Report on the PhD thesis *Digraph Coloring* by Lucas Picasarri-Arrieta

I will first give my overall opinion on the thesis followed by my comments on the individual chapters in the sections below.

The thesis is well-written and contains a large number of deep results which are of great interest to the community. The results are nice and interesting and many of the proofs and examples are deep and insightful. Lucas has already written 13 papers (according to dblp) and several of these have already been published in very good journals. This includes an article published in Journal of Graph Theory, where Lucas is the sole author. It is clear that Lucas is a very strong researcher with a bright future ahead of him. And the thesis is evidence of this. So, to conclude, this is a very well-written thesis that bears evidence of some very strong results. I therefore definitely find the thesis of good enough quality for a PhD.

Recommendation: I recommend that (subject to the defence) a PhD degree is awarded based on the thesis.

The thesis has eight chapters which I will comment on below.

1 Chapter 1: Introduction

The introduction is a nice chapter introducing the concepts of digraph coloring. Both known results for graph coloring (in undirected graphs) and for digraph coloring are introduced. And an outline of the thesis is presented. The chapter is well written and shows a very good understanding of the field.

2 Chapter 2: Dichromatic number of chordal graphs

In this chapter results on digraphs whose underlying graph is a chordal graph is presented. These results are based on the paper [26] (S. Bessy, F. Havet, and L. Picasarri-Arrieta, *Dichromatic number of chordal graphs*). Several upper bounds are mentioned and examples showing that these are best (or almost best) possible. The paper that this chapter is based on is not published yet, but the results are nice and I am sure it will be published in a good journal.

3 Chapter 3: On the Directed Brooks Theorem

This chapter is devoted to generalisations of the Directed Brooks Theorem (Theorem 1.2.12 in the thesis). Furthermore a linear algorithm is presented. A second generalisation of the Directed Brooks Theorem is also given and necessary conditions for a digraph D to have dichromatic number exactly $\Delta_{min}(D) + 1$ are given. The results in this chapter are based on [139] (L. Picasarri-Arrieta, *Strengthening the Directed Brooks' Theorem for oriented graphs and consequences on digraph redicolouring.* J. Graph Theory, 2023) and [82] (D. Goncalves, L. Picasarri-Arrieta, and A. Reinald, *Brooks-type colourings of digraphs in linear time*, To appear, 2024). The fact that Journal of Graph Theory is one of the top journals in the area is evidendence of the strength of the results in this chapter.

4 Chapter 4: Minimum density of dicritical digraphs

This chapter is devoted to dicritical graphs. That is, digraphs whose dichromatic number drops if any arc or vertex is removed. Exact values of $d_k(n)$ ($d_k(n)$ is the minimum number of arcs in n-vertex k-dicritical digraph) are given when $k \geq 4$ and $k+2 \leq n \leq 2k-1$, which generalises a result of Gallai. A couple of conjectures are proved and a bound for $o_k(n)$ is given (i.e. for oriented digraphs). The results in this chapter are based on [141] (L. Picasarri-Arrieta and M. Stiebitz, *Minimum number of arcs in k-critical digraphs with order at most $2k-1$.*) and [97] (F. Havet, L. Picasarri-Arrieta, and C. Rambaud, *On the minimum number of arcs in 4-dicritical oriented graphs*). The results in this paper are very nice and will definitely be published in a good journal.

5 Chapter 5: The three-dicritical semi-complete digraphs

It is an open problem to decide if the number of k-dicritical tournaments is finite when $k \geq 3$. In this chapter this problem is answered for $k = 3$, where it is shown that the number of 3-dicritical semicomplete digraphs is finite. Furthermore, using a computer, it is shown that there are only 8 such graphs, 2 of which are tournaments. An upper bound on the maximum number of arcs in a 3-dicritical digraph is also given. The results in this chapter are based on [96] (F. Havet, F. Hörsch, and L. Picasarri-Arrieta, *The 3-dicritical semi-complete digraphs*) and are again very nice and moves this research area forwards.

6 Chapter 6: Subdivisions in dicritical digraphs with large order or digirth

In this chapter it is shown that for every $k \geq 2$ there exists a function $f_k : N \to N$ such that every k-dicritical digraph on at least $f_k(l)$ vertices contains an oriented path of length l and a similar result is shown for oriented cycles. Furthermore results on subdivision of digraphs are also given. The results in this chapter are based on [140] (L. Picasarri-Arrieta and C. Rambaud, *Subdivisions in dicritical digraphs with large order or digirth*). Again these results fit very nicely into the scope of this thesis and are definitely publishable in a good journal.

7 Chapter 7: Redicolouring digraphs

In this chapter the analog of graph recoloring is considered for digraphs. That is, one is interested in knowing if one can achieve one k-coloring from another k-coloring by changing the color of one vertex at a time such that all intermediate colorings are proper k-colorings. It is shown that the problem of determining if such a recoloring exists is PSPACE-complete. Then c-degeneracy is defined and a nice conjecture involving c-degeneracy is mentioned. And finally results on when all k-colorings can or cannot be recolored in order to reach all other k-colorings are mentioned. Chapter 7 is based on [41] (N. Bousquet, F. Havet, N. Nisse, L. Picasarri-Arrieta, and A. Reinald, *Digraph redicolouring. Eur. J. Comb,* 116:103876, 2024.), [139] (L. Picasarri-Arrieta, *Strengthening the Directed Brooks' Theorem for oriented graphs and consequences on digraph redicolouring* J. Graph Theory, 2023) and [136] (N. Nisse, L. Picasarri-Arrieta, and I. Sau, *Redicolouring digraphs: directed treewidth and cycle-degeneracy*).

8 Chapter 8: Conclusion and perspectives

In this section a number of nice conjectures are mentioned. This shows that the student has a good knowledge of the area and that the area is an iteresting area with many interesting problems. This section is well written and adds to the strength of the thesis.

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