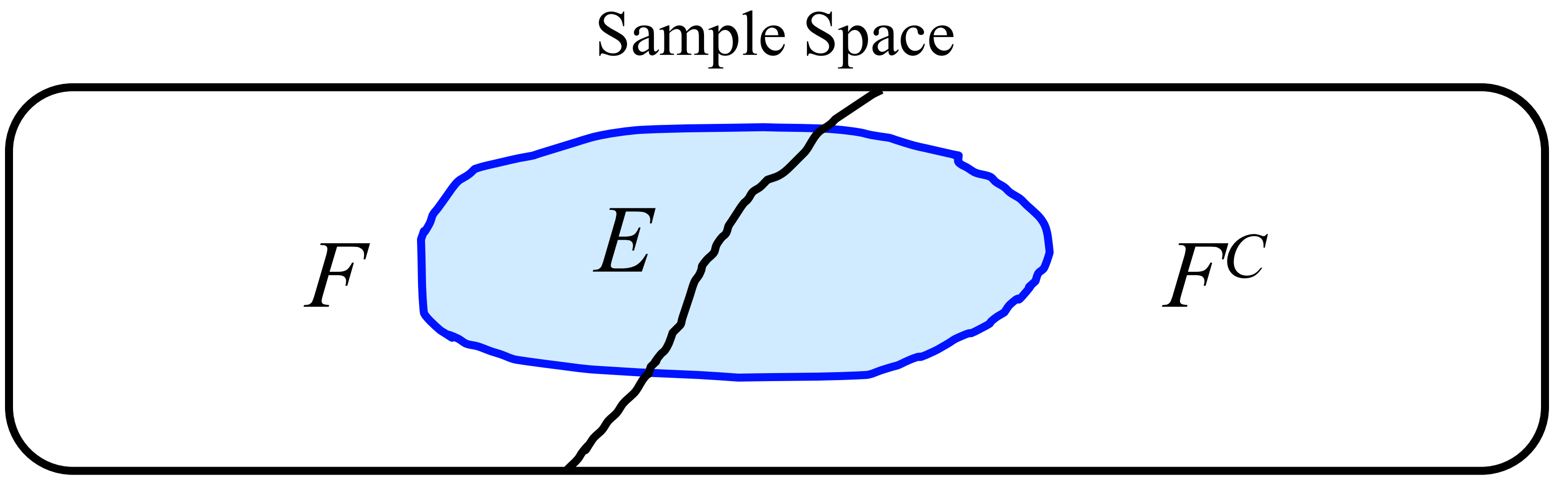
Thm Let F be an event where $P(F) > 0$. For any event E,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$$

Proof

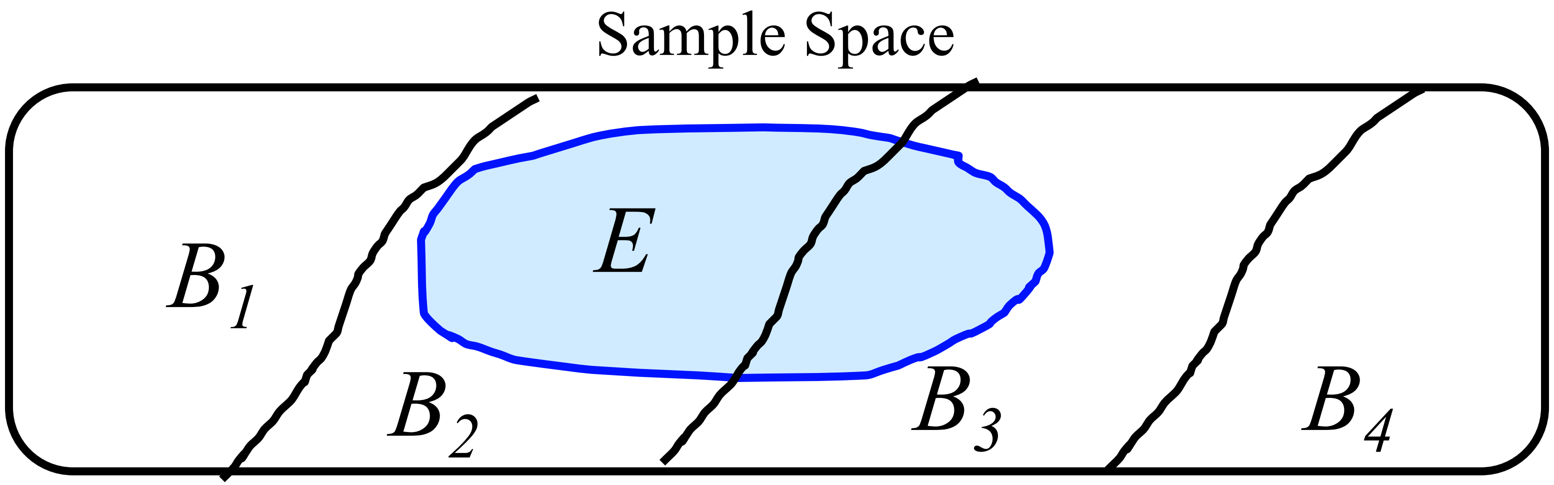
1. $E = (EF)$ or (EF^C) — Since F and F^C are disjoint
2. $P(E) = P(EF) + P(EF^C)$ — Probability of **or** for disjoint
3. $P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ — Chain rule (product rule)

Law of Total Probability



$$
\begin{aligned}
P(E) &= P(EF) + P(EF^C) \\
&= P(E|F)P(F) + P(E|F^C)P(F^C)
\end{aligned}
$$

Law of Total Probability



<u>Thm</u> For mutually exclusive events $B_1, B_2, \ldots, B_n$ s.t. $B_1 \cup B_2 \cup \cdots \cup B_n = S$,

$$P(E) = \sum_i P(B_i \cap E)$$

$$= \sum_i P(E|B_i)P(B_i)$$

Evolution of Bacteria

$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C)$ Law of Total Probability

You have bacteria in your gut which is causing a disease.
10% have a mutation which makes them resistant to anti-biotics
You take half a course of anti-biotics...

Probability a bacteria survives given it has the mutation: 20%
Probability a bacteria survives given it doesn't have the mutation: 1%
What is the probability that a randomly chosen bacteria survives?

Let E be the event that our bacterium survives. Let M be the event that a bacteria has the mutation. By the By Law of Total Probability (LOTP):

$$
\begin{aligned}
\Pr(E) &= \Pr(E \text{ and } M) + \Pr(E \text{ and } M^C) && \text{LOTP} \\
&= \Pr(E|M)\Pr(M) + \Pr(E|M^C)\Pr(M^C) && \text{Chain Rule} \\
&= 0.20 \cdot 0.10 + 0.01 \cdot 0.90 && \text{Substituting} \\
&= 0.029
\end{aligned}
$$

Real question. What is the probability that a surviving bacteria has the mutation?

$$\Pr(\text{Mutation} \mid \text{Survives})$$

$$\Pr(M \mid E)$$

Real Question: Pr(M | E)?

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\end{aligned}$$

Real Question: Pr(M | E)?

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10% have a mutation which makes them resistant to anti-biotics
You take half a course of anti-biotics…

$\Pr(E \mid M) = 0.20$
$\Pr(E \mid M^C) = 0.01$
What is the probability that a randomly chosen bacteria survives?

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$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

Chain rule
(Product rule)

Definition of
conditional probability

$$P(E|F)$$

Law of Total
Probability

$$P(E)$$

Relationship Between Probabilities

$$P(E \text{ and } F)$$

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Definition of conditional probability

$$P(E|F)$$

Law of Total Probability

Bayes' Theorem

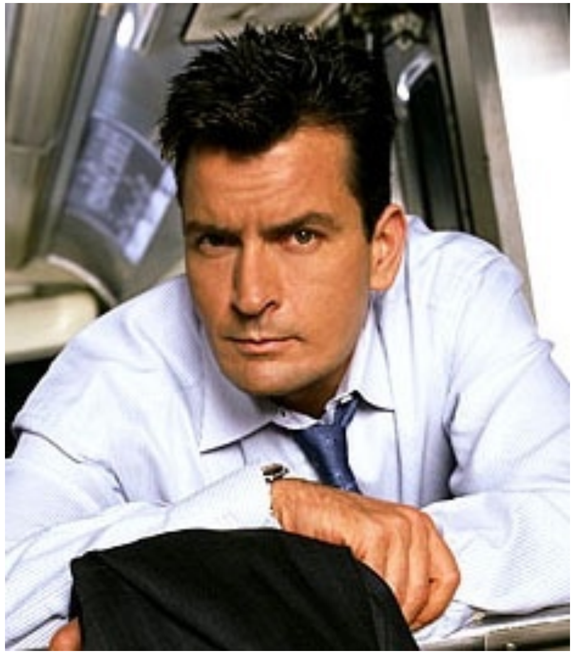
$$P(E)$$

$$P(F|E)$$

Bayes’ Theorem

Thomas Bayes

Rev. Thomas Bayes (~1701-1761):
British mathematician and Presbyterian minister



He looked remarkably similar to Charlie Sheen

(but that's not important right now)

Thomas Bayes

$P(F \mid E)$

I want to calculate
P(State of the world F | Observation E)
It seems so tricky!…

The other way around is easy
P(Observation E | State of the world F)
What options to I have, chief?

$P(E \mid F)$

Thomas Bayes

Want $P(F \mid E)$. Know $P(E \mid F)$

$$P(F|E) = \frac{P(EF)}{P(E)} \quad \text{Def. of Conditional Prob.}$$

A little while later…

$$= \frac{P(E|F)P(F)}{P(E)} \quad \text{Chain Rule}$$

Bayes' Theorem

$P(E|F) \Rightarrow P(F|E)$

Thm For any events E and F where $P(E) > 0$ and $P(F) > 0$,

$$\boxed{P(F|E) = \frac{P(E|F)P(F)}{P(E)}}$$

Proof

2 steps! See board

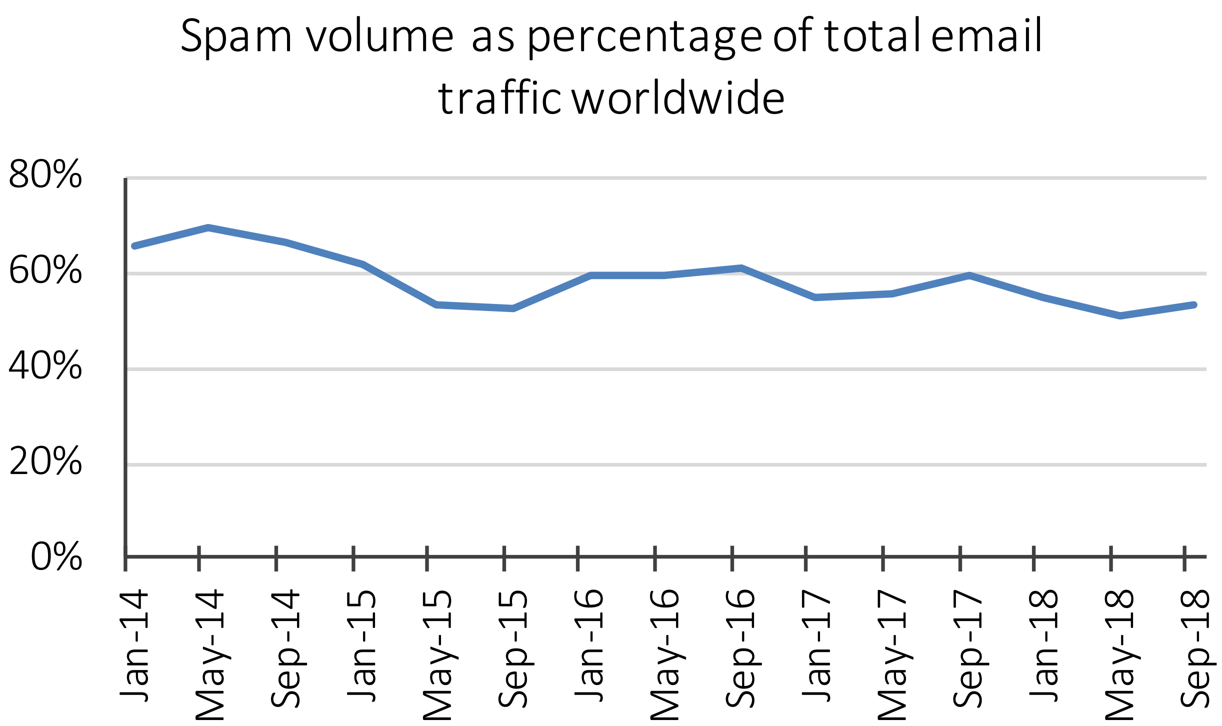
Expanded form:

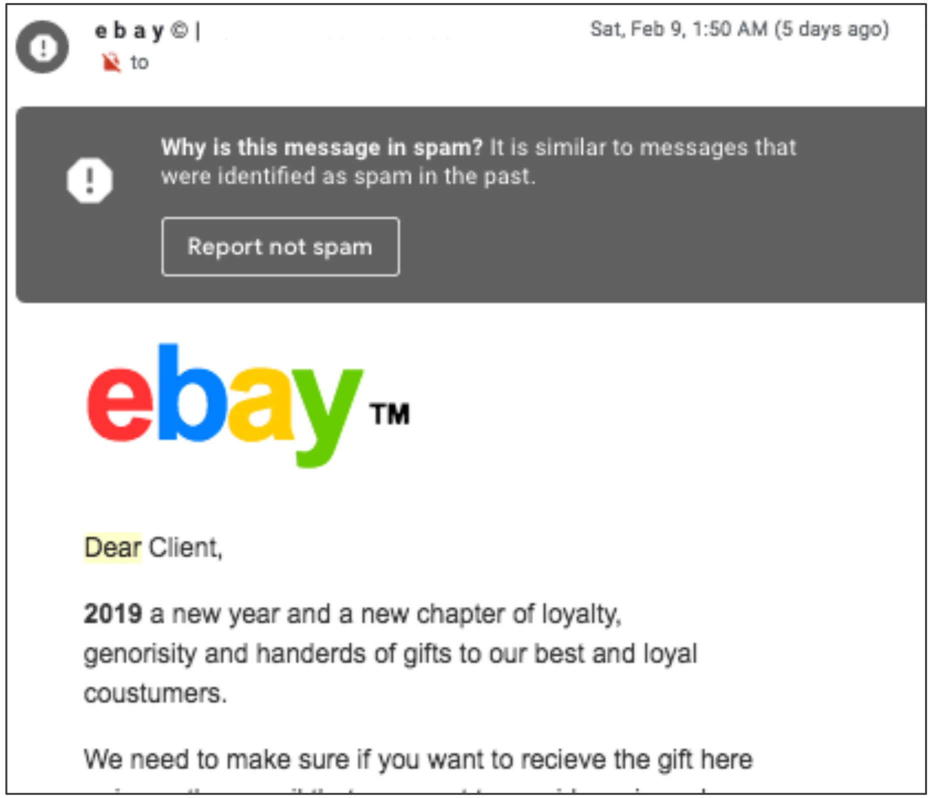
$$\boxed{P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}}$$

Proof

1 more step! See board

Detecting spam email





We can easily calculate how many spam emails contain "Dear":

$$P(E|F) = P\left(\text{“Dear”} \,\middle|\, \begin{matrix}\text{Spam}\\ \text{email}\end{matrix}\right)$$

But what is the probability that an email containing "Dear" is spam?

$$P(F|E) = P\left(\begin{matrix}\text{Spam}\\ \text{email}\end{matrix} \,\middle|\, \text{“Dear”}\right)$$

(silent drumroll)

Detecting spam email

$$P(F|E) = \frac{P(E|F)P(F)}{P(E|F)P(F) + P(E|F^C)P(F^C)}$$ Bayes' Theorem

- 60% of all email in 2016 is spam.
- 20% of spam has the word "Dear"
- 1% of non-spam (aka ham) has the word "Dear"

You get an email with the word "Dear" in it.

What is the probability that the email is spam?

1. Define events & state goal
2. Identify <u>known</u> probabilities
3. Solve

Let: E: "Dear", F: spam

Want: P(spam | "Dear")

$= P(F|E)$

Bayes' Theorem terminology

- 60% of all email in 2016 is spam. $P(F)$
- 20% of spam has the word "Dear" $P(E|F)$
- 1% of non-spam (aka ham) has the word "Dear" $P(E|F^C)$

You get an email with the word "Dear" in it.

What is the probability that the email is spam? Want: $P(F|E)$

$$\underbrace{P(F|E)}_{\text{posterior}} = \frac{\overbrace{P(E|F)}^{\text{likelihood}}\overbrace{P(F)}^{\text{prior}}}{\underbrace{P(E)}_{\text{normalization constant}}}$$

SARS Virus Testing

A test is 98% effective at detecting SARS

- However, test has a "false positive" rate of 1%
- 0.5% of US population has SARS
- Let E = you test positive for SARS with this test
- Let F = you actually have SARS
- What is P(F | E)?

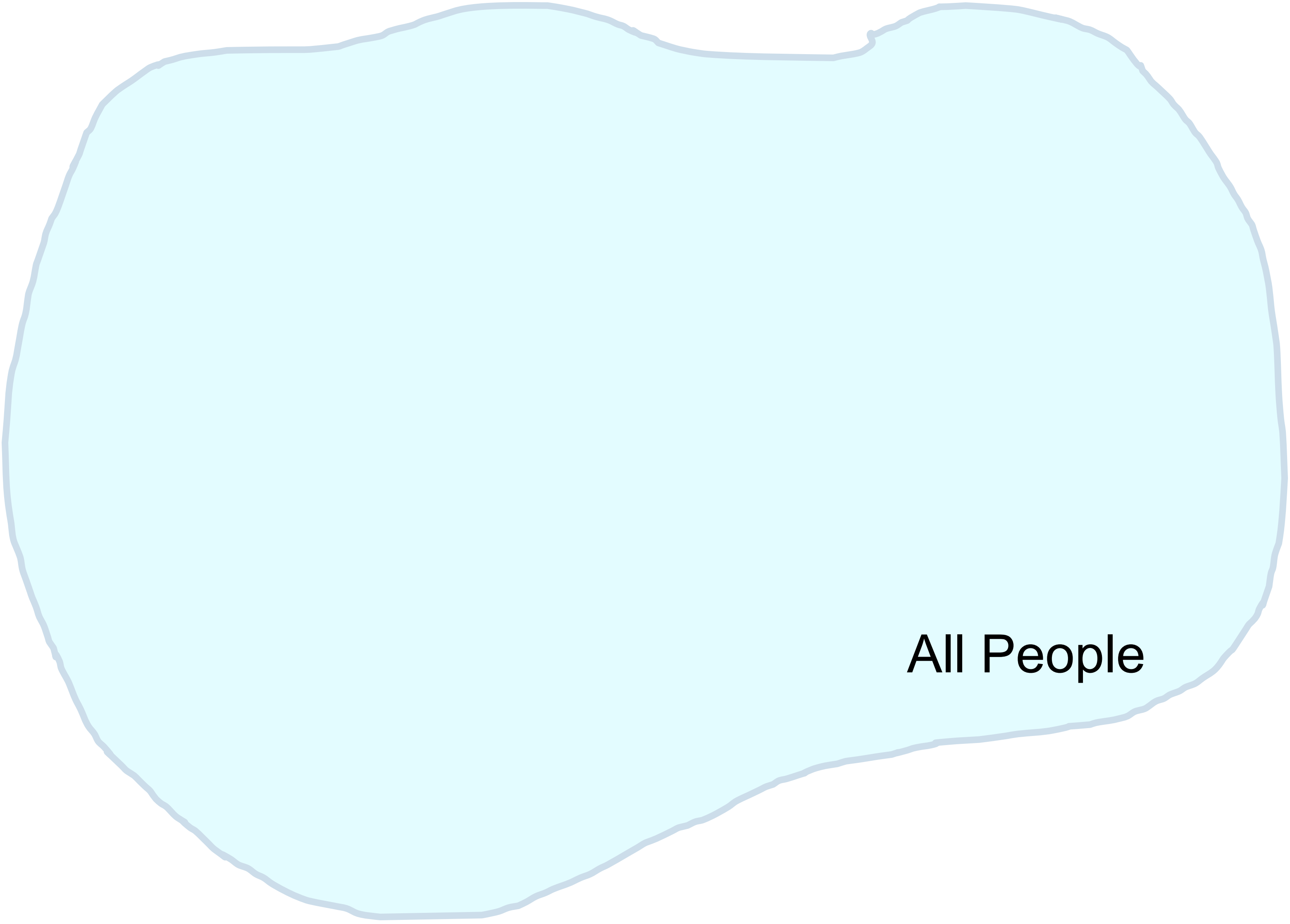
Solution:

$$P(F \mid E) = \frac{P(E \mid F)\, P(F)}{P(E \mid F)\, P(F) + P(E \mid F^c)\, P(F^c)}$$

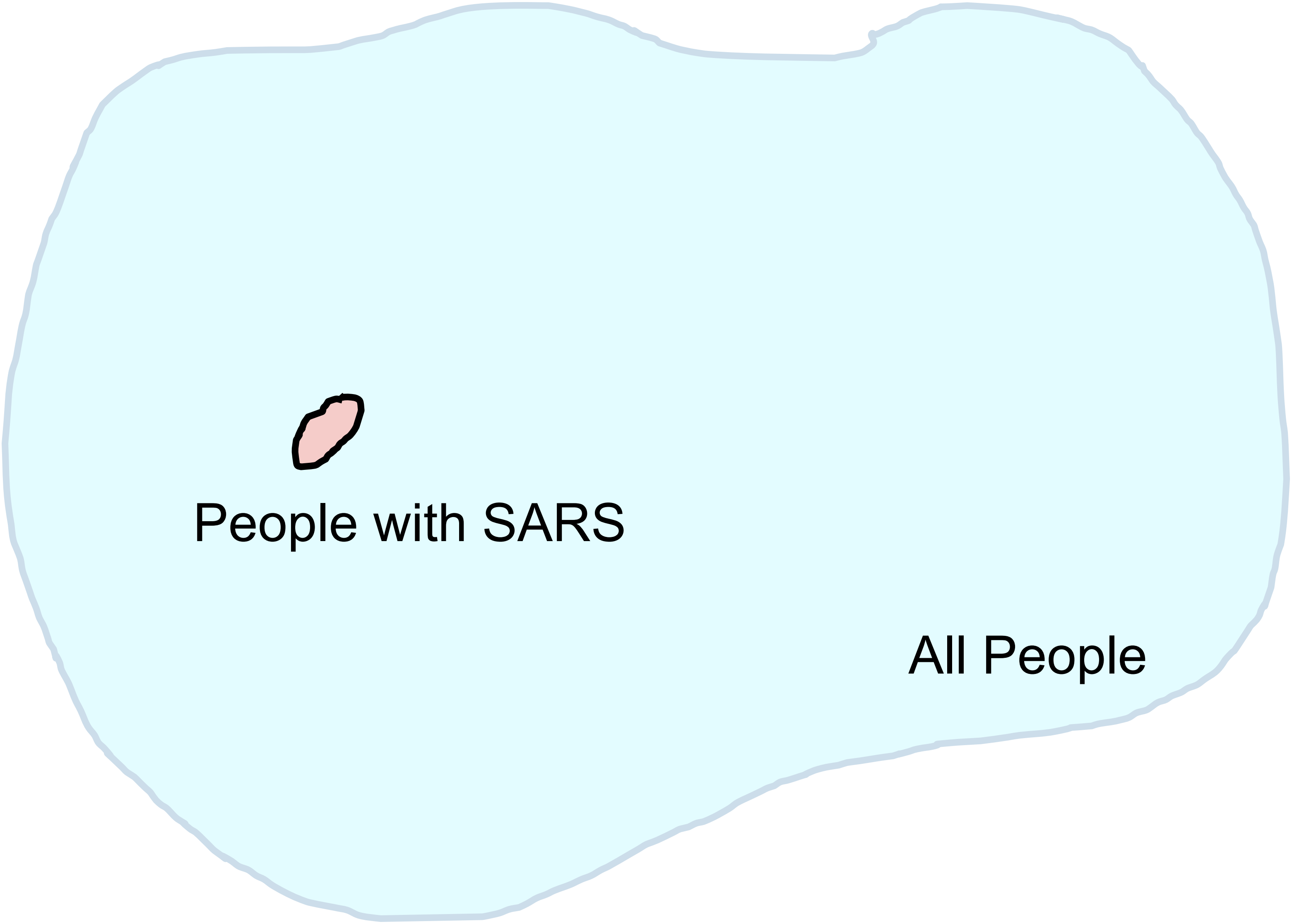
$$P(F \mid E) = \frac{(0.98)(0.005)}{(0.98)(0.005) + (0.01)(1 - 0.005)} \approx 0.330$$

Intuition Time

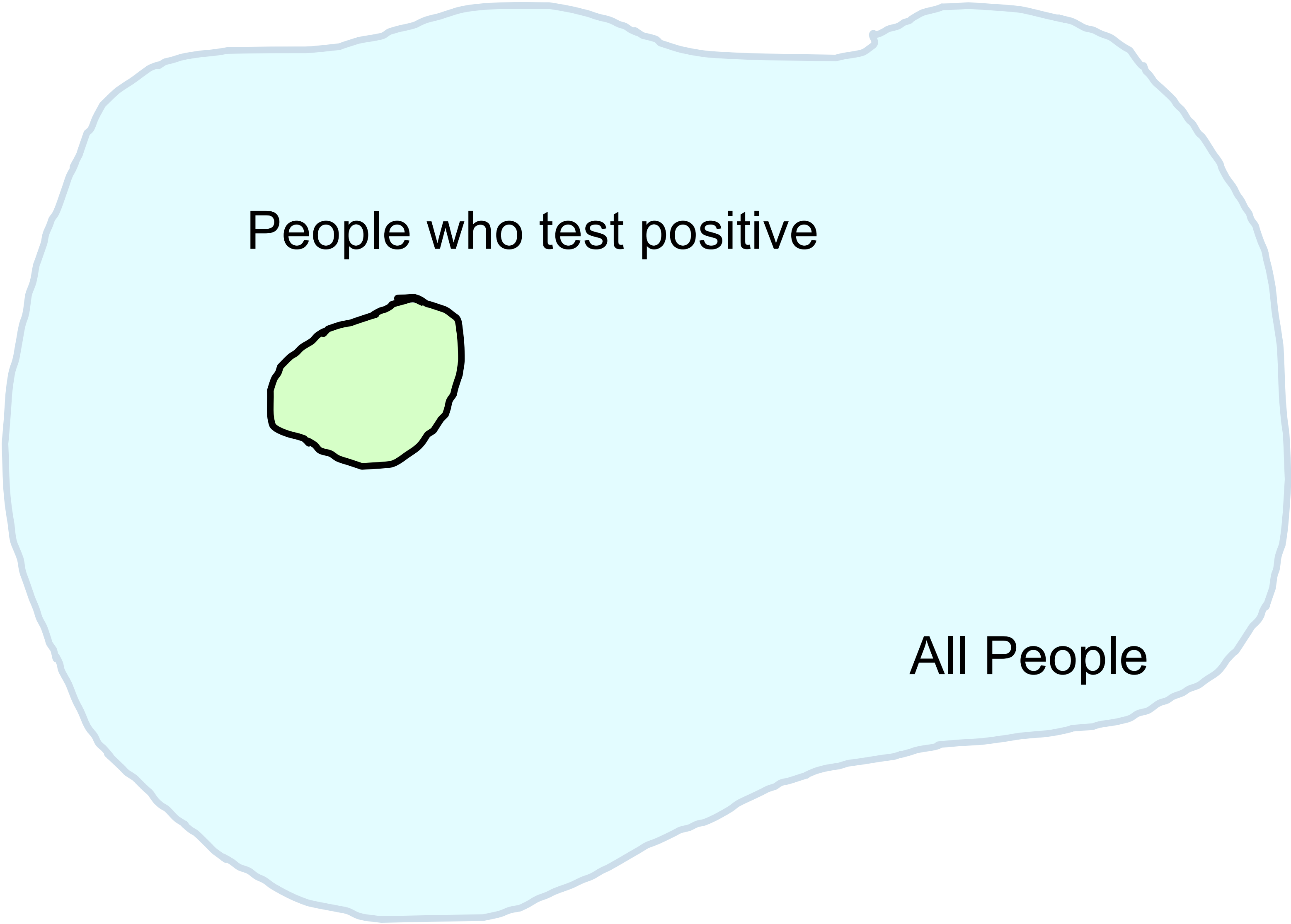
Bayes Thorem Intuition



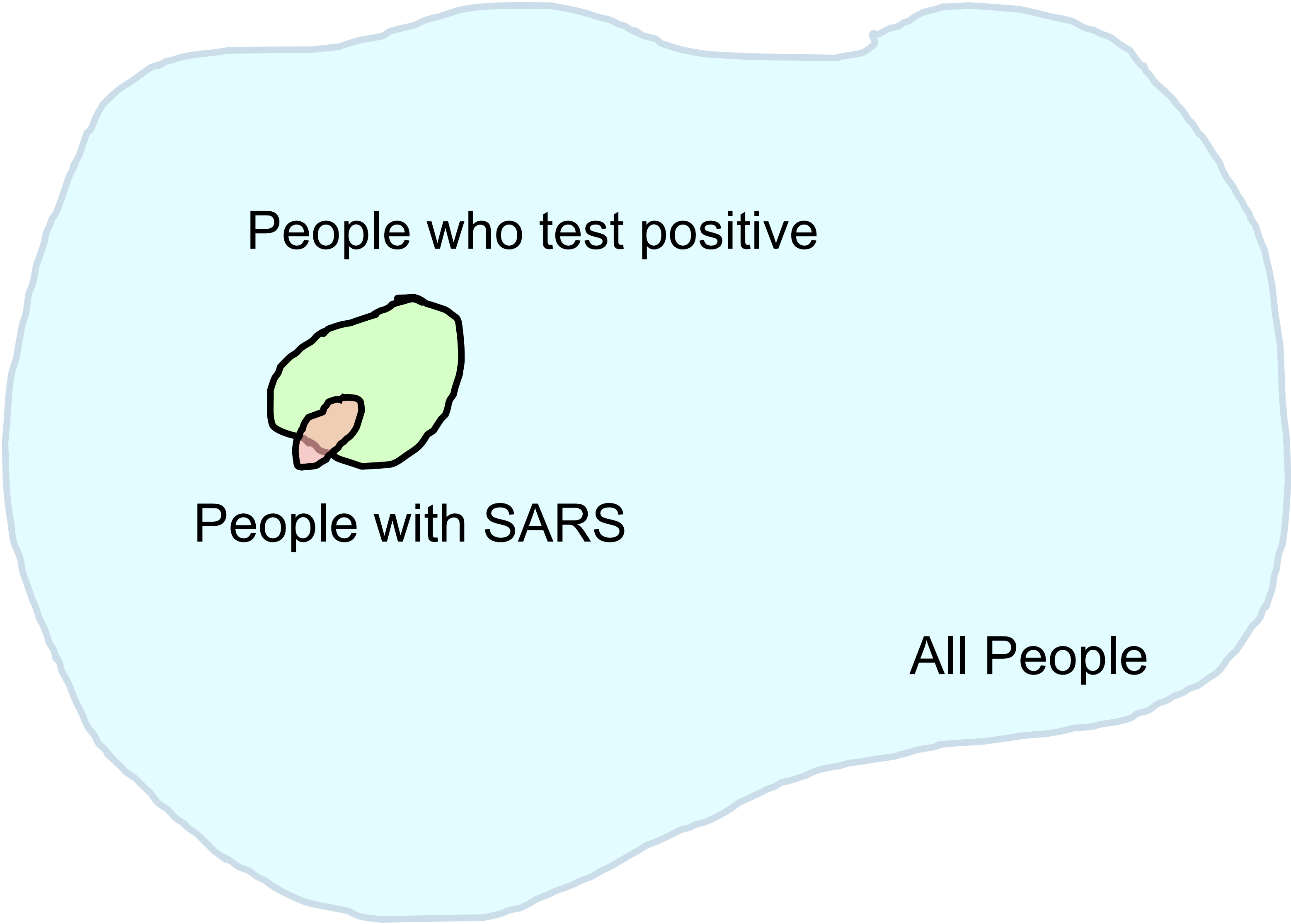
Bayes Thorem Intuition



Bayes Thorem Intuition

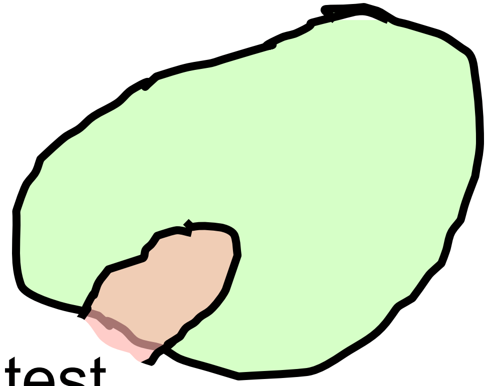


Bayes Thorem Intuition



Bayes Thorem Intuition

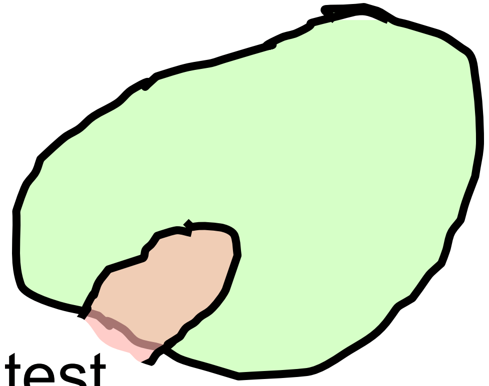
Conditioning on a positive result changes the sample space to this:



≈ 0.330

Bayes Thorem Intuition

Conditioning on a positive result changes the sample space to this:



People who test positive

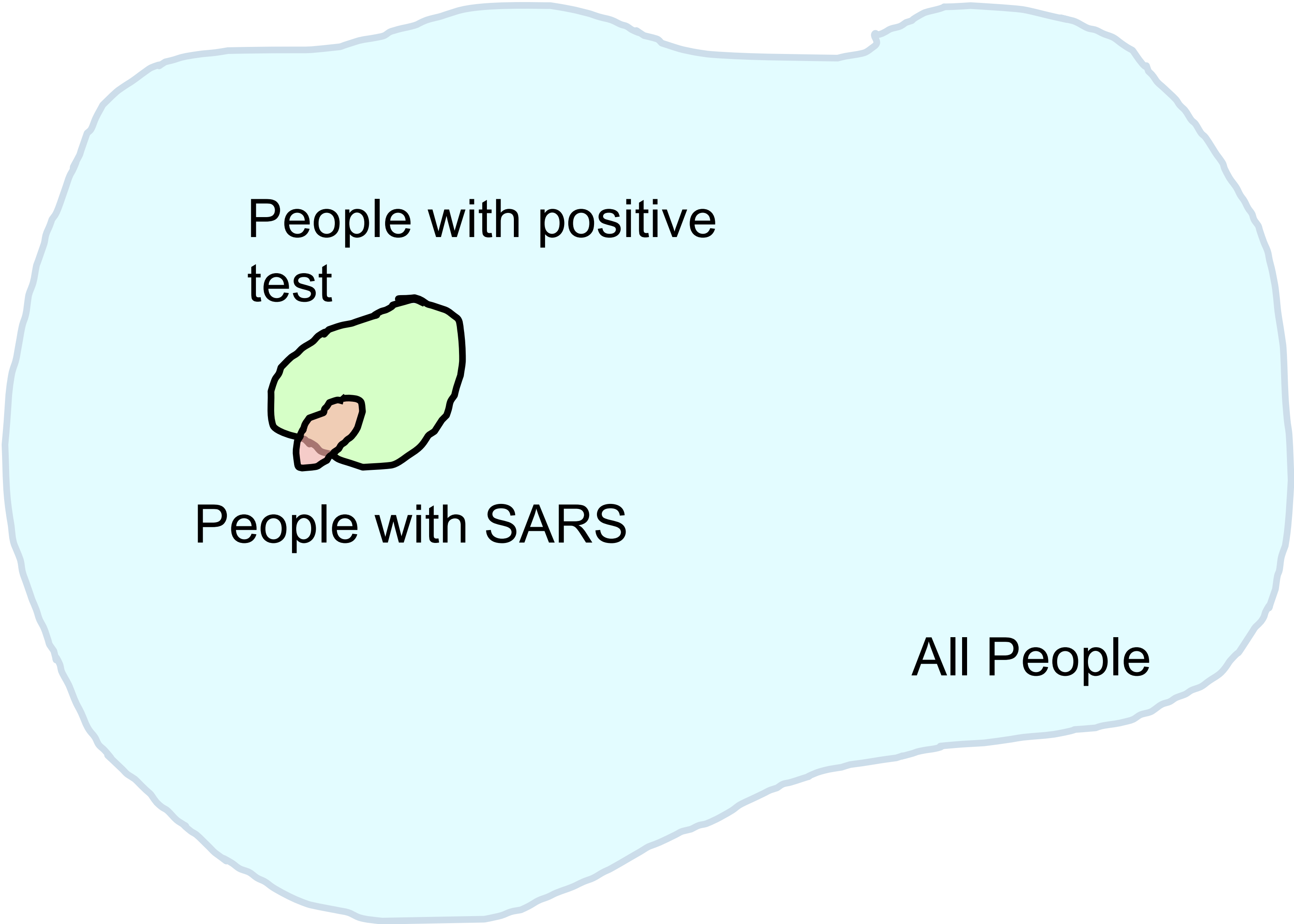
$$P(F)P(E|F) + P(F^c)P(E|F^c)$$

People who test positive and have SARS

$$P(F)P(E|F)$$

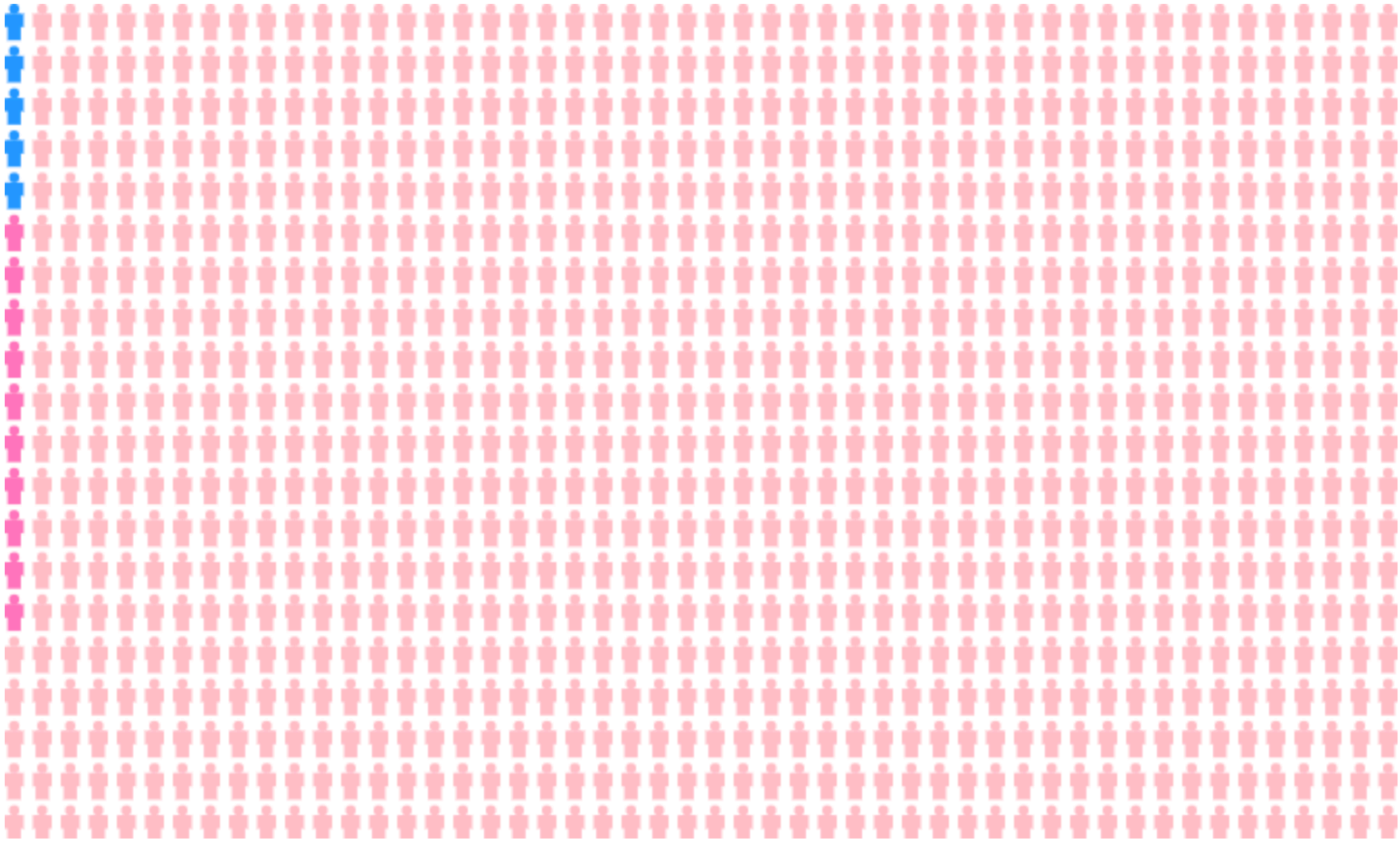
≈ 0.330

Bayes Thorem Intuition



Bayes Thorem Intuition

Say we have 1000 people:



5 have SARS and test positive, 985 **do not** have SARS and test negative.
10 **do not** have SARS and test positive.

≈ 0.333

Why it is still good to get tested

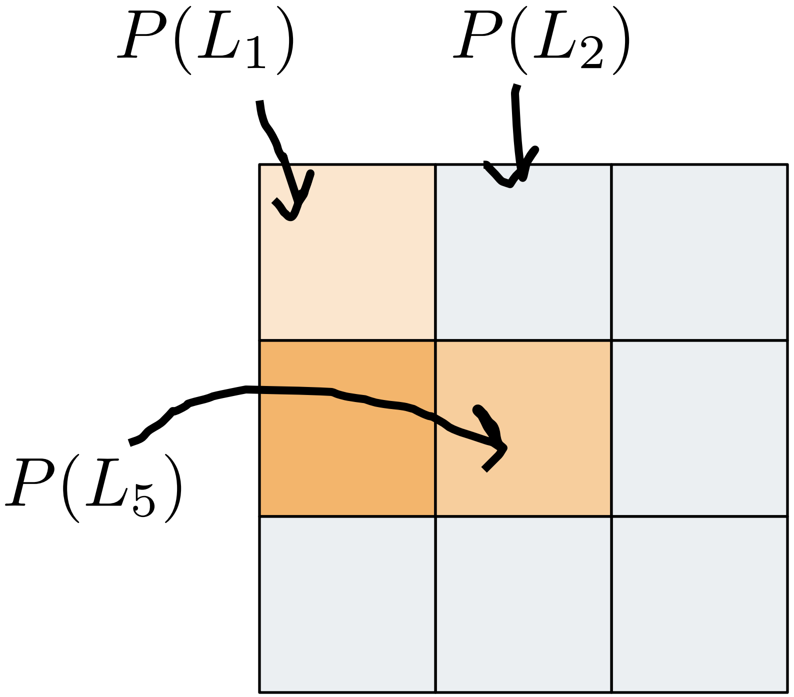
	SARS +	SARS –
Test +	$0.98 = P(E \mid F)$	$0.01 = P(E \mid F^c)$
Test –	$0.02 = P(E^c \mid F)$	$0.99 = P(E^c \mid F^c)$

- Let E^c = you test <u>negative</u> for SARS with this test
- Let F = you actually have SARS
- What is $P(F \mid E^c)$?

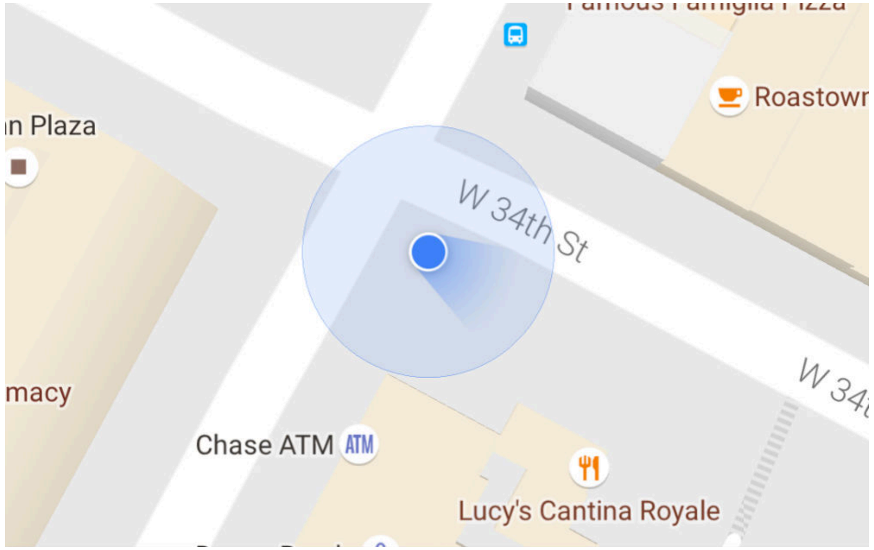
$$P(F \mid E^c) = \frac{P(E^c \mid F)\, P(F)}{P(E^c \mid F)\, P(F) + P(E^c \mid F^c)\, P(F^c)}$$

$$P(F \mid E^c) = \frac{(0.02)(0.005)}{(0.02)(0.005) + (0.99)(1 - 0.005)} \approx 0.0001$$

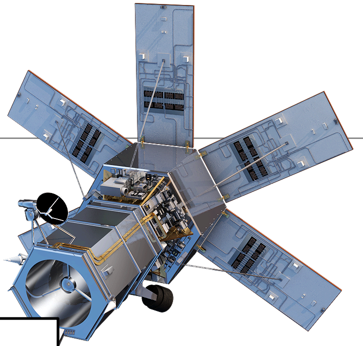
Bayes' Theorem and Location



Before Observation

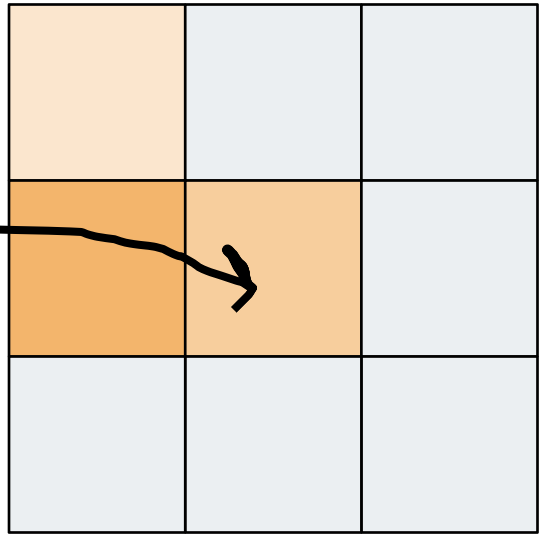


Bayes' Theorem and Location

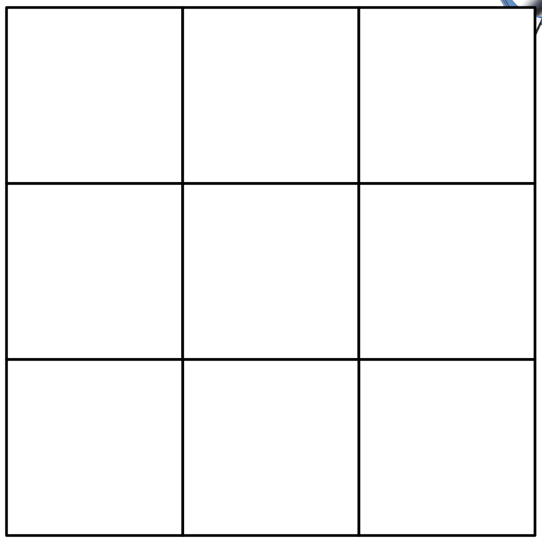


Know: $P(O|L_i)$

$P(L_5)$

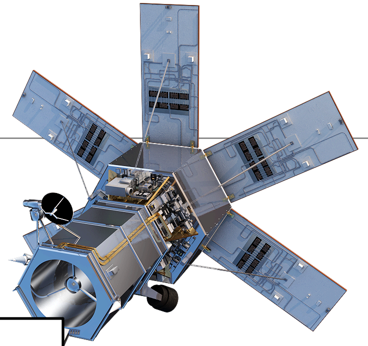


Before Observation

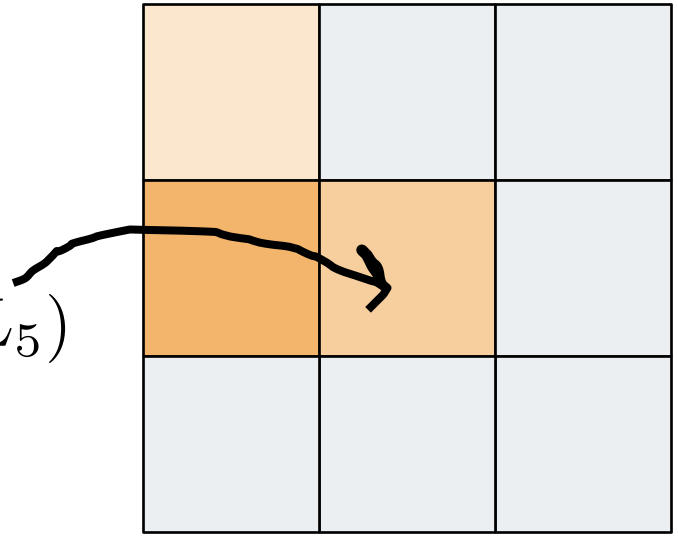


After Observation

Bayes' Theorem and Location

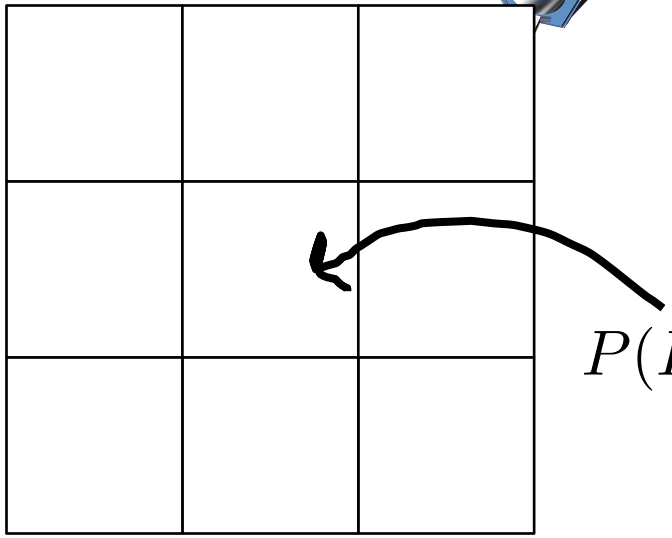


Know: $P(O|L_i)$



$P(L_5)$

Before Observation

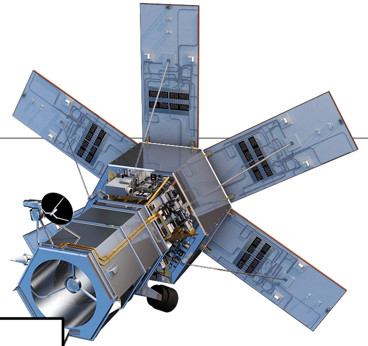


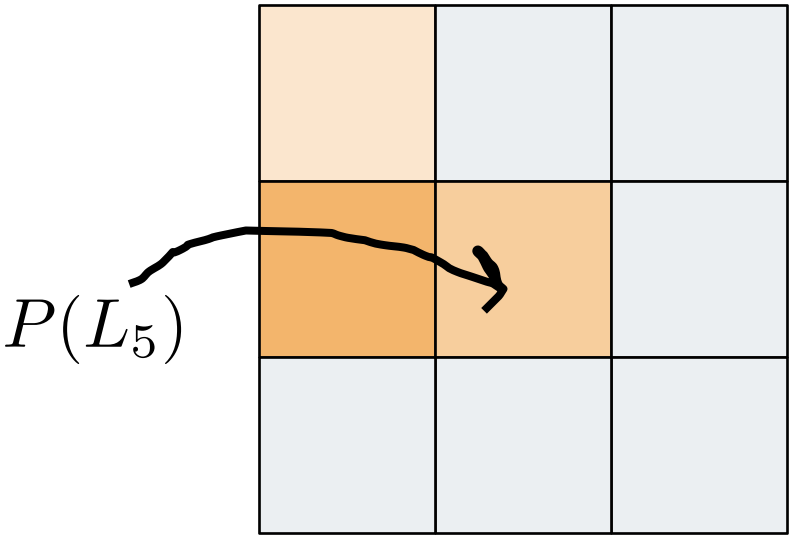
$P(L_5|O)$

After Observation

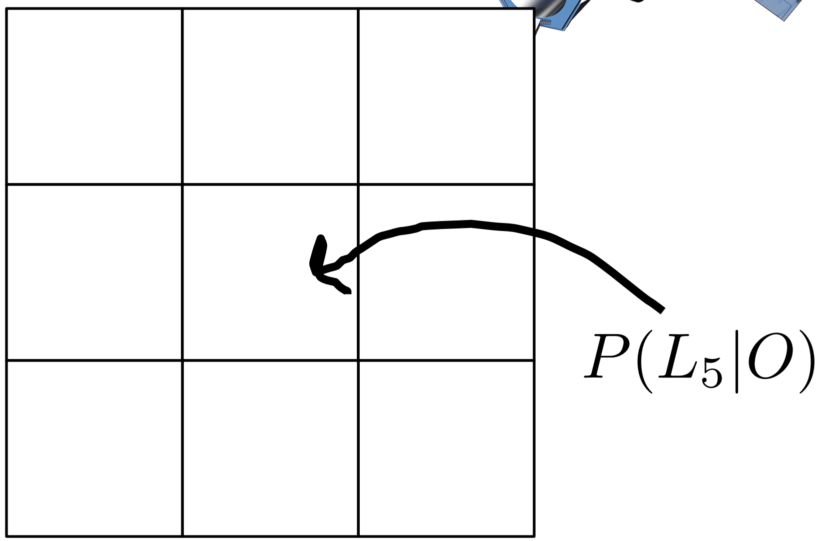
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{P(O)}$$

Bayes' Theorem and Location





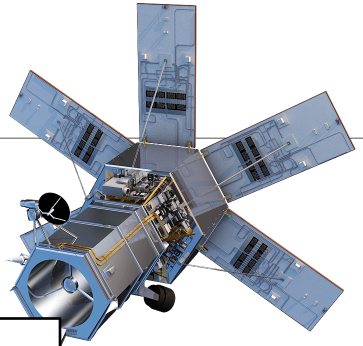
Before Observation

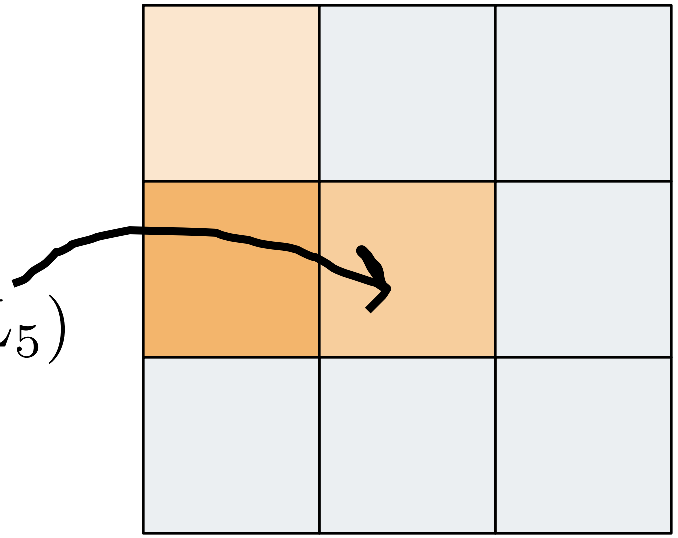


After Observation

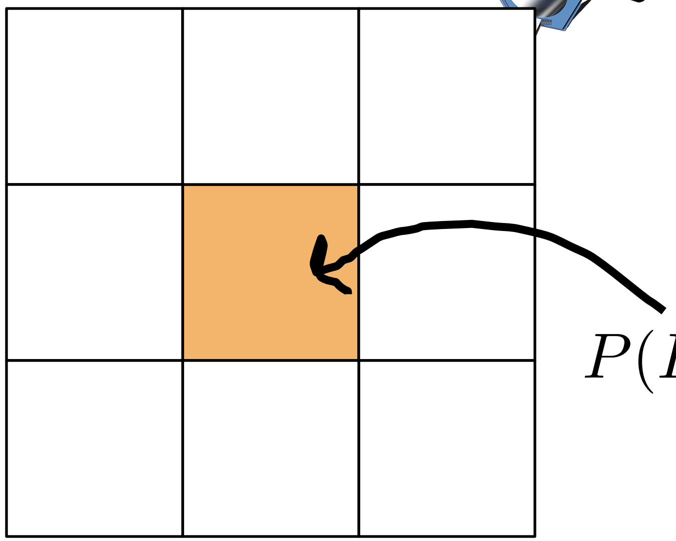
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Bayes' Theorem and Location





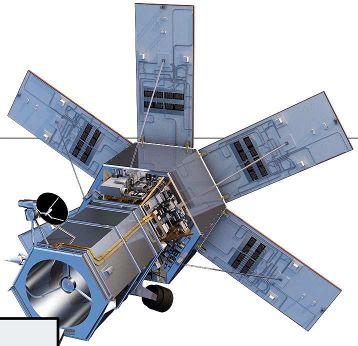
Before Observation

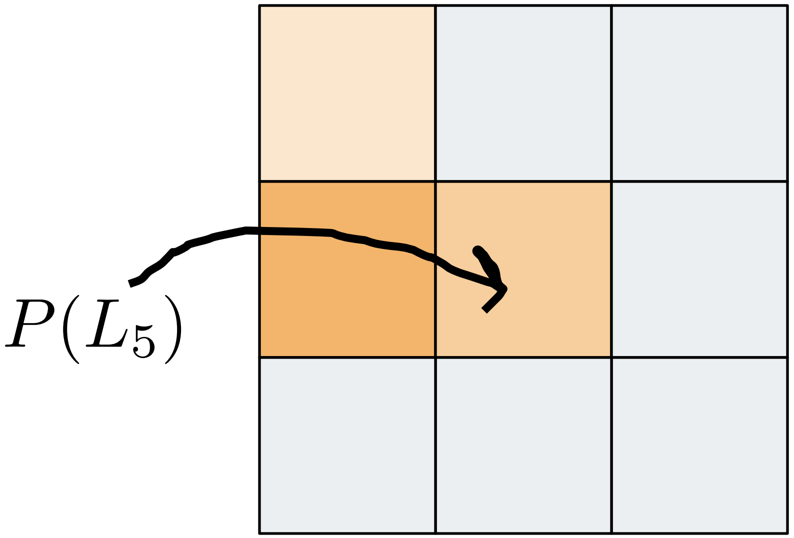


After Observation

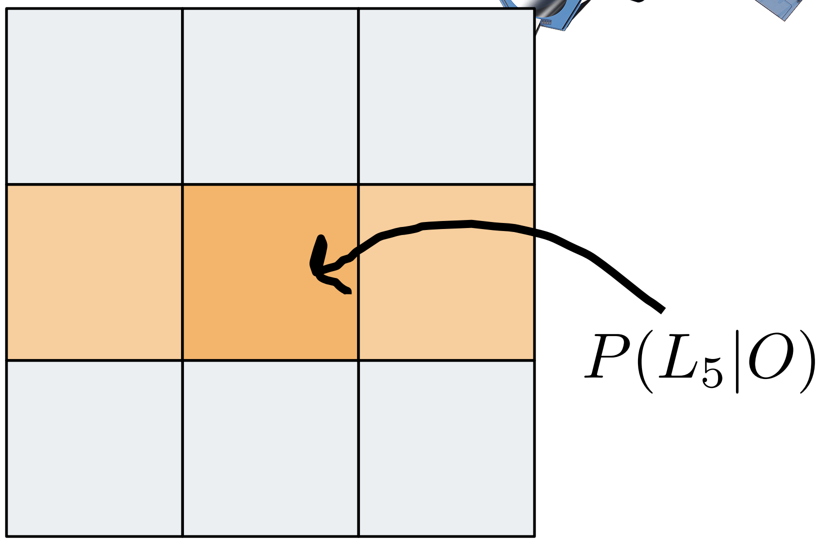
$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Bayes' Theorem and Location





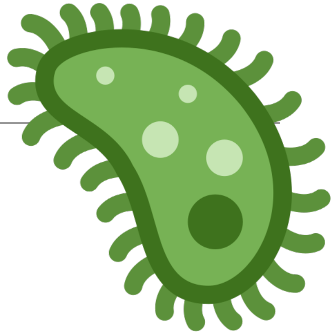
Before Observation



After Observation

$$P(L_5|O) = \frac{P(O|L_5)P(L_5)}{\sum_i P(O|L_i)P(L_i)}$$

Smell Test for Coronavirus?



- What is the probability of having corona virus given that you **can** still smell? Pr(Anosmia | Covid) = 0.7, Pr(Anosmia) = 0.03 among patients concerned about having Covid.

Let A be the event of Anosmia, C be the event of Covid

$$\Pr(C|A^C) = \frac{\Pr(A^C|C)\Pr(C)}{\Pr(A^C)} \qquad \text{Bayes}$$

$$= \frac{0.3 \cdot \Pr(C)}{0.97} \qquad \text{Substitute}$$

$$= 0.31 \cdot \Pr(C)$$

Monty Hall Problem

Monty Hall Problem

and Wayne Brady

Monty Hall Problem aka Let's Make a Deal

Behind one door is a prize (equally likely to be any door).

Behind the other two doors is nothing

1. We choose a door
2. Host opens 1 of other 2 doors, revealing nothing
3. We are given an option to change to the other door.

Doors A,B,C

Should we switch?

Note: If we don't switch, P(win) = 1/3 (random)

We are comparing P(win) and P(win|switch).

If we switch

Without loss of generality, say we pick A (out of Doors A,B,C).

1/3 1/3 1/3

A = prize

- Host opens B or C
- We switch
- We always lose

P(win | A prize,
picked A,
switched) = 0

B = prize

- Host must open C
- We switch to B
- We always win

P(win | B prize,
picked A,
switched) = 1

C = prize

- Host must open B
- We switch to C
- We always win

P(win | C prize,
picked A,
switched) = 1

P(win | picked A, switched) = 1/3 * 0 + 1/3 * 1 + 1/3 * 1 = 2/3

You should switch.

Monty Hall, 1000 envelope version

Start with 1000 envelopes (of which 1 is the prize).

1. You choose 1 envelope.

$$\left\{\begin{array}{l} \frac{1}{1000} = \text{P(envelope is prize)} \\ \frac{999}{1000} = \text{P(other 999 envelopes have prize)} \end{array}\right.$$

2. I open 998 of remaining 999 (showing they are empty).

$$\frac{999}{1000} = \text{P(998 empty envelopes had prize)} + \text{P(last other envelope has prize)}$$

$$= \text{P(last other envelope has prize)}$$

3. Should you switch?

No: P(win without switching) = $\frac{1}{\text{original \# envelopes}}$

Yes: P(win with new knowledge) = $\frac{\text{original \# envelopes - 1}}{\text{original \# envelopes}}$