Continuous Joint

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GO PLACIDLY amid the noise and the haste, and remember what peace there may be in silence. As far as possible, without surrender, be on good terms with all persons. Speak your truth quietly and clearly; and listen to others, even to the dull and the ignorant; they too have their story. If you compare yourself with others, you become vain or bitter, for always there will be greater and lesser persons than yourself.

Enjoy your achievements as well as your plans. Keep interested in your own career, however humble; it is a real possession in the changing fortunes of time. Exercise caution in your affairs, for the world is full of trickery. But let this not blind you to what virtue there is; many persons strive for high ideals, and everywhere life is full of heroism. Neither be cynical about love; for in the face of all aridity and disenchantment, it is as perennial as the grass.

Nurture strength of spirit to shield you in sudden misfortune. But do not distress yourself with dark imaginings. Many fears are born of fatigue and loneliness. Be gentle with yourself. You are a child of the universe no less than the trees and the stars; you have a right to be here.

And whether or not it is clear to you, no doubt the universe is unfolding as it should. Therefore be at peace with God, whatever you conceive Him to be. And whatever your labors and aspirations, in the noisy confusion of life, keep peace in your soul. With all its sham, drudgery and broken dreams, it is still a beautiful world. – Desiderata, Ehrmann

What is Gaussian blur all about?

<Review>

Probability Density Function

$\mathcal{N}(\mu, \sigma^2)$

"exponential"

the distance to the mean

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

probability density at x

a constant

sigma shows up twice

Cumulative Density Function

$$\mathcal{N}(\mu, \sigma^2)$$

CDF of Standard Normal: A function that has been solved for numerically

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

The cumulative density function (CDF) of any normal

Sum of Independent Normals

Let X and Y be independent random variables

- $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$
- $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Generally, have *n* independent random variables $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2, \ldots, n$:

$$\left(\sum_{i=1}^{n} X_i\right) \sim N\left(\sum_{i=1}^{n} \mu_i,\ \sum_{i=1}^{n} \sigma_i^2\right)$$

Sum of independent Normals

$X \sim \mathcal{N}(\mu_1, \sigma_1^2)$,
$Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$
X, Y independent

$$X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

(proof left to Wikipedia)

Holds in general case:

$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$
X_i independent for $i = 1, \dots, n$

$$\sum_{i=1}^{n} X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right)$$

Back for another playoffs game

$$P(A_W > A_B)$$

This is a probability of an event involving ***two*** random variables!

We will compute:

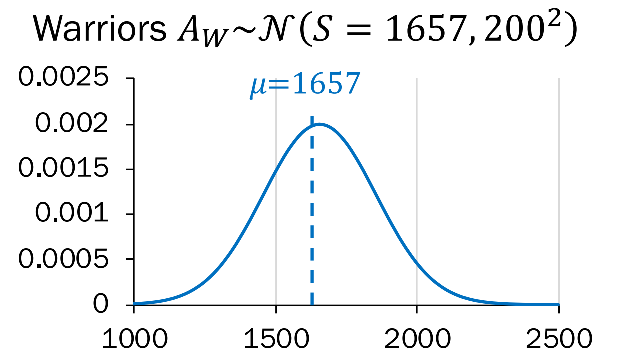
$$P(\underbrace{A_W - A_B}_{\text{A sum of Normals!}} > 0)$$

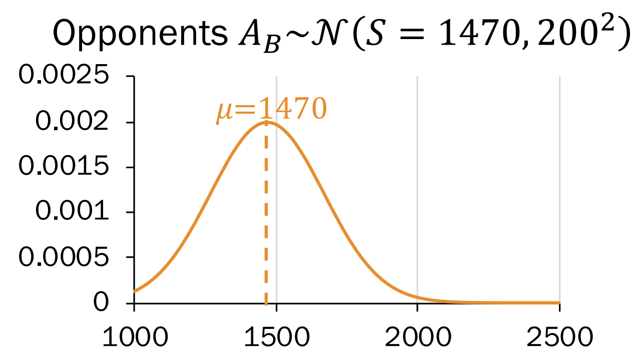
What is the probability that the Warriors win?
How do you model zero-sum games?

Motivating idea: Zero sum games

Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

Assume A_W, A_B are independent.
Let $D = A_W - A_B$.





What is the distribution of D?

A. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 - 200^2)$
B. $D \sim \mathcal{N}(1657 - 1470,\ 200^2 + 200^2)$
C. $D \sim \mathcal{N}(1657 + 1470,\ 200^2 + 200^2)$
D. Dance, Dance, Convolution
E. None/other

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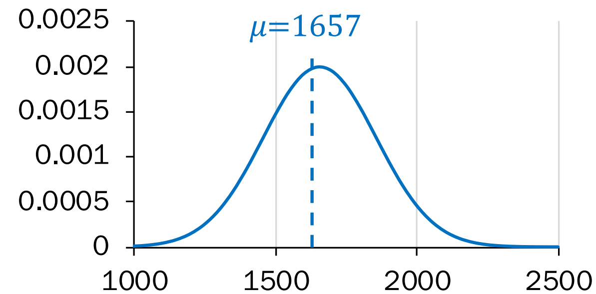
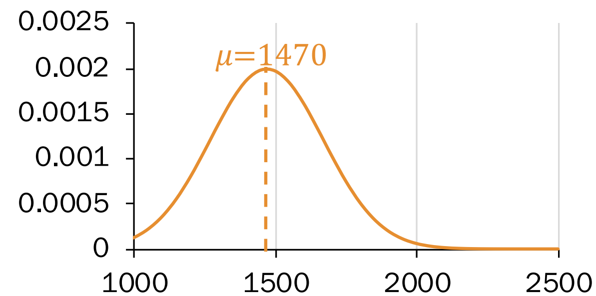
E. None/other

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$



If $X \sim \mathcal{N}(\mu_1, \sigma^2)$,
then $(-X) \sim \mathcal{N}(-\mu, (-1)^2\sigma^2 = \sigma^2)$.

Motivating idea: Zero sum games

Want: $P(\text{Warriors win}) = P(A_W - A_B > 0)$

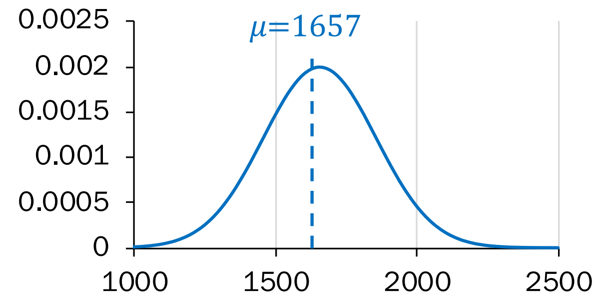
Assume A_W, A_B are independent.
Let $D = A_W - A_B$.

$$D \sim \mathcal{N}(1657 - 1470,\ \ 200^2 + 200^2)$$
$$\sim \mathcal{N}(187, 2 \cdot 200^2) \qquad \sigma \approx 283$$

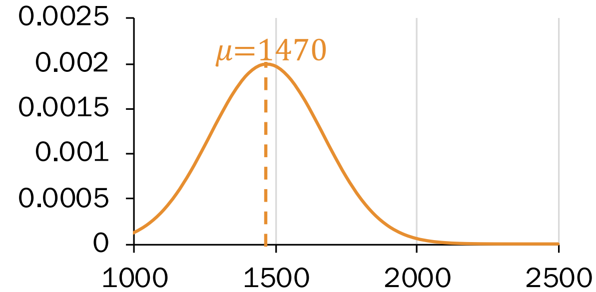
$$P(D > 0) = 1 - F_D(0) = 1 - \Phi\left(\frac{0 - 187}{283}\right)$$
$$\approx 0.7454$$

Compare with 0.7488, calculated by sampling!

Warriors $A_W \sim \mathcal{N}(S = 1657, 200^2)$



Opponents $A_B \sim \mathcal{N}(S = 1470, 200^2)$



Sum of Normal Approximations

Say you are working on getting volunteers to sign up for your day of action

- Advertise to two groups
- P1: 50 people, each independently join with $p = 0.1$
- P2: 100 people, each independently join with $p = 0.4$
- Question: Probability of more than 40 join?

Sum of Normal Approximations

- Two groups
- P1: 50 people, each independently infected with $p = 0.1$
- P2: 100 people, each independently infected with $p = 0.4$

n · p

n · p · (1 – p)

- A = # join in P1 $\quad A \sim \text{Bin}(50, 0.1) \approx X \sim N(5, 4.5)$
- B = # join in P2 $\quad B \sim \text{Bin}(100, 0.4) \approx Y \sim N(40, 24)$
- What is P(≥ 40 people join)?
- $P(A + B \geq 40) \approx P(X + Y \geq 39.5)$
- $X + Y = W \sim N(5 + 40 = 45, 4.5 + 24 = 28.5)$

$$P(W \geq 39.5) = P\left(\frac{W-45}{\sqrt{28.5}} > \frac{39.5-45}{\sqrt{28.5}}\right) = 1 - \Phi(-1.03) \approx 0.8485$$

Great Question: Linear Transform or Sum?

$$X \sim N(\mu, \sigma^2)$$

$$Y = X + X = 2 \cdot X$$

$$Y \sim N(2\mu, 4\sigma^2)$$

$$Y = X + X = 2 \cdot X$$

$$X + X \sim N(\mu + \mu, \sigma^2 + \sigma^2)$$

$$Y \sim N(2\mu, 2\sigma^2)$$

X is not independent of X

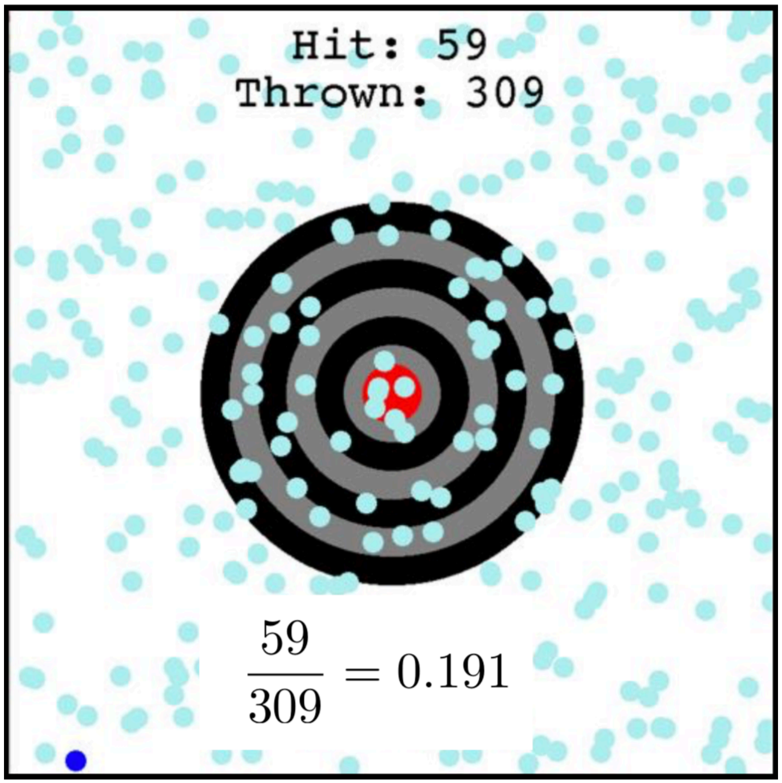
</Review>

Quick slide reference

16a_cont_joint

Continuous joint distributions

Remember target?



Good times...

CS109 logo with darts

The CS109 logo was created by throwing 500,000 darts according to a joint distribution.

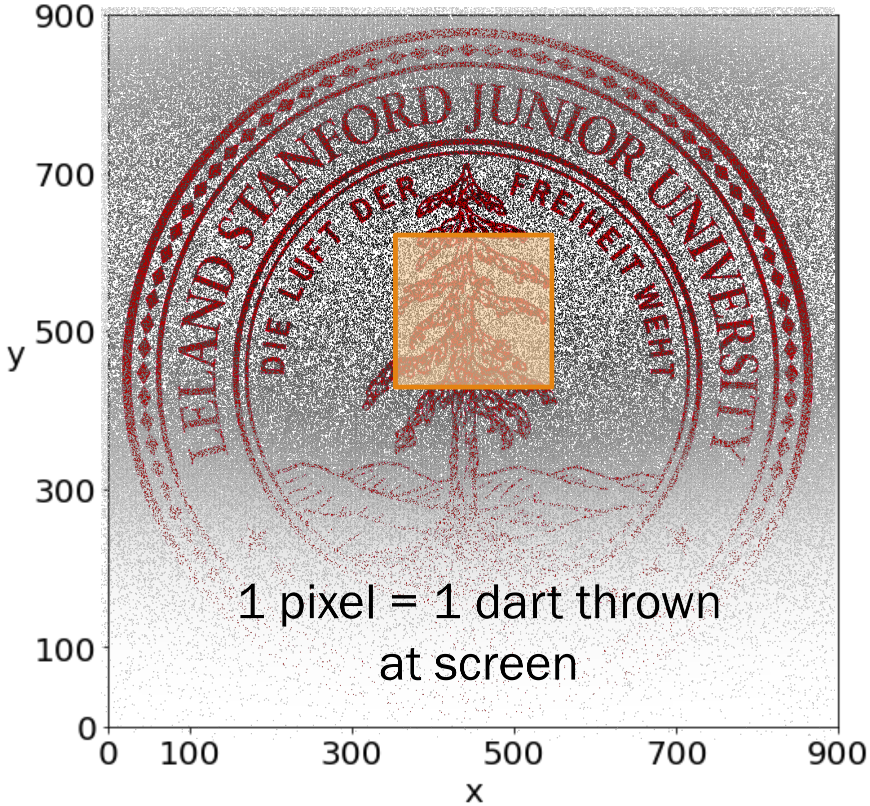
If we throw another dart according to the same distribution, what is

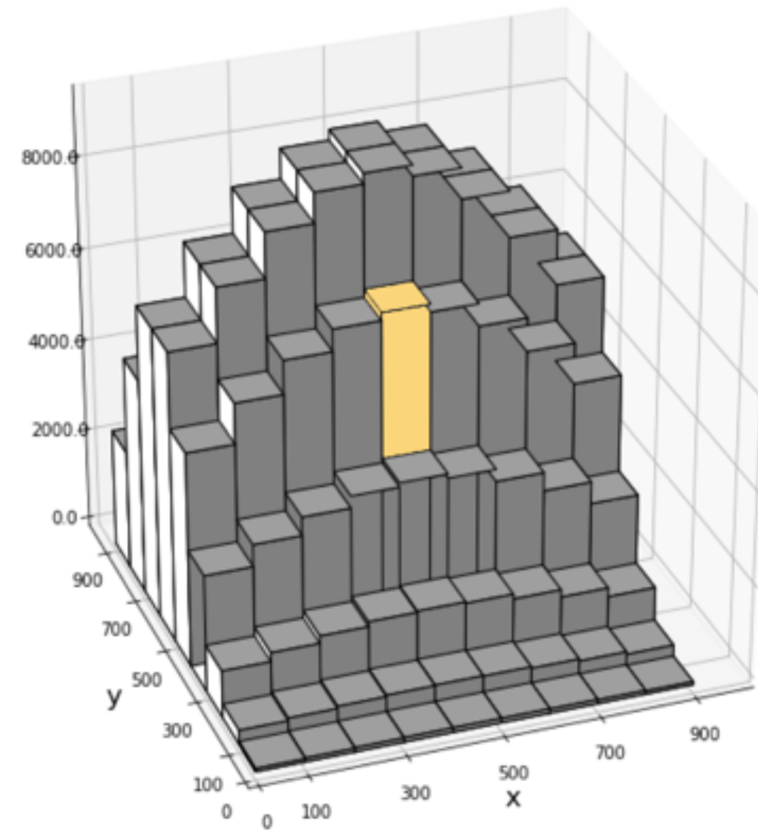
P(dart hits within r pixels of center)?

Quick check: What is the probability that a dart hits at (456.2344132343, 532.1865739012)?

CS109 logo with darts

P(dart hits within r pixels of center)?

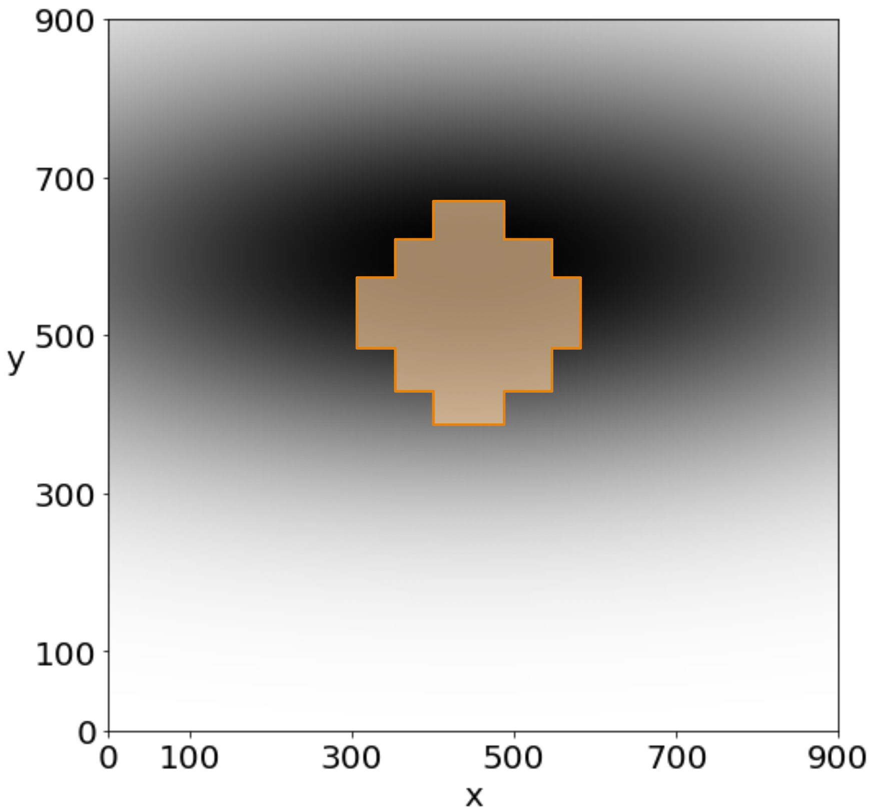


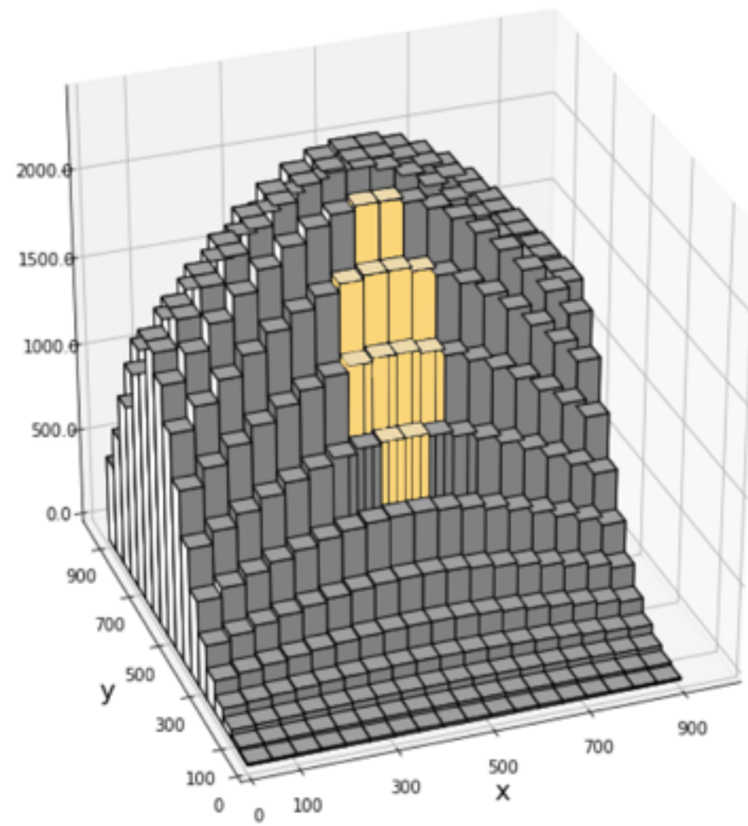


Possible dart counts (in 100x100 boxes)

CS109 logo with darts

P(dart hits within r pixels of center)?

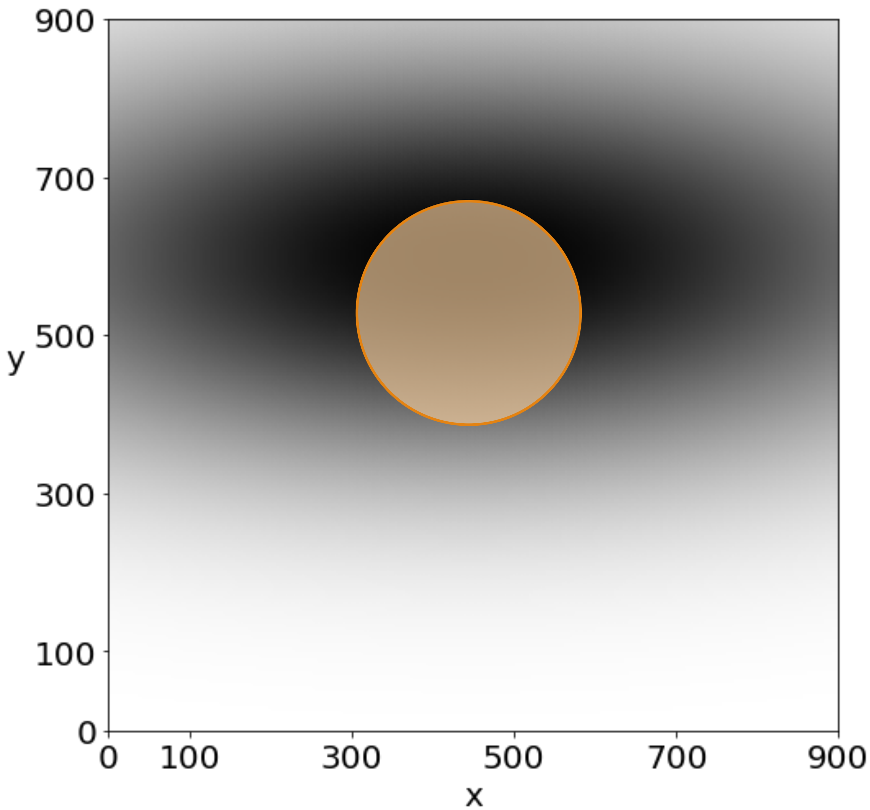


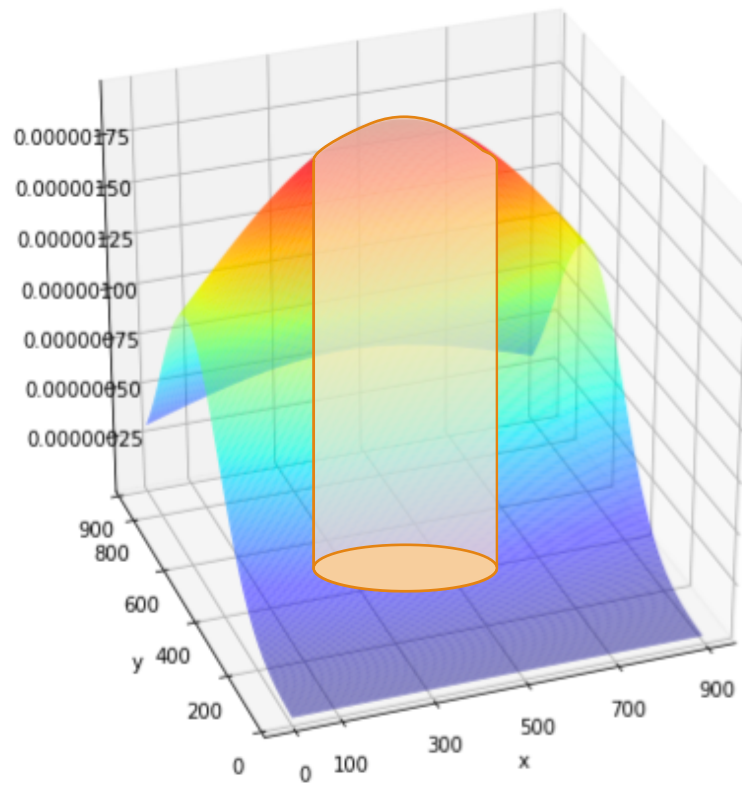


Possible dart counts (in 50x50 boxes)

CS109 logo with darts

P(dart hits within r pixels of center)?





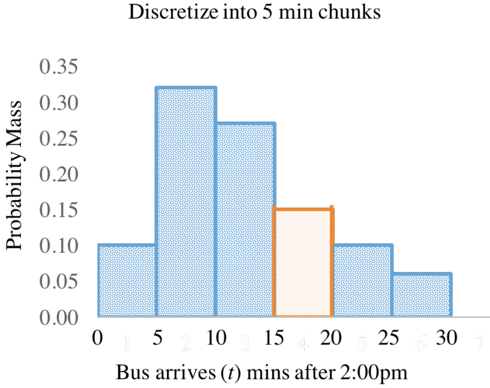
Possible dart counts

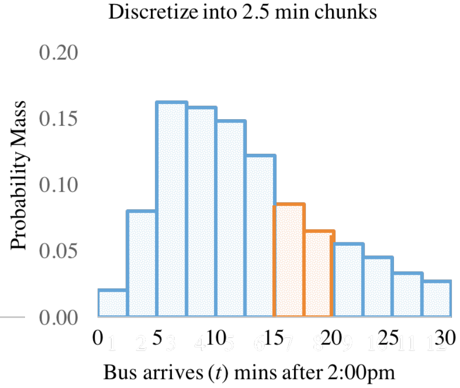
(in infinitesimally small boxes)

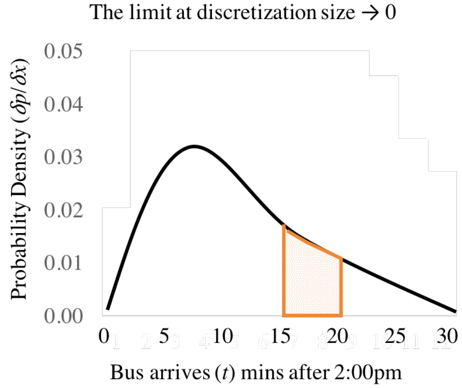
Riding the Marguerite

You are running to the bus stop. You don't know exactly when the bus arrives. You arrive at 2:20pm.

What is P(wait < 5 min)?







Continuous joint probability density functions

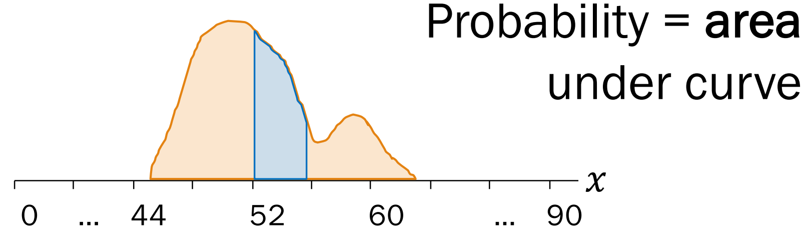
If two random variables X and Y are jointly continuous, then there exists a joint probability density function $f_{X,Y}$ defined over $-\infty < x, y < \infty$ such that:

$$P\left(a_1 \le X \le a_{2,}\ \ b_1 \le Y \le b_2\right) = \int_{a_1}^{a_2} \int_{b_1}^{b_2} f_{X,Y}(x, y) dy\, dx$$

From one continuous RV to jointly continuous RVs

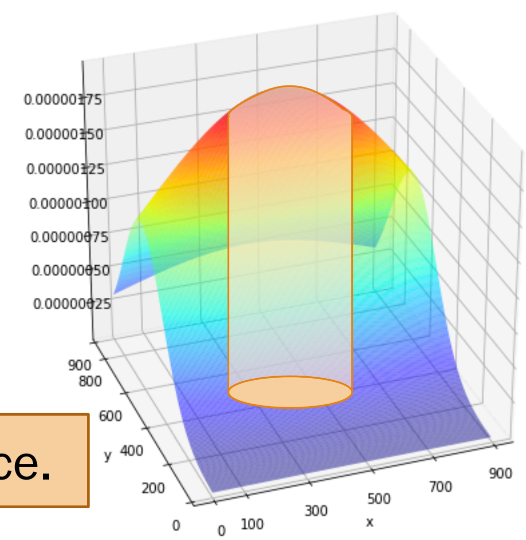
Single continuous RV X

- PDF f_X such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- Integrate to get probabilities



Jointly continuous RVs X and Y

- PDF $f_{X,Y}$ such that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\,dx = 1$
- Double integrate to get probabilities



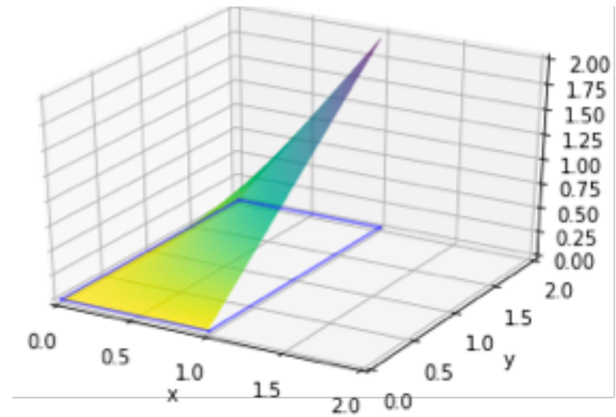
Probability for jointly continuous RVs is **volume** under a surface.

Double integrals without tears

Let X and Y be two continuous random variables.

- Support: $0 \leq X \leq 1$, $0 \leq Y \leq 2$.

Is $g(x, y) = xy$ a valid joint PDF over X and Y?



Write down the definite double integral that must integrate to 1:

Double integrals without tears

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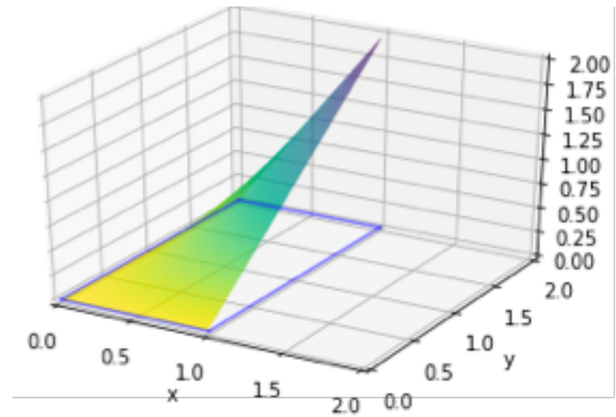
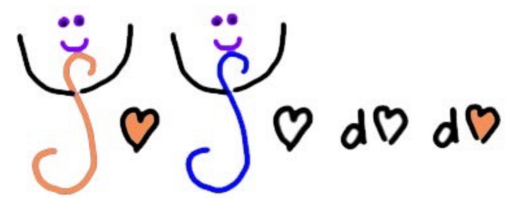


Write down the definite double integral that must integrate to 1:

$$\underbrace{\int_{y=0}^{2} \int_{x=0}^{1} xy \, dx \, dy = 1}_{\text{(used in next slide)}} \quad \text{or} \quad \int_{x=0}^{1} \int_{y=0}^{2} xy \, dy \, dx = 1$$



Double integrals without tears

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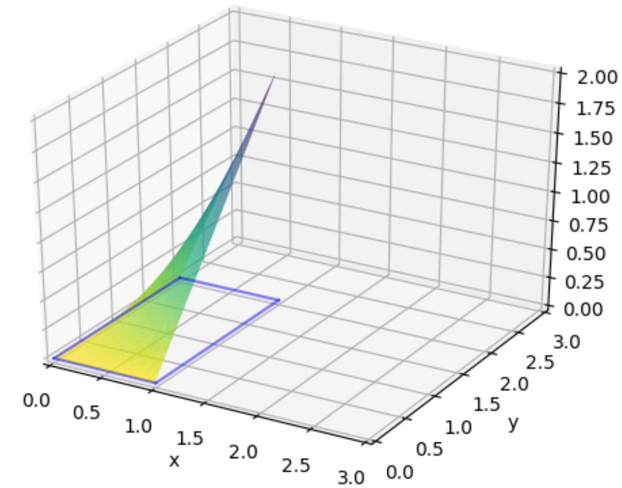
- Support: $0 \le X \le 1$, $0 \le Y \le 2$.

Is $g(x, y) = xy$ a valid joint PDF over X and Y?



0. Set up integral:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) dx\, dy = \int_{y=0}^{2}\int_{x=0}^{1} xy\, dx\, dy$$

1. Evaluate inside integral by treating y as a constant:

$$\int_{y=0}^{2}\left(\int_{x=0}^{1} xy\, dx\right) dy = \int_{y=0}^{2} y\left(\int_{x=0}^{1} x\, dx\right) dy = \int_{y=0}^{2} y\left[\frac{x^2}{2}\right]_0^1 dy = \int_{y=0}^{2} y\frac{1}{2}\, dy$$

2. Evaluate remaining (single) integral:

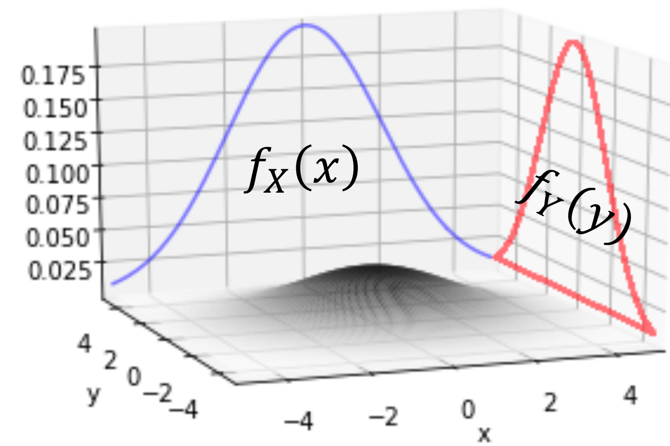
$$\int_{y=0}^{2} y\frac{1}{2}\, dy = \left[\frac{y^2}{4}\right]_{y=0}^{2} = 1 - 0 = 1$$

Yes, $g(x, y)$ is a valid joint PDF because it integrates to 1.

Marginal distributions

Suppose X and Y are continuous random variables with joint PDF:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy\,dx = 1$$

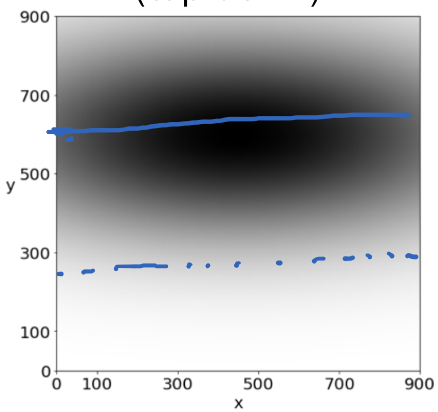


The marginal density functions (**marginal PDFs**) are therefore:

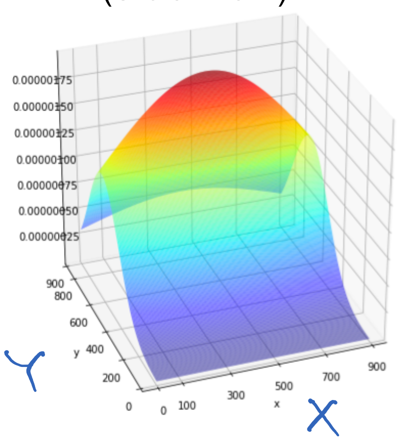
$$f_X(a) = \int_{-\infty}^{\infty} f_{X,Y}(a,y)dy \qquad f_Y(b) = \int_{-\infty}^{\infty} f_{X,Y}(x,b)dx$$

Back to darts!

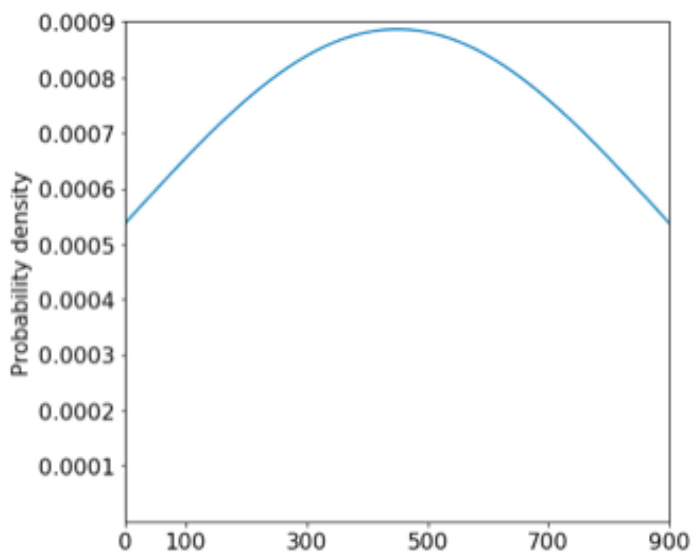
(top-down)

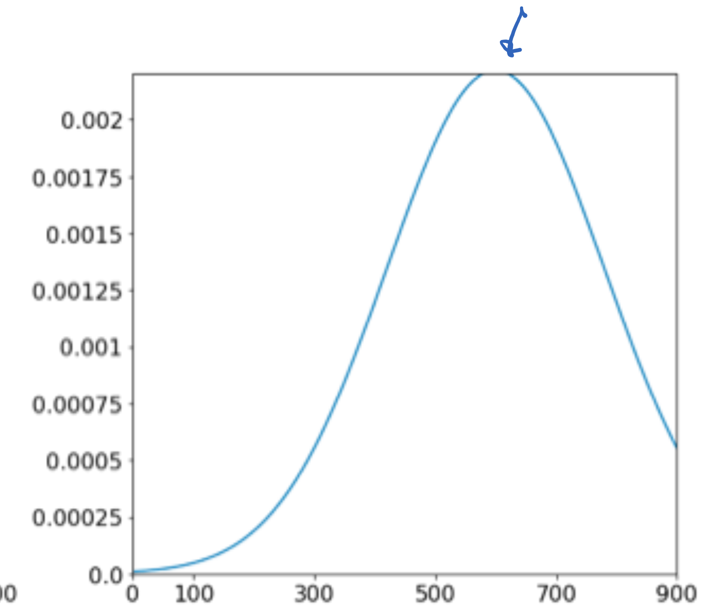


(side view)



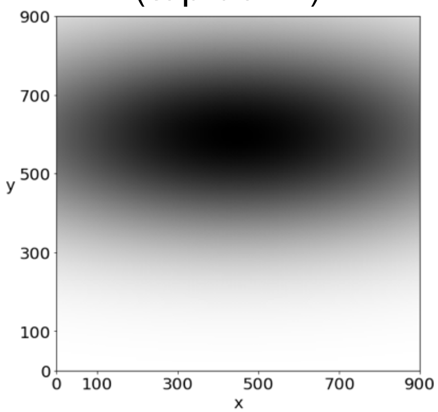
Match X and Y to their respective marginal PDFs:





Back to darts!

(top-down)



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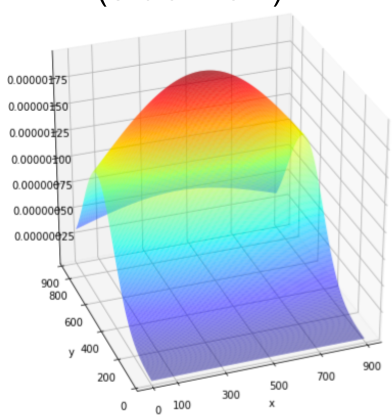

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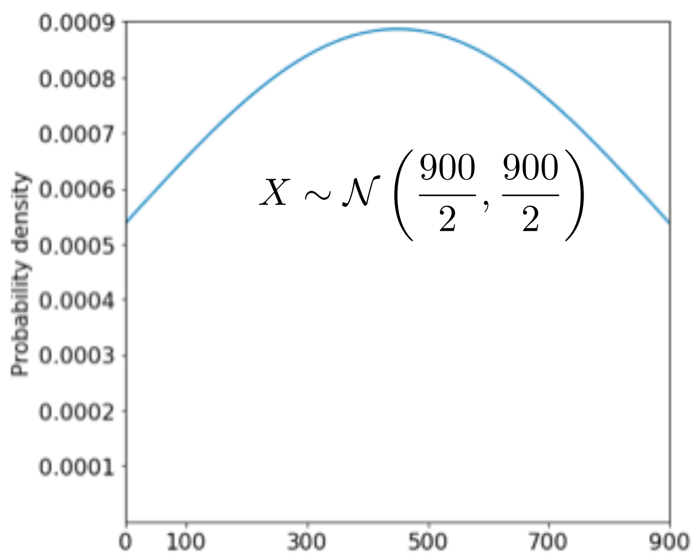
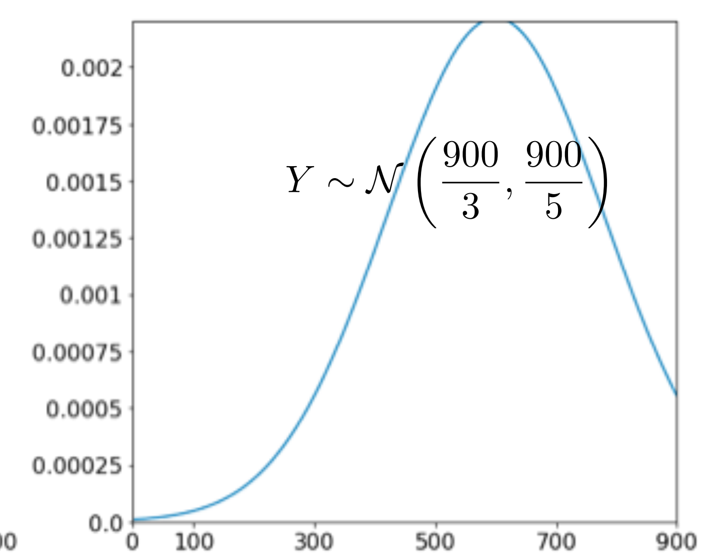
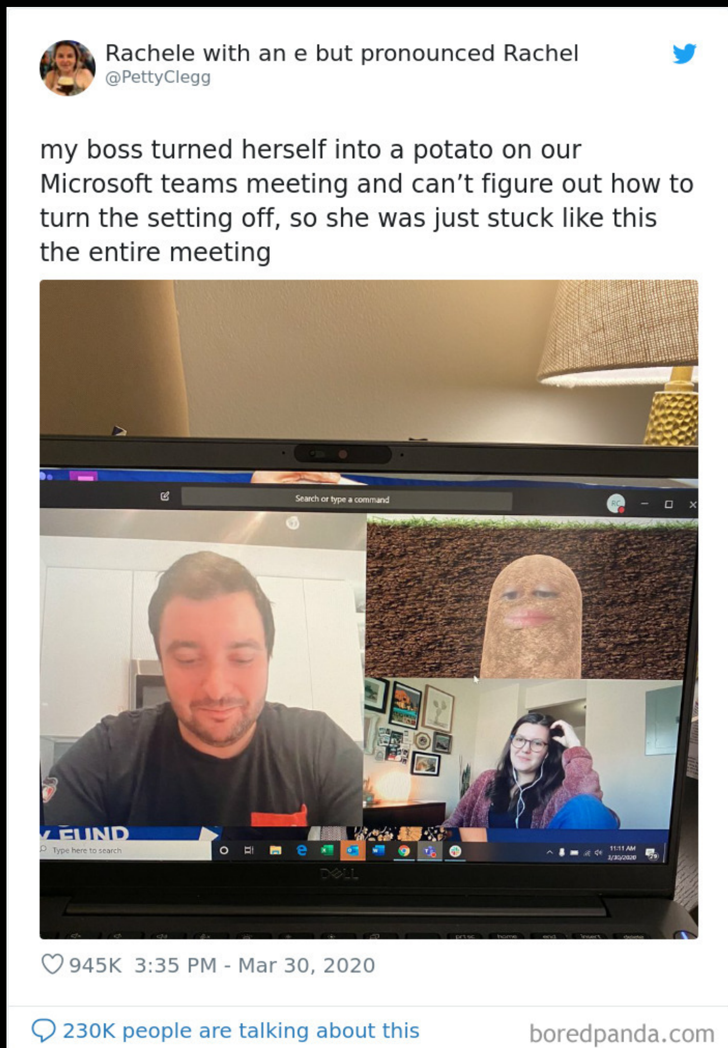


pixel x



pixel y



The Joy of Meetings ☺

The joy of meetings

Two people set up a meeting time. Each arrives independently at a time uniformly distributed between 12pm and 12:30pm.

Define X = # minutes past 12pm that person 1 arrives. $X \sim \text{Uni}(0, 30)$
Y = # minutes past 12pm that person 2 arrives. $Y \sim \text{Uni}(0, 30)$

What is the probability that the first to arrive waits >10 mins for the other?

Compute: $P(X + 10 < Y) + P(Y + 10 < X) = 2P(X + 10 < Y)$ (by symmetry)

1. What is "symmetry" here?
2. How do we integrate to compute this probability?

Double integrals: A guide

From last slide: $$2P(X+10<Y) \quad = 2 \cdot \iint\limits_{\substack{x+10<y,\\ 0\le x,y,\le 30}} (1/30)^2 dxdy$$ (by symmetry, independence)

<u>Steps</u>:

1. Draw a picture.
2. Set bounds "from outside in."
   - Outer integral bounds should be full range possible
   - Inner integral can depend on integration variable of outer integral

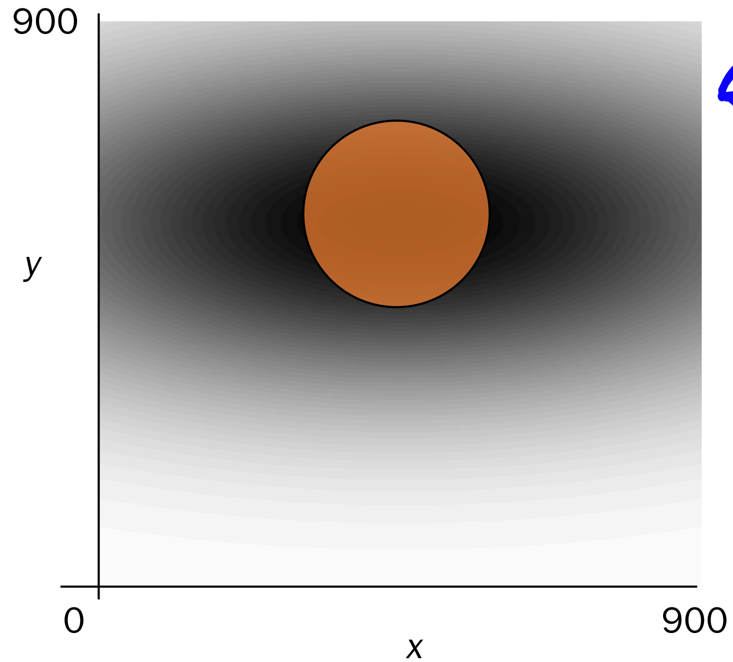
$$\boxed{= \frac{2}{30^2}\int_{10}^{30}\int_{0}^{y-10} dxdy}$$

$$= \frac{2}{30^2}\int_{10}^{30}(y-10)dy \qquad = \cdots = \frac{4}{9}$$

Extra slides

If you want more practice with double integrals,
I've included two exercises at the end of this lecture.

$$f(X = x, Y = y)$$



Joint CDFs

An observation: Connecting CDF to PDF

For a continuous random variable X with PDF f, the CDF (cumulative distribution function) is

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

The density f is therefore the derivative of the CDF, F:

$$\boxed{f(a) = \frac{d}{da}F(a)}$$

(Fundamental Theorem of Calculus)

Joint cumulative distribution function

For two random variables X and Y, there can be a **joint cumulative distribution function** $F_{X,Y}$:

$$F_{X,Y}(a, b) = P(X \leq a, Y \leq b)$$

For discrete X and Y:

$$F_{X,Y}(a, b) = \sum_{x \leq a} \sum_{y \leq b} p_{X,Y}(x, y)$$

For continuous X and Y:

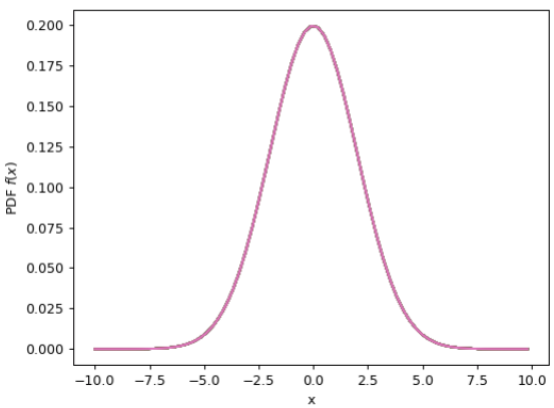
$$F_{X,Y}(a, b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(x, y) dy\, dx$$

$$f_{X,Y}(a, b) = \frac{\partial^2}{\partial a\, \partial b} F_{X,Y}(a, b)$$

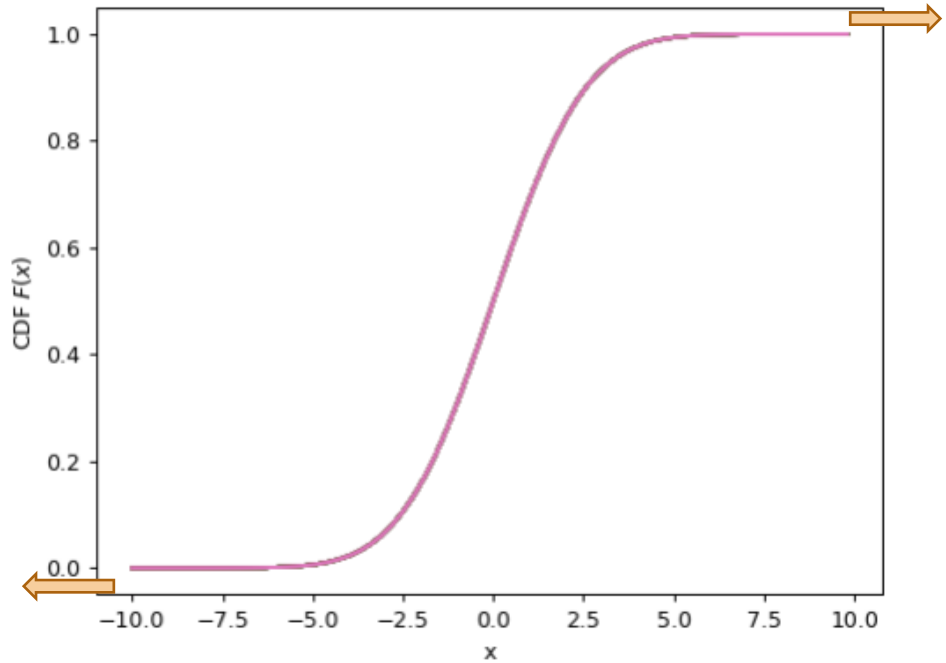
Single variable CDF, graphically

Review

$$\lim_{x \to +\infty} F_X(x) = 1$$



$f_X(x)$

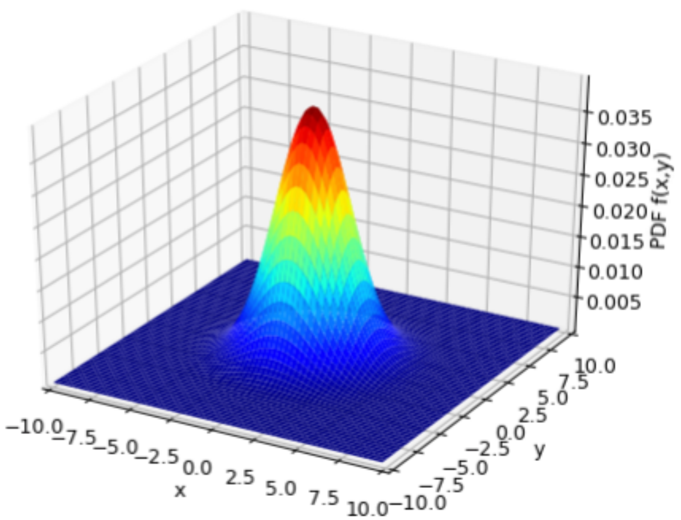


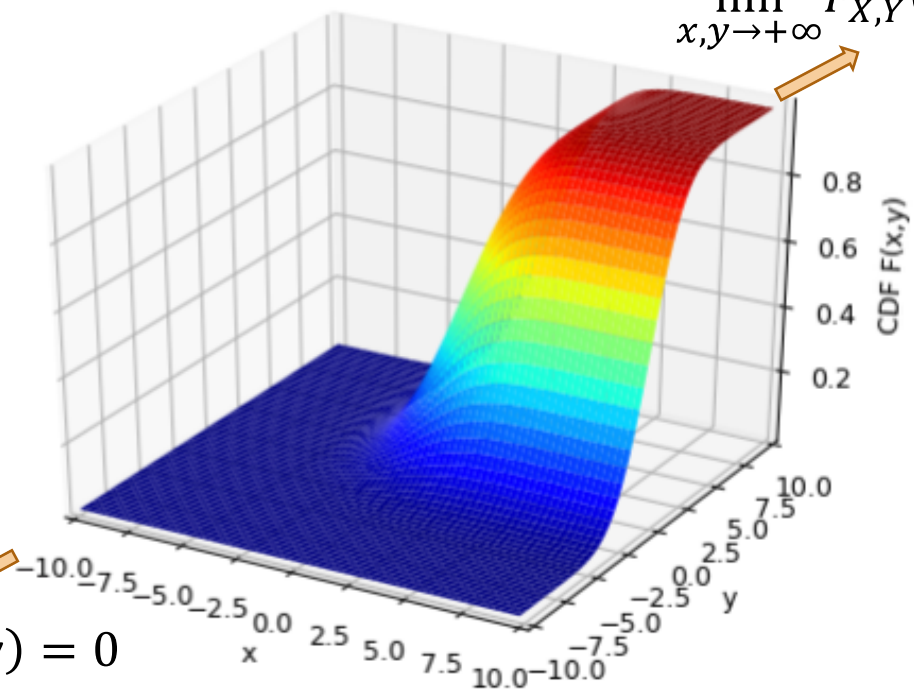
$$\lim_{x \to -\infty} F_X(x) = 0$$

$$F_X(x) = P(X \leq x)$$

Joint CDF, graphically

$$\lim_{x,y\to+\infty} F_{X,Y}(x,y) = 1$$





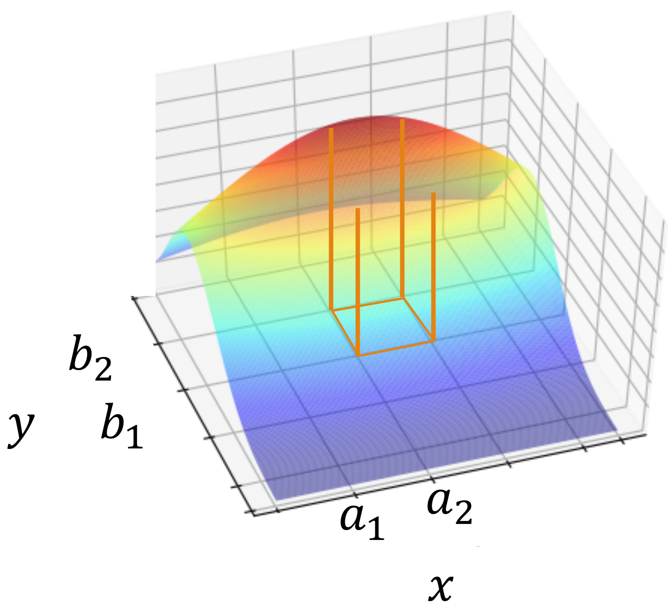
$$\lim_{x,y\to-\infty} F_{X,Y}(x,y) = 0$$

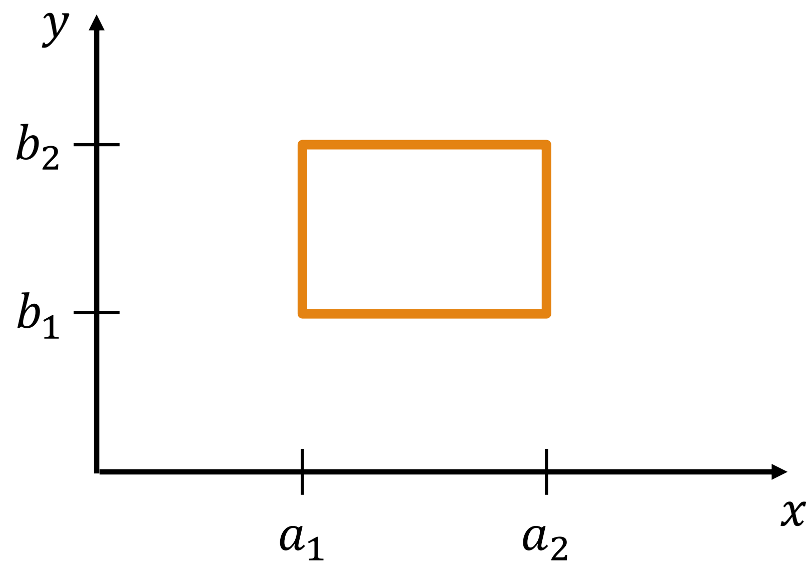
$f_{X,Y}(x,y)$

$F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

Probabilities from joint CDFs

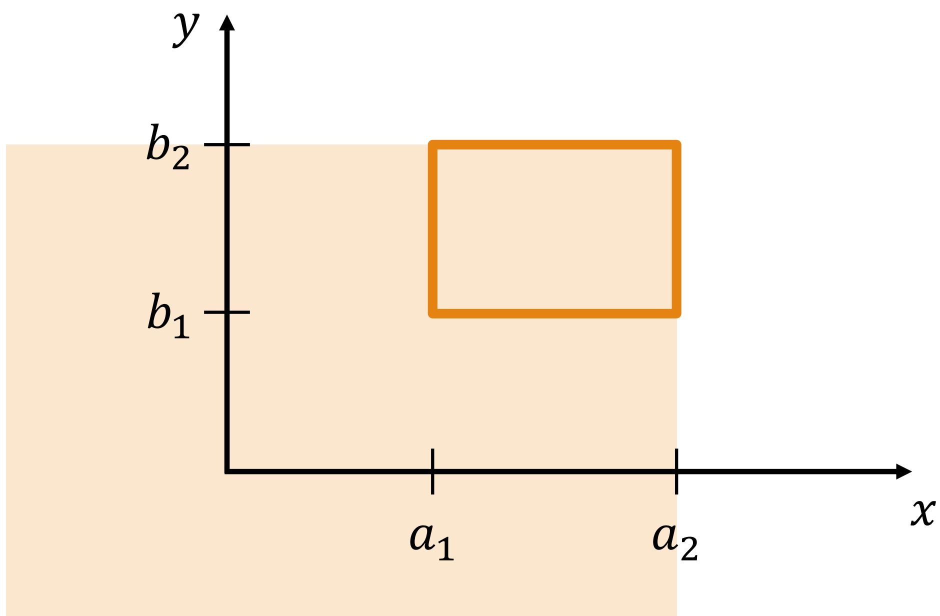
$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$





Probabilities from joint CDFs

$$P(a_1 < X \le a_2, b_1 < Y \le b_2) =$$
$$\boxed{F_{X,Y}(a_2, b_2)} - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$



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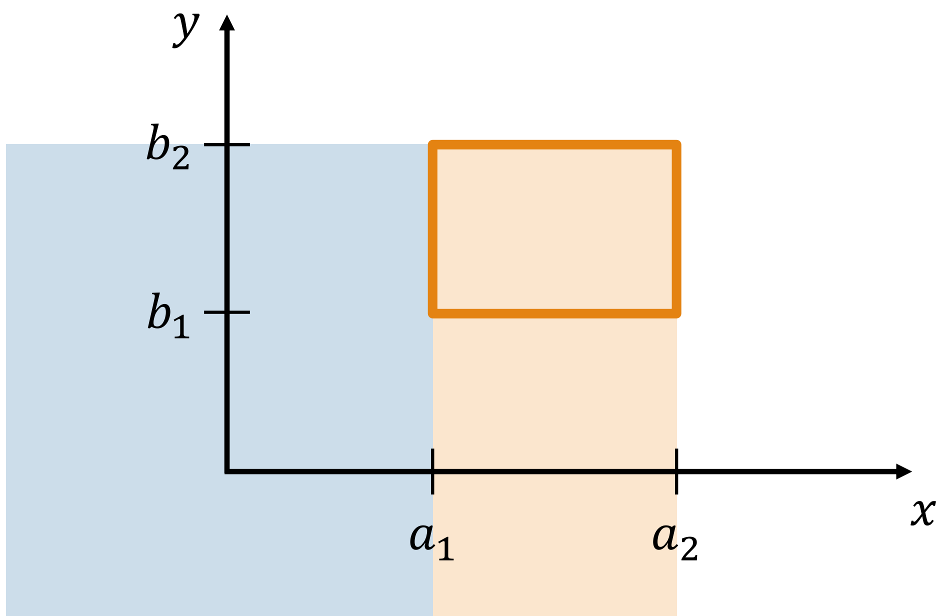

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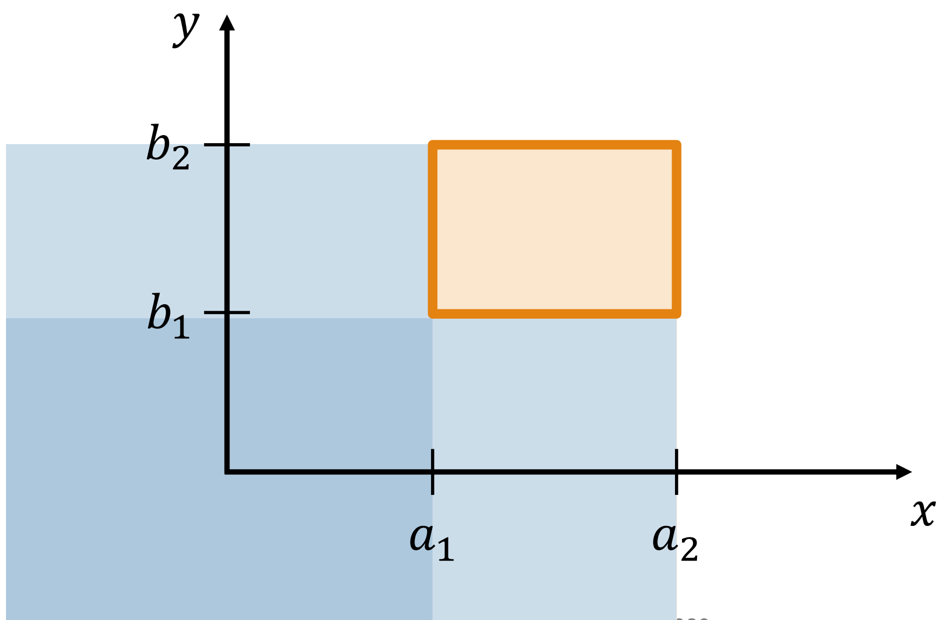

Probabilities from joint CDFs

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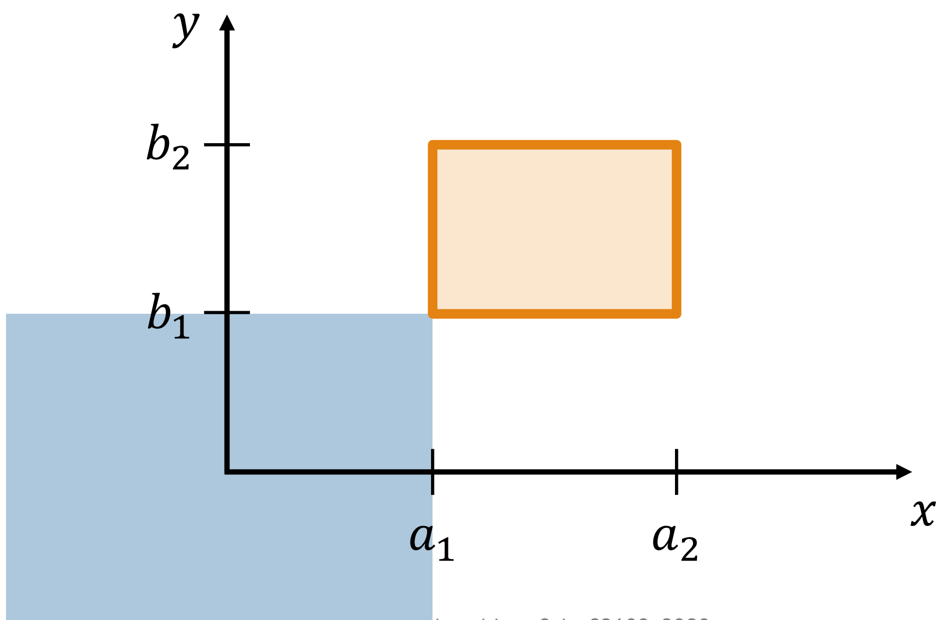

Probabilities from joint CDFs

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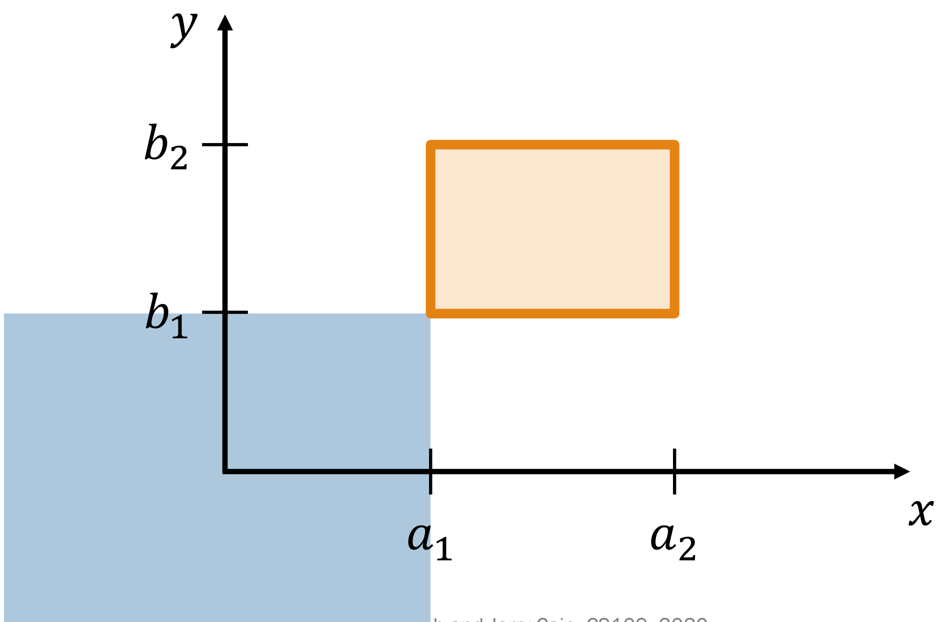

Probabilities from joint CDFs

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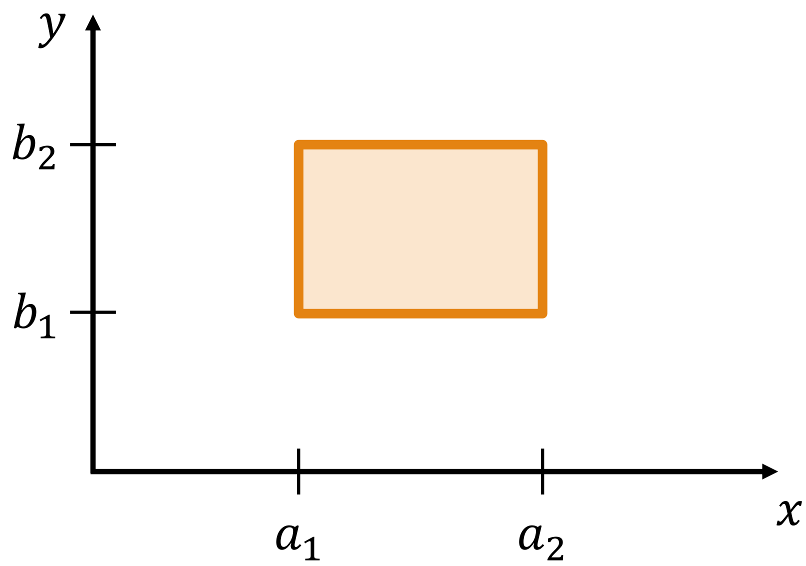


Probabilities from joint CDFs

Recall for a single RV X with CDF F_X:

CDF: $P(X \leq x) = F_X(x)$

$$P(a < X \leq b) = F_X(b) - F(a)$$

For two RVs X and Y with joint CDF $F_{X,Y}$:

Joint CDF: $P(X \leq x, Y \leq y) = F_{X,Y}(x, y)$

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) =$$
$$F_{X,Y}(a_{2,} b_2) - F_{X,Y}(a_{1,} b_2) - F_{X,Y}(a_{2,} b_1) + F_{X,Y}(a_{1,} b_1)$$

Note strict inequalities; these properties hold for both discrete and continuous RVs.

Independent Continuous RVs

Independent continuous RVs

Two continuous random variables X and Y are independent if:

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y) \qquad \forall x, y$$

Equivalently:

$$\boxed{\begin{aligned} F_{X,Y}(x,y) &= F_X(x)F_Y(y) \\ f_{X,Y}(x,y) &= f_X(x)f_Y(y) \end{aligned}} \qquad \forall x, y$$

Proof of PDF:

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_X(x)F_Y(y)$$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_X(x)F_Y(y) \qquad = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

$$= f_X(x)f_Y(y)$$

Independent continuous RVs

Two continuous random variables X and Y are **independent** if:

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

Equivalently:

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$
$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

More generally, X and Y are independent if joint density factors separately:

$$f_{X,Y}(x, y) = g(x)h(y), \text{ where } -\infty < x, y < \infty$$

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ → independent X and Y

Are X and Y independent in the following cases?

1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$
   where $0 < x, y < \infty$

2. $f_{X,Y}(x,y) = 4xy$
   where $0 < x, y < 1$

3. $f_{X,Y}(x,y) = 24xy$
   where $0 < x + y < 1$

Pop quiz! (just kidding)

$f_{X,Y}(x,y) = g(x)h(y)$, where $-\infty < x, y < \infty$ → independent X and Y

Are X and Y independent in the following cases?

✅ 1. $f_{X,Y}(x,y) = 6e^{-3x}e^{-2y}$ where $0 < x, y < \infty$

Separable functions: $g(x) = 3e^{-3x}$, $h(y) = 2e^{-2y}$

✅ 2. $f_{X,Y}(x,y) = 4xy$ where $0 < x, y < 1$

Separable functions: $g(x) = 2x$, $h(y) = 2y$

❌ 3. $f_{X,Y}(x,y) = 24xy$ where $0 < x + y < 1$

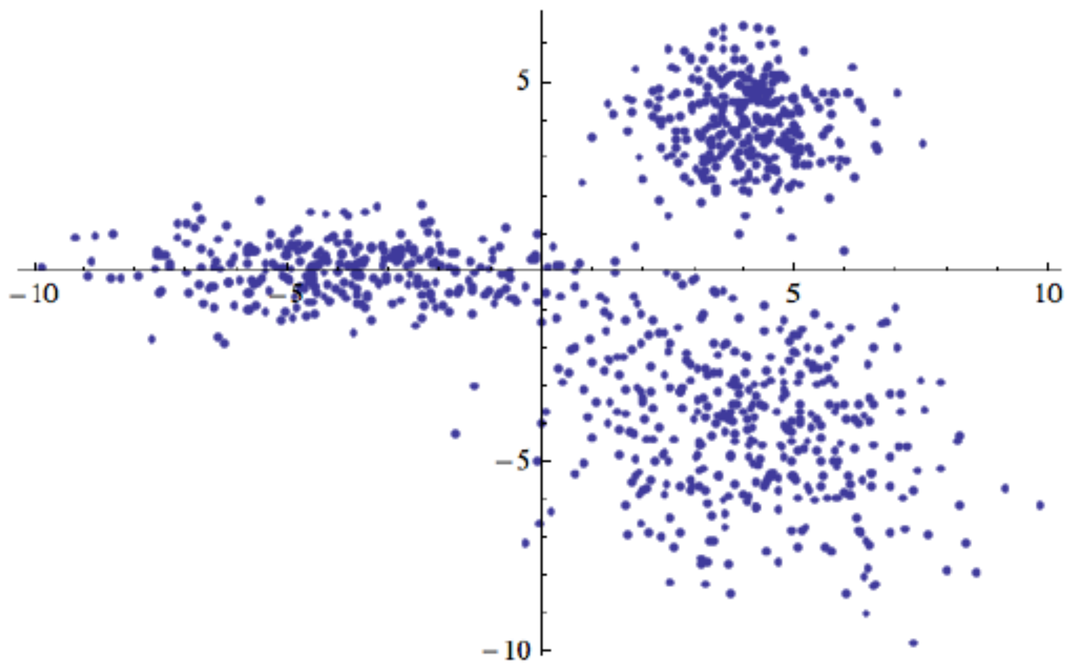
Cannot capture constraint on $x + y$ into factorization!

If you can factor densities over all of the support, you have independence.

16d_bivariate_normal

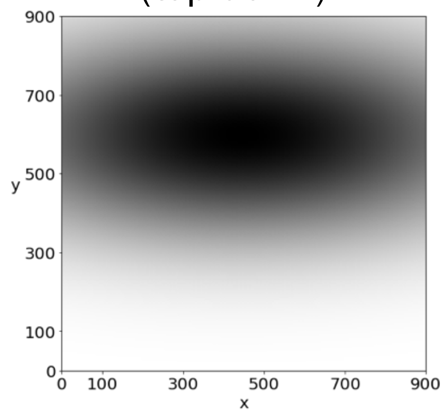
Multivariate Normal Distribution

How could you model this data?

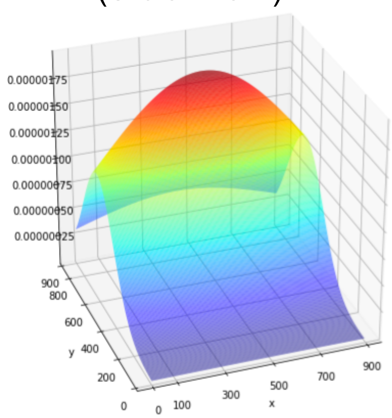


Back to darts

(top-down)



(side view)



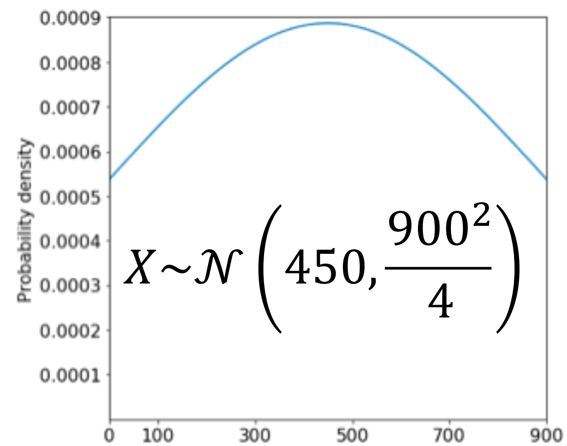
These darts were actually thrown according to a bivariate normal distribution:

$$(X, Y) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\mu} = (450, 600)$$

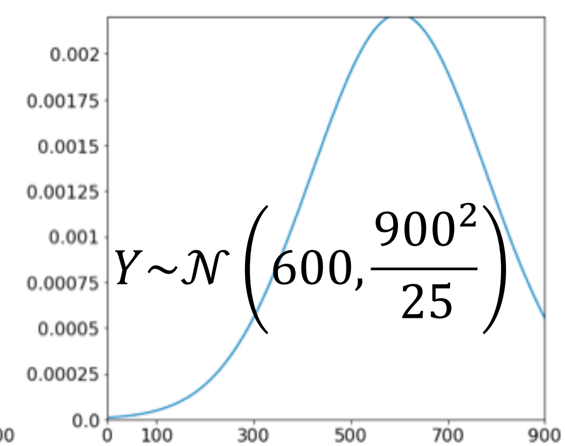
$$\boldsymbol{\Sigma} = \begin{bmatrix} 900^2/4 & 0 \\ 0 & 900^2/25 \end{bmatrix}$$

Marginal PDFs:



$$X \sim \mathcal{N}\left(450, \frac{900^2}{4}\right)$$

pixel x



$$Y \sim \mathcal{N}\left(600, \frac{900^2}{25}\right)$$

pixel y

(Bi)-variate Normal Distribution

X_1 and X_2 follow a bivariate normal distribution if their joint PDF f is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

Can show that $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$

Often written as:

$$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Vector $\boldsymbol{X} = (X_1, X_2)$
- Mean vector $\boldsymbol{\mu} = (\mu_1, \mu_2)$, Covariance matrix: $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

Recall correlation: $\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$

We will focus on understanding the **shape** of a bivariate Normal RV.

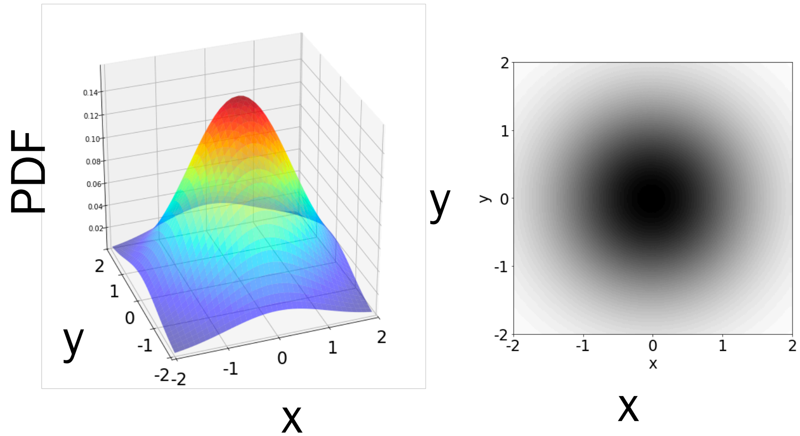
(X, Y) Matching (all have $\boldsymbol{\mu} = (0, 0)$)

A. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ B. $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

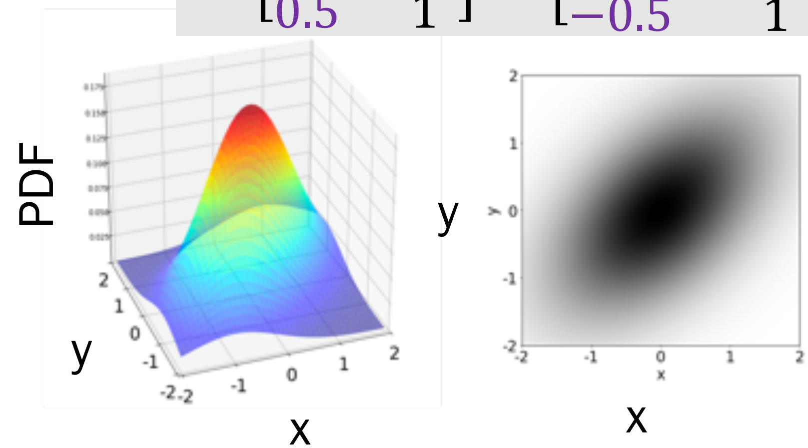
C. $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ D. $\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

recall this purple term is: $\rho\sigma_1\sigma_2$

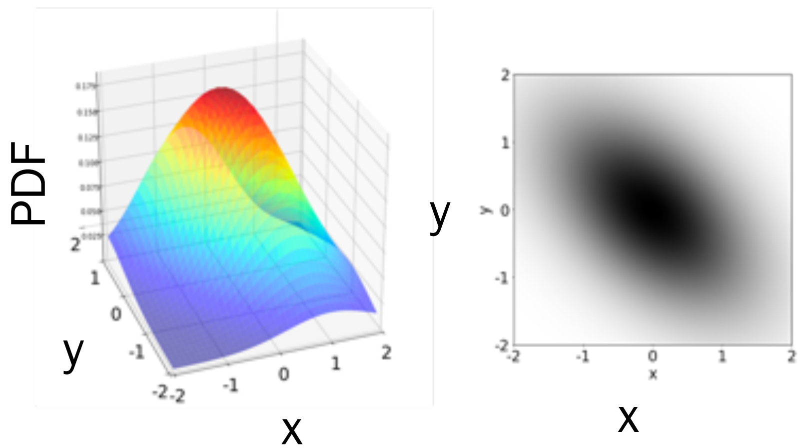
1.



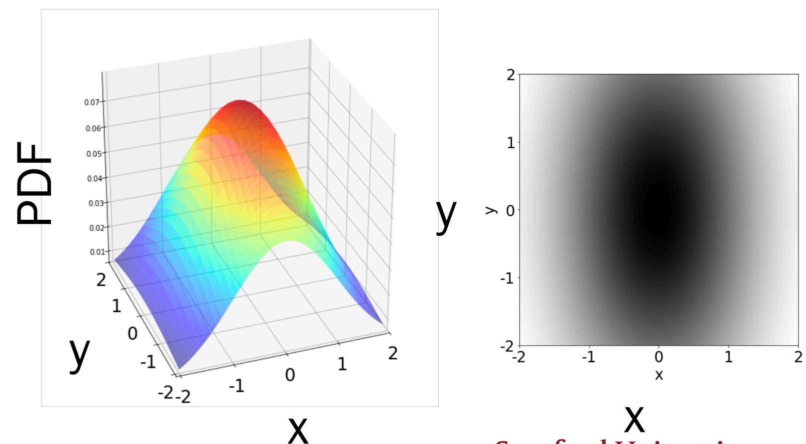
2.



3.



4.



(X, Y) Matching (all have $\boldsymbol{\mu} = (0, 0)$)

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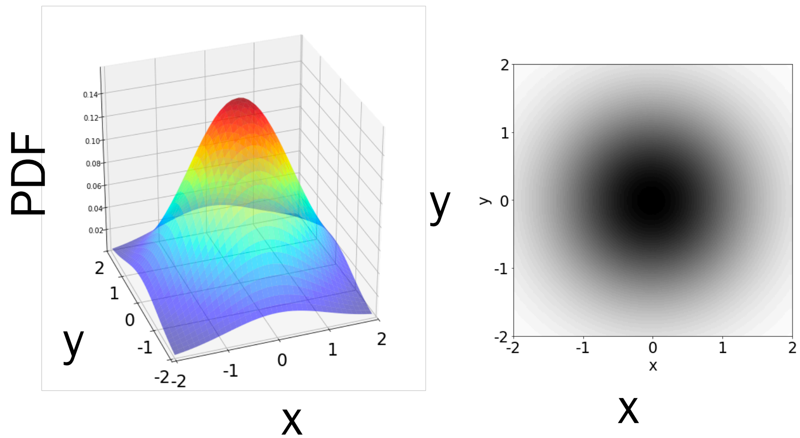
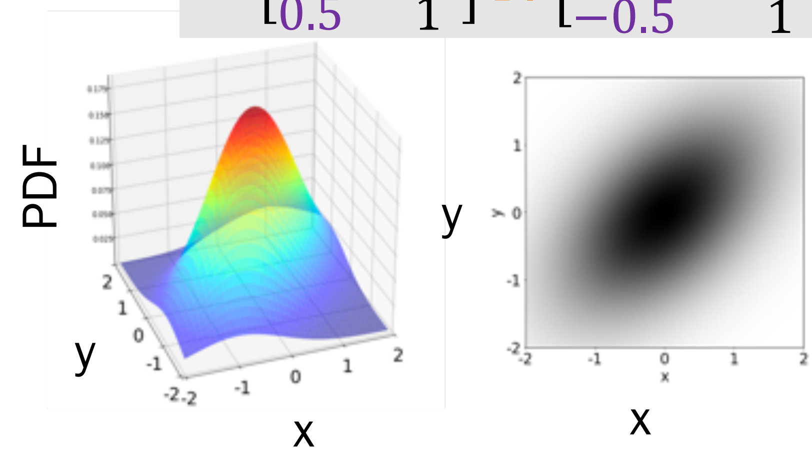
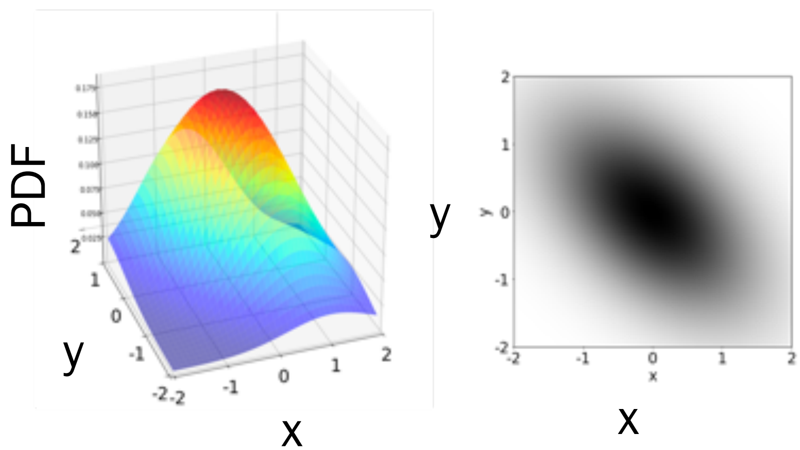


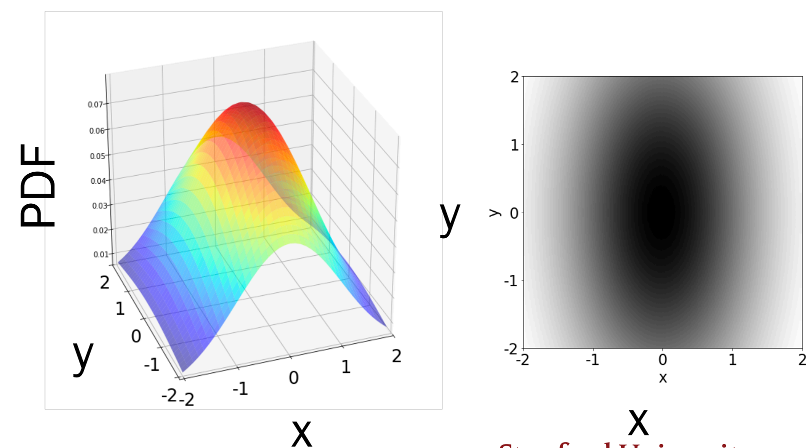
2.



3.



4.



Independent Multivariate Gaussian

Let $\boldsymbol{X} = (X_1, X_2)$ follow a bivariate normal distribution $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = (\mu_1, \mu_2), \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

recall this term is: $\boldsymbol{\rho\sigma_1\sigma_2}$

Are X_1 and X_2 independent?

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

(Note covariance: $\rho\sigma_1\sigma_2 = 0$)

$$= \frac{1}{\sigma_1\sqrt{2\pi}} e^{-(x_1-\mu_1)^2/2\sigma_1^2} \frac{1}{\sigma_2\sqrt{2\pi}} e^{-(x_2-\mu_2)^2/2\sigma_2^2}$$ ✅

X_1 and X_2 are **independent** with marginal distributions $X_1 \sim \mathcal{N}(\mu_1\ \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2\ \sigma_2^2)$

Independent Multivariate Gaussians

X_1 and X_2 are **independent** with marginal distributions
$X_1 \sim \mathcal{N}(\mu_1\ \sigma_1^2), X_2 \sim \mathcal{N}(\mu_2\ \sigma_2^2)$

Joint PDF

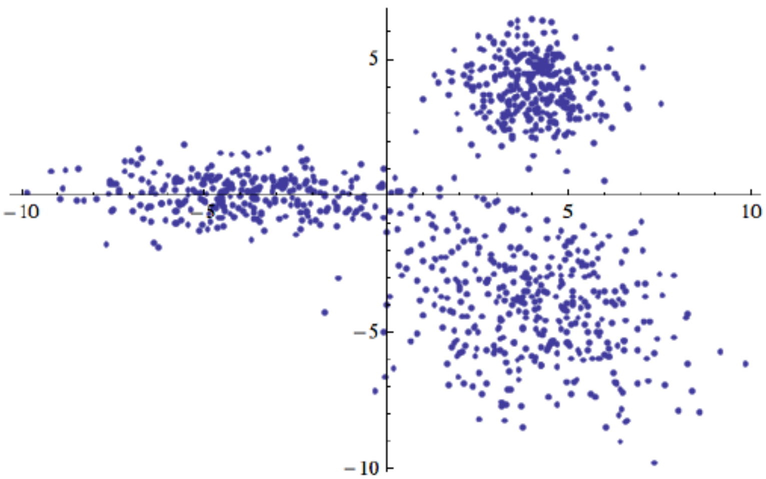
$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

Joint CDF

$$F(x_1, x_2) = \Phi\left(\frac{x_1 - \mu_1}{\sigma_1}\right) \cdot \Phi\left(\frac{x_2 - \mu_2}{\sigma_2}\right)$$

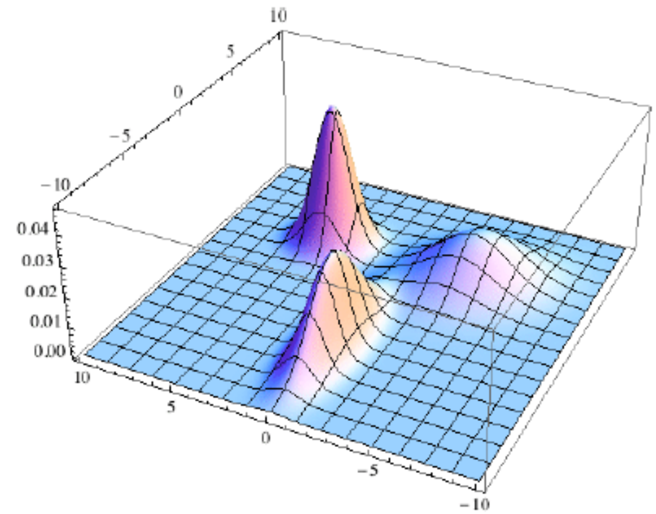
Multivariate Normal Distributions are Rad ☺

The Data



A 2D scatter plot with three "clusters"

The Joint Model



A mixture of 3 bi-variate Normal distributions

See you in CS221

Probability with Instagram!

$$P(a_1 < X \leq a_2, b_1 < Y \leq b_2) = F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$$

In image processing, a Gaussian blur is the result of blurring an image by a Gaussian function. It is a widely used effect in graphics software, typically to reduce image noise.

Gaussian blur (for next time)

$P(a_1 < X \le a_2, b_1 < Y \le b_2) =$
$F_{X,Y}(a_2, b_2) - F_{X,Y}(a_1, b_2) - F_{X,Y}(a_2, b_1) + F_{X,Y}(a_1, b_1)$

In a Gaussian blur, for every pixel:

- Weight each pixel by the probability that X and Y are both within the pixel bounds
- The weighting function is a Bivariate Gaussian (Normal) standard deviation parameter σ

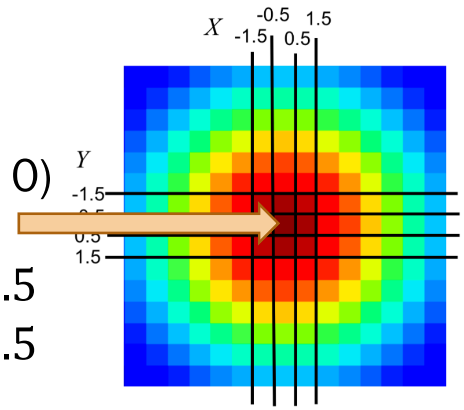
Weight matrix:

Center pixel: (0, 0)

Pixel bounds:

$-0.5 < x \le 0.5$

$-0.5 < y \le 0.5$



Gaussian blurring with $\sigma = 3$:

$$f_{X,Y}(x, y) = \frac{1}{2\pi \cdot 3^2} e^{-(x^2+y^2)/2\cdot 3^2}$$

→ Independent $X \sim \mathcal{N}(0, 3^2), Y \sim \mathcal{N}(0, 3^2)$

→ Joint CDF: $F_{X,Y}(x, y) = \Phi\left(\frac{x}{3}\right)\Phi\left(\frac{y}{3}\right)$

What is the weight of the center pixel?

$P(-0.5 < X \le 0.5, -0.5 < Y \le 0.5) =$

$= 0.206$

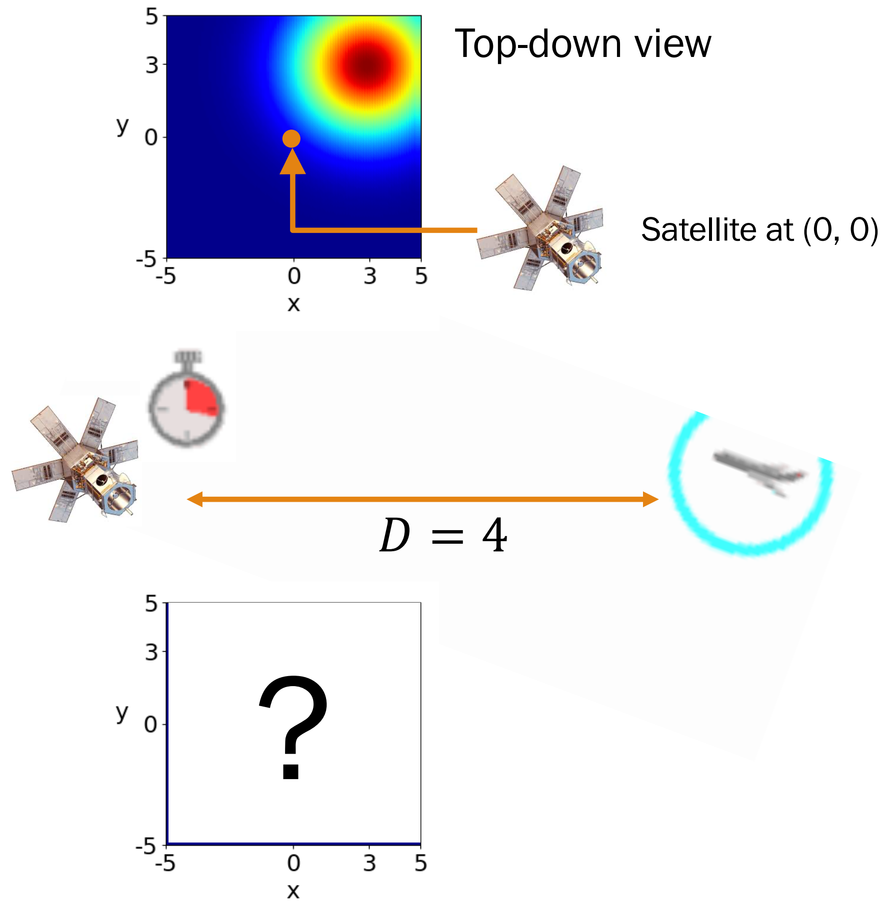

Give this a try!

Tracking in 2-D space

- Before measuring, we have some prior belief about the 2-D location of an object, (X, Y).

- We observe some noisy measurement $D = 4$, the Euclidean distance of the object to a satellite.

- After the measurement, what is our updated (posterior) belief of the 2-D location of the object?

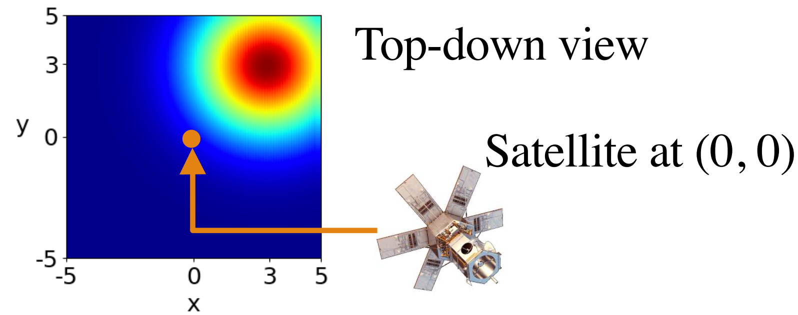


Tracking in 2-D space

- Before measuring, we have some **prior belief** about the 2-D location of an object, (X, Y).

X,Y ~ Independent Bivariate normal

$$f_{X,Y}(x, y) = \frac{1}{2\pi 2^2} e^{-\frac{[(x-3)^2+(y-3)^2]}{2(2^2)}}$$



Tracking in 2-D space

- Before measuring, we have some **prior belief** about the 2-D location of an object, (X, Y).

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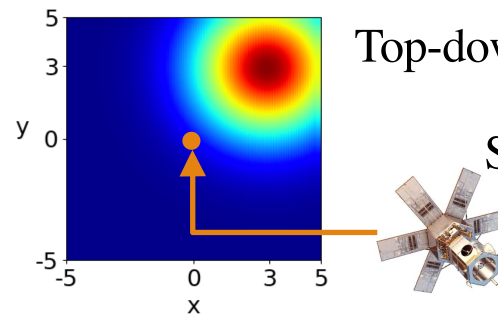
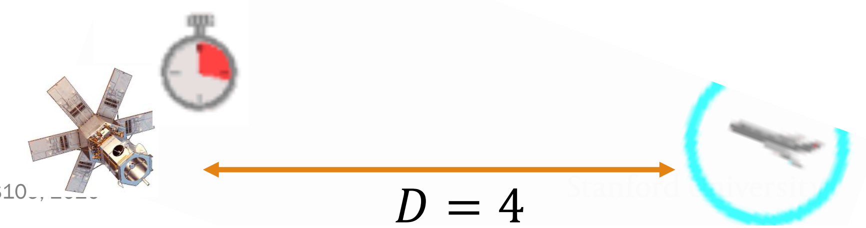


Top-down view

Satellite at $(0, 0)$

- We observe some noisy **measurement** $D = 4$, the Euclidean distance of the object to a satellite.

Let D = observed distance from the satellite.
Observed distance is true distance plus noise.
Noise is a **standard normal**.




Give this a try!

Extra

1. Integral practice

Let X and Y be two continuous random variables with joint PDF:

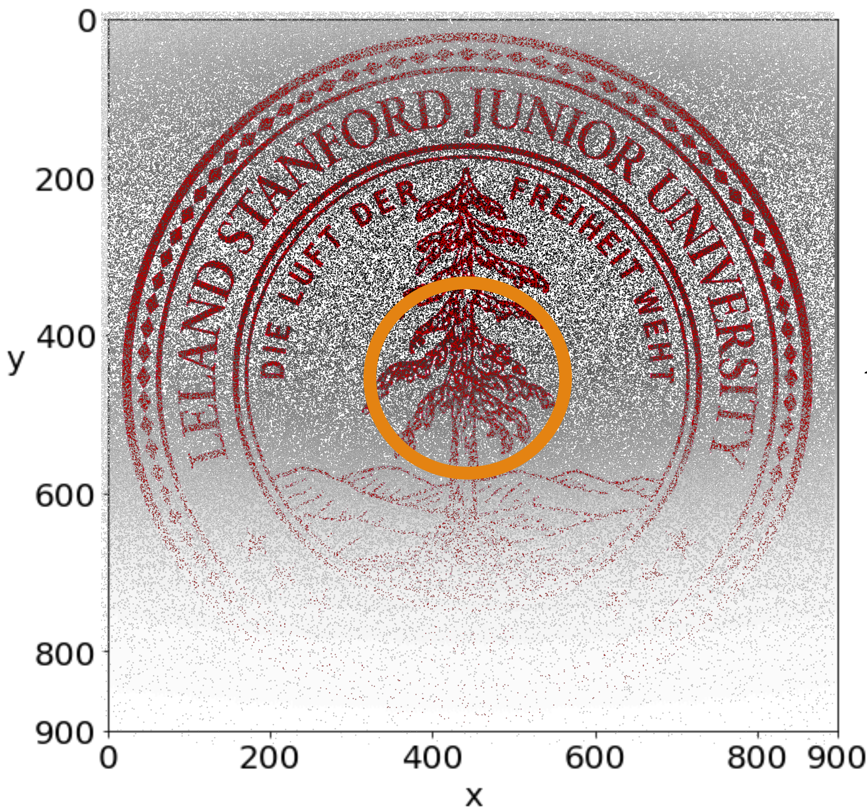
$$f(x,y) = \begin{cases} 4xy & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X \le Y)$?

$$P(X \le Y) = \iint\limits_{\substack{x \le y, \\ 0 \le x, y \le 1}} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x \le y} 4xy \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{y} 4xy \, dx \, dy$$

$$= \int_{y=0}^{1} 4y \left[\frac{x^2}{2}\right]_0^y dy = \int_{y=0}^{1} 2y^3 dy = \left[\frac{2}{4} y^4\right]_0^1 = \frac{1}{2}$$

2. How do you integrate over a circle?



P(dart hits within $r = 10$ pixels of center)?

$$P(x^2 + y^2 \le 10^2) = \iint_{x^2 + y^2 \le 10^2} f_{X,Y}(x, y)\,dy\,dx$$

Let's try an example that doesn't involve integrating a Normal RV ☺

2. Imperfection on Disk

You have a disk surface, a circle of radius R. Suppose you have a single point imperfection uniformly distributed on the disk.

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & x^2 + y^2 \le R^2 \\ 0 & \text{otherwise} \end{cases}$$

What are the marginal distributions of X and Y? Are X and Y independent?

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \frac{1}{\pi R^2} \int_{x^2+y^2 \le R^2} dy \qquad \text{where } -R \le x \le R$$

$$= \frac{1}{\pi R^2} \int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \quad = \frac{2\sqrt{R^2 - x^2}}{\pi R^2}$$

$$f_Y(y) = \frac{2\sqrt{R^2 - y}}{\pi R^2} \qquad \text{where } -R \le y \le R \text{, by symmetry}$$

No, X and Y are **dependent.**

$$f_{X,Y}(x,y) \ne f_X(x) f_Y(y)$$