CS109: Poisson and More

Poisson RV

Before we start

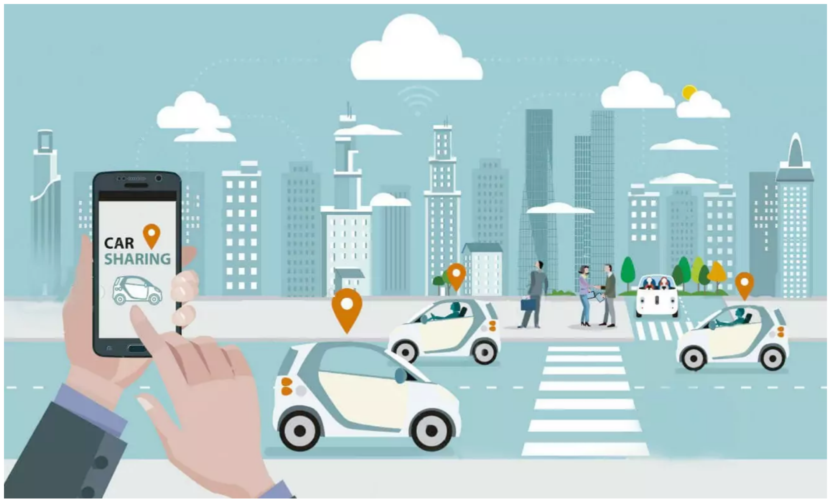
The natural exponent e:

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

https://en.wikipedia.org/wiki/E_(mathematical_constant)

Jacob Bernoulli while studying compound interest in 1683

Algorithmic ride sharing



Probability of k requests from this area in the next 1 minute?

Suppose we know: On average, $\lambda = 5$ requests per minute

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60 seconds:

0	0	1	0	1	…	0	0	0	0	1
1	2	3	4	5						60

At each second:

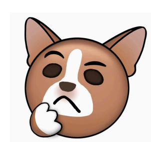
- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n = 60,\ p = 5/60)$$

$$P(X = k) = \binom{60}{k}\left(\frac{5}{60}\right)^k \left(1 - \frac{5}{60}\right)^{n-k}$$



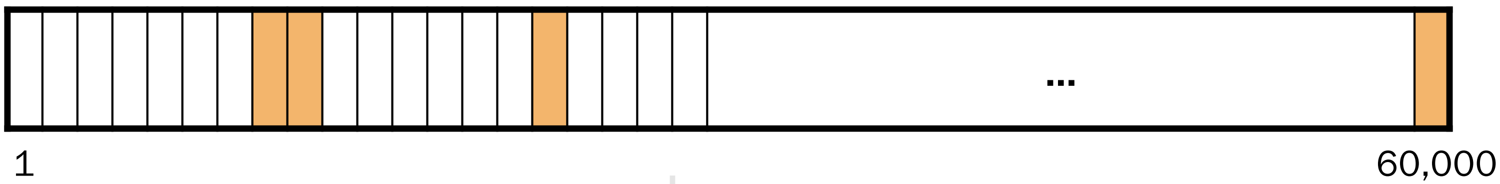
But what if there are *two* requests in the same second?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into 60,000 milliseconds:



At each millisecond:

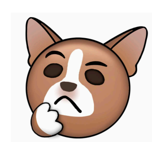
- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n = 60000,\ p = \lambda/n)$$

$$P(X = k) = \binom{n}{k}\left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$



But what if there are *two* requests in the same millisecond?

Algorithmic ride sharing, approximately

Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

Break a minute down into infinitely small buckets:

1 ∞

For each time bucket:

- Independent trial
- You get a request (1) or you don't (0).

Let X = # of requests in minute.

$E[X] = \lambda = 5$

$$X \sim \text{Bin}(n,\ p = \lambda/n)$$

$$P(X = k) = \lim_{n \to \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Who wants to see some cool math?

Binomial in the limit

$$\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$P(X=k) = \lim_{n\to\infty}\binom{n}{k}\left(\frac{\lambda}{n}\right)^k\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Expand

$$= \lim_{n\to\infty}\frac{n!}{k!(n-k)!}\frac{\lambda^k}{n^k}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Rearrange

$$= \lim_{n\to\infty}\frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}$$

Def natural exponent

$$= \lim_{n\to\infty}\frac{n!}{n^k(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty}\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{(n-k)!}{(n-k)!}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^k}$$

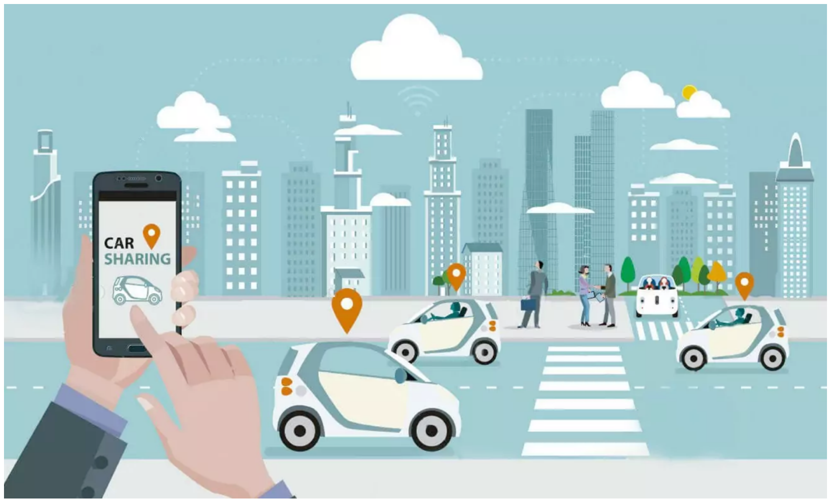
Limit analysis + cancel

$$= \lim_{n\to\infty}\frac{n^k}{n^k}\frac{\lambda^k}{k!}\frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!}e^{-\lambda}$$

Algorithmic ride sharing



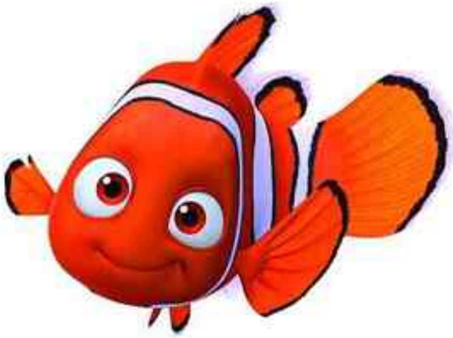
Probability of k requests from this area in the next 1 minute?

On average, $\lambda = 5$ requests per minute

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Poisson distribution

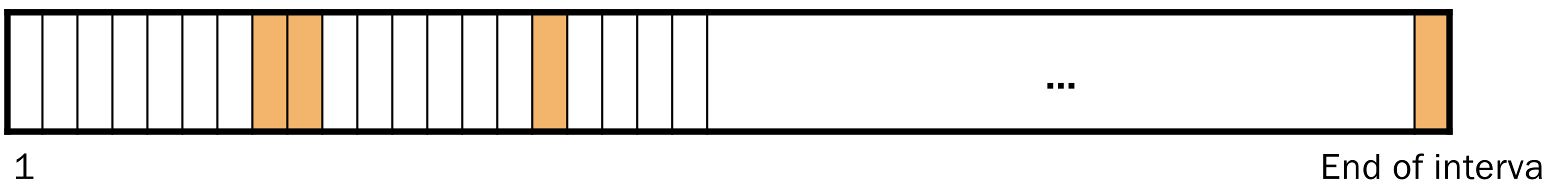
Poisson, continued



Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of successes over the experiment duration, assuming **the time that each success occurs is independent** and the average # of requests over time is constant.



Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

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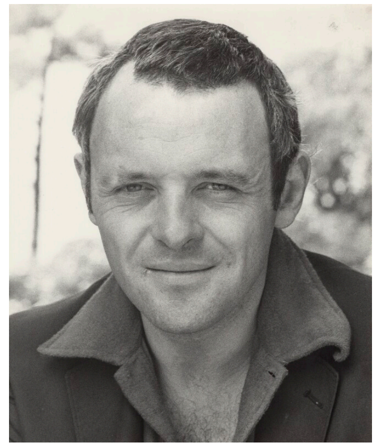
$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0, 1, 2, \ldots\}$	Variance	$\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Yes, expectation == variance for Poisson RV! More later.

Simeon-Denis Poisson



French mathematician (1781 – 1840)

- Published his first paper at age 18
- Professor at age 21
- Published over 300 papers

"Life is only good for two things: doing mathematics and teaching it."

Earthquakes

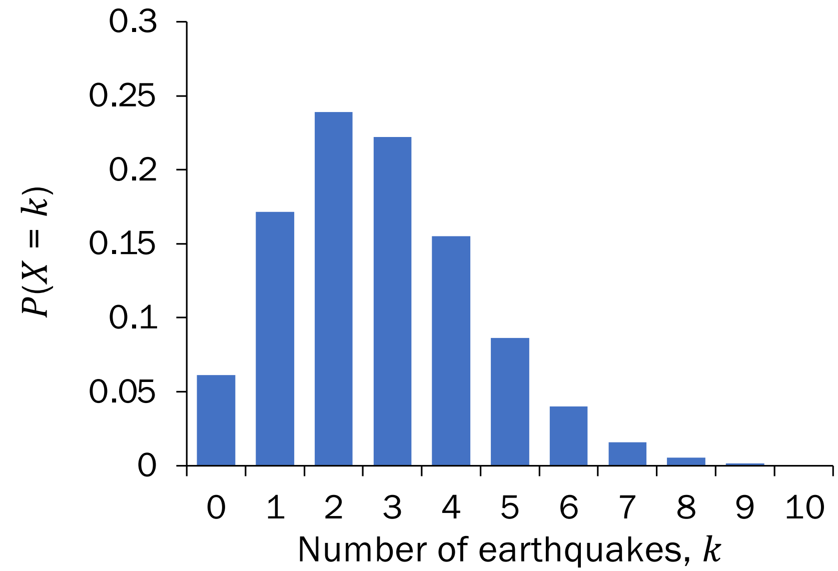
$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$
$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

There are an average of 2.79 major earthquakes in the world each year, and major earthquakes occur independently.

What is the probability of 3 major earthquakes happening next year?

1. Define RVs

2. Solve



Are earthquakes really Poissonian?

Bulletin of the
Seismological Society of America

Vol. 64 October 1974 No. 5

IS THE SEQUENCE OF EARTHQUAKES IN SOUTHERN CALIFORNIA, WITH AFTERSHOCKS REMOVED, POISSONIAN?

By J. K. Gardner and L. Knopoff

Abstract

Yes.

Other Discrete RVs

Grid of random variables

	Number of successes	Time until success	
One trial	$\text{Ber}(p)$		One success
Several trials	$n = 1$ $\text{Bin}(n, p)$		Several successes
Interval of time	$\text{Poi}(\lambda)$	(next time)	Interval of time to first success

Geometric RV

Consider an experiment: independent trials of Ber(p) random variables.

def A **Geometric** random variable X is the # of trials until the first success.

$X \sim \text{Geo}(p)$	PMF	$P(X = k) = (1 - p)^{k-1} p$
	Expectation	$E[X] = \frac{1}{p}$
Support: $\{1, 2, \ldots\}$	Variance	$\text{Var}(X) = \frac{1-p}{p^2}$

Examples:

- Flipping a coin ($P(\text{heads}) = p$) until first heads appears
- Generate bits with $P(\text{bit} = 1) = p$ until first 1 generated

Negative Binomial RV

Consider an experiment: independent trials of Ber(p) random variables.

def A **Negative Binomial** random variable X is the # of trials until r successes.

$$X \sim \text{NegBin}(r, p)$$

Support: $\{r, r+1, \dots\}$

PMF	$P(X = k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r$
Expectation	$E[X] = \frac{r}{p}$
Variance	$\text{Var}(X) = \frac{r(1-p)}{p^2}$

Examples:

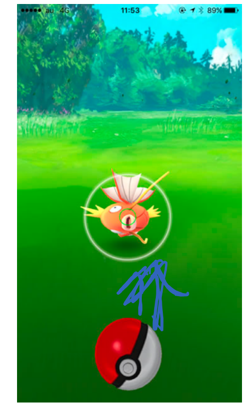
- # coin flips until r^{th} head appears
- # of strings hashes until bucket 1 has r entries

$\text{Geo}(p) = \text{NegBin}(1, p)$

Grid of random variables

	Number of successes	Time until success	
One trial	$\text{Ber}(p)$	$\text{Geo}(p)$	One success
	$\uparrow$ $n = 1$	$\uparrow$ $r = 1$	
Several trials	$\text{Bin}(n, p)$	$\text{NegBin}(r, p)$	Several successes
Interval of time	$\text{Poi}(\lambda)$	(next time)	Interval of time to first success

Catching Pokemon



Wild Pokemon are captured by throwing Pokeballs at them.

- Each ball has probability p = 0.1 of capturing the Pokemon.
- Each throw is independent of previous ones.

What is the probability that you catch the Pokemon on the 5th try?

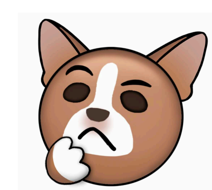
1. Define events/ RVs & state goal

   $X \sim$ some distribution

   Want: $P(X = 5)$

2. Solve

A. $X \sim \text{Bin}(5, 0.1)$
B. $X \sim \text{Poi}(0.5)$
C. $X \sim \text{NegBin}(5, 0.1)$
D. $X \sim \text{NegBin}(1, 0.1)$
E. $X \sim \text{Geo}(0.1)$
F. None/other



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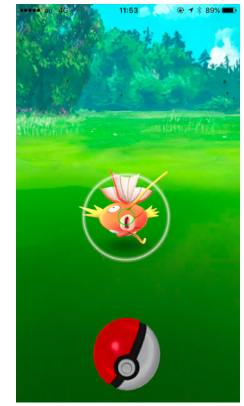

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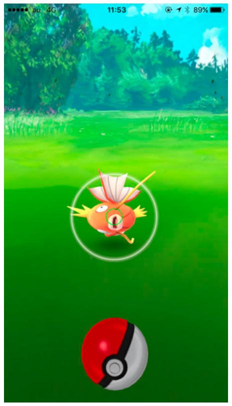
D. $X \sim \text{NegBin}(1, 0.1)$
E. $X \sim \text{Geo}(0.1)$
F. None/other

Catching Pokemon

$X \sim \text{Geo}(p) \quad p(k) = (1-p)^{k-1}p$



Wild Pokemon are captured by throwing Pokeballs at them.

- Each ball has probability p = 0.1 of capturing the Pokemon.
- Each throw is independent of previous ones.

What is the probability that you catch the Pokemon on the 5th try?

1. Define events/ RVs & state goal
2. Solve

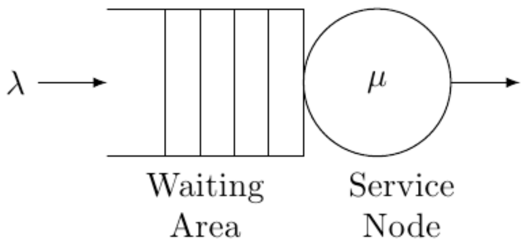
$X \sim \text{Geo}(0.1)$

Want: $P(X = 5)$

CS109 Learning Goal: Use new RVs

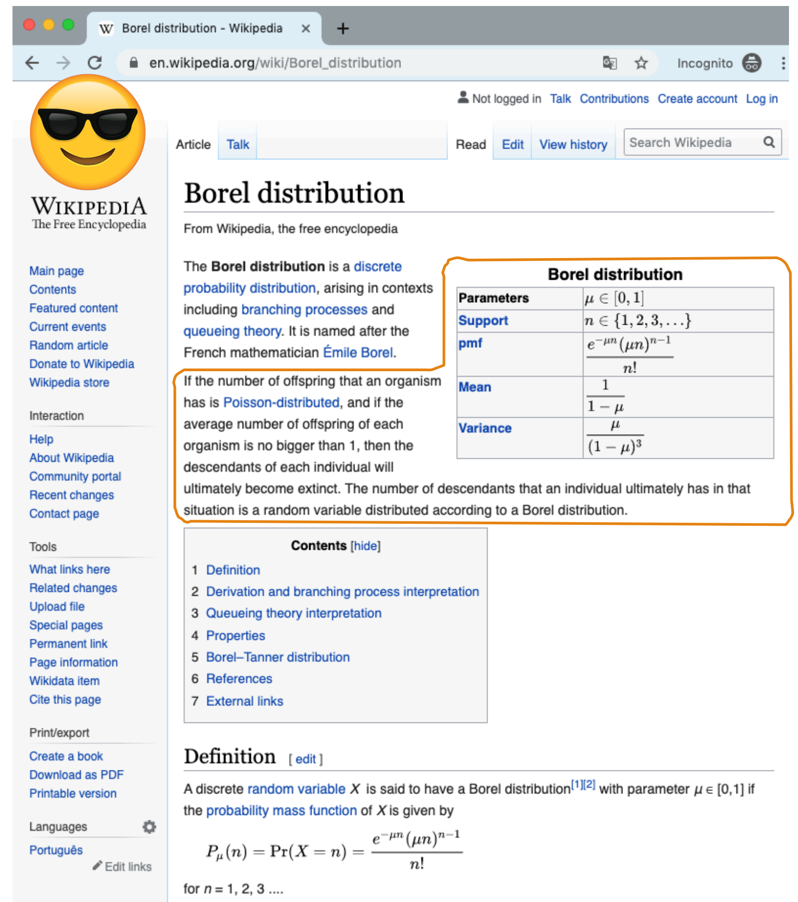
Let's say you are learning about servers/networks.

You read about the M/D/1 queue:



"The service time busy period is distributed as a Borel with parameter $\mu = 0.2$."

Goal: You can recognize terminology and understand experiment setup.



Big Q: Fixed parameter or random variable?

Parameter	What is common among all outcomes of our experiment?

Examples so far:
- Prob. success
- # total trials
- # target successes
- Average rate of success

Random variable	What differentiates our event from the rest of the sample space?

Examples so far:
- # of successes
- Time until success (for some definition of time)

Grid of random variables

	Number of successes	Time until success	
...in one trial	Ber(p)	Geo(p)	...until one success
..in several trials	$n = 1$ Bin(n, p)	$r = 1$ NegBin(r, p)	...until several successes
...in a fixed interval of time	Poi(λ)	(next time!)	Interval of time until first success

Grid of random variables

	Number of successes	Time until success	
...in one trial	Ber(p)	Geo(p)	...until one success
	$\uparrow n = 1$	$\uparrow r = 1$	
...in several trials	Bin(n, p)	NegBin(r, p)	...until several successes
...in a fixed interval of time	Poi(λ)	(next time)	Interval of time until first success

Ponder Models

Let's take a two minute siesta.

Slide 29 presents five scenarios to dream about while you sleep. We'll press through them once we all wake up.

Kickboxing with RVs

Choose from:

A. Ber(p)

B. Bin(n, p)

C. Poi(λ)

D. Geo(p)

E. NegBin(r, p)

How would you model the following?

1. # of friend requests you receive in a day
2. # of children born to the same parent until one is born with brown eyes.
3. If stock ends up higher at end of trading.
4. # of probability problems you try until you get 5 correct (if you are randomly correct)
5. # of years in a decade with more than 6 Atlantic hurricanes

Kickboxing with RVs

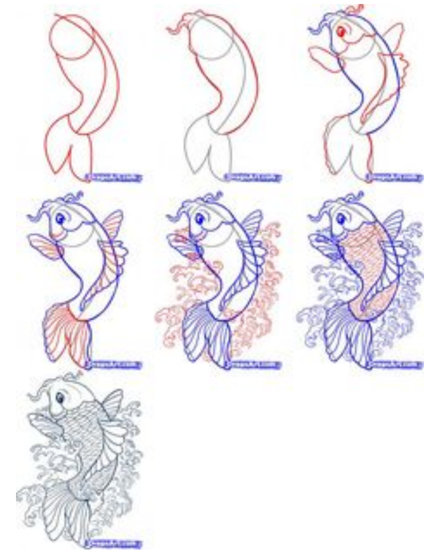
Choose from:

A. Ber(p)
B. Bin(n, p)
C. Poi(λ)
D. Geo(p)
E. NegBin(r, p)

How would you model the following?

1. # of friend requests you receive in a day — C. Poi(λ)
2. # of children born to the same parent until one is born with brown eyes. — D. Geo(p) or E. NegBin$(1, p)$
3. If stock ends up higher at end of trading. — A. Ber(p) or B. Bin$(1, p)$
4. # of probability problems you try until you get 5 correct (if you are randomly correct) — E. NegBin$(r = 5, p)$
5. # of years in a decade with more than 6 Atlantic hurricanes — B. Bin$(n = 10, p)$, where $p = P(\geq 6$ hurricanes in a year$)$ calculated from C. Poi(λ)

Note: These exercises are designed to build intuition; in a problem statement, you will generally have more clues.



Poisson Approximation

Poisson Random Variable

Review

$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0, 1, 2, \dots\}$	Variance	$\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed interval of time, where successes are independent
   $\lambda = E[X]$, average success/interval

1. Web server load

$X \sim \text{Poi}(\lambda)$
$E[X] = \lambda$
$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}$

Consider requests to a web server in 1 second.

- In the past, server load averages 2 hits/second, where hits arrive independently.
- Let X = # hits the server receives in a second.

What is $P(X < 5)$?

Define RVs

Solve

Poisson Random Variable

$X \sim \text{Poi}(\lambda)$

Support: $\{0, 1, 2, \ldots\}$

PMF $P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$

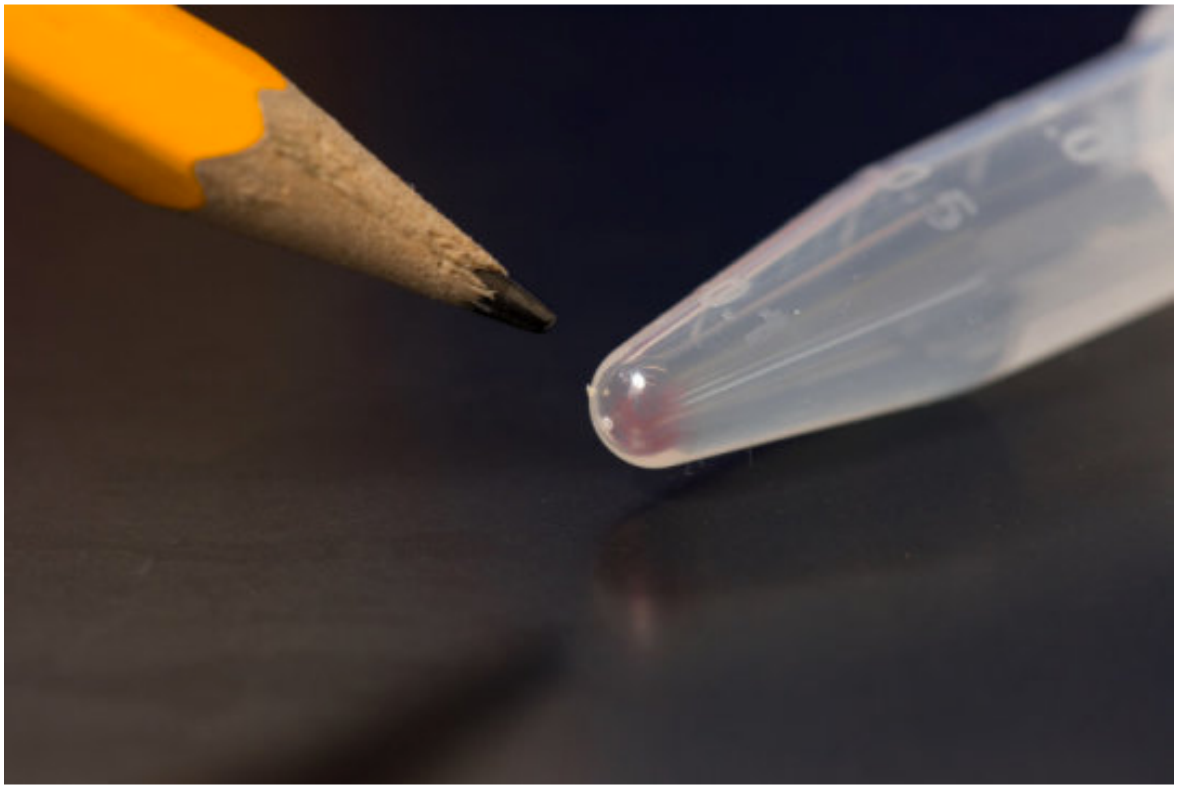
Expectation $E[X] = \lambda$

Variance $\text{Var}(X) = \lambda$

In CS109, a Poisson RV $X \sim \text{Poi}(\lambda)$ most often models

1. # of successes in a fixed interval of time, where successes are independent
   $\lambda = E[X]$, average success/interval
2. Approximation of $Y \sim \text{Bin}(n, p)$ where n is large and p is small.
   $\lambda = E[Y] = np$
   Approximation works even when trials not entirely independent.

2. DNA



All the movies, images, emails and other digital data from more than 600 smartphones (10,000 GB) can be stored in the faint pink smear of DNA at the end of this test tube.

What is the probability that DNA storage stays uncorrupted?

2. DNA

What is the probability that DNA storage stays uncorrupted?

- In DNA (and real networks), we store large strings.
- Let string length be long, e.g., $n \approx 10^4$
- Probability of corruption of each base pair is very small, e.g., $p = 10^{-6}$
- Let $X = \#$ of corruptions.

What is P(DNA storage is uncorrupted) = $P(X = 0)$?

1. Approach 1:

$$X \sim \text{Bin}(n = 10^4, p = 10^{-6})$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

unwieldy! ⚠️ $= \binom{10^4}{0} 10^{-6 \cdot 0} (1 - 10^{-6})^{10^4 - 0}$

≈ 0.99049829

2. Approach 2:

$$X \sim \text{Poi}(\lambda = 10^4 \cdot 10^{-6} = 0.01)$$

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} = e^{-0.01} \frac{0.01^0}{0!}$$

$$= e^{-0.01}$$

≈ 0.99049834 a good approximation! ✅

Ruminate Over Limits

Let’s take a thirty second break to quickly order a pizza.

Slide 38 presents one thought problem that we’ll work through once we’re confident the pizza is on its way.

When is a Poisson approximation appropriate?

$$P(X = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} = \cdots$$

Def natural exponent

$$= \lim_{n\to\infty} \frac{n!}{n^k (n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

Expand

$$= \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \frac{(n-k)!}{(n-k)!} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{\left(1 - \frac{\lambda}{n}\right)^k}$$

Limit analysis

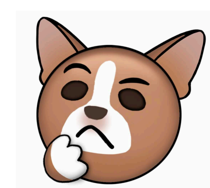
$$= \lim_{n\to\infty} \frac{n^k}{n^k} \frac{\lambda^k}{k!} \frac{e^{-\lambda}}{1}$$

Simplify

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

Under which conditions will $X \sim \text{Bin}(n, p)$ behave like $\text{Poi}(\lambda)$, where $\lambda = np$?

A. Large n, large p
B. Small n, small p
C. Large n, small p
D. Small n, large p
E. Other



Poisson approximation

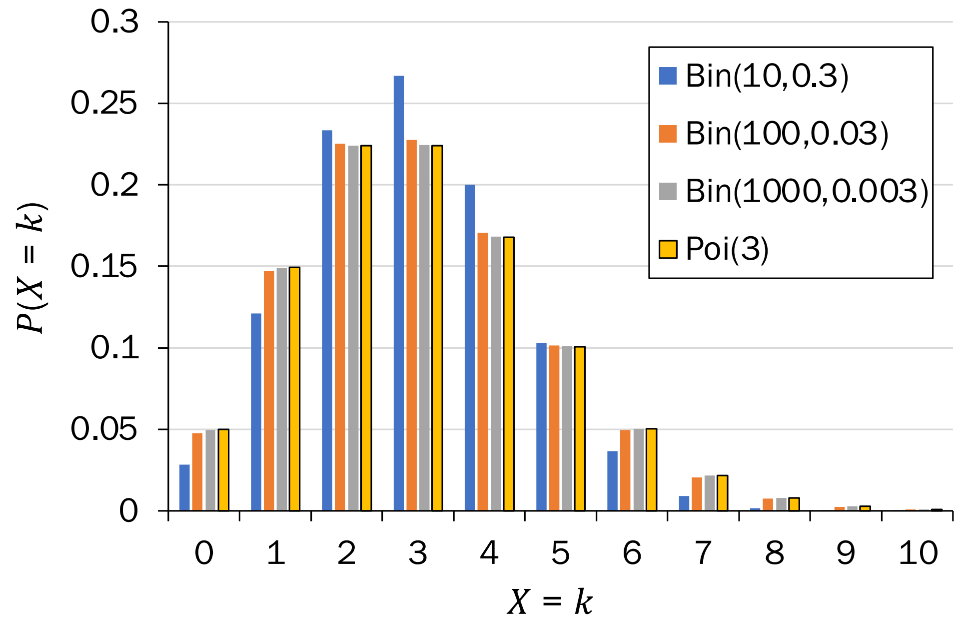
$X \sim \text{Poi}(\lambda)$ $\quad$ $Y \sim \text{Bin}(n, p)$

$E[X] = \lambda$ $\quad$ $E[Y] = np$

Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate".

Different interpretations of "moderate":

- $n > 20$ and $p < 0.05$
- $n > 100$ and $p < 0.1$



Poisson is Binomial in the limit:

- $\lambda = np$, where $n \to \infty, p \to 0$

Poisson Random Variable

Consider an experiment that lasts a fixed interval of time.

def A **Poisson** random variable X is the number of occurrences over the experiment duration.

$X \sim \text{Poi}(\lambda)$	PMF	$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$
	Expectation	$E[X] = \lambda$
Support: $\{0, 1, 2, \ldots\}$	Variance	$\text{Var}(X) = \lambda$

Examples:

- # earthquakes per year
- # server hits per second
- # of emails per day

Time to show intuition for why expectation == variance!

Properties of Poi(λ) with the Poisson paradigm

Recall the Binomial:

$Y \sim \text{Bin}(n, p)$	Expectation	$E[Y] = np$
	Variance	$\text{Var}(Y) = np(1-p)$

Consider $X \sim \text{Poi}(\lambda)$, where $\lambda = np$ $(n \to \infty, p \to 0)$:

$X \sim \text{Poi}(\lambda)$	Expectation	$E[X] = \lambda$
	Variance	$\text{Var}(X) = \lambda$

Proof:

$$E[X] = np = \lambda$$
$$\text{Var}(X) = np(1-p) \to \lambda(1-0) = \lambda$$

Poisson Approximation, approximately

Poisson can still provide a good approximation of the Binomial, even when assumptions are "mildly" violated.

You can apply the Poisson approximation when:

- "Successes" in trials are not entirely independent 👉
  e.g.: # entries in each bucket in large hash table.
- Probability of "success" in each trial varies (slightly), like a small relative change in a very small p
  e.g.: Average # requests to web server/sec may fluctuate slightly due to load on network

We won't explore this too much, but we want you to know it exists.

Entertain What's Possible

Let's take a forty second break to answer the door to get your pizza.

Slide 44 presents three scenarios that we'll consider as we chew.

Can these Binomial RVs be approximated?

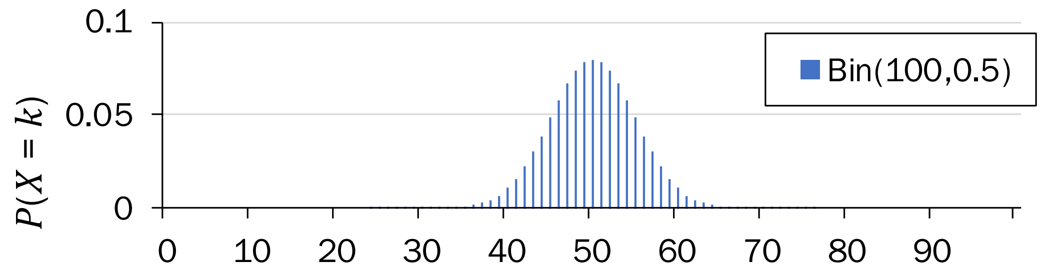
Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate".

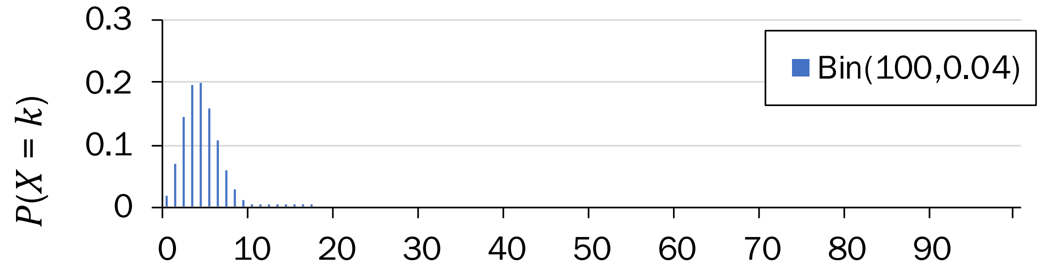
Different interpretations of "moderate":

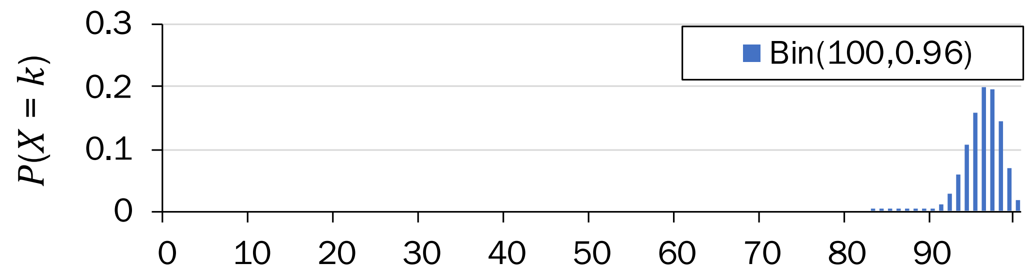
- $n > 20$ and $p < 0.05$
- $n > 100$ and $p < 0.1$

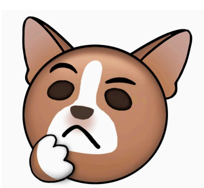
Poisson is Binomial in the limit:

- $\lambda = np$, where $n \to \infty, p \to 0$









Can these Binomial RVs be approximated?

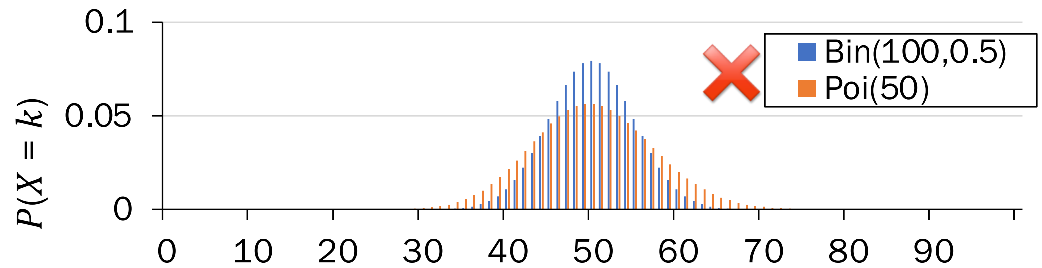
Poisson approximates Binomial when n is large, p is small, and $\lambda = np$ is "moderate".

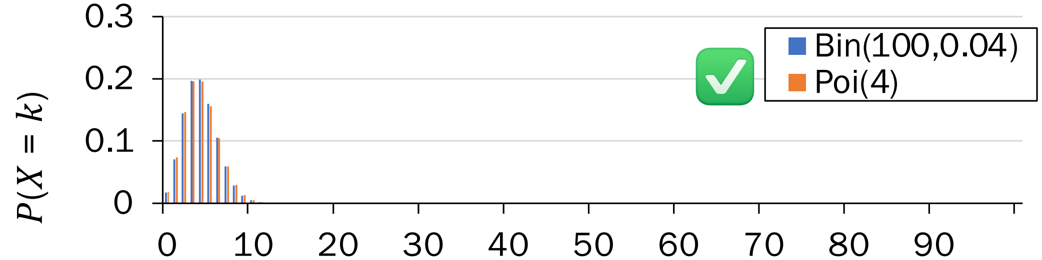
Different interpretations of "moderate":

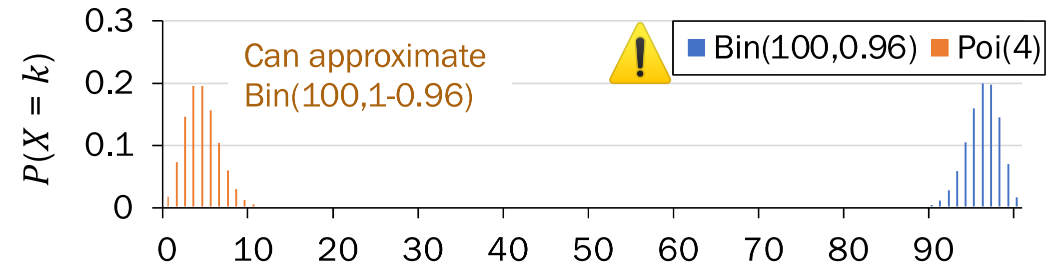
- $n > 20$ and $p < 0.05$
- $n > 100$ and $p < 0.1$

Poisson is Binomial in the limit:

- $\lambda = np$, where $n \to \infty, p \to 0$





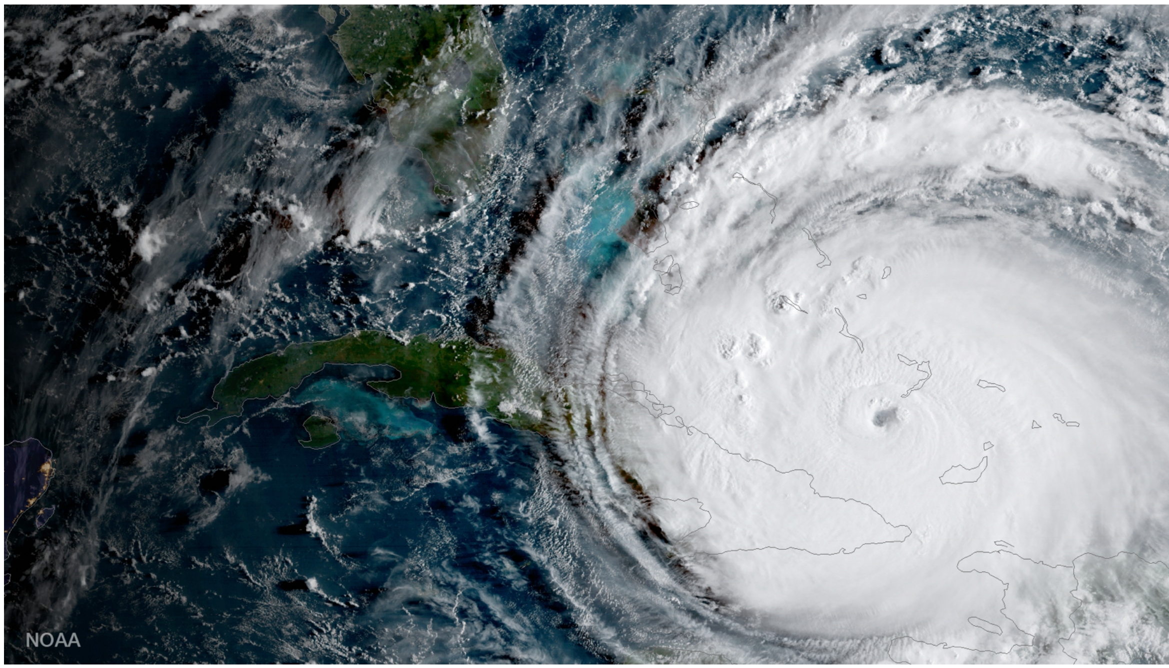


A Real License Plate Seen at Stanford

No, it's not mine…
but I kind of wish it was.

Modeling Hurricanes (extra)

Hurricanes



What is the probability of an extreme weather event?

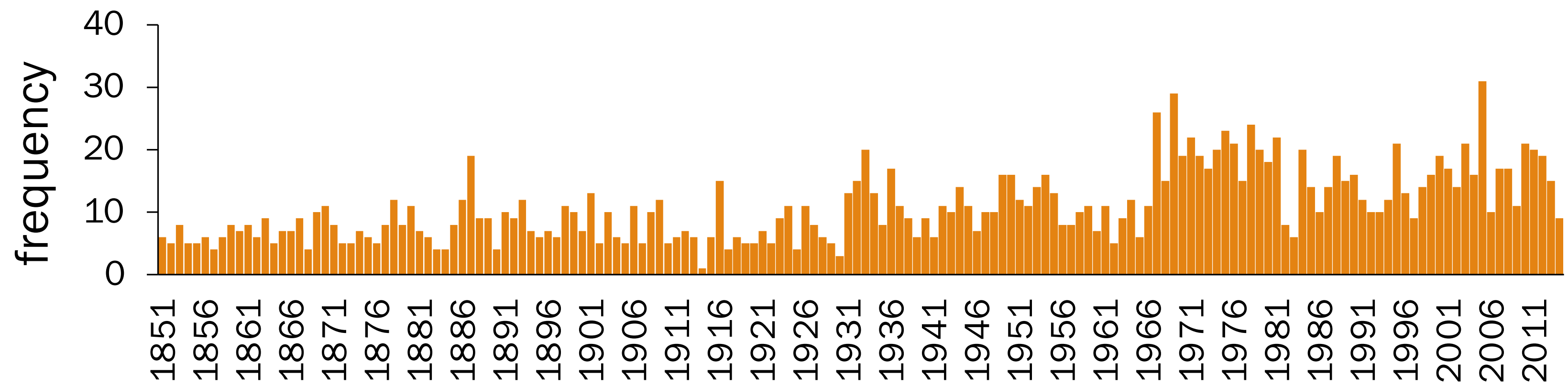
How do we model the number of hurricanes in a season (year)?

1. Graph your distribution.

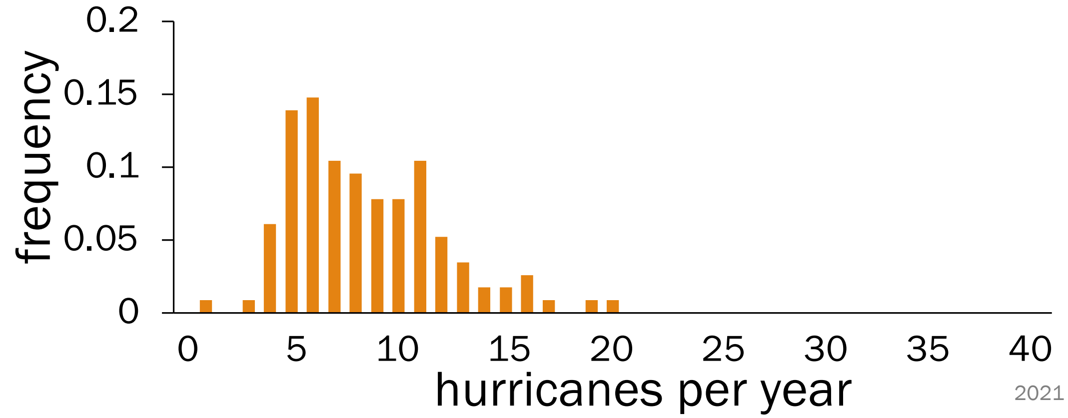
1. Graph: Hurricanes per year since 1851

Which graph is a histogram (i.e., distribution) of frequency (# of hurricanes per year)?

A.



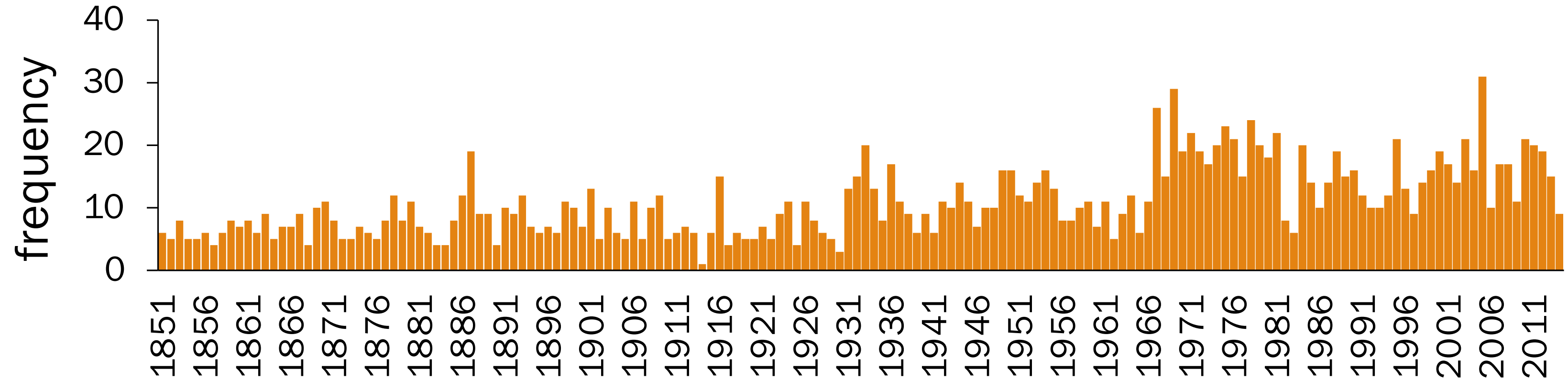
B.



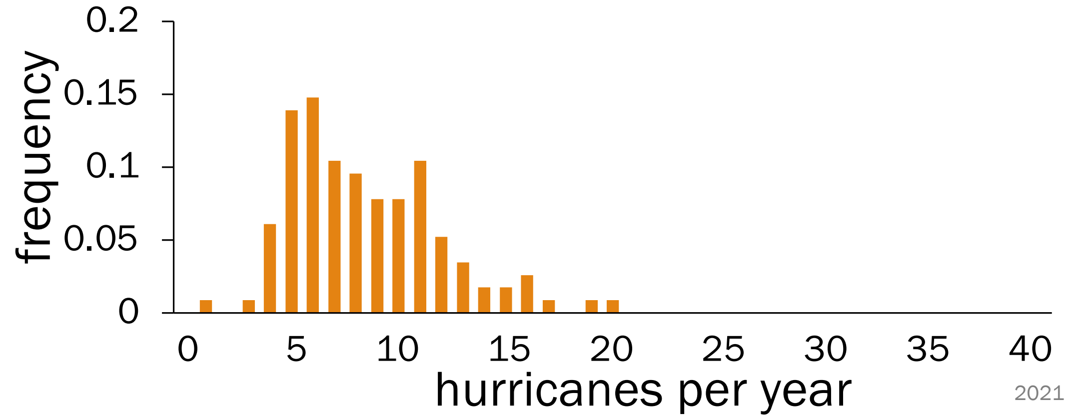
1. Graph: Hurricanes per year since 1851

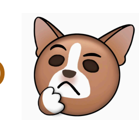
Which graph is a histogram (i.e., distribution) of frequency (# of hurricanes per year)?

A.

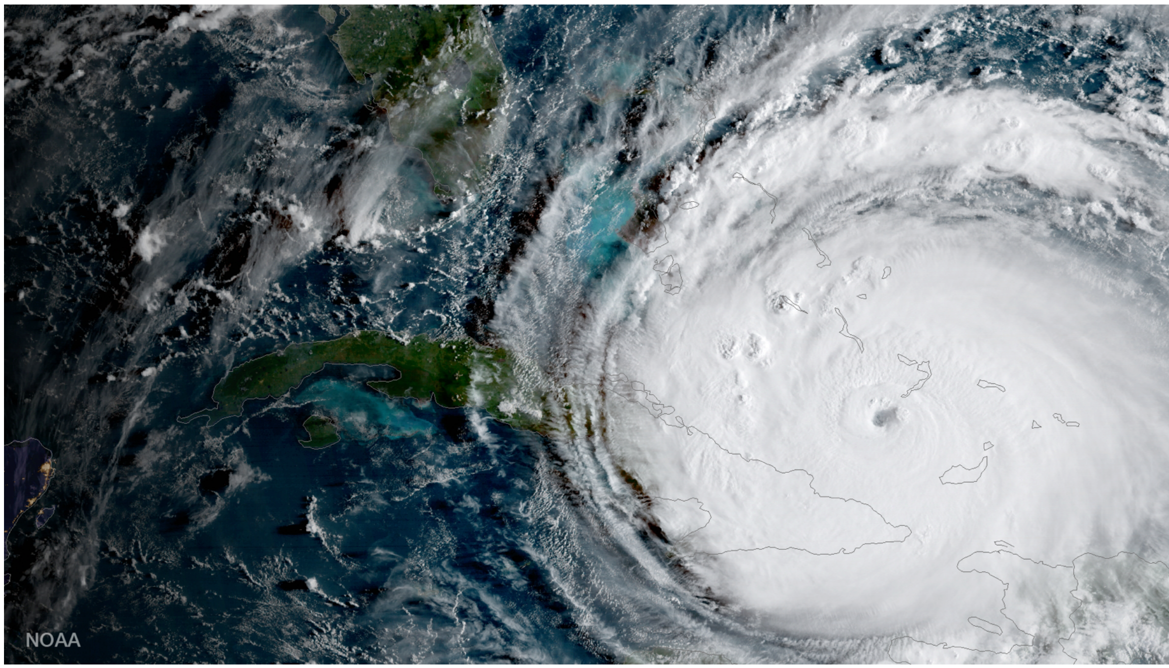


B.



Looks Poissonian to  !

Hurricanes



How do we model the number of hurricanes in a season (year)?

2. Find a reasonable distribution and compute parameters.

2. Find a distribution: Python SciPy RV methods

```
from scipy import stats       # great package
X = stats.poisson(8.5)        # X ~ Poi(λ = 8.5)
X.pmf(2)                      # P(X = 2)
```

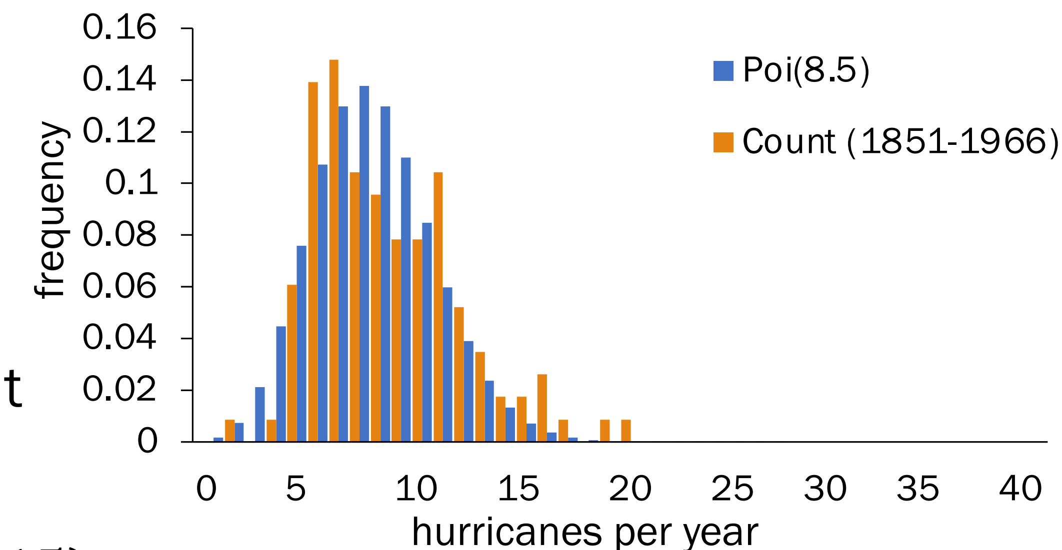
Function	Description
`X.pmf(k)`	$P(X = k)$
`X.cdf(k)`	$P(X \leq k)$
`X.mean()`	$E[X]$
`X.var()`	$\text{Var}(X)$
`X.std()`	$\text{SD}(X)$

SciPy reference: https://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.poisson.html

2. Find a distribution

Until 1966, things look pretty Poisson.

What is the probability of over 15 hurricanes in a season (year) given that the distribution doesn't change?



$$P(X > 15) = 1 - P(X \leq 15)$$

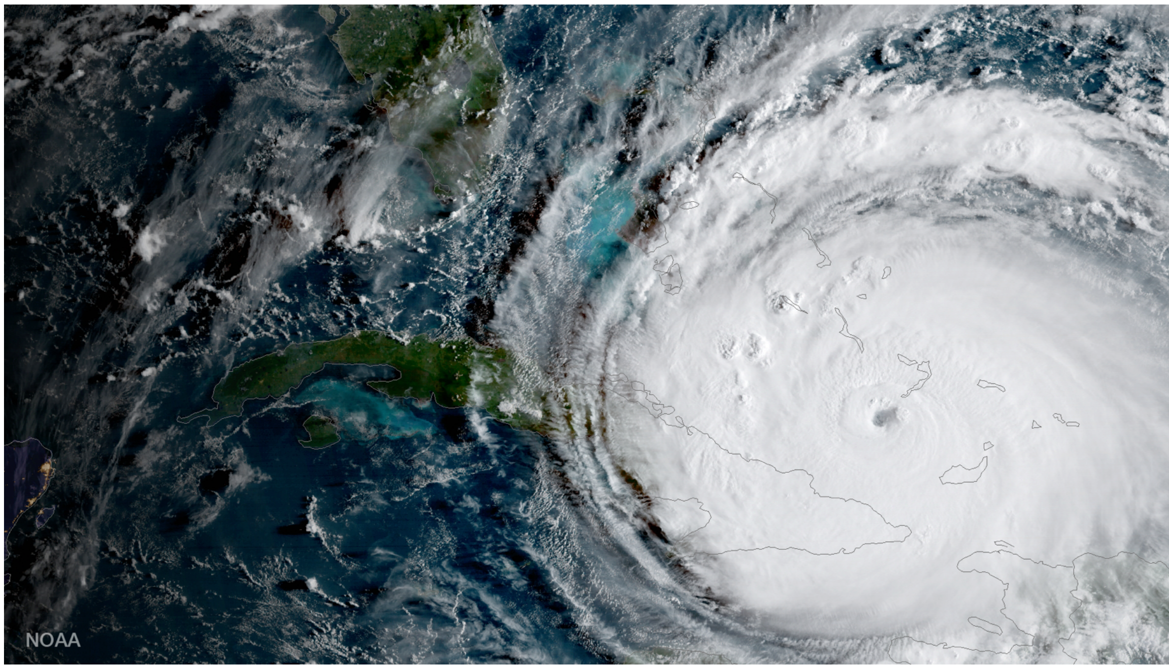
$$= 1 - \sum_{k=0}^{15} P(X = k)$$

$X \sim \text{Poi}(\lambda = 8.5)$

You can calculate this PMF using your favorite programming language. Or use Python3.

$$= 1 - 0.986 = 0.014$$

Hurricanes



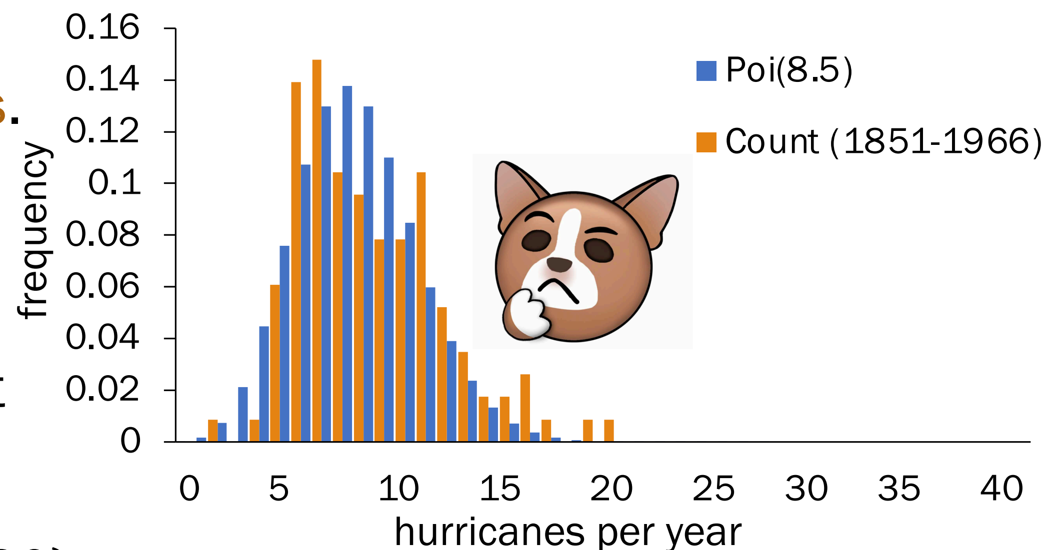
How do we model the number of hurricanes in a season (year)?

3. Identify and explain outliers.

3. Improbability

Since 1966, there have been <u>**two years**</u> with over 30 hurricanes.

What is the probability of over 30 hurricanes in a season (year) given that the distribution doesn't change?



$$P(X > 30) = 1 - P(X \leq 30)$$

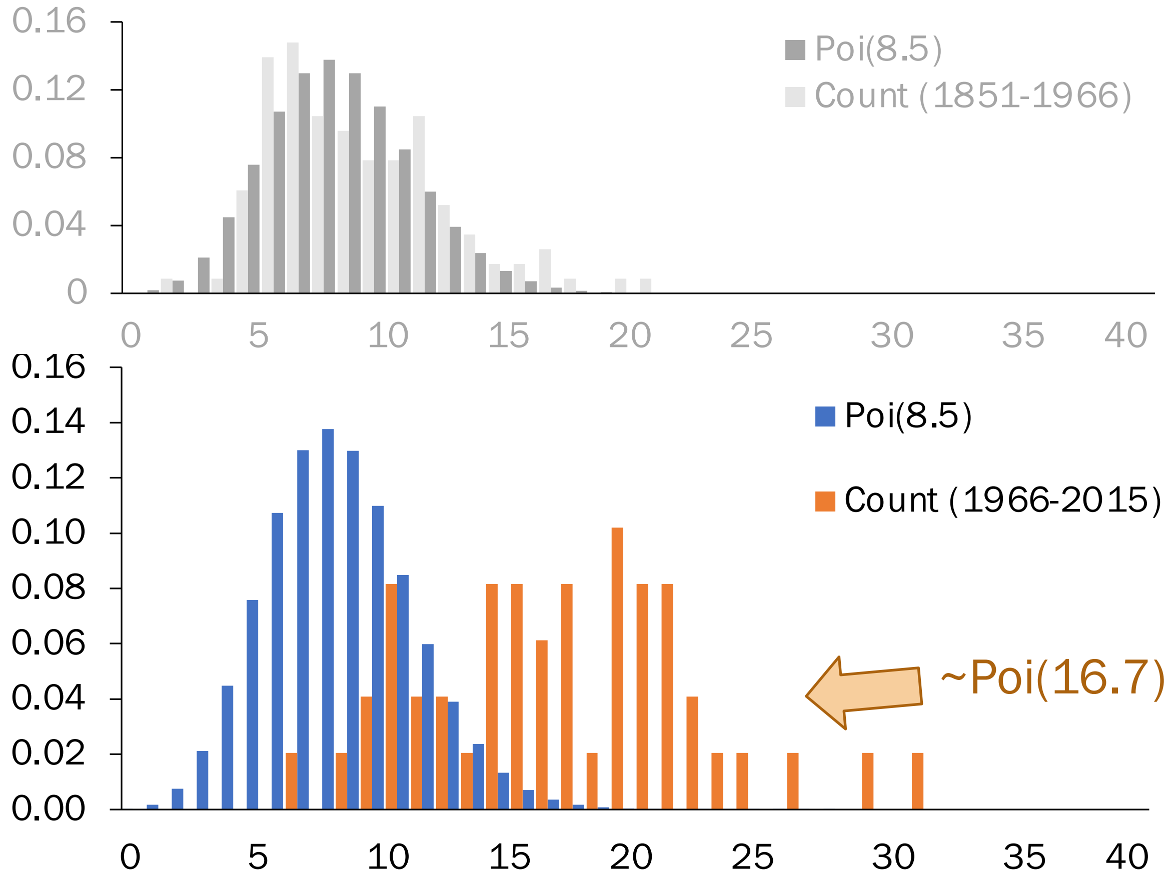
$$= 1 - \sum_{k=0}^{30} P(X = k) \qquad X \sim \text{Poi}(\lambda = 8.5)$$

$$= 2.2\text{E} - 09$$

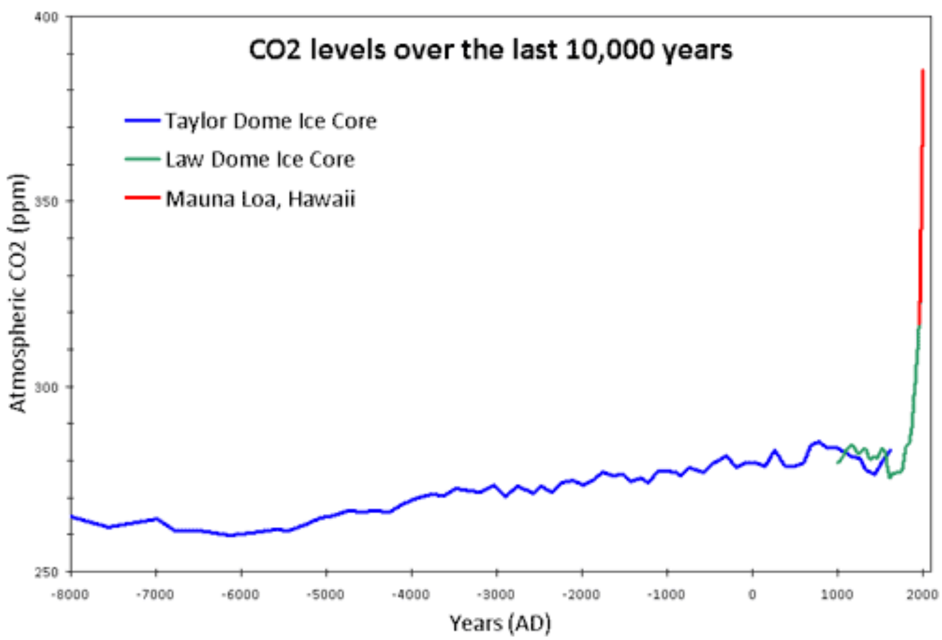
3. The distribution has changed.

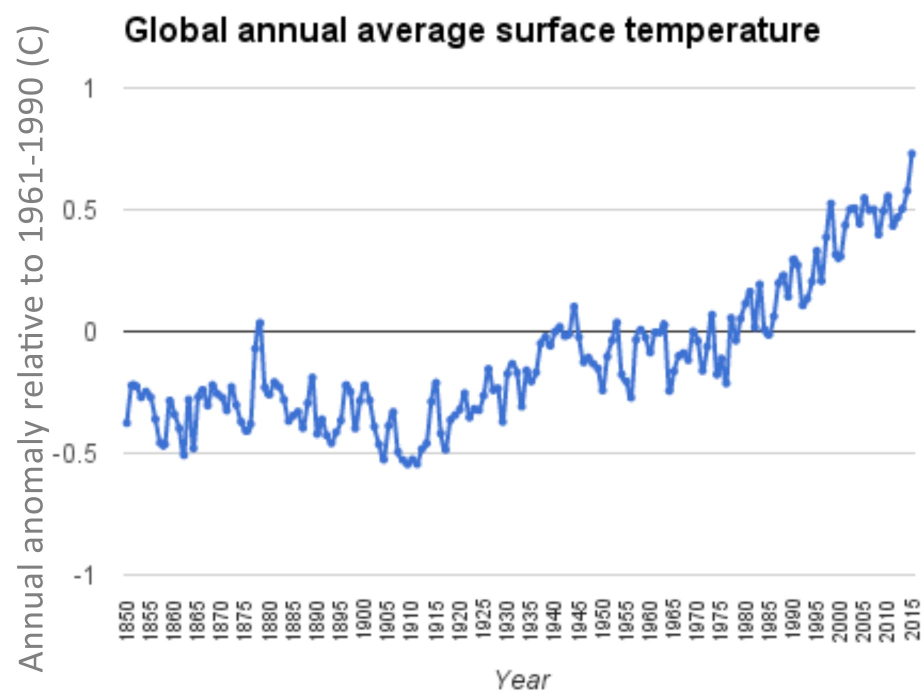
1851–
1966

Since
1966

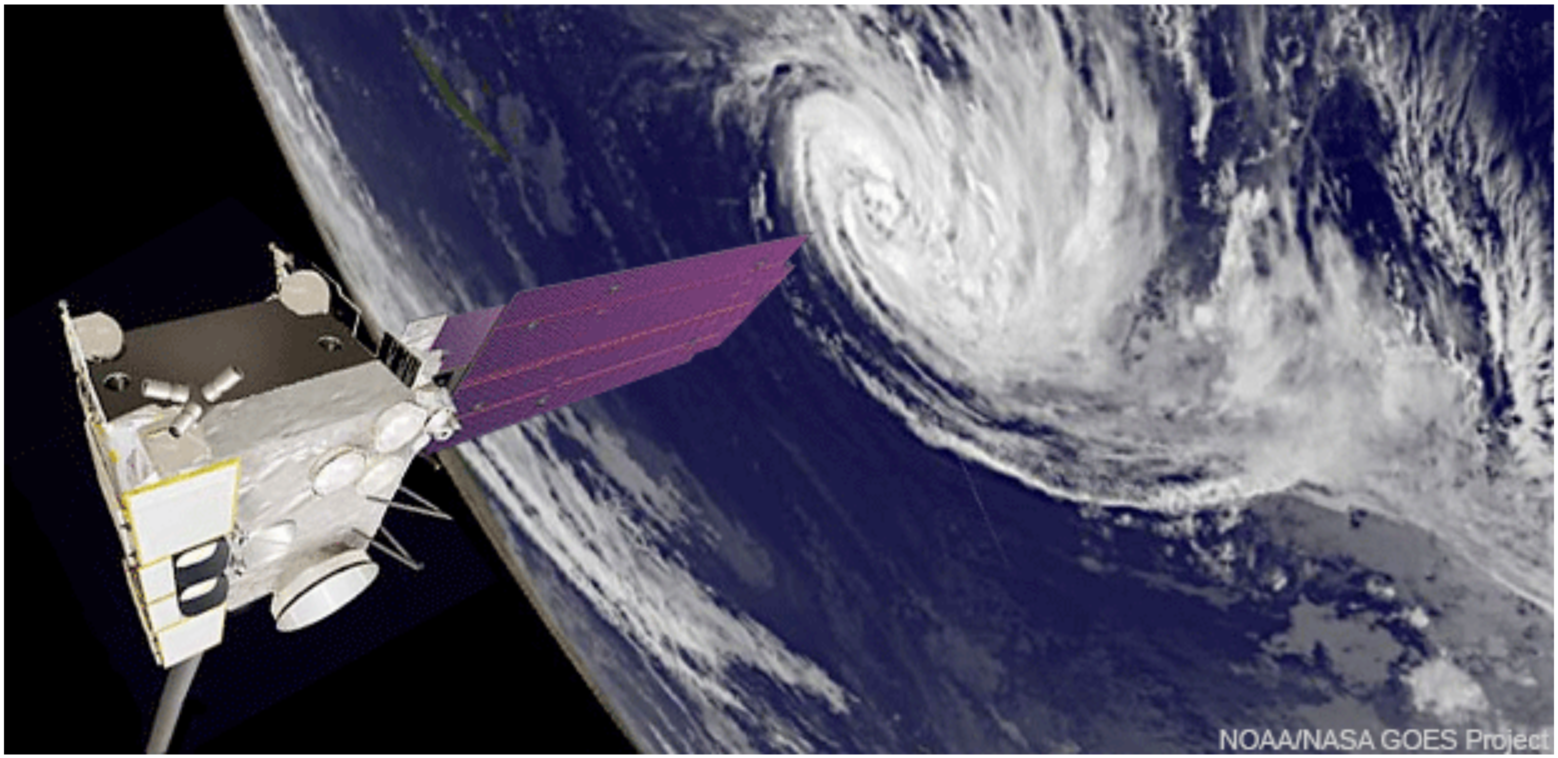


3. What changed?





3. What changed?



It's not just climate change. We also have tools for better data collection.