Quantum Physics II. Exercise 2.8

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Enunciate

Consider a system of two particles with angular momenta $j_1 = 1$ and $j_2 = \frac{1}{2}$. Suppose the system is initially in the state $|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\rangle$, expressed in the total angular momentum basis $\{|j_1, j_2, j, m\rangle\}$. A rotation R by an angle ϕ about the y-axis is applied to the entire system.

a) Using the appropriate rotation matrices for each particle, express the rotated state

$$R(\hat{y}, \phi)\,|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\rangle$$

in the uncoupled basis $\{|j_1, j_2, m_1, m_2\rangle\}$

b) Recombine the result using the Clebsch-Gordan coefficients to rewrite the rotated state in the coupled basis $\{|j_1, j_2, j, m\rangle\}$

c) Determine the probability that a measurement of the total z-component of the angular momentum yields $m = \frac{1}{2}$ for a given angle ϕ.

1. Rotation of the uncoupled state

First, we have to re-express the coupled state in the uncoupled basis $\{|j_1, j_2, m_1, m_2\rangle\}$. Looking at the Clebsch-Gordan coefficients chart, we get that the state can be expressed as:

$$|\psi\rangle = |1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\rangle = |1, \frac{1}{2}, 1, \frac{1}{2}\rangle = |1, 1\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle$$

The rotation matrix is expressed as:

$$R(\hat{n},\phi) = e^{-i\frac{\vec{J}\cdot\hat{n}}{\hbar}}$$

So for a rotation in the y-axis the rotation matrix will be:

$$R(\hat{y},\phi) = e^{-i\frac{J_y}{\hbar}}$$

Because we are working with a two particle system there will be two rotation matrices expressed with the operator $\frac{J_{iy}}{\hbar}$, where $i = 1, 2$. This operator can be defined with the ladder operators:

$$\frac{J_{iy}}{\hbar} = \frac{i}{2\hbar}(J_{i-} - J_{i+})$$

And when applied to a state we will get the following formula:

$$\frac{J_{iy}}{\hbar}\left|j_i, m_i\right\rangle = \frac{i}{2}(\sqrt{(j_i+m_i)(j_i-m_i+1)}\left|j_i, m_i-1\right\rangle - \sqrt{(j_i-m_i)(j_i+m_i+1)}\left|j_i, m_i+1\right\rangle)$$

With all this defined we can calculate now the components of both matrices. For J_{1y} we will get these values where $m_1 = \{1, 0, 1\}$:

$$\begin{aligned}
\frac{J_{1y}}{\hbar}\left|1,-1\right\rangle &= -\frac{i}{2}\sqrt{2}\left|1,0\right\rangle \\
\frac{J_{1y}}{\hbar}\left|1,0\right\rangle &= \frac{i}{2}\sqrt{2}(\left|1,-1\right\rangle - \left|1,1\right\rangle) \\
\frac{J_{1y}}{\hbar}\left|1,1\right\rangle &= \frac{i}{2}\sqrt{2}\left|1,0\right\rangle
\end{aligned}$$

By the spectral theorem, we can define the matrix $\frac{J_{iy}}{\hbar}$ using the obtained values as the components:

$$\frac{J_{iy}}{\hbar} = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{1}$$

And we can obtain the rotation matrix for the particle $j_1 = 1$:

$$R^{(1)}(\hat{y},\phi) = \begin{pmatrix} 0 & e^{-\frac{\sqrt{2}}{2}\phi} & 0 \\ e^{\frac{\sqrt{2}}{2}\phi} & 0 & e^{-\frac{\sqrt{2}}{2}\phi} \\ 0 & e^{\frac{\sqrt{2}}{2}\phi} & 0 \end{pmatrix} \tag{2}$$

Now we can repeat the previous operation with J_{2y} where $m_2 = \{\frac{1}{2}, -\frac{1}{2}\}$:

$$\frac{J_{2y}}{\hbar} |\frac{1}{2}, -\frac{1}{2}\rangle = -\frac{i}{2}\sqrt{\frac{3}{2}} |\frac{1}{2}, \frac{1}{2}\rangle$$
$$\frac{J_{2y}}{\hbar} |\frac{1}{2}, \frac{1}{2}\rangle = \frac{i}{2}\sqrt{\frac{3}{2}} |\frac{1}{2}, -\frac{1}{2}\rangle$$

In the same way as before, we can define the rotation matrix for $j_2 = \frac{1}{2}$ as:

$$R^{(1/2)}(\hat{y}, \phi) = \begin{pmatrix} 0 & e^{-\frac{\phi}{2}\sqrt{\frac{3}{2}}} \\ e^{\frac{\phi}{2}\sqrt{\frac{3}{2}}} & 0 \end{pmatrix} \tag{3}$$

Finally we apply the matrices in our uncoupled state:

$$|\psi'\rangle = R^{(1)}(\hat{y}, \phi) |1, 0\rangle \otimes R^{(1/2)}(\hat{y}, \phi) |\frac{1}{2}, \frac{1}{2}\rangle$$

First we operate with the left term in the tensor product:

$$R^{(1)}(\hat{y}, \phi) |1, 1\rangle = \begin{pmatrix} 0 & e^{-\frac{\sqrt{2}}{2}\phi} & 0 \\ e^{\frac{\sqrt{2}}{2}\phi} & 0 & e^{-\frac{\sqrt{2}}{2}\phi} \\ 0 & e^{\frac{\sqrt{2}}{2}\phi} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e^{\frac{\sqrt{2}}{2}\phi} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e^{\frac{\sqrt{2}}{2}\phi} |1, 0\rangle$$

And now with the right term:

$$R^{(1/2)}(\hat{y}, \phi) |\frac{1}{2}, \frac{1}{2}\rangle = \begin{pmatrix} 0 & e^{-\frac{\phi}{2}\sqrt{\frac{3}{2}}} \\ e^{\frac{\phi}{2}\sqrt{\frac{3}{2}}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{\frac{\phi}{2}\sqrt{\frac{3}{2}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\frac{\phi}{2}\sqrt{\frac{3}{2}}} |\frac{1}{2}, -\frac{1}{2}\rangle$$

At last, if we return to $|\psi'\rangle$ we obtain that the rotated state is :

$$|\psi'\rangle = e^{\frac{\sqrt{2}}{2}\phi} |1, 0\rangle \otimes e^{\frac{\phi}{2}\sqrt{\frac{3}{2}}} |\frac{1}{2}, -\frac{1}{2}\rangle = e^{\frac{\phi}{2}(\sqrt{2} + \sqrt{\frac{3}{2}})} |1, \frac{1}{2}, 0, -\frac{1}{2}\rangle \tag{4}$$

2. Rotated State in the coupled basis

For rewriting the result in the coupled basis is necessary to look at the Clebsch-Gordan coefficients chart for $1 \times 1/2$ looking for the correspondent coefficients for $m_1 = 0, m_2 = -\frac{1}{2}$.

$1 \times 1/2$	j m	
m_1 m_2	3/2 -1/2	1/2 -1/2
0 -1/2	2/3	1/3
-1 +1/2	1/3	-2/3

Cuadro 1: Clebsch-Gordan Coefficients for 1 x 1/2

We can conclude with the following expression of the state in the coupled basis is :

$$|\psi'\rangle = e^{\frac{\phi}{2}(\sqrt{2}\ +\sqrt{\frac{3}{2}})}\,|1,\frac{1}{2},0,-\frac{1}{2}\rangle = e^{\frac{\phi}{2}(\sqrt{2}\ +\sqrt{\frac{3}{2}})}(\sqrt{\frac{2}{3}}\,|1,\frac{1}{2},\frac{3}{2},-\frac{1}{2}\rangle+\frac{1}{\sqrt{3}}\,|1,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\rangle) \tag{5}$$

3. Probability of measuring $m = 1/2$

The probability of measuring an event or physical magnitude is defined as:

$$P(a \to b) = |\,\langle a|b\rangle\,|^2$$

So the probability of measuring $|1,\frac{1}{2},j,m=\frac{1}{2}\rangle$ in the state $|\psi'\rangle$ will be null as the value given for m isn't presented in any of the eigenstates that compose it, meaning that they are orthonormal:

$$\begin{aligned} P\left(m=\frac{1}{2}\right) &= \left|\langle 1,\tfrac{1}{2},j,\tfrac{1}{2}|\,\psi'\rangle\right|^2 \\ &= \left| e^{\frac{\phi}{2}(\sqrt{2}+\sqrt{\frac{3}{2}})}\,\langle 1,\tfrac{1}{2},j,\tfrac{1}{2}|\left(\sqrt{\tfrac{2}{3}}\,|1,\tfrac{1}{2},\tfrac{3}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}\,|1,\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)\right|^2 \\ &= 0 \end{aligned}$$