Randomness properties of $\mathbb{Z}_v$ ElGamal sequences

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ElGamal Permutations

For p prime, $\mathbb{Z}_p^* = \{1, \ldots, p-1\}$ is a cyclic group of order $p-1$ under multiplication. For g a generator, the ElGamal map $\gamma = g^x : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ is a permutation

In 2016 Joachim von zur Gathen posed this research challenge:

$$\text{How random is the map } \gamma(x) = g^x?$$

Cycle sizes in ElGamal Permutations

Example: The generators of $\mathbb{Z}_5^*$ are 2 and 3.

x	g^x
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 3$
4	$2^4 = 1$

$$\gamma = (1, 2, 4)(3)$$

x	g^x
1	$3^1 = 3$
2	$3^2 = 4$
3	$3^3 = 2$
4	$3^4 = 1$

$$\gamma = (1, 2, 3, 4)$$

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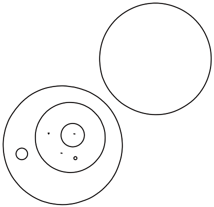
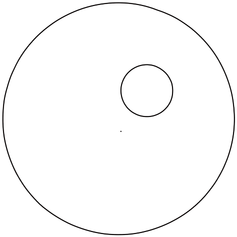
$\gamma = (1, 2, 4)(3)$ $\qquad$ $\gamma = (1, 2, 3, 4)$

- Distinct g produce distinct permutations;
- Distinct g affect the cyclic structures.

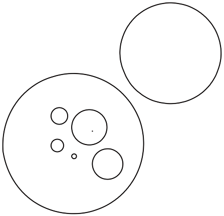
$p = 1009$



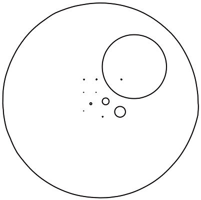
$g = 11$



$g = 17$

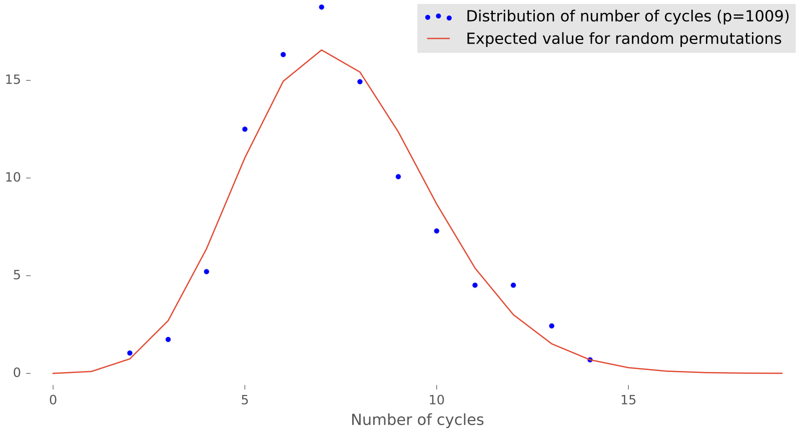


$g = 22$



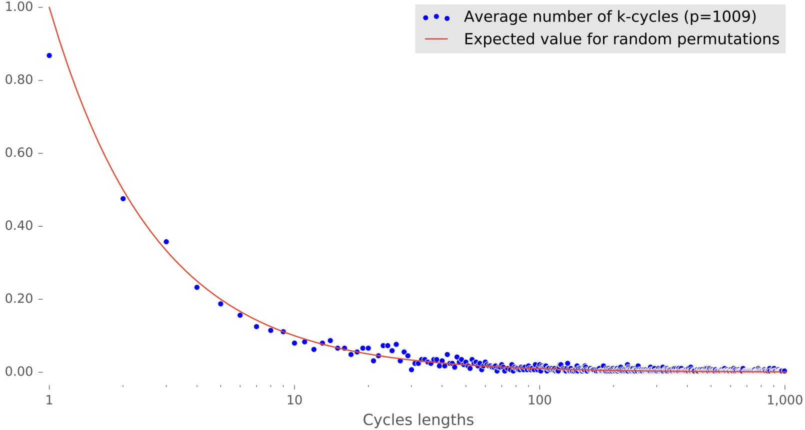
$g = 26$

Number of cycles



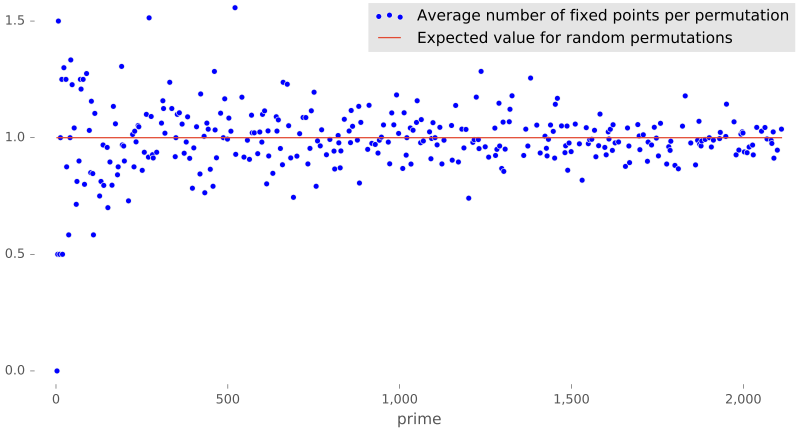
Distribution of number of cycles for all 288 generators of $\mathbb{F}_{1009}$

Number of k-cycles



Average number of k-cycles in $\mathbb{F}_{1009}$

Number of fixed points ($k = 1$)



Average number of fixed points in the generators of $\mathbb{F}_p$

Results with Sidon Sets

Sequences from permutations

For any permutation π in $\mathbb{Z}_p^*$, make a sequence

$$\pi_v = (\pi_1 \% v, \ldots, \pi_{p-1} \% v).$$

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For any permutation π in $\mathbb{Z}_p^*$, make a sequence

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Example: $p = 5$ and $g = 2$

$$\begin{aligned}\gamma &= ((2^0)\%5)\%2, \ldots, (2^3)\%5)\%2) \\ &= (1, 2, 4, 3) \\ \gamma_2 &= (1, 0, 0, 1) \in \mathbb{Z}_2^4\end{aligned}$$

Randomness properties of ElGamal Sequences?

How closely do ElGamal sequences compare to sequences from random permutations?

- Balance
- Period length
- Distribution of fixed t-*tuples* $z \in \mathbb{Z}_v^t$:

$$\lambda(z) = \#\{i \in [0, p-1] : \gamma_v(i +_n \iota) = z(\iota),\ 0 \leq \iota < t\}$$

- Distribution of *runs* of $b \in \mathbb{Z}_v$ and of length t:

$$\begin{aligned}\rho(b,t) = &\#\{i \in [0, p-1] : \\ &\gamma_v(i -_n 1), \gamma_v(i +_n t) \neq b = \gamma_v(i +_n \iota),\ 0 \leq \iota < t\}\end{aligned}$$

- $\rho(t) = v\rho(t+1)$

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Balance

Proposition

Let π be a permutation in $\mathbb{Z}_p^*$, then π_v is a balanced sequence over $\mathbb{Z}_v$ if and only if $v \mid p-1$.

Proof.

The number of $x \equiv a \mod v$ in $[1, p-1]$ is

$$|\pi_v|_a = \lceil (p - 1 - ((a-1) \mod v))/v \rceil$$

□

Period

Lemma

If $p \equiv \alpha \neq 1 \pmod{v}$, then π_v has period $N = p - 1$ for any π permutation of $\mathbb{Z}_p^*$.

Proof.

The difference in the number of occurrences of any two symbols must be a multiple of $(p-1)/N$. But

$$|\pi_v|_a = \begin{cases} \lceil (p-1)/v \rceil & 0 \leq a < \alpha - 1, \\ \lfloor (p-1)/v \rfloor & \text{otherwise.} \end{cases}$$

□

Period

Theorem

For every $\epsilon > 0$ there exists an n_ϵ so that for all $p \geq n_\epsilon$, the number T of balanced sequences π_v with period $p - 1$ satisfies

$$(p-1)!(1-\epsilon) \leq T \leq (p-1)!. \tag{1}$$

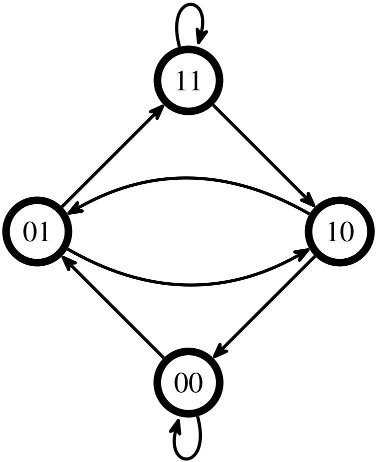
Special case

When q is prime and $p = vq + 1$,

$$\frac{(p-1)! - T}{(p-1)!} = \frac{v!(q!)^v}{(p-1)!}$$

This includes the case of Sophie Germain primes.

de Bruijn graph



Transfer Matrix

Transfer matrix is directed adjacency matrix of de Bruijn graph with variables

$$T = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} ux_0 & ux_0 & 0 & 0 \\ 0 & 0 & x_0 & x_0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

$$C = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\sum_{\mathbf{k} \in \mathbb{N}^t} a_n(\mathbf{k}) x^{\mathbf{k}} = \sum_{z', z'' \in \mathbb{Z}_v^t} C_{z',z''} T^n_{z',z''}.$$

Asymptotic Normality

Theorem (Bender, Richmond, Williamson 1983)

Suppose $a_n(k)$ *is admissible at 1 for* $n \equiv n_0 \pmod{d}$ *and that* Λ *is d-dimensional. Then* $a_n(k)$ *satisfies a central limit theorem for* $n \equiv n_0 \pmod{d}$ *with means and covariance matrix asymptotically proportional to* n*. Let* q *be such that* $qc \in \Lambda$ *for all* $c \in \mathbb{Z}^v$*. Then* $a_n(k)$ *satisfies a local limit theorem modulo* Λ *for* $n \equiv n_0 \pmod{dq}$

Asymptotic Normality

Theorem

Let $z \in \mathbb{Z}_v^t$ and $t(\kappa)$ be the number of balanced circular sequences of length n over $\mathbb{Z}_v$ for which $\lambda(z) = \kappa$. There exists a $m_\lambda, b_\lambda, c_\lambda \in \mathbb{R}$ such that

$$\sup_\kappa \left| \frac{\sqrt{2\pi b_\lambda}\, t(\kappa)}{\binom{vl}{l,l,\ldots,l}} - c_\lambda e^{(\kappa - m_\lambda)^2/b_\lambda} \right| = o(1).$$

Let $b \in \mathbb{Z}_v$, $t \in \mathbb{N}$ and $r(\kappa)$ be the number of balanced circular sequences of length n over $\mathbb{Z}_v$ for which $\rho(b,t) = \kappa$. There exists a $m_\rho, b_\rho, c_\rho \in \mathbb{R}$ such that

$$\sup_\kappa \left| \frac{\sqrt{2\pi b_\rho}\, r(\kappa)}{\binom{vl}{l,l,\ldots,l}} - c_\rho e^{(\kappa - m_\rho^2)/b_\rho} \right| = o(1).$$

Mean for tuples

$$\frac{n}{v^t}\left(1+\frac{-(t^2-2tv+v^2-t)(v-1)}{2n}\right)+O\left(\frac{1}{n}\right)$$
$$\leq E(\lambda(z)) \leq$$
$$\frac{n}{v^t}\left(1+\frac{t(v-1)}{2n}\right)+O\left(\frac{1}{n}\right)$$

Variance for tuples

$$
\begin{aligned}
&\frac{n}{v^{2t}}\left(\frac{2v^t}{2}+\frac{-12t^2v^t}{24n}\right)+O\left(\frac{1}{n}\right)\\
&\qquad\qquad\qquad\qquad \lesssim \mathrm{VAR}(\lambda) \lesssim\\
&\qquad\qquad\qquad\qquad \frac{n}{v^{2t}}\left(\frac{2v^t(v+1)}{2(v-1)}+\frac{12v^{t+2}t}{24n(v-1)}\right)+O\left(\frac{1}{n}\right)
\end{aligned}
$$

Runs

$$
\begin{aligned}
E(\rho(b,t)) &= \frac{(l(v-1)-1)(v-1)l(l)_t}{(n-1)_{t+1}}, \\
\mathrm{VAR}(\rho(b,t)) &= \frac{(l(v-1)-1)(v-1)l(l)_t}{(n-1)_{t+1}} \\
&\quad + \frac{(v-1)l(l)_{2t}(l(v-1)-1)^2(l(v-1)-2)}{(n-1)_{2t+2}} \\
&\quad - \left(\frac{(l(v-1)-1)(v-1)l(l)_t}{(n-1)_{t+1}}\right)^2 .
\end{aligned}
$$

Where $l = n/v$.

Runs

$$E(\rho(b,t)) = \frac{n(v-1)}{v^{t+2}}\left((v-1) - \frac{(v-1)^2t^2 - (v+3)(v-1)t + 2}{2n}\right) + O\left(\frac{1}{n}\right)$$

$$\mathrm{VAR}(\rho(b,t)) \approx \frac{n(v-1)^2}{v^{t+2}}\left(1 + \frac{-(v-1)t^2}{2n}\right) + O\left(\frac{1}{n}\right)$$

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Let π *be a permutation in* $\mathbb{Z}_p^*$*, then* π_v *is a balanced sequence over* $\mathbb{Z}_v$ *if and only if* $v \mid p-1$.

Period

Theorem

The ElGamal sequence γ_v has period $N = p - 1$.

Proof.

1. $p \not\equiv 1 \pmod{v}$: Use Balance
2. $p \equiv 1 \pmod{v}$: Suppose period $N < p - 1$: $g^{i+N}\%p \equiv_v g^i\%p$
3. Let $i = 0$: $g' = g^N\%p \equiv_v 1$.
4. Let $p = kg' + r$, $x = k + 1$ $(p < xg' < 2p)$. Let $i = \log_g(x)$:

$$x \equiv_v xg'\%p = xg' - p \equiv_v xg' - 1$$

5. $x(g' - 1) \equiv_v 1 \equiv_v g'$ is a contradiction.

$\square$

Tuples

Theorem

Let γ_v be an ElGamal sequence and $p = qg^{t-1} + r$, then

$$\left\lfloor \frac{g}{v} \right\rfloor^{t-1} \left\lfloor \frac{q}{v} \right\rfloor \leq \lambda(z) \leq \left\lceil \frac{g}{v} \right\rceil^{t-1} \left(\left\lfloor \frac{q}{v} \right\rfloor + 1 \right).$$

Proof

$$X = \left\{x \in [1, p-1] : (g^i x)\% p \equiv_v z_i, 0 \le i < t\right\}$$

Let $c_i = g^i z_0 - z_i, 0 \le i < t$.

$D = \{d \in \mathbb{Z}^t : d_0 = 0, d_i \equiv_v \alpha^{-1} c_i$ and $g d_{i-1} \le d_i < g(d_{i-1}+1)$ for $0 < i < t\}$.

For $d \in D$, let

$$X_d = \left\{x \in \mathbb{Z} : x \equiv_v z_0,\ \frac{d_i p}{g^i} \le x < \frac{(d_i+1)p}{g^i},\ \text{for } 0 \le i < t\right\}.$$

Claim:

$$X = \bigcup_{d \in D} X_d$$

$X_d \subset X$

If $x \in X_d$, then $x \equiv_v z_0$ and

$$d_i p \le g^i x < (d_i + 1)p$$

Thus

$$g^i x \% p = g^i x - d_i p \equiv_v g^i x - \alpha d_i \equiv_v g^i z_0 - c_i \equiv_v g^i z_0 - (g^i z_0 - z_i) = z_i,$$

So $x \in X$.

$X \subset \cup X_d$

For $x \in X$, define $g^i x = q_i p + r_i$:

$$q_0 = 0$$
$$r_i = g^i x - q_i p = (g^i x)\%p \equiv_v z_i$$
$$\frac{q_i p}{g^i} \leq x < \frac{(q_i + 1)p}{g^i}$$

So $x \in X_{(q_0,\ldots,q_{t-1})}$

$X \subset \cup X_d$

$$q_i \equiv_v \alpha^{-1} q_i p = \alpha^{-1}(g^i x - r_i) \equiv_v \alpha^{-1}(g^i z_0 - z_i) = \alpha^{-1} c_i.$$

Then,

$$\begin{aligned} q_i &= \frac{g^i x - r_i}{p} = \frac{g(g^{i-1}x) - r_i}{p} = \frac{g(q_{i-1}p + r_{i-1}) - r_i}{p} \\ &= g q_{i-1} + g\frac{r_{i-1}}{p} - \frac{r_i}{p} < g(q_{i-1} + 1), \end{aligned}$$

and

$$g q_{i-1} = \frac{g q_{i-1} p}{p} \leq \frac{g(q_{i-1}p + r_{i-1})}{p} = \frac{g(g^{i-1}x)}{p} = \frac{g^i x}{p} = q_i + \frac{r_i}{p}.$$

Since $g q_{i-1}, q_i \in \mathbb{Z}$ and $r_i/p < 1$, $\Rightarrow q_i \geq g q_{i-1}$.
Thus $(q_0, \ldots, q_{t-1}) \in D$.

Final step

$$X = \bigcup_{d \in D} X_d = \bigcup_{d \in D} \left(\{x \equiv_v z_0\} \bigcap \left(\bigcap_{0 \le i < t} \left\{ \frac{d_i p}{g^i} \le x < \frac{(d_i + 1)p}{g^i} \right\} \right) \right).$$
$$= \bigcup_{d \in D} \left(\{x \equiv_v z_0\} \cap \left\{ \frac{d_{t-1} p}{g^{t-1}} \le x < \frac{(d_{t-1} + 1)p}{g^{t-1}} \right\} \right).$$

$$\begin{array}{rcccl} \lfloor g/v \rfloor^{t-1} & \le & \# D & \le & \lceil g/v \rceil^{t-1} \\ q & \le & \#[d_{t-1}p/g^{t-1}, (d_{t-1}+1)p/g^{t-1}) & \le & q+1 \\ \lfloor q/v \rfloor & \le & \# X_d & \le & \lceil (q+1)/v \rceil \end{array}$$

$$\left\lfloor \frac{g}{v} \right\rfloor^{t-1} \left\lfloor \frac{q}{v} \right\rfloor \le \lambda(z) \le \left\lceil \frac{g}{v} \right\rceil^{t-1} \left(\left\lfloor \frac{q}{v} \right\rfloor + 1 \right).$$

□

Observations

- When $g = mv$ bounds differ by at most m^t
- When $g = v$, $\left\lfloor \frac{q}{v} \right\rfloor \leq \lambda(z) \leq \left\lfloor \frac{q}{v} \right\rfloor + 1$
- If $p \geq vg^{t-1}$ and $g \geq v$, then $\lambda(z) > 0$ for all $z \in \mathbb{Z}_v^t$
- If $\lambda(z) > 0$ for all $z \in \mathbb{Z}_v^t$, then $g \geq v$ and $p \geq v^t + 1$.
- Coincide when $g = v$
- $\gamma_v(i+1) \equiv_v g\gamma_v(i) - s$ for some $0 \leq s < g$.

Runs

Theorem

Let γ_v be an ElGamal sequence and $p = qg^{t-1} + r$. For $z \in \mathbb{Z}_v^t$, let

$$\mu(z) = \#\{i \in [1, p-1] : \ g^{i+j}\%p \equiv_v z_j, \ 0 \leq j < t-1, \ g^{i+t-1}\%p \not\equiv_v z_{t-1}\}.$$

Then

$$\left\lfloor \frac{g}{v} \right\rfloor^{t-2} \left\lfloor \frac{(v-1)g}{v} \right\rfloor \left\lfloor \frac{q}{v} \right\rfloor \leq \mu(z) \leq \left\lceil \frac{g}{v} \right\rceil^{t-2} \left\lceil \frac{(v-1)g}{v} \right\rceil \left(\left\lfloor \frac{q}{v} \right\rfloor + 1 \right).$$

Corollary

Let $p = q_t g^t + r_t$ and $p = q_{t+1} g^{t+1} + r_{t+1}$. Then

$$\left\lfloor \frac{g}{v} \right\rfloor^{t-1} \left\lfloor \frac{(v-1)g}{v} \right\rfloor \left\lfloor \frac{q_t}{v} \right\rfloor - \left\lceil \frac{g}{v} \right\rceil^{t} \left\lceil \frac{(v-1)g}{v} \right\rceil \left\lceil \frac{q_{t+1}+1}{v} \right\rceil$$
$$\leq \rho(b,t) \leq$$
$$\left\lceil \frac{g}{v} \right\rceil^{t-1} \left\lceil \frac{(v-1)g}{v} \right\rceil \left\lceil \frac{q_t+1}{v} \right\rceil - \left\lfloor \frac{g}{v} \right\rfloor^{t} \left\lfloor \frac{(v-1)g}{v} \right\rfloor \left\lfloor \frac{q_{t+1}}{v} \right\rfloor,$$

and

$$(v-1) \left\lfloor \frac{g}{v} \right\rfloor^{t} \left\lfloor \frac{(v-1)g}{v} \right\rfloor \left\lfloor \frac{q}{v} \right\rfloor \leq \rho(b,t) \leq (v-1) \left\lceil \frac{g}{v} \right\rceil^{t} \left\lceil \frac{(v-1)g}{v} \right\rceil \left\lceil \frac{q+1}{v} \right\rceil.$$

Comparison to random balanced sequences

From theoretical results

- Balance matches exactly
- Periodicity matches very closely
- To first order, the number of tuples and runs matches
- To first order $\rho(t) \approx v\rho(t+1)$

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ElGamal Run ratio Experiment

If we consider $\rho(t)$ as the occurrences of all runs of length t, we expect that

$$\rho(t+1)/\rho(t) = \frac{1}{v}$$

from Golomb's postulates of randomness.

ElGamal Sequences run ratio Experiment

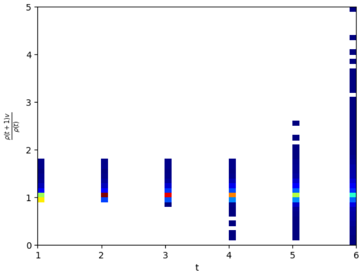

Figure 1: Distribution of $\rho(t+1)v/\rho(t)$ as a heatmap

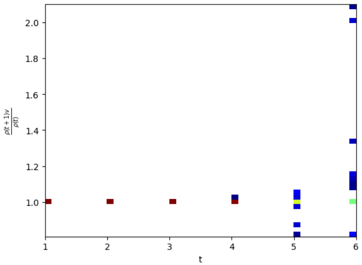

Figure 2: Distribution of $\rho(t+1)v/\rho(t)$ with $v = g$ as a heatmap

ElGamal Sequences run ratio Experiment

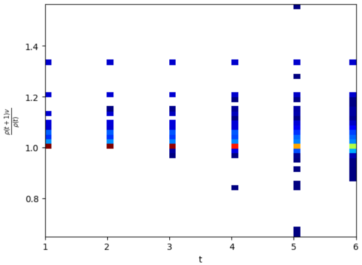

Figure 3: Distribution of $\rho(t+1)v/\rho(t)$ and $v = 2$ as a heatmap

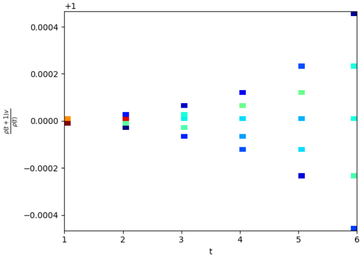

Figure 4: Distribution of $\rho(t+1)v/\rho(t)$ with $v = g = 2$ as a heatmap

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