Quantum Physics II. Exercise 2.8

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Enunciate

Given the singlet

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+,-\rangle - |-,+\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \tag{1}$$

expressed in the $\{|+\rangle, |-\rangle\}$ basis of spin eigenstates along the $\hat{z}$ axis:

a) Using the results from the rotation matrix $\hat{\vec{S}}_{\vec{n}}$ (see Problem 2.4), express the state

$$|-\rangle_{\vec{n'}} \otimes |+\rangle_{\vec{n}} \tag{2}$$

in the $\{|+\rangle, |-\rangle\}$ basis, where the measurement directions $\vec{n}$ and $\vec{n'}$ are given by the angles (θ, ϕ) and (θ', ϕ'), respectively.

b) Calculate the probability for measuring $|-\rangle_{\vec{n'}} \otimes |+\rangle_{\vec{n}}$ when the sistem is originally in the singlet state $|0,0\rangle$.

c) Verify that the singlet state is invariant under a simultaneus rotation of both spins; that is, show that for any rotation R,

$$(U(R) \otimes U(R))\, |0,0\rangle = e^{i\delta}\, |0,0\rangle\,, \tag{3}$$

with $e^{i\delta}$ an irrelevant global phase.

1. State $|-\rangle_{\vec{n}'} \otimes |+\rangle_{\vec{n}}$ in the basis $\{|+\rangle\,, |-\rangle\}$

First, in order to get the kets $|-\rangle_{\vec{n}'}$ and $|+\rangle_{\vec{n}}$ we will act the matrix $\hat{\vec{S}}_{\vec{n}}$ over the basis $\{|+\rangle\,, |-\rangle\}$.

$$\hat{\vec{S}}_{\vec{n}}\,|+\rangle = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\left[\cos\theta\,|+\rangle + e^{i\theta}\sin\theta\,|-\rangle\right]$$

$$\hat{\vec{S}}_{\vec{n}'}\,|-\rangle = \frac{\hbar}{2}\begin{pmatrix} \cos\theta' & e^{-i\phi'}\sin\theta' \\ e^{i\phi'}\sin\theta' & -\cos\theta' \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2}\left[e^{-i\phi'}\sin\theta'\,|+\rangle - \cos\theta'\,|-\rangle\right]$$

now we only have to perform the tensorial product:

$$\begin{aligned}
|-\rangle_{\vec{n}'} \otimes |+\rangle_{\vec{n}} &= \frac{\hbar^2}{4}\left[e^{-i\phi'}\sin\theta'\,|+\rangle - \cos\theta'\,|-\rangle\,\right] \otimes \left[\,\cos\theta\,|+\rangle + e^{i\phi}\sin\theta\,|-\rangle\,\right] \\
&= \frac{\hbar^2}{4}\left[e^{-i\phi'}\sin\theta'\cos\theta\,|+\rangle \otimes |+\rangle + e^{i(\phi-\phi')}\sin\theta'\sin\theta\,|+\rangle \otimes |-\rangle \right. \\
&\qquad \left. - \cos\theta'\cos\theta\,|-\rangle \otimes |+\rangle - e^{i\phi}\sin\theta\cos\theta'\,|-\rangle \otimes |-\rangle\,\right]
\end{aligned}$$

which can be expressed as:

$$\begin{aligned}
|-,+\rangle_{\vec{n}'\vec{n}} = \frac{\hbar^2}{4}\Big[&e^{-i\phi'}\sin\theta'\cos\theta\,|+,+\rangle \\
&+ e^{i(\phi-\phi')}\sin\theta'\sin\theta\,|+,-\rangle \\
&- \cos\theta'\cos\theta\,|-,+\rangle \\
&- e^{i\phi}\sin\theta\cos\theta'\,|-,-\rangle\,\Big]
\end{aligned}$$

2. Probability of measuring $|-,+\rangle_{\vec{n}'\vec{n}}$ on $|0,0\rangle$

For obtaining the probability of measuring the new state $|-,+\rangle_{\vec{n}'\vec{n}}$ in the singlet state we must perform a squared bra-ket or inner product.

$$P\left(|0,0\rangle \to |-,+\rangle_{\vec{n}'\vec{n}}\right) = ||_{\vec{n}'\vec{n}}\,\langle +,-|0,0\rangle\,||^2$$

First we will do the regular inner product. It will only survive the bra-kets with the same eigenstate.

$$
\begin{aligned}
{}_{\vec{n}'\vec{n}}\langle +,-|0,0\rangle &= \frac{\hbar^2}{4\sqrt{2}}\left(e^{-i(\phi-\phi')}\sin\theta'\sin\theta\,\langle -,+|+,-\rangle + \cos\theta\cos\theta'\,\langle +,-|-,+\rangle\right)\\
&= \frac{\hbar^2}{4\sqrt{2}}\left(e^{-i(\phi-\phi')}\sin\theta'\sin\theta + \cos\theta\cos\theta'\right)
\end{aligned}
$$

Now we just have to square the result of the inner product, which is multiply it with its conjugate.

$$
\begin{aligned}
P\left(|0,0\rangle \to |-,+\rangle_{\vec{n}'\vec{n}}\right) &= \frac{\hbar^4}{32}\left(e^{-i(\phi-\phi')}\sin\theta'\sin\theta + \cos\theta\cos\theta'\right)\cdot\left(e^{i(\phi-\phi')}\sin\theta'\sin\theta + \cos\theta\cos\theta'\right) =\\
&= \frac{\hbar^4}{32}\Big(\sin^2\theta'\sin^2\theta + \left(e^{-i(\phi-\phi')} + e^{i(\phi-\phi')}\right)\sin\theta'\sin\theta\cos\theta\cos\theta'\\
&\qquad + \cos^2\theta\cos^2\theta'\Big)
\end{aligned}
$$

Finally, we can simplify the result if we apply some trigonometrical properties such as: $\sin 2\theta = 2\sin\theta\cos\theta$ and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.

$$
P\left(|0,0\rangle \to |-,+\rangle_{\vec{n}'\vec{n}}\right) = \frac{\hbar^4}{32}\left(\sin^2\theta'\sin^2\theta + \cos(\phi-\phi')\frac{1}{2}\sin 2\theta'\sin 2\theta + \cos^2\theta\cos^2\theta'\right)
$$

3. Verify that $|0,0\rangle$ is invariant under any rotation R

In order to verify that the singlet is invariant for any rotation R we will apply the distributive properties from tensor products:

$$
\left(U(R)\otimes U(R)\right)|0,0\rangle = \frac{1}{\sqrt{2}}\left(\left(U(R)\otimes U(R)\right)|+,-\rangle - \left(U(R)\otimes U(R)\right)|-,+\rangle\right)
$$

Secondly, if we expand the rotation matrix as $U(R) = cos\frac{\alpha}{2}I - i\sin\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})$ can multiply it with

$$\frac{1}{\sqrt{2}}\Big[\Big(\cos\frac{\alpha}{2}\left|+,-\right\rangle - i\sin\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})\left|+,-\right\rangle\Big)\otimes\Big(\cos\frac{\alpha}{2}\left|+,-\right\rangle - i\sin\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})\left|+,-\right\rangle\Big)$$
$$-\Big(\cos\frac{\alpha}{2}\left|-,+\right\rangle - i\sin\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})\left|-,+\right\rangle\Big)\otimes\Big(\cos\frac{\alpha}{2}\left|-,+\right\rangle - i\sin\frac{\alpha}{2}(\vec{\sigma}\cdot\vec{n})\left|-,+\right\rangle\Big)\Big]$$
$$=\frac{1}{\sqrt{2}}\Big[\cos^2\frac{\alpha}{2}\left|+,-\right\rangle\otimes\left|+,-\right\rangle - \sin^2\frac{\alpha}{2}\left|+,-\right\rangle\otimes\left|+,-\right\rangle - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left|+,-\right\rangle\otimes\left|+,-\right\rangle\Big]$$
$$-\cos^2\frac{\alpha}{2}\left|-,+\right\rangle\otimes\left|-,+\right\rangle + \sin^2\frac{\alpha}{2}\left|-,+\right\rangle\otimes\left|-,+\right\rangle + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left|-,+\right\rangle\otimes\left|-,+\right\rangle\Big]$$

Now we will take common factor of the tensor products $\left|+,-\right\rangle\otimes\left|+,-\right\rangle$ and $\left|-,+\right\rangle\otimes\left|-,+\right\rangle$:

$$\frac{1}{\sqrt{2}}\Big[\cos^2\frac{\alpha}{2}(\left|+,-\right\rangle\otimes\left|+,-\right\rangle - \left|-,+\right\rangle\otimes\left|-,+\right\rangle) - \sin^2\frac{\alpha}{2}(\left|+,-\right\rangle\otimes\left|+,-\right\rangle -$$
$$\left|-,+\right\rangle\otimes\left|-,+\right\rangle) - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}(\left|+,-\right\rangle\otimes\left|+,-\right\rangle - \left|-,+\right\rangle\otimes\left|-,+\right\rangle)\Big]$$
$$=\frac{1}{\sqrt{2}}\Big[\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\Big](\left|+,-\right\rangle\otimes\left|+,-\right\rangle - \left|-,+\right\rangle\otimes\left|-,+\right\rangle)$$

After this simplification we can see that the operation between the eigenstates represents the entanglement of two 1/2 spin particles so we can simplify it as a single singlet.

$$\frac{1}{\sqrt{2}}\Big[\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\Big](\left|+,-\right\rangle - \left|-,+\right\rangle)$$
$$=\frac{1}{\sqrt{2}}\Big[\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\Big]\left|0,0\right\rangle$$

And if we apply the following properties $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we can confirm that the singlet is followed by a phase:

$$\frac{1}{\sqrt{2}}\Big[\cos\alpha - i\sin\alpha\Big]\left|0,0\right\rangle = e^{i\alpha}\frac{1}{\sqrt{2}}\left|0,0\right\rangle$$