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Coloration de Graphes Dirigés

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COLORATION DE GRAPHES DIRIGÉS

Digraph Colouring

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Coloration de Graphes Dirigés

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Coloration de Graphes Dirigés

Résumé

Cette thèse est dédiée à l'étude de la dicoloration, une notion de coloration pour les digraphes introduite par Erdős et Neumann-Lara à la fin des années 1970, ainsi que le paramètre qui lui est associé, à savoir le nombre dichromatique. Lors des dernières décennies, ces deux notions ont permis de généraliser de nombreux résultats classiques de coloration de graphes.

Nous commençons par donner différentes bornes sur le nombre dichromatique des digraphes dont le graphe sous-jacent est un graphe cordal. Ensuite, nous améliorons la borne donnée par le théorème de Brooks pour les digraphes sans arcs antiparallèles et introduisons une notion de dégénérescence variable pour les digraphes, ce qui nous permet de prouver une version plus générale du théorème de Brooks.

Nous étudions ensuite les digraphes k-dicritiques, c'est-à-dire les obstructions minimales à la $(k-1)$-dicolorabilité. En particulier, nous généralisons un résultat de Gallai au cas dirigé, et nous prouvons une conjecture de Kostochka et Stiebitz dans le cas particulier $k = 4$. Nous discutons également la densité maximum de tels digraphes, et prouvons qu'il n'y a qu'un nombre fini de digraphes semi-complets 3-dicritiques. On donne par la suite certains résultats structurels sur les digraphes dicritiques de grand ordre.

Enfin, nous étudions la notion de redicoloration pour les digraphes. En particulier, nous prouvons que de nombreux résultats soutenant la conjecture de Cereceda se généralisent au cas dirigé.

Mots-clés : Digraphes, dicoloration, nombre dichromatique, reconfiguration, digraphes dicritiques.

Digraph Colouring

Abstract

This thesis focuses on a notion of colouring of digraphs introduced by Erdős and Neumann-Lara in the late 1970s, namely the dicolouring, and its associated digraph parameter: the dichromatic number. It appears in the last decades that many classical results on graph colouring have directed counterparts using these notions.

We first give a collection of bounds on the dichromatic number of digraphs for which the underlying graph is chordal. We then introduce a notion of variable degeneracy for digraphs which leads to a more general version of Brooks Theorem. We also strengthen this theorem on a large class of digraphs which contains digraphs without antiparallel arcs.

Next we prove a collection of results on k-dicritical digraphs, the digraphs that are minimal obstructions for the $(k-1)$-dicolourability. We first generalise a result of Gallai to the directed case, and then prove a conjecture of Kostochka and Stiebitz in the particular case $k = 4$. We also discuss the maximum density of such digraphs and prove that the number of 3-dicritical semi-complete digraphs is finite. We then give a collection of results on the substructures in large dicritical digraphs.

We finally study the notion of redicolouring for digraphs. In particular, we prove that a large collection of evidences for Cereceda's conjecture admit a directed counterpart.

Keywords: Digraphs, dicolouring, dichromatic number, reconfiguration, dicritical digraphs.

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Contents

Chapter 1

Introduction

1.1 Graph colouring

1.1.1 Preliminaries

A *graph* is an ordered pair $G = (V, E)$ where V is a finite set of *vertices* and E is a set of unordered pairs of V called *edges*. The vertex-set of G is denoted by $V(G)$ and its edge-set is denoted by $E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n(G)$, and its number of edges is denoted by $m(G)$. When G is clear from the context, we simply write n and m for $n(G)$ and $m(G)$ respectively. Informally, a graph represents pairwise relations between the objects of a finite set. These objects can be of any kind. For instance, one can model a road network with a graph, in which case the vertices represent cities and there is an edge between two vertices if and only if a road connects the corresponding cities.

Some graphs are of particular interest because of their particular structure. For every integer $n \in \mathbb{N}$, the *complete graph* on n vertices, denoted by K_n, is the graph on n vertices with all possible edges. The *path* on n vertices, denoted by P_n, is the graph with vertex-set $\{v_1, \dots, v_n\}$ and edge-set $\{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\}$. When $n \geq 3$, if we further add the edge $\{v_1, v_n\}$ to P_n, we obtain the *cycle* on n vertices that we denote by C_n. The *length* of a path or a cycle is its number of edges. A path or a cycle is *odd* if its length is odd, and it is *even* otherwise. The particular cycle of length 3 is called the *triangle*.

Graphs are easy to draw and visualise: given a graph, we represent its vertices with points, and we connect two points with a line when there is an edge between the corresponding vertices. Figure 1.1 illustrates three different graphs.

Let u, v be two vertices of a graph G. We say that u and v are *adjacent* if and only if $\{u, v\}$ is an edge of G. We also say that v is a *neighbour* of u and that the edge $\{u, v\}$ is *incident* to u. With a slight abuse of notations, we usually denote by uv the edge $\{u, v\}$. The set of neighbours of u is

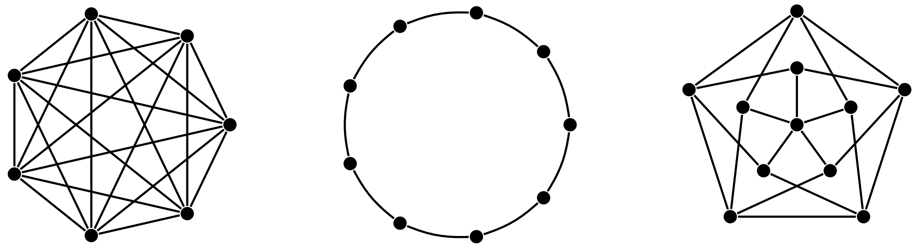


Figure 1.1: Examples of graphs. From left to right: the complete graph on 7 vertices, the cycle on 9 vertices, and the Grötzsch graph.

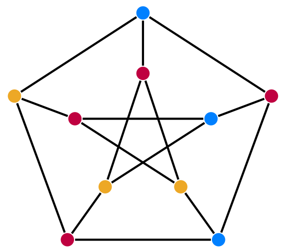


Figure 1.2: A 3-colouring of the Petersen graph.

called the *neighbourhood* of u and is denoted by $N(u)$. The *closed neighbourhood* $N(u) \cup \{u\}$ of u is denoted by $N[u]$.

We denote by $[k]$ the set of integers $\{1, \ldots, k\}$. Given an integer $k \in \mathbb{N}$, a k-*colouring* of G is a function $\alpha \colon V(G) \to [k]$. It is *proper* if and only if every pair of adjacent vertices $\{u, v\}$ satisfies $\alpha(u) \neq \alpha(v)$. When G admits a proper k-colouring, we say that G is k-*colourable*. Observe that every graph G is $n(G)$-colourable: one just has to label arbitrarily the vertices of G from 1 to $n(G)$ to obtain a proper $n(G)$-colouring. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colourable. Figure 1.2 illustrates a proper 3-colouring of a particular graph known as the Petersen graph.

The notions of proper colouring and chromatic number are known to have a lot of real-world applications. We detail two very classical examples.

Example 1.1.1 – We first consider the following general scheduling problem. Assume that we have a set of tasks $\tau_1, \ldots, \tau_n$, where every task τ_i starts at time s_i and ends at time $t_i \geq s_i$. Every task has to be handled by an agent, and every agent handles at most one task at once. Now the question is: how many agents do we need to handle all the tasks? One can formulate this problem in terms of graph colouring as follows. Let $G_\tau = (V, E)$ be the graph where V is the set of tasks and $\{\tau_i, \tau_j\}$ is an edge of G_τ if and only if $[s_i, t_i]$ and $[s_j, t_j]$ intersect. Then, for any $k \in \mathbb{N}$, all the tasks may be handled by k agents if and only if G_τ is k-colourable. In particular, the minimum number of agents we need is exactly the chromatic number of G_τ.

Example 1.1.2 – We now consider the following very classical telecommunication problem. Assume we have a set of antennas on the plane, and we want to assign to each of them a frequency. However, when two antennas are close to each other, they must receive different frequencies so they do not interfere. Now the question is the following: how many frequencies do we need to find such an assignment? Again, this problem can be formulated in terms of graph colouring. Let $G = (V, E)$ be the graph where V is the set of antennas, and there is an edge between two of them if they are close to each other. We obtain that there exists an assignment of k frequencies if and only if G is k-colourable. In particular, the chromatic number of G is exactly the minimum number of frequencies we need.

Computing the chromatic number of a graph is an NP-hard problem. It means that this problem is intractable unless P=NP. In fact, Karp [104] proved that it is even NP-complete to decide whether a graph is k-colourable for every fixed integer $k \geq 3$ (see [79]). One can then ask for an approximation of the chromatic number. Again, unless NP = ZPP, it is intractable to approximate the chromatic number of a graph $G = (V, E)$ to within $n(G)^{1-\varepsilon}$ for every fixed $\varepsilon > 0$ [75].

Figure 1.3: An example of a 4-coloured map.

On the positive side, it appears that the chromatic number can be computed when restricted to some well-structured classes of graphs. For instance, the problem presented in Example 1.1.1 can be solved in linear time in the size of the graph. This is because the graph we obtain has a specific structure: it is formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. We call such a graph an *interval graph*, and not all graphs are interval graphs. In the particular case of interval graphs, the chromatic number can be computed in linear time in the number of vertices (see [81, Theorem 4.17]).

Analogously, the graphs obtained in Example 1.1.2 have a specific structure. They are formed from a set of unit disks on the plane, with a vertex for each unit disk and an edge between two vertices whenever the corresponding unit disks intersect. On this specific class of graphs, computing the chromatic number is still NP-hard [52] but it can be approximated by a constant factor [84] (the positions of the unit disks on the plane must be known).

A general question consists then of finding how the structure of a graph is related to its chromatic number. In the next section, we present a few classical results and conjectures on this topic.

1.1.2 A few classical results on graph colouring

Four Colour Theorem

The most widely known result on graph colouring is probably the Four Colour theorem. This result answers a question of Guthrie in 1852, who asked if we can colour the regions of any map with four colours in such a way that regions sharing a common boundary (of non-zero length) do not share the same colour (see Figure 1.3).

A *planar graph* is a graph that can be drawn on the plane in such a way that no edges cross each other. Given a map, we can construct a planar graph by associating exactly one vertex to each region and putting an edge between two vertices whenever the corresponding regions share a common boundary. In terms of graph colouring, we can thus reformulate the question raised by Guthrie as follows: is every planar graph 4-colourable? It was proved to be true in 1976 by Appel and Haken [14], and is now known as the Four Colour theorem.

Theorem 1.1.1 (FOUR COLOUR THEOREM)**.** *Every planar graph is 4-colourable.*

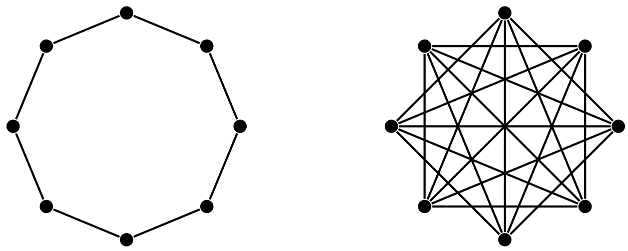


Figure 1.4: Illustration of a hole (on the left) and an antihole (on the right).

Moreover, as shown by Robertson, Sanders, Seymour, and Thomas in [147], there exists an algorithm that computes a proper 4-colouring of a planar graph G in $O(n^2)$ time.

Strong Perfect Graph theorem

Another celebrated result on graph colouring is known as the Strong Perfect Graph Theorem, for which we first need a few definitions.

Let G and H be two graphs. We say that H is a *subgraph* of G, and we denote it by $H \subseteq G$, if H can be obtained from G by deleting a (possibly empty) set of vertices and edges. We also say that G *contains* H as a subgraph. It formally means that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say that H is a *proper subgraph* of G, and we denote it by $H \subsetneq G$, if H is a subgraph of G and $H \neq G$. We say that H is an *induced subgraph* of G if it can be obtained by removing a (possibly empty) set of vertices X and the set of edges incident to at least one vertex in X. The graph H is a *spanning subgraph* of G if it is a subgraph of G with $V(H) = V(G)$. In this case, we denote H by $G - X$. When $X = \{x\}$, with a slight abuse of notations, we denote H by $G - x$. We also say that H is the subgraph of G induced by $Y = V(G) \setminus X$ and we denote it by $G\langle Y \rangle$. If H is obtained from G by removing a set of edges F, we denote H by $G \setminus F$. When $F = \{e\}$, with a slight abuse of notation, we denote H by $G \setminus e$.

We say that H is *isomorphic* to G if there exists a bijection $\psi \colon V(H) \to V(G)$ such that, for every pair of vertices $u, v \in V(H)$, $\{u, v\}$ is an edge of H if and only if $\{\psi(u), \psi(v)\}$ is an edge of G. When we say that G contains H as an (induced) subgraph, we mean that G has an (induced) subgraph which is isomorphic to H, hence not necessarily maintaining vertex labels. In case we want vertex labels to be maintained, we speak of an *(induced) labelled subgraph*.

A graph G is *connected* if and only if, for any pair of vertices $\{u, v\}$ of G, there exists a path between them in G. Formally, it means that for every such pair, G contains as a subgraph a copy of a path containing both u and v. A *forest* is a graph that does not contain any cycle as a subgraph. A *tree* is a connected forest.

Given two graphs H_1, H_2 such that $V(H_1) \cap V(H_2) = \varnothing$, the *disjoint union* of H_1 and H_2 is the graph with vertex-set $V(H_1) \cup V(H_2)$ and edge-set $E(H_1) \cup E(H_2)$.

The *complement* of $G = (V, E)$, denoted by $\overline{G}$, is the graph with vertex-set V in which, for every pair of vertices $\{u, v\}$, uv is an edge if and only if it is not an edge of G. A *hole* in a graph G is an induced cycle of length at least 4 and an *antihole* of G is an induced subgraph of G whose complement is a hole in $\overline{G}$. See Figure 1.4 for an illustration.

The *clique number* of G, denoted by $\omega(G)$, is the largest integer ℓ such that K_ℓ is a subgraph of G. Observe that every proper colouring of K_ℓ uses exactly ℓ colours. Therefore, we easily deduce that every graph G satisfies $\chi(G) \geq \omega(G)$. Note that equality does not occur in general:

the smallest counter-example is C_5, the cycle on five vertices, for which the clique number is 2, but the chromatic number is 3.

One can then ask for a characterisation of graphs for which the chromatic number is equal to the clique number. Unfortunately, this class of graphs is not well-structured at all. To see this, consider any graph H, and let G be the disjoint union of H and $K_{\chi(H)}$. Then H is an induced subgraph of G and G satisfies $\chi(G) = \omega(G)$. This shows that every graph H is an induced subgraph of a larger graph G for which the chromatic number is equal to the clique number. To address this issue, one can ask for a characterisation of graphs G such that $\chi(H) = \omega(H)$ holds not only for $H = G$, but for every induced subgraph H of G. Such a graph G is called a *perfect graph*.

The characterisation of perfect graphs was initiated by Berge in 1961 [23] when he first gave a necessary condition for being a perfect graph. A graph G is a *berge graph* if every hole and antihole of G has even length. Let us show that being a berge graph is indeed a necessary condition for being a perfect graph. If G is not a berge graph, either G or $\overline{G}$ contains an induced odd cycle of length at least five. In the former case, since an odd cycle has clique number 2 and chromatic number 3, we deduce that G is not perfect. In the latter case, the following implies that G is not perfect.

Proposition 1.1.2. *For every $k \geq 2$, let $\overline{C_{2k+1}}$ be the complement of C_{2k+1}, then*

$$\omega(\overline{C_{2k+1}}) = k \quad \textit{and} \quad \chi(\overline{C_{2k+1}}) \geq k + 1.$$

The interested reader may note that $\chi(\overline{C_{2k+1}})$ is actually equal to $k + 1$, but the inequality is sufficient for our purpose.

Proof. Let $v_0, \dots, v_{2k}$ be an ordering of the vertices of $\overline{C_{2k+1}}$ such that $v_i v_{i+1}$ is a non-edge for every $i \in \{0, \dots, 2k\}$ (with indices taken modulo $2k + 1$).

We first show that $\omega(\overline{C_{2k+1}}) = k$. We have $\omega(\overline{C_{2k+1}}) \geq k$ because $S = \{v_{2i} \mid 0 \leq i \leq k-1\}$ is a set of pairwise adjacent vertices. On the other hand, every set of $k + 1$ vertices contains two successive vertices v_i and v_{i+1} for some $i \in [2k]$ which are non-adjacent. This shows $\omega(\overline{C_{2k+1}}) = k$.

Let us now show that $\chi(\overline{C_{2k+1}}) \geq k + 1$. Let α be any proper colouring of $\chi(\overline{C_{2k+1}})$, we will show that it uses at least $k + 1$ colours. Since $\overline{C_{2k+1}}$ contains an edge, α uses at least two distinct colours, implying that there exists an index $i \in [2k]$ for which $\alpha(v_i) \neq \alpha(v_{i+1})$. Free to relabel the vertices, we assume without loss of generality that $\alpha(v_0) \neq \alpha(v_1)$. Consider the set of vertices $R = \{v_{3+2i} \mid 0 \leq i \leq k - 2\}$. Then both $R_0 = R \cup \{v_0\}$ and $R_1 = R \cup \{v_1\}$ induce a complete graph on k vertices on $\overline{C_{2k+1}}$. Since $\alpha(v_0) \neq \alpha(v_1)$, $R \cup \{v_0, v_1\}$ is a set of $k + 1$ vertices using pairwise distinct colours in α, implying that α uses at least $k + 1$ colours as desired. □

Berge conjectured that indeed this necessary condition for being a perfect graph is sufficient. This conjecture received a lot of attention during 40 years and was finally proved by Chudnovsky, Robertson, Seymour, and Thomas in 2006 [51]. This result is now known as the Strong Perfect Graph Theorem.

Theorem 1.1.3 (STRONG PERFECT GRAPH THEOREM)**.** *A graph is perfect if and only if it is a berge graph.*

χ-boundedness

The notion of χ-boundedness was introduced and widely investigated by Gyárfás (see [86, 87]). It attempts to be a generalisation of graph perfectness.

Since $\chi(G) \geq \omega(G)$ holds for every graph G but equality does not occur in general, a natural question is to ask whether the chromatic number is bounded above by a function of the clique number. Let $\mathcal{G}$ be a class of graphs, we say that $\mathcal{G}$ is *χ-bounded* if there exists a function f (depending on $\mathcal{G}$) such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. For instance, the class of perfect graphs is χ-bounded (with f being the identity function).

Let us first mention that the class of all graphs is not χ-bounded, since there exist triangle-free graphs (*i.e.* graphs with clique number 2) of arbitrarily large chromatic number. This was first proved by Tutte [55] (writing as Blanche Descartes). Then people proposed many other constructions of triangle-free graphs with large chromatic number. See for instance the ones due to Zykov [171], Mycielsky [132], and Burling [45]. The interested reader is also referred to [35], in which Bonnet et al. recently proved that a simple class of triangle-free graphs surprisingly has unbounded chromatic number.

As shown by Theorem 1.1.3, perfect graphs can be characterised in terms of forbidden induced subgraphs. Given two graphs G and H, we say that G is *H-free* (resp. *H-induced-free*) if it does not contain H as a subgraph (resp. as an induced subgraph). Therefore, the following question naturally arises: what are the graphs H such that the class of H-induced-free graphs is χ-bounded?

We define the *girth* of a graph G, denoted by $\mathrm{girth}(G)$, as the length of its shortest cycles (with the convention $\mathrm{girth}(G) = +\infty$ if G is a forest). The following celebrated result of Erdős strengthens the results above on triangle-free graphs.

Theorem 1.1.4 (Erdős [72]). *For every fixed integers $k, \ell \in \mathbb{N}$, there exists a graph G such that $\chi(G) \geq k$ and $\mathrm{girth}(G) \geq \ell$.*

For every graph H, if the class of H-induced-free graph is χ-bounded, Erdős' result implies that H is a forest. Indeed, if H is not a forest, it contains a cycle of length at most $n(H)$ and Erdős' result implies that there exist graphs of arbitrarily large chromatic number with girth at least $n(H)+1$ (which a fortiori do not contain H). Gyárfás [86] and Sumner [164] independently conjectured that this necessary condition on H is indeed sufficient.

Conjecture 1.1.5 (GYÁRFÁS-SUMNER CONJECTURE). *For every fixed forest H, the class of H-induced-free graphs is χ-bounded.*

This conjecture is still widely open. We refer the interested reader to the recent survey of Scott and Seymour on χ-boundedness [155].

Brooks Theorem

The celebrated Brooks Theorem makes a connection between the chromatic number of a graph and its maximum degree. Let u be a vertex of a graph $G = (V, E)$. The *degree* of u, denoted by $d(u)$, is the number of vertices adjacent to u in G. The *maximum degree* of G, denoted by $\Delta(G)$, is the maximum of $\{d(u) \mid u \in V\}$. Analogously, the *minimum degree* of G, denoted by $\delta(G)$, is the minimum of $\{d(u) \mid u \in V\}$. If $\delta(G) = \Delta(G) = d$, then G is *d-regular*. We say that G is *subcubic* if $\Delta(G) = 3$ and that it is *cubic* if it is 3-regular.

Every graph G satisfies $\chi(G) \leq \Delta(G) + 1$. To see this, take any graph G and an arbitrary ordering of its vertices. Then, according to this ordering, consider the vertices one after the other.

At each step, we may choose a colour of $[\Delta(G) + 1]$ that is not already appearing in the neighbourhood of the considered vertex. Following this easy greedy procedure, we finally find a proper colouring of G using at most $\Delta(G) + 1$ colours. This implies $\chi(G) \leq \Delta(G) + 1$. Brooks [44] characterised the connected graphs for which equality holds.

Theorem 1.1.6 (BROOKS THEOREM)**.** *A connected graph G satisfies $\chi(G) = \Delta(G) + 1$ if and only if G is an odd cycle or a complete graph.*

We give a short proof of this theorem due to Rabern [144], for which we first need a few definitions. A *connected component* of a graph G is a maximal connected subgraph H of G. The *distance* between two vertices u, v in G, denoted $\mathrm{dist}_G(u, v)$ is defined as the length of a shortest path in G containing both u and v, with the convention $\mathrm{dist}_G(u, v) = +\infty$ if u and v belong to different connected components of G. An *independent set* of G is a set of pairwise non-adjacent vertices in G. A *clique* is a set of pairwise adjacent vertices. A *matching* of G is a set of pairwise disjoint edges of G. If every vertex of G belongs to at least one edge of M, then M is called a *perfect matching*.

We say that is *bipartite* if and only if it is 2-colourable. We need the following well-known characterisation of bipartite graphs (see for instance [58, Proposition 1.6.1]).

Proposition 1.1.7. *A graph is bipartite if and only if it contains no odd cycle.*

We are now ready to prove Theorem 1.1.6.

Proof of Theorem 1.1.6. We will show by induction on $\Delta \geq 0$ that every connected graph G with maximum degree at most Δ is Δ-colourable, unless G is a complete graph on $\Delta + 1$ vertices or $\Delta = 2$ and G is an odd cycle.

The result is trivial when $\Delta \leq 1$. When $\Delta = 2$, a connected graph with maximum degree 2 is either a path or a cycle, so the result follows from Proposition 1.1.7. Henceforth we assume that $\Delta \geq 3$ and that the result holds for $\Delta - 1$. For the purpose of contradiction, let G be a minimum counter-example. In other words, G is not the complete graph on $\Delta + 1$ vertices, it satisfies both $\Delta(G) \leq \Delta$ and $\chi(G) = \Delta + 1$, and $n(G)$ is minimum for these properties.

We first show that $\delta(G) = \Delta$. Assume this is not the case, that is G contains a vertex v of degree at most $\Delta - 1$. Since G is connected, there is an ordering $v_1, \dots, v_n$ of $V(G)$ such that $v_n = v$ and every vertex v_i, $i \in [n-1]$, has at least one neighbour in $\{v_{i+1}, \dots, v_n\}$. Consider the vertices one after the other, starting from v_1 and moving forward to v_n. At each step, we may choose for v_i a colour of $[\Delta]$ that is not appearing in its neighbourhood. When $i < n$, this is because v_i has at most Δ neighbours, and one of them is not coloured yet because it belongs to $v_{i+1}, \dots, v_n$. When $i = n$, this is because $v_n = v$ has at most $\Delta - 1$ neighbours. Henceforth we assume that every vertex of G has degree exactly Δ.

An induced subgraph H of G is said to be bad if it is an odd cycle when $\Delta = 3$ or the complete graph on Δ vertices when $\Delta \geq 4$. Assume first that G contains no bad subgraph. Let S be a maximal (for the inclusion) independent set of G. By maximality of S, every vertex in $V(G) \setminus S$ has at least one neighbour in S. Let G' be $G - S$. The remark above implies that $\Delta(G') \leq \Delta(G) - 1 \leq \Delta - 1$. By induction on Δ, and because G does not contain any bad subgraph, every connected component of G' is $(\Delta-1)$-colourable. This implies that G' is $(\Delta-1)$-colourable. Using one additional colour for S, we obtain a Δ-colouring of G, a contradiction.

Assume now that H is a bad subgraph of G. By definition, $\Delta(H) = \delta(H) = \Delta - 1$, so every vertex of $V(H)$ has exactly one neighbour in $V(G) \setminus V(H)$. Let $X \subseteq (V(G) \setminus V(H))$ be the set of vertices having at least one neighbour in H. We claim that $|X| \geq 2$. Assume not, then $X = \{x\}$ and x is adjacent to every vertex in H. If H is a complete graph on Δ vertices, then $V(H) \cup \{x\}$ induces a complete graph on $\Delta + 1$ vertices on G, a contradiction. Then $\Delta = 3$ and H is an odd cycle of length at least 5. This is a contradiction because x has degree at least 5. Hence, let x, y be two distinct vertices in X. Let G' be the graph obtained from $G - V(H)$ by adding the edge $\{x, y\}$ (if it does not already exist, otherwise we just take $G' = G - V(H)$). Since both x and y have a neighbour in H, we obtain that $\Delta(G') \leq \Delta$. By minimality of G, either G' has a connected component isomorphic to $K_{\Delta+1}$ or it is Δ-colourable.

In the former case, there exists $R \subseteq (V(G) \setminus V(H))$ such that the subgraph of G induced by R is a complete graph on $\Delta + 1$ vertices minus one edge (namely $\{x, y\}$). Note that this forces both x and y to have exactly one neighbour in $V(H)$ (otherwise they have degree at least $\Delta + 1$). Let H' be the bad subgraph of G induced by $R \setminus \{y\}$ and x' be the only neighbour of x in H. Since y has at least two neighbours in R, we reduced to the latter case with H', x' and y playing the roles of H, x and y respectively.

Henceforth assume G' is Δ-colourable and let α be a proper Δ-colouring of G'. Hence, α is also a partial proper Δ-colouring of G. We now show that α can be extended to a proper Δ-colouring of G, yielding the contradiction. Since H is either a complete graph or a cycle, and because $\alpha(x) \neq \alpha(y)$, there exist two adjacent vertices u, v in H such that the neighbour u' of u outside H and the neighbour v' of v outside H satisfy $\alpha(v') \neq \alpha(u')$. We can also take an ordering $u = v_1, \ldots, v_r = v$ of $V(H)$ such that every vertex v_i (for $i \in [r-1]$) has at least one neighbour in $\{v_{i+1}, \ldots, v_r\}$. We then extend α as follows: we start by setting $\alpha(u) = \alpha(v')$. Then we move forward from v_2 to v_{r-1} and at each step we choose for $\alpha(v_i)$ a colour of $[\Delta]$ that is not appearing in the neighbourhood of v_i. At the end, all the neighbours of v are coloured, but two of them (namely u and v') use the same colour. Hence, one colour of $[\Delta]$ is not used in its neighbourhood, and we can set $\alpha(v)$ to this colour. This yields the contradiction. □

As we will see in Section 1.3.2 and Chapter 3, Brooks Theorem has been generalised and strengthened in many ways. We only mention here the celebrated result of Johansson which asymptotically improves the bound of Brooks Theorem for triangle-free graphs.

Theorem 1.1.8 (Johansson [102]). *Every triangle-free graph G satisfies $\chi(G) = O\left(\frac{\Delta(G)}{\ln \Delta(G)}\right)$.*

Johansson's bound is conjectured to be true not only for triangle-free graphs but for H-free graphs in general, for every fixed H.

Conjecture 1.1.9 (Alon, Krivelevich, and Sudakov [12]). *For every fixed graph H, there exists a positive constant c_H such that every H-free graph G satisfies $\chi(G) \leq c_H \frac{\Delta(G)}{\ln \Delta(G)}$.*

Density of critical graphs

Since the chromatic number is non-decreasing with respect to the subgraph relation, it is natural to consider the minimal graphs (for this relation) which are not $(k-1)$-colourable. Following this idea, Dirac introduced in the 1950s the concept of *critical graphs* and established the basic properties of such graphs in a series of papers [59, 60, 62].

A graph G is *k-critical* if it has chromatic number k and each of its proper subgraph $H \subsetneq G$ has chromatic number at most $k - 1$. In other words, removing every vertex or any edge of G decreases the chromatic number. See Figure 1.5 for an illustration of a critical graph.

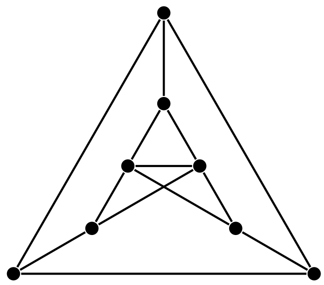


Figure 1.5: A 4-critical graph on eight vertices with $g_4(8) = 13$ edges.

For a fixed integer k, Dirac also started the study of the function $g_k(n)$, defined as the minimum number of edges in n-vertex k-critical graphs. Formally, we have:

$$g_k(n) = \min \{m(G) \mid G \text{ is } k\text{-critical and has order } n\},$$

with the convention $g_k(n) = +\infty$ if there exists no such graph. A first property of k-critical graphs is that their minimum degree is at least $k-1$. Indeed, if a vertex v has degree at most $k-2$, then a proper $(k-1)$-colouring of $G - v$ can be extended to G by choosing for v a colour that is not appearing in its neighbourhood, contradicting the fact that $\chi(G) = k$. As a consequence, we obtain $g_k(n) \geq \frac{1}{2}(k-1)n$. This bound is tight for complete graphs and odd cycles, but Dirac [62] proved an inequality of the form $g_k(n) \geq \frac{1}{2}(k - 1 + \varepsilon_k)n - c_k$, for some c_k and $\varepsilon_k > 0$. This shows that, for n sufficiently large, the average degree of a k-critical graph is at least $k - 1 + \varepsilon_k$. This initiated the quest after the best lower bound on $g_k(n)$. This problem was almost completely solved by Kostochka and Yancey [114] in 2014.

Theorem 1.1.10 (Kostochka and Yancey [114])**.** *Every k-critical graph on n vertices has at least $\frac{1}{2}(k - \frac{2}{k-1})n - \frac{k(k-3)}{2(k-1)}$ edges.*

For every k, this bound is tight for infinitely many values of n. Kostochka and Yancey [116] later characterised k-critical graphs for which this inequality is an equality.

Reconfiguration of graph proper colourings

The reconfiguration of graph proper colourings is a widely studied notion related to graph colouring. In this context, given a graph G, an integer k, and two proper k-colourings α and β of G, the main question is whether it is possible to transform α into β by changing the colour of exactly one vertex at a time while maintaining a proper k-colouring at each step. We formally define these notions via the introduction of an auxiliary graph, namely the *k-colouring graph* of a graph.

Given a graph $G = (V, E)$ and an integer $k \geq \chi(G)$, the *k-colouring graph* of G, denoted by $\mathcal{C}_k(G)$, is the graph whose vertices are the proper k-colourings of G and in which two proper k-colourings are adjacent if they differ on exactly one vertex. A *walk* in a graph is a sequence of vertices $u_1, \ldots, u_r$ such that, for every $i \in [r-1]$, $u_i u_{i+1}$ is an edge of the graph. Note that the u_is are not necessarily distinct. A *recolouring sequence* between two proper k-colourings of G is a walk between them in $\mathcal{C}_k(G)$. If $\mathcal{C}_k(G)$ is connected, we say that G is *k-mixing*. Given a graph G, one may ask for which values of k it is k-mixing, and when it is, how many steps are required at most to get from one colouring to another.

Determining if a graph is k-mixing has applications in statistical physics, where proper colourings represent states of the antiferromagnetic Potts model at temperature zero. The questions above were first addressed by researchers studying the Glauber dynamics for sampling proper k-colourings of a given graph. This is a Markov chain used to obtain efficient algorithms for approximately counting or almost uniformly sampling proper k-colourings of a graph, and the connectedness of the k-colouring graph is a necessary condition for such a Markov chain to be rapidly mixing. In graph theory, the study of recolouring has been rapidly developing in the past fifteen years, since the works of Cereceda, van den Heuvel, and Johnson [48, 49].

We refer the reader to the Ph.D. thesis of Bartier [21] for a complete overview on graph recolouring and to the surveys of van den Heuvel [169] and Nishimura [135] for reconfiguration problems in general.

The remaining of this section is dedicated to Cereceda's conjecture, which is one of the most widely open conjecture on graph recolouring. The *degeneracy* of a graph G, denoted by $\delta^*(G)$, is the largest minimum degree of all subgraphs of G. We say that a graph G is d-*degenerate* for every integer $d \geq \delta^*(G)$. Bonsma and Cereceda [36] and Dyer et al. [67] independently proved the following.

Theorem 1.1.11 (Bonsma and Cereceda [36] ; Dyer et al. [67])**.** *Let $k \in \mathbb{N}$ and G be a graph. If $k \geq \delta^*(G) + 2$, then G is k-mixing.*

Cereceda conjectured that not only G is k-mixing when k is at least $\delta^*(G) + 2$ but also the shortest recolouring sequence between two proper k-colourings is always bounded by a quadratic function in $n(G)$. We define the *diameter* of a graph to be the maximum length of a shortest path in G. Formally, Cereceda conjectured the following.

Conjecture 1.1.12 (Cereceda [47])**.** *Let $k \in \mathbb{N}$ and G be a graph on n vertices.*
If $k \geq \delta^*(G) + 2$, then the diameter of $\mathcal{C}_k(G)$ is at most $O(n^2)$.

Cereceda [47] proved that this is true when $k \geq 2\delta^*(G) + 1$. This was improved by Bousquet and Heinrich [42], who showed the following.

Theorem 1.1.13 (Bousquet and Heinrich [42])**.** *Let $k \in \mathbb{N}$ and G be a graph. Then $\mathcal{C}_k(G)$ has diameter at most:*

- *Cn^2 if $k \geq \frac{3}{2}(\delta^*(G) + 1)$ (where C is a constant independent from k),*
- *$C_\varepsilon n^{\lceil \frac{1}{\varepsilon} \rceil}$ if $k \geq (1 + \varepsilon)(\delta^*(G) + 2)$ (where C_ε is a constant independent from k),*
- *$(Cn)^{\delta^*(G)+1}$ for any $k \geq \delta^*(G) + 2$ (where C is a constant independent from k).*

The third item of Theorem 1.1.13 is currently the best known result towards Conjecture 1.1.12.

1.2 Digraph colouring

1.2.1 From graphs to digraphs

Our notation on directed graphs follow [18]. A *directed graph*, or *digraph* for short, is an ordered pair $D = (V, A)$ where V is a finite set of *vertices* and A is a set of ordered pairs of V called *arcs*. The vertex-set of D is denoted by $V(D)$ and its arc-set is denoted by $A(D)$. The number of vertices of D is called the *order* of D and is denoted by $n(D)$, and its number of arcs is denoted by $m(D)$. For conciseness, we denote an arc (u, v) by uv.

In a digraph, between two distinct vertices u, v, there might be two arcs in opposite directions, namely uv and vu. Such a pair of arcs is called a *digon* and is denoted by $[u, v]$. A *simple arc* is an arc which is not in a digon. The *underlying graph* of a digraph D, denoted by $\mathrm{UG}(D)$, is the undirected graph with vertex-set $V(D)$ in which uv is an edge if and only if uv or vu is an arc of D. A *bidirected graph* is a digraph with no simple arc, and an *oriented graph* is a digraph with no digon. We also say that an oriented graph D is an *orientation* of $\mathrm{UG}(D)$. Along this thesis, we will often denote an oriented graph by $\vec{G}$ to emphasise the fact that it is an orientation of an undirected graph G (and thus it does not contain any digon). Given a graph G, the bidirected graph $\overleftrightarrow{G}$ is the digraph obtained by replacing every edge uv of G by a digon $[u, v]$. A *tournament* is an orientation of a complete graph. The only acyclic tournament on n vertices is called the *transitive tournament* on n vertices and is denoted by TT_n.

For every arc $uv \in A(D)$, v is said to be an *out-neighbour* of u and u is said to be an *in-neighbour* of v. Vertices u and v are said to be *adjacent* to each other, and uv is *incident* to both u and v. Vertices u and v are called respectively the *tail* and the *head* of uv.

The sets of out-neighbours and in-neighbours of u, denoted respectively by $N^+(u)$ and $N^-(u)$, are called respectively the out-neighbourhood and the in-neighbourhood of u. We denote by $N(u) = N^-(u) \cup N^+(u)$ the set of neighbours of u and by $N[u]$ the *closed neighbourhood* $N(u) \cup \{u\}$ of u. Moreover, for every set X of vertices, we write $N^+(X) = \bigcup_{x \in X} N^+(x)$, $N^-(X) = \bigcup_{x \in X} N^-(x)$ and $N(X) = N^+(X) \cup N^-(X)$.

The *out-degree* and the *in-degree* of u, respectively denoted by $d^+(u)$ and $d^-(u)$, are respectively the number of out-neighbours and the number of in-neighbours of u. Vertex u is called a *source* if $d^-(u) = 0$, and it is called a *sink* if $d^+(u) = 0$. The *degree* of u, denoted by $d(u)$, is the sum of its in-degree and its out-degree. A digraph D is called *eulerian* if every vertex $u \in V(D)$ satisfies $d^+(u) = d^-(u)$. If further there exists an integer d such that, for every vertex u, $d^+(u) = d^-(u) = d$, then D is *d-diregular*.

Let D and H be two digraphs. Analogously to the undirected case, we say that H is a *subdigraph* of D if both $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$ hold, and we denote it by $H \subseteq D$. Also H is a *proper subdigraph* if H is a subdigraph of D and $H \neq D$, and we denote it by $H \subsetneq D$. We say that H is a *spanning subdigraph* of D if it is a subdigraph of D with $V(H) = V(D)$. Let $X \subseteq V(D)$ be a set of vertices and $F \subseteq A(D)$ be a set of arcs. We denote by $D - X$ the digraph obtained from D by removing every vertex in X and every arc incident to at least one vertex in X, and we say that $D - X$ is an *induced subdigraph* of D. Let $Y = V(D) \setminus X$, then $D - X$ is also called the subdigraph induced by Y and is denoted by $D\langle Y \rangle$. When $X = \{x\}$, we let $D - x$ be the digraph $D - X$. Also we let $D \setminus F$ denote the digraph $(V(D), A(D) \setminus F)$ and if $F = \{e\}$ we let $D \setminus e$ be $D \setminus F$. Finally, if $F \subseteq V(D) \times V(D)$, then $D \cup F$ denotes the digraph $(V(D), A(D) \cup F)$.

We say that H is *isomorphic* to D if there exists a bijection $\psi\colon V(H) \to V(D)$ such that, for every $u, v \in V(H)$, (u, v) is an arc of H if and only if $(\psi(u), \psi(v))$ is an arc of D. Again, when we say that D contains H as an (induced) subdigraph, we mean that D has an (induced) subdigraph which is isomorphic to H, hence not necessarily maintaining vertex labels. In case we want vertex labels to be maintained, we speak of an *(induced) labelled subdigraph*.

Digraphs are naturally used to model many real-world problems. Indeed, many graphs modelling networks are by essence directed: the web graph is such an example. However, for various reasons, digraph theory is a lot less developed than (undirected) graph theory. One such reason is that every undirected graph can be seen as a bidirected graph. Hence, plenty of problems on graphs can be considered as a particular case of a more general problem on digraphs, and digraph problems appear to be harder.

To illustrate this difference, let us consider the problem of partitioning the edges of a graph $G = (V, E)$ into k parts $E_1, \dots, E_k$ in such a way that every subgraph $G_i = (V, E_i)$ is connected. If the considered graph models a network architecture, it is related to the network's fault tolerance. Indeed, if at most $k - 1$ failures appear on the network (a failure corresponds to the removal of an edge), all the nodes remain connected by at least one of the E_is. Deciding if a graph admits such a partition can be solved in polynomial time, and the following celebrated theorem, proved independently by Nash-Williams and Tutte, guarantees the existence of such a partition. For $\lambda \in \mathbb{N}^*$, a graph G is *λ-edge-connected* if it has at least λ vertices and for any set F of at most $\lambda - 1$ edges, $G \setminus F$ is connected. For $\kappa \in \mathbb{N}^*$, a graph G is *κ-connected* if it has at least κ vertices and the removal of any set of at most $\kappa - 1$ vertices does not disconnect the graph.

Theorem 1.2.1 (Nash-Williams [133] ; Tutte [168])**.** *Every $2k$-edge-connected graph has k edge-disjoint spanning trees.*

The directed path on n vertices, denoted by $\vec{P}_n$, is the oriented graph with vertex-set $\{v_1, \dots, v_n\}$ and arc-set $\{(v_i, v_{i+1}) \mid i \in [n-1]\}$.

We denote by $\mathrm{init}(\vec{P}_n)$ the *initial vertex* of $\vec{P}_n$, which is its unique source, and by $\mathrm{term}(P)$ its terminal one, which is its unique sink. The vertices in $V(P) \setminus \{\mathrm{init}(P), \mathrm{term}(P)\}$ are called the *internal vertices* of P. If U and V are two sets of vertices in D, then a *(U, V)-path* in D is a directed path P in D with $\mathrm{init}(P) \in U$ and $\mathrm{term}(P) \in V$, and we also say that P is a directed path from U to V. If $U = \{u\}$ (resp. $V = \{v\}$), then we simply write u for U (resp. v for V) in these notations. When $n \geq 2$, adding the arc (v_n, v_1) to $\vec{P}_n$ gives the directed cycle $\vec{C}_n$. A digraph is *acyclic* if it does not contain any directed cycle.

A digraph is *connected* (resp. *κ-connected*) if its underlying graph is connected (resp. κ-connected). It is *strongly connected* if, for every ordered pair (u, v) of its vertices, there exists a directed path from u to v. It is *λ-arc-strong*, for $\lambda \in \mathbb{N}^*$, if it has at least λ vertices and it remains strongly connected after the removal of any set of at most $\lambda - 1$ arcs. It is *κ-strong*, for $\kappa \in \mathbb{N}^*$, if it has at least κ vertices and it remains strongly connected after the deletion of any set of at most $\kappa - 1$ vertices. A *strongly connected component* of a digraph D is a maximal strongly connected subdigraph of D.

Assume now that, in the problem above, the connections in the network are not all bidirectional. It is then modelled by a digraph $D = (V, A)$, and the problem consists of deciding whether its arc-set can be partitioned into k parts $A_1, \dots, A_k$ such that every subdigraph $D_i = (V, A_i)$ is strongly connected. As shown by Bang-Jensen and Yeo [20], this problem turns out to be NP-complete even for $k = 2$. Furthermore, the existence of an analogue of Theorem 1.2.1 for digraphs is widely open and motivated the following conjecture of Bang-Jensen and Yeo, which is still open even for $k = 2$.

Conjecture 1.2.2 (Bang-Jensen and Yeo [20])**.** *For every fixed $k \in \mathbb{N}$, there exists $f(k) \in \mathbb{N}$ such that every $f(k)$-arc-strong digraph contains k arc-disjoint spanning subdigraphs.*

This thesis is about the generalisation of graph proper colouring results to digraphs. To this purpose, we use the notions of *dicolouring* and *dichromatic number*. The dichromatic number

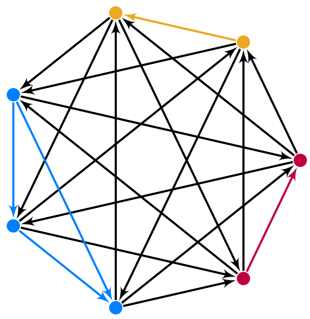


Figure 1.6: An example of a 3-dicolouring of the Paley tournament on seven vertices, which is known to have dichromatic number exactly 3. Each colour class induces an acyclic subdigraph.

of a digraph was introduced by Erdős and Neumann-Lara in the late 1970s [134, 69]. It was rediscovered by Mohar in the 2000s [129, 28] and received a lot of attention since then. We refer the reader to the Ph.D. thesis of Harutyunyan [89] and the one of Aubian [16], both dedicated to this colouring parameter for digraphs.

Along this thesis, we will show that a lot of graph colouring results have a directed counterpart when using dicolouring and dichromatic number. In this sense, we contribute to show that these two notions are the appropriate extensions of the ones of proper colouring and chromatic number.

1.2.2 Dicolouring and dichromatic number of digraphs

For a positive integer k, a *k-colouring* of a digraph $D = (V, A)$ is a function $\alpha \colon V \to [k]$. It is *proper* if every pair of adjacent vertices are coloured differently. It is a *k-dicolouring* if $\alpha^{-1}(i)$ induces an acyclic subdigraph in D for every $i \in [k]$. Equivalently, α is a dicolouring if and only if D, coloured with α, does not contain any monochromatic directed cycle. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the smallest k such that D admits a k-dicolouring. When a digraph admits a k-dicolouring, we say that it is *k-dicolourable*.

Observe that any proper colouring of D is indeed a dicolouring of D, since an independent set necessarily induces an acyclic subdigraph. Hence we always have $\vec{\chi}(D) \leq \chi(\mathrm{UG}(D))$. See Figure 1.6 for an illustration of a dicolouring.

Since every digon induces a directed cycle of length 2, there is a one-to-one correspondence between the proper k-colourings of a graph G and the k-dicolourings of its associated bidirected graph $\overleftrightarrow{G}$, and in particular $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$. Hence every result on graph proper colourings can be seen as a result on dicolourings of bidirected graphs. Two main questions then arise:

1. Is this result true for every digraph and not only for bidirected graphs ?

2. Can it be strengthened on the class oriented graphs ?

This thesis focuses on these two questions for several classical results on graph colouring. Many researchers already considered them before, for different results, and this lead to a lot of generalisations and open problems. We illustrate it with some results presented in Section 1.1.2.

Neumann-Lara's conjecture

If D is a planar digraph, that is a digraph that can be drawn on the plane without crossing arcs, then it is 4-dicolourable because its underlying graph is 4-colourable by the Four Color Theorem (Theorem 1.1.1). This is best possible as $\overleftrightarrow{K_4}$ is planar. However, if D is an oriented planar graph (*i.e.* a planar digraph with no digon), this result can be easily improved as follows.

Proposition 1.2.3. *Let $\vec{G}$ be an oriented planar graph, then $\vec{\chi}(\vec{G}) \leq 3$.*

Proof. We proceed by induction on the order of $\vec{G}$. If $n(\vec{G}) \leq 3$, then $\vec{G}$ is trivially 3-dicolourable, so assume $n(\vec{G}) \geq 4$. Let G be the underlying graph of $\vec{G}$. Since $\vec{G}$ is planar, Euler's formula (see [58, Theorem 4.2.9]) implies that G contains a vertex v of degree at most 5. Hence v is a vertex of out-degree or in-degree at most 2 in $\vec{G}$.

By induction let α be a 3-dicolouring of $\vec{G} - v$. If $d^+(v) \leq 2$, we set $\alpha(v)$ to a colour that it not appearing in its out-neighbourhood. Otherwise, the remark above implies $d^-(v) \leq 2$ and we set $\alpha(v)$ to a colour that is not appearing in its in-neighbourhood.

This operation extends α into a dicolouring of $\vec{G}$. Indeed, if α is not a dicolouring of $\vec{G}$, it contains a monochromatic directed cycle C. By induction, α is a dicolouring of $\vec{G} - v$ so C must contain v. This is a contradiction because v does not share its colour with its in-neighbour or its out-neighbour in C. □

In [93] Harutyunyan and Mohar proved that not only an oriented planar graph is 3-dicolourable, but it also admits exponentially many different 3-dicolourings. Also we do not know if the upper bound given by the proposition above is tight. Neumann-Lara conjectured that it is not, and this is probably the main open conjecture on the dichromatic number.

Conjecture 1.2.4 (Neumann-Lara [134])**.** *Every oriented planar graph is 2-dicolourable.*

The current best result approaching this conjecture is due to Li and Mohar. The *digirth* of a digraph D, denoted by $\mathrm{digirth}(D)$, is the length of its shortest cycle. Note that an oriented graph is a digraph with digirth at least 3.

Theorem 1.2.5 (Li and Mohar [121])**.** *Every oriented planar graph with digirth at least 4 is 2-dicolourable.*

Perfect digraphs

The *clique number* of a digraph D, denoted by $\overleftrightarrow{\omega}(D)$, is the largest integer k such that $\overleftrightarrow{K_k}$ is a subdigraph of D. As in the undirected case, every digraph satisfies $\vec{\chi}(D) \geq \overleftrightarrow{\omega}(D)$. Note that the clique number of $\mathrm{UG}(D)$ is not a lower bound on the dichromatic number of D because TT_k has dichromatic number 1 but $\omega(\mathrm{UG}(TT_k)) = k$.

The definition of perfect digraphs is the natural extension of the ones of perfect graphs. That is, a digraph D is *perfect* if $\vec{\chi}(H) = \overleftrightarrow{\omega}(H)$ holds for every induced subdigraph H of D. Using the Strong Perfect Graph Theorem (Theorem 1.1.3), Andres and Hochstättler characterised exactly the perfect digraphs. We define the *symmetric part* of a digraph D, denoted by $S(D)$, as the undirected graph with vertex-set $V(D)$ in which uv is an edge if and only if $[u, v]$ is a digon of D.

Theorem 1.2.6 (Andres and Hochstättler [13])**.** *A digraph D is perfect if and only if $S(D)$ is perfect and D does not contain an induced directed cycle of length at least 3.*

Proof. Assume first that D is perfect. Then D does not contain any induced directed cycle C of length at least 3, for otherwise C would be an induced subdigraph of D satisfying $\overleftrightarrow{\omega}(C) = 1$ and $\vec{\chi}(C) = 2$, a contradiction. Assume for a contradiction that $S(D)$ is not perfect, then it contains an induced subgraph H with $\chi(H) > \omega(H)$. Let H' be the subdigraph of D induced by $V(H)$. We obtain that $\vec{\chi}(H') \geq \chi(S(H')) = \chi(H) > \omega(H) = \overleftrightarrow{\omega}(H')$, a contradiction.

Conversely, assume that $S(D)$ is perfect and that D does not contain any induced directed cycle of length at least 3. Let H be any induced subdigraph of D. Let α be a proper colouring of $S(H)$ using exactly $\omega(S(H)) = \overleftrightarrow{\omega}(H)$ colours (the existence of which is guaranteed as $S(D)$ is perfect). Assume that α is not a dicolouring of H, then it contains a monochromatic directed cycle. Among all such cycles, let C be a shortest one. Then C must be an induced directed cycle of D, and therefore it must be of length 2. This yields a contradiction since α is a proper colouring of $S(H)$. Since α uses $\overleftrightarrow{\omega}(H)$ colours, We conclude that $\vec{\chi}(H) = \overleftrightarrow{\omega}(H)$ holds for every induced subdigraph H of D. Hence D is perfect. $\square$

$\vec{\chi}$-boundedness

There are several definition of $\vec{\chi}$-boundedness for digraphs. We only present one of them here, but the interested reader is especially refereed to [3] in which the Aboulker, Aubian, Charbit, and Lopes gave a nice and different approach of $\vec{\chi}$-boundedness for digraphs.

We say that a class of digraphs $\mathcal{D}$ is *$\vec{\chi}$-bounded* if and only if there exists some function f (depending only on $\mathcal{D}$) such that every digraph $D \in \mathcal{D}$ has dichromatic number at most $f(\omega(\mathrm{UG}(D)))$. In order to generalise Gyárfás–Sumner Conjecture (Conjecture 1.1.5) to digraphs, one can ask for the digraphs H such that the class of H-induced-free digraphs is $\vec{\chi}$-bounded. We will show that such digraphs are exactly the *arcless digraphs*, that is the digraphs H satisfying $A(H) = \varnothing$.

Assume that such a digraph H exists. Then it must be bidirected, for otherwise one can take any undirected triangle-free graph G of arbitrarily large chromatic number, and $\overleftrightarrow{G}$ is an H-induced-free digraph with arbitrarily large dichromatic number satisfying $\omega(G) = 2$. This shows that H must be bidirected. The following result shows that H must also be an oriented graph. Since H is both a bidirected graph and an oriented graph, it must be arcless.

Theorem 1.2.7. *For every fixed $k \in \mathbb{N}$, there exists an oriented graph $\vec{G}_k$, $\mathrm{UG}(\vec{G}_k) = G_k$, such that G_k is triangle-free and $\vec{\chi}(\vec{G}_k) \geq k$.*

We prove this result with a nice construction due to Carbonero et al. [46]. They actually show a stronger result so we simplify their construction for our purpose.

Proof. We prove the result by induction on k. When $k = 1$ we let $\vec{G}_k$ be the single-vertex oriented graph. Assume now that $k \geq 1$ and let $\vec{G}_k$ be an orientation of a triangle-free graph G_k with dichromatic number at least k.

Let n be the order of $\vec{G}_k$ and $v_1, \dots, v_n$ be any ordering of its vertices. In order to construct $\vec{G}_{k+1}$, we will recursively build, for every $i \in \{0, \dots, n\}$, an orientation $\vec{H}_i$ of a graph H_i satisfying each of the following properties.

(*i*) there is a partition $V_1, \dots, V_r$ of $V(\vec{H}_i)$ such that each V_i induces a copy $\vec{G}_k^i$ of $\vec{G}_k$, where $r = 4^i$.

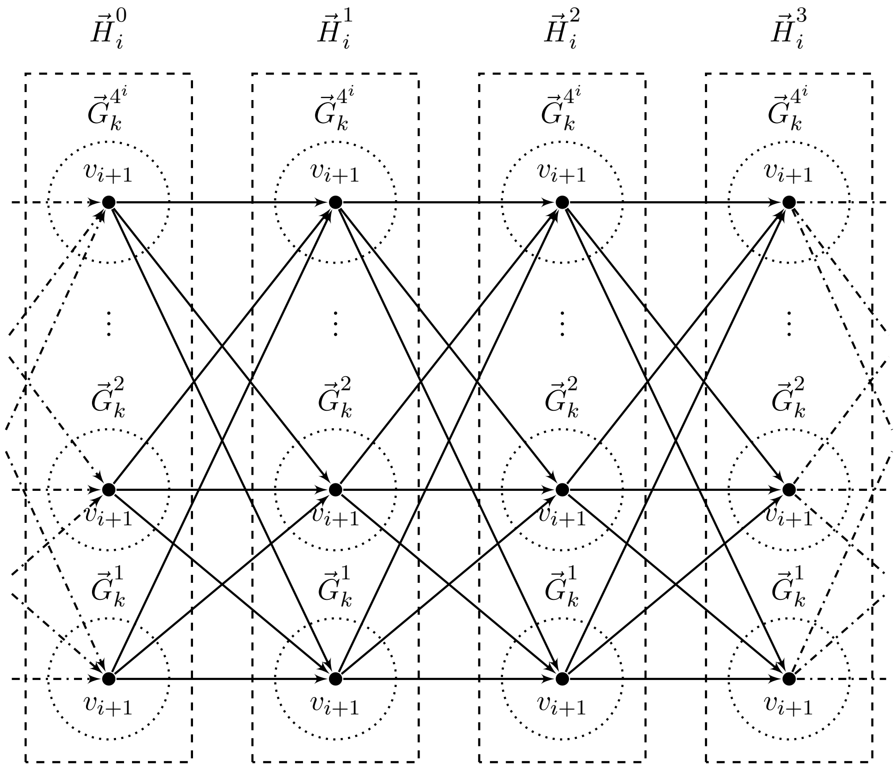


Figure 1.7: The construction of $\vec{H}_{i+1}$ from $\vec{H}_i$. Dotted circles represent copies of $\vec{G}_k$ and dashed rectangles represent copies of $\vec{H}_i$.

(ii) For every vertex $v_j \in V(\vec{G}_k)$, $j \geq i+1$, the set of copies of v_j (with respect to (i)) is an independent set.

(iii) H_i is triangle-free.

(iv) In every k-dicolouring α of $\vec{H}_i$, there exists a copy $\vec{G}_k^\ell$ of $\vec{G}_k$ in which the copies of $v_1, \dots, v_i$ are not coloured k.

Once we have proved this, we can take $\vec{G}_{k+1}$ to be $\vec{H}_n$: H_n is triangle-free by (iii) and $\vec{H}_n$ has dichromatic number at least $k+1$ by (iv). Indeed, if H_n admits a k-dicolouring α, then condition (iv) implies that it contains a copy of $\vec{G}_k$ in which no vertex is coloured k, a contradiction since $\vec{\chi}(\vec{G}_k) \geq k$ by induction.

Let $\vec{H}_0$ be $\vec{G}_k$, note that all the properties trivially hold. Assume now that we already built $\vec{H}_i$ for some $i \in [n-1]$. We construct $\vec{H}_{i+1}$ by taking four disjoint copies $\vec{H}_i^0, \dots, \vec{H}_i^3$ of $\vec{H}_i$ and adding every arc uv such that u is a copy of v_{i+1} in $\vec{H}_i^j$ and v is a copy of v_{i+1} in $\vec{H}_i^{j+1}$ (we take $j+1$ modulo 4). Note that we did not create any digon. See Figure 1.7 for an illustration.

Let us show that $\vec{H}_{i+1}$ satisfies the four conditions above.

(i) $\vec{H}_{i+1}$ is actually the vertex-disjoint union of four copies of $\vec{H}_i$, implying by induction that it is the vertex-disjoint union of 4^{i+1} copies of $\vec{G}_k$.

(ii) For every $j \geq i+2$, the set of copies of v_j is an independent set because we only added arcs between copies of v_{i+1}.

(iii) Assume that H_{i+1} contains a triangle. Since H_i is triangle-free, this triangle must contain an edge uv that we added between two copies of H_i. Assume without loss of generality that u belongs to H_i^1 and v belongs to H_i^2. Let w be a common neighbour of u and v. Then w does not belong to H_i^3, for otherwise it is not adjacent to u, and it does not belong to H_i^4, for otherwise it is not adjacent to v. Assume it belongs to H_i^1, the case of H_i^2 being symmetric. Since we added the edge wv, it means that w is a copy of v_i in H_i^1. Then, by induction, uw is not an edge since w and u are both copies of v_i in H_i^1. This yields the contradiction, and shows that H_{i+1} is triangle-free.

(iv) Let us fix a k-dicolouring α of $\vec{H}_{i+1}$ (if $\vec{H}_{i+1}$ is not k-dicolourable, condition (iv) holds vacuously, so we assume such a dicolouring exists). By induction, for every $j \in \{0,1,2,3\}$, $\vec{H}_i^j$ contains a copy $\vec{G}_k^{\ell,j}$ of $\vec{G}_k$ in which the copies of $v_1,\dots,v_i$ are not coloured k. Then at least one of these copies does not use colour k for v_{i+1}, for otherwise D, coloured with α, would contain a monochromatic directed cycle of length four. This shows condition (iv). □

We have shown that if a digraph H is such that the class of H-induced-free digraphs is $\vec{\chi}$-bounded, then H is arcless. Conversely, assume that H is an arcless digraph and let $\mathcal{D}_H$ be the class of H-induced-free digraphs. A seminal result of Ramsey [145] shows that, for every fixed integers k,ℓ, there exists an integer $R(k,\ell)$ such that every undirected graph on at least $R(k,\ell)$ vertices either contains a clique of size k or an independent set of size ℓ. This implies, for every fixed ω, that there are finitely many digraphs D in $\mathcal{D}_H$ satisfying $\omega(\mathrm{UG}(D)) \leq \omega$. Therefore, $\mathcal{D}_H$ is $\vec{\chi}$-bounded since every digraph $D \in \mathcal{D}_H$ satisfies $\vec{\chi}(D) \leq f_H(\omega(\mathrm{UG}(D)))$, where f_H is defined as follows:

$$f_H(\omega) = \max\left\{\vec{\chi}(D) \mid n(D) < R(\omega+1, n(H))\right\}.$$

This easy characterisation of such digraphs H motivates the more general study of the finite sets of digraphs $\mathcal{H} = \{H_1,\dots,H_r\}$ such that the class of $\mathcal{H}$-induced-free digraphs is $\vec{\chi}$-bounded (a digraph D is $\mathcal{H}$-induced-free if it does not contain any digraph H_i as an induced subdigraph). Let us fix such a set $\mathcal{H} = \{H_1,\dots,H_r\}$. Erdős' result on undirected graphs of large chromatic number and large girth (Theorem 1.1.4) implies that at least one of the H_is must be a bidirected forest and Theorem 1.2.7 implies that at least one of them is an oriented graph. The following result due to Harutyunyan and Mohar is the directed analogue of Erdős' result. It strengthens Theorem 1.2.7 and implies that at least one of the H_is must actually be an orientation of a forest. The *girth* of a digraph is 2 if it contains a digon, and it is the girth of its underlying graph otherwise.

Theorem 1.2.8 (Harutyunyan and Mohar [94])**.** *For every fixed $k,\ell \in \mathbb{N}$, there exists an oriented graph $\vec{G}$ such that $\vec{\chi}(\vec{G}) \geq k$ and $\mathrm{girth}(\vec{G}) \geq \ell$.*

Hence, if $\mathcal{H}$ is a pair of digraphs $\{H_1,H_2\}$, such that none of H_1,H_2 is arcless, we know that one of them is a bidirected forest and the other one is an orientation of a forest. An orientation of a forest (resp. a tree) is called an *oriented forest* (resp. an *oriented tree*). Thus, Gyárfás-Sumner Conjecture (Conjecture 1.1.5) can be generalised to digraphs as follows.

Conjecture 1.2.9. *For every bidirected forest F_1 and every oriented forest F_2, the class of $\{F_1, F_2\}$-induced-free digraphs is $\vec{\chi}$-bounded.*

The following weaker conjecture, obtained by setting $F_1 = \overleftrightarrow{K_2}$, has been posed by Aboulker, Charbit, and Naserasr. It is still widely open and is known for a very few restricted cases (see [7, 53, 4]).

Conjecture 1.2.10 (Aboulker, Charbit, and Naserasr [7])**.** *For every oriented forest F, the class of F-induced-free oriented graphs is $\vec{\chi}$-bounded.*

Conjecture 1.2.10 appears to be the oriented counterpart of Gyárfás-Sumner Conjecture (Conjecture 1.1.5), as shown by the following.

Proposition 1.2.11. *Conjecture 1.2.9 holds if and only if both Conjecture 1.1.5 and Conjecture 1.2.10 hold.*

Proof. One direction is clear, since Conjecture 1.1.5 is a special case of Conjecture 1.2.9 for $F_2 = TT_2$, and Conjecture 1.2.10 is a special case of Conjecture 1.2.9 for $F_1 = \overleftrightarrow{K_2}$.

Assume now that both Conjecture 1.1.5 and Conjecture 1.2.10 hold. Let us fix a bidirected forest F_1 and an oriented forest F_2. Let $D = (V, A)$ be any digraph that is $\{F_1, F_2\}$-induced-free, and let ω be the clique number of $\mathrm{UG}(D)$. Let (A_1, A_2) be the partition of A where A_1 contains all the arcs in a digon of D and A_2 contains all the simple arcs of D. Since $D_1 = (V, A_1)$ is an F_1-induced-free bidirected graph, and because we assume that Conjecture 1.1.5 holds, it admits a proper $f_1(\omega)$-colouring α_1 for some function f_1 depending only on F_1. On the other hand, if Conjecture 1.2.10 holds, since $D = (V, A_2)$ is an F_2-induced-free oriented graph, it admits an $f_2(\omega)$-dicolouring α_2 where f_2 depends only on F_2. We can then define $\alpha(v) = (\alpha_1(v), \alpha_2(v))$ to obtain a dicolouring of D using at most $f_1(\omega) \times f_2(\omega)$ colours. This shows that $\vec{\chi}(D) \leq f(\omega(\mathrm{UG}(D)))$ holds for every $\{F_1, F_2\}$-induced-free digraph D, where f depends only on $\{F_1, F_2\}$, implying Conjecture 1.2.9. $\square$

Directed Brooks Theorem

The directed version of Brooks Theorem was first proved by Harutyunyan and Mohar in [91]. Aboulker and Aubian also gave four new proofs of this theorem in [2]. Before we state it, we need a few definitions. Let D be a digraph and v be a vertex of D. The *maximum degree* of v, denoted by $d_{\max}(v)$, is the maximum between its in-degree and its out-degree. We can then define the corresponding maximum degree of D:

$$\Delta_{\max}(D) = \max\left\{d_{\max}(v) \mid v \in V(D)\right\}.$$

Note that it is actually an extension of the maximum degree of an undirected graph, since every graph G satisfies $\Delta(G) = \Delta_{\max}(\overleftrightarrow{G})$. Let us show that $\vec{\chi}(D) \leq \Delta_{\max}(D) + 1$ holds by a simple greedy procedure: fix an arbitrary ordering $v_1, \dots, v_n$ of $V(D)$. Along this ordering, consider the vertices one after the other. At step i, choose for v_i a colour that is not already appearing in its out-neighbourhood. This operation does not create any monochromatic directed cycle because v is now a sink in its colour class. At the end, we obtain a $(\Delta_{\max}(D)+1)$-dicolouring of D. As in the undirected case, the Directed Brooks Theorem, due to Harutyunyan and Mohar [91], characterises the connected digraphs D for which $\vec{\chi}(D) = \Delta_{\max}(D) + 1$. See Figure 1.8 for an illustration.

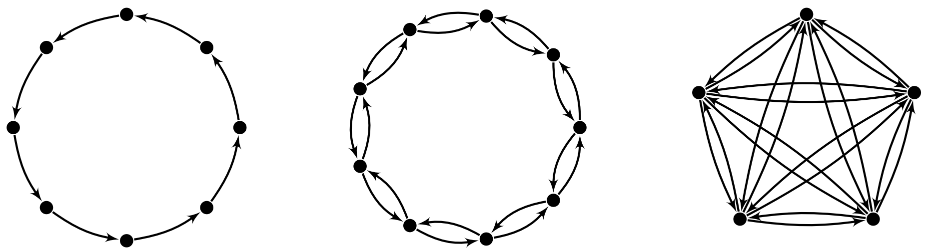


Figure 1.8: The only connected digraphs D for which $\vec{\chi}(D) = \Delta_{\max}(D) + 1$. From left to right: a directed cycle, a bidirected odd cycle, and a bidirected complete graph.

Theorem 1.2.12 (DIRECTED BROOKS THEOREM). *A connected digraph satisfied $\vec{\chi}(D) = \Delta_{\max}(D) + 1$ if and only if D is a directed cycle, a bidirected odd cycle, or a bidirected complete graph.*

In Section 1.3.2 and in Chapter 3, we discuss different generalisations and strengthenings of this result. We only mention here a celebrated conjecture due to Erdős and Neumann-Lara (see [69]), which is the analogue of Johansson's result (Theorem 1.1.8) for digraphs. The *maximum degree* of a digraph D, denoted by $\Delta(D)$, is the maximum of $\{d(v) = d^+(v) + d^-(v) \mid v \in V(D)\}$. Analogously, the *minimum degree* of D, denoted by $\delta(D)$, is the minimum of $\{d(v) \mid v \in V(D)\}$.

Conjecture 1.2.13 (Erdős and Neumann-Lara [69]). *Every oriented graph $\vec{G}$ satisfies:*

$$\vec{\chi}(\vec{G}) = O\left(\frac{\Delta(\vec{G})}{\ln \Delta(\vec{G})}\right).$$

1.3 Contributions and outline of this thesis

1.3.1 Dichromatic number of chordal graphs

We say that a digraph D is a *super-orientation* of an undirected graph G if G is the underlying graph of D. By definition, a super-orientation can contain digons, while an orientation cannot. A graph is *chordal* if it does not contain any hole. In Chapter 2, which is based on [26], we give both lower and upper bounds on the dichromatic number of super-orientations of chordal graphs. In general, the dichromatic number of such digraphs is bounded above by the clique number of the underlying graph because chordal graphs are perfect. However, this bound can be improved when we restrict the symmetric part of such a digraph.

Let $D = (V, A)$ be a super-orientation of a chordal graph G. An easy greedy procedure shows $\vec{\chi}(D) \leq \left\lceil \frac{\omega(G)+\Delta(S(D))}{2} \right\rceil$. We show that this bound is best possible by constructing, for every fixed k, ℓ with $k \geq \ell + 1$, a super-orientation $D_{k,\ell}$ of a chordal graph $G_{k,\ell}$ such that $\omega(G_{k,\ell}) = k$, $\Delta(S(D_{k,\ell})) = \ell$ and $\vec{\chi}(D_{k,\ell}) = \left\lceil \frac{k+\ell}{2} \right\rceil$. When $\Delta(S(D)) = 0$ (*i.e.* D is an orientation of G), we give another construction showing that this is tight even for orientations of interval graphs.

Next, we show that $\vec{\chi}(D) \leq \frac{1}{2}\omega(G) + O(\sqrt{\mathrm{Mad}(S(D)) \cdot \omega(G)})$ where $\mathrm{Mad}(G) = \max_{H \subseteq G} \left(\frac{2m(H)}{n(H)}\right)$ is the *maximum average degree* of a graph G. Finally, we show that if $S(D)$

contains no C_4 as a subgraph, then $\vec{\chi}(D) \leq \left\lceil \frac{\omega(G)+3}{2} \right\rceil$. We justify that this is almost best possible by constructing, for every fixed k, a super-orientation D_k of a chordal graph G_k with clique number k such that $S(D_k)$ is a disjoint union of paths and $\vec{\chi}(D_k) = \left\lfloor \frac{k+3}{2} \right\rfloor$.

We also show a family of orientations of cographs, which is another class of perfect graphs, for which the dichromatic number is equal to the clique number of the underlying graph.

1.3.2 On the Directed Brooks Theorem

The first main result of Chapter 3 is a generalisation of the Directed Brooks Theorem (Theorem 1.2.12). Brooks Theorem has been generalised in many ways. One of the most general results of this kind is due to Borodin, Kostochka, and Toft, who introduced the notion of variable degeneracy. An extension of this result to digraphs has been recently proposed by Bang-Jensen, Schweser, and Stiebitz. We introduce a new extension of variable degeneracy for digraphs, that we call bi-variable degeneracy. With this new definition, we prove a more general result, with a new proof based on ear-decompositions. Moreover, we justify the existence of a linear-time algorithm for deciding whether a digraph admits a colouring with "degenerate colour classes", for this notion of degeneracy, and under some specific conditions. It can thus be derived into linear-time algorithms for plenty of intermediate generalisations of Brooks Theorem, such as list (di)colouring and partitioning into (weakly-)degenerate sub(di)graphs.

The second main result of Chapter 3 is a strengthening of the Directed Brooks Theorem on a large class of digraphs containing oriented graphs. Let D be a digraph and v be a vertex of D. The *minimum degree* of v, denoted by $d_{\min}(v)$, is the minimum between its in-degree and its out-degree. We can define the corresponding maximum degree of D:

$$\Delta_{\min}(D) = \max_{v \in V(D)} (d_{\min}(v))$$

Note that, by definition, we have $\Delta_{\min}(D) \leq \Delta_{\max}(D)$. Observe also that $\vec{\chi}(D) \leq \Delta_{\min}(D)+1$: we can find a $(\Delta_{\min}(D)+1)$-dicolouring of D with the greedy procedure consisting of choosing for every vertex a colour that is not already appearing either in its in-neighbourhood or in its out-neighbourhood. Hence, it seems natural to ask for a generalisation of the Directed Brooks Theorem using $\Delta_{\min}(D)$ instead of $\Delta_{\max}(D)$. Unfortunately, as shown by Aboulker and Aubian in [2], deciding whether $\vec{\chi}(D) \leq \Delta_{\min}(D)$ holds for a given digraph D is an NP-complete problem. Hence, unless P=NP, there is no easy characterisation of digraphs satisfying $\vec{\chi}(D) = \Delta_{\min}(D)+1$. As a partial result, in Chapter 3 we give a necessary condition for a digraph D to have dichromatic number exactly $\Delta_{\min}(D) + 1$. In particular, our result implies that every oriented graph $\vec{G}$ with $\Delta_{\min}(\vec{G}) \geq 2$ has dichromatic number at most $\Delta_{\min}(\vec{G})$. The very first consequence of this result is that every orientation of a graph with maximum degree at most 5 is 2-dicolourable, answering a question of Harutyunyan [90].

1.3.3 Density of dicritical digraphs

Exactly as in the undirected case, one can define k-*dicritical* digraphs to be the digraphs D such that $\vec{\chi}(D) = k$ and $\vec{\chi}(H) < k$ for every proper subdigraph H of D. For fixed k, n, we then define the function $d_k(n)$ to be the minimum number of arcs in n-vertex k-dicritical digraphs, with the convention $d_k(n) = +\infty$ if no such digraph exists. For every k-critical graph G, observe that $\overleftrightarrow{G}$ is a k-dicritical digraph. Therefore, for every fixed integers k, n, we have $d_k(n) \leq 2 \cdot g_k(n)$.

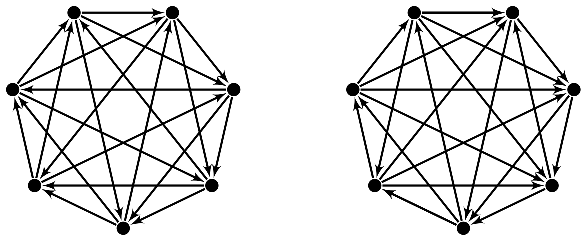


Figure 1.9: The only 3-dicritical tournaments.

The study of $d_k(n)$ was initiated by Kostochka and Stiebitz [113], who conjectured that in fact equality always holds except when $k = n + 1$. Moreover, they conjectured that all the k-dicritical digraphs on n vertices with $d_k(n)$ arcs are bidirected. They proved the first part of the conjecture when $k = 4$ (*i.e.* $g_4(n) = 2 \cdot d_4(n)$).

Restricted to oriented graphs, we define the function $o_k(n)$ to be the minimum number of arcs in n-vertex k-dicritical oriented graphs, with the convention $o_k(n) = +\infty$ if there exists no such oriented graph. By definition, we have $o_k(n) \geq d_k(n)$. In the same paper, Kostochka and Stiebitz conjectured that there exists a constant $c > 1$ such that $o_k(n) \geq c \cdot d_k(n)$ for $k \geq 3$ and n large enough. This was proved to be true when $k = 3$ by Aboulker, Bellitto, Havet, and Rambaud [6].

In Chapter 4, which is highly based on [141] and [97], we first generalise a result of Gallai [76, 77] and give the exact value of $d_k(n)$ when $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$. This implies the first conjecture of Kostochka and Stiebitz in this particular case. We then prove both conjectures for $k = 4$ using the potential method. We also exhibit a construction of dicritical oriented graphs, which shows $o_k(n) \leq (2k - \frac{7}{2}) \cdot n$ for every fixed k and infinitely many values of n.

Alternatively, for fixed k, n, one may ask for the maximum number of edges (or arcs) in k-critical graphs (or k-dicritical digraphs) on n vertices. A really few is known about this question, even in the undirected case. Aboulker [1] asked if the number of k-dicritical tournaments is finite when $k \geq 3$. This is still an open question. A *semi-complete* digraph is a super-orientation of a complete graph. In Chapter 5, which is based on [96], we answer this question for $k = 3$ by giving a simple hand-made proof that the number of 3-dicritical semi-complete digraphs is finite. We then give a more involved computer-assisted proof to show that there are only eight 3-dicritical semi-complete digraphs, and only two of them are tournaments, illustrated in Figure 1.9. We finally give a general upper bound on the maximum number of arcs in a 3-dicritical digraph.

1.3.4 Substructures in dicritical digraphs with large order or large digirth

In Chapter 6, which is based on [140], we give sufficient conditions on a dicritical digraph of large order or large digirth to contain a specific substructure.

We define an *oriented path* as the orientation of a path. Similarly, an *oriented cycle* is either an orientation of a cycle or $\vec{C}_2$. Note that a directed path (resp. cycle) is a particular oriented path (resp. oriented cycle). The *length* of an oriented path or an oriented cycle $\vec{G}$ is its number of arcs and is denoted by $\mathrm{length}(\vec{G})$.

We first extend a result of Kelly and Kelly [105] to the directed case by showing the existence, for every fixed integer $k \geq 2$, of a function $f_k\colon \mathbb{N} \to \mathbb{N}$ such that every k-dicritical digraph on at least $f_k(\ell)$ vertices contains an oriented path of length ℓ. Informally, this means that the length of a longest oriented path in a dicritical digraph grows with the order. Using a result of Dirac, we obtain the analogue result for the length of a longest oriented cycle. We also justify that the analogue result for directed paths does not hold by constructing, for every fixed $k \geq 3$, infinitely many k-dicritical digraphs that do not contain $\vec{P}_{3k+1}$ as a subdigraph.

Let us fix a digraph F. A *subdivision* of F is any digraph obtained from F by replacing every arc uv by a directed path (of length at least 1) from u to v. We say that a digraph D contains F as a subdivision if D contains a digraph F' which is a subdivision of F. Aboulker et al. [8] proved, for every fixed digraph F, the existence of a constant c_F such that every digraph with dichromatic number at least c_F contains F as a subdivision.

Let us now fix a subdivision F^* of F. When restricted to digraphs of arbitrarily large digirth, we strengthen the result of Aboulker et al. by showing that the value of c_{F^*} actually depends only on F. Formally, we show the existence of functions f, g such that for every subdivision F^* of a digraph F, digraphs with digirth at least $f(F^*)$ and dichromatic number at least $g(F)$ contain a subdivision of F^*. When F is a tree, we give the exact value of $g(F) = n(F)$.

We finally show the existence of a function f such that for every subdivision F^* of TT_3, digraphs with digirth at least $f(F^*)$ and minimum out-degree at least 2 contain F^* as a subdivision. In particular, this implies that every digraph with arbitrarily large digirth and dichromatic number at least 3 contains F^* as a subdivision. This confirms a very particular case of the following general conjecture we pose.

Conjecture 1.3.1. *There is a function f such that for every digraph F with maximum degree Δ, there is an integer g such that every digraph D satisfying* $\mathrm{digirth}(D) \geq g$ *and* $\vec{\chi}(D) \geq f(\Delta)$ *contains a subdivision of F.*

1.3.5 Redicolouring digraphs

In Chapter 7, which is based on [41, 139, 136], we study the directed analogue of graph recolouring. For the sake of conciseness, we briefly describe the results of Chapter 7 here. For a more detailed overview, the reader is referred to Section 7.1.

For any $k \geq \vec{\chi}(D)$, the *k-dicolouring graph* of a digraph D, denoted by $\mathcal{D}_k(D)$, is the graph whose vertices are the k-dicolourings of D and in which two k-dicolourings are adjacent if they differ on exactly one vertex. Observe that $\mathcal{C}_k(G) = \mathcal{D}_k(\overleftrightarrow{G})$ for every graph G. A *redicolouring sequence* between two dicolourings is a walk between these dicolourings in $\mathcal{D}_k(D)$. The digraph D is *k-mixing* if $\mathcal{D}_k(D)$ is connected.

We first prove, for every fixed $k \geq 2$, that the following problem is PSPACE-complete.

> k-DICOLOURING PATH
> **Input:** A digraph D along with two k-dicolourings α and β of D.
> **Output:** Is there a redicolouring sequence between α and β?

Given a digraph $D = (V, A)$ and a vertex $v \in V$, we define the *cycle-degree* of v, denoted by $d_c(v)$, as the minimum size of a set $S \subseteq V(D) \setminus \{v\}$ intersecting every directed cycle of D containing v. From this definition of cycle-degree, we define the c-degeneracy of D, which we denote by $\delta^*_c(D)$. It appears to be a nice generalisation of the undirected degeneracy, especially when dealing with directed treewidth. Using this new definition of degeneracy for digraphs, we extend a collection of evidence for Cereceda's conjecture to digraphs. These results lead us to believe that the following stronger version of Cereceda's conjecture holds.

Conjecture 1.3.2. *Let $k \in \mathbb{N}$ and D be a digraph. If $k \geq \delta_c^*(D) + 2$, then the diameter of $\mathcal{D}_k(D)$ is at most $O(n^2)$.*

We finally turn our focus to the density of non-mixing graphs and digraphs. We first provide a construction witnessing that there exist $(k-1)$-regular graphs of arbitrarily large girth that are not k-mixing, which was first shown by Bonamy, Bousquet, and Perarnau [32] using probabilistic arguments. However, this is not the case for digraphs with arbitrary large digirth. In fact, we show that this is not even the case for oriented graphs. We pose a conjecture on the minimum density of non-mixing oriented graphs and provide some support for it.

Chapter 2

Dichromatic number of chordal graphs

This chapter contains joint work with Stéphane Bessy and Frédéric Havet and is based on [26].

2.1 Introduction

An undirected graph is *chordal* if it does not contain any induced cycle of length at least 4. Proper colourings of chordal graphs have been largely studied and it is well-known that chordal graphs are perfect (see [58, Proposition 5.5.2]). Let us recall a few definitions that we will especially use in this chapter. Let D be a digraph. We say that D is a *super-orientation* of $\mathrm{UG}(D)$ and that it is an *orientation* of $\mathrm{UG}(D)$ if further D is an oriented graph. The *symmetric part* of D, denoted by $S(D)$, is the undirected graph G with vertex-set $V(D)$ in which uv is an edge if and only if uv is a digon of D. We denote by $\overleftrightarrow{\omega}(D)$ the size of a largest bidirected clique of D, *i.e.* the size of the largest clique of $S(D)$.

In this chapter, we look for lower and upper bounds on the dichromatic number of orientations and super-orientations of chordal graphs. Dicolourings of such digraphs have also been studied in [5], in which the authors characterise exactly the digraphs H for which there exists $c_H \in \mathbb{N}$ such that every orientation $\vec{G}$ of a chordal graph with $\vec{\chi}(\vec{G}) \geq c_H + 1$ contains H as an induced subdigraph.

We refer the interested reader to [127], in which the authors define a class of chordal digraphs, which extends the class of undirected chordal graphs. One can easily prove that every digraph D in this class is actually a perfect digraph, so it satisfies $\vec{\chi}(D) = \overleftrightarrow{\omega}(D)$ by Theorem 1.2.6.

The very first interesting class of super-orientations of chordal graphs are tournaments for which the question has been settled by Erdős, Gimbel and Kratsch in [70]. They showed that the dichromatic number of a tournament T on n vertices is always in $O\left(\frac{n}{\log n}\right)$, and that this bound is best possible (up to a constant factor). One can ask if this result is true not only for tournaments but for all orientations of chordal graphs. That is, do we always have $\vec{\chi}(\vec{G}) = O\left(\frac{\omega(G)}{\log \omega(G)}\right)$ when $\vec{G}$ is an orientation of a chordal graph G ? We answer this by the negative. Indeed, we show in Section 2.3 that it is not even true for orientations of interval graphs. Recall that an *interval graph* is obtained from a set of intervals on the real line: the intervals are the vertices, and there is an edge between two intervals if and only if they intersect. It is well-known that interval graphs are chordal graphs (see [58, Exercise 5.42]).

Theorem 2.1.1. *For every fixed $k \in \mathbb{N}$, there exists an interval graph G_k and an orientation $\vec{G}_k$ of this graph such that $\omega(G_k) = k$ and $\vec{\chi}(\vec{G}_k) \geq \left\lceil \frac{k}{2} \right\rceil$.*

On the positive side, if $\vec{G}$ is the orientation of a proper interval graph G (which is an interval graph where each interval has length exactly 1), then $\vec{\chi}(\vec{G}) = O\left(\frac{\omega(G)}{\log(\omega(G))}\right)$, as proved in [5]. The key-idea is that a proper interval graph G admits a partition (V_1, V_2) of its vertex-set such that both $G\langle V_1\rangle$ and $G\langle V_2\rangle$ are disjoint union of complete graphs.

Another well-known class of perfect graphs is the one of cographs. The *Dirac join* of two undirected graphs G_1 and G_2 is the graph built from the disjoint union of G_1 and G_2 where every edge between vertices of G_1 and vertices of G_2 are added. Cographs form the smallest class of graphs containing the single-vertex graph that is closed under disjoint union and the Dirac join operation. One can easily prove that the oriented graphs built in the proof of Theorem 2.1.1 are indeed orientations of cographs. In Section 2.4, we improve this result for cographs in general.

Theorem 2.1.2. *For every fixed $k \in \mathbb{N}$, there exists a cograph G_k and an orientation $\vec{G}_k$ of this graph such that $\vec{\chi}(\vec{G}_k) = \omega(G_k) = k$.*

Next we consider super-orientations of chordal graphs. If D is a super-orientation of a chordal graph G, then obviously $\vec{\chi}(D) \leq \omega(G)$ because $\vec{\chi}(D) \leq \chi(G) = \omega(G)$. Note that we cannot expect any improvement of this bound in general, because if D is the bidirected graph $\overleftrightarrow{G}$ then $\vec{\chi}(D) = \omega(G)$. But one can ask what happens if we restrict the structure of $S(D)$, the symmetric part of D.

In Section 2.5, we consider digraphs for which the symmetric part has bounded maximum degree. Using the degeneracy of the underlying graph, we show the following easy proposition.

Proposition 2.1.3. *Let D be a super-orientation of a chordal graph G, then*

$$\vec{\chi}(D) \leq \left\lceil \frac{\omega(G) + \Delta(S(D))}{2} \right\rceil.$$

This proposition is best possible when $\Delta(S(D)) = 0$ by Theorem 2.1.1. In the following, we show that it is indeed best possible for every fixed value of $\Delta(S(D))$.

Theorem 2.1.4. *For every fixed $k, \ell \in \mathbb{N}$ such that $k \geq \ell + 1$, there exists a chordal graph $G_{k,\ell}$ and a super-orientation $D_{k,\ell}$ of $G_{k,\ell}$ such that $\omega(G_{k,\ell}) = k$, $\Delta(S(D_{k,\ell})) = \ell$ and $\vec{\chi}(D_{k,\ell}) = \left\lceil \frac{k+\ell}{2} \right\rceil$.*

We define the *maximum average degree* of an undirected graph G, denoted by $\mathrm{Mad}(G)$, as follows:

$$\mathrm{Mad}(G) = \max\left\{\frac{2m(H)}{n(H)} \mid H \subseteq G\right\}.$$

In Section 2.6, we show the following bound on digraphs D for which $\mathrm{Mad}(S(D))$ is bounded.

Theorem 2.1.5. *Let D be a super-orientation of a chordal graph G. If $\mathrm{Mad}(S(D)) \leq d$, then*

$$\vec{\chi}(D) \leq \frac{1}{2}\omega(G) + O\left(\sqrt{d \cdot \omega(G)}\right).$$

Finally in Section 2.7 we show the following bound on super-orientations D of chordal graphs that do not contain $\overleftrightarrow{C_4}$.

Theorem 2.1.6. *Let D be a super-orientation of a chordal graph G. If $S(D)$ is C_4-free, then*

$$\vec{\chi}(D) \leq \left\lceil \frac{\omega(G)+3}{2} \right\rceil.$$

We also prove that the bound of Theorem 2.1.6 is almost tight by proving the following.

Theorem 2.1.7. *For every fixed $k \geq 3$ and every $n \geq \mathbb{N}$, there exists a super-orientation $D_{k,n}$ of a chordal graph $G_{k,n}$ on at least n vertices such that $S(D_{k,n})$ is a disjoint union of paths, $\omega(G_{k,n}) = k$ and $\vec{\chi}(D_{k,n}) = \left\lfloor \frac{k+3}{2} \right\rfloor$.*

A *tree-decomposition* of a graph $G = (V, E)$ is a pair $(T, \mathcal{X})$ where $T = (I, F)$ is a tree, and $\mathcal{X} = (B_i)_{i \in I}$ is a family of subsets of $V(G)$, called *bags* and indexed by the vertices of T, such that:

1. each vertex $v \in V$ appears in at least one bag, *i.e.* $\bigcup_{i \in I} B_i = V$,
2. for each edge $e = xy \in E$, there is an $i \in I$ such that $x, y \in B_i$, and
3. for each $v \in V$, the set of nodes indexed by $\{i \mid i \in I, v \in B_i\}$ forms a subtree of T.

The *width* of a tree decomposition is defined as $\max_{i \in I}\{|B_i| - 1\}$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. It is well-know that every graph G is a subgraph of a chordal graph G' with $\omega(G') = \mathrm{tw}(G) + 1$ (see [57, Corollary 12.3.12]). Hence, the following is a direct consequence of Proposition 2.1.3 and Theorems 2.1.5 and 2.1.6.

Corollary 2.1.8. *Let D be a super-orientation of G. Then we have:*

- $\vec{\chi}(D) \leq \left\lceil \frac{\mathrm{tw}(G)+\Delta(S(D))+1}{2} \right\rceil$, *and*
- $\vec{\chi}(D) \leq \frac{1}{2}\mathrm{tw}(G) + O(\sqrt{\mathrm{Mad}(S(D)) \cdot \mathrm{tw}(G)})$, *and*
- $\vec{\chi}(D) \leq \left\lceil \frac{\mathrm{tw}(G)+4}{2} \right\rceil$ *if $S(D)$ is C_4-free.*

2.2 Preliminaries

Let $G = (V, E)$ be an undirected graph. A *perfect elimination ordering* of G is an ordering $v_1, \dots, v_n$ of its vertex-set such that, for every $i \in [n]$, the set $N(v_i) \cap \{v_{i+1}, \dots, v_n\}$ is a clique of G. We skip the proofs of the following two well-known results.

Proposition 2.2.1 (Rose [148]). *A graph G is chordal if and only if G admits a perfect elimination ordering.*

Proposition 2.2.2 (see [58, Corollary 12.4.4]). *The treewidth of a chordal graph G is exactly $\omega(G) - 1$.*

A tree-decomposition $(T, \mathcal{X})$ is *reduced* if, for every $tt' \in E(T)$, $X_t \setminus X_{t'}$ and $X_{t'} \setminus X_t$ are non-empty. It is easy to see that any graph G admits an optimal (*i.e.*, of width $\text{tw}(G)$) tree-decomposition which is reduced (indeed, if $X_t \subseteq X_{t'}$ for some edge $tt' \in E(T)$, then contract this edge and remove X_t from $\mathcal{X}$).

A tree-decomposition $(T, \mathcal{X})$ of a graph G of width $k \geq 0$ is *full* if every bag has size exactly $k + 1$. It is *valid* if $|X_t \setminus X_{t'}| = |X_{t'} \setminus X_t| = 1$ for every $tt' \in E(T)$. Note that any valid tree-decomposition is full and reduced.

The following result is well-known, see for instance [27]. We give here a short proof for sake of completeness.

Lemma 2.2.3. *Every graph $G = (V, E)$ admits a valid tree-decomposition of width* $\text{tw}(G)$.

Proof. Let $(T, \mathcal{X})$ be an optimal reduced tree-decomposition of $G = (V, E)$, which exists by the remark above the lemma. We will progressively modify $(T, \mathcal{X})$ in order to make it first full and then valid.

While the current decomposition is not full, let $tt' \in E(T)$ such that $|X_t| < |X_{t'}| = \text{tw}(G)+1$ and let $v \in X_{t'} \setminus X_t$. Add v to X_t. The obtained decomposition is still a tree-decomposition. Moreover, the updated decomposition remains reduced all along the process, as since $|X_t| < |X_{t'}|$ and the initial decomposition is reduced, $X_{t'}$ must contain another vertex $u \neq v$ with $u \notin X_t$. At the end of the process, we obtain an optimal decomposition $(T, \mathcal{X})$ that is full.

Now, while $(T, \mathcal{X})$ is not valid, let $tt' \in E(T)$, $x, y \in X_t \setminus X_{t'}$ and $u, v \in X_{t'} \setminus X_t$ (such an edge of T and four distinct vertices of V must exist since $(T, \mathcal{X})$ is full and reduced but not valid). Then, add a new node t'' to T, with corresponding bag $X_{t''} = (X_{t'} \setminus \{u\}) \cup \{x\}$ and replace the edge tt' in T by the two edges tt'' and $t''t'$. Clearly, subdividing the edge tt' by adding a bag $X_{t''} = X_{t'} \setminus \{u\} \cup \{x\}$ still leads to an optimal full tree-decomposition of the same width.

Note that, after the application of each step as described above, either the maximum of $|X_t \setminus X_{t'}|$ over all edges $tt' \in E(T)$, or the number of edges $tt' \in E(T)$ that maximise $|X_t \setminus X_{t'}|$, strictly decreases, and none of these two quantities increases. Therefore, the process terminates, and eventually $(T, \mathcal{X})$ becomes an optimal valid tree-decomposition. □

Let D_1 and D_2 be two digraphs. Let u_1v_1 be an arc of D_1 and v_2u_2 be an arc of D_2. The *directed Hajós join* of D_1 and D_2, denoted by $D_1 \triangledown D_2$, is the digraph obtained from the union $D_1 \cup D_2$ by deleting the arcs u_1v_1 as well as v_2u_2, identifying the vertices v_1 and v_2 into a new vertex v and adding the arc u_1u_2.

Theorem 2.2.4 (Bang-Jensen et al. [17] (see also [99])). *Let D_1 and D_2 be two digraphs, then*

$$\vec{\chi}(D_1 \triangledown D_2) \geq \min\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}.$$

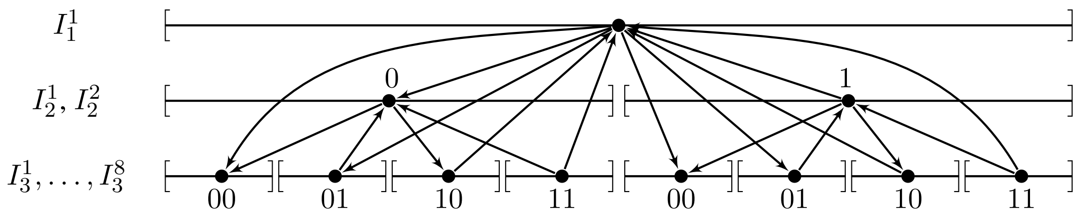


Figure 2.1: The oriented interval graph D_3 (bits of b_i^ℓ are read from left to right).

2.3 Orientations of interval graphs with large dichromatic number

This section is devoted to the proof of Theorem 2.1.1.

Theorem 2.1.1. *For every fixed $k \in \mathbb{N}$, there exists an interval graph G_k and an orientation $\vec{G}_k$ of this graph such that $\omega(G_k) = k$ and $\vec{\chi}(\vec{G}_k) \geq \left\lceil \frac{k}{2} \right\rceil$.*

Proof. Let us fix $k \in \mathbb{N}$, we will build an orientation D_k of an interval graph G_k such that $\omega(G_k) = k$ and $\vec{\chi}(D_k) \geq \left\lceil \frac{k}{2} \right\rceil$.

We start from one interval I_1^1. Then, for every i from 2 to k, we do the following: for each interval I_{i-1}^s we added at step $i-1$, we add 2^{i-1} new pairwise disjoint intervals whose union is included in I_{i-1}^s, and we associate to each of these new intervals I_i^ℓ a distinct binary number b_i^ℓ on $i-1$ bits. By construction, every new interval intersects exactly $i-1$ other intervals (one for each step).

Let G_k be the interval graph made of the intervals built above. By construction, $\omega(G_k) = k$. Now we consider D_k the orientation of G_k defined as follows. For every pair $j < i$, we orient the edge $I_j^s I_i^\ell$ from I_i^ℓ to I_j^s if the j^{th} bit of b_i^ℓ is 1, and from I_j^s to I_i^ℓ otherwise. Figure 2.1 illustrates the construction of D_3.

Let us prove that $\vec{\chi}(D_k) \geq \lceil \frac{k}{2} \rceil$. To do this, let ϕ be any optimal dicolouring of D_k. We will find a tournament T of size k in D_k such that, for each colour c in ϕ, c appears at most twice in T. This will prove that ϕ uses at least $\lceil \frac{k}{2} \rceil$ colours, implying the result.

Start from the universal vertex I_1^1. Then, for $i \in \{2, \dots, k\}$, we do the following : let I_{i-1}^s be the last vertex added to T, we will extend T with a vertex I_i^ℓ so that $I_i^\ell \subseteq I_{i-1}^s$. For each colour $c \in \phi$ that appears exactly twice in T, let $x_c y_c$ be a monochromatic arc of T coloured c. Then we choose I_i^ℓ so for each such colour c, $x_c y_c I_i^\ell$ is a directed triangle. The existence of I_i^ℓ is guaranteed by construction. This implies that the colour of I_i^ℓ in ϕ appears at most twice in T. $\square$

2.4 Orientations of cographs with large dichromatic number

This section is devoted to the proof of Theorem 2.1.2.

Theorem 2.1.2. *For every fixed $k \in \mathbb{N}$, there exists a cograph G_k and an orientation $\vec{G}_k$ of this graph such that $\vec{\chi}(\vec{G}_k) = \omega(G_k) = k$.*

Proof. We define $\vec{G}_1$ as the only orientation of G_1, the graph on one vertex. We obviously have $\vec{\chi}(\vec{G}_1) = \omega(G_1) = 1$, and G_1 is a cograph.

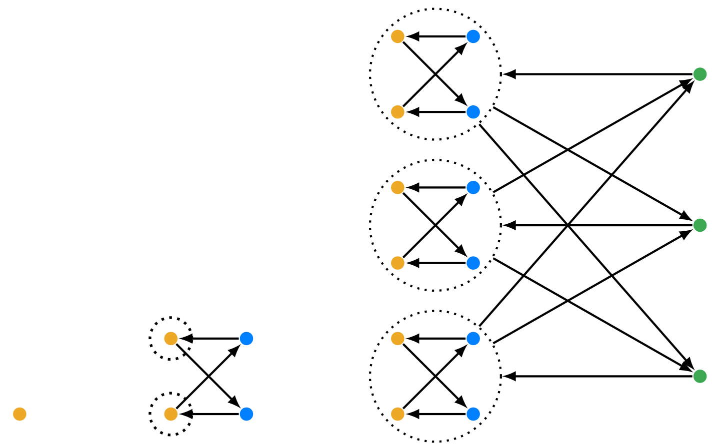


Figure 2.2: The oriented graphs $\vec{G}_1$, $\vec{G}_2$, and $\vec{G}_3$.

Let us fix $k \geq 1$, we build $\vec{G}_{k+1}$ from $\vec{G}_k$ as follows. Start from $k+1$ disjoint copies $\vec{G}_k^1, \dots, \vec{G}_k^{k+1}$ of $\vec{G}_k$ and $k+1$ new vertices $v_1, \dots, v_{k+1}$. Then, for every $i \in [k+1]$, we add all arcs from v_i to $V(\vec{G}_k^i)$ and all arc from $\bigcup_{j \neq i} V(\vec{G}_k^j)$ to v_i. Let $\vec{G}_{k+1}$ be the obtained oriented graph and G_{k+1} be its underlying graph. Figure 2.2 illustrates the construction of $\vec{G}_3$.

Note first that G_{k+1} is a cograph: the disjoint union of $G_k^1, \dots, G_k^k$ is a cograph, the independent set $v_1, \dots, v_k$ is a cograph, and G_{k+1} is the join of these two cographs. Let us prove by induction on k that $\vec{\chi}(\vec{G}_k) = \omega(G_k) = k$. For $k = 1$, the result is immediate, and assume it holds for $k \geq 1$. Note first that $\omega(G_{k+1}) = k+1$ since every clique of G_{k+1} contains at most one vertex of $\{v_1, \dots, v_{k+1}\}$ and do not contain two vertices from distinct copies of G_k. So every maximum clique of G_{k+1} is made of a maximum clique of G_k and one additional vertex v_i.

Moreover $\vec{\chi}(\vec{G}_{k+1}) \leq \chi(G_{k+1}) = \omega(G_{k+1}) = k+1$. Let us now show that the dichromatic number of $\vec{G}_{k+1}$ is at least $k+1$. Assume for the purpose of contradiction that $\vec{G}_{k+1}$ admits a k-dicolouring ϕ. Then there exist $i \neq j$ such that $\phi(v_i) = \phi(v_j)$. Since $\vec{\chi}(\vec{G}_k) \geq k$, there exist $x \in V(\vec{G}_k^i)$ and $y \in V(\vec{G}_k^j)$ such that $\phi(x) = \phi(y) = \phi(v_i) = \phi(v_j)$. Hence $v_i x v_j y v_i$ is a monochromatic $\vec{C}_4$ of $\vec{G}_{k+1}$ coloured with ϕ, a contradiction. □

2.5 Digraphs with a symmetric part having bounded maximum degree

This section is devoted to the proofs of Proposition 2.1.3 and Theorem 2.1.4.

Proposition 2.1.3. *Let D be a super-orientation of a chordal graph G, then*

$$\vec{\chi}(D) \leq \left\lceil \frac{\omega(G) + \Delta(S(D))}{2} \right\rceil .$$

Proof. Let $v_1, \dots, v_n$ be a perfect elimination ordering of G (which exists by Proposition 2.2.1). Then, in G, every vertex v_i has at most $\omega(G) - 1$ neighbours in $\{v_{i+1}, \dots, v_n\}$. Hence, in $D\langle\{v_i, \dots, v_n\}\rangle$, $d^+(v_i) + d^-(v_i) \leq \omega(G) - 1 + \Delta(S(D))$.

Thus, considering the vertices from v_n to v_1, we can greedily find a dicolouring of D using at most $\left\lceil \frac{\omega(G)+\Delta(S(D))}{2} \right\rceil$ by choosing for v_i a colour that is not appearing in $N^+(v_i) \cap \{v_{i+1}, \dots, v_n\}$ or in $N^-(v_i) \cap \{v_{i+1}, \dots, v_n\}$. □

Theorem 2.1.4. *For every fixed $k, \ell \in \mathbb{N}$ such that $k \geq \ell+1$, there exists a chordal graph $G_{k,\ell}$ and a super-orientation $D_{k,\ell}$ of $G_{k,\ell}$ such that $\omega(G_{k,\ell}) = k$, $\Delta(S(D_{k,\ell})) = \ell$ and $\vec{\chi}(D_{k,\ell}) = \left\lceil \frac{k+\ell}{2} \right\rceil$.*

Proof. Let us fix $\ell \in \mathbb{N}$. We define $D_{\ell+1,\ell}$ as the bidirected complete graph on $\ell+1$ vertices. Note that $D_{\ell+1,\ell}$ clearly satisfies the desired properties.

Then, for every $k \geq \ell+2$, we iteratively build $D_{k,\ell}$ from $D_{k-1,\ell}$ or $D_{k-2,\ell}$ as follows:

- If $k+\ell$ is even, we just add a dominating vertex to $D_{k-1,\ell}$ to construct $D_{k,\ell}$. We obtain that $\omega(\mathrm{UG}(D_{k,\ell})) = 1 + \omega(\mathrm{UG}(D_{k-1,\ell})) = k$, $\Delta(S(D_{k,\ell})) = \Delta(S(D_{k-1,\ell})) = \ell$ and $\vec{\chi}(D_{k,\ell}) = \vec{\chi}(D_{k-1,\ell}) = \left\lceil \frac{k+\ell-1}{2} \right\rceil = \left\lceil \frac{k+\ell}{2} \right\rceil$ (the last equality holds because $k+\ell$ is even).
- If $k+\ell$ is odd (implying that k is at least $\ell+3$), we start from T, a copy of $TT_{\frac{k+\ell+1}{2}}$, the transitive tournament on $\frac{k+\ell+1}{2}$ vertices. Note that $\frac{k+\ell+1}{2} \leq k-1$ because $k \geq \ell+3$.

  For each arc xy in T, we add a copy D^{xy} of $D_{k-2,\ell}$ with all arcs from y to D^{xy} and all arcs from D^{xy} to x. Let $D_{k,\ell}$ be the obtained digraph.

  First, $\mathrm{UG}(D_{k,\ell})$ is chordal because it has a perfect elimination ordering: we first eliminate each copy D^{xy} of $D_{k-2,\ell}$, which is possible because $\mathrm{UG}(D_{k-2,\ell})$ is chordal, and x, y are adjacent to every vertex of D^{xy}. When every copy of $D_{k-2,\ell}$ is eliminated, the remaining digraph is T, which is clearly chordal because it is a tournament.

  Next, we have $\omega(\mathrm{UG}(D_{k,\ell})) = \max(\omega(\mathrm{UG}(T)), \omega(\mathrm{UG}(D_{k-2,\ell})) + 2) = k$, and $\Delta(S(D_{k,\ell})) = \Delta(S(D_{k-2,\ell})) = \ell$.

  Finally, let us show that $\vec{\chi}(D_{k,\ell}) \geq \frac{k+\ell+1}{2}$ (the equality then comes from Proposition 2.1.3). In order to get a contradiction, assume that ϕ is a dicolouring of $D_{k,\ell}$ that uses at most $\frac{k+\ell-1}{2}$ colours. We know by induction that each copy of $D_{k-2,\ell}$ uses all the colours in ϕ. Since T is a tournament on $\frac{k+\ell+1}{2}$ vertices, we know that it must contain a monochromatic arc xy. Now let z be a vertex in D^{xy} such that $\phi(x) = \phi(y) = \phi(z)$, then xyz is a monochromatic triangle, a contradiction.

Figure 2.3 illustrates the construction of $D_{1,0}$, $D_{3,0}$ and $D_{5,0}$. □

2.6 Digraphs with a symmetric part having bounded maximum average degree

This section is devoted to the proof of Theorem 2.1.5. We first need to prove the following.

Lemma 2.6.1. *Let $G = (V, E)$ be a chordal graph. There exists an ordering $a_1, \dots, a_n$ of V such that for any $k \in [n]$:*

$$|N(a_k)| \leq \omega(G) + k - 2 \tag{P1}$$

$$\text{and} \quad \left| \bigcup_{i=1}^{k} N[a_i] \right| \leq \omega(G) + 2k - 1 \tag{P2}$$

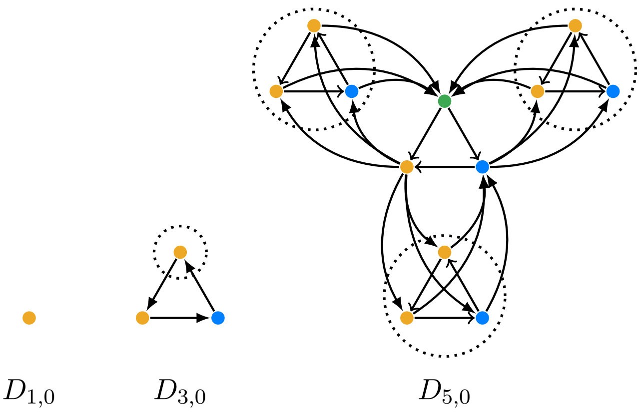


Figure 2.3: The digraphs $D_{1,0}$, $D_{3,0}$, and $D_{5,0}$.

Proof. Let $(T = (I, F), \mathcal{X} = (B_u)_{u\in I})$ be a valid tree-decomposition of G of width $\omega(G) - 1$, which exists by Lemma 2.2.3 (recall that $\text{tw}(G) = \omega(G) - 1$ by Proposition 2.2.2). One can easily show that, since T is valid, $|I| = n - \omega(G) + 1$ (see [27, Lemma 2.5]).

Let $P = u_0, \dots, u_r$ be a longest path in T. We root T in u_r. For any vertex u of T different from u_r, $\text{father}(u)$ denotes the father of u in T.

We now consider a Depth-First Search of T from u_r. The vertices of P have the priority. Along this route, we label the vertices of T. A vertex is labelled when all of its children are labelled. We denote by $v_1, \dots, v_{n-\omega(G)+1}$ the vertices of T in this labelling. Note that v_1 corresponds to u_0 and $v_{n-\omega(G)+1}$ corresponds to u_r.

Now, for each $i \in [n - \omega(G)]$, we denote by a_i the unique vertex of G that belongs to B_{v_i} but not to $\text{father}(B_{v_i})$ (recall that $T, \mathcal{X}$ is valid so a_i is well-defined). We finally label $a_{n-\omega(D)+1}, \dots, a_n$ the remaining vertices of G in B_{u_r} in an arbitrary way. See Figure 2.4 for an illustration.

We will now prove that $(a_i)_{1\le i\le n}$ satisfies the two properties of the statement. First observe that, for every $i \in [n]$, $N(a_i) \subseteq \{a_1, \dots, a_{i-1}\} \cup X_{v_i}$ because $a_i \notin \bigcup_{j=i+1}^{n-\omega(G)+1} X_{v_j}$. Hence we have $|N(a_i)| \le i - 1 + \omega - 1 = \omega(G) - 2 + i$, which shows (P1).

To show that (P2) holds, we fix $k \in [n]$. Note that the result is trivially true when $k \ge n - \omega + 1$, thus we assume that $k \le n - \omega$. Hence, both v_k and $\text{father}(v_k)$ are well-defined. We set $X_T = \{v_1, \dots, v_k\}$, $X_G = \{a_1, ..., a_k\}$ and we let T' be the smallest subtree of T that contains all vertices of X_T. Let ℓ be the largest integer such that u_ℓ belongs to $V(T')$ (ℓ is well-defined because T' contains $v_1 = u_0$). We root T' in u_ℓ.

We will now show that T' contains at most $2k$ vertices. If $u_\ell = v_k$, then the vertices of T' are exactly $\{v_1, \dots, v_k\}$ and this is clear. Otherwise, let us show that $T'' = T' \setminus X_T$ contains at most k vertices, and we will get the result since $|X_T| = k$. By construction, we know that every descendant of a vertex v_i is labelled less than i. Hence, $T'' = T' \setminus X_T$ is a tree rooted in u_ℓ.

Assume first that T'' contains at least two leaves f_1 and f_2 different from u_ℓ (u_ℓ may be a leaf if it has only one child). We denote by P_1 and P_2 two paths from their lowest common ancestor. Without loss of generality, we assume that f_1 is before f_2 in $(v_1, \dots, v_n)$. Since f_2 has a child g_2 in X_T and by construction of $(v_i)_{1\le i\le n}$, the internal vertices of P_1 are before g_2 in $(B_1, \dots, B_n)$. This implies that all internal vertices in P_1 must belong to X_T, which contradicts the existence of f_1. This shows that T'' must have exactly two leaves (one of them is u_ℓ) and then T'' is a path

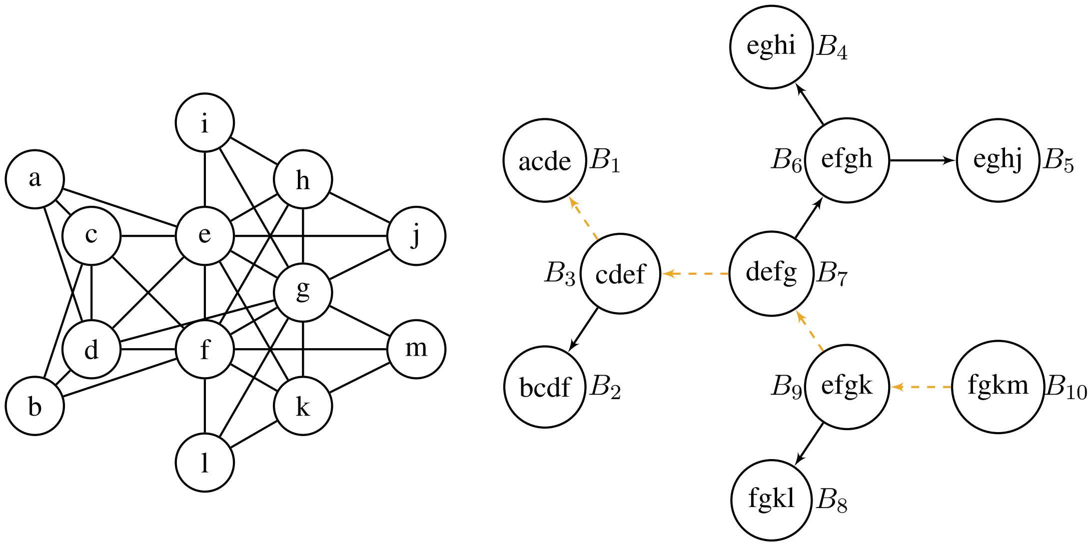


Figure 2.4: A chordal graph G (on the left) and its valid tree-decomposition T (on the right). The orange dashed arcs represent the chosen maximum path P. The ordering $a_1, \dots, a_n$ of $V(G)$ we built is $a, b, c, i, j, h, d, l, e, f, g, k, m$.

rooted in u_ℓ. Since P is a longest path in T, we get that $n(T'') \leq \ell \leq k$ and T' contains at most $2k$ vertices as desired.

We now consider the set $N_G = \{a_j \in V(G) \mid v_j \in V(T') \setminus \{u_\ell\}\}$. Let x be any vertex in X_G. Then every neighbour of x must belong to some bag in T'. Moreover, if a vertex belongs to a bag of T', then either it belongs to B_{u_ℓ} or it belongs to N_G. Then the neighbourhood of x is a subset of $N_G \cup B_{u_\ell}$. Furthermore, x itself belongs to N_G. Since x is any vertex in X_G, we have:

$$\bigcup_{x \in X_G} N[x] \subseteq (N_G \cup B_{u_\ell})$$

Since $|N_G| \leq 2k - 1$ and $|B_{u_\ell}| = \omega - 1$, we get (P2). □

In order to prove Theorem 2.1.5, we prove the more general following result.

Theorem 2.6.2. *Let D be a super-orientation of a chordal graph G such that $\mathrm{Mad}(S(D)) \leq d$. For every $\varepsilon > 0$, we have*

$$\vec{\chi}(D) \leq \left(\frac{1+\varepsilon}{2}\right)\omega(G) + \frac{d}{\varepsilon} + 1.$$

Proof. Let $\varepsilon > 0$ and $d \geq 1$, we assume that $\varepsilon \leq 1$ for otherwise the result is trivial. We fix $c_{d,\varepsilon} = \max\left(\left\lceil \frac{d}{2\varepsilon} \right\rceil, \frac{3}{4}d + \frac{d}{8\varepsilon} + \frac{1}{2}\right)$. Easy calculations imply $c_{d,\varepsilon} \leq \frac{d}{\varepsilon} + 1$. We will show that every super-orientation D of a chordal graph G with $\mathrm{Mad}(S(D)) \leq d$ satisfies

$$\vec{\chi}(D) \leq \left(\frac{1+\varepsilon}{2}\right)\omega(G) + c_{d,\varepsilon}.$$

We prove it by contradiction, so assume that $D = (V, A)$ is a smaller counterexample, meaning that $\vec{\chi}(D) > \left(\frac{1+\varepsilon}{2}\right)\omega(G) + c_{d,\varepsilon}$. Thus, D must be vertex-dicritical (meaning that $\vec{\chi}(H) <$

$\vec{\chi}(D)$ for every induced subdigraph H of D), for otherwise there exists a vertex $x \in V$ such that $\vec{\chi}(D-x) = \vec{\chi}(D)$, and $D-x$ would be a smaller counterexample.

For the simplicity of notations, from now on, we write ω for $\omega(G)$. Let v be any vertex of D and α be any optimal dicolouring of $D-v$ (meaning that α uses exactly $\vec{\chi}(D)-1$ colours). Then α cannot be extended to D without using a new colour for v (because D is dicritical). Since every digon (incident to v) may forbid at most one colour at v, and each pair of simple arcs (incident to v) may forbid at most one colour at v, we get the following inequalities with $d^{\pm}(v)$ denoting the number of digons incident to v:

$$
\begin{aligned}
& d^{\pm}(v) + \frac{|N(v)| - d^{\pm}(v)}{2} \geq \vec{\chi}(D) - 1 > \left(\frac{1+\varepsilon}{2}\right)\omega + c_{d,\varepsilon} - 1 \\
\Rightarrow \quad & d^{\pm}(v) > (1+\varepsilon)\omega + 2c_{d,\varepsilon} - 2 - |N(v)| \qquad (2.1)
\end{aligned}
$$

Note that these inequalities hold for every vertex v of D. By Lemma 2.6.1, there is an ordering $a_1, \ldots, a_n$ of $V(D)$ such that, for any $i \in [n]$,

$$|N(a_i)| \leq \omega + i - 2 \qquad \text{(P1)}$$

$$\text{and} \quad \left|\bigcup_{j=1}^{i} N(a_j)\right| \leq \omega + 2i - 1 \qquad \text{(P2)}$$

Let us fix $i = \left\lceil \frac{d}{2\varepsilon} \right\rceil$. Note that $i \leq c_{d,\varepsilon}$. Thus, since $\vec{\chi}(D) > c_{d,\varepsilon}$, we obviously have $i \leq n$. Let $X = \{a_j \mid j \leq i\}$ and $W = \bigcup_{j=1}^{i} N[a_j]$. Together with inequality (2.1), property (P1) implies, for every $j \in [i]$, $d^{\pm}(a_j) > \varepsilon\omega + 2c_{d,\varepsilon} - j$. Hence we get:

$$\sum_{v \in X} d^{\pm}(v) = \sum_{j=1}^{i} d^{\pm}(a_j) > \varepsilon\omega i + 2c_{d,\varepsilon} i - \frac{i(i+1)}{2} \qquad (2.2)$$

By (P2), we know that $|W| \leq \omega + 2i - 1$. Thus $D\langle W \rangle$ contains at most $\frac{d}{2}(\omega + 2i - 1)$ digons. Similarly, since $|X| = i$, $D\langle X \rangle$ contains at most $\frac{di}{2}$ digons. When we sum $d^{\pm}(v)$ over all vertices v in X, we count exactly once every digon between X and $W \setminus X$, and exactly twice every digon in X. Then, the following is a consequence of (2.2), with $\mathrm{dig}(H)$ denoting the number of digons in a digraph H.

$$
\begin{aligned}
\varepsilon\omega i + 2c_{d,\varepsilon} i - \frac{i(i+1)}{2} < \sum_{v \in X} d^{\pm}(v) & \leq \mathrm{dig}(D\langle W \rangle) + \mathrm{dig}(D\langle X \rangle) \\
& \leq \frac{d}{2}(\omega + 2i - 1) + \frac{di}{2}
\end{aligned}
$$

Since $i = \left\lceil \frac{d}{2\varepsilon} \right\rceil$, we conclude that $c_{d,\varepsilon} < \frac{3}{4}d + \frac{d}{8\varepsilon} + \frac{1}{2}$, a contradiction. □

The proof of Theorem 2.1.5 now follows.

Theorem 2.1.5. *Let D be a super-orientation of a chordal graph G. If $\mathrm{Mad}(S(D)) \leq d$, then*

$$\vec{\chi}(D) \leq \frac{1}{2}\omega(G) + O\left(\sqrt{d \cdot \omega(G)}\right).$$

Proof. This is a direct consequence of Theorem 2.6.2 applied for $\varepsilon = \sqrt{\frac{d}{\omega(G)}}$. □

2.7 Digraphs with no bidirected cycle of length four

This section is devoted to the proof of Theorems 2.1.6 and 2.1.7.

Theorem 2.1.6. *Let D be a super-orientation of a chordal graph G. If $S(D)$ is C_4-free, then*

$$\vec{\chi}(D) \leq \left\lceil \frac{\omega(G)+3}{2} \right\rceil.$$

Proof. We assume that $\omega = \omega(G)$ is odd, for otherwise we select an independent set I of D such that $D' = D - I$ satisfies $\omega(\mathrm{UG}(D')) = \omega - 1$, so $\omega(\mathrm{UG}(D'))$ is odd and $\vec{\chi}(D) \leq \vec{\chi}(D') + 1$ (the existence of I is guaranteed because G is chordal).

Let $(T, \mathcal{X} = (B_u)_{u \in V(T)})$ be a valid tree-decomposition of G, that is each bag $B \in \mathcal{X}$ has size exactly ω and, for every two adjacent bags B and B', $|B \setminus B'| = 1$. Recall that the existence of such a tree-decomposition is guaranteed by Lemma 2.2.3. We assume that each bag induces a clique on G, otherwise we just add the missing arcs (oriented in an arbitrary direction). Note that this operation does not increase ω nor decrease $\vec{\chi}(D)$ and does not create any $\overleftrightarrow{C_4}$. Note also that D remains chordal after this operation.

Let $k = \frac{\omega+3}{2}$. A k-dicolouring ϕ of D is *balanced* if, for each bag B and colour $c \in [k]$, $0 \leq |\phi^{-1}(c) \cap B| \leq 2$. Note that every balanced k-dicolouring satisfies $|\phi^{-1}(c) \cap B| = 1$ for either 1 or 3 colours. Moreover, in the former case, exactly one colour of $[k]$ is missing in $\phi(B)$. We will show that $\vec{\chi}(D) \leq k$ by proving the existence of a balanced k-dicolouring ϕ of D such that, for each bag B, we have:

(i) $|\phi^{-1}(c) \cap B| = 1$ holds for exactly one colour c, or

(ii) $|\phi^{-1}(c_i) \cap B| = 1$ holds for exactly three distinct colours c_1, c_2, c_3 and two vertices of $\{v_1, v_2, v_3\}$ are connected by a $\overleftrightarrow{P_3}$ in D (where $\{v_i\} = \phi^{-1}(c_i) \cap B$ and a $\overleftrightarrow{P_3}$ is a bidirected path on 3 vertices).

We will say that a bag B is of type (i) or (ii), depending if ϕ satisfies condition (i) or (ii) respectively on B.

We show the existence of ϕ by induction on the number of bags in the tree-decomposition. If $n(T) = 1$, let $\mathcal{X} = \{B\}$, then D is a semi-complete digraph on ω vertices which is $\overleftrightarrow{C_4}$-free. We construct ϕ greedily as follows: choose a simple arc uv such that both u and v have not been coloured yet, and use a new colour for them. At the end, there are either one or three uncoloured vertices. If there is only one, we just use a new colour for it and B is of type (i), for otherwise the three remaining vertices induce a bidirected triangle on D and we can use one new colour for each of them, so B is of type (ii).

Assume now that $n(T) \geq 2$. Let x be a leaf of T and y its only neighbour in T. Let $\{u\} = B_y \setminus B_x$ and $\{v\} = B_x \setminus B_y$. By induction, with $D - v$ and $(T - x, \mathcal{X} \setminus B_x)$ playing the role of D and $(T, \mathcal{X})$ respectively, there exists a balanced k-dicolouring ϕ of $D - v$ for which each bag is of type (i) or (ii). We will show by a case analysis that ϕ can be extended to v.

- Assume first that B_y is of type (i), and let r be the only vertex alone in its colour class in $D\langle B_y \rangle$. If $r = u$, then we set $\phi(v) = \phi(u)$ and ϕ is a balanced k-dicolouring of D with B_x being of type (i). Henceforth assume $u \neq r$. Let w be the neighbour of u in B_y such that $\phi(w) = \phi(u)$. Since u and v are not adjacent, setting $\phi(v) = \phi(u)$ yields a balanced

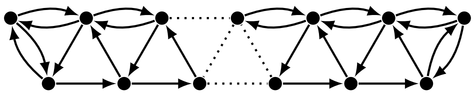


Figure 2.5: The digraph $D_{3,n}$.

k-dicolouring of D, with B_x being of type *(i)*, except if w and v are linked by a digon. Analogously, setting $\phi(v) = \phi(r)$ yields a balanced k-dicolouring of D, with B_x being of type *(i)* since $|\phi^{-1}(c) \cap B_x| = 1$ holds only for $c = \phi(w)$, except if r and v are linked by a digon.

But then, if both $[v, w]$ and $[v, r]$ are digons, we can set $\phi(v)$ to the missing colour of $\phi(B_y)$. Then ϕ is a balanced k-dicolouring of D with B_x being of type *(ii)*, since $|\phi^{-1}(c) \cap B_x| = 1$ holds exactly for every $c \in \{\phi(w), \phi(v), \phi(r)\}$ with r, w being connected by a $\overleftrightarrow{P_3}$ in D.

- Henceforth, assume that B_y is of type *(ii)* and let r, s, t be the only vertices alone in their colour class in $D\langle B_y \rangle$ such that s and t are connected by a $\overleftrightarrow{P_3}$ in $D - v$. If $u = r$, then we set $\phi(v) = \phi(u)$ and ϕ is a balanced k-dicolouring of D with B_x being of type *(ii)*.

  Assume now that $u \in \{s, t\}$. Without loss of generality, we assume that $u = s$. If r and v are not linked by a digon, we can set $\phi(v) = \phi(r)$ and ϕ is a balanced k-dicolouring of D with B_x being of type *(i)*. The same argument holds if t and v are not linked by a digon. But if both $[v, r]$ and $[v, t]$ are digons, we can set $\phi(v) = \phi(s)$. Then ϕ is a balanced k-dicolouring of D with B_x being of type *(ii)*, since $|\phi^{-1}(c) \cap B_x| = 1$ holds exactly for every $c \in \{\phi(v), \phi(r), \phi(t)\}$ with r, t being connected by a $\overleftrightarrow{P_3}$ in D.

  Assume finally that $u \notin \{r, s, t\}$ and let w be the neighbour of u in B_y such that $\phi(w) = \phi(u)$. If r and v are not linked by a digon, we can set $\phi(v) = \phi(r)$ and ϕ is a balanced k-dicolouring of D with B_x being of type *(ii)*, where $|\phi^{-1}(c) \cap B_x| = 1$ holds exactly for every $c \in \{\phi(w), \phi(s), \phi(t)\}$ with s, t being connected by a $\overleftrightarrow{P_3}$ in $D - v$. The same argument holds if v and w are not linked by a digon. Henceforth we assume that both $[v, w]$ and $[v, r]$ are digons. Since D is $\overleftrightarrow{C_4}$-free, and because s, t are connected by a $\overleftrightarrow{P_3}$ in $D - v$, we know that either $[v, s]$ or $[v, t]$ is not a digon of D. Assume without loss of generality that $[v, s]$ is not, then we set $\phi(v) = \phi(s)$. Then ϕ is a balanced k-dicolouring of D with B_x being of type *(ii)*, since $|\phi^{-1}(c) \cap B_x| = 1$ holds exactly for every $c \in \{\phi(w), \phi(r), \phi(t)\}$ with w, r being connected by a $\overleftrightarrow{P_3}$ in D. □

Theorem 2.1.7. *For every fixed $k \geq 3$ and every $n \geq \mathbb{N}$, there exists a super-orientation $D_{k,n}$ of a chordal graph $G_{k,n}$ on at least n vertices such that $S(D_{k,n})$ is a disjoint union of paths, $\omega(G_{k,n}) = k$ and $\vec{\chi}(D_{k,n}) = \left\lfloor \frac{k+3}{2} \right\rfloor$.*

Proof. We only have to prove it for $k = 3$. For larger values of k, we build $D_{k,n}$ from $D_{k-1,n}$ or $D_{k-2,n}$ as in the proof of Theorem 2.1.4. The digraph $D_{3,n}$, depicted in Figure 2.5, is clearly a super-orientation of a 2-tree. As a consequence of Theorem 2.2.4, it has dichromatic number 3, since it is obtained from successive Hajós joins applied on $\overleftrightarrow{K_3}$. □

2.8 Further research directions

In this chapter, we gave both lower and upper bounds on the dichromatic number orientations and super-orientations of different classes of chordal graphs and cographs. Plenty of questions arise and we detail a few of them.

First, we do not know if the bound of Theorem 2.1.5 is optimal, and we ask the following.

Question 2.8.1. *Does there exist a computable function f such that every super-orientation D of a chordal graph G satisfies $\vec{\chi}(D) \le \frac{1}{2}\omega(G) + f(\mathrm{Mad}(S(D)))$?*

We also ask if Theorem 2.1.6 is true not only for $\overleftrightarrow{C_4}$-free digraphs but for every $\overleftrightarrow{C_\ell}$-free digraphs.

Question 2.8.2. *For every $\ell \ge 3$, does there exist $k_\ell \in \mathbb{N}$ such that every $\overleftrightarrow{C_\ell}$-free super-orientation D of a chordal graph G satisfies $\vec{\chi}(D) \le \frac{1}{2}\omega(G) + k_\ell$?*

A famous class of graphs is the class of claw-free graphs (a graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph). Line-graphs and proper interval graphs are examples of claw-free graphs. We ask the following.

Problem 2.8.3. *Let $\vec{G}$ be an orientation of a claw-free graph G, then $\vec{\chi}(\vec{G}) = O\left(\frac{\omega(G)}{\log \omega(G)}\right)$.*

Recall that a celebrated conjecture of Erdős and Neumann-Lara (Conjecture 1.2.13) states that every orientation $\vec{G}$ of a graph G satisfies $\vec{\chi}(\vec{G}) = O\left(\frac{\Delta(G)}{\log \Delta(G)}\right)$. Since every claw-free graph G satisfies $\Delta(G) \le 2\omega(G) - 2$, the problem above is a consequence of Erdős and Neumann-Lara's conjecture.

CHAPTER 3

On the Directed Brooks Theorem

This chapter contains joint work with Daniel Gonçalves and Amadeus Reinald and is based on [139, 82].

3.1 Brooks Theorem and its generalisations

As mentioned in the introduction of this thesis, a simple greedy procedure shows that every graph G satisfies $\chi(G) \leq \Delta(G)+1$. Brooks Theorem [44] is a fundamental theorem in graph colouring that characterises exactly the connected graphs for which equality holds.

Theorem 3.1.1 (BROOKS THEOREM)**.** *A connected graph G satisfies $\chi(G) = \Delta(G) + 1$ if and only if G is an odd cycle or a complete graph.*

Brooks Theorem has been generalised in many ways. We refer the interested reader to the recent book of Stiebitz, Schweser, and Toft [162] dedicated to Brooks Theorem and its generalisations. One such generalisation is about list colouring. Given a graph G, a *list assignment* L is a function which associates a list of colours to every vertex v of G. An *L-colouring* of G is a proper colouring α of G such that $\alpha(v) \in L(v)$ for every vertex v.

A *block* B in a graph G (resp. a digraph D) is a maximal 2-connected subgraph of G (resp. subdigraph of D). Recall that K_2 is 2-connected by definition. A *cut-vertex* of G (resp. D) is a vertex x such that $G - v$ (resp. $D - v$) is disconnected. A block B of G (resp. D) is an *end-block* if it contains exactly one cut-vertex of G (resp. D). A *Gallai tree* is a connected graph in which every block is either a complete graph or an odd cycle. The following was obtained independently by Borodin [38] and by Erdős, Rubin, and Taylor [71].

Theorem 3.1.2 (Borodin [38] ; Erdős et al. [71])**.** *Let G be a connected graph and L be a list assignment of G such that, for every vertex $v \in V$, $|L(v)| \geq d(v)$. If G is not L-colourable, then G is a Gallai tree and $|L(v)| = d(v)$ for every vertex v.*

Note that this is actually a generalisation of Brooks Theorem: if G is a connected graph of maximum degree Δ, then G is Δ-colourable if and only if it is L-colourable where $L(v) = [\Delta]$ for every vertex v of G. Hence, if it is not Δ-colourable, Theorem 3.1.2 implies that G must be a Δ-regular Gallai tree, that is a complete graph or an odd cycle.

Another independent generalisation of Brooks Theorem is about partitioning into degenerate subgraphs. Given an integer d, a graph G is *d-degenerate* if every non-empty subgraph of G contains a vertex of degree at most d. Note that X is an independent set of a graph G if and only if $G\langle X\rangle$ is 0-degenerate. Let $P = (p_1, \dots, p_s)$ be a sequence of positive integers. A graph G is *P-colourable* if there exists an s-colouring α of G such that, for every $i \in [s]$, the subgraph of G induced by the colour class $\alpha^{-1}(i)$ is $(p_i - 1)$-degenerate. When $p_1 = \cdots = p_s = 1$, observe that a P-colouring is exactly a proper s-colouring. Borodin [37], and Bollobás and Manvel [30] independently proved the following.

Theorem 3.1.3 (Borodin [37] ; Bollobás and Manvel [30])**.** *Let G be a connected graph with maximum degree Δ and $P = (p_1, \dots, p_s)$ be a sequence of $s \geq 2$ non-negative integers such that $\sum_{i=1}^{s} p_i \geq \Delta$. If G is not P-colourable, then $\sum_{i=1}^{s} p_i = \Delta$ and G is a complete graph or an odd cycle.*

Brooks Theorem follows from Theorem 3.1.3 by setting $s = \Delta$ and $p_i = 1$ for every $i \in [\Delta]$.

Given these two independent generalisations of Brooks Theorem, one can naturally ask for an even more general theorem subsuming both Theorems 3.1.2 and Theorem 3.1.3. Such a result has been proved by Borodin, Kostochka, and Toft when they introduced the notion of variable degeneracy in [39].

Let G be a graph and $f\colon V \to \mathbb{N}$ be a function. We say that G is *strictly-f-degenerate* if every subgraph H of G contains a vertex v such that $d(v) < f(v)$. Given an integer $s \geq 1$ and a sequence of functions $F = (f_1, \dots, f_s)$, we say that G is *F-colourable* if there exists an s-colouring α of G such that, for every $i \in [s]$, the subgraph of G induced by $\alpha^{-1}(i)$ is strictly-f_i-degenerate. Also we say that (G, F) is a *hard pair* if one of the following conditions holds:

- G is 2-connected and there exists $i \in [s]$ such that, for every vertex $v \in V(G)$, $f_i(v) = d(v)$ and $f_k(v) = 0$ when $k \neq i$.

- G is an odd cycle and all the f_is are constant equal to 0, except exactly two that are constant equal to 1.

- G is a complete graph, all the f_is are constant and for every vertex v, $f_1(v) + \dots + f_s(v) = n(G) - 1$.
- (G, F) is obtained from two hard pairs (G^1, F^1) and (G^2, F^2) by identifying two vertices $x_1 \in V(G^1)$ and $x_2 \in V(G^2)$ into a new vertex $x \in V(G)$, such that for every vertex $v \in V(G)$ and every $k \in [s]$ we have:

$$f_k(v) = \begin{cases} f_k^1(v) & \text{if } v \in V(G_1) \setminus \{x_1\} \\ f_k^2(v) & \text{if } v \in V(G_2) \setminus \{x_2\} \\ f_k^1(v) + f_k^2(v) & \text{if } v = x. \end{cases}$$

where $F^1 = (f_1^1, \dots, f_s^1)$ and $F^2 = (f_1^2, \dots, f_s^2)$.

The following generalises both Theorems 3.1.2 and 3.1.3.

Theorem 3.1.4 (Borodin, Kostochka, and Toft [39]). *Let G be a connected graph and $F = (f_1, \dots, f_s)$ be a sequence of functions such that, for every vertex $v \in V(G)$, $\sum_{i=1}^{s} f_i(v) \geq d(v)$. Then G is F-colourable if and only if (G, F) is not a hard pair.*

Note that Theorem 3.1.3 is obtained from the result above by setting f_i to the constant function equal to $p_i + 1$ for every $i \in [s]$. On the other hand, given a graph G and a list assignment L of G, one can set $f_i(v)$ to 1 when $i \in L(v)$ and 0 otherwise to obtain Theorem 3.1.2.

We mention that Theorem 3.1.4 has been extended to hypergraphs, the interested reader on these questions is referred to [154] and [153]. It has also been generalised to correspondence colouring (which is also known as DP-colouring), see [110].

As mentioned in the introduction of this thesis, Brooks theorem has been first generalised to digraphs by Harutyunyan and Mohar in [91] as follows.

Theorem 3.1.5 (Directed Brooks Theorem). *Let D be a connected digraph. Then $\vec{\chi}(D) \leq \Delta_{\max}(D) + 1$ and equality holds if and only if one of the following occurs:*

- *D is a directed cycle, or*
- *D is a bidirected odd cycle, or*
- *D is a bidirected complete graph (of order at least 4).*

Harutyunyan and Mohar actually extended Theorem 3.1.2 to digraphs. A *directed Gallai tree* is a digraph in which every block is a directed cycle, a bidirected odd cycle or a bidirected complete graph. Given a list assignment of a digraph D, an *L-dicolouring* α is a dicolouring of D such that $\alpha(v) \in L(v)$ holds for every vertex v of D. If D admits an L-dicolourable, then D is *L-dicolourable*.

Theorem 3.1.6 (Harutyunyan and Mohar [91]). *Let D be a connected digraph and L a list assignment of D such that $|L(v)| \geq d_{\max}(v)$. If D is not L-dicolourable, then D is a directed Gallai tree and $|L(v)| = d_{\max}(v)$ for every vertex v.*

The notion of variable degeneracy and Theorem 3.1.4 have been extended to digraphs by Bang-Jensen, Schweser, and Stiebitz [19]. In Section 3.2, we propose another extension of variable degeneracy for digraphs, namely the *bivariable degeneracy*. With this new definition, we prove a

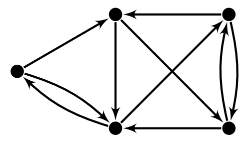


Figure 3.1: A digraph which is strictly-$(2,0)$-degenerate while it is not strictly-$(0,2)$-degenerate.

more general result than the one of [19], with a new proof based on ear decompositions. Moreover, our algorithmic proof justifies the existence of a linear-time algorithm for deciding whether a digraph verifying some conditions is F-colourable and computing such a colouring when it exists. It can thus be derived into linear-time algorithms for all the results presented in this section and their extensions to digraphs, which was known for Theorem 3.1.3 [54], but not for Theorem 3.1.4 to the best of the author's knowledge.

In Section 3.3, we discuss strengthenings of Theorem 3.1.5 when restricted to oriented graphs. On this class of digraphs, we prove that an analogue of Theorem 3.1.5 can be obtained when replacing $\Delta_{\max}(D)$ by $\Delta_{\min}(D)$. We actually prove a more general statement on the list colouring of digraphs with no large bidirected cliques. We also give an NP-hardness result to justify that our result is in some sense best possible.

3.2 Partitioning digraphs into degenerate subdigraphs in linear time

Let $D = (V, A)$ be a digraph and $f : V \to \mathbb{N}^2$ be a function. We denote by f^- (resp. f^+) the projection of f on the first (resp. second) coordinate. Such a function is said to be *symmetric* if $f^-(v) = f^+(v)$ for every $v \in V$. For two elements $p = (p_1, p_2)$ $q = (q_1, q_2)$ of $\mathbb{N}^2$, we denote by $p \le q$ the relation $p_1 \le q_1$ and $p_2 \le q_2$. We denote by $p < q$ the relation $p \le q$ and $p \ne q$.

We say that D is *strictly-f-degenerate* if every subdigraph H of D has a vertex v such that $d_H^-(v) < f^-(v)$ or $d_H^+(v) < f^+(v)$. Let s be a positive integer and $F = (f_1, \dots, f_s)$ be a sequence of functions $f_i : V \to \mathbb{N}^2$. The digraph D if *F-dicolourable* if there exists an s-colouring α of D such that, for every $i \in [s]$, the subdigraph of D induced by the colour class $\alpha^{-1}(i)$ is strictly-f_i-degenerate. Such a colouring α is called an *F-dicolouring*.

Observe that a digraph is acyclic if and only if it is strictly-$(0,1)$-degenerate (*i.e.* strictly-f-degenerate for f being the constant function equal to $(0,1)$). Although acyclic digraphs are also exactly the strictly-$(1,0)$-degenerate digraphs, this does not generalise to higher values. For instance, the digraph illustrated in Figure 3.1 is strictly-$(2,0)$-degenerate while it is not strictly-$(0,2)$-degenerate.

Deciding if a digraph is F-dicolourable is clearly NP-hard because it includes a large collection of NP-hard problems for specific values of F. For instance, deciding if a digraph D has dichromatic number at most 2, which is shown to be NP-hard in [50], consists exactly of deciding whether D is (f_1, f_2)-dicolourable where f_1, f_2 are the constant functions equal to $(1,0)$. We thus restrict ourselves to pairs (D, F) verifying the following property:

$$\forall v \in V(D), \quad \sum_{i=1}^{s} f_i^-(v) \ge d^-(v) \quad \text{and} \quad \sum_{i=1}^{s} f_i^+(v) \ge d^+(v). \tag{V}$$

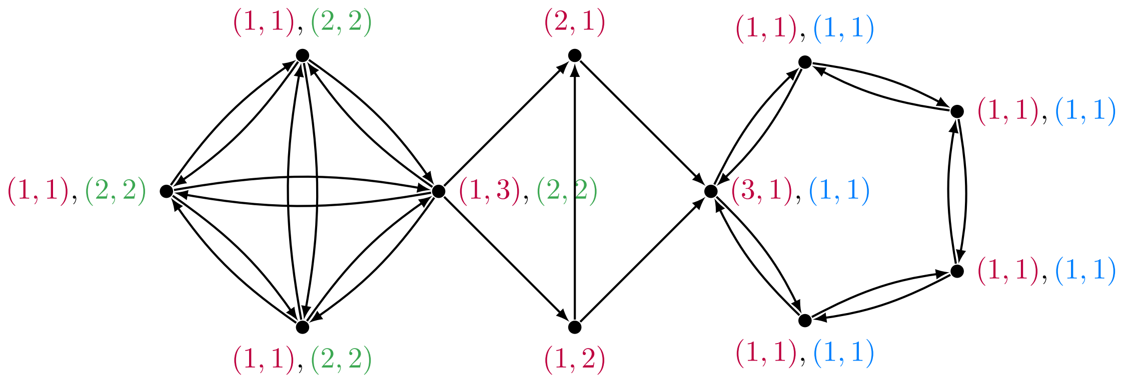


Figure 3.2: An example of a hard pair (D, F) where F is a sequence of three functions f_1, f_2, f_3 the values of which are respectively represented in red, green, and blue. For the sake of readability, the value of f_i is missing when it is equal to $(0, 0)$.

If (D, F) is a pair satisfying the validity property (V) and furthermore D is connected, then (D, F) is called a *valid pair*. Some valid pairs have a particular structure, and we call them *hard pairs*. We say that (D, F) is a *hard pair* if one of the following four conditions holds:

(i) D is a 2-connected digraph, and there exists $i \in [s]$ such that, for every vertex $v \in V$, $f_i(v) = (d^-(v), d^+(v))$ and $f_k(v) = (0, 0)$ when $k \neq i$.

We refer to such a hard pair as a *monochromatic hard pair*.

(ii) D is a bidirected odd cycle, and the functions $f_1, \ldots, f_s$ are all constant equal to $(0, 0)$ except exactly two that are constant equal to $(1, 1)$.

We refer to such a hard pair as a *bicycle hard pair*.

(iii) D is a bidirected complete graph, the functions $f_1, \ldots, f_s$ are all constant and symmetric, and for every vertex v we have $\sum_{i=1}^{s} f_i^+(v) = n(D) - 1$.

We refer to such a hard pair as a *complete hard pair*.

(iv) (D, F) is obtained from two hard pairs (D^1, F^1) and (D^2, F^2) by identifying two vertices $x_1 \in V(D^1)$ and $x_2 \in V(D^2)$ into a new vertex $x \in V(D)$, such that for every vertex $v \in V(D)$ we have:

$$f_k(v) = \begin{cases} f_k^1(v) & \text{if } v \in V(D_1) \setminus \{x_1\} \\ f_k^2(v) & \text{if } v \in V(D_2) \setminus \{x_2\} \\ f_k^1(v) + f_k^2(v) & \text{if } v = x. \end{cases}$$

where $F^1 = (f_1^1, \ldots, f_s^1)$, $F^2 = (f_1^2, \ldots, f_s^2)$.

We refer to such a hard pair as a *join hard pair*.

See Figure 3.2 for an illustration of a hard pair.

The following is the main result of this section.

Theorem 3.2.1. *Let (D, F) be a valid pair. Then D is F-dicolourable if and only if (D, F) is not a hard pair. Moreover, there is an algorithm running in time $O(n(D) + m(D))$ that decides if (D, F) is a hard pair, and that outputs an F-dicolouring if it is not.*

As mentioned in the introduction, the first part of this theorem generalises a result of [19]. Indeed, their result corresponds to restricting ourselves to symmetric functions. As this result generalised Theorem 3.1.4 and Theorem 3.1.6, Theorem 3.2.1 also does. The algorithm of Theorem 3.2.1 also improves on the result of [19] in terms of complexity. Their algorithm is shown to be polynomial, without more details on the exponent. Note that due to F, the input has size $O(n(D) + m(D) + s \cdot n(D))$. The algorithm of Theorem 3.2.1 can thus be sublinear in the input size if $(n(D) + m(D))$ is asymptotically dominated by $(s \cdot n(D))$. In the remaining of this section, "linear time complexity" means linear in the number of vertices and arcs of the considered digraph.

In Section 3.2.1, we give a short proof that if (D, F) is a hard pair, then D is not F-dicolourable. In Section 3.2.2 we prove the converse and the second part of Theorem 3.2.1.

3.2.1 Hard pairs are not dicolourable

Lemma 3.2.2. *Let (D, F) be a hard pair, then D is not F-dicolourable.*

Proof. We proceed by induction. Let $(D, F = (f_1, \dots, f_s))$ be a hard pair. Assume for a contradiction that it admits an F-dicolouring α with colour classes $V_1, \dots, V_s$. We distinguish four cases, depending on the kind of hard pair (D, F) is.

(i) If (D, F) is a monochromatic hard pair, let $i \in [s]$ be such that, for every vertex $v \in V$, $f_i(v) = (d^-(v), d^+(v))$ and $f_k(v) = (0, 0)$ when $k \neq i$. Then, for every $k \neq i$, V_k must be empty, for otherwise $D\langle V_k \rangle$ is not strictly-f_k-degenerate. Therefore, we have $V = V_i$ and D must be strictly-f_i-degenerate. Hence D contains a vertex v such that $d^-(v) < f_i^-(v)$ or $d^+(v) < f_i^+(v)$, a contradiction.

(ii) If (D, F) is a bicycle hard pair, let $i, j \in [s]$ be distinct integers such that, for every vertex $v \in V$, $f_i(v) = f_j(v) = (1, 1)$ and $f_k(v) = (0, 0)$ when $k \notin \{i, j\}$. Again, V_k must be empty when $k \notin \{i, j\}$, so (V_i, V_j) partitions V. Since D is a bidirected odd cycle, it is not bipartite, so $D\langle V_i \rangle$ or $D\langle V_j \rangle$ contains a digon between vertices u and v. Assume that $D\langle V_i \rangle$ does, then $H = D\langle \{u, v\} \rangle$ must contain a vertex x such that $d_H^-(x) < f_i^-(x)$ or $d_H^+(x) < f_i^+(x)$, a contradiction since $f_i(x) = (d_H^-(x), d_H^+(x)) = (1, 1)$ for every $x \in \{u, v\}$.

(iii) If (D, F) is a complete hard pair, then for each $i \in [s]$, since $D\langle V_i \rangle$ is strictly-f_i-degenerate, we have $|V_i| \leq f_i^-(v)$ where v is any vertex (recall that the functions $f_1, \dots, f_s$ are constant and symmetric). Then $\sum_{i=1}^s |V_i| \leq \sum_{i=1}^s f_i^-(v) = n(D) - 1$, a contradiction since $(V_1, \dots, V_s)$ partitions V.

(iv) Finally, if (D, F) is a join hard pair, (D, F) is obtained from two hard pairs (D^1, F^1) and (D^2, F^2) by identifying two vertices $x_1 \in V(D^1)$ and $x_2 \in V(D^2)$ into a new vertex $x \in V(D)$, such that for every vertex $v \in V(D)$ we have:

$$f_k(v) = \begin{cases} f_k^1(v) & \text{if } v \in V(D_1) \setminus \{x_1\} \\ f_k^2(v) & \text{if } v \in V(D_2) \setminus \{x_2\} \\ f_k^1(v) + f_k^2(v) & \text{if } v = x. \end{cases}$$

where $F^1 = (f_1^1, \dots, f_s^1)$, $F^2 = (f_1^2, \dots, f_s^2)$.

By induction, we may assume that D^1 is not F^1-dicolourable and D^2 is not F^2-dicolourable. Let α^1 and α^2 be the following s-colourings of D^1 and D^2.

$$\alpha^1(v) = \begin{cases} \alpha(v) \text{ if } v \neq x_1 \\ \alpha(x) \text{ otherwise.} \end{cases} \quad \text{and} \quad \alpha^2(v) = \begin{cases} \alpha(v) \text{ if } v \neq x_2 \\ \alpha(x) \text{ otherwise.} \end{cases}$$

We denote respectively by $U_1, \dots, U_s$ and $W_1, \dots, W_s$ the colour classes of α^1 and α^2. Since D^1 is not F^1-dicolourable, there exist $i \in [s]$ and a subdigraph H^1 of $D^1\langle U_i \rangle$ such that every vertex $u \in V(H^1)$ satisfies $d^-_{H^1}(u) \geq {f_i^1}^-(u)$ and $d^+_{H^1}(u) \geq {f_i^1}^+(u)$. We claim that $x_1 \in V(H^1)$. Assume not, then H^1 is a subdigraph of $D\langle V_i \rangle$ and every vertex u in H^1 satisfies $f_i(u) = f_i^1(u)$, and so it satisfies $d^-_{H^1}(u) \geq f_i^-(u)$ and $d^+_{H^1}(u) \geq f_i^+(u)$. This is a contradiction to $D\langle V_i \rangle$ being strictly-f_i-degenerate.

Hence, we know that $x_1 \in V(H^1)$. Swapping the roles of D^1 and D^2, there exists an index $j \in [s]$ and a subdigraph H^2 of $D^2\langle W_j \rangle$ such that every vertex $w \in V(H^2)$ satisfies $d^-_{H^2}(w) \geq {f_j^2}^-(w)$ and $d^+_{H^2}(w) \geq {f_j^1}^+(w)$. Analogously, we must have $x_2 \in V(H^2)$.

By definition of α^1 and α^2, we have $i = j$ and $x \in V_i$. Let H be the subdigraph of $D\langle V_i \rangle$ obtained from H_1 and H_2 by identifying x_1 and x_2 into x. For every vertex $u \in V(H) \cap V(H^1)$, we have $d^-_H(u) = d^-_{H^1}(u) \geq f_i^-(u)$ and $d^+_H(u) = d^+_{H^1}(u) \geq f_i^+(u)$. Analogously, for every vertex $w \in V(H) \cap V(H^2)$, we have $d^-_H(w) = d^-_{H^2}(w) \geq {f_i^2}^-(w)$ and $d^+_H(w) = d^+_{H^2}(w) \geq {f_i^1}^+(w)$. Finally, we have

$$\begin{aligned} & d^-_H(x) = d^-_{H^1}(x_1) + d^-_{H^2}(x_2) \geq {f_i^1}^-(x_1) + {f_i^2}^-(x_2) = f_i^-(x) \\ \text{and} \quad & d^+_H(x) = d^+_{H^1}(x_1) + d^+_{H^2}(x_2) \geq {f_i^1}^+(x_1) + {f_i^2}^+(x_2) = f_i^+(x). \end{aligned}$$

This is a contradiction to $D\langle V_i \rangle$ being strictly-f_i-degenerate. □

3.2.2 Dicolouring non-hard pairs in linear time

In this section, we prove Theorem 3.2.1 by describing the mentioned algorithm, proving its correctness, and its time complexity. Essentially, this algorithm consists in first testing if (D, F) is a hard pair or not, and then if it is not, in pipelining various algorithms reducing the initial instance until a solution can be found.

We first discuss the data structure encoding the input in Section 3.2.2.1. We give some technical definitions used in the proof and some preliminary remarks in Section 3.2.2.2. In the following subsections, we describe different reduction steps unfolding in our algorithm. We consider valid pairs for which the constraints are loose in Section 3.2.2.4 based on a simple greedy algorithm presented in Section 3.2.2.3. Then, we consider "tight" valid pairs. In Section 3.2.2.5, we show how to reduce a tight and valid pair (D, F) into another one (D', F') for which D' is 2-connected and smaller than D. At this point, we may use the hardness of (D', F') to detect whether (D, F) is hard, otherwise the remainder of the section consists in exhibiting an F'-dicolouring of D', yielding an F-dicolouring of D. Then in Section 3.2.2.6, we reduce the instance to one with two colours. After these reductions, we show how to solve 2-connected instances with two colours in Section 3.2.2.7 and in Section 3.2.2.8, through the use of ear-decompositions. We conclude with the proof of Theorem 3.2.1 in Section 3.2.2.9.

3.2.2.1 Data structures

We need appropriate data structures to process the entry pair (D, F). The following structures are standard, but let us list their properties. The digraph D is encoded in space $O(n(D) + m(D))$ with a data structure allowing, for every vertex v,

- to access the values $n(D)$, $m(D)$, $d^-(v)$, $d^+(v)$, and $|N^-(v) \cap N^+(v)|$ in $O(1)$ time,
- to enumerate the vertices of the sets $V(D)$, $N^-(v)$, $N^+(v)$ in $O(1)$ time per vertex,
- to delete a vertex v (and update all the related values and sets) in $O(d(v))$ time, and
- to compute a spanning tree rooted at a specified root in $O(n(D) + m(D))$ time.

The functions $F = (f_i)_{i \in [s]}$ are encoded in $O(s \cdot n(D))$ space in a data structure allowing, for every vertex v,

- to read or modify $f_i^-(v)$ and $f_i^+(v)$ in $O(1)$ time, and
- to enumerate (only) the colours i such that $f_i(v) \neq (0, 0)$, in $O(1)$ time per such colour.

This data structure is simply a $s \times n(D)$ table with pointers linking the cells (i, v) and (j, v) if $f_i(v) \neq (0, 0)$, $f_j(v) \neq (0, 0)$, and $f_k(v) = (0, 0)$ for every k such that $i < k < j$.

The output is a vertex colouring which we build along the different steps of our algorithm, and is simply encoded in a table.

3.2.2.2 Preliminaries

Let $(D, F = (f_1, \ldots, f_s))$ be a (non-necessarily valid) pair, let $X \subseteq V(D)$ be a subset of vertices of D, and $\alpha \colon X \to [s]$ be a partial s-colouring of D. Let $D' = D - X$ and $F' = (f'_1, \ldots, f'_s)$ be defined as follows:

$$\begin{aligned} & f_i'^-(u) = \max(0, f_i^-(u) - |\alpha^{-1}(i) \cap N^-(u)|) \\ \text{and} \quad & f_i'^+(u) = \max(0, f_i^+(u) - |\alpha^{-1}(i) \cap N^+(u)|) \end{aligned}$$

We call (D', F') the pair *reduced* from (D, F) by colouring α.

Lemma 3.2.3. *Consider a (non-necessarily valid) pair (D, F) and an F-dicolouring α of $D\langle X \rangle$, for a subset $X \subseteq V(D)$. Let (D', F') be the pair reduced from (D, F) by α. Then, combining α with any F'-dicolouring of D' yields an F-dicolouring of D.*

Furthermore, there is an algorithm that given a pair (D, F), and a colouring α of some set $X \subseteq V(D)$, outputs the reduced pair (D', F') in $O(\sum_{v \in X} d(v))$ time.

Proof. Let β be any F'-dicolouring of D'. We will show that the combination γ of α and β is necessarily an F-dicolouring of D. We formally have

$$\gamma(v) = \begin{cases} \alpha(v) & \text{if } v \in X \\ \beta(v) & \text{otherwise.} \end{cases}$$

Let $i \in [s]$ be any colour, we will show that $D\langle V_i \rangle$ is strictly-f_i-degenerate, where $V_i = \gamma^{-1}(i)$, which implies the first part of the statement. To this purpose, let H be any subdigraph of $D\langle V_i \rangle$, we will show that H contains a vertex v satisfying $d^-_H(v) < f^-_i(v)$ or $d^+_H(v) < f^+_i(v)$. Observe first that, if $V(H) \subseteq X$, the existence of v is guaranteed since α is an F-dicolouring of $D\langle X\rangle$. Henceforth assume that $V(H) \setminus X \neq \varnothing$, and let H' be $H - X$. Since $V(H') \subseteq \beta^{-1}(i)$, by hypothesis on β, it is strictly-f'_i-degenerate. Hence there must be a vertex $v \in V(H')$ such that $d^-_{H'}(v) < f'^-_i(v)$ or $d^+_{H'}(v) < f'^+_i(v)$. If $d^-_{H'}(v) < f'^-_i(v)$, we obtain

$$d^-_H(v) \leq d^-_{H'}(v) + |\alpha^{-1}(i) \cap N^-(v)| < f'^-_i(v) + |\alpha^{-1}(i) \cap N^-(v)| = f^-_i(v).$$

Symmetrically, $d^+_{H'}(v) < f'^+_i(v)$ implies $d^+_H(v) < f^+_i(v)$. We are thus done.

The algorithm is elementary. It consists on a loop over vertices $v \in X$, updating a table storing the colouring with colour $\alpha(v)$ for v, deleting v from D, and visiting every neighbour u of v in order to update its adjacency list and $f_c(u)$. This is clearly linear in $\sum_{v\in X} d(v)$. □

Let (D, F) be a valid pair. Let X be a subset of vertices of D, $\alpha\colon X \to [s]$ be a partial colouring of D, and (D', F') be the pair reduced from (D, F) by α. We say that the colouring α of $D\langle X\rangle$ is *safe* if each of the following holds:

- α is an F-dicolouring of $D\langle X\rangle$,
- D' is connected, and
- (D, F) is a hard pair if and only if (D', F') is a hard pair.

For the particular case $X = \{v\}$, we thus say that colouring v with c is *safe* if $f_c(v) > (0, 0)$, $D - x$ is connected, and the reduced pair (D', F') is a hard pair if and only if (D, F) is.

The following lemma tells us how directed variable degeneracy relates to vertex orderings.

Lemma 3.2.4. *Given a digraph $D = (V, A)$, and a function $f\colon V \to \mathbb{N}^2$, D is strictly-f-degenerate if and only if there exists an ordering $v_1, \ldots, v_n$ of the vertices of D such that*

$$f^-(v_i) > |N^-(v_i) \cap \{v_j \mid j \leq i\}| \quad \textit{or} \quad f^+(v_i) > |N^+(v_i) \cap \{v_j \mid j \leq i\}|$$

Proof. ($\Longrightarrow$) We proceed by induction on the number of vertices n. This implication clearly holds for $n = 1$. Assume now that $n \geq 2$. Let v_n be a vertex of D such that $f^-(v_n) > d^-(v_n)$ or $f^+(v_n) > d^+(v_n)$, which exists by strict degeneracy of D. Since $d^-(v_n) \geq |N^-(v_n) \cap S|$ and $d^+(v_n) \geq |N^+(v_n) \cap S|$ for any set $S \subseteq V$, the property holds for v_n. By induction there is an ordering $v_1, \ldots, v_{n-1}$ of the vertices of $D' = D - v_n$ such that for every vertex v_i, with $1 \leq i \leq n - 1$, we have

$$f^-(v_i) > |N^-_{D'}(v_i) \cap \{v_j \mid j \leq i\}| \quad \text{or} \quad f^+(v_i) > |N^+_{D'}(v_i) \cap \{v_j \mid j \leq i\}|$$

As $N^-_{D'}(v) \cap \{v_j \mid j \leq i\} = N^-_D(v) \cap \{v_j \mid j \leq i\}$ and $N^+_{D'}(v) \cap \{v_j \mid j \leq i\} = N^+_D(v) \cap \{v_j \mid j \leq i\}$, the property still holds in D and we are done.

($\Longleftarrow$) For any subdigraph $H = (V_H, A_H)$ of D let v_k be the vertex of V_H with largest index k. Since $N^-_H(v_k) = N^-_D(v_k) \cap V_H \subseteq N^-_D(v_k) \cap \{v_j \mid j \leq k\}$, and $N^+_H(v_k) = N^+_D(v_k) \cap V_H \subseteq N^+_D(v_k) \cap \{v_j \mid j \leq k\}$, we have that

$$f^-(v_k) > |N^-_D(v) \cap \{v_j \mid j \leq k\}| \geq d^-_H(v_k) \quad \text{or} \quad f^+(v_i) > |N^+_D(v) \cap \{v_j \mid j \leq i\}| \geq d^+_H(v_k).$$

□

3.2.2.3 Greedy algorithm

In this subsection, we consider an algorithm that greedily colours D, partially or entirely. We are going to give conditions ensuring that this approach succeeds in providing a (partial) F-dicolouring.

Lemma 3.2.5. *Given a (non-necessarily valid) pair (D,F) and an ordered list of vertices $v_1,\dots,v_\ell$ of $V(D)$, there is an algorithm, running in time $O\left(\sum_{i=1}^{\ell} d(v_i)\right)$, that tries to colour the vertices $v_1,\dots,v_\ell$ in order to get an F-dicolouring of $D\langle\{v_1,\dots,v_\ell\}\rangle$. If the algorithm succeeds, it also computes the reduced pair (D',F').*

If F is such that for every $v_i \in \{v_1,\dots,v_\ell\}$, we have

$$\sum_{c=1}^{s} f_c^-(v_i) > |N^-(v_i) \cap \{v_j \mid j \le i\}| \quad \text{or} \quad \sum_{c=1}^{s} f_c^+(v_i) > |N^+(v_i) \cap \{v_j \mid j \le i\}| \tag{R}$$

the algorithm succeeds. If condition (R) is fulfilled, if (D,F) is valid, and if $D - \{v_1,\dots,v_\ell\}$ is connected, the reduced pair (D',F') is a valid pair.

Note that as $\sum_{v\in D} d(v) = 2m(D)$ the time complexity here is $O(n(D)+m(D))$, even when the whole digraph is coloured.

Proof. The algorithm simply consists in considering the vertices $v_1,\dots,v_\ell$ in this order and, if possible, to colour v_i with a colour c such that

$$\begin{aligned} f_c^-(v_i) &> \left|\{u \in N^-(v_i) \cap \{v_1,\dots,v_i\} \mid u \text{ is coloured } c\}\right|, \\ \text{or such that } \; f_c^+(v_i) &> \left|\left\{u \in N^+(v_i) \cap \{v_1,\dots,v_i\} \mid u \text{ is coloured } c\right\}\right|. \end{aligned}$$

The complexity of the algorithm holds because, it actually consists in a loop over vertices $v_1,\dots,v_\ell$, where 1) it looks for a colour c such that $f_c(v_i) \neq (0,0)$ (where F is updated after each vertex colouring), and if it finds such a colour, 2) colours v_i and updates (D,F) into the corresponding reduced pair (D',F'). Step 1) is done in constant time (see Section 3.2.2.1), and step 2) is done in $O(d(v_i))$ time (by Lemma 3.2.3). Hence, the complexity clearly follows. Lemma 3.2.4, applied to $D\langle\{v_1,\dots,v_\ell\}\rangle$, implies that if the algorithm succeeds, the obtained colouring is an F-dicolouring of $D\langle\{v_1,\dots,v_\ell\}\rangle$.

To show the second statement, let us show the following invariant of the algorithm:

> *At the beginning of the i^{th} iteration of the main loop, for any vertex v_k with $i \le k \le \ell$, at least one of the following occurs:*
>
> - *its number of uncoloured in-neighbours with index at most k is less than $\sum_{c=1}^{s} f_c^-(v_k)$, or*
> - *its number of uncoloured out-neighbours with index at most k is less than $\sum_{c=1}^{s} f_c^+(v_k)$,* (♣)
>
> *where the sums are made on the updated functions f_c.*

Note that, by (R), Invariant (♣) holds for the first iteration. Let us show that if (♣) holds at the beginning of the i^{th} iteration for some vertex v_k with $i < k$, then it still holds at the beginning of the $i+1^{\text{th}}$ iteration. Indeed, during the i^{th} iteration, if the sum $\sum_{c=1}^{s} f_c^-(v_k)$ decreases, it decreases by exactly one, and in that case we have $v_i \in N^-(v_k)$. Hence, the number of uncoloured in-neighbours of v_k with index at most k also decreases by one, and (♣) sill holds. The same holds for the out-neighbourhood of v_i.

By (♣), at the beginning of the i^{th} iteration, we have $\sum_{i=1}^{s} f_i(v) \neq (0,0)$. Hence, there is a colour c such that $f_c(v_i) \neq (0,0)$, so there is always a colour (*e.g.* c) available for colouring v_i, and the algorithm thus succeeds in colouring the whole digraph $D\langle\{v_1,\dots,v_\ell\}\rangle$.

Finally for the last statement, if (R) holds, then the algorithm succeeds in producing an F-dicolouring. Furthermore, if (D,F) is valid, for any vertex $u \in V(D) \setminus \{v_1,\dots,v_\ell\}$, we have $d^-_D(u) \leq \sum_{c=1}^{s} f^-_c(v_i)$ and $d^+_D(u) \leq \sum_{c=1}^{s} f^+_c(v_i)$. For each of these inequalities, and at each iteration of the main loop, the left-hand side decreases if and only if the right-hand side does, in the reduced pair (D',F'). Hence, the validity property (V) holds in (D',F'), and since D' is connected, the pair (D',F') is valid. □

3.2.2.4 Solving loose instances

If the input digraph D as a whole is strictly-$\widetilde{f}$-degenerate for some $\widetilde{f}$, it may be easy to produce an F-dicolouring, under some conditions on $\widetilde{f}$. Note that, in what follows, we do not ask for (D,F) to be a valid pair, so D may be disconnected and vertices v do not necessarily satisfy the validity property (V).

Lemma 3.2.6. *Let $D = (V,A)$ be a digraph, $F = (f_1,\dots,f_s)$ be a sequence of functions $f_i \colon V \to \mathbb{N}^2$, and $\widetilde{f} = \sum_{i=1}^{s} f_i$. If D is strictly-$\widetilde{f}$-degenerate, then D is F-dicolourable and an F-dicolouring can be computed in linear time.*

Proof. Let $(D = (V,A), F = (f_1,\dots,f_s))$ be such a pair. By Lemma 3.2.4, there exists an ordering $\sigma = v_1,\dots,v_n$ of $V(D)$ such that, for every $i \in [s]$, v_i satisfies $d^+_{D_i}(v_i) < f^+(v_i)$ or $d^-_{D_i}(v_i) < f^-(v_i)$, where D_i is the subdigraph of D induced by $\{v_1,\dots,v_i\}$.

We prove that such an ordering σ can be computed in linear time. Our proof follows a classical linear-time algorithm for computing the classical degeneracy of a graph, but we give the proof for completeness. We create two tables `out_gap` and `in_gap`, both of size n and indexed by V. We iterate once over the vertices of V and, for every vertex v, we set `out_gap`(v) to $\widetilde{f}^+(v) - d^+(v)$ and `in_gap` to $\widetilde{f}^-(v) - d^-(v)$. We also store in a set S all vertices v for which `out_gap`$(v) \leq -1$ or `in_gap`$(v) \leq -1$.

Then for i going from n to 1, we choose a vertex u in S that has not been treated before, we set v_i to u, and for every in-neighbour (resp. out-neighbour) w of u that has not been treated before, we decrease `in_gap`(w) (resp. `out_gap`(w)) by one. If `out_gap`$(w) \leq -1$ or `in_gap`$(w) \leq -1$, we add w to S. We then remember (using a boolean table for instance) that u has been treated.

Following this linear-time algorithm, at the beginning of each step $i \in [n]$, for every non-treated vertex u, we have `in_gap` $= f^-(u) - d^-_{D_i}(u)$ and `out_gap` $= f^+(u) - d^+_{D_i}(u)$. We also maintain that S contains all the vertices u satisfying `in_gap`$(u) \leq -1$ or `out_gap`$(u) \leq -1$. Since D is strictly-$\widetilde{f}$-degenerate, at step i, there exists a non-treated vertex $u \in S$.

We now admit that such an ordering $\sigma = (v_1,\dots,v_n)$ has been computed, and we greedily colour the vertices from v_1 to v_n. The result follows from Lemma 3.2.5 and the fact that $\widetilde{f} = \sum_{i=1}^{s} f_i$. □

A valid pair is *tight*, if for every vertex the two inequalities of the validity property (V) are equalities. The following particular case of Lemma 3.2.6 treats the case of instances that are non-tight, which we also refer to as *loose*.

Lemma 3.2.7. *Let (D, F) be a valid pair that is loose. Then D is F-dicolourable and an F-dicolouring can be computed in linear time.*

Proof. Let $(D = (V, A), F = (f_1, \dots, f_s))$ be such a pair, and let $\widetilde{f}\colon V \to \mathbb{N}^2$ be $\sum_{i=1}^{s} f_i$. It is sufficient to prove that D is necessarily strictly-$\widetilde{f}$-degenerate, so the result follows from Lemma 3.2.6.

Assume for a contradiction that D is not strictly-$\widetilde{f}$-degenerate, so by definition there exists an induced subdigraph H of D such that, for every vertex $v \in V(H)$, $d_H^+(v) \geq \widetilde{f}^+(v)$ and $d_H^-(v) \geq \widetilde{f}^-(v)$. By definition of $\widetilde{f}$ and because (D, F) is a valid pair, we thus have $d_H^+(v) = \sum_{i=1}^{s} f_i^+(v)$ and $d_H^-(v) = \widetilde{f}^-(v)$. We directly deduce that $H \neq D$ as (D, F) is loose.

Since D is connected (as (D, F) is a valid pair) and because $H \neq D$, there exists in D an arc between vertices u and v such that $u \in V(H)$ and $v \in V(D) \setminus V(H)$. Depending on the orientation of this arc, we have $d_H^+(u) < d_D^+(u) = \sum_{i=1}^{s} f_i^+(u)$ or $d_H^-(u) < d_D^-(u) = \sum_{i=1}^{s} f_i^-(u)$, a contradiction. □

3.2.2.5 Reducing to a block and detecting hard pairs

We have just shown how to solve loose instances. In this subsection, we thus consider a tight instance (D, F), and show how to obtain a reduced instance (B, F') where B is a block of D, by safely colouring $V(D) \setminus V(B)$. Then, (B, F') is a hard pair if and only if (D, F) is, in which case we may terminate the algorithm. Otherwise, the following subsections show that B may be F'-dicoloured, and together with the colouring of $V(D) \setminus V(B)$ fixed at this step, this yields an F-dicolouring of D.

Our reduction of (D, F) proceeds by considering the end-blocks of D one after the other. If an end-block B, together with F, may correspond to a monochromatic hard pair, a bicycle hard pair, or a complete hard pair glued to the rest of the digraph, then we safely colour $V(B) \setminus V(D)$ and move to the next end-block. If B cannot be such a block, then we will show that we can safely colour $V(D) \setminus V(B)$.

Before formalising this strategy, we need a few definitions. Given a valid pair (D, F), an end-block B of D, with cut-vertex x, is a *hard end-block* if it is of one of the following types:

(i) There exists a colour $i \in [s]$ such that $f_i(x) \geq (d_B^-(x), d_B^+(x))$, and for every $v \in V(B) \setminus \{x\}$ we have $f_i(v) = (d_B^-(v), d_B^+(v))$ and $f_k(v) = (0, 0)$ when $k \neq i$.

We refer to such a hard end-block as a *monochromatic hard end-block*.

(ii) B is a bidirected odd cycle and there exists colours $i \neq j$ such that, $f_i(x) \geq (1, 1)$, $f_j(x) \geq (1, 1)$ and for every $v \in V(B) \setminus \{x\}$, we have $f_c(v) = (1, 1)$ if $c \in \{i, j\}$ and $f_c(v) = (0, 0)$ otherwise.

We refer to such a hard end-block as a *bicycle hard end-block*.

(iii) B is a bidirected complete graph, the functions $f_1, \dots, f_s$ are constant and symmetric on $V(B) \setminus \{x\}$, and $f_i(x) \geq f_i(u)$ for every $i \in [s]$ and every $u \in V(B) \setminus \{x\}$.

We refer to such a hard end-block as a *complete hard end-block*.

Let $(D, F = (f_1, \ldots, f_s))$ be a tight valid pair such that D is not 2-connected and let B be a hard end-block of D with cut-vertex x. Let u be any vertex of B distinct from x. We define the *contraction* $(D', F' = (f'_1, \ldots, f'_s))$ of (D, F) with respect to B, as follows:

- $D' = D - (V(B) \setminus \{x\})$;
- for every vertex $v \in V(D') \setminus \{x\}$ and every $i \in [s]$, $f'_i(v) = f_i(v)$;
- if B is a monochromatic hard end-block, let c be the unique colour such that $f_c(u) \neq (0,0)$, then $f'_c(x) = f_c(x) - (d^-_B(x), d^+_B(x))$ and $f'_i(x) = f_i(x)$ for every $i \in [s] \setminus \{c\}$;
- otherwise, B a bicycle or a complete hard end-block, and $f'_i(x) = f_i(x) - f_i(u)$ for every $i \in [s]$.

Lemma 3.2.8. *Let $(D, F = (f_1, \ldots, f_s))$ be a tight valid pair such that D is not 2-connected and let B be a hard end-block of D with cut-vertex x. Let $(D', F' = (f'_1, \ldots, f'_s))$ be the contraction of (D, F) with respect to B. Then (D, F) is a hard pair if and only if (D', F') is a hard pair.*

Proof. If (D', F') is a hard pair, (D, F) could be defined as the join hard pair obtained from two hard pairs, (D', F') and $(B, \widetilde{F} = (\widetilde{f}_1, \ldots, \widetilde{f}_s))$, where $\widetilde{f}_i = f_i - f'_i$ for every $i \in [s]$.

Conversely, let us show by induction that for every hard pair (D, F), and every end-block B, then B is a hard end-block and (D', F'), the contraction of (D, F) with respect to B, is a hard pair. This is trivial if D is 2-connected as D does not have any end-block. We thus assume that (D, F) is a join hard pair obtained from two hard pairs (D_1, F_1) and (D_2, F_2). Assume without loss of generality that B is a block of D_1. If B is not an end-block in D_1, since it is an end-block of D, we necessarily have $D_1 = B$. Hence (D', F') is exactly (D_2, F_2), which is a hard pair. Furthermore, irrespective of the type of hard pair (D_1, F_1), by the definition of hard join, $D_1 = B$ is clearly a hard end-block of D. If B is an end-block of D_1, By induction hypothesis, B is a hard end-block of (D_1, F_1) and the contraction of (D_1, F_1) with respect to B, (D'_1, F'_1), is a hard pair. It is easy to check that (D', F') is exactly the join hard pair obtained from (D'_1, F'_1) and (D_2, F_2). □

Before describing our algorithm reducing the instance to a single block, we need the following subroutine testing if an end-block is hard.

Lemma 3.2.9. *Given a tight valid pair $(D, F = (f_1, \ldots, f_s))$, and an end-block B of D with cut-vertex x, testing whether B is a hard end-block can be done in time $O(n(B) + m(B))$.*

Proof. The algorithm takes block B with its cut-vertex x, and considers any $u \in V(B) \setminus \{x\}$ as a reference vertex.

We first check whether B is a monochromatic hard end-block. To do so, we first check whether exactly one colour i is available for u (*i.e.* $\{j \mid f_j(u) \neq (0,0)\} = \{i\}$). We then check whether i is indeed the only available colour for every vertex $v \in V(B) \setminus \{u, x\}$. We finally check whether $f_i(x) \geq (d^-_D(x), d^+_D(x))$. Since (D, F) is tight, B is a monochromatic hard end-block if and only if all these conditions are met.

Assume now that B is not a monochromatic hard end-block. Now, B is a hard end-block if and only if it is a bicycle hard end-block or a complete hard end-block. In both cases, the functions $f_1, \ldots, f_s$ must be symmetric, and constant on $V(B) \setminus \{x\}$. Since (D, F) is tight, for every vertex $v \in V(B) \setminus \{x\}$, we can iterate over $\{j \mid f_j(v) \neq (0,0)\}$ in time $O(d(v))$. This allows us to check in linear time whether the functions $f_1, \ldots, f_s$ are symmetric and constant on $V(B) \setminus \{x\}$. If this is not the case, B is not a hard end-block. We can assume now that $f_1, \dots, f_s$ are symmetric and constant on $V(B) \setminus \{x\}$. We also assume that B is either a bidirected odd cycle or a bidirected complete graph, for otherwise B is clearly not a hard end-block.

If B is a bidirected odd cycle, we check in constant time that $f_i(u) \neq (0,0)$ holds for exactly two colours. If this does not hold, then B is not a hard end-block. If this is the case, denote these two colours i, j, and note (by tightness and symmetry) that $f_i(u) = f_j(u) = (1,1)$, and that $f_k(u) = (0,0)$ for $k \notin \{i, j'\}$. Now B is a bicycle hard end-block if and only if $f_k(x) \geq (1,1)$ for $k \in \{i,j\}$, which can be checked in constant time.

Assume finally that B is a bidirected complete graph. Since (D,F) is tight, and because the functions $f_1, \dots, f_s$ are symmetric and constant on $V(B) \setminus \{x\}$, then B is a complete hard end-block if and only if $f_i(x) \geq f_i(u)$ for every colour $i \in [s]$. This can be done in $O(n(B))$ time, since there are at most $n(B)$ colours i such that $f_i(u) \neq (0,0)$ (as (D,F) is tight). $\square$

We now turn to showing the main lemma of this subsection, which reduces any tight valid pair to a 2-connected one.

Lemma 3.2.10. *Let (D,F) be a tight valid pair. There exists a block B of D such that $V(D) \setminus V(B)$ may be safely coloured, yielding the reduced pair (B,F'). Moreover, (B,F') and the colouring of $V(D) \setminus V(B)$ can be computed in linear time.*

Proof. Let (D,F) be such a pair. We first compute, in linear time [165], an ordering $B^1, \dots, B^r$ of the blocks of D and an ordered set of vertices $(x_2, \dots, x_r)$ in such a way that for every $2 \leq \ell \leq r$, x_ℓ is the only cut-vertex of B^ℓ in $D^\ell = D\langle \bigcup_{j=1}^{\ell} V(B^\ell) \rangle$. If D is 2-connected, that is $r = 1$, the result follows for $B = D$ and there is nothing to do. We now assume $r \geq 2$.

For ℓ going from r to 2, we proceed as follows. We consider the block B^ℓ, which is an end-block of D^ℓ. We will either safely colour $V(D^\ell) \setminus V(B^\ell)$ and output the reduced pair (B^ℓ, F'), or safely colour $V(B^\ell) \setminus \{x_\ell\}$, compute the reduced pair $(D^{\ell-1}, F^{\ell-1})$, and go on with the end-block $B^{\ell-1}$ of $D^{\ell-1}$. If we colour all the blocks B^ℓ with $\ell \geq 2$, we output the reduced pair (D^1, F^1).

More precisely, when considering B^ℓ, we first check in time $O(n(B^\ell) + m(B^\ell))$ whether B^ℓ is a hard end-block of (D^ℓ, F^ℓ) (by Lemma 3.2.9). We distinguish two cases, depending on whether B^ℓ is a hard end-block.

Case 1: *B^ℓ is a hard end-block of D^ℓ.*

In this case, we safely colour the vertices of $V(B^\ell) \setminus \{x_\ell\}$ in time $O(n(B^\ell) + m(B^\ell))$. To do so, we first compute an ordering $\sigma = (v_1, \dots, v_b = x_\ell)$ of $V(B^\ell)$ corresponding to a leaves-to-root ordering of a spanning tree of B^ℓ rooted in x_ℓ, in time $O(n(B^\ell) + m(B^\ell))$. We then greedily colour the vertices $v_1, \dots, v_{b-1}$ in this order. For every $j \in [b-1]$, the vertex v_j has at least one neighbour in $\{v_{j+1}, \dots, v_b\}$ (its parent in the spanning tree), so condition (R) is fulfilled. Hence, Lemma 3.2.5 ensures that the algorithm succeeds and that the reduced pair $(D^{\ell-1}, F^{\ell-1})$ is a valid pair. We will now show that $(D^{\ell-1}, F^{\ell-1})$ is necessarily tight, and that the colouring of $V(B^\ell) \setminus \{x_\ell\}$ we computed is safe, by showing that $(D^{\ell-1}, F^{\ell-1})$ is exactly the contraction of (D^ℓ, F^ℓ) with respect to B^ℓ.

Assume first that B^ℓ is a monochromatic hard end-block. Then, all the vertices of $V(B) \setminus \{x_\ell\}$ are coloured with the same colour i, and by definition of a monochromatic hard end-block, we have $f_i^{\ell-1}(x_\ell) = f_i^\ell(x_\ell) - (d^-_{B^\ell}(x_\ell), d^+_{B^\ell}(x_\ell))$ and $f_j^{\ell-1}(x_\ell) = f_j^\ell(x_\ell)$ for every colour $j \neq i$. Hence, $(D^{\ell-1}, F^{\ell-1})$ is tight (as the degrees from D^ℓ to $D^{\ell-1}$ only change for x_ℓ, for which they decrease exactly by $d^-_{B^\ell}(x_\ell)$, and $d^+_{B^\ell}(x_\ell)$) and Lemma 3.2.8 implies that the colouring we computed is safe.

Assume now that B^ℓ is a bicycle hard end-block. Let $i \neq j$ be such that $f^\ell_i(u) = f^\ell_j(u) = (1,1)$ for every vertex $u \in V(B^\ell) \setminus \{x_\ell\}$. Recall that, by definition of a bicycle hard end-block, $f^\ell_k(x_\ell) \geq (1,1)$ for $k \in \{i,j\}$. Let u, v be the two neighbours of x_ℓ in B^ℓ, then u and v are connected by a bidirected path of odd length in $B^\ell - x_\ell$. This implies that u and v are coloured differently, and $f^{\ell-1}_k(x_\ell) = f^\ell_k(x_\ell) - (1,1)$ for $k \in \{i,j\}$ and $f^{\ell-1}_k(x_\ell) = f^\ell_k(x_\ell)$ for $k \notin \{i,j\}$. Since $d^+_{B^\ell}(x_\ell) = d^-_{B^\ell}(x_\ell) = 2$, we obtain that $(D^{\ell-1}, F^{\ell-1})$ is tight, and Lemma 3.2.8 implies that the colouring we computed is safe.

Assume finally that B^ℓ is a complete hard end-block, and let u be any neighbour of x_ℓ in B. The functions $f^\ell_1, \dots, f^\ell_s$ are constant and symmetric on $V(B) \setminus \{x_\ell\}$. We also have $f^\ell_i(x_\ell) \geq f^\ell_i(u)$ for every $i \in [s]$. By construction of the greedy colouring, for every $i \in [s]$, at most $f^{\ell+}_i(u)$ vertices of $V(B^\ell) \setminus \{x_\ell\}$ are coloured i. Since $n(B^\ell) - 1$ vertices are coloured in total, and because $\sum_{i=1}^s f^{\ell+}_i(u) = n(B^\ell) - 1$, we conclude that, for every $i \in [s]$, exactly $f^{\ell+}_i(u)$ vertices of $V(B^\ell) \setminus \{x_\ell\}$ are coloured i. Hence, we obtain that $(D^{\ell-1}, F^{\ell-1})$ is tight, and Lemma 3.2.8 implies that the colouring we computed is safe.

Case 2: *B^ℓ is not a hard end-block of D^ℓ.*

In this case, we colour the vertices of $V(D^{\ell-1}) \setminus \{x_\ell\}$ in time $O(n(D) + m(D))$ as follows. We first compute an ordering $\sigma = (v_1, \dots, v_{n'} = x_\ell)$ of $V(D^{\ell-1})$, that is a leaves-to-root ordering of a spanning tree of $D^{\ell-1}$ rooted in x_ℓ, in time $O(n(D^\ell) + m(D^\ell))$. We greedily colour vertices $v_1, \dots, v_{n'-1}$ in this order. For every $j \in [n'-1]$, the vertex v_j has at least one neighbour in $\{v_{j+1}, \dots, v_{n'}\}$ (its parent in the spanning tree), so condition (R) is fulfilled. Hence, by Lemma 3.2.5, the greedy colouring succeeds and provides a reduced pair $(B^\ell, F' = (f'_1, \dots, f'_s))$ that is valid.

Finally, observe that (B^ℓ, F') is not a hard pair since, for every $i \in [s]$, we have $f'_i(x_\ell) \leq f^\ell_i(x_\ell)$. So if (B^ℓ, F') is a hard pair, B^ℓ is necessarily a hard end-block of (D^ℓ, F^ℓ), a contradiction. The result follows.

Note that Case 2 may only be reached once, at which point it outputs a reduced pair. Therefore, the running time of the algorithm described above is bounded by $O(\sum_\ell (n(B^\ell) + m(B^\ell)) + n(D) + m(D))$, which is linear in $n(D) + m(D)$. Assume finally that, in the process above, the second case is never attained. Then the result follows as B_1 is a block of D, and we found a safe colouring of $V(D) \setminus V(B^1)$. □

With the last two lemmas at our disposal, we are ready to test whether a tight valid pair is hard.

Lemma 3.2.11. *Given a tight valid pair $(D, F = (f_1, \dots, f_s))$, testing whether (D, F) is a hard pair can be done in time linear time.*

Proof. We first consider the pair (B, F') reduced from (D, F), where B is a block of D, obtained by safely colouring $V(D) \setminus V(B)$ through Lemma 3.2.10. In particular, (B, F') is a hard pair if and only if (D, F) is, and as B is 2-connected, we may only check whether (B, F') is a 2-connected hard pair, either monochromatic, bicycle or complete. This amounts to verifying the conditions of the definition. The fact that this can be done in linear time is already justified in Lemma 3.2.9.

Indeed, testing (B, F') for a hard 2-connected pair amounts to testing the conditions for a hard end-block, except for the condition on the cut-vertex. Since there is no cut-vertex in B, we test the same conditions on all vertices of B. □

3.2.2.6 Reducing to pairs with two colours

We have just shown how to reduce any tight valid pair into a 2-connected one, after which we may decide whether the initial pair was a hard one. In this subsection, we therefore consider any tight valid pair (D, F) that is not hard, and such that D is 2-connected. We show how the problem of F-dicolouring D boils down to $\widetilde{F}$-dicolouring D where $\widetilde{F}$ involves only $s = 2$ colours.

Lemma 3.2.12. *Let (D, F) be a valid pair that is not a hard pair and such that D is 2-connected. In linear time, we can either find an F-dicolouring of D or compute a valid pair $(D, \widetilde{F} = (\widetilde{f}_1, \widetilde{f}_2))$ such that:*

- *$(D, \widetilde{F})$ is not hard, and*
- *given an $\widetilde{F}$-dicolouring of D, we can compute an F-dicolouring of D in linear time.*

Proof. We assume that (D, F) is tight, otherwise we compute an F-dicolouring of D in linear time by Lemma 3.2.7. Since (D, F) is tight, we can iterate over $\{i \mid f_i(v) \neq (0, 0)\}$ in time $O(d(u))$. This allows us to check, in linear time, which of the cases below apply to (D, F).

Case 1: *There exist $u \in V(D)$ and $i \in [s]$ such that $f_i^+(u) \neq f_i^-(u)$.*

We define $\widetilde{f}_1 = f_i$ and $\widetilde{f}_2 = \sum_{j \in [s], j \neq i} f_j$. Let us show that $(D, \widetilde{F})$ is not a hard pair. Since D is 2-connected, $(D, \widetilde{F})$ is not a join hard pair. Since $\widetilde{f}_1$ is not symmetric by choice of $\{u, c\}$, $(D, \widetilde{F})$ is neither a bicycle hard pair nor a complete hard pair. Finally, assume for a contradiction that $(D, \widetilde{F})$ is a monochromatic hard pair. Since $\widetilde{f}_1(u) \neq (0, 0)$, we thus have $\widetilde{f}_2(v) = (0, 0)$ and $f_i(v) = (d^-(v), d^+(v))$ for every vertex v. We conclude that (D, F) is also a monochromatic hard pair, a contradiction.

Case 2: *There exist $u, v \in V(D)$ and $i \in [s]$ such that $f_i(u) \neq f_i(v)$.*

As in Case 1, we set $\widetilde{f}_1 = f_i$ and $\widetilde{f}_2 = \sum_{j \in [s], j \neq i} f_j$. Since D is 2-connected, $(D, \widetilde{F} = (\widetilde{f}_1, \widetilde{f}_2))$ is not a join hard pair. Moreover, since $f_i(v) \neq f_i(u)$, $\widetilde{f}_1$ is not constant, so $(D, \widetilde{F})$ is neither a bicycle hard pair nor a complete hard pair. Again, assume for a contradiction that $(D, \widetilde{F})$ is a monochromatic hard pair. Since $\widetilde{f}_1(u)$ or $\widetilde{f}_1(v)$ is distinct from $(0, 0)$, we thus have $\widetilde{f}_2(v) = (0, 0)$ and $f_i(v) = (d^-(v), d^+(v))$ for every vertex v. We conclude that (D, F) is also a monochromatic hard pair, a contradiction.

Case 3: *None of the cases above is matched.*

Therefore, for each $i \in [s]$, f_i is a symmetric constant function. Thus, since (D, F) is not a hard pair, and because (D, F) is tight, D is not a bidirected odd cycle or a bidirected complete graph. Let $i \in [s]$ be such that f_i is not the constant function equal to $(0, 0)$. We set $\widetilde{f}_1 = f_i$ and $\widetilde{f}_2 = \sum_{j \in [s], j \neq i} f_j$. Since (D, F) is not a monochromatic hard pair, there is a colour $k \neq i$ such that f_k is not the constant function equal to $(0, 0)$. Hence, none of $\widetilde{f}_1, \widetilde{f}_2$ is the constant function equal to $(0, 0)$, and $(D, (\widetilde{f}_1, \widetilde{f}_2))$ is not a hard pair.

In each case, we have built a valid pair $(D, \widetilde{F} = (\widetilde{f}_1, \widetilde{f}_2))$ where $\widetilde{f}_1 = f_i$ and $\widetilde{f}_2 = \sum_{j \neq i} f_j$, in such a way that $(D, \widetilde{F})$ is not hard. We finally prove that, given an $\widetilde{F}$-dicolouring of D, we can compute an F-dicolouring of D in linear time. Let $\widetilde{\alpha}$ be an $\widetilde{F}$-dicolouring of D. Let X be the set of vertices coloured 1 in $\widetilde{\alpha}$ and $\widehat{D}$ be $D - X$ ($\widehat{D}$ is built in linear time by successively removing vertices in X from D). We define $\widehat{F} = (\widehat{f}_1, \dots, \widehat{f}_s)$, $\widehat{f}_i \colon V(\widehat{D}) \to \mathbb{N}^2$ as follows:

$$\widehat{f}_k(v) = \begin{cases} f_k(v) & \text{if } k \neq i \\ (0,0) & \text{otherwise.} \end{cases}$$

Since $\widetilde{f}_2$ is exactly $\sum_{i=1}^{s} \widehat{f}_i$, by definition of $\widetilde{\alpha}$ and $\widehat{D}$, we know that $\widehat{D}$ is strictly-($\sum_{i=1}^{s} \widehat{f}_i$)-degenerate. Then applying Lemma 3.2.6 yields an $\widehat{F}$-dicolouring $\widehat{\alpha}$ of $\widehat{D}$ in linear time. The colouring α defined as follows is thus an F-dicolouring of D obtained in linear time:

$$\alpha(v) = \begin{cases} i & \text{if } \widetilde{\alpha}(v) = 1 \\ \widehat{\alpha}(v) & \text{otherwise.} \end{cases}$$

□

3.2.2.7 Solving blocks with two colours - particular cases

We are now left to deal with pairs (D, F) such that D is 2-connected, and F only involves two colours. Eventually, our strategy consists in finding a suitable decomposition for D, and colouring parts of it inductively, which will be done in Section 3.2.2.8. In this subsection, we first deal with some specific forms (D, F) may take, showing D can be F-dicoloured if the instance in those cases. Along the way, this allows us to only deal with increasingly restricted instances, later enabling us to constrain the instances considered in the induction. In Lemma 3.2.13 and Lemma 3.2.14, we exhibit an F-dicolouring of D if there exists a vertex x which does not satisfy some conditions relating its neighbourhood and F. Then, we solve instances where D is a bidirected complete graph in Lemma 3.2.15, or an orientation of a cycle in Lemma 3.2.16. Lastly, we solve those that are constructed by a star attached to a cycle in Lemma 3.2.17. These will serve as base cases for the induction.

Given a valid pair $(D, F = (f_1, f_2))$ that is tight, non-hard, and such that D is 2-connected, let us define a set of properties that (D, F) may or may not fulfil. Each of these properties is easy to check in linear time, and we will see in the following that when they are not met, there is a linear-time algorithm providing an F-dicolouring of D. The first property, (E), guarantees that both f_1 and f_2 *exceed* some lower bound. It states that both f_1^-, f_2^- (f_1^+, f_2^+) are non-zero on vertices having at least one in-neighbour (out-neighbour).

$$\forall x, c \quad \text{we have} \quad (d^-(x) > 0 \Longrightarrow f_c^-(x) \geq 1) \quad \text{and} \quad (d^+(x) > 0 \Longrightarrow f_c^+(x) \geq 1). \tag{E}$$

Given a digraph $D = (V, A)$, we define the function $\mathbb{1}_A \colon V \times V \to \mathbb{N}$ as follows:

$$\mathbb{1}_A(u, v) = \begin{cases} 1 & \text{if } uv \in A \\ 0 & \text{otherwise.} \end{cases}$$

Note in particular, that if (E) holds, then the following holds.

$$\forall x, y, z, c \quad \text{we have} \quad f_c(x) \geq (\mathbb{1}_A(y, x), \mathbb{1}_A(x, z)). \tag{E'}$$

Lemma 3.2.13. *Let $(D, F = (f_1, f_2))$ be a valid tight non-hard pair such that D is 2-connected. If the pair does not fulfil property (E), then D is F-dicolourable. Furthermore, there is a linear-time algorithm checking property (E), and if the property is not met, that computes an F-dicolouring of D.*

Proof. We first prove that, for some arc uv and for some colour $c \in \{1, 2\}$, we have:

$$(f_c^+(u) = 0 \wedge f_c(v) \neq (0, 0)) \quad \text{or} \quad (f_c^-(v) = 0 \wedge f_c(u) \neq (0, 0)). \tag{3.1}$$

Note that, if it exists, such an arc is found in linear time. Assume for a contradiction that no such arc exists. By assumption, $\exists x \in V(D)$, $c \in \{1, 2\}$ such that $(d^+(x) > 0 \wedge f_c^+(x) = 0)$ or $(d^-(x) > 0 \wedge f_c^-(x) = 0)$. If $(d^+(x) > 0 \wedge f_c^+(x) = 0)$, let y be any out-neighbour of x, then we must have $f_c(y) = (0, 0)$ for otherwise xy clearly satisfies (3.1). Symmetrically, if $(d^-(x) > 0 \wedge f_c^-(x) = 0)$, let y be any in-neighbour of x, then we have $f_c(y) = (0, 0)$ for otherwise yx satisfies (3.1). In both cases, we conclude on the existence of a vertex y for which $f_c(y) = (0, 0)$. Let Y be the non-empty set of vertices y for which $f_c(y) = (0, 0)$ and Z be $V(D) \setminus Y$, that is the set of vertices z for which $f_c(z) \neq (0, 0)$. Since (D, F) is valid and tight, Z must also be non-empty, for otherwise (D, F) is a monochromatic hard pair. The connectivity of D guarantees the existence of an arc between a vertex in Y and another one in Z. This arc satisfies (3.1).

Once we found an arc uv satisfying (3.1), we proceed as follows to compute an F-dicolouring of D. If $f_c^+(u) = 0 \wedge f_c(v) > (0, 0)$, we colour v with c, otherwise we have $f_c^-(v) = 0 \wedge f_c(u) > (0, 0)$ and we colour u with c. Let $(D', F' = (f'_1, f'_2))$ be the pair reduced from this colouring, and let c' be the colour distinct from c. In the former case, that is v is coloured with c, we obtain ${f'_{c'}}^+(u) = f_{c'}^+(u) \geq d_D^+(u) > d_{D'}^+(u)$. In the latter case, u is coloured with c and ${f'_{c'}}^-(v) = f_{c'}^-(v) \geq d_D^-(v) > d_{D'}^-(v)$. In both cases, we obtain that (D', F') is loose, so the result follows from Lemma 3.2.7. □

The next property, (DS), guarantees that the vertices incident only to digons have symmetric constraints.

$$\forall x, c \quad \text{we have} \quad N^-(x) \neq N^+(x), \quad \text{or} \quad f_c^-(x) = f_c^+(x). \tag{DS}$$

Again, we may solve the instance at this point if (D, F) doesn't satisfy (DS).

Lemma 3.2.14. *Let $(D, F = (f_1, f_2))$ be a valid tight non-hard pair such that D is 2-connected. If the pair does not fulfil property (DS), then D is F-dicolourable. Furthermore, there is a linear-time algorithm checking property (DS), and if the property is not met, computing an F-dicolouring of D.*

Proof. Recall that the data structure encoding D allows checking $N^+(u) = N^-(u)$ in constant time (as this is equivalent to checking $|N^+(u) \cap N^-(u)| = d^-(u) = d^+(u)$), so in linear time we may find a vertex x such that $N^-(x) = N^+(x)$ and $f_c^-(x) \neq f_c^+(x)$ for some $c \in \{1, 2\}$.

Let T be a spanning tree rooted in x and let $\sigma = (v_1, \dots, v_n = x)$ be a leaves-to-root ordering of $V(D)$ with respect to T. For every $j \in [n-1]$, the vertex v_j has at least one neighbour in $\{v_{j+1}, \dots, v_n\}$ (its parent in the spanning tree), so condition (R) is fulfilled. Hence, by Lemma 3.2.5, there is an algorithm that computes an F-dicolouring of $D - x$. It remains to show that the reduced pair (D', F') is non-hard. As D' is the single vertex x, this is equivalent to showing that $f'_c(x) > (0, 0)$ for some c.

Towards a contradiction, suppose $f'_1(x) = f'_2(x) = (0,0)$. For $i \in \{1,2\}$, let d_i be the number of neighbours of x coloured i, and note that $d_1 + d_2 = |N(x)|$. Since $f_i(x) - (d_i, d_i) \leq f'_i(x) = (0,0)$ we have that

$$\forall i \quad f_i(x) \leq (d_i, d_i). \tag{3.2}$$

By tightness, we then have that

$$(|N(x)|, |N(x)|) = f_1(x) + f_2(x) \leq (d_1 + d_2, d_1 + d_2) = (|N(x)|, |N(x)|).$$

Thus, the inequalities of (3.2) are equalities, and both f_1, f_2 are symmetric on x, a contradiction. □

At this point, we have proven that if one of (E) or (DS) doesn't hold, an F-dicolouring of D may be computed in linear time. From now on, we thus consider only instances satisfying both conditions, and move on to solving the base cases of our induction, corresponding to D having a certain structure. We first prove that if D is a bidirected complete graph, and the instance isn't hard, an F-dicolouring of D exists and can be computed in linear time.

Lemma 3.2.15. *Let $(D, F = (f_1, f_2))$ be a valid tight non-hard pair such that D is a bidirected complete graph. Then D is F-dicolourable and there is a linear-time algorithm providing an F-dicolouring of D.*

Proof. We assume that f_1, f_2 are symmetric (i.e. $\forall x, c\ f_c^+(x) = f_c^-(x)$), for otherwise we are done by Lemma 3.2.14. Let v be any vertex such that $f_1^+(v) = \min\{f_1^+(x) \mid x \in V(D)\}$. Let u be any vertex such that $f_1^+(u) > f_1^+(v)$. The existence of u is guaranteed, for otherwise the tightness of (D, F) implies that (D, F) is a complete hard pair, a contradiction. Let $X \subseteq (V(D) \setminus \{u, v\})$ be any set of $f_1^+(v)$ vertices (we have $f_1^+(u) \leq n - 1$, which implies $f_1^+(v) \leq n - 2$, and the existence of X is guaranteed). Note that X, u, and v can be computed in linear time. We colour all the vertices of X with 1, and note that this is an F-dicolouring of $D\langle X \rangle$, as every vertex $x \in X$ satisfies $f_1^+(x) \geq f_1^+(v) = |X| > |X| - 1 = d^+_{D\langle X \rangle}(x)$. Let $(D', F' = (f'_1, f'_2))$ be the pair reduced from this colouring. Observe that we may have $X = \varnothing$ in which case $(D', F') = (D, F)$. Then $f'_1(v) = (0,0)$ and $f'_1(u) > (0,0)$, so (D', F') is not a hard pair. If (D', F') is loose, we are done by Lemma 3.2.7, otherwise we are done by Lemma 3.2.13 (since (D', F') does not fulfil (E)). □

The following shows how to compute, in linear time, an F-dicolouring of D if the underlying graph of D is a cycle.

Lemma 3.2.16. *Let $(D, F = (f_1, f_2))$ be a valid tight non-hard pair such that $\mathrm{UG}(D)$ is a cycle. Then D is F-dicolourable and there is a linear-time algorithm providing an F-dicolouring of D.*

Proof. We assume that (E) holds as otherwise we are done by Lemma 3.2.13. Let us show that for any vertex $x \in V(D)$, we have $d^-(x), d^+(x) \in \{0, 2\}$. Towards a contradiction, and by symmetry, assume that some vertex x verifies $d^-(x) = 1$. By tightness and by (E), we have that $1 = d^-(x) = f_1^-(v) + f_2^-(v) \geq 1 + 1$, a contradiction. We distinguish two cases, depending on whether D contains a digon or not.

Case 1: *D contains a digon.*

In this case, there is only one orientation avoiding in- or out-degree 1, the one with D fully bidirected. We now claim that, by (E'), for every vertex $x \in V(D)$ and every colour c,

$f_c(x) \geq (1,1)$. Hence, by tightness, f_1, f_2 are constant functions equal to $(1,1)$. Since (D,F) is not a hard pair, we conclude that $n(D)$ is even, and any proper 2-colouring of $\mathrm{UG}(D)$ is indeed an F-dicolouring.

Case 2: *D does not contain any digon.*

In this case, the only possible orientation avoiding in- and out-degree 1 is the antidirected cycle, that is an orientation of a cycle in which every vertex is either a source or a sink. In that case we colour every vertex with colour one, and note that for a sink x (resp. a source) we have that $f_1^+(x) = 1 > 0 = d^+(x)$ (resp. $f_1^-(x) = 1 > 0 = d^-(x)$). Hence, this colouring is indeed an F-dicolouring of D. □

A *subwheel* is a graph G such that, for some vertex $v \in V(G)$, $G - v$ is a cycle. The following shows how to compute, in linear time, an F-dicolouring of D if the underlying graph of D is a subwheel.

Lemma 3.2.17. *Let $(D, F = (f_1, f_2))$ be a valid tight non-hard pair, and let $v \in V$ be such that $\mathrm{UG}(D - v)$ is a cycle C. Then D is F-dicolourable and there is a linear-time algorithm providing an F-dicolouring of D.*

Proof. We assume that (E) and (DS) hold as otherwise we are done by Lemma 3.2.13 or Lemma 3.2.14. Let $v_1, \ldots, v_{n-1}$ be an ordering of $V(D) \setminus \{v\}$ along C, which can be obtained in linear time. We distinguish several cases, according to the structure of (D,F). It is straightforward to check, in linear time, which of the following cases matches the structure of (D,F).

Case 1: $|N(v)| < n(D) - 1$.

Let w be any neighbour of v and let y be any vertex that is not adjacent to v. Let ρ_w be $(f_1^-(w) - \mathbb{1}_A(v,w), f_1^+(w) - \mathbb{1}_A(w,v))$, informally, ρ_w corresponds to the new value of $f_1(w)$ if we were to colour v with 1. If $\rho_w \neq (1,1)$, we colour v with 1, otherwise we colour v with 2. Let $(D', F' = (f'_1, f'_2))$ be the pair reduced from colouring v, we claim that (D', F') is not a hard pair.

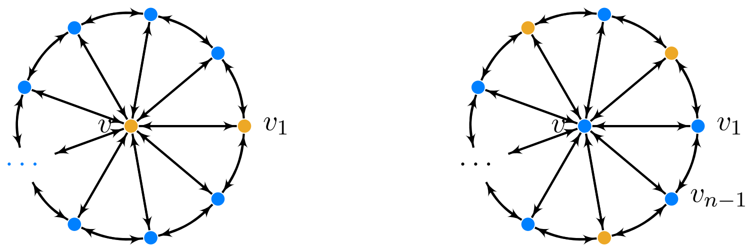
For every $c \in \{1,2\}$, we have $f'_c(y) = f_c(y)$ because y is not adjacent to v, and by (E) $f'_c(y) = f_c(y) > (0,0)$. Therefore (D', F') cannot be a monochromatic hard pair. It is also not a join hard pair because D' is 2-connected.

Assume that (D', F') is a bicycle hard pair. Hence, $f'_1(w) = (1,1)$, so v is coloured 2, as otherwise we would have $f'_1(w) = \rho_w \neq (1,1)$. But if v is coloured 2, then $(1,1) = f'_1(w) = f_1(w)$ and in that case $\rho_w \neq (1,1)$ (as w is adjacent to v), and we should have coloured v with colour 1, a contradiction.

Assume that (D', F') is a complete hard pair, then D' is a bidirected complete graph. Since $\mathrm{UG}(D')$ is a cycle, D' is necessarily $\overleftrightarrow{K_3}$. Such a complete hard pair is also, either a monochromatic hard pair or a bicycle hard pair, but we already discarded both cases. Hence (D', F') is not a hard pair, and the result follows from Lemma 3.2.16, as $\mathrm{UG}(D')$ is a cycle.

Case 2: *D is bidirected, $d^+(v) = n(D) - 1$, $n(D)$ is even, and f_1 is constant on $V(D) \setminus \{v\}$.*

Assume $D \neq \overleftrightarrow{K_4}$ as otherwise the result follows from Lemma 3.2.15. Observe that $D - v$ is a bidirected odd cycle, and that it contains at least 5 vertices, for otherwise D is exactly



(a) (b)

Figure 3.3: The F-dicolourings for Case 2. The partition on the left corresponds to the case $f_1^+(v) \geq 2$ and the one on the right corresponds to the case $f_1^+(v) \leq 1$. Vertices coloured 1 are represented in orange and vertices coloured 2 are represented in blue.

$\overleftrightarrow{K_4}$. Since D is bidirected, by (DS) and (E) we have $f_c^+(x) = f_c^-(x) > 0$ for every $x \in V$ and every $c \in \{1, 2\}$.

Since f_1 is constant on $V(D) \setminus \{v\}$ by assumption, so is f_2, as (D, F) is tight and as $d^-(u) = d^+(u) = 3$ for every $u \in V(D) \setminus \{v\}$. Assume without loss of generality that $f_1(u) = (1, 1)$ and $f_2(u) = (2, 2)$ for every $u \in V(D) \setminus \{v\}$.

If $f_1^+(v) \geq 2$, we colour v and v_1 with 1, and all other vertices with 2, see Figure 3.3(a). The digraph induced by the vertices coloured 2 is a bidirected path, which is strictly-f_2-degenerate since on these vertices, f_2 is the constant function equal to $(2, 2)$. The digraph induced by the vertices coloured 1 is $\overleftrightarrow{K_2}$ and contains v. Since $f_1^+(v) \geq 2$ and $f_1(v_1) = (1, 1)$, it is strictly-f_1-degenerate.

Else, we have $f_1^+(v) \leq 1$, and we colour $v \cup \{v_{2i} \mid i \in \left[\frac{n(D)}{2}\right]\}$ with 2 and $\{v_{2i-1} \mid i \in \left[\frac{n-2}{2}\right]\}$ with 1, see Figure 3.3(b). Vertices coloured 1 form an independent set, so the digraph induced by them is strictly-f_1-degenerate (as f_1 is constant equal to $(1, 1)$ on these vertices). Since $f_1^+(v) \leq 1$, and because (D, F) is valid, we have $f_2^+(v) \geq n - 2$. Since $n(D) \geq 6$ it implies $f_2^+(v) \geq \frac{n(D)}{2} + 1$. Let H be the digraph induced by the vertices coloured 2. Then the out-degree of v in H is exactly $\frac{n(D)}{2}$. Since $f_2^+(v) \geq \frac{n(D)}{2} + 1$, we obtain that the digraph H is strictly-f_2-degenerate if and only if $H - v$ is strictly-f_2-degenerate. Observe that $H - v$ is made of a copy of $\overleftrightarrow{K_2}$ and isolated vertices, so it is strictly-f_2-degenerate since f_2 is constant equal to $(2, 2)$ on $V(H) \setminus \{v\}$.

Case 3: *$N^+(v) = N^-(v) = V(D) \setminus \{v\}$ and $D - v$ is a directed cycle.*

Note that in this case, for every vertex $u \in V(D) \setminus \{v\}$ we have $d^-(u) = d^+(u) = 2$, and by (E) and by tightness of (D, F), we have that $f_1(u) = f_2(u) = (1, 1)$. Since $d^+(v) \geq 3$, there is a colour $c \in \{1, 2\}$ such that $f_c^+(v) \geq 2$. Assume $f_2^+(v) \geq 2$. We are now ready to give an F-dicolouring explicitly. We colour v and v_1 with 2, and all other vertices with 1, see Figure 3.4 for an illustration.

The digraph induced by the vertices coloured 1 is a directed path, which is strictly-f_1-degenerate since on these vertices, f_1 is the constant function equal to $(1, 1)$. The digraph induced by the vertices coloured 2 is $\overleftrightarrow{K_2}$ and contains v. Since $f_2^+(v) \geq 2$ and $f_2(v_1) = (1, 1)$, it is strictly-f_2-degenerate.

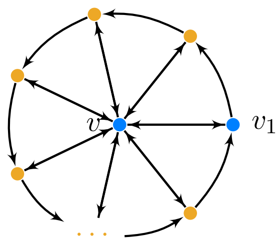


Figure 3.4: The F-dicolouring for Case 3. Vertices coloured 1 are represented in orange and vertices coloured 2 are represented in blue.

Case 4: *None of the previous cases apply.*

We first prove the existence of a vertex $u \neq v$ and a colour $c \in \{1,2\}$ such that $f_c(u) > (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v))$. By (E'), for any vertex $u \neq v$ we have $f_c(u) \geq (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v))$, we thus look for a vertex $u \neq v$ such that $f_c(u) \neq (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v))$. Assume for a contradiction that such a pair does not exist, so for every vertex $x \in V(D) \setminus \{v\}$ and every colour $c \in \{1,2\}$, $f_c(x)$ is exactly $(\mathbb{1}_A(v,x), \mathbb{1}_A(x,v))$. Assume first that there exists a simple arc xv, then $f_1(x) = f_2(x) = (0,1)$. By (E), we deduce that x is a source. Since $\mathrm{UG}(D-v)$ is a cycle, we then have $d^+(x) = 3 > f_1^+(x) + f_2^+(x)$, a contradiction to (D,F) being a valid pair. The existence of a simple arc vx is ruled out symmetrically, so we now assume $N^+(v) = N^-(v)$. Then every vertex $x \in V(D) \setminus \{v\}$ satisfies $f_1(x) = f_2(x) = (\mathbb{1}_A(v,x), \mathbb{1}_A(x,v)) = (1,1)$. Since (D,F) is a tight valid pair, this implies that $d^+(x) = d^-(x) = 2$. Hence $D-v$ is a directed cycle, so Case 3 is matched, a contradiction. This proves the existence of u and c such that $f_c(u) \neq (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v))$. From now on, we assume $c = 1$ without loss of generality, and $f_1(u) > (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v))$.

Consider the following property, which can be checked straightforwardly in linear time:

$$\exists x \in V(D) \setminus \{v\}, \text{ such that } f_1(x) \neq (1 + \mathbb{1}_A(v,x), 1 + \mathbb{1}_A(x,v)) \text{ or } f_2(x) \neq (1,1). \tag{3.3}$$

If $n(D)$ is odd or if (3.3) holds, we colour v with 1. Otherwise, we colour v with 2. In both cases, we claim that the reduced pair $(D', F' = (f_1', f_2'))$ is not hard.

Assume first that $n(D)$ is odd or (3.3) is satisfied, so we have coloured v with 1. Hence, $f_1'(u) = f_1(u) - (\mathbb{1}_A(v,u), \mathbb{1}_A(u,v)) \neq (0,0)$ and $f_2'(u) = f_2(u) > (0,0)$ (by (E)), so (D', F') is not a monochromatic hard pair. If $n(D)$ is odd, then $n(D')$ is even so D' is not a bidirected odd cycle. If a vertex x satisfies (3.3), we have $f_1'(x) = f_1(x) - (\mathbb{1}_A(v,x), \mathbb{1}_A(x,v)) \neq (1,1)$ or $f_2'(x) = f_2(x) \neq (1,1)$. In both cases (D', F') is not a bicycle hard pair, as desired. Since $UG(D')$ is a cycle, if (D', F') was a complete hard pair it would be either a monochromatic or a bicycle hard pair, hence (D', F') is not a hard pair.

Assume finally that $n(D)$ is even and (3.3) is not satisfied. Hence v is coloured with 2 and, for every vertex $x \in V(D) \setminus \{v\}$, $f_1(x) = (1 + \mathbb{1}_A(v,x), 1 + \mathbb{1}_A(x,v))$ and $f_2(x) = (1,1)$. Since every vertex $x \neq v$ is adjacent to v, otherwise Case 1 would match, we have $f_2'(x) \neq f_2(x) = (1,1)$ so (D', F') is not a bicycle hard pair. Assume for a contradiction that it is a monochromatic hard pair. Since $f_1'(x) = f_1(x) \neq (0,0)$ for every vertex $x \neq v$, we have $f_2'(x) = (0,0)$. Since $f_2(x) = (1,1)$, we deduce that $N^+(v) = N^-(v) = V(D) \setminus \{v\}$.

Hence every vertex x satisfies $f_1(x) = (2,2)$ and $f_2 = (1,1)$. Since (D,F) is tight, we conclude that D is bidirected, $n(D)$ is even and f_1 is constant on $V(D) \setminus \{v\}$. Thus Case 2 matches, a contradiction.

Since (D', F') is not a hard pair, and because $\mathrm{UG}(D')$ is a cycle, the result follows from Lemma 3.2.16. □

3.2.2.8 Solving blocks with two colours - general case

The goal of this subsection is to provide an algorithm which, given a pair $(D, F = (f_1, f_2))$ that is not hard and such that D is 2-connected, computes an F-dicolouring of D in linear time. The idea is to find a decomposition of the underlying graph $\mathrm{UG}(D)$, similar to an ear-decomposition, and successively colour the "ears" in such a way that the successive reduced pairs remain non-hard. Let us begin with the definition of the decomposition.

A *CSP-decomposition* (CSP stands for Cycle, Stars, and Paths) of a 2-connected graph G is a sequence $(H_0, \dots, H_r)$ of subgraphs of G partitioning the edges of G, such that H_0 is a cycle, and such that for any $i \in [r]$ the subgraph H_i is either:

- a star with a central vertex v_i of degree at least 2 in H_i, and such that $V(H_i) \cap \left(\bigcup_{0 \le j < i} V(H_j)\right)$ is the set of leaves of H_i, or
- a path $(v_0, \dots, v_\ell)$ of length $\ell \ge 3$, and such that $V(H_i) \cap \left(\bigcup_{0 \le j < i} V(H_j)\right) = \{v_0, v_\ell\}$.

Lemma 3.2.18. *Every 2-connected graph admits a CSP-decomposition. Furthermore, computing such decomposition can be done in linear time.*

Proof. An *ear-decomposition* is similar to a CSP-decomposition except that paths of length one or two are allowed, and that there are no stars. It is well-known that every 2-connected graph G admits an ear-decomposition, and that it can be computed in linear time (see [152] and the references therein). To obtain a CSP-decomposition of G, we first compute an ear-decomposition $(H_0, \dots, H_r)$, which we modify in order to get a CSP-decomposition. We do this in two steps.

Before describing these steps, we have to explain how a decomposition $(H_0, \dots, H_r)$ is encoded. The sequence is encoded as a doubly-linked chain whose cells contain 1) a copy of the subgraph H_i 2) a mapping from the vertices of this copy to their corresponding ones in G, and 3) an integer n_i equal to $|\bigcup_{0 \le j < i} V(H_j)|$. This last integer will allow us, given two cells, corresponding to H_i and H_j, to decide whether $i \le j$ or not. Indeed, as the decomposition will evolve along the execution, maintaining indices from $\{0, \dots, r\}$ seems hard in linear time.

The first step consists in modifying the ear-decomposition in order to get an ear-decomposition avoiding triples $i \le j < k$ such that $H_k = (u_0, u_1)$ is of length one, where u_0, u_1 are inner vertices of H_i, H_j respectively, and where H_j is an ear of length at least three. Towards this, we first go along every ear, in increasing order, to compute $\mathtt{birth}(v)$, the copy of vertex v appearing first in the current ear-decomposition. Note that $\mathtt{birth}(v)$ is always an inner vertex of its ear. Then, we go along every ear in decreasing order, and when we have a length one ear $H_k = (u_0, u_1)$, we consider the ears of $\mathtt{birth}(u_0)$ and $\mathtt{birth}(u_1)$, say H_i and H_j respectively.

If $0 \le i < j$, and H_j is a path of length at least three, we combine H_j and H_k into two ears, H' and H'', each of length at least two (see Figure 3.5 (left)). In the ear-decomposition, we insert H' and H'' at the position of H_j, and delete H_j and H_k.

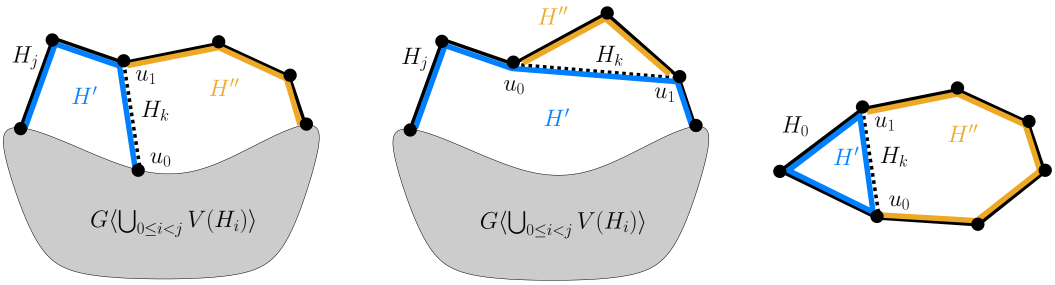


Figure 3.5: Ears H_j and H_k recombined into H' (blue) and H'' (orange), each of length at least two. (left) Case where H_k has one endpoint in H_j. (middle) Case where H_k has its two endpoints in H_j. (right) Case where H_k has its two endpoints in the cycle $H_j = H_0$.

If $0 < i = j$, then $H_j = H_i$ must be a path of length at least four. In that case, we combine H_j and H_k into two ears, H' and H'', each of length at least two (see Figure 3.5 (middle)). In the ear-decomposition, we insert H' and H'' at the position of H_j, and delete H_j and H_k.

If $0 = i = j$, then $H_j = H_0$ must be a cycle of length at least four. In that case, we combine H_j and H_k into two ears, H' and H'', each of length at least two (see Figure 3.5 (right)). In the ear-decomposition, we insert H' and H'' at the position of H_j, and delete H_j and H_k.

Along this process, the ears that are modified are always shortened while still having length at least two. Hence, we do not create new couples H_j, H_k, with the second of length one, that could be recombined into ears of length at least two. Hence, we are done with the first step.

Before proceeding to the second step, recall that we now have an ear-decomposition without triples $i \le j < k$ such that $H_k = (u_0, u_1)$ is of length one, where u_0, u_1 are inner vertices of H_i, H_j respectively, and where H_j is an ear of length at least three. In particular, this implies that for any ear of length one $H_k = (u_0, u_1)$, the ears containing u_0 and u_1 as inner vertices are distinct. We denote them H_i and H_j, respectively, and assume without loss of generality that $i < j$. Note that as H_j and H_k cannot be recombined, we have that H_j has length exactly two and that u_1 is its only inner vertex. The second step then simply consists, for each such ear H_k, in including H_k into H_j, redefining the latter to form a star centred in u_1.

After this process, no ear can be a path of length one. Assimilating the paths of length two as stars, we thus have a CSP-decomposition. □

We move to showing that (D, F) may be inductively coloured following this decomposition (on the underlying graph of D). Again, we only consider instances that are not dealt with in the previous subsection, which in particular satisfy (E) and (DS). Then, in the induction, Lemma 3.2.19 corresponds to colouring a star, and Lemma 3.2.20 corresponds to colouring a path, when those attach to the rest of the graph as in the decomposition. Eventually, bidirected complete graphs, cycles, and subwheels, which were solved in the previous subsection, form the base of our decomposition. The induction is formalised in Lemma 3.2.21, which achieves to solve the case of 2-connected graphs with two colours.

Lemma 3.2.19. *Let $(D = (V, A), F = (f_1, f_2))$ be a valid tight non-hard pair, and let $v \in V(D)$ be such that both D and $D - v$ are 2-connected, and $\mathrm{UG}(D)$ is neither a cycle nor a subwheel centred in v. Then there exists a colour $c \in \{1, 2\}$ such that colouring v with c is safe. Moreover,* *there is an algorithm that either finds c in time $O(d(v))$ or computes an F-dicolouring of D in linear time.*

Proof. We assume that (E) and (DS) hold as otherwise we are done by Lemma 3.2.13 and Lemma 3.2.14. Observe that $d(v) > 0$ and (E) implies that $f_1(v) \neq (0,0)$ and $f_2(v) \neq (0,0)$, so both colours 1 and 2 are available for v. Note also that since $\mathrm{UG}(D-v)$ is not a cycle, in particular colouring v with any colour $c \in \{1,2\}$ may never reduce it to a bicycle hard pair. We distinguish three cases.

Case 1: $|N(v)| \leq n(D) - 2$.

Let u be any neighbour of v and w be any vertex that is not adjacent to v. Since $|N(w)| > 0$, we have $f_1(w) \neq (0,0)$ and $f_2(w) \neq (0,0)$ by (E). Hence colouring v with any colour $c \in \{1,2\}$ does not reduce to a monochromatic hard pair. If $f_1(u)$ is distinct from $(f_1^-(w) + \mathbb{1}_A(vu), f_1^+(w) + \mathbb{1}_A(uv))$, we colour v with 1, otherwise we colour v with 2. By choice of u and w, the reduced pair $(D', F' = (f_1', f_2'))$ satisfies $f_1'(u) \neq f_1'(w)$, so in particular it is not a complete hard pair.

Case 2: $|N(v)| = n(D) - 1$ *and* $D - v$ *is distinct from* $\overleftrightarrow{K}_{n(D)-1}$.

We only have to guarantee the existence of $c \in \{1,2\}$ such that colouring v with c does not reduce to a monochromatic hard pair. Also note that $|N(v)| = n(D) - 1$ allows us $O(n(D))$ time for computing c.

If there exists a vertex $x \neq v$ and a colour $c \in \{1,2\}$ such that $f_c(x) > (\mathbb{1}_A(vx), \mathbb{1}_A(xv))$, then colouring v with c is safe as it does not reduce to a monochromatic hard pair as $f_c'(x) = f_c(x) - (\mathbb{1}_A(vx), \mathbb{1}_A(xv)) > (0,0)$ and as $f_{c'}'(x) = f_{c'}(x) \neq (0,0)$ by (E) for the colour $c' \neq c$. Note that, if it exists, such a vertex x can be found in time $O(n(D))$.

We now prove the existence of such a vertex x. Assume for a contradiction that $f_c(x) \not> (\mathbb{1}_A(vx), \mathbb{1}_A(xv))$ for every $x \neq v$ and every $c \in \{1,2\}$. Observe that, for every $x \neq v$, $f_c(x) \geq (\mathbb{1}_A(vx), \mathbb{1}_A(xv))$ by (E'), so we have $f_1(x) = f_2(x) = (\mathbb{1}_A(vx), \mathbb{1}_A(xv))$. Let $x \neq v$ be any vertex. If xv is a simple arc, then $f_1(x) = f_2(x) = (0,1)$ so $d^+(x) = 2$ and $d^-(x) = 0$. In particular, x has degree 1 in $\mathrm{UG}(D-v)$, a contradiction since $\mathrm{UG}(D-v)$ is 2-connected and distinct from K_2 (as $\mathrm{UG}(D)$ is not a cycle). The case of vx being a simple arc is symmetric, so we now assume $N^+(v) = N^-(v) = V \setminus \{v\}$, which implies $f_1(x) = f_2(x) = (1,1)$ for every vertex $x \neq v$. Hence in $D - v$ every vertex has in- and out-degree 1. Since $\mathrm{UG}(D-v)$ is 2-connected and distinct from K_2, $D - v$ is necessarily a directed cycle, a contradiction.

Case 3: $|N(v)| = n(D) - 1$ *and* $D - v$ *is* $\overleftrightarrow{K}_{n(D)-1}$.

If $N^+(v) = N^-(v)$ then D is $\overleftrightarrow{K}_{n(D)}$, and since (D,F) is not hard, Lemma 3.2.15 yields that it is F-dicolourable and an F-dicolouring is computed in linear time.

Henceforth we assume that there exists a simple arc between v and some $u \in V(D) \setminus \{v\}$. The vertex u being incident to some digon, (E) implies that $f_c(u) \geq (1,1)$ for every colour c. This guarantees that colouring v with any colour $c \in \{1,2\}$ does not reduce to a monochromatic hard pair, since in the reduced pair the constraints for u are unchanged on one coordinate. We thus have to guarantee that for some $c \in \{1,2\}$, colouring v with c does not reduce to a complete hard pair.

We assume that the simple arc between v and u goes from u to v, the other case being symmetric. If v has at least one out-neighbour w, then either $f_1^-(u) = f_1^-(w)$ and colouring v with 1 does not reduce to a complete hard pair, or $f_1^-(u) \neq f_1^-(w)$ and colouring v with 2 does not reduce to a complete hard pair.

Finally, if v is a sink and colouring v with 1 reduces to a complete hard pair, then $f_1^+(x) = f_1^-(x) + 1$ for every vertex $x \neq v$. Therefore, colouring v with 2 does not reduce to a complete hard pair, as in the reduced pair we have $f_1'(x) = f_1(x)$ and $f_1^+(x) \neq f_1^-(x)$. $\square$

Lemma 3.2.20. *Let $(D = (V, A), F = (f_1, f_2))$ be a valid pair satisfying* (E) *and let $v_0, \dots, v_\ell$ be a path of $G = \mathrm{UG}(D)$ of length $\ell \geq 3$ such that $d_G(v_i) = 2$ for $i \in [\ell - 1]$, and such that both $G = \mathrm{UG}(D)$ and $G' = \mathrm{UG}(D - \{v_1, \dots, v_{\ell-1}\})$ are 2-connected and contain a cycle. There is an algorithm computing an F-dicolouring of $D\langle\{v_1, \dots, v_{\ell-1}\}\rangle$ in time $O(\ell)$ in such a way that the reduced pair (D', F') is not hard.*

Proof. Since (E) is satisfied, in particular, for any vertex x and colour c, we have $f_c(x) \neq (0, 0)$. Note that $V \setminus \{v_0, \dots, v_\ell\} \neq \varnothing$ for otherwise G' does not contain a cycle, a contradiction. Let u be any vertex in $V \setminus \{v_0, \dots, v_\ell\}$. If $f_1(v_0) = f_1(u)$, we colour v_1 with 1, otherwise we colour v_1 with 2. We then (greedily) colour $v_2, \dots, v_{\ell-1}$ using the algorithm of Lemma 3.2.5. For every $j \in \{2, \dots, \ell - 1\}$, the vertex v_j has at least one neighbour in $\{v_{j+1}, \dots, v_\ell\}$, v_{j+1}, so condition (R) is fulfilled, and we are ensured that the obtained colouring is an F-dicolouring $D\langle\{v_1, \dots, v_{\ell-1}\}\rangle$.

Let $(D', F' = (f_1', f_2'))$ be the pair reduced from this colouring. As D' is 2-connected, (D', F') is not a join hard pair. By construction, we have $f_1'(v_0) \neq f_1'(u)$ as v_0 is adjacent to v_1 but u is not adjacent to any v_i with $i \in [1, \ell - 1]$, which we just coloured. Hence (D', F') is neither a bicycle hard pair nor a complete hard pair, as f_1' is not constant on $V(D')$. Finally, (E) ensures that $f_1'(u) = f_1(u) \neq (0, 0)$ and $f_2'(u) = f_2(u) \neq (0, 0)$. Hence (D', F') is not a monochromatic hard pair. $\square$

With these lemmas in hand, we are ready to prove the main result of this subsection, that non-hard 2-connected pairs are F-dicolourable.

Lemma 3.2.21. *Let $(D = (V, A), F = (f_1, f_2))$ be a valid non-hard pair. There exists an algorithm providing an F-dicolouring of D in linear time.*

Proof. We denote by $\mathcal{P}$ the property of (D, F) being tight, fulfilling (E) and (DS), not being a bidirected complete graph.

We first check in linear time that each property of $\mathcal{P}$ holds. If one does not, we are done by one of Lemmas 3.2.7, 3.2.13, 3.2.14 and 3.2.15.

We then compute a CSP-decomposition $(H_0, \dots, H_r)$ of $\mathrm{UG}(D)$ in linear time, which is possible by Lemma 3.2.18. If $r = 0$ then $\mathrm{UG}(D)$ is a cycle and the result follows from Lemma 3.2.16, assume now that $r \geq 1$.

For each i going from r to 2, we proceed as follows (if $r = 1$ we skip this part). If H_i is a path of length $\ell \geq 3$, we colour $v_1, \dots, v_{\ell-1}$ in time $O(\ell)$ in such a way that the reduced pair (D', F') is not hard, which is possible by Lemma 3.2.20. We then check in constant time that (D', F') satisfies $\mathcal{P}$ (we only have to check that it still holds for v_0 and v_ℓ, and this is done in constant time). If it does not, we are done by one of Lemmas 3.2.7, 3.2.13, 3.2.14 and 3.2.15. If H_i is a star with central vertex v, then we compute a safe colouring of v in time $O(d(v))$, which is possible by Lemma 3.2.19. Note that, at this step, we may directly find an F-dicolouring in linear time, in which case we stop here. Assume we do not stop, then we obtain a reduced pair (D', F'). We then check in time $O(d(v))$ that (D', F') satisfies $\mathcal{P}$ (we only have to check that it still holds for the neighbours of v in H_i). Again, if it does not, we are done by one of Lemmas 3.2.7, 3.2.13, 3.2.14 and 3.2.15.

Assume we have not already found an F-dicolouring by the end of this process, and consider H_1. If H_1 is a star, we are done by Lemma 3.2.17. Otherwise, H_1 is a path, and we colour it as in the process above, then we conclude by colouring H_0 through Lemma 3.2.16.

Since $(H_0, \dots, H_r)$ partitions the edges of $\mathrm{UG}(D)$, the total running time of the described algorithm is linear in the size of D. □

3.2.2.9 Proof of Theorem 3.2.1

We are now ready to prove Theorem 3.2.1, that we first recall here for convenience.

Theorem 3.2.1. *Let (D, F) be a valid pair. Then D is F-dicolourable if and only if (D, F) is not a hard pair. Moreover, there is an algorithm running in time $O(n(D) + m(D))$ that decides if (D, F) is a hard pair, and that outputs an F-dicolouring if it is not.*

Proof. Let (D, F) be a valid pair. We first check in linear time whether (D, F) is tight. If it is not, we are done by Lemma 3.2.7. Henceforth assume that (D, F) is tight. We then check whether (D, F) is a hard pair, which is possible in linear time by Lemma 3.2.11. If (D, F) is a hard pair, then D is not F-dicolourable by Lemma 3.2.2 and we can stop. Then, we may assume that (D, F) is not a hard pair. In linear time, we find a block B of D and compute a safe colouring α of $V(D) \setminus V(B)$, which is possible by Lemma 3.2.10. Let (B, F_B) be the reduced pair, which is not hard, then every F_B-dicolouring of B, together with α, extends to an F-dicolouring of D. Then, in linear time, we either find an F_B-dicolouring of B, or we compute a valid pair $(B, \widetilde{F}_B = (\widetilde{f}_1, \widetilde{f}_2))$ such that $(B, \widetilde{F})$ is not hard, and an F_B-dicolouring of B can be computed in linear time from any $\widetilde{F}_B$-dicolouring of B. This is possible by Lemma 3.2.12.

If we have not yet found an F_B-dicolouring of B, we compute an $\widetilde{F}_B$-dicolouring of B in linear time, which is possible by Lemma 3.2.21. From this we compute an F_B-dicolouring of B in linear time. As mentioned before, this F_B-dicolouring together with α gives an F-dicolouring of D, obtained in linear time. □

3.3 Strengthening the Directed Brooks Theorem on oriented graphs

Recall that every digraph D satisfies $\vec{\chi}(D) \leq \Delta_{\min}(D)+1$ (this is obtained via an easy greedy procedure). Hence, one can wonder if Brooks Theorem can be extended to digraphs using $\Delta_{\min}(D)$ instead of $\Delta_{\max}(D)$. Unfortunately, Aboulker and Aubian [2] proved that, given a digraph D, deciding whether D is $\Delta_{\min}(D)$-dicolourable is NP-complete. Thus, unless P=NP, we cannot expect an easy characterisation of digraphs satisfying $\vec{\chi}(D) = \Delta_{\min}(D) + 1$. In the reduction of [2], the built digraphs have large bidirected cliques. Hence it is natural to ask what is happening for digraphs with no large bidirected complete subgraphs, and it particular what is happening for oriented graphs.

Let the *maximum geometric mean* of a digraph D be $\widetilde{\Delta}(D) = \max\{\sqrt{d^+(v)d^-(v)} \mid v \in V(D)\}$. By definition, we have $\Delta_{\min}(D) \leq \widetilde{\Delta}(D) \leq \Delta_{\max}(D)$. Restricted to oriented graphs, Harutyunyan and Mohar [92] have strengthened Theorem 3.1.5 by proving the following.

Theorem 3.3.1 (Harutyunyan and Mohar [92]). *There is an absolute constant Δ_1 such that every oriented graph $\vec{G}$ with $\tilde{\Delta}(\vec{G}) \geq \Delta_1$ satisfies $\vec{\chi}(\vec{G}) \leq (1 - e^{-13})\tilde{\Delta}(\vec{G})$.*

We here give another strengthening of Theorem 3.1.5 on a large class of digraphs which contains oriented graphs. We first need a few definitions. Given two digraphs H_1 and H_2, let $H_1 \Rightarrow H_2$ denote the *directed join* of H_1 and H_2, that is the digraph obtained from disjoint copies of H_1 and H_2 by adding all arcs from the copy of H_1 to the copy of H_2 (H_1 or H_2 may be empty). For every fixed integer $k \geq 1$, let $\mathscr{F}_k$ be the finite class of digraphs defined as follows.

$$\mathscr{F}_k = \begin{cases} \left\{\overleftrightarrow{K_k}\right\} & \text{if } k \leq 2 \\ \left\{\overleftrightarrow{K_s} \Rightarrow \overleftrightarrow{K_{k+1-s}} \mid 0 \leq s \leq k+1\right\} & \text{otherwise.} \end{cases}$$

Given a vertex v in a digraph D, we define the $\mathscr{F}$-degree of v, denoted by $d_{\mathscr{F}}(v)$, as the largest integer k such that v belongs to a copy of $H \in \mathscr{F}_k$ in D. The following is the main result of this section.

Theorem 3.3.2. *Let $D = (V, A)$ be a digraph and L be a list assignment of D such that, for every vertex $v \in V$,*

$$|L(v)| \geq \max\left(d_{\mathscr{F}}(v) + 1, d_{\min}(v)\right).$$

Then D is L-dicolourable.

The proof of Theorem 3.3.2 is given in Section 3.3.1. We discuss its consequences in the remaining of this section. Since in an oriented graph, every vertex v satisfies $d_{\mathscr{F}}(v) = 1$, we directly obtain the following.

Corollary 3.3.3. *Let $\vec{G}$ be an oriented graph and L be a list assignment of $\vec{G}$ such that, for every vertex $v \in V$,*

$$|L(v)| \geq \max\left(2, d_{\min}(v)\right).$$

Then $\vec{G}$ is L-dicolourable.

Interestingly, Harutyunyan and Mohar [91] proved that deciding if an oriented graph $\vec{G}$ is L-dicolourable is NP-complete even if L satisfies $|L(v)| \geq d_{\min}(v)$. Corollary 3.3.3 shows that the hardness of this problem only comes from vertices with in- or out-degree exactly 1.

The following is another consequence of Theorem 3.3.2.

Corollary 3.3.4. *Let D be a digraph. If $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, then one of the following holds:*

- $\Delta_{\min}(D) \leq 1$,
- $\Delta_{\min}(D) = 2$ *and D contains $\overleftrightarrow{K_2}$, or*
- $\Delta_{\min}(D) \geq 3$ *and D contains $\overleftrightarrow{K_r} \Rightarrow \overleftrightarrow{K_s}$, for some $r, s \geq 0$ such that $r + s = \Delta_{\min}(D) + 1$.*

Proof. Let D be a digraph such that $\vec{\chi}(D) = \Delta_{\min}(D) + 1 \geq 3$. If $\Delta_{\min}(D) = 2$, then Corollary 3.3.3 implies that D is not an oriented graph, and in particular it contains $\overleftrightarrow{K_2}$.

Henceforth assume that $\Delta_{\min}(D) \geq 3$. Let L be the constant list assignment which associates $[\Delta_{\min}(D)]$ to every vertex $v \in V(D)$. Since $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, Theorem 3.3.2 implies that some vertex v satisfies $d_{\mathscr{F}}(v) \geq |L(v)| = \Delta_{\min}(D)$. By definition of $d_{\mathscr{F}}(v)$, we conclude that D contains $\overleftrightarrow{K_r} \Rightarrow \overleftrightarrow{K_s}$, for some $r, s \geq 0$ such that $r + s = \Delta_{\min}(D) + 1$. □

In particular, we conclude that every digraph D satisfying $\vec{\chi}(D) = \Delta_{\min}(D) + 1$ contains a large complete bidirected graph as a subdigraph.

Corollary 3.3.5. *Let D be a digraph. If $\vec{\chi}(D) = \Delta_{\min}(D) + 1$, then D contains the complete bidirected graph on $\left\lceil \frac{\Delta_{\min}+1}{2} \right\rceil$ vertices as a subdigraph.*

Moreover, since an oriented graph does not contain any digon, Corollary 3.3.5 implies the following.

Corollary 3.3.6. *Let $\vec{G}$ be an oriented graph. If $\Delta_{\min}(\vec{G}) \geq 2$, then $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.*

As an interesting particular case, every orientation $\vec{G}$ of a graph G satisfying $\Delta(G) = 5$ is 2-dicolourable, which answers by the affirmative a question of Harutyunyan [90].

We denote by k-DICOLOURABILITY the problem of deciding if a given digraph D is k-dicolourable. We finally prove the following in Section 3.3.2, which shows that Corollary 3.3.5 is in some sense best possible.

Theorem 3.3.7. *For every fixed $k \geq 2$, k-DICOLOURABILITY remains NP-complete when restricted to digraphs D satisfying $\Delta_{\min}(D) = k$ and not containing the bidirected complete graph on $\left\lceil \frac{k+1}{2} \right\rceil + 1$ vertices.*

3.3.1 Proof of Theorem 3.3.2

This section is devoted to the proof of Theorem 3.3.2, which we first recall here for convenience.

Theorem 3.3.2. *Let $D = (V, A)$ be a digraph and L be a list assignment of D such that, for every vertex $v \in V$,*

$$|L(v)| \geq \max\left(d_{\mathscr{F}}(v) + 1, d_{\min}(v)\right).$$

Then D is L-dicolourable.

Proof. Let D be a digraph and L be a list assignment such that, for every vertex $v \in V(D)$, we have

$$|L(v)| \geq \max\left(d_{\mathscr{F}}(v) + 1, d_{\min}(v)\right).$$

Assume for a contradiction that D is not L-dicolourable. Let (X, Y) be the following partition of $V(D)$:

$$X = \{v \in V(D) \mid d^+(v) \leq d^-(v)\} \quad \text{and} \quad Y = \{v \in V(D) \mid d^-(v) < d^+(v)\}.$$

We define an auxiliary digraph $\widetilde{D}$ as follows:

- $V(\widetilde{D}) = V(D)$,
- $A(\widetilde{D}) = A(D\langle X\rangle) \cup A(D\langle Y\rangle) \cup \{xy, yx \mid xy \in A(D), x \in X, y \in Y\}$.

Claim 3.3.8. *$\widetilde{D}$ is not L-dicolourable.*

Proof of claim. Assume for a contradiction that there exists an L-dicolouring α of $\widetilde{D}$. Then D, coloured with α, must contain a monochromatic directed cycle C. Now C is not contained in X nor Y, for otherwise C, coloured with α, would be a monochromatic directed cycle of $D\langle X \rangle$ or $D\langle Y \rangle$ and so a monochromatic directed cycle of $\widetilde{D}$. Thus C contains an arc xy from X to Y. But $[x, y]$, coloured with α, is a monochromatic digon in $\widetilde{D}$, a contradiction. $\diamondsuit$

We let $H \subseteq \widetilde{D}$ be an induced subdigraph of $\widetilde{D}$ that is not L-dicolourable and such that $n(H)$ is minimum for this property (by H being L-dicolourable we mean that H is L_H-dicolourable where L_H is the restriction of L to $V(H)$). By choice of H, for every vertex $v \in V(H)$, we have $d^+_H(v) \geq |L(v)|$ and $d^-_H(v) \geq |L(v)|$, for otherwise an L-dicolouring of $H - v$ could be extended to H by choosing for v a colour $c \in L(v)$ that is not appearing in its out-neighbourhood or in its in-neighbourhood. We define X_H as $X \cap V(H)$ and Y_H as $Y \cap V(H)$. Note that both $H\langle X_H \rangle$ and $H\langle Y_H \rangle$ are subdigraphs of D.

Claim 3.3.9. *For every vertex $v \in V(H)$, we have $d^+_H(v) = d^-_H(v) = |L(y)|$.*

Proof of claim. Let ℓ be the number of digons between X_H and Y_H in H. Recall that every vertex $x \in X$ satisfies $d^+_D(x) \leq d^-_D(x)$, so $d^+_D(x) = d_{\min}(x) \leq |L(v)|$. In the construction of $\widetilde{D}$ from D, we only added arcs from Y to X. Since H is a subdigraph of $\widetilde{D}$, we thus have $d^+_H(x) \leq |L(x)|$ for every vertex $x \in X_H$. As observed above, $d^+_H(x) \geq |L(x)|$, so $d^+_H(x) = |L(x)|$ for every vertex $x \in X_H$.

Note also that, in H, ℓ is exactly the number of arcs leaving X_H and exactly the number of arcs entering X_H. We get:

$$\begin{aligned} \sum_{x \in X_H} |L(x)| &= \sum_{x \in X_H} d^+_H(x) \\ &= \ell + m(H\langle X_H \rangle) \\ &= \sum_{x \in X_H} d^-_H(x). \end{aligned}$$

Since $d^-_H(x) \geq |L(x)|$ for every $x \in X_H$, this implies $d^+_H(x) = d^-_H(x) = |L(x)|$ for every vertex $x \in X_H$. Using a symmetric argument, we obtain $d^+_H(y) = d^-_H(y) = |L(y)|$ for every vertex $y \in Y_H$. $\diamondsuit$

In particular, Claim 3.3.9 implies $|L(v)| \geq d_{\max}(v)$ for every vertex $v \in V(H)$. Since H is not L-dicolourable, by Theorem 3.1.6, H must be a directed Gallai tree. Let B be an end-block of H, recall that B is a bidirected complete graph, a bidirected odd cycle or a directed cycle. We distinguish these three possible cases.

Case 1: *B is a directed cycle.*

Let $v \in V(B)$ be any vertex that is not a cut-vertex of H. Then $d^+_H(v) = d^-_H(v) = 1$, so $|L(v)| = 1$. This is a contradiction because every vertex v satisfies $d_{\mathscr{F}}(v) \geq 1$ (in D) and by assumption $|L(v)| \geq d_{\mathscr{F}}(v) + 1$.

Case 2: *B is a bidirected odd cycle.*

Then at least one digon $[u, v]$ of B belongs to $H\langle X_H \rangle$ or $H\langle Y_H \rangle$, for otherwise B would be bipartite (with bipartition $(X_H \cap V(B), Y_H \cap V(B))$). In particular, $[u, v]$ is a digon in D. By definition of an end-block, at most one of $\{u, v\}$ is a cut-vertex of H. Assume without loss of generality that v is not. This implies $|L(v)| = d^+_H(v) = d^-_H(v) = 2$. Since v is incident to a digon in D we have $d_{\mathscr{F}}(v) \geq 2$ in D. This contradicts $|L(v)| \geq d_{\mathscr{F}}(v) + 1$.

Case 3: *B is a bidirected complete graph on $k+1 \geq 4$ vertices.*

Let $v \in V(B)$ be any vertex that is not a cut-vertex of H. We hence have $d^+_H(v) = d^-_H(v) = |L(v)| = k$. Let X_B and Y_B denote respectively $X \cap V(B)$ and $Y \cap V(B)$. Both $B\langle X_B \rangle$ and $B\langle Y_B \rangle$ are subdigraphs of D, and every arc from X_B to Y_B belongs to D. Hence the subdigraph of D induced by $V(B)$ contains a copy of some digraph in $\mathscr{F}_k$. Since $v \in V(B)$, we have $d_{\mathscr{F}}(v) \geq k$ in D. This contradicts $|L(v)| \geq d_{\mathscr{F}}(v) + 1$. □

3.3.2 Proof of Theorem 3.3.7

This section is devoted to the proof of Theorem 3.3.7 which we recall here for convenience.

Theorem 3.3.7. *For every fixed $k \geq 2$, k-DICOLOURABILITY remains NP-complete when restricted to digraphs D satisfying $\Delta_{\min}(D) = k$ and not containing the bidirected complete graph on $\left\lceil \frac{k+1}{2} \right\rceil + 1$ vertices.*

Proof. We reduce from k-DICOLOURABILITY, which is known to be NP-complete for every fixed $k \geq 2$ (see [50]). Let us fix $k \geq 2$ and let $D = (V, A)$ be an instance of k-DICOLOURABILITY. We build $D' = (V', A')$ as follows:

- For each vertex $x \in V$, we associate a copy of $S^-_x \Rightarrow S^+_x$ where S^-_x is the bidirected complete graph on $\left\lfloor \frac{k+1}{2} \right\rfloor$ vertices, and S^+_x is the bidirected complete graph on $\left\lceil \frac{k+1}{2} \right\rceil$ vertices.
- For each arc $xy \in A$, we associate all possible arcs x^+y^- in A', such that $x^+ \in S^+_x$ and $y^- \in S^-_y$.

First observe that $\Delta_{\min}(D') = k$. Let v be a vertex of D', if v belongs to some S^+_x, then $d^-(v) = k$, otherwise it belongs to some S^-_x and then $d^+(v) = k$. Then observe that D' does not contain the bidirected complete graph on $\left\lceil \frac{k+1}{2} \right\rceil + 1$ vertices since every digon in D' is contained in some S^+_x or S^-_x. Thus we only have to prove that $\vec{\chi}(D) \leq k$ if and only if $\vec{\chi}(D') \leq k$ to get the result.

Let us first prove that $\vec{\chi}(D) \leq k$ implies $\vec{\chi}(D') \leq k$.

Assume that $\vec{\chi}(D) \leq k$. Let $\phi\colon V \to [k]$ be a k-dicolouring of D. Let ϕ' be the k-dicolouring of D' defined as follows: for each vertex $x \in V$, choose arbitrarily $x^- \in S^-_x$, $x^+ \in S^+_x$, and set $\phi'(x^-) = \phi'(x^+) = \phi(x)$. Then choose a distinct colour for every other vertex v in $S^-_x \cup S^+_x$, and set $\phi'(v)$ to this colour. We obtain that ϕ' must be a k-dicolouring of D': for each $x \in V$, every vertex but x^- in S^-_x must be a sink in its colour class, and every vertex but x^+ in S^+_x must be a source in its colour class. Thus if D', coloured with ϕ', contains a monochromatic directed cycle C', then C' must be of the form $x_1^- x_1^+ x_2^- x_2^+ \cdots x_\ell^- x_\ell^+ x_1^-$. But then $C = x_1 x_2 \cdots x_\ell x_1$ is a monochromatic directed cycle in D coloured with ϕ, a contradiction.

Reciprocally, let us prove that $\vec{\chi}(D') \leq k$ implies $\vec{\chi}(D) \leq k$, so assume that $\vec{\chi}(D') \leq k$. Let $\phi'\colon V' \to [k]$ be a k-dicolouring of D'. Let ϕ be the k-dicolouring of D defined as follows. For each vertex $x \in V$, we know that $|S^+_x \cup S^-_x| = k+1$, thus there must be two vertices x^+ and x^- in $S^+_x \cup S^-_x$ such that $\phi'(x^+) = \phi'(x^-)$. Moreover, since both S^+_x and S^-_x are bidirected, one of these two vertices belongs to S^+_x and the other one belongs to S^-_x. We assume without loss of generality $x^+ \in S_x^+$ and $x^- \in S_x^-$. Then we set $\phi(x) = \phi'(x^+)$. We get that ϕ must be a k-dicolouring of D. If D, coloured with ϕ, contains a monochromatic directed cycle $C = x_1 x_2 \cdots x_\ell x_1$, then $C' = x_1^- x_1^+ x_2^- x_2^+ \cdots x_\ell^- x_\ell^+ x_1^-$ is a monochromatic directed cycle in D' coloured with ϕ', a contradiction. $\square$

CHAPTER 4

Minimum density of dicritical digraphs

This chapter contains joint work with Frédéric Havet, Clément Rambaud, and Michael Stiebitz and is based on [97, 141].

4.1 Minimum number of edges in critical graphs

An easy observation is that if a graph is k-colourable, then so is each of its subgraphs. This leads to the study of graphs that are in some way minimal obstructions to $(k-1)$-colourability, which are exactly k-critical graphs. For a non-negative integer k, recall that a graph G is *critical* and *k-critical* if every proper subgraph G' of G satisfies $\chi(G') < \chi(G) = k$.

The only graph with chromatic number 0 is the empty graph, and a graph is 1-colourable if and only if it is edgeless. Therefore, for $k \leq 2$, K_k is the only k-critical graph. Furthermore, it is well-known that a graph is 3-critical if and only if it is an odd cycle (see [58, Proposition 1.6.1]). When $k \geq 4$, the class of k-critical graphs is very dense and an easy characterisation of such graphs is very unlikely.

The concept of critical graphs is due to Dirac. In the 1950s he established the basic properties of critical graphs in a series of papers [59, 60, 61, 62] and started to investigate the function $g_k(n)$, defined as

$$g_k(n) = \min \{m(G) \mid G \text{ is } k\text{-critical and has order } n\},$$

with the convention $g_k(n) = +\infty$ if there exists no such graph. For $k \geq 4$, he showed in [61] that there exists a k-critical graph of order n if and only if $n \geq k$ and $n \neq k+1$ (see also [137, Theorem 11.7.5]).

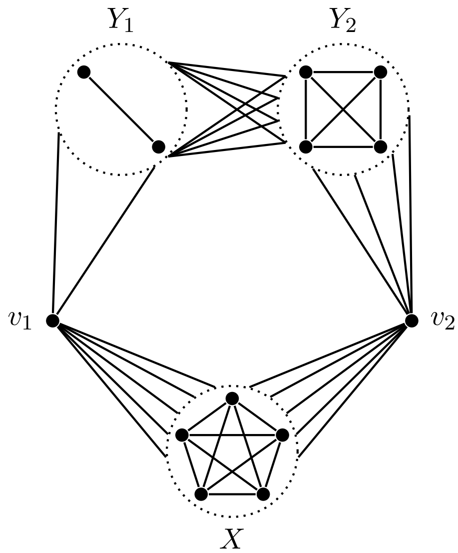


Figure 4.1: An example of a graph in $\mathcal{DG}(7)$.

It is easy to show that every k-critical graph has minimum degree at least $k-1$, which gives the trivial lower bound $2g_k(n) \geq (k-1)n$. By Brooks Theorem (Theorem 3.1.1) it follows that $2g_k(n) = (k-1)n$ if and only if $n = k$ or $k = 3$ and n is odd. For every $k \geq 3$ and $n \geq k+2$, Dirac proved in [62] (see also [64]) that

$$2g_k(n) \geq (k-1)n + k - 3.$$

For $k \geq 3$, we denote by $\mathcal{DG}(k)$ the finite class of graphs defined as follows. A graph G belongs to $\mathcal{DG}(k)$ if and only if $V(G)$ consists of three non-empty pairwise disjoint sets X, Y_1, Y_2 with $|Y_1| + |Y_2| = |X| + 1 = k - 1$ and two additional vertices v_1 and v_2 such that both $G\langle X \rangle$ and $G\langle Y_1 \cup Y_2 \rangle$ are complete graphs not joined by any edge in G, and $X \cup Y_i$ is exactly the neighbourhood of v_i for $i \in [2]$. See Figure 4.1 for an illustration.

The class $\mathcal{DG}(k)$ was introduced by Dirac [64] and Gallai [76]. For $k \geq 4$ and $n \geq k+2$, after proving the inequality above, Dirac also proved in [64] that a k-critical graph on n vertices has exactly $(k-1)n + k - 3$ edges if and only if it belongs to $\mathcal{DG}(k)$.

Given two graphs G_1 and G_2, the *Dirac join* of G_1 and G_2, denoted by $G_1 \boxplus G_2$, is the graph obtained from disjoint copies of G_1 and G_2 by adding all the edges between $V(G_1)$ and $V(G_2)$. It is straightforward that $G_1 \boxplus G_2$ is $(\chi(G_1) + \chi(G_2))$-critical if and only if both G_1 and G_2 are critical. In 1963, Gallai published two fundamental papers [76, 77] about the structure of critical graphs. In particular, he proved the two following remarkable results (see [131, 158] for alternative proofs of Theorem 4.1.1).

Theorem 4.1.1 (Gallai)**.** *Let G be a k-critical graph of order n. If $n \leq 2k-2$, then $\overline{G}$ is disconnected.*

Theorem 4.1.2 (Gallai)**.** *Let $n = k + p$ be an integer, where $k, p \in \mathbb{N}$ and $2 \leq p \leq k-1$, then*

$$g_k(n) = \binom{n}{2} - (p^2 + 1).$$

Moreover, a k-critical graph G of order n has exactly $g_k(n)$ edges if and only if $G = G_1 \boxplus G_2$ where $G_1 = K_{k-p-1}$ and $G_2 \in \mathcal{DG}(p+1)$.

In 2014 Kostochka and Yancey established the following lower bound for $g_k(n)$ when they first popularised the potential method.

Theorem 4.1.3 (Kostochka and Yancey [114])**.** *Let n and k be two integers such that $n \geq k \geq 4$ and $n \neq k+1$, then*

$$g_k(n) \geq \frac{1}{2}\left(k - \frac{2}{k-1}\right)n - \frac{k(k-3)}{2(k-1)}$$

This lower bound is sharp when $k \geq 4$ and $n = 1 \bmod (k-1)$, because of a construction due to Hajós [88]. They gave a much simpler proof of the case $k = 4$ in a second work [115]. It is of interest that $g_4(n) = \left\lceil \frac{5n-2}{3} \right\rceil$ implies Grötzsch's famous theorem [85] stating that every triangle-free planar graph is 3-colourable. This result of Kostochka and Yancey confirmed a conjecture of Gallai [76] and improved on earlier results [117, 112] (see also [109]).

In 2018 Kostochka and Yancey [116] refined their result by describing all n-vertex k-critical graphs G with $m(G) = \frac{1}{2}\left(k - \frac{2}{k-1}\right)n - \frac{k(k-3)}{2(k-1)}$. All of them contain a copy of K_{k-2}, which motivated the following conjecture of Postle [143].

Conjecture 4.1.4 (Postle [143])**.** *For every integer $k \geq 4$, there exists $\varepsilon_k > 0$ such that every k-critical K_{k-2}-free graph G on n vertices has at least $\frac{1}{2}\left(k - \frac{2}{k-1} + \varepsilon_k\right)n - \frac{k(k-3)}{2(k-1)}$ edges.*

For $k = 4$, the conjecture trivially holds, as there is no K_2-free 4-critical graph. Moreover, this conjecture has been confirmed for $k = 5$ by Postle [143], for $k = 6$ by Gao and Postle [78], and for $k \geq 33$ by Gould, Larsen, and Postle [83].

4.2 Minimum number of arcs in dicritical digraphs

Analogously to the undirected case, recall that a digraph is *dicritical* and *k-dicritical* if it is not $(k-1)$-dicolourable, but all of its proper subdigraphs are. The interest in k-dicritical graphs arises in a similar way as the interest in k-critical graphs. While the only 1-dicritical digraph is the digraph on one vertex and a graph is 2-dicritical if and only if it is a directed cycle, already 3-dicritical digraphs have a very diverse structure. Analogues of Hajós' construction have been found by Bang-Jensen, Bellitto, Schweser, and Stiebitz [17] (Theorem 2.2.4). Again, it is natural to consider $d_k(n)$, the minimum number of arcs of a k-dicritical digraph of order n, with the convention $d_k(n) = +\infty$ if no such digraph exists. Observe that a graph G is k-critical if and only if its associated bidirected graph $\overleftrightarrow{G}$ is k-dicritical.

Let us briefly justify that $d_k(n) < +\infty$ whenever $n \geq k \geq 2$. Given two digraphs H_1 and H_2, recall that $H_1 \Rightarrow H_2$ denotes the *directed join* of H_1 and H_2, that is the digraph obtained from disjoint copies of H_1 and H_2 and by adding all arcs from the copy of H_1 to the copy of H_2. If we further add all the arcs from H_2 to H_1, we obtain the *Dirac join* of H_1 and H_2, denoted by $H_1 \boxplus H_2$. It is straightforward that $\vec{\chi}(H_1 \boxplus H_2) = \vec{\chi}(H_1) + \vec{\chi}(H_2)$ (see [17]), and that $H_1 \boxplus H_2$ is dicritical if and only if both H_1 and H_2 are dicritical. Hence, for every pair of integers k, n satisfying $n \geq k \geq 2$, the digraph $\overleftrightarrow{K_{k-2}} \boxplus \vec{C}_{n-k+2}$ is k-dicritical and has order n, which implies $d_k(n) < +\infty$.

Since for every k-critical graph G, the associated bidirected graph $\overleftrightarrow{G}$ is a k-critical digraph of the same order with $2m(G)$ arcs, we directly have $d_k(n) \leq 2 \cdot g_k(n)$ for every pair of integers k, n. Kostochka and Stiebitz [113] conjectured that indeed the sparsest k-dicritical digraphs are bidirected.

Conjecture 4.2.1 (Kostochka and Stiebitz [113]). *Let k, n be two integers such that $n \geq k+2 \geq 6$, then*

$$d_k(n) = 2 \cdot g_k(n).$$

Moreover, every k-dicritical digraph on n vertices with $d_k(n)$ arcs is bidirected.

If the first part of the conjecture above is true, then every bound of the form "$g_k(n) \geq f_k(n)$" has a directed counterpart of the form "$d_k(n) \geq 2 \cdot f_k(n)$". In particular, every lower bound on $g_k(n)$ presented in the last section should have a counterpart in digraphs. Aboulker and Vermande [10] showed that it is the case for Dirac's bound. Let us define $\overrightarrow{\mathcal{DG}}(k)$ as follows: $\overrightarrow{\mathcal{DG}}(2) = \{\vec{C}_3\}$ and $\overrightarrow{\mathcal{DG}}(k) = \{\overleftrightarrow{G} \mid G \in \mathcal{DG}(k)\}$ for every $k \geq 3$.

Theorem 4.2.2 (Aboulker and Vermande [10]). *If D is a k-dicritical digraph and $n(D) > k \geq 4$ then*

$$m(D) \geq (k-1)n(D) + k - 3.$$

Moreover, equality holds if and only if $D \in \overrightarrow{\mathcal{DG}}(k)$.

Let $D = (V, A)$ be a digraph. The *complement* of D, denoted by $\overline{D}$, is the digraph with vertex-set V and arc-set $(V \times V) \setminus A(D)$. We call D *decomposable* if it is the Dirac join of two non-empty subdigraphs (*i.e.* $\overline{D}$ is disconnected); otherwise D is called *indecomposable*. Stehlík [159] proved that every k-dicritical digraph with few vertices is decomposable, thereby generalising Theorem 4.1.1 and answering a question proposed in [17].

Theorem 4.2.3 (Stehlík [159]). *If D is an indecomposable dicritical digraph, then*

$$n(D) \geq 2\vec{\chi}(D) - 1.$$

Stehlík's proof uses matching theory, but it can also be proved using the hypergraph version of Theorem 4.1.1 obtained by Stiebitz, Storch, and Toft [163] (see [141]).

In Section 4.3 we give several structural results on dicritical digraphs D whose order is close to $\vec{\chi}(D)$. Our main result is the following generalisation of Theorem 4.1.2, which implies Conjecture 4.2.1 when $k+2 \leq n \leq 2k-1$. For a digraph H and a class of digraphs $\mathcal{D}$, we define $H \boxplus \mathcal{D}$ as the class $\{H \boxplus D \mid D \in \mathcal{D}\}$, which is empty if $\mathcal{D}$ is.

Theorem 4.2.4. *Let $n = k + p$ be an integer, where $k, p \in \mathbb{N}$ and $1 \leq p \leq k-1$, then*

$$d_k(n) = \begin{cases} 2\binom{n}{2} - 3 & \text{if } p = 1 \\ 2\left(\binom{n}{2} - (p^2+1)\right) & \text{otherwise.} \end{cases}$$

Moreover, if D is a k-dicritical digraph of order n, then $m(D) = d_k(n)$ if and only if

$$D \in \overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1).$$

Finally, observe that Theorem 4.1.3 together with Conjecture 4.2.1 implies the following slightly weaker one.

Conjecture 4.2.5. *Let n and k be two integers such that $n \geq k \geq 4$, then*

$$d_k(n) \geq \left(k - \frac{2}{k-1}\right) n - \frac{k(k-3)}{k-1}.$$

Kostochka and Stiebitz showed this weaker version of the conjecture when $k = 4$, by showing $d_4(n) \geq \frac{10n-4}{3}$. It remains open for every $k \geq 5$, and the best result approaching it is due to Aboulker and Vermande [10].

Theorem 4.2.6 (Aboulker and Vermande [10])**.** *Let n and k be two integers such that $n \geq k \geq 5$, then*

$$d_k(n) \geq \left(k - \frac{1}{2} - \frac{1}{k-1}\right) n - \frac{k(k-3)}{2(k-1)}.$$

Kostochka and Stiebitz [111] showed that if a k-critical graph G is triangle-free, then $m(G)/n(G) \geq k - o(k)$ as $k \to +\infty$. Informally, this means that the minimum average degree of a k-critical triangle-free graph is (asymptotically) twice the minimum average degree of a k-critical graph. Similarly to this undirected case, it is expected that the minimum number of arcs in a k-dicritical digraph of order n is larger than $d_k(n)$ if we impose this digraph to have no short directed cycles, and in particular if the digraph is an oriented graph. Let $o_k(n)$ denote the minimum number of arcs in a k-dicritical oriented graph of order n (with the convention $o_k(n) = +\infty$ if there is no k-dicritical oriented graph of order n). Clearly $o_k(n) \geq d_k(n)$, but Kostochka and Stiebitz conjectured that $o_k(n)$ is asymptotically significantly larger than $d_k(n)$.

Conjecture 4.2.7 (Kostochka and Stiebitz [113])**.** *There exists $\varepsilon > 0$ such that, for every $k \geq 3$ and n sufficiently large,*

$$o_k(n) \geq (1 + \varepsilon) \cdot d_k(n).$$

For $k = 3$, this conjecture has been confirmed by Aboulker, Bellitto, Havet, and Rambaud [6] who proved that $o_3(n) \geq (2 + \frac{1}{3})n + \frac{2}{3}$. In view of Conjecture 4.1.4, Conjecture 4.2.7 can be generalised to $\overleftrightarrow{K}_{k-2}$-free digraphs.

Conjecture 4.2.8. *For any $k \geq 4$, there is a constant $\beta_k > 0$ such that every k-dicritical $\overleftrightarrow{K}_{k-2}$-free digraph D on n vertices has at least $(1 + \beta_k) d_k(n)$ arcs.*

Together with Conjecture 4.2.5, this conjecture would imply the following generalisation of Conjecture 4.1.4.

Conjecture 4.2.9. *For every integer $k \geq 4$, there exists $\varepsilon_k > 0$ and $c_k > 0$ such that every k-dicritical $\overleftrightarrow{K}_{k-2}$-free digraph D on n vertices has at least $(k - \frac{2}{k-1} + \varepsilon_k)n - c_k$ arcs.*

It is easy to see that there are infinitely many 4-dicritical oriented graphs. Thus, while Conjecture 4.1.4 holds vacuously for $k = 4$, this is not the case for Conjecture 4.2.9. In Section 4.4, we prove that Conjectures 4.2.7, 4.2.8, and 4.2.9 hold for $k = 4$.

Theorem 4.2.10. *If $\vec{G}$ is a 4-dicritical oriented graph, then*

$$m(\vec{G}) \geq \left(\frac{10}{3} + \frac{1}{51}\right) n(\vec{G}) - 1.$$

To prove Theorem 4.2.10, we will use an approach similar to the proof of the case $k = 5$ of Conjecture 4.1.4 by Postle [143]. Using the potential method, we will prove a more general result on all 4-dicritical digraphs that takes into account the digons. Recall that Kostochka and Stiebitz proved $d_4(n) \geq \frac{10n-4}{3}$. As a consequence of our general result, we also characterise the 4-dicritical digraphs with exactly $\frac{10n-4}{3}$ arcs, which slightly improves on the result of Kostochka and Stiebitz. These digraphs are all bidirected, so we also prove the second part of Conjecture 4.2.1 when $k = 4$ and $n = 1 \bmod 3$.

Finally, we give in Section 4.5 a construction of dicritical oriented graphs, which implies $o_k(n) \leq (2k - \frac{7}{2})n$ for every fixed k and infinitely many values of n.

4.3 Dicritical digraphs with order close to the dichromatic number

In this section, we give a collection of structural results on critical digraphs whose order is close to the dichromatic number. We first give a result which makes a connection between the order of a dicritical digraph and its number of universal vertices and universal directed cycles, and discuss the consequences of this result. Then, as mentioned in the introduction of this chapter, we will generalise Theorem 4.1.1.

We will make use of the result of Stehlík about decomposable digraphs, and of the result of Aboulker and Vermande which generalises Dirac's bound. We recall both of them here for convenience.

Theorem 4.2.3 (Stehlík [159]). *If D is an indecomposable dicritical digraph, then*

$$n(D) \geq 2\vec{\chi}(D) - 1.$$

Theorem 4.2.2 (Aboulker and Vermande [10]). *If D is a k-dicritical digraph and $n(D) > k \geq 4$ then*

$$m(D) \geq (k-1)n(D) + k - 3.$$

Moreover, equality holds if and only if $D \in \overrightarrow{\mathcal{DG}}(k)$.

4.3.1 Universal vertices and cycles in dicritical digraphs

Let D be a digraph. A non-empty subdigraph D' of D is called a *universal subdigraph* of D is there exists a non-empty subdigraph $\widetilde{D}$ such that $D = D' \boxplus \widetilde{D}$. A vertex v of D is called a *universal vertex* of D if $D\langle\{v\}\rangle$ is a universal subdigraph of D. By definition, observe that a universal subdigraph D' of D is necessarily induced. Observe also that if $X \subseteq V(D)$ is a set of $p \geq 0$ universal vertices of D, then $D = \overleftrightarrow{K_p} \boxplus (D - X)$.

Let $\overrightarrow{\mathrm{CRI}}(k)$ denote the class of k-dicritical digraphs, and for an integer n, let $\overrightarrow{\mathrm{CRI}}(k,n)$ denote $\{D \in \overrightarrow{\mathrm{CRI}}(k) \mid n(D) = n\}$.

Theorem 4.3.1. *Let D be a k-dicritical digraph of order n with $k \geq 1$. Let p be the number of universal vertices of D and q be the number of universal directed cycles of D having order at least three. Then the following statements hold:*

(a) $p \geq 3k - 2n$ and equality holds if and only if $D = \overleftrightarrow{K_p} \boxplus D'$ where D' is the Dirac join of $\frac{1}{2}(k-p)$ copies of $\vec{C}_3$.

(b) $p + 2q \geq 5k - 3n$ *and equality holds if and only if* $D = \overleftrightarrow{K_p} \boxplus D_1 \boxplus D_2$ *where* D_1 *is the Dirac join of* q *copies of* $\vec{C}_3$ *and* D_2 *is the Dirac join of* $\frac{1}{3}(k - p - 2q)$ *disjoint subdigraphs of* D *each of which belong to* $\overrightarrow{\mathrm{CRI}}(3, 5)$.

Proof. Let us fix a k-dicritical digraph D on n vertices. Let $\overline{D_1}, \overline{D_2}, \ldots, \overline{D_s}$ be the connected components of $\overline{D}$. In particular, we have

$$D = D_1 \boxplus D_2 \boxplus \ldots \boxplus D_s.$$

For every $i \in [s]$, let $k_i = \vec{\chi}(D_i)$ and $n_i = n(D_i)$. We thus have $k = k_1 + k_2 + \cdots + k_s$ and $D_i \in \overrightarrow{\mathrm{CRI}}(k_i, n_i)$ for every $i \in [s]$. Since $\overline{D_i}$ is connected, by Theorem 4.2.3 we also have

$$n_i \geq 2k_i - 1 \quad \text{for every } i \in [s]. \tag{4.1}$$

In particular, for every $i \in [s]$, we obtain that one of the following holds:

- $k_i = 1$ and $D_i = K_1$, or
- $k_i = 2$ and $n(D_i) \geq 3$, or
- $k_i \geq 3$ and $n(D_i) \geq 5$.

For a subset I of $[s]$, let $D_I = \boxplus_{i \in I} D_i$ be the Dirac join of the digraphs D_i with $i \in I$, and let $k_I = \sum_{i \in I} k_i$, where $D_\varnothing$ is the empty digraph and $k_\varnothing = 0$. Note that, in particular, $D_I \in \overrightarrow{\mathrm{CRI}}(k_I)$. Let $P = \{i \in [s] \mid k_i = 1\}$, $Q = \{i \in [s] \mid k_i = 2\}$, $R = [s] \setminus (P \cup Q)$, $p = |P|$, $q = |Q|$, and $r = |R|$. Then P, Q, and R are pairwise disjoint sets whose union is $[s]$. We thus obtain

$$D = D_P \boxplus D_Q \boxplus D_R, \text{ where } D_P = \overleftrightarrow{K_p} \text{ and } D_Q \in \overrightarrow{\mathrm{CRI}}(2q).$$

Note that p is the number of universal vertices of D and q is the number of universal directed cycles of D.

We are now going to prove (a). For every $i \in Q \cup R$, we have $k_i \geq 2$ and so, by (4.1), $n_i \geq 2k_i - 1 \geq \frac{3}{2}k_i$, where equality holds if and only if $k_i = 2$ and $D_i = \vec{C}_3$. Therefore, we have

$$n = p + \sum_{i \in Q \cup R} n_i \geq p + \frac{3}{2} \sum_{i \in Q \cup R} k_i = p + \frac{3}{2}(k - p),$$

which is equivalent to $p \geq 3k - 2n$. Also equality holds if and only if $D_{Q \cup R}$ is the Dirac join of $\frac{1}{2}(k - p)$ disjoint copies of $\vec{C}_3$. This proves (a).

We are now going to prove (b). For every $i \in R$, we have $k_i \geq 3$ so (4.1) implies $n_i \geq 2k_i - 1 \geq \frac{5}{3}i$, where equality holds if and only if $D_i \in \overrightarrow{\mathrm{CRI}}(3, 5)$. We thus obtain

$$n = p + \sum_{i \in Q} n_i + \sum_{i \in R} n_i \geq p + 3q + \frac{5}{3} \sum_{i \in R} k_i = p + 3q + \frac{5}{3}(k - p - 2q),$$

which is equivalent to $2p + q \geq 5k - 3n$. Also equality holds if and only if $D_i = \vec{C}_3$ for every $i \in Q$ and $D_i \in \overrightarrow{\mathrm{CRI}}(3, 5)$ for every $i \in R$. Thus (b) is proved. $\square$

In the remaining of this section, we discuss some consequences of Theorem 4.3.1. From Theorem 4.3.1(a) it follows that every digraph $D \in \overrightarrow{\mathrm{CRI}}(k, k+1)$ contains a universal vertex. Since $\overrightarrow{\mathrm{CRI}}(2,3) = \{\vec{C}_3\}$, we have the following result by induction on k.

Proposition 4.3.2. *For every integer $k \geq 2$,*

$$\overrightarrow{\mathrm{CRI}}(k, k+1) = \{\overleftrightarrow{K}_{k-2} \boxplus \vec{C}_3\}.$$

Theorem 4.3.1(b) implies that a 4-critical digraph on 6 vertices either contains a universal vertex or two disjoint universal directed triangles. Hence, we have

$$\overrightarrow{\mathrm{CRI}}(4,6) = (K_1 \boxplus \overrightarrow{\mathrm{CRI}}(3,5)) \cup \{\vec{C}_3 \boxplus \vec{C}_3\}.$$

If $k \geq 5$, Theorem 4.3.1(a) implies that every k-dicritical digraph on $k+2$ vertices contains at least one universal vertex. Hence, $\overrightarrow{\mathrm{CRI}}(k, k+2) = K_1 \boxplus \overrightarrow{\mathrm{CRI}}(k-1, k+1)$ if $k \geq 5$ and we obtain the following by induction on k.

Corollary 4.3.3. *For every integer $k \geq 4$,*

$$\overrightarrow{\mathrm{CRI}}(k, k+2) = \left(\overleftrightarrow{K}_{k-3} \boxplus \overrightarrow{\mathrm{CRI}}(3,5)\right) \cup \left(\overleftrightarrow{K}_{k-4} \boxplus \vec{C}_3 \boxplus \vec{C}_3\right).$$

Let $\overrightarrow{\mathrm{CRI}}^*(k,n)$ be the class consisting of all digraphs in $\overrightarrow{\mathrm{CRI}}(k,n)$ with no universal vertex. For k-dicritical digraphs on $k+p$ vertices, the following is a nice consequence of Theorem 4.3.1. In particular, it implies that the number of k-dicritical digraphs on $k+p$ vertices (up to isomorphism) is bounded by a function depending only on p (when $2p < k$).

Corollary 4.3.4. *For every integer k, p such that $k > 2p \geq 4$,*

$$\overrightarrow{\mathrm{CRI}}(k, k+p) = \bigcup_{\ell=2}^{2p} \left\{\overleftrightarrow{K}_{k-\ell} \boxplus \overrightarrow{\mathrm{CRI}}^*(\ell, \ell+p)\right\}$$

Proof. Let D be a k-dicritical digraph on $k+p$ vertices. Then $n(D) = k + p < \frac{3}{2}k$. Let s be the number of universal vertices of D. By definition, $D = \overleftrightarrow{K}_s \boxplus D'$ where D' belongs to $\overrightarrow{\mathrm{CRI}}^*(\ell, \ell+p)$, for $\ell = k - s$. It remains to show that $2 \leq \ell \leq 2p$. Since $p \geq 2$, we obviously have $\ell \geq 2$. On the other hand, D' contains no universal vertex, so $n(D') \geq \frac{3}{2}\ell$ (by applying Theorem 4.3.1(a) on D'). We deduce

$$n(D') = \ell + p \geq \frac{3}{2}\ell.$$

This implies $\ell \leq 2p$ as desired. □

4.3.2 Generalisation of a result of Gallai

This section is devoted to the proof of Theorem 4.2.4. We recall it here for convenience.

Theorem 4.2.4. *Let $n = k + p$ be an integer, where $k, p \in \mathbb{N}$ and $1 \leq p \leq k-1$, then*

$$d_k(n) = \begin{cases} 2\binom{n}{2} - 3 & \text{if } p = 1 \\ 2\left(\binom{n}{2} - (p^2+1)\right) & \text{otherwise.} \end{cases}$$

Moreover, if D is a k-dicritical digraph of order n, then $m(D) = d_k(n)$ if and only if

$$D \in \overleftrightarrow{K}_{k-p-1} \boxplus \overrightarrow{\mathcal{DG}}(p+1).$$

Proof. For every fixed k, p, easy calculations prove that every digraph $D \in \overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ contains exactly $f_k(n)$ arcs, where $n = k + p$ and

$$f_k(n) = \begin{cases} 2\binom{n}{2} - 3 & \text{if } p = 1, \\ 2\left(\binom{n}{2} - (p^2+1)\right) & \text{otherwise.} \end{cases}$$

Furthermore, it is clear that every digraph $D \in \overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ is k-dicritical since it is either the bidirected graph associated to a k-critical graph or the Dirac join of a bidirected complete graph and a directed triangle. Therefore, it remains to prove that every k-dicritical digraph D of order $n = k + p$, $1 \leq p \leq k - 1$, satisfying $m(D) = d_k(n)$ belongs to $\overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$.

We proceed by induction on k. Assume first that $k = 2$, then $p = 1$ and $\vec{C}_3$ is clearly the only 2-dicritical digraph on 3 vertices. Assume now that $k = 3$, then $1 \leq p \leq 2$. If $p = 1$, by Proposition 4.3.2 $K_1 \boxplus \vec{C}_3$ is the only 3-dicritical digraph on 4 vertices. If $p = 2$ then we are done by the second part of Theorem 4.2.2, which implies that the k-dicritical digraphs of order $2k - 1$ with $d_k(2k-1)$ arcs are exactly the digraphs in $\overrightarrow{\mathcal{DG}}(k)$.

We now assume $k \geq 4$. Again, if $p = 1$ we are done by Proposition 4.3.2, and if $p = k - 1$ we are done by the second part of Theorem 4.2.2. So we assume $2 \leq p \leq k - 2$. Let us fix a k-dicritical digraph D on $n = k + p$ vertices such that $m(D) = d_k(n)$, we will show that D necessarily belongs to $\overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$. We distinguish three cases, depending on the structure of D.

Case 1: *D contains a universal vertex.*

Then $D = K_1 \boxplus H$, where $H = D - v \in \overrightarrow{\mathrm{CRI}}(k-1, n-1)$. Furthermore, we must have $m(H) = d_{k-1}(n-1)$, for otherwise there exists a digraph $H' \in \overrightarrow{\mathrm{CRI}}(k-1, n-1)$ such that $m(H') < m(H)$, and $D' = K_1 \boxplus H'$ would be a k-dicritical digraph on n vertices with $m(D') < m(D)$, a contradiction to the choice of D.

Since $k + 2 \leq n \leq 2k - 2$, we have $(k-1) + 1 \leq n - 1 \leq 2(k-1) - 1$, which allows us to apply the induction hypothesis on H. Hence H belongs to $\overleftrightarrow{K_{k-p-2}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$. Consequently, $D = K_1 \boxplus H$ belongs to $\overleftrightarrow{K_{k-p-1}} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ as desired.

Case 2: *D contains a universal $\vec{C}_3$ but no universal vertex.*

Then $D = \vec{C}_3 \boxplus H$ and $H \in \overrightarrow{\mathrm{CRI}}(k-2, n-3)$. We have $n - 3 \geq k - 1$, for otherwise H is the bidirected complete graph on $k - 2$ vertices, which implies that D contains a universal vertex. We also have $n - 3 \leq 2(k-2) - 1$ since $n \leq 2k - 2$. Analogously to the previous case, we must also have $m(H) = d_{k-2}(n-3)$. Hence, we may apply the induction on H, which implies that $m(H) \geq 2\left(\binom{n-3}{2} - ((p-1)^2 + 1)\right)$ arcs. Consequently, we obtain that

$$\begin{aligned} m(D) &= m(H) + 3 + 6(n-3) \\ &\geq 2\left(\binom{n-3}{2} - ((p-1)^2+1)\right) + 3 + 6(n-3) \\ &= 2\left(\binom{n}{2} - (p^2+1)\right) - 5 + 4p \\ &\geq 2\left(\binom{n}{2} - (p^2+1)\right) + 3, \end{aligned}$$

where in the last inequality we used $p \geq 2$. This is a contradiction to the minimality of D, since every digraph in $\overleftrightarrow{K}_{k-p-1} \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ contains exactly $2\left(\binom{n}{2} - (p^2+1)\right)$ arcs.

Case 3: *D does not contain a universal vertex, nor does it contain a universal $\vec{C}_3$.*

Let $\overline{D_1}, \overline{D_2}, \ldots, \overline{D_s}$ be the connected components of $\overline{D}$. In particular, we have

$$D = D_1 \boxplus D_2 \boxplus \ldots \boxplus D_s.$$

For every $i \in [s]$, let $k_i = \vec{\chi}(D_i)$ and $n_i = n(D_i)$. We thus have $k = k_1 + k_2 + \cdots + k_s$ and $D_i \in \overrightarrow{\mathrm{CRI}}(k_i, n_i)$ for every $i \in [s]$. Since $n \leq 2k-2$, it follows from Theorem 4.2.3 that $s \geq 2$. Also none of the D_is is decomposable, which implies $n_i \geq 2k_i - 1$ for every $i \in [s]$. Let t be the number of indices i for which $n_i = 2k_i - 1$. Assume first that $t \leq 1$, then we have

$$n = \sum_{i=1}^{s} n_i \geq \sum_{i=1}^{s} 2k_i - 1 = 2k - 1.$$

This is a contradiction since $n \leq 2k-2$. Assume then that $t \geq 2$. By symmetry, we may assume $n_1 = 2k_1 - 1$ and $n_2 = 2k_2 - 1$.

Since $m(D) = d_k(n)$ we obtain, as in the first case, that $m(D_i) = d_{k_i}(n_i)$ for $i \in [s]$. Since D does not contain any universal vertex nor any universal $\vec{C}_3$, we have $k_i \geq 3$. Hence we have $n_1 \geq k_1 + 2$ and $n_2 \geq k_2 + 2$. Then the induction hypothesis implies that D_i belongs to $\overrightarrow{\mathcal{DG}}(k_i)$ for $i \in \{1, 2\}$. On the one hand, this implies that:

$$\begin{aligned} m(D_1 \boxplus D_2) = 2&\left(\binom{2k_1-1}{2} - ((k_1-1)^2+1)\right) \\ &+ 2\left(\binom{2k_2-1}{2} - ((k_2-1)^2+1)\right) \\ &+ 2(2k_1-1)(2k_2-1). \end{aligned}$$

On the other hand, the minimality of D implies that $m(D_1 \boxplus D_2) = d_{k_1+k_2}(2(k_1+k_2)-2)$. A digraph H in $K_1 \boxplus \overrightarrow{\mathcal{DG}}(k_1+k_2-1)$ belongs to $\overrightarrow{\mathrm{CRI}}(k_1+k_2, 2(k_1+k_2)-2)$ and has $2\left(\binom{2k_1+2k_2-2}{2} - ((k_1+k_2-2)^2+1)\right)$ arcs. We finally contradict $m(D_1 \boxplus D_2) = d_{k_1+k_2}(2(k_1+k_2)-2)$, since

$$m(D_1 \boxplus D_2) - m(H) = 4k_1k_2 - 4k_1 - 4k_2 + 2 \geq 2.$$

This concludes the proof. $\square$

4.4 Minimum density of 4-dicritical oriented graphs

This section is devoted to the proof of Theorem 4.2.10, which we restate here for convenience.

Theorem 4.2.10. *If $\vec{G}$ is a 4-dicritical oriented graph, then*

$$m(\vec{G}) \geq \left(\frac{10}{3} + \frac{1}{51}\right) n(\vec{G}) - 1.$$

As mentioned in the introduction of this chapter, our proof is based on the potential method and the idea is to prove a more general result on all 4-dicritical digraphs that takes into account the digons. A *packing* of digons and bidirected triangles is a set of pairwise vertex-disjoint digons and bidirected triangles. To take into account the digons, we define a parameter $T(D)$ as follows.

$$T(D) = \max\{d + 2t \mid \text{there exists a packing of } d \text{ digons and } t \text{ bidirected triangles in } D\}$$

Clearly, $T(D) = 0$ if and only if D is an oriented graph. Let ε, δ be fixed non-negative real numbers. We define the *potential* (with respect to ε and δ) of a digraph D to be

$$\rho(D) = \left(\frac{10}{3} + \varepsilon\right) n(D) - m(D) - \delta T(D).$$

Theorem 4.2.10 can be rephrased as follows.

Theorem 4.4.1. *Set $\varepsilon = \frac{1}{51}$ and $\delta = 6\varepsilon = \frac{2}{17}$. If $\vec{G}$ is a 4-dicritical oriented graph, then $\rho(\vec{G}) \leq 1$.*

In fact, we prove a more general statement which holds for every 4-dicritical digraph (with or without digons), except for some exceptions called the 4-*Ore digraphs*. Those digraphs, which are formally defined in Section 4.4.1, are the bidirected graphs whose underlying graph is one of the 4-critical graphs reaching equality in Theorem 4.1.3. In particular, every 4-Ore digraph D has exactly $\frac{10}{3}n(D) - \frac{4}{3}$ arcs. Moreover, the statement holds for all non-negative constants ε and δ satisfying the following inequalities:

- $\delta \geq 6\varepsilon$;
- $3\delta - \varepsilon \leq \frac{1}{3}$;

Theorem 4.4.2. *Let $\varepsilon, \delta \geq 0$ be constants satisfying the aforementioned inequalities. If D is a 4-dicritical digraph with n vertices, then*

(i) $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$ *if D is 4-Ore, and*

(ii) $\rho(D) \leq 1$ *otherwise.*

In order to provide some intuition to the reader, let us briefly describe the main ideas of our proof. We will consider a minimum counterexample D to Theorem 4.4.2, and show that every subdigraph of D must have large potential. To do so, we need to construct some smaller 4-dicritical digraphs to leverage the minimality of D. These smaller 4-dicritical digraphs will be constructed by identifying some vertices of D. This is why, in the definition of the potential, we consider $T(D)$ instead of the number of digons: when identifying a set of vertices, the number of digons may be arbitrary larger in the resulting digraph, but $T(D)$ increases at most by 1. Using the fact that every subdigraph of D has large potential, we will prove that some subdigraphs are forbidden in D. Using this, we get the final contradiction by a discharging argument.

In addition to Theorem 4.2.10, Theorem 4.4.2 has also the following consequence when we take $\varepsilon = \delta = 0$.

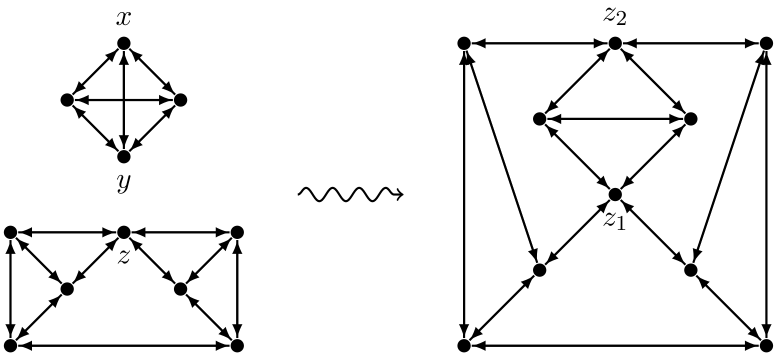


Figure 4.2: An example of a 4-Ore digraph obtained by an Ore-composition of two smaller 4-Ore digraphs, with replaced digon $[x, y]$ and split vertex z.

Corollary 4.4.3. *If D is a 4-dicritical digraph, then $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$. Moreover, equality holds if and only if D is 4-Ore, otherwise $m(D) \geq \frac{10}{3}n(D) - 1$.*

As mentioned in the introduction, this is a slight improvement on a result of Kostochka and Stiebitz [113] who proved the inequality $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$ without characterising the equality case.

Another interesting consequence of our result is the following bound on the number of vertices in a 4-dicritical oriented graph embedded on a fixed surface. Since a graph on n vertices embedded on a surface of Euler characteristic c has at most $3n - 3c$ edges, we immediately deduce the following from Theorem 4.2.10.

Corollary 4.4.4. *If $\vec{G}$ is a 4-dicritical oriented graph embedded on a surface of Euler characteristic c, then $n(\vec{G}) \leq \frac{17}{6}(1 - 3c)$.*

The previous best upper bound was $n(\vec{G}) \leq 4 - 9c$ [113].

In Section 4.4.1 we prove some preliminary results on 4-Ore digraphs, before proving Theorem 4.4.2 in Section 4.4.2.

4.4.1 The 4-Ore digraphs and their properties

Let D_1, D_2 be two bidirected graphs, $[x, y] \subseteq A(D_1)$, and $z \in V(D_2)$. An *Ore-composition* D of D_1 and D_2 with *replaced digon* $[x, y]$ and *split vertex* z is a digraph obtained by removing $[x, y]$ of D_1 and z of D_2, and adding the set of arcs $\{xz_1 \mid zz_1 \in A(D_2) \text{ and } z_1 \in Z_1\}$, $\{z_1x \mid z_1z \in A(D_2) \text{ and } z_1 \in Z_1\}$, $\{yz_2 \mid zz_2 \in A(D_2) \text{ and } z_2 \in Z_2\}$, $\{z_2y \mid z_2z \in A(D_2) \text{ and } z_2 \in Z_2\}$, where (Z_1, Z_2) is a partition of $N_{D_2}(z)$ into non-empty sets. We call D_1 the *digon side* and D_2 the *split side* of the Ore-composition. The class of the 4-*Ore digraphs* is the smallest class containing $\overleftrightarrow{K_4}$ which is stable under Ore-composition. See Figure 4.2 for an example of a 4-Ore digraph. Observe that the 4-Ore-digraphs are all bidirected.

Proposition 4.4.5 (Dirac [63], see also [116])**.** *4-Ore digraphs are 4-dicritical.*

Proof. One can easily show that a bidirected digraph is 4-dicritical if and only if its undirected underlying graph is 4-critical. Then the result follows from the undirected analogous proved by [63]. □

Lemma 4.4.6. *Let D be a 4-dicritical bidirected digraph and $v \in V(D)$. Let (N_1^+, N_2^+) and (N_1^-, N_2^-) be two partitions of $N(v)$. Consider D' the digraph with vertex-set $V(D) \setminus \{v\} \cup \{v_1, v_2\}$ with $N^+(v_i) = N_i^+, N^-(v_i) = N_i^-$ for $i = 1, 2$ and $D'\langle V(D) \setminus \{v\}\rangle = D - v$. Then D' has a 3-dicolouring with v_1 and v_2 coloured the same except if $N_1^+ = N_1^-$ (that is D' is bidirected).*

Proof. Suppose that D' is not bidirected. Consider a vertex $u \in N_D(v)$ such that $v_1u, uv_2 \in A(D')$ or $v_2u, uv_1 \in A(D')$. Without loss of generality, suppose $v_1u, uv_2 \in A(D')$. As D is 4-dicritical, $D \setminus [u, v]$ has a proper 3-dicolouring ϕ. We set $\phi(v_1) = \phi(v_2) = \phi(v)$ and claim that ϕ is a 3-dicolouring of D'. To show that, observe that ϕ is a proper 3-colouring of the underlying undirected graph of $D' \setminus \{v_1u, uv_2\}$, and so ϕ is a 3-dicolouring of D' as wanted. ☐

Lemma 4.4.7. *Let D be a digraph. If v is a vertex of D, then $T(D - v) \geq T(D) - 1$.*

Proof. Let M be a packing of d digons and t bidirected triangles in H such that $d+2t = T(D)$. If v belongs to a digon $[u, v]$ in M, then $M \setminus \{[u, v]\}$ witnesses the fact that $T(D - v) \geq T(D) - 1$. If v belongs to a bidirected triangle u, v, w, u, then $M \setminus \{u, v, w\} \cup [u, w]$ witnesses the fact that $T(D - v) \geq T(D) - 2 + 1$. Otherwise $T(D - v) \geq T(D)$. ☐

Lemma 4.4.8. *If D_1, D_2 are two digraphs, and D is an Ore-composition of D_1 and D_2, then $T(D) \geq T(D_1) + T(D_2) - 2$. Moreover, if D_1 or D_2 is isomorphic to $\overleftrightarrow{K_4}$, then $T(D) \geq T(D_1) + T(D_2) - 1$.*

Proof. Let D be the Ore-composition of D_1 (the digon side with replaced digon $[x, y]$) and D_2 (the split side with split vertex z). One can easily see that $T(D) \geq T(D_1 - x) + T(D - z) \geq T(D_1) + T(D_2) - 2$ by Lemma 4.4.7. Moreover, if D_1 (resp. D_2) is a copy of $\overleftrightarrow{K_4}$, then $T(D_1 - x) = 2 = T(D_1)$ (resp. $T(D_2 - z) = 2 = T(D_2)$) and therefore $T(D) \geq T(D_1) + T(D_2) - 1$. ☐

Lemma 4.4.9. *If D is 4-Ore, then $T(D) \geq \frac{2}{3}(n(D) - 1)$.*

Proof. If D is $\overleftrightarrow{K_4}$, then the result is clear. Suppose now that D is an Ore-composition of D_1 and D_2. Then $n(D) = n(D_1) + n(D_2) - 1$ and, by Lemma 4.4.8, $T(D) \geq T(D_1) + T(D_2) - 2$. By induction, $T(D_1) \geq \frac{2}{3}(n(D_1) - 1)$ and $T(D_2) \geq \frac{2}{3}(n(D_2) - 1)$, and so $T(D) \geq \frac{2}{3}(n(D_1) + n(D_2) - 1 - 1) = \frac{2}{3}(n(D) - 1)$. ☐

Let D be a digraph. A *diamond* in D is a subdigraph isomorphic to $\overleftrightarrow{K_4}$ minus a digon $[u, v]$, with vertices different from u and v having degree 6 in D. An *emerald* in D is a subdigraph isomorphic to $\overleftrightarrow{K_3}$ whose vertices have degree 6 in D.

Let R be an induced subdigraph of D with $n(R) < n(D)$. The *boundary* of R in D, denoted by $\partial_D(R)$, or simply $\partial(R)$ when D is clear from the context, is the set of vertices of R having a neighbour in $V(D) \setminus R$. We say that R is *Ore-collapsible* if the boundary of R contains exactly two vertices u and v and $R \cup [u, v]$ is 4-Ore.

Lemma 4.4.10. *If D is 4-Ore and $v \in V(D)$, then there exists either an Ore-collapsible subdigraph of D disjoint from v or an emerald of D disjoint from v.*

Proof. If D is a copy of $\overleftrightarrow{K_4}$, then $D - v$ is an emerald. Otherwise, D is the Ore-composition of two 4-Ore digraphs: D_1 the digon side with replaced digon $[x, y]$, and D_2 the split side with split vertex z. If $v \in V(D_2 - z)$, then D_1 is an Ore-collapsible subdigraph with boundary $\{x, y\}$. Otherwise, $v \in V(D_1)$ and we apply induction on D_2 to find an emerald or an Ore-collapsible subdigraph in D_2 disjoint from z. □

Lemma 4.4.11. *If $D \neq \overleftrightarrow{K_4}$ is 4-Ore and T is a copy of $\overleftrightarrow{K_3}$ in D, then there exists either an Ore-collapsible subdigraph of D disjoint from T or an emerald of D disjoint from T.*

Proof. As D is not $\overleftrightarrow{K_4}$, it is an Ore-composition of two 4-Ore digraphs: D_1 the digon side with replaced digon $[x, y]$, and D_2 the split side with split vertex z. As x and y are non-adjacent, we have either $T \subseteq D_1$, $T \subseteq D_2 - z$, or T contains a vertex $w \in \{x, y\}$ and two vertices in $V(D_2 - z)$.

If $T \subseteq D_1$, then by Lemma 4.4.10, in D_2 there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from z. In the former case O is an Ore-collapsible subdigraph of D disjoint from T, and in the later one E is an emerald in D disjoint from T.

If $T \subseteq D_2 - z$, then $D_1 \setminus \{x, y\}$ is an Ore-collapsible subdigraph disjoint from T.

Assume now that T contains a vertex $w \in \{x, y\}$ and two vertices in $V(D_2 - z)$. Without loss of generality, we may assume that $y \notin T$. Let z_1 and z_2 be the two vertices of T disjoint from w. Then $\{z, z_1, z_2\}$ induces a bidirected triangle T' in D_2. If $D_2 \neq \overleftrightarrow{K_4}$, then by induction in D_2, there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from T'. In the former case O is an Ore-collapsible subdigraph of D disjoint from T, and in the later one E is an emerald in D disjoint from T.

Henceforth we may assume that $D_2 = \overleftrightarrow{K_4}$. This implies that y has exactly one neighbour in $D_2 - z$ and so its degree is the same in D_1 and D. By Lemma 4.4.10, in D_1 there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from x. In the former case O is an Ore-collapsible subdigraph of D disjoint from T, and in the later one E is an emerald in D disjoint from T even if $y \in V(E)$ because y has the same degree in D_1 and D. □

Lemma 4.4.12. *If R is an Ore-collapsible induced subdigraph of a 4-Ore digraph D, then there exists a diamond or an emerald of D whose vertices lie in $V(R)$.*

Proof. Let D be a digraph. Let R be a minimal counterexample to this lemma, and let $\partial(R) = \{u, v\}$ and $H = D\langle R \rangle \cup [u, v]$. If $H = \overleftrightarrow{K_4}$, then R is a diamond in D. Suppose now that H is the Ore-composition of two 4-Ore digraphs H_1 (the digon side with replaced digon $[x, y]$) and H_2 (the split side with split vertex z). If $\{u, v\} \not\subseteq V(H_2)$, then by Lemma 4.4.10 there exists an Ore-collapsible subdigraph in H_2 disjoint from z. As it is smaller than H, it contains an emerald or a diamond as desired, a contradiction.

Now assume that $\{u, v\} \subseteq V(H_2)$, then H_1 is an Ore-collapsible subdigraph of D smaller than H, and by induction, H_1 contains a diamond or an emerald in D. □

Lemma 4.4.13. *If D is a 4-Ore digraph and v is a vertex in D, then D contains a diamond or an emerald disjoint from v.*

Proof. Follows from Lemmas 4.4.10 and 4.4.12. □

Lemma 4.4.14. *If D is a 4-Ore digraph and T is a bidirected triangle in D, then either $D = \overleftrightarrow{K_4}$ or D contains a diamond or an emerald disjoint from T.*

Proof. Follows from Lemmas 4.4.11 and 4.4.12. □

The following theorem was formulated for undirected graphs, but by replacing every edge by a digon, it can be restated as follows:

Theorem 4.4.15 (Kostochka and Yancey [116, Theorem 6])**.** *Let D be a 4-dicritical bidirected digraph. If $\frac{10}{3}n(D) - m(D) > 1$, then D is 4-Ore and $\frac{10}{3}n(D) - m(D) = \frac{4}{3}$.*

Lemma 4.4.16. *If D is a 4-Ore digraph with n vertices, then $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta\frac{2(n-1)}{3}$.*

Proof. Follows from Theorem 4.4.15 and Lemma 4.4.9. □

Lemma 4.4.17 (Kostochka and Yancey [116, Claim 16])**.** *Let D be a 4-Ore digraph. If $R \subseteq D$ and $0 < n(R) < n(D)$, then $\frac{10}{3}n(R) - m(R) \geq \frac{10}{3}$.*

Lemma 4.4.18. *Let D be a 4-Ore digraph obtained from a copy J of $\overleftrightarrow{K_4}$ by successive Ore-compositions with 4-Ore digraphs, vertices and digons in J being always on the digon side. Let $[u, v]$ be a digon in $D\langle V(J)\rangle$. For every 3-dicolouring ϕ of $D \setminus [u, v]$, vertices in $V(J)$ receive distinct colours except u and v.*

Proof. We proceed by induction on $n(D)$, the result holding trivially when D is $\overleftrightarrow{K_4}$. Now assume that D is the Ore-composition of D_1, the digon side containing J, and D_2, with D_1 and D_2 being 4-Ore digraphs. Let $[x, y] \subseteq A(D_1)$ be the replaced digon in this Ore-composition, and let $z \in V(D_2)$ be the split vertex. Let ϕ be a 3-dicolouring of $D \setminus [u, v]$. Then ϕ induces a 3-dicolouring of $D\langle V(D_2 - z) \cup \{x, y\}\rangle$. Necessarily $\phi(x) \neq \phi(y)$, for otherwise ϕ_2 defined by $\phi_2(v) = \phi(v)$ if $v \in V(D_2 - z)$ and $\phi_2(z) = \phi(x)$ is a 3-dicolouring of D_2, contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 4.4.5. Hence ϕ induces a 3-dicolouring of $D_1 \setminus [u, v]$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in ϕ, except u and v. □

Lemma 4.4.19. *Let D be a 4-Ore digraph obtained from a copy J of $\overleftrightarrow{K_4}$ by successive Ore-compositions with 4-Ore digraphs, vertices and digons in J being always on the digon side. Let v be a vertex in $V(J)$. For every 3-dicolouring ϕ of $D - v$, vertices in J receive distinct colours.*

Proof. We proceed by induction on $n(D)$, the result holding trivially when D is $\overleftrightarrow{K_4}$. Now assume that D is the Ore-composition of D_1, the digon side containing J, and D_2, with D_1 and D_2 being 4-Ore digraphs. Let $[x, y] \subseteq A(D_1)$ be the replaced digon in this Ore-composition, and let $z \in V(D_2)$ be the split vertex. Let ϕ be a 3-dicolouring of $D - v$. If $v \in \{x, y\}$, then ϕ is a 3-dicolouring of $D_1 - v$ and the result follows by induction. Now assume $v \notin \{x, y\}$. Then ϕ induces a 3-dicolouring of $D\langle V(D_2 - z) \cup \{x, y\}\rangle$. Necessarily $\phi(x) \neq \phi(y)$, for otherwise ϕ_2 defined by $\phi_2(v) = \phi(v)$ if $v \in V(D_2 - z)$ and $\phi_2(z) = \phi(x)$ is a 3-dicolouring of D_2, contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 4.4.5. Hence ϕ induces a 3-dicolouring of $D_1 - v$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in ϕ. □

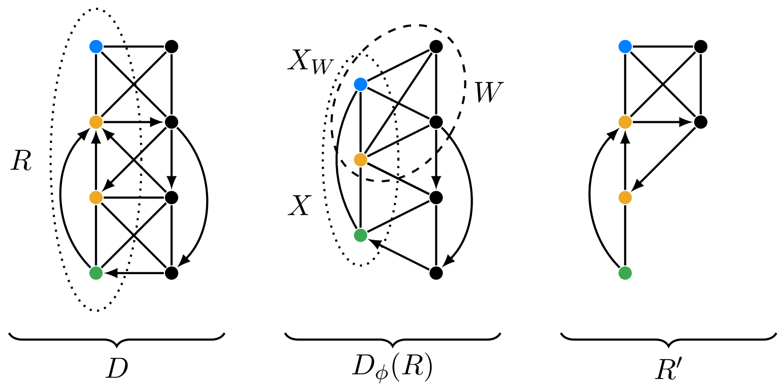


Figure 4.3: A 4-dicritical digraph D together with an induced subdigraph R of D and ϕ a 3-dicolouring of R, the ϕ-identification $D_\phi(R)$ of R in D and the dicritical extension R' of R with extender W and core X_W. For clarity, the digons are represented by undirected edges.

4.4.2 Proof of Theorem 4.4.2

Let D be a 4-dicritical digraph, R be an induced subdigraph of D with $4 \leq n(R) < n(D)$ and ϕ a 3-dicolouring of R. The *ϕ-identification* of R in D, denoted by $D_\phi(R)$ is the digraph obtained from D by identifying for each $i \in [3]$ the vertices coloured i in $V(R)$ to a vertex x_i, adding the digons $[x_i, x_j]$ for all $1 \leq i < j \leq 3$. Observe that $D_\phi(R)$ is not 3-dicolourable. Indeed, assume for a contradiction that $D_\phi(R)$ has a 3-dicolouring ϕ'. Since $\{x_1, x_2, x_3\}$ induces a $\overleftrightarrow{K_3}$, we may assume without loss of generality that $\phi'(x_i) = i$ for $i \in [3]$. Consider the 3-colouring ϕ'' of D defined by $\phi''(v) = \phi'(v)$ if $v \notin R$ and $\phi''(v) = \phi(v)$ if $v \in R$. One easily checks that ϕ'' is a 3-dicolouring of D, a contradiction to the fact that $\vec{\chi}(D) = 4$.

Now let W be a 4-dicritical subdigraph of $D_\phi(R)$ and $X = \{x_1, x_2, x_3\}$. Then we say that $R' = D\langle (V(W) \setminus X) \cup R \rangle$ is the *dicritical extension* of R with *extender* W. We call $X_W = X \cap V(W)$ the *core* of the extension. Note that X_W is not empty because W is not a subdigraph of D. Thus $1 \leq |X_W| \leq 3$. See Figure 4.3 for an example of a ϕ-identification and a dicritical extension.

Let D be a counterexample to Theorem 4.4.2 with minimum number of vertices. By Lemma 4.4.16, D is not 4-Ore. Thus $\rho(D) > 1$.

Claim 4.4.20. *If $\widetilde{D}$ is a 4-dicritical digraph with $n(\widetilde{D}) < n(D)$, then $\rho(\widetilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$.*

Proof of claim. If $\widetilde{D}$ is not 4-Ore, then $\rho(\widetilde{D}) \leq 1$ by minimality of D. Thus, $\rho(\widetilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ because $4\varepsilon - 2\delta \geq \frac{-1}{3}$. Otherwise, by Lemma 4.4.16, $\rho(\widetilde{D}) \leq \frac{4}{3} + \varepsilon n(\widetilde{D}) - \delta\frac{2(n(\widetilde{D})-1)}{3} \leq \frac{4}{3} + 4\varepsilon - 2\delta$ because $\delta \geq \frac{3}{2}\varepsilon$ and $n(\widetilde{D}) \geq 4$. ◊

Claim 4.4.21. *Let R be a subdigraph of D with $4 \leq n(R) < n(D)$. If R' is a dicritical extension of R with extender W and core X_W, then*

$$\rho(R') \leq \rho(W) + \rho(R) - \left(\rho(\overleftrightarrow{K_{|X_W|}}) + \delta \cdot T(\overleftrightarrow{K_{|X_W|}})\right) + \delta \cdot (T(W) - T(W - X_W))$$

and in particular

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Proof of claim. We have

- $n(R') = n(W) - |X_W| + n(R)$,
- $m(R') \geq m(W) + m(R) - m(\overleftrightarrow{K}_{|X_W|})$,
- $T(R') \geq T(W - X_W) + T(R)$

and by summing these inequalities, we get the first result.

Now observe that $T(W) - T(W - X_W) \leq |X_W|$ by Lemma 4.4.7, and that the maximum of $-\left(\rho(\overleftrightarrow{K}_{|X_W|}) + \delta T(\overleftrightarrow{K}_{|X_W|})\right) + \delta|X_W|$ is reached when $|X_W| = 1$, in which case it is equal to $-\frac{10}{3} - \varepsilon + \delta$. The second inequality follows. ♢

Claim 4.4.22. *If R is a subdigraph of D with $4 \leq n(R) < n(D)$, then $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$.*

Proof of claim. We proceed by induction on $n - n(R)$. Let R' be a dicritical extension of R with extender W and core X_W. By Claim 4.4.21, we have

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Either $V(R') = V(D)$ and so $\rho(R') \geq \rho(D)$ or $V(R')$ is a proper subset of $V(D)$ and, since R is a proper subdigraph of R', by induction $\rho(R') \geq \rho(D) + 2 - 3\varepsilon + \delta \geq \rho(D)$. In both cases, $\rho(R') \geq \rho(D)$. Now W is smaller than D so $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20. Thus,

$$\rho(D) \leq \rho(R') \leq \frac{4}{3} + 4\varepsilon - 2\delta + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

This gives $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$ because $\rho(D) > 1$. ♢

As a consequence of Claim 4.4.22, any subdigraph (proper or not) of size at least 4 has potential at least $\rho(D)$.

We say that an induced subdigraph R of D is *collapsible* if, for every 3-dicolouring ϕ of R, its dicritical extension R' (with extender W and core X_W) is D, has core of size 1 (i.e. $|X_W| = 1$), and the border $\partial_D(R)$ of R is monochromatic in ϕ.

Claim 4.4.23. *Let R be an induced subdigraph of D and ϕ a 3-dicolouring of R such that $\partial(R)$ is not monochromatic in ϕ. If D is a dicritical extension of R dicoloured by ϕ with extender W and core X_W with $|X_W| = 1$, then*

$$\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta.$$

Proof of claim. Assume D is a dicritical extension of R dicoloured by ϕ with extender W and core X_W with $|X_W| = 1$. Observe that each of the following inequalities holds:

- $n(D) = n(W) - |X_W| + n(R) = n(W) + n(R) - 1$,
- $m(D) \geq m(W) + m(R) - m(\overleftrightarrow{K}_{|X_W|}) + 1 = m(W) + m(R) + 1$ because $\partial_D(R)$ is not monochromatic in ϕ, and

- $T(D) \geq T(W - X_W) + T(R) \geq T(W) + T(R) - 1$ by Lemma 4.4.7.

By Claim 4.4.20, we have

$$\rho(D) \leq \rho(W) + \rho(R) - \left(\frac{10}{3} + \varepsilon\right) - 1 + \delta \leq \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + \rho(R) - \frac{13}{3} - \varepsilon + \delta$$

and so $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta$. ◊

Claim 4.4.24. *If R is a subdigraph of D with $4 \leq n(R) < n(D)$ and R is not collapsible, then $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$.*

Proof of claim. Let R' be a dicritical extension of R dicoloured by ϕ with extender W and core X_W. We distinguish four cases.

Case 1: *$R' \neq D$.*

Then R' has a dicritical extension R'' with extender W'. By (the consequence of) Claim 4.4.22, we have $\rho(D) \leq \rho(R'')$. By Claim 4.4.21 (applied twice), we have

$$\rho(R'') \leq \rho(R) + \rho(W') + \rho(W) + 2\left(-\frac{10}{3} - \varepsilon + \delta\right).$$

Both W and W' are smaller than D, so, by Claim 4.4.20, $\rho(W), \rho(W') \leq \frac{4}{3} + 4\varepsilon - 2\delta$. Those three inequalities imply

$$\rho(D) \leq \rho(R'') \leq \rho(R) + 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + 2\left(-\frac{10}{3} - \varepsilon + \delta\right) = \rho(R) - 4 + 6\varepsilon - 2\delta$$

and so $\rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

Case 2: *$R' = D$ and $|X_W| = 2$.*

Then $\rho(\overleftrightarrow{K}_{|X_W|}) + \delta T(\overleftrightarrow{K}_{|X_W|}) = \frac{14}{3} + 2\varepsilon$, and, by Lemma 4.4.7, $T(W) - T(W - X_W) \leq |X_w| = 2$. Thus, by Claim 4.4.21,

$$\rho(D) \leq \rho(W) + \rho(R) - \frac{14}{3} - 2\varepsilon + 2\delta$$

Now, since W is smaller than D, $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - \frac{14}{3} - 2\varepsilon + 2\delta = \rho(R) - \frac{10}{3} + 2\varepsilon$$

and so $\rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

Case 3: *$R' = D$ and $|X_W| = 3$.*

Then $\rho(\overleftrightarrow{K}_{|X_W|}) + \delta T(\overleftrightarrow{K}_{|X_W|}) = 4 + 3\varepsilon$, and, by Lemma 4.4.7, $T(W) - T(W - X_W) \leq |X_w| = 3$. Thus, by Claim 4.4.21,

$$\rho(D) \leq \rho(W) + \rho(R) - 4 - 3\varepsilon + 3\delta.$$

Now, since W is smaller than D, $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - 4 - 3\varepsilon + 3\delta = \rho(R) - \frac{8}{3} + \varepsilon + \delta$$

and so $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

Case 4: *$R' = D$, $|X_W| = 1$, and $\partial(R)$ is not monochromatic in ϕ.*

Then, by Claim 4.4.23, we have $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

If R is not collapsible, then by definition it has a dicritical extension R' satisfying the hypothesis of one of the cases above. In any case, $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$. $\Diamond$

A *k-cutset* in a graph G is a set S of k vertices such that $G - S$ is not connected. Recall that a graph is *k-connected* if it has at least k vertices and has no $(k-1)$-cutset, and that a digraph is *k-connected* if its underlying graph is k-connected.

Claim 4.4.25. *D is 2-connected.*

Proof of claim. Suppose for contradiction that $\{x\}$ is a 1-cutset of $\mathrm{UG}(D)$. Let (A_0, B_0) be a partition of $V(D - x)$ into non-empty sets such that there is no edge between A_0 and B_0, and set $A = A_0 \cup \{x\}$ and $B = B_0 \cup \{x\}$.

Since D is 4-dicritical, there exist a 3-dicolouring ϕ_A of $D\langle A\rangle$ and a 3-dicolouring ϕ_B of $D\langle B\rangle$. Free to swap the colours, we may assume $\phi_A(x) = \phi_B(x)$. Let ϕ be defined by $\phi(v) = \phi_A(v)$ if $v \in A$ and $\phi(v) = \phi_B(v)$ if $v \in B$. Since $\vec{\chi}(D) = 4$, D, coloured with ϕ, must contain a monochromatic directed cycle. Such a directed cycle must be contained in $D\langle A\rangle$ or $D\langle B\rangle$, a contradiction. $\Diamond$

Claim 4.4.26. *D is 3-connected. In particular, D contains no diamond.*

Proof of claim. Suppose for contradiction that $\{x, y\}$ is a 2-cutset of $\mathrm{UG}(D)$. Let (A_0, B_0) be a partition of $V(D) \setminus \{x, y\}$ into non-empty sets such that there is no edge between A_0 and B_0, and set $A = A_0 \cup \{x, y\}$ and $B = B_0 \cup \{x, y\}$.

Assume for a contradiction that there exists a 3-dicolouring ϕ_A of $D\langle A\rangle$ and a 3-dicolouring ϕ_B of $D\langle B\rangle$ such that $\phi_A(x) \neq \phi_A(y)$ and $\phi_B(x) \neq \phi_B(y)$. Free to swap the colours, we may assume $\phi_A(x) = \phi_B(x)$ and $\phi_A(y) = \phi_B(y)$. Let ϕ be defined by $\phi(v) = \phi_A(v)$ if $v \in A$ and $\phi(v) = \phi_B(v)$ if $v \in B$. Every directed cycle either is in $D\langle A\rangle$, or is in $D\langle B\rangle$ or contains both x and y. Therefore, it cannot be monochromatic with ϕ because ϕ_A and ϕ_B are 3-dicolourings of $D\langle A\rangle$ and $D\langle B\rangle$ respectively, and $\phi(x) \neq \phi(y)$. Thus ϕ is a 3-dicolouring of D, a contradiction. Hence either $D\langle A\rangle$ or $D\langle B\rangle$ has no 3-dicolouring ϕ such that $\phi(x) \neq \phi(y)$. Suppose without loss of generality that it is $D\langle A\rangle$.

Let $D_A = D\langle A\rangle \cup [x, y]$. D_A is not 3-dicolourable because in every 3-dicolouring of $D\langle A\rangle$, x and y are coloured the same. Let D_B be the digraph obtained from $D\langle B\rangle$ by identifying x and y into a vertex z. Assume for a contradiction that D_B has a 3-dicolouring ψ_B. Set $\psi(x) = \psi(y) = \psi_B(z)$, and $\psi(u) = \psi_B(u)$ for every $u \in B \setminus \{x, y\}$. Then consider a 3-dicolouring ψ_A of $D\langle A\rangle$ such that $\psi_A(x) = \psi(x) = \psi_A(y) = \psi(y)$ (such a colouring exists because A is a proper subdigraph of D) and we set $\psi(u) = \psi_A(u)$ for every $u \in V(A) \setminus \{x, y\}$. As D is not 3-dicolourable, it contains a monochromatic directed cycle C (with respect to ψ). The cycle C is not included in $D\langle A\rangle$ nor in D_B. As a consequence, there is a monochromatic directed path from $\{x, y\}$ to $\{x, y\}$ in B, and so there is a monochromatic directed cycle in D_B for ψ_B, a contradiction. Therefore D_B is not 3-dicolourable.

Now D_A has a 4-dicritical subdigraph W_A which necessarily contains $\{x, y\}$, and D_B has a 4-dicritical subdigraph W_B which necessarily contains z. As W_A and W_B are 4-dicritical digraphs smaller than D, we have $\rho(W_A), \rho(W_B) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20. Let H be the subdigraph of D induced by $V(W_A) \cup V(W_B - z)$.

Note that $n(H) = n(W_A) + n(W_B) - 1$ and $m(H) \geq m(W_A) + m(W_B) - 2$. Moreover $T(H) \geq T(W_A - x) + T(W_B - z) \geq T(W_A) + T(W_B) - 2$, by Lemma 4.4.7. Hence, we have

$$\begin{aligned} \rho(H) &\leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + (m(W_A) + m(W_B) - m(H)) + 2\delta \\ &\leq \rho(W_A) + \rho(W_B) - \frac{10}{3} - \varepsilon + 2 + 2\delta \\ &= \rho(W_A) + \rho(W_B) - \frac{4}{3} - \varepsilon + 2\delta \\ &\leq 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \frac{4}{3} - \varepsilon + 2\delta \\ &= \frac{4}{3} + 7\varepsilon - 2\delta \end{aligned} \tag{4.2}$$

By Claim 4.4.22, if $n(H) < n(D)$ then $\rho(H) > 3 - 3\varepsilon + \delta$. As $10\varepsilon - 3\delta \leq \frac{5}{3}$, we deduce that $H = D$. Hence $1 < \rho(D) = \rho(H) \leq \frac{4}{3} + 7\varepsilon - 2\delta + (m(W_A) + m(W_B) - m(H) - 2)$ and so $m(H) = m(W_A) + m(W_B) - 2$ because $2\delta - 7\varepsilon \leq \frac{2}{3}$. In particular, there is no arc between x and y in D. Moreover, no arc was suppressed when identifying x and y into z to obtain D_B, so x and y have no common out-neighbour (resp. in-neighbour) in B_0.

We first show that either W_A or W_B is not 4-Ore. Assume for contradiction that both W_A and W_B are 4-Ore. If $H = D$ is not bidirected, then by Lemma 4.4.6, $D\langle B\rangle$ admits a 3-dicolouring ϕ_B such that $\phi_B(x) = \phi_B(y)$. Now let ϕ_A be a 3-dicolouring of $D\langle A\rangle$. We have $\phi_A(x) = \phi_A(y)$. Free to exchange colours, we may assume $\phi_A(x) = \phi_A(y) = \phi_B(x) = \phi_B(y)$. Hence, we can define the 3-colouring ϕ of D by $\phi(v) = \phi_A(v)$ if $v \in A$, and $\phi(v) = \phi_B(v)$ if $v \in B$. Observe that, since A is bidirected, all neighbours of x and y in $D\langle A\rangle$ have a colour distinct from $\phi(x)$. Therefore, there is no monochromatic directed cycle in D coloured by ϕ. Thus ϕ is a 3-dicolouring of D, a contradiction. Therefore, $H = D$ is bidirected, and so H is an Ore-composition of W_A and W_B (because D is 2-connected by Claim 4.4.25), and so D is 4-Ore, a contradiction. Henceforth, we may assume that either W_A or W_B is not 4-Ore.

If none of W_A and W_B is a 4-Ore, then by minimality of D, $\rho(W_A) \leq 1$ and $\rho(W_B) \leq 1$. Together with Equation (4.2), this yields

$$\rho(H) \leq \frac{2}{3} - \varepsilon + 2\delta \leq 1$$

because $2\delta - \varepsilon \leq \frac{1}{3}$, a contradiction.

If none of W_A and W_B is $\overleftrightarrow{K_4}$, then $\rho(W_A) + \rho(W_B) \leq 1 + (\frac{4}{3} + 7\varepsilon - 4\delta)$ (recall that if a digraph is 4-Ore but not $\overleftrightarrow{K_4}$, then it has potential at most $\frac{4}{3} + 7\varepsilon - 4\delta$ by Lemma 4.4.16). Thus, with Equation (4.2), we get

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 7\varepsilon - 4\delta\right) - \frac{4}{3} - \varepsilon + 2\delta = 1 + 6\varepsilon - 2\delta \leq 1$$

because $\delta \geq 3\varepsilon$.

Finally, if exactly one of W_A or W_B is isomorphic to $\overleftrightarrow{K_4}$, then either $T(W_A - x) = T(W_A) = 2$ (if $W_A = \overleftrightarrow{K_4}$) or $T(W_B - z) = T(W_B) = 2$ (if $W_B = \overleftrightarrow{K_4}$). Therefore $T(H) \geq T(W_A - x) +$

$T(W_B - z) \geq T(W_A) + T(W_B) - 1$ by Lemma 4.4.7, and so

$$\rho(H) \leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta.$$

Now the non 4-Ore digraph among W_A, W_B has potential at most 1 and the other has potential $\rho(\overleftrightarrow{K_4}) = \frac{4}{3} + 4\varepsilon - 2\delta$. Thus

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta = 1 + 3\varepsilon - \delta \leq 1$$

because $\delta \geq 3\varepsilon$. In all three cases, $\rho(D) = \rho(H) \leq 1$, which is a contradiction. Hence D is 3-connected. $\lozenge$

Claim 4.4.27. *If R is a collapsible subdigraph of D, u, v are in the boundary of R and $D\langle R\rangle \cup [u, v]$ is 4-Ore, then there exists $R' \subseteq R$ such that*

(i) either R' is an Ore-collapsible subdigraph of D, or

(ii) R' is an induced subdigraph of R, $n(R') < n(R)$, and there exist u', v' in $\partial_D(R')$ such that $R' \cup [u', v']$ is 4-Ore.

Proof of claim. If $\partial(R) = \{u, v\}$, then R is Ore-collapsible and we are done. Suppose now that there exists $w \in \partial(R)$ distinct from u and v. Let $H = D\langle R\rangle \cup [u, v]$. Observe that $H \neq \overleftrightarrow{K_4}$ as u, v and w receive the same colour in any 3-dicolouring of $D\langle R\rangle$ because R is collapsible. Hence, H is the Ore-composition of two 4-Ore digraphs H_1 (the digon side with replaced digon $[x, y]$) and H_2 (the split side with split vertex z).

If u or v is in $V(H_2)$, then $R' = D\langle V(H_1)\rangle$ with $u' = x, v' = y$ satisfies (ii). Now we assume that $u, v \in V(H_1) \setminus V(H_2)$. By repeating this argument successively on H_1, and then on the digon-side of H_1, etc, either we find a subdigraph R' satisfying (ii) or u and v are in a copy J of $\overleftrightarrow{K_4}$ such that H is obtained by Ore-compositions between J and some 4-Ore digraphs with J being always in the digon side.

Observe that $w \notin V(J)$ because in any 3-dicolouring of $H \setminus [u, v]$, vertices in J receive different colours by Lemma 4.4.18, except u and v. Hence, at one step in the succession of Ore-compositions, w was in the split-side S when a digon e in J has been replaced. However $e \neq [u, v]$, so either u or v is not in e. Suppose without loss of generality that e is not incident to v.

We claim that $H' = R - v \cup [u, w]$ is not 3-dicolourable. Otherwise, let ϕ be a 3-dicolouring of H'. Then ϕ is a 3-dicolouring of $H - v$ with H 4-Ore, so vertices in $J - v$ must receive pairwise different colours by Lemma 4.4.19. Let ϕ' be a 3-dicolouring of R. Without loss of generality, we may assume that $\phi(x) = \phi'(x)$ for every $x \in V(J - v)$. If $y \in S$, let $\phi''(y) = \phi(y)$, and let $\phi''(y) = \phi'(y)$ if $y \notin S$. Then ϕ'' is a 3-dicolouring of R but with $\phi(u) \neq \phi(w)$, contradicting the fact that R is collapsible. This shows that $H' = R - v \cup [u, w]$ is not 3-dicolourable.

Hence $R - v \cup [u, w]$ contains a 4-dicritical digraph K. By Lemma 4.4.17, $R' = D\langle V(K)\rangle$, as a subdigraph of H which is a 4-Ore, satisfies $\frac{10}{3}n(R') - m(R') \geq \frac{10}{3}$. This implies that $\frac{10}{3}n(K) - m(K) \geq \frac{4}{3}$. Note also that K is bidirected because $R - v$ is bidirected. Thus, by Theorem 4.4.15, K is 4-Ore. Hence R' with u, w satisfies (ii). $\lozenge$

Claim 4.4.28. *If R is a subdigraph of D with $n(R) < n(D)$ and $u, v \in V(R)$, then $R \cup [u, v]$ is 3-dicolourable. As a consequence, there is no collapsible subdigraph in D.*

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest induced subdigraph R for which the statement does not hold. Then $K = R \cup [u, v]$ is 4-vertex-dicritical, that is for every vertex $v \in V(K), \vec{\chi}(K - v) < 4 = \vec{\chi}(K)$. Note that 4-vertex-dicritical digraphs smaller than D satisfy the outcome of Theorem 4.4.2 since adding arcs does not increase the potential. Note that $\rho(R) \leq \rho(K) + 2 + \delta$.

If R is not collapsible, then, by Claim 4.4.24, $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$. But we also have $\rho(R) \leq \rho(K) + 2 + \delta \leq \frac{10}{3} + 4\varepsilon - \delta$ by Claim 4.4.20, which is a contradiction because $5\varepsilon \leq \frac{1}{3}$. Hence R is collapsible.

Let ϕ be a 3-dicolouring of R. Observe that $\phi(u) = \phi(v)$ for otherwise $R \cup [u, v]$ would be 3-dicolourable. Let R' be the dicritical extension of R with extender W and core X_W. We have $R' = D$ and $|X_W| = 1$. Since R is collapsible, for every two vertices u', v' on the boundary of R, $R \cup [u', v']$ is not 3-dicolourable. Hence, free to consider u', v' instead of u, v, we can suppose that u and v are on the boundary of R. If K is 4-Ore, then, by Claim 4.4.27 and by minimality of R, we have that R is Ore-collapsible, and so has boundary of size 2. This contradicts the fact that D is 3-connected. Hence K is not 4-Ore.

By Claim 4.4.21, we have

$$\begin{aligned} 1 < \rho(D) = \rho(R') &\leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta \\ &\leq \rho(W) + (\rho(K) + 2 + \delta) - \frac{10}{3} - \varepsilon + \delta \end{aligned}$$

and as $\rho(K) \leq 1$ (because it is not 4-Ore and by minimality of D) we get

$$1 < 1 + \rho(W) - \left(\frac{4}{3} + \varepsilon - 2\delta\right)$$

that is $\rho(W) > \frac{4}{3} + \varepsilon - 2\delta$. But as W is smaller than D, it satisfies Theorem 4.4.2. Thus, since $\varepsilon - 2\delta \geq \frac{-1}{3}$, W must be 4-Ore. Moreover, W must be isomorphic to $\overleftrightarrow{K_4}$, for otherwise $\rho(W)$ would be at most $\frac{4}{3} + 7\varepsilon - 4\delta$, and $\frac{4}{3} + 7\varepsilon - 4\delta \geq \rho(W) > \frac{4}{3} + \varepsilon - 2\delta$ would contradict $\delta \geq 3\varepsilon$. Hence $\rho(W) = \rho(\overleftrightarrow{K_4}) = \frac{4}{3} + 4\varepsilon - 2\delta$ and $T(W - X_W) = 2 = T(W)$. Thus, by Claim 4.4.21 and because $\delta \geq 3\varepsilon$, we have

$$1 < \rho(D) \leq \rho(W) + \rho(K) + 2 + \delta - \frac{10}{3} - \varepsilon \leq \rho(K) + 3\varepsilon - \delta \leq \rho(K) \leq 1,$$

a contradiction.

This implies that D does not contain any collapsible subdigraph. Indeed, assume for a contradiction that D contains a collapsible subdigraph R, and let u, v be two vertices in its boundary. Then there exists a 3-dicolouring ϕ of $R \cup [u, v]$, for which $\partial(R)$ is not monochromatic, a contradiction. $\lozenge$

Claim 4.4.29. *If R is a subdigraph of D with $n(R) < n(D)$ and $u, v, u', v' \in R$, then $R \cup \{uv, u'v'\}$ is 3-dicolourable. In particular, D contains no copy of $\overleftrightarrow{K_4}$ minus two arcs.*

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest subdigraph R for which the statement does not hold. Then $K = R \cup \{uv, u'v'\}$ is 4-dicritical and smaller than D, so $\rho(K) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20. By Claim 4.4.28, R is not collapsible, so, by Claim 4.4.24, we have $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$. But $\rho(R) \leq \rho(K) + 2 + 2\delta \leq \frac{10}{3} + 4\varepsilon$, which is a contradiction as $5\varepsilon + \delta \leq \frac{1}{3}$. ♢

Claim 4.4.30. *Vertices of degree* 6 *in* D *have either three or six neighbours.*

Proof of claim. Let x be a vertex of degree 6.

If $|N(x)| = 4$, then let a, b, c, d be its neighbours such that $N^+(x) = \{a, b, c\}$ and $N^-(x) = \{a, b, d\}$. Consider $D' = D - x \cup dc$. By Claim 4.4.29, D' has a 3-dicolouring ϕ. If $|\phi(N^-(x))| < 3$, then choosing $\phi(x)$ in $\{1, 2, 3\} \setminus \phi(N^-(x))$, we obtain a 3-dicolouring of D, a contradiction. Hence $\phi(N^-(x)) = \{1, 2, 3\}$. We set $\phi(x) = \phi(d)$. As D is not 3-dicolourable, D contains a monochromatic directed cycle C. This cycle C must contain the arc dx, and an out-neighbour z of x. Since $\phi(a)$, $\phi(b)$ and $\phi(d)$ are all distinct, necessarily $z = c$. But then $C - x \cup dc$ is a monochromatic directed cycle in D', a contradiction.

Similarly, if $|N(x)| = 5$, let $N^+(x) = \{a, b, c\}$ and $N^-(x) = \{a, d, e\}$, and consider $D' = D - x \cup \{db, dc\}$. By Claim 4.4.29, D' has a 3-dicolouring ϕ. If $|\phi(N^-(x))| < 3$, then choosing $\phi(x)$ in $\{1, 2, 3\} \setminus \phi(N^-(x))$, we obtain a 3-dicolouring of D, a contradiction. Hence $\phi(N^-(x)) = \{1, 2, 3\}$. We set $\phi(x) = \phi(d)$. As D is not 3-dicolourable, there is a monochromatic directed cycle C, which must contain the arc dx and an out-neighbour z of x. Note that z must be b or c because $\phi(a) \neq \phi(d)$. Then $C - x \cup dz$ is a monochromatic directed cycle in D', a contradiction. ♢

Claim 4.4.31. *There is no bidirected triangle containing two vertices of degree* 6*. In particular,* D *contains no emerald.*

Proof of claim. Suppose that $D\langle\{x, y, z\}\rangle = \overleftrightarrow{K_3}$ and $d(x) = d(y) = 6$. By Claim 4.4.30, x and y have exactly three neighbours, and $N[x] \neq N[y]$ because D contains no copy of $\overleftrightarrow{K_4}$ minus two arcs by Claim 4.4.29. Let u (resp. v) be the unique neighbour of x distinct from y and z (resp. x and z). Consider $D' = D - \{x, y\} \cup [u, v]$. By Claim 4.4.28, D' has a 3-dicolouring ϕ. Without loss of generality, suppose that $\phi(u) = 1$ and $\phi(v) = 2$. If $\phi(z) = 1$ (resp. $\phi(z) = 2$, $\phi(z) = 3$), we set $\phi(x) = 2$ and $\phi(y) = 3$ (resp. $\phi(x) = 3$ and $\phi(y) = 1$, $\phi(x) = 2$ and $\phi(y) = 1$). In each case, this yields a 3-dicolouring of D, a contradiction. ♢

So now we know that D contains no emerald, and no diamond by Claim 4.4.26.

Claim 4.4.32. *If* R *is an induced subdigraph of* D *with* $4 \leq n(R) < n(D)$*, then* $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$*, except if* $D - R$ *contains a single vertex which has degree* 6 *in* D*.*

Proof of claim. Let R be an induced subdigraph of D with $4 \leq n(R) < n(D)$. By Claim 4.4.28, R is not collapsible. Let ϕ be a 3-dicolouring of R, R' be a dicritical extension of R with extender W and core X_W (with respect to ϕ). By (the consequence of) Claim 4.4.22, we know that $\rho(R') \geq \rho(D)$.

Assume first that $R' \neq D$. Then, by Claims 4.4.22 and 4.4.21,

$$\rho(D) + 2 - 3\varepsilon + \delta \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Since $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20, we have $\rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ because $1 \geq 9\varepsilon - 5\delta$. In the following we suppose that $R' = D$. We distinguish three cases depending on the cardinality of $|X_W|$.

- Assume first that $|X_W| = 2$. Then, by Claim 4.4.21 and Lemma 4.4.7,

  $$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{20}{3} - 2\varepsilon + 2 + 2\delta$$

  and, as $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 4.4.20, we have $\rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ because $5\varepsilon - 3\delta \leq \frac{1}{3}$.

- Assume now that $|X_W| = 3$. If there is a vertex $v \in V(D-R)$ with two out-neighbours (resp. two in-neighbours) in $V(R)$ with the same colour for ϕ, then

  - $n(R') = n(W) - |X_W| + n(R)$,
  - $m(R') \geq m(W) + m(R) - m(\overleftrightarrow{K}_{|X_W|}) + 1$ because v has two in- or out-neighbour in $V(R)$ with the same colour for ϕ,
  - $T(R') \geq T(W - X_W) + T(R)$.

  It follows that

  $$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta - 1$$

  and so

  $$\rho(R) \geq \rho(D) - \frac{4}{3} - 4\varepsilon + 2\delta + 5 + 3\varepsilon - 3\delta \geq \rho(D) + \frac{11}{3} - \varepsilon - \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$$

  because $4\varepsilon - 2\delta \leq \frac{2}{3}$. Now we assume that there is no vertex with two out-neighbours (resp. two in-neighbours) in R with the same colour for ϕ. In other words, the in-degrees and out-degrees of vertices in $D - R$ are the same in D and in W.

  If W is not 4-Ore, then by Claim 4.4.21

  $$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta$$

  and, as $\rho(W) \leq 1$, we have $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$.

  Now suppose W is 4-Ore. If $W \neq \overleftrightarrow{K}_4$, then, by Lemma 4.4.14, W contains a diamond or an emerald disjoint from X, and this gives a diamond or an emerald in D because the degrees of vertices in $D - R$ are the same in D and in W, which is a contradiction. If $W = \overleftrightarrow{K}_4$, then $D - R$ has a single vertex of degree 6 in D.

- Assume finally that $|X_W| = 1$. Since R is not collapsible by Claim 4.4.28, ϕ may have been chosen so that $\partial(R)$ is not monochromatic in ϕ. Then, by Claim 4.4.23, $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ because $6\varepsilon - 4\delta \leq 0$.

$\lozenge$

Claim 4.4.33. *Vertices of degree 7 have seven neighbours.*

Proof of claim. Let x be a vertex of degree 7. We suppose, without loss of generality, that $d^-(x) = 3$ and $d^+(x) = 4$.

If $|N(x)| = 4$, then x has a unique neighbour $a \in (N^+(x) \setminus N^-(x))$. As D is 4-dicritical, $D \setminus xa$ has a 3-dicolouring ϕ. But then every directed cycle is either in $D \setminus xa$ or it contains xa and thus an in-neighbour t of x. In the first case, it is not monochromatic because ϕ is a 3-dicolouring of $D \setminus xa$, and in the second case, it is not monochromatic because $[t, x]$ is a digon and so $\phi(t) \neq \phi(x)$. Hence ϕ is a 3-dicolouring of D, a contradiction.

If $|N(x)| = 5$, let $N^-(x) = \{a, b, c\}$ and $N^+(x) = \{a, b, d, e\}$. By Claim 4.4.29, $D' = D - x \cup \{cd, ce\}$ has a 3-dicolouring ϕ. If $|\phi(N^-(x))| < 3$, then choosing $\phi(x)$ in $\{1, 2, 3\} \setminus \phi(N^-(x))$ gives a 3-dicolouring of D, a contradiction. If $|\phi(N^-(x))| = 3$, then we set $\phi(x) = \phi(c)$. Suppose for a contradiction that there is a monochromatic directed cycle C in D (with ϕ). Necessarily C contains x (since ϕ is a 3-dicolouring of $D - x$) and so it must contain c and one vertex y in $\{d, e\}$ because $\phi(a)$, $\phi(b)$, and $\phi(c)$ are all distinct. Then $C - x \cup cy$ is a monochromatic directed cycle in D', a contradiction. Therefore ϕ is a 3-dicolouring of D, a contradiction.

If $|N(x)| = 6$, let $N^-(x) = \{a, b, c\}$ and $N^+(x) = \{a, d, e, f\}$. Consider $D' = D - x \cup \{bd, be, bf\}$.

We first show that D' is not 3-dicolourable. Assume for a contradiction that there is a 3-dicolouring ϕ of D'. If $|\phi(N^-(x))| < 3$, then choosing $\phi(x)$ in $\{1, 2, 3\} \setminus \phi(N^-(x))$ gives a 3-dicolouring of D, a contradiction. Hence $|\phi(N^-(x))| = 3$. We set $\phi(x) = \phi(b)$. Since D is not 3-dicolourable, there exists a monochromatic directed cycle C in D (with ϕ). Necessarily C contains x (since ϕ is a 3-dicolouring of $D - x$) and so it must contain b and one vertex y in $\{d, e, f\}$ because $\phi(a)$, $\phi(b)$, and $\phi(c)$ are all distinct. Then $C - x \cup by$ is a monochromatic directed cycle in D', a contradiction. This gives a 3-dicolouring of D, a contradiction.

Hence D' is not 3-dicolourable, and so it contains a 4-dicritical digraph $\widetilde{D}$, smaller than D. If $\widetilde{D}$ does not contain the three arcs bd, be, bf, then it can be obtained from a proper induced subdigraph of D by adding at most two arcs, and so it is 3-dicolourable by Claim 4.4.29, a contradiction.

Hence $\{b, d, e, f\} \subseteq V(\widetilde{D})$. Now consider $U = D\langle V(\widetilde{D}) \cup \{x\}\rangle$. We distinguish two cases.

Case 1: *We have $a \notin V(U)$ or $c \notin V(U)$.*

In this case, we have:

- $n(U) = n(\widetilde{D}) + 1$,
- $m(U) \geq m(\widetilde{D}) + 1$ and
- $T(U) \geq T(\widetilde{D} - b) \geq T(\widetilde{D}) - 1$ by Lemma 4.4.7.

Hence

$$
\begin{aligned}
\rho(U) &\leq \rho(\widetilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \\
&\leq \frac{4}{3} + 4\varepsilon - 2\delta + \frac{10}{3} + \varepsilon - 1 + \delta \quad \text{by Claim 4.4.20,} \\
&= 1 + \frac{8}{3} + 5\varepsilon - \delta \\
&< \rho(D) + \frac{8}{3} + 5\varepsilon - \delta \\
&\leq \rho(D) + 3 + 3\varepsilon - 3\delta \qquad \text{because } \frac{1}{3} \geq 2\delta + 2\varepsilon.
\end{aligned}
$$

Hence by Claim 4.4.32, $D - U$ has a single vertex of degree 6 (in D), which must be either a or c. Then we have

- $n(D) = n(\widetilde{D}) + 2$,
- $m(D) \geq m(\widetilde{D}) - 3 + 11$ and
- $T(D) \geq T(\widetilde{D} - b) \geq T(\widetilde{D}) - 1$.

Thus

$$\begin{aligned} \rho(D) &\leq \rho(\widetilde{D}) + 2\left(\frac{10}{3} + \varepsilon\right) - 8 + \delta \\ &\leq \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \frac{4}{3} + 2\varepsilon + \delta && \text{by Claim 4.4.20,} \\ &\leq 1 && \text{because } 6\varepsilon - \delta \leq 1. \end{aligned}$$

This is a contradiction.

Case 2: *Both a and c belong to $V(U)$.*

In this case, we have:

- $n(U) = n(\widetilde{D}) + 1$,
- $m(U) \geq m(\widetilde{D}) + 4$ and
- $T(U) \geq T(\widetilde{D} - b) \geq T(\widetilde{D}) - 1$ by Lemma 4.4.7.

Thus

$$\begin{aligned} \rho(U) &\leq \rho(\widetilde{D}) + \frac{10}{3} + \varepsilon - 4 + \delta \\ &\leq \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + \frac{10}{3} + \varepsilon - 4 + \delta && \text{by Claim 4.4.20,} \\ &\leq 1 && \text{because } 5\varepsilon - \delta \leq \frac{1}{3}. \end{aligned}$$

Together with the consequence of Claim 4.4.22, we get that $\rho(D) \leq \rho(U) \leq 1$, a contradiction.

$\Diamond$

The 8^+-*valency* of a vertex v, denoted by $\nu(v)$, is the number of arcs incident to v and a vertex of degree at least 8.

Let D_6 be the subdigraph of D induced by the vertices of degree 6 incident to digons. Let us describe the connected components of D_6 and their neighbourhoods. Remember that vertices of degree 7 are incident to no digon by Claim 4.4.33, and so they do not have neighbours in $V(D_6)$. If v is a vertex in D_6, we define its *neighbourhood valency* to be the sum of the 8^+-valency of its neighbours of degree at least 8. We denote the neighbourhood valency of v by $\nu_N(v)$.

Claim 4.4.34. *If $[x, y]$ is a digon and both x and y have degree 6, then either*

(i) the two neighbours of y distinct from x have degree at least 8, or

(ii) the two neighbours of x distinct from y have degree at least 8 and $\nu_N(x) \geq 4$.

Proof of claim. Let $[x, y]$ be a digon in D with $d(x) = d(y) = 6$. By Claim 4.4.30 $|N(x)| = |N(y)| = 3$. Let u and v be the two neighbours of x different from y. By Claim 4.4.33, u and v have degree 6 or at least 8.

If u and v are linked by a digon, then by Claim 4.4.31, u and v do not have degree 6, so they have degree 8. Moreover $\nu(u) \geq 2$ and $\nu(v) \geq 2$. Thus $\nu_N(x) = \nu(u) + \nu(v) \geq 4$ and (ii) holds. Henceforth, we may assume that u and v are not linked by a digon.

Let D' the digraph obtained by removing x and y and identifying u and v into a single vertex $u \star v$. We claim that D' is not 3-dicolourable. To see that, suppose for contradiction that there exists a 3-dicolouring ϕ of D'. Then set $\phi(u) = \phi(v) = \phi(u \star v)$, choose $\phi(y)$ in $\{1, 2, 3\} \setminus \phi(N(y) \setminus \{x\})$, and finally choose $\phi(x)$ in $\{1, 2, 3\} \setminus \{\phi(u \star v), \phi(y)\}$. One can easily see that ϕ is now a 3-dicolouring of D, a contradiction. This proves that D' is not 3-dicolourable and so it contains a 4-dicritical digraph $\widetilde{D}$, which must contain $u \star v$ because every subdigraph of D is 3-dicolourable. Let R be the subdigraph of D induced by $(V(\widetilde{D}) \setminus \{u \star v\}) \cup \{u, v, x\}$. We have

- $n(R) = n(\widetilde{D}) + 2$,
- $m(R) \geq m(\widetilde{D}) + 4$ and
- $T(R) \geq T(\widetilde{D} - u \star v) + 1 \geq T(\widetilde{D})$ because $[x, u]$ is a digon, and by Lemma 4.4.7.

If $\widetilde{D}$ is not 4-Ore, then $\rho(\widetilde{D}) \leq 1$ by minimality of D, and so

$$
\begin{aligned}
\rho(R) &\leq \rho(\widetilde{D}) + 2\left(\frac{10}{3} + \varepsilon\right) - 4 \\
&\leq 1 + \frac{8}{3} + 2\varepsilon \\
&< \rho(D) + 3 + 3\varepsilon - 3\delta \qquad \text{because } \varepsilon - 3\delta \geq -\frac{1}{3}.
\end{aligned}
$$

Similarly, if $\widetilde{D}$ is 4-Ore but not $\overleftrightarrow{K_4}$, then

$$
\begin{aligned}
\rho(R) &\leq \rho(\widetilde{D}) + 2\left(\frac{10}{3} + \varepsilon\right) - 4 \\
&\leq \left(\frac{4}{3} + 7\varepsilon - 4\delta\right) + \frac{8}{3} + 2\varepsilon \quad \text{by Lemma 4.4.16,} \\
&= 1 + 3 + 9\varepsilon - 4\delta \\
&< \rho(D) + 3 + 9\varepsilon - 4\delta \\
&\leq \rho(D) + 3 + 3\varepsilon - 3\delta \qquad \text{because } \delta \geq 6\varepsilon.
\end{aligned}
$$

In both cases (that is when $\widetilde{D}$ is not $\overleftrightarrow{K_4}$), by Claim 4.4.32, $D - R$ is a single vertex of degree 6, namely y. Then every neighbour w of y different from x has degree at least 6 in $\widetilde{D}$ (because $\widetilde{D}$ is 4-dicritical) and so has degree at least 8 in D and (i) holds.

Assume now that $\widetilde{D}$ is a copy of $\overleftrightarrow{K_4}$. Let us denote by a, b, c the vertices of $\widetilde{D}$ different from $u \star v$. Suppose for a contradiction that u has degree 6. Then u has exactly three neighbours by

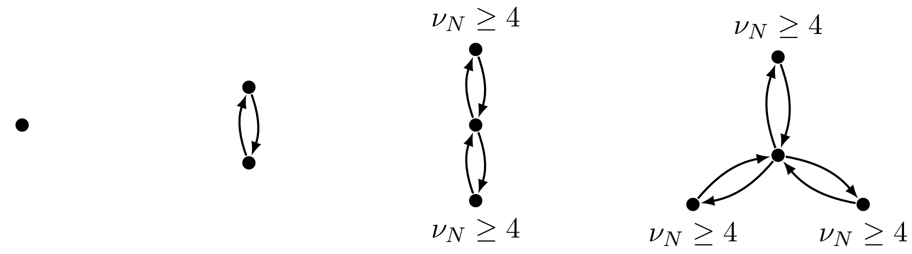


Figure 4.4: The possible connected components of D_6.

Claim 4.4.30. If $|N(u) \cap \{a, b, c\}| = 2$, then $D\langle\{u, a, b, c\}\rangle$ is a copy of $\overleftrightarrow{K_4}$ minus a digon, contradicting Claim 4.4.28. If $|N(u) \cap \{a, b, c\}| \leq 1$, then v must be adjacent to at least two vertices of $\{a, b, c\}$ with a digon, and so $D\langle\{v, a, b, c\}\rangle$ contains a copy of $\overleftrightarrow{K_4}$ minus a digon, contradicting Claim 4.4.28. Hence u has degree at least 8, and by symmetry so does v. Moreover $D\langle\{a, b, c\}\rangle$ is a bidirected triangle, and so by Claim 4.4.31, at least two of these vertices have degree at least 8 (remember that vertices of degree 7 are in no digon by Claim 4.4.33). Hence at least four arcs between $\{u, v\}$ and $\{a, b, c\}$ are incident to two vertices of degree at least 8. In other word, $\nu_N(x) = \nu(u) + \nu(v) \geq 4$, so (ii) holds. $\lozenge$

Claim 4.4.35. *Let C be a connected component of D_6. Then C is one of the following (see Figure 4.4):*

(i) a single vertex, or

(ii) a bidirected path on two vertices, or

(iii) a bidirected path on three vertices, whose extremities have neighbourhood valency at least 4, or

(iv) a star on four vertices, whose non-central vertices have neighbourhood valency at least 4.

Proof of claim. First observe that C does not contain a bidirected path on four vertices x, y, z, w, because otherwise, by Claim 4.4.34 applied on $[y, z]$, either y or z has two neighbours of degree at least 8, a contradiction. Observe also that C contains no bidirected triangle by Claim 4.4.31.

Moreover, if x, y, z is a bidirected path in C on three vertices, then by Claim 4.4.34 applied both on $[y, z]$ and $[z, y]$, x and z have both neighbourhood valency at least 4. The statement of the claim follows. $\lozenge$

An arc xy is said to be *out-chelou* if

(i) $yx \notin A(D)$,

(ii) $d^+(x) = 3$,

(iii) $d^-(y) = 3$, and

(iv) there exists $z \in N^-(y) \setminus N^+(y)$ distinct from x.

Symmetrically, we say that an arc xy is *in-chelou* if yx is out-chelou in the digraph obtained from D by reversing every arc. See Figure 4.5 for an example of an out-chelou arc.

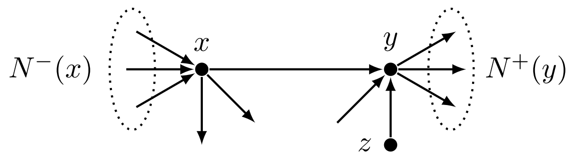


Figure 4.5: An example of an out-chelou arc xy.

Claim 4.4.36. *There is no out-chelou arc and no in-chelou arc in D.*

Proof of claim. By directional duality, it suffices to prove that D has no out-chelou arcs.

Let xy be an out-chelou arc with $z \in N^-(y) \setminus (N^+(y) \cup \{x\})$. Consider $D' = D - \{x, y\} \cup \{zz' \mid z' \in N^+(y) \setminus N^-(y)\}$. We claim that D' is not 3-dicolourable. To see that, suppose for contradiction that there is a 3-dicolouring ϕ of D'. As $d^+(x) = 3$, we can choose $\phi(x)$ in $\{1, 2, 3\} \setminus \phi(N^+(x) \setminus \{y\})$ to obtain a 3-dicolouring of $D - y$. If $|\phi(N^-(y))| < 3$, then choosing $\phi(y)$ in $\{1, 2, 3\} \setminus \phi(N^-(y))$ gives a 3-dicolouring of D, a contradiction. Hence $|\phi(N^-(y)| = 3$. Set $\phi(x) = \phi(z)$. Suppose there is a monochromatic directed cycle C in D. It must contain y and thus z, its unique in-neighbour with its colour. Let z' be the out-neighbour of y in C. It must be in $N^+(y) \setminus N^-(y)$, so zz' is an arc in D'. Thus $C - y \cup zz'$ is a monochromatic directed cycle in D', a contradiction. Therefore ϕ is a 3-dicolouring of D, a contradiction. Hence D' is not 3-dicolourable.

Consequently, D' contains a 3-dicritical digraph $\widetilde{D}$, which is smaller than D and contains z, for otherwise $\widetilde{D}$ would be a subdigraph of D. Consider $U = D\langle V(\widetilde{D}) \cup \{y\}\rangle$. We have

- $n(U) = n(\widetilde{D}) + 1$,
- $m(U) \geq m(\widetilde{D}) + 1$ and
- $T(U) \geq T(\widetilde{D} - z) \geq T(\widetilde{D}) - 1$ by Lemma 4.4.7.

First if $\widetilde{D}$ is not 4-Ore, then by minimality of D we have $\rho(\widetilde{D}) \leq 1$, so

$$\rho(U) \leq \rho(\widetilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{10}{3} + \varepsilon + \delta \leq \frac{11}{3} - \varepsilon - \delta$$

because $2\varepsilon + 2\delta \leq \frac{1}{3}$. Next if $\widetilde{D}$ is 4-Ore, but not isomorphic to $\overleftrightarrow{K_4}$, then $\rho(\widetilde{D}) \leq \frac{4}{3} + 7\varepsilon - 4\delta$ by Lemma 4.4.16, and

$$\rho(U) \leq \rho(\widetilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{11}{3} + 8\varepsilon - 3\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because $9\varepsilon - 2\delta \leq 0$. Finally if $\widetilde{D}$ is isomorphic to $\overleftrightarrow{K_4}$, then we have $T(U) \geq T(\widetilde{D} - z) \geq T(\widetilde{D})$ and $\rho(\widetilde{D}) = \frac{4}{3} + 4\varepsilon - 2\delta$. So the same computation yields

$$\rho(U) \leq \rho(\widetilde{D}) + \frac{10}{3} + \varepsilon - 1 \leq \frac{11}{3} + 5\varepsilon - 2\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because $6\varepsilon - \delta \leq 0$. In all cases, we have $\rho(U) \leq \frac{11}{3} - \varepsilon - \delta$. This contradicts Claim 4.4.24 because U is not collapsible by Claim 4.4.28. $\Diamond$

We now use the discharging method. For every vertex v, let $\sigma(v) = \frac{\delta}{|C|}$ if v has degree 6 and is in a component C of D_6 of size at least 2, and $\sigma(v) = 0$ otherwise. Clearly $T(D)$ is at least the number of connected components of size at least 2 of D_6 so $\sum_{v \in V(D)} \sigma(v) \leq \delta T(D)$. We define the *initial charge* of v to be $w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} - \sigma(v)$. We have

$$\rho(D) \leq \sum_{v \in V(D)} w(v).$$

We now redistribute this total charge according to the following rules:

(R1) A vertex of degree 6 incident to no digon sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its neighbours.

(R2) A vertex of degree 6 incident to digons sends $\frac{2}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ to each neighbour v of degree at least 8 (so $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ via each arc of the digon).

(R3) A vertex of degree 7 with $d^-(v) = 3$ (resp. $d^+(v) = 3$) sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its in-neighbours (resp. out-neighbours).

For every vertex v, let $w^*(v)$ be the final charge of v.

Claim 4.4.37. *If v has degree at least 8, then $w^*(v) \leq 0$.*

Proof of claim. Let v be a vertex of degree at least 8. If v is not adjacent to a vertex of degree at most 7, then $w^*(v) = w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} \leq 0$ (because $\varepsilon \leq \frac{2}{3}$). Otherwise, $d(v) - \nu(v) \geq 1$ and

$$\begin{aligned}\frac{1}{d(v)-\nu(v)}\left(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon\right) &\geq \frac{1}{d(v)}\left(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon\right)\\ &\geq \frac{1}{12} - \frac{\varepsilon}{8}.\end{aligned}$$

Thus v receives at most $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ per arc incident with a vertex of degree 6 or 7. Since there are $d(v) - \nu(v)$ such arcs, $w^*(v) \leq w(v) - \frac{10}{3} - \varepsilon + \frac{d(v)}{2} = 0$. ◊

Claim 4.4.38. *If v has degree 7, then $w^*(v) \leq 0$.*

Proof of claim. By Claim 4.4.33, v has seven neighbours. Without loss of generality, let us suppose that $d^-(v) = 3$ and $d^+(v) = 4$. By Claim 4.4.36, the in-neighbours of v cannot have out-degree 3. In particular, they do not have degree 6, and if they have degree 7, they do not send anything to v by Rule (R3). Hence v receives at most four times the charge $\frac{1}{12} - \frac{\varepsilon}{8}$ by (R1) or (R3), and it sends three times this charge by (R3). Hence

$$\begin{aligned}w^*(v) &\leq w(v) + \frac{1}{12} - \frac{\varepsilon}{8}\\ &= -\frac{1}{12} + \frac{7}{8}\varepsilon\end{aligned}$$

and the result comes because $\varepsilon \leq \frac{2}{21}$. ◊

Claim 4.4.39. *If v is a vertex of degree 6 incident to no digon, then $w^*(v) \leq 0$.*

Proof of claim. The vertex v sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its neighbours by Rule (R2), and it receives no charge as all its in-neighbours (resp. out-neighbours) have out-degree (resp. in-degree) at least 4, by Claim 4.4.36. As a consequence,

$$w^*(v) = w(v) - 6\left(\frac{1}{12} - \frac{\varepsilon}{8}\right) = -\frac{1}{6} + \frac{7\varepsilon}{4}$$

and the result comes because $\varepsilon \leq \frac{2}{21}$. ♢

Claim 4.4.40. *Let v be a vertex in D_6 having at least two neighbours of degree at least 8. Then $w^*(v) \leq 0$. Moreover, if v is not an isolated vertex in D_6 and $\nu_N(v) \geq 4$, then $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$.*

Proof of claim. Observe that v receives no charge and sends the following charge to each of its neighbour u with degree at least 8:

$$\begin{aligned}\frac{2}{d(u) - \nu(u)}\left(-\frac{10}{3} - \varepsilon + \frac{d(u)}{2}\right) &\geq \frac{2}{d(u)}\left(-\frac{10}{3} - \varepsilon + \frac{d(u)}{2}\right)\\ &= 1 - \frac{2}{d(u)}\left(\frac{10}{3} + \varepsilon\right)\\ &\geq \frac{2}{8}\left(-\frac{10}{3} - \varepsilon + 4\right)\\ &= \frac{1}{6} - \frac{\varepsilon}{4}.\end{aligned}$$

Assume first that v is isolated in D_6. By Claim 4.4.33, its three neighbours do not have degree 7, and so have degree at least 8. Thus v sends three times at least $\frac{1}{6} - \frac{\varepsilon}{4}$, and so

$$w^*(v) \leq w(v) - 3\left(\frac{1}{6} - \frac{\varepsilon}{4}\right) = -\frac{1}{6} + \frac{7}{4}\varepsilon$$

and the result comes because $\varepsilon \leq \frac{2}{21}$.

Assume now that v is in a connected component C of D_6 of size at least 2. By Claim 4.4.35, $\sigma(v) \geq \frac{\delta}{4}$, so $w(v) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4}$. Moreover, it sends twice at least $\frac{1}{6} - \frac{\varepsilon}{4}$. Hence

$$w^*(v) \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4}\right) - 2\left(\frac{1}{6} - \frac{\varepsilon}{4}\right) = \frac{3}{2}\varepsilon - \frac{\delta}{4}$$

and the result comes because $\delta \geq 6\varepsilon$. This shows the first part of the statement.

We will now prove the second part of the statement. Assume that v is not an isolated vertex in D_6 and $\nu_N(v) \geq 4$. Let u_1 and u_2 be the two neighbours of v with degree at least 8. For every $i \in \{1, 2\}$ we have

$$\frac{2}{d(u_i) - \nu(u_i)}\left(-\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2}\right) = 1 - \frac{1}{d(u_i) - \nu(u_i)}\left(\frac{20}{3} + 2\varepsilon - \nu(u_i)\right).$$

We distinguish two cases.

Case 1: $\nu(u_i) \geq 7$ *for some* $i \in \{1, 2\}$.

Without loss of generality, suppose $i = 1$. Then we have

$$1 - \frac{1}{d(u_1) - \nu(u_1)} \left(\frac{20}{3} + 2\varepsilon - \nu(u_1) \right) \geq 1$$

because $\nu(u_1) \geq 7 \geq \frac{20}{3} + 2\varepsilon$ as $\varepsilon \leq \frac{1}{6}$. Then the total charge sent by v is at least 1, and thus

$$w^*(v) \leq w(v) - 1 \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - 1 = -\frac{2}{3} + \varepsilon - \frac{\delta}{4}$$

Thus, we have $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$ because $\varepsilon, \delta \geq 0$.

Case 2: $\nu(u_1), \nu(u_2) \leq 6$.

Let $f \colon [0, 6] \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{2}{8 - x} \left(-\frac{10}{3} - \varepsilon + \frac{8}{2} \right) = 1 - \frac{1}{8 - x} \left(\frac{20}{3} - 2\varepsilon - x \right)$$

for every $x \in [0, 6]$. Observe that f is non-decreasing and convex on $[0, 6]$ because $-\frac{10}{3} - \varepsilon + \frac{8}{2} \geq 0$. For $i = 1, 2$, we have

$$\frac{2}{d(u_i) - \nu(u_i)} \left(-\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2} \right) \geq f(\nu(u_i))$$

because the function $d \mapsto 1 - \frac{1}{d - \nu(u_i)} \left(\frac{20}{3} + 2\varepsilon - \nu(u_i) \right)$ is non-decreasing on $[8, +\infty[$ as $\nu(u_i) \leq 6 \leq \frac{20}{3} + 2\varepsilon$. Hence, the charge sent by v to u_i is at least $f(\nu(u_i))$. By hypothesis, we have $\nu_N(v) = \nu(u_1) + \nu(u_2) \geq 4$. It follows that the total charge sent by v is at least

$$\begin{aligned} f(\nu(u_1)) + f(\nu(u_2)) &\geq 2f\left(\frac{\nu(u_1) + \nu(u_2)}{2} \right) && \text{by convexity of } f \\ &\geq 2f(2) && \text{because } f \text{ is non-decreasing} \\ &= \frac{4}{9} - \frac{2}{3}\varepsilon. \end{aligned}$$

Hence

$$w^*(v) \leq w(v) - \left(\frac{4}{9} - \frac{2}{3}\varepsilon \right) \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - \frac{4}{9} + \frac{2}{3}\varepsilon = -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}.$$

showing the second part of the statement.

$\Diamond$

Claim 4.4.41. *If C is a connected component of D_6, then $\sum_{v \in V(C)} w^*(v) \leq 0$.*

Proof of claim. If C has a unique vertex v, then, by Claim 4.4.40, we have $w^*(v) \leq 0$ as wanted.

If C has two vertices x and y, then, again by Claim 4.4.40, $w^*(x), w^*(y) \leq 0$, and so $w^*(x) + w^*(y) \leq 0$.

If C is a bidirected path $[x, y, z]$, then, by Claim 4.4.35, x and z have both neighbourhood valency at least 4 and so by Claim 4.4.40 $w^*(x), w^*(z) \leq -\frac{1}{9} - \frac{\varepsilon}{6}$. Moreover, y sends at least $\frac{2}{8}(-\frac{10}{3} + 4 - \varepsilon) = \frac{1}{6} - \frac{\varepsilon}{4}$ to its neighbour out of C. Hence

$$w^*(y) \leq w(y) - \left(\frac{1}{6} - \frac{\varepsilon}{4}\right) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{3} - \frac{1}{6} + \frac{\varepsilon}{4} = \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3}.$$

Altogether, we get that

$$w^*(x) + w^*(y) + w^*(z) \leq \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3} + 2\left(-\frac{1}{9} - \frac{\varepsilon}{6}\right) = -\frac{1}{18} + \frac{11}{12}\varepsilon - \frac{\delta}{3} \leq 0$$

because $\delta \geq 6\varepsilon$.

Finally, if C is a bidirected star with centre x and three other vertices y, z, w, then $w^*(x) \leq w(x) = \frac{1}{3} + \varepsilon - \frac{\delta}{4}$. Moreover, each of y, z, w has neighbourhood valency at least 4 by Claim 4.4.35 and so has final charge at most $-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$ by Claim 4.4.40. Hence

$$w^*(x) + w^*(y) + w^*(z) + w^*(w) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4} + 3\left(-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}\right) \leq 6\varepsilon - \delta \leq 0$$

because $\delta \geq 6\varepsilon$. ◊

As a consequence of these last claims, we have $\rho(D) \leq \sum_{v \in V(D)} w(v) = \sum_{v \in V(D)} w^*(v) \leq 0 \leq 1$, a contradiction. This proves Theorem 4.4.2. □

4.5 A construction of dicritical oriented graphs with few arcs

In this section, we show that, for every fixed k, there are infinitely many values of n such that $o_k(n) \leq (2k - \frac{7}{2})n$. The proof is strongly based on the proof of [6, Theorem 4.4], which shows $o_k(n) \leq (2k-3)n$ for every k, n (with n large enough). For $k = 4$, the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4-dicritical oriented graphs $\vec{G}$ with $\frac{m(\vec{G})}{n(\vec{G})} \leq 9/2$.

Proposition 4.5.1. *Let $k \geq 3$ be an integer. For infinitely many values of $n \in \mathbb{N}$, there exists a k-dicritical oriented graph $\vec{G}_k$ on n vertices with at most $(2k - \frac{7}{2})n$ arcs.*

Proof. Let us fix $n_0 \in \mathbb{N}$. We will show, by induction on $k \geq 3$, that there exists a k-dicritical oriented graph $\vec{G}_k$ on n vertices with at most $(2k - \frac{7}{2})n$ arcs, such that $n \geq n_0$.

When $k = 3$, the result is known ([6, Corollary 4.3]). We briefly describe the construction for completeness. Start from any orientation of an odd cycle on $2n_0 + 1$ vertices. Then for each arc xy in this orientation, add a directed triangle $\vec{C}_3$ and every arc from y to $V(\vec{C}_3)$ and every arc from $V(\vec{C}_3)$ to x (see Figure 4.6). This gadget forces x and y to have different colours in every 2-dicolouring. Since we started from an orientation of an odd cycle, the result is a 3-dicritical oriented graph on $4(2n_0 + 1)$ vertices and $10(2n_0 + 1)$ arcs.

Let us fix $k \geq 4$ and assume that there exists such a $(k-1)$-dicritical oriented graph $\vec{G}_{k-1}$ on $n_{k-1} \geq n_0$ vertices with $m_{k-1} \leq (2(k-1) - \frac{7}{2})n_{k-1}$ arcs. We start from any tournament T on k vertices. Then we add, for each arc xy of T, a copy $\vec{G}_{k-1}^{xy}$ of $\vec{G}_{k-1}$, all arcs from y to $\vec{G}_{k-1}^{xy}$ and all

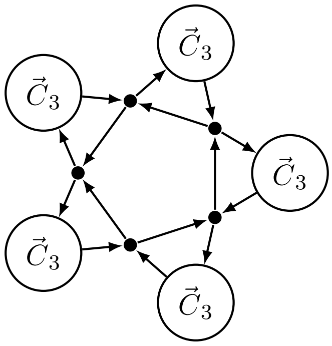


Figure 4.6: A 3-dicritical oriented graph with $\frac{5}{2}n$ arcs.

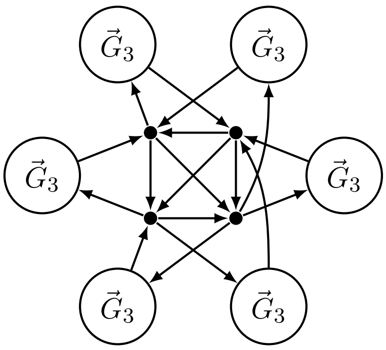


Figure 4.7: A 4-dicritical oriented graph with at most $\frac{9}{2}n$ arcs.

arcs from $\vec{G}^{xy}_{k-1}$ to x. Figure 4.7 illustrates a possible construction of $\vec{G}_4$, where T is the transitive tournament on 4 vertices.

Let $\vec{G}_k$ be the resulting oriented graph. By construction, $n_k = n(\vec{G}_k)$ and $m_k = m(\vec{G}_k)$ satisfy:

$$
\begin{aligned}
n_k &= k + \binom{k}{2} n_{k-1} \\
m_k &= \binom{k}{2} + \binom{k}{2} \times 2 \times n_{k-1} + \binom{k}{2} \times m_{k-1} \\
&\leq \binom{k}{2} + \binom{k}{2}\left(2 + 2(k-1) - \frac{7}{2}\right) n_{k-1} \\
&= \binom{k}{2} + \binom{k}{2}\left(2k - \frac{7}{2}\right) n_{k-1} \\
&= \binom{k}{2} + \left(2k - \frac{7}{2}\right)(n_k - k) \\
&\leq \left(2k - \frac{7}{2}\right) n_k
\end{aligned}
$$

where in the last inequality we used $k\left(2k - \frac{7}{2}\right) \geq \binom{k}{2}$, which holds when $k \geq 2$. We will now prove that $\vec{G}_k$ is indeed k-dicritical.

We first prove that $\vec{\chi}(\vec{G}_k) = k$. Assume that there exists a $(k-1)$-dicolouring α of $\vec{G}_k$. Then there exist $x, y \in V(T)$ such that $\alpha(x) = \alpha(y)$. Since $\vec{\chi}(\vec{G}_{k-1}) = k-1$, there exists $z \in V(\vec{G}^{xy}_{k-1})$ such that $\alpha(z) = \alpha(x)$. But then (x, y, z, x) is a monochromatic directed triangle in α: a contradiction.

Let us now prove that $\vec{\chi}(\vec{G}_k \setminus \{uv\}) \leq k-1$ for every arc $uv \in A(\vec{G}_k)$. This implies immediately that $\vec{\chi}(\vec{G}_k = k$ and shows the result.

Consider first an arc uv in $A(T)$. We colour each copy $\vec{G}^{xy}_{k-1}$ of $\vec{G}_{k-1}$ with a $(k-1)$-dicolouring of $\vec{G}_{k-1}$. We then choose a distinct colour for every vertex in T, except u and v which receive the same colour. This results in a $(k-1)$-dicolouring of $\vec{G}_k \setminus \{uv\}$.

Consider now an arc uv of $\vec{G}^{xy}_{k-1}$ for some $xy \in A(T)$. Because $\vec{G}_{k-1}$ is $(k-1)$-dicritical, there exists a $(k-2)$-dicolouring ξ of $\vec{G}^{xy}_{k-1} \setminus \{uv\}$. Hence we colour $\vec{G}^{xy}_{k-1} \setminus \{uv\}$ with ξ, every other copy of $\vec{G}_{k-1}$ a $(k-1)$-dicolouring of $\vec{G}_{k-1}$, and we choose a distinct colour for every vertex in T, except x and y which both receive colour $k-1$. This results in a $(k-1)$-dicolouring of $\vec{G}_k \setminus \{uv\}$.

Consider finally an arc uv arc from $u \in V(T)$ to $v \in V(\vec{G}^{uy}_{k-1})$ (the case of $u \in V(\vec{G}^{xv}_{k-1})$ and $v \in V(T)$ being symmetric). Because $\vec{G}_{k-1}$ is dicritical, there exists a $(k-1)$-dicolouring γ of $\vec{G}^{uy}_{k-1}$ in which v is the only vertex coloured $k-1$. Hence, we colour $\vec{G}^{uy}_{k-1}$ with γ, every other copy of $\vec{G}_{k-1}$ with a $(k-1)$-dicolouring of $\vec{G}_{k-1}$, and we choose a distinct colour for every vertex in T, except u and y which both receive colour $k-1$. This results in a $(k-1)$-dicolouring of $\vec{G}_k \setminus \{uv\}$. □

Chapter 5

The three-dicritical semi-complete digraphs

This chapter contains joint work with Frédéric Havet and Florian Hörsch and is based on [96].

5.1 Introduction

Erdős [68] asked, for every fixed $k \geq 4$, whether there exist arbitrarily large k-critical graphs G with at least $c_k \cdot n(G)^2$ edges for some constant $c_k > 0$. This was proved by Dirac [59] when $k \geq 6$ and then by Toft [166] when $k \in \{4, 5\}$. This initiated the quest after the supremum c_k^*, for fixed $k \geq 4$, of all values c_k for which the statement holds. The following lower bound on c_k^* follows from the explicit construction given in [166] and is still the best current bound.

Theorem 5.1.1 (Toft [166])**.** *For every integer $k \geq 4$ and infinitely many values of n, there exists a k-critical graph with n vertices and at least $\frac{1}{2}\left(1 - \frac{3}{k-\delta_k}\right) n^2$ edges, where $\delta_k = 0$ if $k = 0 \bmod 3$, $\delta_k = \frac{4}{7}$ if $k = 1 \bmod 3$, and $\delta_k = \frac{22}{23}$ if $k = 2 \bmod 3$.*

Concerning the upper bounds on c_k^*, observe that a k-critical graph does not contain any copy of K_k as a subgraph. A seminal result of Turán [167] implies that such a graph G of order n has at most $\frac{1}{2}\left(1 - \frac{1}{k-1}\right) n^2$ edges (when $n = 0 \bmod k$). Hence we have $c_k^* \leq \frac{1}{2}\left(1 - \frac{1}{k-1}\right)$. In 1987, Stiebitz [161] improved on this lower bound.

Theorem 5.1.2 (Stiebitz [161])**.** *For every integer $k \geq 4$ and sufficiently large integers n, every k-critical graph G of order n has at most $\frac{1}{2}\left(1 - \frac{1}{k-2}\right) n^2$ edges.*

This remained the best upper bound on c_k^* for many years, until Luo, Ma, and Yang [124] proved the following in 2023.

Theorem 5.1.3 (Luo, Ma, and Yang [124]). *For every integer $k \geq 4$, there exists $\varepsilon_k \geq \frac{1}{18(k-1)^2}$ such that for sufficiently large integers n, every k-critical graph G of order n has at most $\frac{1}{2}\left(1 - \frac{1}{k-2} - \varepsilon_k\right) n^2$ edges.*

It remains an open problem to find the exact value of c_k^*. However, the analogue of c_k^* is well-understood for triangle-free graphs when $k \geq 6$. Indeed, for $k \geq 6$, Pegden [138] proved that there exist infinitely many k-critical triangle-free graphs G with $\left(\frac{1}{4} - o(1)\right) n(G)^2$ edges. This is asymptotically best possible because of Turán's result.

We now turn our attention to the maximum density of k-dicritical digraphs. For every $k \geq 3$, Hoshino and Kawarabayashi [99] constructed an infinite family of k-dicritical oriented graphs $\vec{G}$ on n vertices which satisfy $m(\vec{G}) \geq (\frac{1}{2} - \frac{1}{2^k - 1})n^2$, and they conjectured that this bound is tight.

Conjecture 5.1.4 (Hoshino and Kawarabayashi [99]). *Let $k \geq 3$ be an integer. If $\vec{G}$ is a k-dicritical oriented graph, then $m(\vec{G}) \leq (\frac{1}{2} - \frac{1}{2^k - 1})n(\vec{G})^2$.*

Aboulker [1] observed that, since a tournament has $\frac{1}{2}n(n-1)$ arcs, this conjecture implies that the number of k-dicritical tournaments is finite, and he asked whether this latter statement holds. It trivially does for the case $k = 2$.

In this chapter, we positively answer Aboulker's question in the case $k = 3$ by showing that the collection of 3-dicritical semi-complete digraphs is finite, and hence so is the subcollection of 3-dicritical tournaments.

Theorem 5.1.5. *There is a finite number of 3-dicritical semi-complete digraphs.*

While the proof of Theorem 5.1.5 is fully human-readable, the result is obtained by showing that the number of vertices of any 3-dicritical semi-complete digraph does not exceed a pretty large number which originates from a Ramsey-type argument.

We after use a computer-assisted proof to provide the following characterisation of all 3-dicritical semi-complete digraphs.

Theorem 5.1.6. *There are exactly eight 3-dicritical semi-complete digraphs. They are depicted in Figure 5.1.*

In particular, we can characterise all 3-dicritical tournaments.

Corollary 5.1.7. *There are exactly two 3-dicritical tournaments, namely $\mathcal{R}(\vec{C}_3, \vec{C}_3)$ and $\mathcal{P}_7$.*

We finally investigate the maximum density of 3-dicritical digraphs. Recall that the *symmetric part* of a digraph D is the graph $S(D)$ with vertex-set $V(D)$ in which two vertices are linked by an edge if and only if there is a digon between them in D. We prove in Proposition 5.5.1 that $S(D)$ is a forest for every 3-dicritical digraph D that is not a bidirected odd cycle. From this result, one can easily deduce that $m(D) \leq \binom{n}{2} + n - 1$ holds for every 3-dicritical digraph D different from $\overleftrightarrow{K_3}$. We slightly improve on this upper bound on $m(D)$ as follows (the digraph $\vec{W}_3$ is depicted in Figure 5.1).

Theorem 5.1.8. *If D is a 3-dicritical digraph distinct from $\overleftrightarrow{K_3}$ and $\vec{W}_3$, then $m(D) \leq \binom{n}{2} + \frac{2}{3}n$.*

The rest of this chapter is structured as follows: in Section 5.2, we give a collection of preliminary results which will be used in the proofs of Theorems 5.1.5, 5.1.6, and 5.1.8. In Section 5.3, we prove Theorem 5.1.5. In Section 5.4, we prove Theorem 5.1.6, with all code we use being shifted to Appendix A. Section 5.5 is devoted to the proof of Theorem 5.1.8. Finally, in Section 5.6, we conclude by giving some directions for further research.

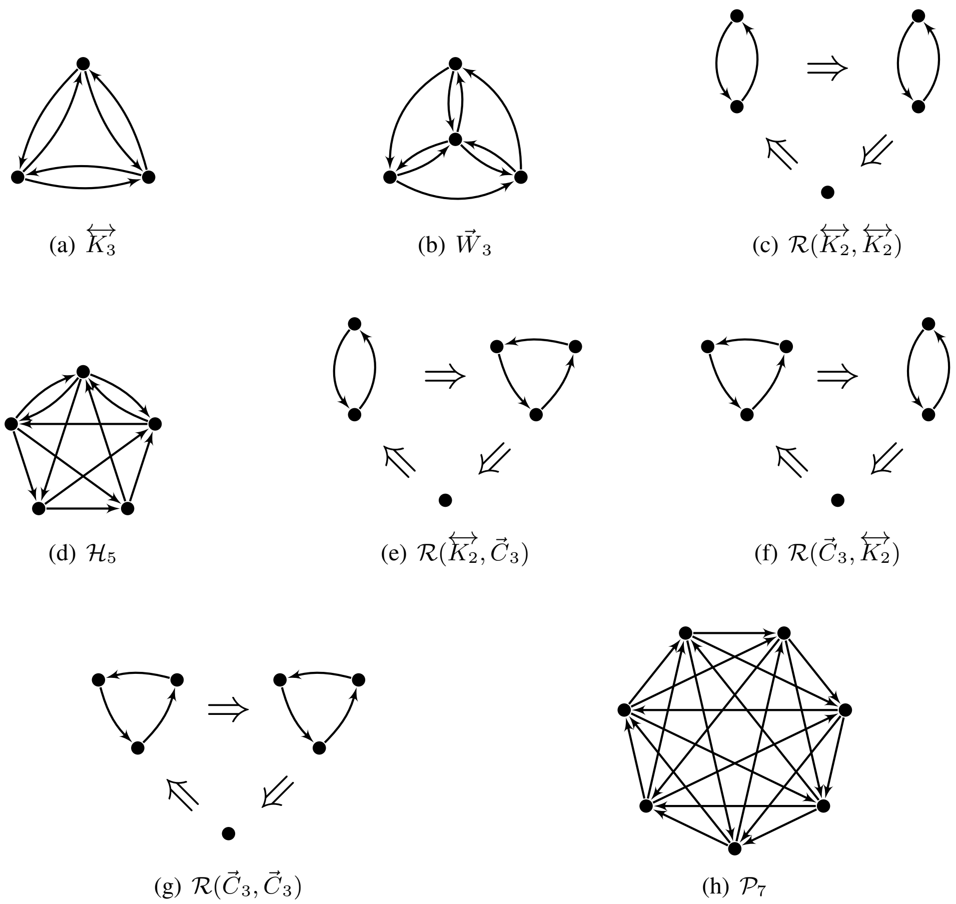


Figure 5.1: The 3-dicritical semi-complete digraphs, namely the bidirected complete graph $\overleftrightarrow{K_3}$, the directed wheel $\vec{W}_3$, the digraph $\mathcal{H}_5$, the rotative digraphs $\mathcal{R}(H_1, H_2)$ for every $H_1, H_2 \in \{\overleftrightarrow{K_2}, \vec{C}_3\}$ and the Paley tournament on seven vertices $\mathcal{P}_7$. A big arrow linking two sets of vertices indicates that there is exactly one arc from every vertex in the first set to every vertex in the second set.

5.2 Useful lemmas

In this section, we give a collection of preliminary results we need in the proof of Theorem 5.1.5. Most of them will be reused in the proof of Theorem 5.1.6. We first describe 2-colourings with some important extra properties.

Let D be a digraph and uv be an arc of D. A *uv-colouring* of D is a 2-colouring $\phi\colon V(D) \to [2]$ such that:

- ϕ is a 2-dicolouring of $D \setminus uv$,
- $\phi(u) = \phi(v) = 1$, and
- $D \setminus uv$, coloured with ϕ, does not contain any monochromatic (u, v)-path.

There is a close relationship between 3-dicritical digraphs and uv-colourings.

Lemma 5.2.1. *Let D be a 3-dicritical digraph and uv be an arc of D. Then D admits a uv-colouring.*

Proof. As D is 3-dicritical, there is a 2-dicolouring $\phi\colon V(D) \to [2]$ of $D \setminus uv$. By symmetry, we may suppose $\phi(u) = 1$. As ϕ is not a 2-dicolouring of D, we obtain that $\phi(v) = 1$ and there is a (v, u)-path P in D such that $\phi(x) = 1$ for all $x \in V(P)$. If there is also a (u, v)-path Q in $D \setminus uv$ such that $\phi(x) = 1$ for all $x \in V(Q)$, then the subdigraph of $D \setminus uv$ induced by $V(P) \cup V(Q)$ contains a monochromatic directed cycle. This contradicts ϕ being a 2-dicolouring of $D \setminus uv$. $\square$

The next result showing that every arc of a 3-dicritical semi-complete digraph is contained in a short directed cycle will play a crucial role in the upcoming proofs.

Lemma 5.2.2. *Let D be a 3-dicritical semi-complete digraph. Then every arc $a \in A(D)$ either belongs to a digon or is contained in an induced directed triangle.*

Proof. As D is 3-dicritical, there is a 2-dicolouring ϕ of $D \setminus a$. As ϕ is not a 2-dicolouring of D, there exists a directed cycle C in D such that C is monochromatic with respect to ϕ. We may suppose that C is chosen to be of minimum length with this property. As D is semi-complete, we obtain that C is either a digon or an induced directed triangle. As C is not a monochromatic directed cycle of $D \setminus a$ with respect to ϕ, we obtain that $a \in A(C)$. $\square$

We define O_5 as the oriented graph which consists of a directed triangle xyz and two additional vertices u, v, one arc from u to every vertex of the directed triangle, one arc from every vertex of the directed triangle to v, and the arc uv. An illustration can be found in Figure 5.2.

The following result is a consequence of Lemma 5.2.1.

Lemma 5.2.3. *Let D be a 3-dicritical digraph. Then D does not contain O_5 as a subdigraph.*

Proof. Assume for a contradiction that D contains O_5 as a subdigraph and let $V(O_5) = \{u, v, x, y, z\}$ be the labelling depicted in Figure 5.2. By Lemma 5.2.1, there exists a uv-colouring ϕ of D. Since there exists no monochromatic (u, v)-path, we have $\phi(x) = \phi(y) = \phi(z) = 2$. Hence $D \setminus uv$ contains a monochromatic directed triangle with respect to ϕ, a contradiction. $\square$

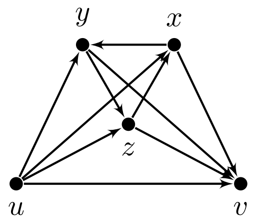


Figure 5.2: The oriented graph O_5.

Let S be a transitive subtournament of a digraph $D = (V, A)$. We denote by $v_1, \dots, v_s$ the unique acyclic ordering of S. For some $i, j \in [s]$, we say that $\{v_i, \dots, v_j\}$ is an *interval* of S. Observe that $\varnothing$ is an interval. For $i_0, j_0, i_1, j_1 \in [s]$ with $j_0 < i_1$, we say that the interval $\{v_{i_0}, \dots, v_{j_0}\}$ is *smaller* than the interval $\{v_{i_1}, \dots, v_{j_1}\}$. By convention, $\varnothing$ is both smaller and greater than any other interval. A sequence of intervals $P_1, \dots, P_t$ is called *increasing* if P_i is smaller than P_j for all $i, j \in [t]$ with $i < j$.

Lemma 5.2.4. *Let T be a subtournament of a 3-dicritical digraph D and let S be a transitive subtournament of T with acyclic ordering $v_1, \dots, v_s$. For any $x \in V(T) \setminus S$, there is an increasing sequence of intervals (I_1, I_2, I_3, I_4) with $\bigcup_{i=1}^{4} I_i = S$ such that, in T, x dominates $I_1 \cup I_3$ and is dominated by $I_2 \cup I_4$.*

Proof. Assume this is not the case. Then there exists an increasing sequence of indices (i_1, i_2, i_3, i_4) such that, in T, x is dominated by v_{i_1} and v_{i_3} and dominates v_{i_2} and v_{i_4}. Then the subdigraph of D induced by $\{v_{i_1}, v_{i_2}, x, v_{i_3}, v_{i_4}\}$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. □

We finally need a well-known theorem which can be found in many basic textbooks on graph theory, see for example [58, Theorem 9.1.3].

Theorem 5.2.5 (MULTI-COLOUR RAMSEY THEOREM)**.** *Let a and b be positive integers. There exists a smallest integer $R_a(b)$ such that for G being a copy of $K_{R_a(b)}$ and for every mapping $\psi \colon E(G) \to [a]$, there is a set $S \subseteq V(G)$ of cardinality b and $i \in [a]$ such that $\psi(e) = i$ for all $e \in E(G\langle S \rangle)$.*

5.3 A simple proof for finiteness

In this section, we prove that the number of 3-dicritical semi-complete digraphs is finite. Let us first restate this result.

Theorem 5.1.5. *There is a finite number of 3-dicritical semi-complete digraphs.*

Proof. Let $D = (V, A)$ be a 3-dicritical semi-complete digraph. We will show that $n(D) \leq 12R_6(3) + 1$, where $R_6(3)$ refers to the Ramsey number in Theorem 5.2.5. Assume for the sake of a contradiction that $n(D) \geq 12R_6(3) + 2$. Let $S \subseteq V$ be a maximum set of vertices such that $D\langle S \rangle$ is acyclic. Let $v_1, \dots, v_s$ be the unique acyclic ordering of S. Since D is 3-dicritical, for an arbitrary vertex $x \in V$, we have that $D - x$ is 2-dicolourable. This yields $s \geq \left\lceil \frac{n(D)-1}{2} \right\rceil \geq 6R_6(3) + 1$.

By Lemma 5.2.2, for every $i \in [s-1]$, the arc $v_i v_{i+1}$ belongs to a digon or an induced directed triangle. Therefore, since $D\langle S\rangle$ is acyclic, we know that there exists a vertex $x_i \in V \setminus S$ such that $v_i v_{i+1} x_i v_i$ is an induced directed triangle C_i.

Let T be an arbitrary spanning subtournament of D. Observe that $T\langle S\rangle = D\langle S\rangle$ as $D\langle S\rangle$ is acyclic. Further, the directed triangle C_i is contained in T for $i \in [s-1]$ as C_i is induced in D. For any vertex x in $V \setminus S$ and $i \in [s-1]$, we say that x *switches at* i if x dominates v_i and is dominated by v_{i+1} in T or x is dominated by v_i and dominates v_{i+1} in T.

Let H be the digraph with vertex-set $V(H) = [s-1]$ and arc-set $A(H) = A_1 \cup A_2$ with $A_1 = \{(i, i+1) \mid i \in [s-2]\}$ and $A_2 = \{(i,j) \mid i \neq j \text{ and } x_i \text{ switches at } j.\}$.

By Lemma 5.2.4, for $i \in [s-1]$, we have that x_i switches at at most three indices in $[s-1]$. Further, as C_i is a directed triangle, x_i switches at i which yields that x_i switches at at most two indices in $[s-1] \setminus \{i\}$. Thus every $i \in [s-1]$ is the tail of at most two arcs in A_2.

For every subset J of $[s-1]$, observe that $H\langle J\rangle$ contains at most $|J|-1$ arcs in A_1 and at most $2|J|$ arcs in A_2, hence at most $3|J|-1$ arcs in total. Thus $\mathrm{UG}(H)\langle J\rangle$ has a vertex of degree at most 5. Hence $\mathrm{UG}(H)$ is 5-degenerate, and so it is 6-colourable. Therefore H has an independent set I of size $\left\lceil \frac{1}{6}(s-1)\right\rceil \geq R_6(3)$.

By definition of I, for any $i, j \in I$ with $i \neq j$, we have that $V(C_i)$ and $V(C_j)$ are disjoint. Moreover, either $\{v_j, v_{j+1}\}$ dominates x_i in T or $\{v_j, v_{j+1}\}$ is dominated by x_i in T. Hence, if $i < j$, the subdigraph of T induced by $V(C_i) \cup V(C_j)$ is one of the eight tournaments depicted in Figure 5.3. For $(\alpha) \in \{(a), \dots, (h)\}$, we say that (i,j) is an (α)-*configuration* if $T\langle V(C_i) \cup V(C_j)\rangle$ is the tournament depicted in Figure 5.3 (α).

Let us fix a pair $i, j \in I$ with $i < j$. We know that it is not a (g)-configuration, for otherwise $D\langle v_{i+1}, v_j, v_{j+1}, x_j, x_i\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. We also know that it is not an (h)-configuration, for otherwise $D\langle x_j, v_i, v_{i+1}, x_i, v_j\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3.

Since $|I| \geq R_6(3)$, and by definition of $R_6(3)$, we know that there exist $\{i,j,h\} \subseteq I, i < j < h$, and $(\alpha) \in \{(a), \dots, (e)\}$ such that the three pairs $(i,j), (j,h), (i,h)$ are (α)-configurations. We show that each of the six cases yields a contradiction, implying the result.

- If $(\alpha) = (a)$, let ϕ be a $v_{i+1}v_h$-colouring of D, the existence of which is guaranteed by Lemma 5.2.1. Recall that $\phi(v_{i+1}) = \phi(v_h) = 1$, ϕ is a 2-dicolouring of $D \setminus v_{i+1}v_h$ and D coloured with ϕ contains no monochromatic (v_{i+1}, v_h)-path. Then $\phi(v_j) = \phi(v_{j+1}) = 2$ because $\{v_j, v_{j+1}\} \subseteq N_D^+(v_{i+1}) \cap N_D^-(v_h)$. Thus, since C_j is not monochromatic in ϕ, we have $\phi(x_j) = 1$. We obtain that $\phi(x_i) = \phi(v_i) = 2$, for otherwise $v_{i+1}x_i x_j v_{i+1}$ or $v_i v_h x_j v_i$ is monochromatic. We deduce that $v_i v_j x_i v_i$ is monochromatic, a contradiction.

- If $(\alpha) = (b)$, then $D\langle x_h, v_j, v_{j+1}, x_j, x_i\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3.

- If $(\alpha) = (c)$, then $D\langle x_i, v_j, v_{j+1}, x_j, v_h\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3.

- If $(\alpha) = (d)$, then $D\langle v_i, v_j, v_{j+1}, x_j, x_h\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3.

- If $(\alpha) = (e)$, then $D\langle v_i, v_j, v_{j+1}, x_j, v_h\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3.

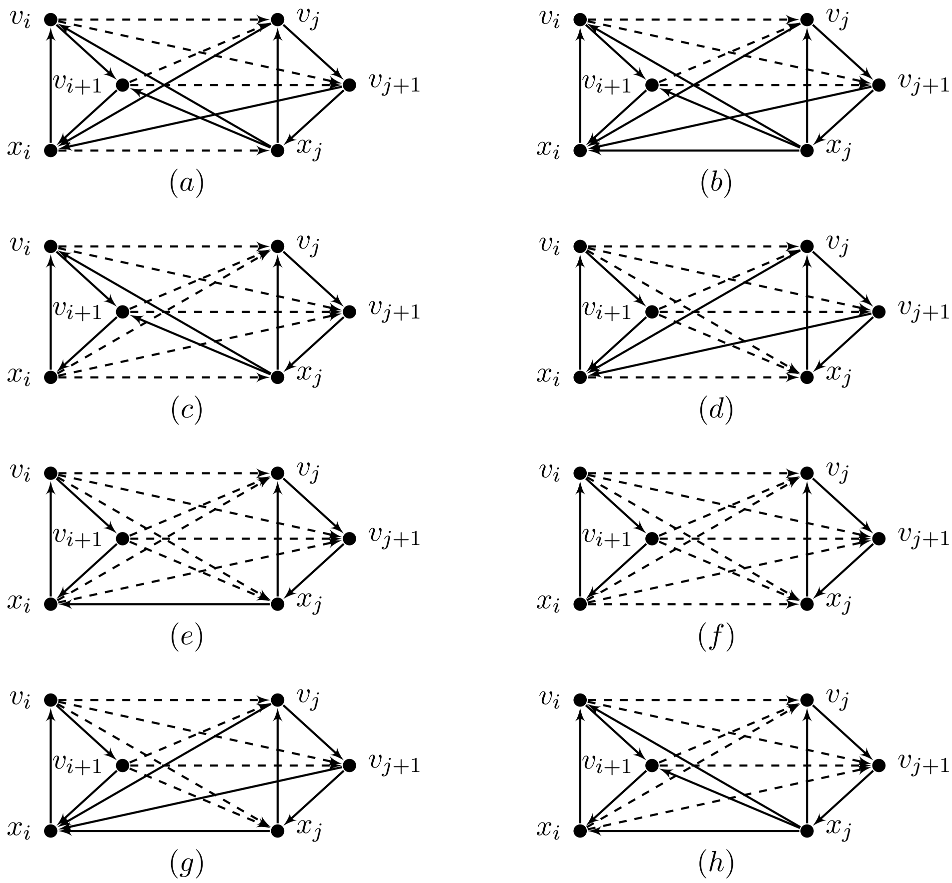


Figure 5.3: A listing of all possible configurations for $i, j \in I$ with $i < j$. For the sake of better readability, the arcs in $A(C_i) \cup A(C_j)$ and the arcs from $V(C_j)$ to $V(C_i)$ are solid, and the arcs from $V(C_i)$ to $V(C_j)$ are dashed.

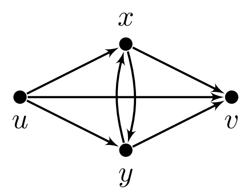


Figure 5.4: The digraph O_4.

- If $(\alpha) = (f)$, then $D\langle v_i, v_j, v_{j+1}, x_j, v_h\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. □

5.4 The 3-dicritical semi-complete digraphs

This section is devoted to a computer-assisted proof of Theorem 5.1.6. It follows a similar line as the one of Theorem 5.1.5, but it needs some refined arguments. Further, due to the significant number of necessary computations, several parts of the proof are computer-assisted. We use codes implemented using SageMath. They are accessible online and are given in Appendix A.

We first restrain the structure of 3-dicritical semi-complete digraphs. To prove Theorem 5.1.5, we only needed the fact that O_5 does not occur as a subdigraph. To prove Theorem 5.1.6, we need to prove that several other digraphs cannot be subdigraphs or induced subdigraphs of a 3-dicritical digraph. One of these digraphs is the transitive tournament of size at least 8. While already parts of this proof are computer-assisted, the most intense computation part is carried out after. We generate all semi-complete digraphs satisfying these properties and check that none of them has dichromatic number 3, except the ones depicted in Figure 5.1.

Before dealing with the collection of digraphs which are not contained in 3-dicritical semi-complete digraphs as subdigraphs, we first give the following simple observation on matchings in graphs on seven vertices which will prove useful later on.

Lemma 5.4.1. *Let H be a graph that is obtained from a path $w_1 \dots w_7$ by adding the edges of a matching M on $\{w_1, \dots, w_7\}$. Then there is a stable set $S \subseteq V(H)$ with $|S| = 3$ and $\{w_1, w_7\} \setminus S \neq \varnothing$.*

Proof. Suppose otherwise. We claim that there exists an edge e_1 between w_1 and $\{3, 5, 6\}$ in M. If this is not the case, then, as none of $\{w_1, w_3, w_5\}$ and $\{w_1, w_3, w_6\}$ is an independent set, we obtain that $w_3w_5, w_3w_6 \in E(M)$, a contradiction to M being a matching. Similarly, M contains an edge e_7 between w_7 and $\{2, 3, 5\}$. Hence, $\{1, 4, 7\}$ is an independent set of H, a contradiction. □

We now start excluding some subdigraphs of 3-dicritical semi-complete digraphs. We first define O_4 as the digraph which consists of a copy of $\overleftrightarrow{K_2}$ and two additional vertices u, v, one arc from u to every vertex of $\overleftrightarrow{K_2}$, one arc from every vertex of $\overleftrightarrow{K_2}$ to v, and the arc uv. An illustration can be found in Figure 5.4.

The digraph O_4 plays a similar role as O_5. Also, the proof of the following result is similar to the one of Lemma 5.2.3.

Lemma 5.4.2. *Let D be a 3-dicritical digraph. Then D does not contain O_4 as a subdigraph.*

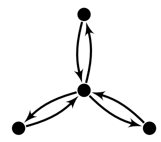


Figure 5.5: The bidirected star on 4 vertices $\overleftrightarrow{S_4}$.

Proof. Assume for the purpose of contradiction that D contains O_4 as a subdigraph and let $V(O_4) = \{u, v, x, y\}$ be the labelling depicted in Figure 5.4. By Lemma 5.2.1, there exists a 2-dicolouring ϕ of $D \setminus uv$ with $\phi(x) = \phi(y)$. Hence $D \setminus uv$ contains a monochromatic digon with respect to ϕ, a contradiction. $\square$

In the following, let $\overleftrightarrow{S_4}$ be the bidirected star on 4 vertices, see Figure 5.5. The following result shows that $\overleftrightarrow{S_4}$ cannot be the subdigraph of any large 3-dicritical semi-complete digraph.

Lemma 5.4.3. *Let D be a semi-complete digraph containing $\overleftrightarrow{S_4}$ as a subdigraph. Then D is 3-dicritical if and only if D is $\vec{W}_3$.*

Proof. It is easy to see that $\vec{W}_3$ is 3-dicritical and contains $\overleftrightarrow{S_4}$. For the other direction, let D be a 3-dicritical semi-complete digraph such that D contains a vertex u linked by digons to three distinct vertices x, y, z.

Then, as D is semi-complete, we have that $D\langle\{x, y, z\}\rangle$ needs to contain $\vec{C}_3$ or TT_3 as a subdigraph. If it is TT_3, then D contains O_4 as a subdigraph, a contradiction to Lemma 5.4.2. Hence $D\langle\{u, x, y, z\}\rangle$ contains $\vec{W}_3$ as a subdigraph. Since both D and $\vec{W}_3$ are 3-dicritical, we have $D = \vec{W}_3$. $\square$

We now prove a similar result for a collection of four digraphs. Given two digraphs H_1 and H_2, let $H_1 \Rightarrow H_2$ denote the *directed join* of H_1 and H_2, that is the digraph obtained from disjoint copies of H_1 and H_2 by adding all arcs from the copy of H_1 to the copy of H_2. If we further add all the arcs from H_2 to H_1, we obtain the *bidirected join* of H_1 and H_2, denoted by $H_1 \boxplus H_2$. It is straightforward that $\vec{\chi}(H_1 \boxplus H_2) = \vec{\chi}(H_1) + \vec{\chi}(H_2)$, see [17].

Lemma 5.4.4. *Let H_1, H_2 be two digraphs in $\{\overleftrightarrow{K_2}, \vec{C}_3\}$ and let D be a semi-complete digraph containing $H_1 \Rightarrow H_2$ as a subdigraph. Then D is 3-dicritical if and only if D is exactly $\mathcal{R}(H_1, H_2)$.*

Proof. It is easy to see that $\mathcal{R}(H_1, H_2)$ is 3-dicritical. For the other direction, let us fix $H_1, H_2 \in \{\overleftrightarrow{K_2}, \vec{C}_3\}$ and let D be a 3-dicritical semi-complete digraph containing $H_1 \Rightarrow H_2$.

Let $X = V(H_1)$ and $Y = V(H_2)$. Let us first prove that $V(D) \setminus (X \cup Y) \neq \varnothing$, so assume for a contradiction that $V(D) = X \cup Y$. We claim that there exists a simple arc uv from X to Y. If this is not the case, then D is exactly $H_1 \boxplus H_2$, so it has dichromatic number 4, a contradiction. This simple arc uv belongs to an induced directed triangle by Lemma 5.2.2. This directed triangle uses an arc from Y to X, which is necessarily in a digon, a contradiction since it must be induced. Henceforth we assume that $V(D) \setminus (X \cup Y) \neq \varnothing$.

First suppose that there exists some $v \in V(D) \setminus (X \cup Y)$ having at least one in-neighbour and one out-neighbour in both X and Y. Since H_1 and H_2 are strongly connected, there exist four distinct vertices $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $\{x_1x_2, y_1y_2, x_1v, vx_2, y_1v, vy_2\}$ are all arcs of D. Then $D\langle\{x_1, v, x_2, y_1, y_2\}\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. Henceforth we may assume that every vertex $v \in V(D) \setminus (X \cup Y)$ has no out-neighbour or no in-neighbour in one of $\{X, Y\}$.

Now suppose that there exists some $v \in V(T) \setminus (X \cup Y)$ that dominates X. If v has an out-neighbour y in Y, then $D\langle X \cup \{v, y\}\rangle$ contains O_4 as a subdigraph if $H_1 = \overleftrightarrow{K_2}$ and O_5 otherwise, a contradiction to Lemma 5.4.2 or 5.2.3, respectively. Hence v has no out-neighbour in Y. Since D is semi-complete, this implies that Y dominates v. Hence D contains $\mathcal{R}(H_1, H_2)$, implying that D is exactly $\mathcal{R}(H_1, H_2)$ since both D and $\mathcal{R}(H_1, H_2)$ are 3-dicritical.

Henceforth we assume that for every vertex $v \in V(D) \setminus (X \cup Y)$, there exists in D a simple arc from X to v. By directional duality, there also exists a simple arc from v to Y. Recall that every vertex $v \in V(D) \setminus (X \cup Y)$ has no out-neighbour or no in-neighbour in one of $\{X, Y\}$. We conclude on the existence of a partition (V_1, V_2) of $V(D) \setminus (X \cup Y)$ such that there is no arc from V_1 to X and there is no arc from Y to V_2.

By symmetry, we may assume that V_2 is non-empty. Let us fix $v_2 \in V_2$ and $y_1 \in Y$. Since v_2y_1 is a simple arc, by Lemma 5.2.2, there exists a vertex v_1 such that $v_2y_1v_1v_2$ is an induced directed triangle in D. Note that $v_1 \notin Y$ since there is no arc from Y to V_2. Also note that $v_1 \notin X$ for otherwise $v_2y_1v_1v_2$ is not induced, since X dominates Y. Further note that $v_1 \notin V_2$ since it is an out-neighbour of y_1. This implies $v_1 \in V_1$. As v_2 does not dominate X, there is some vertex in X, say x_1, that dominates v_2. Note that x_1 dominates v_1 by definition of V_1. Let y_2 be the unique out-neighbour of y_1 in Y.

If v_1 dominates y_2, we obtain that $D\langle\{x_1, v_1, v_2, y_1, y_2\}\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. We may hence suppose that y_2 dominates v_1. As Y does not dominate v_1, this implies that H_2 is $\vec{C}_3$ and the out-neighbour y_3 of y_2 is dominated by v_1. Then $D\langle\{x_1, v_1, v_2, y_2, y_3\}\rangle$ contains O_5 as a subdigraph, a contradiction to Lemma 5.2.3. □

The rest of the preparatory results before the main proof of Theorem 5.1.6 aims to exclude a collection of tournaments $\mathcal{T}_8$ as induced subdigraphs and another digraph F as a (not necessarily induced) subdigraph. As the proofs of these results contain several common preliminaries, we give them together. While the exact definition of $\mathcal{T}_8$ is postponed, we now give the definition of F. Let F be the oriented graph with vertex-set $\{u_1, \dots, u_6, x_1, x_2, x_3\}$ such that:

- $\{u_1, \dots, u_6\}$ induces a copy of TT_6 the unique acyclic ordering of which is exactly $u_1, \dots, u_6$, and
- for every $i \in [3]$, F contains the arcs $u_{2i}x_i$ and x_iu_{2i-1}.

See Figure 5.6 for an illustration of F.

We let $\mathcal{F}$ be the set of tournaments T with vertex-set $V(T) = V(F)$ and such that $A(F) \subseteq A(T)$. Note that $\mathcal{F}$ contains 2^{15} tournaments since F has exactly 15 pairs of non-adjacent vertices. Four of them are of special interest and we denote them by $T^1, \dots, T^4$. We give their adjacency matrices in Appendix A.1.

Lemma 5.4.5. *None of the tournaments in $\mathcal{F} \setminus \{T^1, T^2, T^3, T^4\}$ is a subdigraph of a 3-dicritical semi-complete digraph.*

Proof. For every tournament $T \in \mathcal{F}$, we check, using the code of Appendix A.3, if it contains $\vec{C}_3 \Rightarrow \vec{C}_3$ as a subdigraph or if it admits no uv-colouring for an arc uv. This is always the case, except when $T \in \{T^1, T^2, T^3, T^4\}$. The claim then follows by Lemmas 5.2.1 and 5.4.4. □

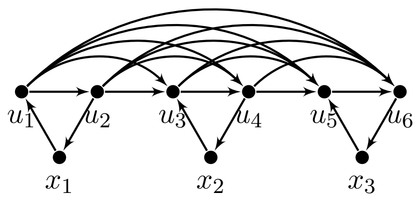


Figure 5.6: The oriented graph F.

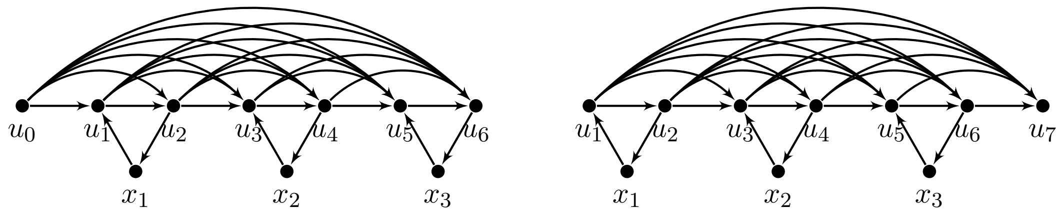


Figure 5.7: The oriented graphs F^+ (left) and F^- (right).

Let F^+ be the oriented graph obtained from F by adding a vertex u_0 and the arcs of $\{u_0u_i \mid i \in [6]\}$. Analogously, let F^- be the oriented graph obtained form F by adding a vertex u_7 and the arcs of $\{u_iu_7 \mid i \in [6]\}$. See Figure 5.7 for an illustration.

Lemma 5.4.6. *Let D be a 3-dicritical semi-complete digraph. Then D does not contain a digraph in $\{F^+, F^-\}$ as a subdigraph.*

Proof. Observe that the digraph obtained from F^- by reversing all its arcs is isomorphic to F^+. As the digraph obtained from a 3-dicritical, semi-complete digraph by reversing all arcs is 3-dicritical and semi-complete, it suffices to prove the statement for F^+.

In order to do so, suppose, for the sake of a contradiction, that there is a 3-dicritical semi-complete digraph D containing F^+. By Lemma 5.4.5, $D - u_0$ contains some $T' \in \{T^1, T^2, T^3, T^4\}$. Now consider the collection $\mathcal{T}$ of tournaments on $\{u_0, \dots, u_6, v_1, v_2, v_3\}$ that have one of T^1, T^2, T^3, T^4 as a labelled subdigraph and in which u_0 dominates $\{u_1, \dots, u_6\}$. Observe that, by assumption, D contains a tournament in $\mathcal{T}$ as a spanning subdigraph. Further, $\mathcal{T}$ contains exactly $4 \times 2^3 = 32$ digraphs. Using the code in Appendix A.4, we check that each of them contains $\vec{C}_3 \Rightarrow \vec{C}_3$ or contains an arc uv with no uv-colouring. We conclude that the same holds for D, a contradiction to Lemmas 5.2.1 or 5.4.4. □

We are now ready to show that 3-dicritical semi-complete digraphs do not contain large transitive tournaments as induced subdigraphs.

Lemma 5.4.7. *Let D be a 3-dicritical semi-complete digraph. Then D does not contain TT_8 as an induced subdigraph.*

Proof. For a contradiction, assume that $D = (V, A)$ is a 3-dicritical semi-complete digraph containing TT_8 as an induced subdigraph. We will prove that D contains F^+ or F^-, which is a contradiction to Lemma 5.4.6.

Let $S \subseteq V$ be such that $D\langle S\rangle$ is isomorphic to TT_8. Let $v_1, \dots, v_8$ be the unique acyclic ordering of S. By Lemma 5.2.2, for every $i \in [7]$, there exists a vertex $x_i \in V \setminus S$ such that $v_i v_{i+1} x_i v_i$ forms an induced directed triangle C_i.

Let H be the graph with vertex-set $V(H) = [7]$ and that contains an edge linking i and j if $V(C_i) \cap V(C_j) \neq \varnothing$. For any $i, j \in [7]$ with $ij \in E(H)$ and $|i - j| \geq 2$, we have $x_i = x_j$. By Lemma 5.2.4, there is no set $\{i, j, k\} \subseteq [7]$ such that $x_i = x_j = x_k$. This yields that H is obtained from a path on 7 vertices by adding a matching. We deduce from Lemma 5.4.1 that there is a set $I \subseteq [7]$ with $|I| = 3$ such that the following holds:

(a) $\{1, 7\} \setminus I \neq \varnothing$, and

(b) C_i and C_j are vertex-disjoint for all $\{i, j\} \subseteq I$.

This shows that D contains F^+ or F^-, yielding a contradiction to Lemma 5.4.6. □

Given an integer k and a semi-complete digraph D, a *k-extension* of D is a semi-complete digraph on $n(D) + k$ vertices containing D as an induced subdigraph. Given a set S of semi-complete digraphs, a *k-extension* of S is a semi-complete digraph that is a k-extension of some $D \in S$. We are now ready to prove that no 3-dicritical semi-complete digraph contains F as a subdigraph.

Lemma 5.4.8. *Let D be a 3-dicritical semi-complete digraph. Then D does not contain F as a subdigraph.*

Proof. By Lemma 5.4.5, it remains to show that D does not contain a graph in $\{T^1, T^2, T^3, T^4\}$ as a subdigraph. Assume for a contradiction that D contains at least one of $\{T^1, T^2, T^3, T^4\}$ as a subtournament.

We use the code in Appendix A.5. In a first part, we compute the set $\mathscr{L}$ of all semi-complete digraphs L on nine vertices such that each of the following holds:

(i) L contains some $T \in \{T^1, T^2, T^3, T^4\}$ as a subdigraph,

(ii) L does not contain any digraph in $\{\overleftrightarrow{S_4}, \overleftrightarrow{K_2} \Rightarrow \overleftrightarrow{K_2}, \overleftrightarrow{K_2} \Rightarrow \vec{C}_3, \vec{C}_3 \Rightarrow \overleftrightarrow{K_2}, \vec{C}_3 \Rightarrow \vec{C}_3, O_4, O_5\}$ as a subdigraph,

(iii) L admits a uv-colouring for every arc $uv \in A(L)$, and

(iv) L does not contain TT_8 as an induced subdigraph.

By Lemmas 5.2.1, 5.2.3, 5.4.2, 5.4.3, and 5.4.4, we know that D contains some $L \in \mathscr{L}$ as an induced subdigraph. In the second part of the code, we check that every 2-extension L' of $\mathscr{L}$ does not satisfy at least one of the properties (ii), (iii) and (iv).

This shows, by Lemmas 5.2.1, 5.2.3, 5.4.2 and 5.4.4, that either $D \in \mathscr{L}$ or D is a 1-extension of $\mathscr{L}$. Finally, we check that every $L \in \mathscr{L}$ has dichromatic number at most 2, and that every 1-extension L' satisfying (ii), (iii) and (iv) has dichromatic number at most 2. This yields a contradiction. □

We now give the definition of $\mathcal{T}_8$ and show that no digraph in $\mathcal{T}_8$ can be contained in a 3-dicritical semi-complete digraph as an induced subdigraph. Let $\mathcal{T}_8$ be the set of tournaments obtained from TT_8 by reversing exactly one arc. Observe that TT_8 belongs to $\mathcal{T}_8$.

Lemma 5.4.9. *Let D be a 3-dicritical semi-complete digraph. Then D does not contain any digraph in $\mathcal{T}_8$ as an induced subdigraph.*

Proof. Assume for a contradiction that D contains some $T' \in \mathcal{T}_8$ as an induced subtournament. Let $X \subseteq V(T)$ be such that $D\langle X\rangle$ is isomorphic to T'. By definition of $\mathcal{T}_8$, let $x_1, \dots, x_8$ be an ordering of X such that D contains every arc $x_i x_j$ when $i < j$, except for exactly one pair $\{k, \ell\}$, $k < \ell$.

Assume first that $k = \ell - 1$. Then observe that T' is isomorphic to TT_8, with the acyclic ordering obtained from $x_1, \dots, x_8$ by swapping x_ℓ and $x_{\ell-1}$. This contradicts Lemma 5.4.7.

Henceforth assume that $k \le \ell - 2$. If $k \ge 2$ and $\ell \le 7$ then $D\langle\{x_1, x_k, x_{\ell-1}, x_\ell, x_8\}\rangle$ is isomorphic to O_5, a contradiction to Lemma 5.2.3. Henceforth we assume that $k = 1$ or $\ell = 8$. By directional duality, we assume without loss of generality that $k = 1$. Let S be the transitive induced subtournament of D on vertices $X \setminus \{x_1, x_2, x_\ell\}$. We denote its acyclic ordering by $y_1, \dots, y_5$, which exactly corresponds to $x_3, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_8$. By Lemma 5.2.2, for every $k \in [4]$, there exists a vertex z_k such that $y_k y_{k+1} z_k y_k$ forms a directed triangle C_k. As S is induced, z_k must be in $V \setminus V(S)$. Moreover, $z_k \notin \{x_1, x_2, x_\ell\}$ because both $X \setminus \{x_1\}$ and $X \setminus \{x_\ell\}$ are acyclic.

Let H be the graph with vertex-set $V(H) = [4]$ and that contains an edge linking i and j if $V(C_i) \cap V(C_j) \neq \varnothing$. For any $i, j \in [4]$ with $ij \in E(H)$ and $|i - j| \ge 2$, we have $z_i = z_j$. By Lemma 5.2.4, there is no set $\{h, i, j\} \subseteq [4]$ such that $z_i = z_j = z_h$. This yields that H is obtained from a path on 4 vertices by adding a matching containing at most 2 edges. Hence H contains two non-adjacent vertices, corresponding to two disjoint directed triangles C_i and C_j in D. Together with the directed cycle $C_h = x_1 x_2 x_\ell$, we deduce that D contains F as a subdigraph. This contradicts Lemma 5.4.8. □

We have now proved all necessary structural properties of 3-dicritical semi-complete digraphs. The following result contains the decisive step of the proof and it requires heavy computation. For every $i \in [7]$, let $\mathscr{D}_i$ be the set of semi-complete digraphs D such that each of the following holds:

- the maximum acyclic set $S \subseteq V(D)$ of D has size exactly i,
- for every arc uv of D, D admits a uv-colouring,
- D does not contain any digraph of $\{\overleftrightarrow{S_4}, \overleftrightarrow{K_2} \Rightarrow \overleftrightarrow{K_2}, \overleftrightarrow{K_2} \Rightarrow \vec{C}_3, \vec{C}_3 \Rightarrow \overleftrightarrow{K_2}, \vec{C}_3 \Rightarrow \vec{C}_3, O_4, O_5, F\}$ as a subdigraph,
- D does not contain any digraph of $\mathcal{T}_8$ as an induced subdigraph,

Lemma 5.4.10. *The 3-dicritical digraphs in $\bigcup_{i=1}^{7} \mathscr{D}_i$ are exactly $\overleftrightarrow{K_3}$, $\mathcal{H}_5$, and $\mathcal{P}_7$.*

Proof. For every $i \in [7]$, we compute $\mathscr{D}_i$ by starting from the singleton $\{TT_i\}$ which is clearly the only digraph in $\mathscr{D}_i$ on at most i vertices. Using the code in Appendix A.6, we first successively compute the digraphs in $\mathscr{D}_i$ on $j \ge i$ vertices by generating every possible 1-extension of the digraphs in $\mathscr{D}_i$ on $j - 1$ vertices, and saving only the ones satisfying the conditions on $\mathscr{D}_i$. When j is large enough, it turns out that the set of digraphs in $\mathscr{D}_i$ on j vertices is empty, implying that $\mathscr{D}_i$ is finite.

We then consider every digraph $D \in \mathscr{D}_i$ and check whether D is 2-dicolourable. When it is not, since it admits a uv-colouring for every arc uv, we conclude that D is 3-dicritical. □

We are now ready to conclude the proof of Theorem 5.1.6.

Theorem 5.1.6. *There are exactly eight 3-dicritical semi-complete digraphs. They are depicted in Figure 5.1.*

Proof. By Lemmas 5.2.1, 5.2.3, 5.4.2, 5.4.3, 5.4.4, 5.4.8, and 5.4.9, we have that every 3-dicritical semi-complete digraph that is not contained in $\bigcup_{i=1}^{7} \mathscr{D}_i$ is one of $\vec{W}_3$, $\mathcal{R}(\overleftrightarrow{K_2}, \overleftrightarrow{K_2})$, $\mathcal{R}(\overleftrightarrow{K_2}, \vec{C}_3)$, $\mathcal{R}(\vec{C}_3, \overleftrightarrow{K_2})$, and $\mathcal{R}(\vec{C}_3, \vec{C}_3)$. The statement then follows directly from Lemma 5.4.10. □

5.5 Maximum number of arcs in 3-dicritical digraphs

This section is devoted to the proof of Theorem 5.1.8. We need a collection of intermediate results. We first show that the bidirected part of a 3-dicritical digraph is a forest unless D is a bidirected odd cycle.

Proposition 5.5.1. *Let D be a 3-dicritical digraph that is not a bidirected odd cycle. Then $S(D)$ is a forest.*

Proof. Assume for a contradiction that $S(D)$ is not a forest. Then it contains a cycle $C = u_1u_2 \dots u_pu_1$. Let $\overleftrightarrow{C}$ be the bidirected cycle in D corresponding to C. The cycle C cannot be odd, for otherwise $\overleftrightarrow{C}$ would be a bidirected odd cycle, and $D = \overleftrightarrow{C}$ because a bidirected odd cycle is 3-dicritical, a contradiction. Hence C is an even cycle. By Lemma 5.2.1, there exists a 2-dicolouring ϕ of $D \setminus \{u_1u_p\}$. Necessarily, u_1 and u_p are coloured differently because there is a bidirected path of odd length between u_1 and u_p. Thus ϕ is a 2-dicolouring of D, a contradiction. □

For the remainder of this section we need a few specific definitions. Let T be a tree and $V_3(T)$ be the set of vertices of degree at least 3 in T. Two vertices $u, v \in V(T)$ form an *odd pair* if they are non-adjacent and $\mathrm{dist}_T(u, v)$ is odd, where $\mathrm{dist}_T(u, v)$ denotes the length of the unique path between u and v in T. The set of odd pairs of T is denoted by $\mathrm{OP}(T)$ and its cardinality is denoted by $\mathrm{op}(T)$. We finally define the *dearth* of T as follows:

$$\mathrm{dearth}(T) = \sum_{v \in V_3(T)} \frac{1}{6} d(v)(d(v) - 1) + \mathrm{op}(T).$$

We first prove that the dearth of a tree is always at least a fraction of its order.

Lemma 5.5.2. *Let T be a tree on n vertices for some positive integer n. Then* $\mathrm{dearth}(T) \geq \frac{1}{3}n - 1$.

Proof. For the sake of a contradiction, suppose that T is a counterexample to the statement whose number of vertices is minimum. Clearly, we have $n \geq 4$. The following claim excludes a collection of simple structures of T.

Claim 5.5.3. *T is neither a path nor a tree.*

Proof of claim. The statement follows from the following simple case distinction.

Case 1: *T is a path of even length.*

For every odd $i \in [n-3]$, as $n \geq 4$, there are exactly i distinct pairs of vertices at distance exactly $n - i$ in T. Hence $\mathrm{dearth}(T) \geq \mathrm{op}(T) = \sum_{i=1}^{\frac{n-2}{2}} (2i - 1) = \left(\frac{n-2}{2}\right)^2 \geq \frac{1}{3}n - 1$.

Case 2: *T is a path of odd length.*

For every odd $i \in [n-2]$, as $n \geq 4$, there are exactly i distinct pairs of vertices at distance exactly $n - i$ in T. Hence $\mathrm{dearth}(T) \geq \mathrm{op}(T) = \sum_{i=1}^{\frac{n-3}{2}} 2i = \binom{\frac{n-3}{2}}{}\binom{\frac{n-1}{2}}{} \geq \frac{1}{3}n - 1$.

Case 3: *T is a star on $n \geq 4$ vertices.*

As $n \geq 4$, we obtain that $\mathrm{dearth}(T)$ is exactly $\frac{1}{6}(n-1)(n-2)$, and so $\mathrm{dearth}(T) \geq \frac{1}{3}n - 1$.

In either case, we obtain a contradiction to the choice of T. $\Diamond$

By Claim 5.5.3, we obtain that T is neither a path nor a star. In particular, it follows that T contains an edge uv such that $d_T(u) \geq 2$ and $d_T(v) \geq 3$. Let $v_1, \ldots, v_r$ be the neighbours of v in T, where $v_1 = u$ and $r = d_T(v) \geq 3$. For each $i \in [r]$, let T_i be the component of $T - v$ containing v_i. By the choice of T, we have $\mathrm{dearth}(T_i) \geq \frac{1}{3}n(T_i) - 1$. Since the T_is are pairwise disjoint and none of them contains v, and because u has a neighbour in T_1 at distance exactly 3 from $v_2, \ldots, v_r$ we obtain:

$$\begin{aligned}
\mathrm{dearth}(T) &\geq \sum_{i=1}^{r} \mathrm{dearth}(T_i) + \frac{1}{6}r(r-1) + (r-1) \\
&\geq \frac{1}{3}(n(T) - 1) - r + \frac{1}{6}r(r-1) + (r-1) \\
&\geq \frac{1}{3}n(T) - 1,
\end{aligned}$$

where in the last inequality we used $r \geq 3$. This contradicts the choice of T. $\square$

Lemma 5.5.4. *Let D be a 3-dicritical digraph distinct from $\{\overleftrightarrow{K_3}, \vec{W}_3\}$ and $\overleftrightarrow{T}$ be a bidirected tree contained in D. Then we have*

$$|\{\{u,v\} \subseteq V(T) \mid \{uv, vu\} \cap A(D) = \varnothing\}| \geq \mathrm{dearth}(T).$$

Proof. Set $\mathcal{O} = \{\{u,v\} \subseteq V(T) \mid \{uv, vu\} \cap A(D) = \varnothing\}$. For every vertex $v \in V_3(T)$, let $\mathcal{O}_v = \mathcal{O} \cap (N_T(v) \times N_T(v))$. Finally let $\mathcal{O}_{\mathrm{odd}} = \mathcal{O} \cap \mathrm{OP}(T)$.

Let us first show that these sets are pairwise disjoint. Let $u, v \in V_3(T)$ be two vertices of degree at least 3 in T. Since T is a tree, we have that $N_T(u) \cap N_T(v)$ contains at most one vertex, implying that $\mathcal{O}_u \cap \mathcal{O}_v = \varnothing$. Also note that vertices in $N_T(v)$ are at distance exactly 2 from each other, so $\mathcal{O}_v \cap \mathcal{O}_{\mathrm{odd}} = \varnothing$. This implies

$$|\mathcal{O}| \geq \sum_{v \in V_3(T)} |\mathcal{O}_v| + |\mathcal{O}_{\mathrm{odd}}|.$$

Hence it is sufficient to prove $|\mathcal{O}_v| \geq \frac{1}{6}d_T(v)(d_T(v)-1)$ for every $v \in V_3(T)$ and $\mathcal{O}_{\mathrm{odd}} = \mathrm{OP}(T)$ to prove Lemma 5.5.4.

Let $v \in V_3(T)$ and u, x, z be three distinct vertices in $N_T(v)$. We claim that $D\langle\{u,x,z\}\rangle$ contains at most two arcs. If this is not the case, then $D\langle\{u,x,z\}\rangle$ contains a digon, a directed triangle or a transitive tournament on three vertices. This implies that $D\langle\{u,x,z,v\}\rangle$ contains $\overleftrightarrow{K_3}$, $\vec{W}_3$ or O_4. By Theorem 5.1.6 and Lemma 5.4.2, in each case, we obtain a contradiction to the choice of D. Since this holds for every choice of three distinct vertices in $N_T(v)$ and each pair of vertices in $N_T(v)$ is contained in $d_T(v) - 2$ triples, we deduce the following inequality

$$m(D\langle N_T(v)\rangle) \cdot (d_T(v) - 2) = \sum_{\substack{X \subseteq N_T(v),\\ |X|=3}} m(D\langle X\rangle) \leq 2 \cdot \binom{d_T(v)}{3},$$

implying that $m(D\langle N_T(v)\rangle) \leq \frac{1}{3}d_T(v)(d_T(v) - 1)$. Therefore, we obtain $|\mathcal{O}_v| = \binom{d_T(v)}{2} - m(D\langle N_T(v)\rangle) \geq \frac{1}{6}d_T(v)(d_T(v) - 1)$ as desired.

To show $\mathcal{O}_{\mathrm{odd}} = \mathrm{OP}(T)$, it is sufficient to show that if $\{u, v\}$ is an odd pair then $\{uv, vu\} \cap A(D) = \varnothing$. Assume this is not the case, then by Lemma 5.2.1 $D' = D \setminus \{uv, vu\}$ admits a 2-dicolouring ϕ in which $\phi(u) = \phi(v)$, a contradiction since u and v are connected by a bidirected odd path in D'. This shows the claim. $\square$

We are now ready to prove Theorem 5.1.8 which we first restate here for convenience.

Theorem 5.1.8. *If D is a 3-dicritical digraph distinct from $\overleftrightarrow{K_3}$ and $\vec{W}_3$, then $m(D) \leq \binom{n}{2} + \frac{2}{3}n$.*

Proof. Let D be such a digraph. If D is a bidirected odd cycle, we have $n \geq 5$ and hence D has $2n \leq \binom{n}{2} + \frac{2}{3}n$ arcs, so the result trivially holds. Henceforth assume D is not a bidirected cycle. Then, by Proposition 5.5.1, $S(D)$ is a forest. Let $T_1, \ldots, T_s$ be the connected components of $S(D)$. For every $i \in [s]$, the number of digons in $D\langle V(T_i)\rangle$ is exactly $n(T_i) - 1$, whereas the number of pairs of non-adjacent vertices is at least $\mathrm{dearth}(T_i)$ by Claim 5.5.4. Hence, since there is no digon between the T_is, we obtain

$$\begin{aligned}
m(D) &\leq \binom{n}{2} + \sum_{i=1}^{s} \left((n(T_i) - 1) - \mathrm{dearth}(T_i)\right) \\
&\leq \binom{n}{2} + \sum_{i=1}^{s} \left((n(T_i) - 1) - (\frac{1}{3}n(T_i) - 1)\right) \quad \text{by Claim 5.5.2} \\
&= \binom{n}{2} + \frac{2}{3}n,
\end{aligned}$$

which concludes the proof. $\square$

5.6 Further research directions

In this chapter, we showed that the number of 3-dicritical semi-complete digraphs is finite and with a computer-assisted proof, we gave a full characterisation of them. We also gave a general upper bound on the number of arcs in 3-dicritical digraphs. These results seem to be only the tip of an iceberg, and natural generalisations in several directions can be considered.

First, the conjecture of Hoshino and Kawarabayashi on the maximum density of 3-dicritical oriented graphs remains widely open. We recall it here.

Conjecture 5.1.4 (Hoshino and Kawarabayashi [99]). *Let $k \geq 3$ be an integer. If $\vec{G}$ is a k-dicritical oriented graph, then $m(\vec{G}) \leq (\frac{1}{2} - \frac{1}{2^k-1})n(\vec{G})^2$.*

We believe that almost all 3-dicritical digraphs are sparser than tournaments. We thus propose the following conjecture, which would imply Theorem 5.1.5 and asymptotically improve on Theorem 5.1.8.

Conjecture 5.6.1. *There is only a finite number of 3-dicritical digraphs D on n vertices that satisfy $m(D) \geq \binom{n}{2}$.*

It is an interesting challenge to generalise the results obtained in this chapter to $k \geq 4$. In particular, we would be interested in a confirmation of the following statement.

Conjecture 5.6.2. *For every $k \geq 4$, there is only a finite number of k-dicritical semi-complete digraphs.*

Finally, it is also natural to consider a different notion of criticality. A digraph D is called *3-vertex-dicritical* if D is not 3-dicolourable, but $D - v$ is for all $v \in V(D)$. Observe that every 3-dicritical digraph is 3-vertex-dicritical, but the converse is not necessarily true. One can hence wonder whether an analogue of Theorem 5.1.5 is true for 3-vertex-dicritical digraphs. However, this turns out not to be the case. Chen et al. proved in [50] that it is NP-hard to decide whether a given tournament is 2-dicolourable. An infinite collection of 3-vertex-dicritical tournaments can easily be derived from their proof.

CHAPTER 6

Subdivisions in dicritical digraphs with large order or digirth

This chapter contains joint work with Clément Rambaud and is based on [140].

6.1 Introduction

We first recall the following central result which deals with substructures in digraphs of large dichromatic number. Recall that a *subdivision* of a digraph F is any digraph obtained from F by replacing every arc uv by a directed path from u to v.

Theorem 6.1.1 (Aboulker et al. [8]). *Let F be a digraph on n vertices, m arcs and c connected components. Every digraph D satisfying $\vec{\chi}(D) \geq 4^{m-n+c}(n-1)+1$ contains a subdivision of F.*

For every digraph F, we denote by $\mathrm{mader}_{\vec{\chi}}(F)$ the least integer c_F for which every digraph D either contains a subdivision of F or has dichromatic number at most $c_F - 1$. Note that $\mathrm{mader}_{\vec{\chi}}(F)$ is well-defined by Theorem 6.1.1. The result above was generalised in a recent work of Steiner [160] (see also [119]) who extended it to subdivisions with modular constraints.

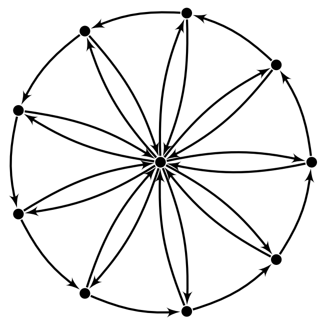


Figure 6.1: The digraph D_{10}.

Since every digraph is a subdigraph of $\overleftrightarrow{K_n}$, the bidirected complete graph on n vertices, it is natural to look for the value of $\mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_n})$. The result above implies that $\mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_n}) \leq 4^{n^2-2n+1}$. A more precise computation using the tools developed in [8] yields the following best known upper bound, whose proof is presented in Section 6.2.

Proposition 6.1.2. $\mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_n}) \leq 4^{\frac{2}{3}n^2+2n-\frac{8}{3}}$.

For every digraph F on n vertices, we have $\mathrm{mader}_{\vec{\chi}}(F) \geq n$. This is because $\overleftrightarrow{K_{n-1}}$ has dichromatic number $n-1$ and does not contain any subdivision of F. For some digraphs F, it then appears that the value of $\mathrm{mader}_{\vec{\chi}}(F)$ does not capture the structure of F but only its order. For instance, $\mathrm{mader}_{\vec{\chi}}(\vec{C}_n) \geq n$, but every 2-dicritical digraph on at least n vertices actually contains a subdivision of $\vec{C}_n$. In order to have a more profound understanding of digraphs forced to contain subdivisions of F, one may then ask for the minimum k such that there is a finite number of k-dicritical digraphs which do not contain any subdivision of F. The following question then naturally arises.

Question 6.1.3. *Let F^* be a subdivision of a digraph F. Is it true that the set of $(\mathrm{mader}_{\vec{\chi}}(F))$-dicritical digraphs that do not contain any subdivision of F^* is finite?*

Unfortunately, the answer to this question is negative. To see that, consider for every positive integers k and ℓ the digraph $C(k,\ell)$, which is the union of two internally disjoint directed paths from a vertex x to a vertex y of lengths respectively k and ℓ. Observe that $\mathrm{mader}_{\vec{\chi}}(C(1,2)) = 3$ because $\overleftrightarrow{K_2}$ does not contain any subdivision of $C(1,2)$ and every 3-dicritical digraph is 2-arc-strong. The following answers Question 6.1.3 by the negative for $F = C(1,2)$ and $F^* = C(3,3)$.

Proposition 6.1.4. *For every integer $n \geq 3$, there exists a 3-dicritical digraph D_n on n vertices that does not contain any subdivision of $C(3,3)$.*

Proof. Let D_n be the digraph obtained from a directed cycle on $n-1$ vertices $\vec{C}_{n-1}$ by adding a new vertex x and all possible digons between x and $V(\vec{C}_{n-1})$ (see Figure 6.1 for an illustration). This digraph is 3-dicritical (it follows from Lemma 6.3.1 and the fact that directed cycles are 2-dicritical) and does not contain any subdivision of $C(3,3)$. □

In Section 6.3, we show that the answer to Question 6.1.3 is actually negative not only for $C(1,2)$ but for every digraph F on at least three vertices with at least one arc. Let F be such a digraph and F^* be a subdivision of F in which an arc has been subdivided at least $3 \cdot \mathrm{mader}_{\vec{\chi}}(F)+1$ times. Since $\mathrm{mader}_{\vec{\chi}}(F) \geq n(F) \geq 3$, the following result implies that the set of $(\mathrm{mader}_{\vec{\chi}}(F))$-dicritical digraphs that do not contain any subdivision of F^* is infinite.

Theorem 6.1.5. *For every integer $k \geq 3$, there are infinitely many k-dicritical digraphs without any directed path on $3k+1$ vertices.*

Theorem 6.1.5 establishes a distinction between the directed and undirected cases. In the undirected case, for every fixed $k \geq 3$, there exists a non-decreasing function $f_k \colon \mathbb{N} \to \mathbb{N}$ such that every k-critical graph on at least $f_k(\ell)$ vertices contains a path on ℓ vertices. This was first proved by Kelly and Kelly [105] in 1954, answering a question of Dirac. The bound on f_k was then improved by Alon, Krivelevich and Seymour [11] and finally settled by Shapira and Thomas [156], who proved that the largest cycle in a k-critical graph on n vertices has length at least $c_k \cdot \log(n)$, where c_k is a constant depending only on k. This bound is best possible up to the multiplicative constant c_k, as shown by a construction of Gallai [76, 77] (see [156]).

On the positive side, we adapt the proof of Alon et al. [11] and show that, in k-dicritical digraphs, the length of the longest oriented cycle (*i.e.* the longest cycle in the underlying graph) grows with the number of vertices, and so does the length of its longest oriented path (*i.e.* the longest path in the underlying graph).

Theorem 6.1.6. *For every fixed integers $k \geq 2$ and $\ell \geq 3$, there are finitely many k-dicritical digraphs with no oriented cycle on at least ℓ vertices.*

Since Question 6.1.3 turns out to be almost always false, we propose as an alternative to restrict to digraphs with large digirth. Recall that there exist digraphs of arbitrarily large girth and dichromatic number (Theorem 1.2.8). For every integer g, we can thus define $\mathrm{mader}^{(g)}_{\vec{\chi}}(F)$ as the least integer k such that every digraph D satisfying $\vec{\chi}(D) \geq k$ and $\mathrm{digirth}(D) \geq g$ contains a subdivision of F. Note that $\mathrm{mader}_{\vec{\chi}}(F) = \mathrm{mader}^{(2)}_{\vec{\chi}}(F)$ and that $\mathrm{mader}^{(g)}_{\vec{\chi}}(F)$ is non-increasing in g. We conjecture that the analogue of Question 6.1.3 for digraphs of large digirth is actually true.

Conjecture 6.1.7. *For every digraph F and every subdivision F^* of F, there exists g such that $\mathrm{mader}^{(g)}_{\vec{\chi}}(F^*) \leq \mathrm{mader}_{\vec{\chi}}(F)$.*

This conjecture seems to be challenging to prove, since the exact value of $\mathrm{mader}_{\vec{\chi}}(F)$ is known for very few classes of digraphs. In order to provide some support to this conjecture, in Section 6.4 we show that the value of $\mathrm{mader}^{(g)}_{\vec{\chi}}(F^*)$ depends only on F when g is large enough. Our proof is strongly based on the key-lemma of [8].

Theorem 6.1.8. *Let $k \geq 1$ be an integer. For every non-empty digraph F, if F^* is obtained from F by subdividing every arc at most $k-1$ times, then $\mathrm{mader}^{(k)}_{\vec{\chi}}(F^*) \leq \frac{1}{3}(4^{m(F)+1}n(F)-1)$.*

In Section 6.5, we prove that Conjecture 6.1.7 holds for every digraph F whose underlying graph $\mathrm{UG}(F)$ is a forest.

Theorem 6.1.9. *Let $k \geq 1$ be an integer and let T be a bidirected tree. If T^* is obtained from T by subdividing every arc at most $k-1$ times, then $\mathrm{mader}^{(2k)}_{\vec{\chi}}(T^*) \leq \mathrm{mader}_{\vec{\chi}}(T) = n(T)$.*

In the case of oriented trees, we improve on Theorem 6.1.9 as follows.

Theorem 6.1.10. *Let $k \geq 1$ be an integer and let T be an oriented tree. If T^* is obtained from T by subdividing every arc at most $k-1$ times, then* $\mathrm{mader}^{(k)}_{\vec{\chi}}(T) \leq \mathrm{mader}_{\vec{\chi}}(T) = n(T)$.

Observe that every digraph D contains a subdigraph H such that $\delta^+(H) \geq \vec{\chi}(D) - 1$ (by taking H a $\vec{\chi}(D)$-dicritical subdigraph of D). Hence, for every integer k, if a digraph F is such that every digraph D with $\delta^+(D) \geq k-1$ contains a subdivision of F, then $\mathrm{mader}_{\vec{\chi}}(F) \leq k$. In Section 6.6, we look for similar results using δ^+ instead of $\vec{\chi}$. Conjecture 6.1.7 for $F = C(1,2)$ appears to be a consequence of the following theorem (recall that $\mathrm{mader}_{\vec{\chi}}(C(1,2)) = 3$).

Theorem 6.1.11. *For every integer $k \geq 2$, every digraph D with $\delta^+(D) \geq 2$ and* $\mathrm{digirth}(D) \geq 8k-6$ *contains a subdivision of $C(k,k)$.*

When $k = 2$, we improve Theorem 6.1.11 as follows.

Theorem 6.1.12. *Every oriented graph D with $\delta^+(D) \geq 2$ contains a subdivision of $C(2,2)$.*

We also consider out-stars. For integers k, ℓ, let $S_k^{+(\ell)}$ be the digraph consisting of k directed paths of length ℓ sharing their origin (and no other common vertices). The *centre* of $S_k^{+(\ell)}$ is its unique source.

Theorem 6.1.13. *Let $k \geq 2$ and $\ell \geq 1$ be two integers. Every digraph D with $\delta^+(D) \geq k$ and* $\mathrm{digirth}(D) \geq \frac{k^\ell - 1}{k-1} + 1$ *contains a copy of $S_k^{+(\ell)}$ with centre u for every chosen vertex u.*

When $k = 2$, Theorem 6.1.13 can be improved by reducing the bound on $\mathrm{digirth}(D)$ down to 2ℓ.

Theorem 6.1.14. *Every digraph D with $\delta^+(D) \geq 2$ and* $\mathrm{digirth}(D) \geq 2\ell$ *contains a copy of* $S_2^{+(\ell)}$.

We conclude in Section 6.7 with some open problems and further research directions.

Notation on paths and cycles

We first give and recall some definitions and notation specific to this chapter. An *antidirected path* is an orientation of a path where every vertex v satisfies $d_{\min}(v) = 0$. If u is a vertex of a digraph D with $d^-(u) = 1$, we define the *predecessor* of u in D, denoted by $\mathrm{pred}_D(u)$, as the unique in-neighbour of u in D. Similarly, if $d^+(u) = 1$, we define the *successor* of u in D, denoted by $\mathrm{succ}_D(u)$, as the unique out-neighbour of u in D.

Given two vertices a, b in a directed cycle C (with possibly $a = b$), we denote by $C[a,b]$ the directed path from a to b along C (which is the single-vertex path when $a = b$). Moreover, we define $C[a,b[= C[a,b] - b$, $C]a,b] = C[a,b] - a$, and $C]a,b[= C[a,b] - \{a,b\}$. Note that these subpaths may be empty. Given a directed path P and two vertices a, b in $V(P)$, we use similar notations $P[a,b]$, $P[a,b[$, $P]a,b]$ and $P]a,b[$. We denote by $\mathrm{init}(P)$ the initial vertex of P (*i.e.* the unique vertex with in-degree 0) and $\mathrm{term}(P)$ its terminal one. Given two directed paths P, Q such that $V(P) \cap V(Q) = \{x\}$ where $x = \mathrm{term}(P) = \mathrm{init}(Q)$, the *concatenation* of P and Q, denoted by $P \cdot Q$, is the digraph $(V(P) \cup V(Q), A(P) \cup A(Q))$. If U and V are two sets of vertices in D, then a *(U,V)-path* in D is a directed path P in D with $\mathrm{init}(P) \in U$ and $\mathrm{term}(P) \in V$, and we also say that P is a directed path from U to V. If $U = \{u\}$ (resp. $V = \{v\}$), then we simply write u for U (resp. v for V) in these notations.

The *distance* from u to v, denoted by $\mathrm{dist}(u,v)$, is the length of a shortest (u,v)-path, with the convention $\mathrm{dist}(u,v) = +\infty$ if no such path exists.

6.2 An improved bound for the bidirected complete graph

This section is devoted to the proof of Proposition 6.1.2. We need the two following lemmas.

Lemma 6.2.1 (Aboulker et al. [8, Lemma 31])**.** *For every integer k and every digraph D with $\vec{\chi}(D) \geq 4k-3$, there is a subdigraph H of D with $\vec{\chi}(H) \geq k$ such that for every pair u, v of distinct vertices in H, there is a directed path from u to v in D whose internal vertices are in $V(D) \setminus V(H)$.*

In particular, for every digraph F and every arc e in F,

$$\mathrm{mader}_{\vec{\chi}}(F) \leq 4 \cdot \mathrm{mader}_{\vec{\chi}}(F \setminus e) - 3.$$

We skip the proof of the following easy lemma.

Lemma 6.2.2. *If $F_1 + F_2$ denotes the disjoint union of two digraphs F_1 and F_2, then* $\mathrm{mader}_{\vec{\chi}}(F_1 + F_2) \leq \mathrm{mader}_{\vec{\chi}}(F_1) + \mathrm{mader}_{\vec{\chi}}(F_2)$.

We are now ready to prove Proposition 6.1.2, let us first restate it.

Proposition 6.1.2. $\mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_n}) \leq 4^{\frac{2}{3}n^2+2n-\frac{8}{3}}$.

Proof. Let $f(n) = \mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_n})$ for every $n \geq 1$. Clearly $f(1) = 1$. Let $g(x) = 4^{\frac{2}{3}x^2+2x-\frac{8}{3}}$ for every positive real x. Observe that g is non-decreasing. We will show by induction on n that $f(n) \leq g(n)$ for every positive integer n. For $n = 1$, $f(1) = 1 = g(1)$. Now suppose $n \geq 2$.

If n is even, by Lemmas 6.2.1 and 6.2.2 we deduce the following inequalities.

$$\begin{aligned}
f(n) &\leq 4^{\frac{n^2}{2}} \cdot \mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_{\frac{n}{2}}} + \overleftrightarrow{K_{\frac{n}{2}}}) \\
&\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot f\left(\frac{n}{2}\right) \\
&\leq 4^{\frac{n^2}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\
&\leq 4^{\frac{n^2}{2}+n} \cdot g\left(\frac{n}{2}\right).
\end{aligned}$$

If n is odd, then $f(n) \leq 4^{2(n-1)}\, \mathrm{mader}_{\vec{\chi}}\left(\overleftrightarrow{K_1} + \overleftrightarrow{K_{n-1}}\right) \leq 4^{2(n-1)}(1 + f(n-1)) \leq 4^{2n-1} f(n-1)$. Hence:

$$\begin{aligned}
f(n) &\leq 4^{2n-1} f(n-1) \\
&\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot \mathrm{mader}_{\vec{\chi}}\left(\overleftrightarrow{K_{\frac{n-1}{2}}} + \overleftrightarrow{K_{\frac{n-1}{2}}}\right) \\
&\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot f\left(\frac{n-1}{2}\right) \\
&\leq 4^{2n-1} \cdot 4^{\frac{(n-1)^2}{2}} \cdot 2 \cdot g\left(\frac{n-1}{2}\right) \\
&\leq 4^{\frac{n^2}{2}+n-\frac{1}{2}} \cdot 2 \cdot g\left(\frac{n}{2}\right) \\
&= 4^{\frac{n^2}{2}+n} \cdot g\left(\frac{n}{2}\right).
\end{aligned}$$

In both cases we have:

$$f(n) \leq 4^{\frac{n^2}{2}+n} \cdot g\left(\frac{n}{2}\right) = 4^{n^2(\frac{1}{2}+\frac{1}{6})+n(1+1)-\frac{8}{3}} = g(n).$$

$\square$

6.3 Paths and cycles in large dicritical digraphs

This section is devoted to the proofs of Theorems 6.1.5 and 6.1.6. We first prove the following useful observation.

Lemma 6.3.1. *For every integer k, if D is a k-dicritical digraph and if D' is obtained from D by adding a vertex u with $N^+(u) = N^-(u) = V(D)$, then D' is $(k+1)$-dicritical.*

Proof. First we show that $\chi(D') \geq k+1$. Indeed, if $\phi\colon V(D') \to [k]$ is a k-dicolouring of D', then $\phi(v) \neq \phi(u)$ for every $v \in V(D)$, and so ϕ induces a $(k-1)$-dicolouring of D, a contradiction.

It remains to show that for every arc vw in D', $D' \setminus vw$ is k-dicolourable. If $vw \in A(D)$, then since D is k-dicritical, $D \setminus vw$ admits a $(k-1)$-dicolouring $\phi\colon V(D) \to [k-1]$. Then extending ϕ to $V(D')$ by $\phi(u) = k$ yields a k-dicolouring of D'. If $u \in \{v,w\}$, then consider a k-dicolouring $\phi\colon V(D) \setminus \{v,w\} \to [k-1]$ of $D' - \{v,w\}$, which exists since $D' - \{v,w\}$ is a proper subdigraph of D. Now set $\phi(v) = \phi(w) = k$. Colour k induces an acyclic digraph of $D' \setminus vw$, and this yields a k-dicolouring of $D' \setminus vw$. $\square$

Theorem 6.1.5. *For every integer $k \geq 3$, there are infinitely many k-dicritical digraphs without any directed path on $3k+1$ vertices.*

Proof. Let n be an odd integer. Let $D_{k,n}$ be the digraph constructed as follows. Start with the antidirected path $P = p_1 \dots p_n$ on n vertices in which $d^+(p_1) = 1$. Add the digon $[p_1, p_n]$, and two vertices x_1, x_2 with a digon $[x_1, x_2]$. For every arc uv of P, add the arcs vx_i, x_iu for every $i \in [2]$. Finally, add $k-3$ vertices $x_3, \dots, x_{k-1}$ inducing a copy of $\overleftrightarrow{K}_{k-3}$ and add the digon $[x_{i+2}, u]$ for every $i \in [k-3]$ and every $u \in V(P) \cup \{x_1, x_2\}$. See Figure 6.2 for an illustration.

Let us show that $D_{k,n}$ is k-dicritical. Since $D_{k,n}$ is obtained from $D_{3,n}$ by adding $k-3$ vertices linked to all other vertices by a digon, by Lemma 6.3.1 it is enough to show that $D_{3,n}$ is 3-dicritical.

We first show that $\vec{\chi}(D_{3,n}) > 2$. Suppose for contradiction that there is a 2-dicolouring $\phi\colon V(D_{3,n}) \to [2]$ of $D_{3,n}$. Without loss of generality, $\phi(x_1) = 1$ and $\phi(x_2) = 2$. For every arc uv of P, for every $i \in [2]$, uvx_iu is a directed triangle. This implies that $\phi(u) \neq \phi(v)$. Since P has an odd number of vertices, $\phi(p_1) = \phi(p_n)$, which is a contradiction, as $[p_1, p_n]$ is a digon in $D_{3,n}$.

Let uv be an arc in $D_{3,n}$. We show that $\vec{\chi}(D_{3,n} \setminus uv) \leq 2$. If $\{u,v\} = \{x_1, x_2\}$, set $\phi(x_1) = \phi(x_2) = \phi(p_1) = 1$, $\phi(p_2) = \cdots = \phi(p_n) = 2$. If $\{u,v\} = \{p_1, p_n\}$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, $\phi(p_i) = 1$ if i is even, $\phi(p_i) = 2$ if i is odd. If $uv \in A(P)$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, and colour $V(P)$ such that the only monochromatic pair of adjacent vertices in P is $\{u,v\}$. If $u \in V(P)$ and $v = x_i$, set $\phi(x_1) = 1$, $\phi(x_2) = 2$, $\phi(u) = i$, and colour $V(P-u)$ such that two adjacent vertices in $V(P-u)$ receive distinct colours. The other cases are symmetric. In each case, one can check that this gives a proper 2-dicolouring of $D_{3,n} \setminus uv$.

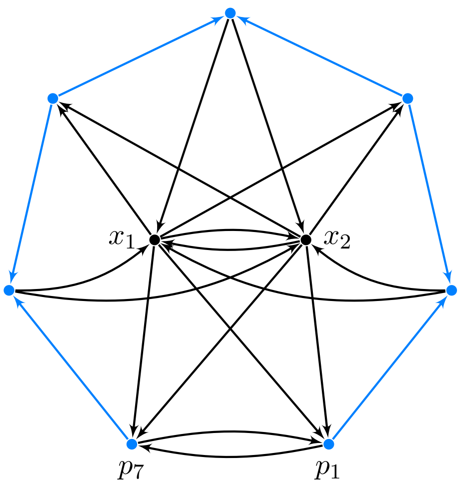


Figure 6.2: The digraph $D_{3,7}$. The antidirected path P is in blue.

It remains to prove that $D_{k,n}$ does not contain a directed path on $3k+1$ vertices. Let Q be a directed path in $D_{k,n}$. Let $y_1, \dots, y_\ell$ be the vertices of $V(Q) \cap \{x_1, \dots, x_{k-1}\}$ in order of appearance along Q. Let Q_i be the subpath $Q]y_i, y_{i+1}[$ of Q for every $i \in [\ell - 1]$ and let $Q_0 = Q[\text{init}(Q), y_1[$ and $Q_\ell = Q]y_\ell, \text{term}(Q)])$. Note that some Q_is may be empty.

Then observe that at most one of the Q_is intersects both p_1 and p_n. Except this one, which has at most three vertices, all the Q_js have at most two vertices (because P is anti-directed). We conclude that the number of vertices in P is at most $3\ell + 3 \leq 3k$. □

Theorem 6.1.6. *For every fixed integers $k \geq 2$ and $\ell \geq 3$, there are finitely many k-dicritical digraphs with no oriented cycle on at least ℓ vertices.*

Proof. Let $k \geq 2$ be a fixed integer. We will show the existence of a function $f_k \colon \mathbb{N} \to \mathbb{N}$ such that every k-dicritical digraph on at least $f_k(\ell)$ vertices contains an oriented path on ℓ vertices. We will then use a result of Dirac to show that every k-dicritical digraph on at least $f_k(\frac{1}{4}\ell^2)$ vertices contains an oriented cycle on ℓ vertices, implying the result.

Given a digraph H, $\text{cc}(H)$ is the number of connected components of H (*i.e.* the number of connected components of $\text{UG}(H)$). Our proof is strongly based on the following claim.

Claim 6.3.2. *Let $D = (V, A)$ be a k-dicritical digraph and $S \subseteq V$, then*

$$\text{cc}(D - S) \leq (k-1)^{|S|} \cdot 3^{\binom{|S|}{2}}.$$

Proof of claim. Assume this is not the case, *i.e.* there exists a k-dicritical digraph D and a subset of its vertices S such that $\text{cc}(D-S) > (k-1)^{|S|} \cdot 3^{\binom{|S|}{2}}$. We denote by $H_1, \dots, H_r$ the connected components of $D - S$.

For every $i \in [r]$, let α_i be a $(k-1)$-dicolouring of $D - V(H_i)$, the existence of which is guaranteed by the dicriticality of D. Let $s = |S|$ and $v_1, \dots, v_s$ be any fixed ordering of S. For every $i \in [r]$, we let σ_i^1 be the ordered set $(\alpha_i(v_1), \dots, \alpha_i(v_s))$. We also define σ_i^2 as the set of all ordered pairs $(u, v) \in S^2$ such that $D - V(H_i)$, coloured with α_i, contains a monochromatic directed path from u to v. We finally define the i^{th} configuration σ_i as the ordered pair (σ_i^1, σ_i^2).

For every pair of vertices u, v in S and every $i \in [r]$, note that at most one of the ordered pairs $(u, v), (v, u)$ actually belongs to σ_i^2, for otherwise $D - V(H_i)$, coloured with α_i, contains a monochromatic directed cycle. Hence, the number of distinct configurations is at most $(k-1)^s \cdot 3^{\binom{s}{2}}$. By the pigeonhole principle, since $r > (k-1)^s \cdot 3^{\binom{s}{2}}$, there exist two distinct integers $i, j \in [r]$ such that $\sigma_i = \sigma_j$. Let α be the colouring of D defined as follows:

$$\alpha(v) = \begin{cases} \alpha_i(v) & \text{if } v \in (V(D) \setminus V(H_i)) \\ \alpha_j(v) & \text{otherwise.} \end{cases}$$

We claim that α is a $(k-1)$-dicolouring of D. Assume for a contradiction that it is not, so D, coloured with α, contains a monochromatic directed cycle. Among all such cycles C, we choose one for which the size of $V(C) \cap V(H_i)$ is minimised. If $V(C) \cap V(H_i) = \varnothing$, then C is a monochromatic directed cycle of $D - H_i$, a contradiction to the choice of α_i. Analogously, we have $V(C) \setminus V(H_i) \neq \varnothing$ by choice of α_j.

Assume first that $|V(C) \setminus V(H_i)| = 1$, implying that C contains exactly one vertex s in S and $(V(C) \setminus \{s\}) \subseteq V(H_i)$. Since $\sigma_i^1 = \sigma_j^1$, we have $\alpha_i(s) = \alpha_j(s)$, which implies that C is a monochromatic directed cycle of $D - H_j$ coloured with α_j, a contradiction to the choice of α_j.

Henceforth we can assume that C contains a directed path P on at least three vertices, with initial vertex u and terminal vertex v, such that $V(P) \cap S = \{u, v\}$ and $V(P) \subseteq (V(H_i) \cup \{u, v\})$. The existence of P ensures that (u, v) belongs to σ_j^2. Hence, since $\sigma_i = \sigma_j$, there exists a monochromatic directed path P' in $D - V(H_i)$ coloured with α_i, from u to v, and with the same colour as P. Hence, replacing P by P' in C, we obtain a closed walk which, coloured with α, contains a monochromatic directed cycle C' such that $|V(C') \cap V(H_i)| < |V(C) \cap V(H_i)|$, a contradiction to the choice of C. $\Diamond$

We are now ready to prove the existence of f_k. Let D be a k-dicritical digraph whose underlying graph G does not contain any path on ℓ vertices. Let v be any vertex of D and T be a spanning DFS-tree of G rooted in v (recall that D is connected since it is dicritical). Let h be the depth of T (*i.e.* the maximum number of vertices in a branch of T), then h is at most ℓ since G does not contain any path of length ℓ.

For every vertex x, let S_x be the ancestors of x (including x itself) in T and $d_T(x)$ be the number of children of x in T. Since T is a DFS-tree, note that for every neighbour y of x, x and y must belong to the same branch. Hence, $d_T(x) \leq \mathrm{cc}(D - S_x)$. Since $|S_x| \leq h \leq \ell$, we deduce from Claim 6.3.2 that $d_T(x) \leq (k-1)^\ell \cdot 3^{\binom{\ell}{2}}$. Since T is spanning, we obtain that $n(D) \leq \left((k-1)^\ell \cdot 3^{\binom{\ell}{2}}\right)^{\ell-1} = f_k(\ell) - 1$.

Dirac proved that every 2-connected graph that contains a path of length t actually contains a cycle of length at least $2\sqrt{t}$ (see [123, Problem 10.29]). It is straightforward to show that every k-dicritical digraph is 2-connected. Hence, if D is a k-dicritical digraph on at least $f_k(\frac{1}{4}\ell^2)$ vertices, then D contains an oriented cycle of length at least ℓ, implying the result. $\square$

6.4 Subdivisions in digraphs with large digirth

This section is devoted to the proof of Theorem 6.1.8.

Theorem 6.1.8. *Let $k \geq 1$ be an integer. For every non-empty digraph F, if F^* is obtained from F by subdividing every arc at most $k-1$ times, then* $\mathrm{mader}_{\vec{\chi}}^{(k)}(F^*) \leq \frac{1}{3}(4^{m(F)+1}n(F) - 1)$.

Proof. We proceed by induction on $m(F)$, the result being trivial when $m(F) = 0$. Let F be any digraph with $m > 0$ arcs, and let F^* be a digraph obtained from F by subdividing every arc at most $k-1$ times.

Let $uv \in A(F)$ be any arc, and $P = x_1, \dots, x_r$ its corresponding directed path in F^* (where $u = x_1$ and $v = x_r$). Then we only have to prove that $\mathrm{mader}_{\vec{\chi}}^{(k)}(F^* \setminus x_1x_2) \leq \frac{4^{m(F)}n(F)-1}{3} + 1$. If this is true, then by Lemma 6.2.1, we get that $\mathrm{mader}_{\vec{\chi}}^{(k)}(F^*) \leq 4\left(\frac{4^{m(F)}n(F)-1}{3} + 1\right) - 3$ which shows the result.

Let D be any digraph with dichromatic number at least $\frac{4^{m(F)}n(F)-1}{3} + 1$ and digirth at least k and let $B \subseteq V(D)$ be a maximal acyclic set in D. Then $\vec{\chi}(D - B) \geq \frac{4^{m(F)}n(F)-1}{3}$, so by induction $D - B$ must contain a subdivision of $F \setminus uv$ where each arc has been subdivided at least $k-1$ times. This is also a subdivision of $F^* - \{x_2, \dots, x_{r-1}\}$. Let y be the vertex in $D - B$ corresponding to x_r. By maximality of B, there must be a directed cycle C in D such that $V(C) \cap V(D - B) = \{y\}$. Note that C has length at least k. Thus, ignoring the leaving arc of y in C, we have found a subdivision of $F^* \setminus x_1x_2$ in D, showing the result. □

6.5 Subdivisions of trees in digraphs with large digirth

This section is devoted to the proofs of Theorems 6.1.9 and 6.1.10.

Theorem 6.1.9. *Let $k \geq 1$ be an integer and let T be a bidirected tree. If T^* is obtained from T by subdividing every arc at most $k-1$ times, then* $\mathrm{mader}_{\vec{\chi}}^{(2k)}(T^*) \leq \mathrm{mader}_{\vec{\chi}}(T) = n(T)$.

Proof. We proceed by induction on $n(T)$. Suppose $n(T) \geq 2$, the result being trivial when $n(T) = 1$.

Let f be a leaf of T with neighbour p, and we denote by $(T-f)^*$ the bidirected tree $T - f$ with every arc subdivided exactly $k-1$ times. By induction hypothesis $\mathrm{mader}_{\vec{\chi}}^{(2k)}((T-f)^*) \leq \mathrm{mader}_{\vec{\chi}}(T-f) = n(T)-1$. Let D be a digraph with $\mathrm{digirth}(D) \geq 2k$ and $\vec{\chi}(D) \geq n(T)$, and consider a maximal acyclic set A in D. Then $\vec{\chi}(D-A) \geq n(T)-1$ and so by induction hypothesis, $D - A$ contains a subdivision of $(T-f)^*$. Let $y \in V(D) \setminus A$ be the vertex corresponding to $p \in V(T)$ in the subdivision of $(T-f)^*$ contained in $D - A$. By maximality of A, $A + y$ contains a directed cycle C with $V(C) \setminus A = \{y\}$. As $\mathrm{digirth}(D) \geq 2k$, C has length at least $2k$. Then the subdivision of $(T-f)^*$ in $D - A$ together with C gives the desired subdivision of T^*. □

Theorem 6.1.10. *Let $k \geq 1$ be an integer and let T be an oriented tree. If T^* is obtained from T by subdividing every arc at most $k-1$ times, then* $\mathrm{mader}_{\vec{\chi}}^{(k)}(T) \leq \mathrm{mader}_{\vec{\chi}}(T) = n(T)$.

Proof. We proceed by induction on $n(T)$. Suppose $n(T) \geq 2$, the result being trivial when $n(T) = 1$.

For every arc e of T, we denote by $s(e)$ the number of subdivisions of e in T^*. Let f be a leaf of T with neighbour p, and we denote by $(T-f)^*$ the oriented tree $T - f$ where every arc e is subdivided $s(e) \leq k-1$ times.

Let D be a digraph with $\mathrm{digirth}(D) \geq k$ and $\vec{\chi}(D) \geq n(T)$, and consider a maximal acyclic set A in D. We have $\vec{\chi}(D - A) \geq n(T) - 1$ and, by the induction hypothesis, $D - A$ contains a copy of $(T-f)^*$. Let $y \in V(D) \setminus A$ be the vertex corresponding to $p \in V(T)$ in the copy of $(T-f)^*$ contained in $D - A$. By maximality of A, $A + y$ contains a directed cycle C with $C \setminus A = \{y\}$. As $\mathrm{digirth}(D) \geq k$, C has length at least k.

If the arc between p and f goes from p to f then we define P as the directed path on $s(pf)$ vertices starting from y along C. Otherwise, it goes from f to p and then we define P as the directed path on $s(fp)$ vertices, along C, ending on y. In both cases, the copy of $(T - f)^*$ in $D - A$ together with P gives the desired copy of T^*. $\square$

6.6 Subdivisions in digraphs of large out-degree and large digirth

6.6.1 Subdivisions of $C(k, k)$

This section is devoted to the proof of Theorem 6.1.11. We will use the following classical theorem due to Menger.

Theorem 6.6.1 (Menger [128] (see [18, Theorem 5.4.1]))**.** *Let D be a digraph and $U, V \subseteq V(D)$. Then the minimum number of vertices intersecting every (U, V)-path is equal to the maximum number of vertex-disjoint (U, V)-paths.*

Theorem 6.1.11. *For every integer $k \geq 2$, every digraph D with $\delta^+(D) \geq 2$ and $\mathrm{digirth}(D) \geq 8k - 6$ contains a subdivision of $C(k, k)$.*

Proof. We will prove the following stronger statement: for every digraph D with $\mathrm{digirth}(D) \geq 8k-6$ and $v_0 \in V(D)$, if $d^+(v_0) \geq 1$ and $d^+(v) \geq 2$ for every $v \in V(D) \setminus \{v_0\}$, then D contains a subdivision of $C(k, k)$. We now consider a counterexample to this statement with minimum number of vertices, and minimum number of arcs if equality holds.

Claim 6.6.2. *D is strongly connected.*

Proof of claim. Let C be a terminal strongly connected component of D, that is a strongly connected component such that there is no arc going out of C. Then C is also a counterexample, so by minimality of D we have $D = C$, and D is strongly connected. $\Diamond$

Claim 6.6.3. *$d^+(v) = 2$ for every vertex $v \neq v_0$ and $d^+(v_0) = 1$.*

Proof of claim. If $v \neq v_0$ is a vertex with at least 3 out-neighbours w_1, w_2, w_3, then $D \setminus vw_3$ is a smaller counterexample. Similarly, if $d^+(v_0) > 1$, then v_0 has at least two distinct out-neighbours w_1, w_2, and $D \setminus vw_2$ is a smaller counterexample. $\Diamond$

Given two vertices u, v of D, a (u, v)-*vertex-cut* is a vertex $x \in V(D) \setminus \{u, v\}$ which intersects every (u, v)-path of D.

Claim 6.6.4. *Let u, v be two vertices in D. If $\mathrm{dist}(u, v) \leq 7k - 6$, then there exists a (v, u)-vertex-cut.*

Proof of claim. Suppose the contrary for contradiction. Then by Menger's theorem, there exist two internally vertex-disjoint (v, u)-paths P_1 and P_2. As $\mathrm{digirth}(D) \geq 8k - 6$, both P_1 and P_2 have length at least k, and so $P_1 \cup P_2$ is a subdivision of $C(k, k)$ with source v and sink u. $\Diamond$

For every directed cycle C in D, let $\rho(C)$ be the number of vertices in the largest connected component of $D - V(C)$. We say that C is isometric if for every $u, v \in V(C)$, C contains a shortest (u, v)-path in D. Clearly D contains an isometric cycle (it is enough to take a minimum directed cycle), and we consider among them an isometric cycle C which maximises $\rho(C)$.

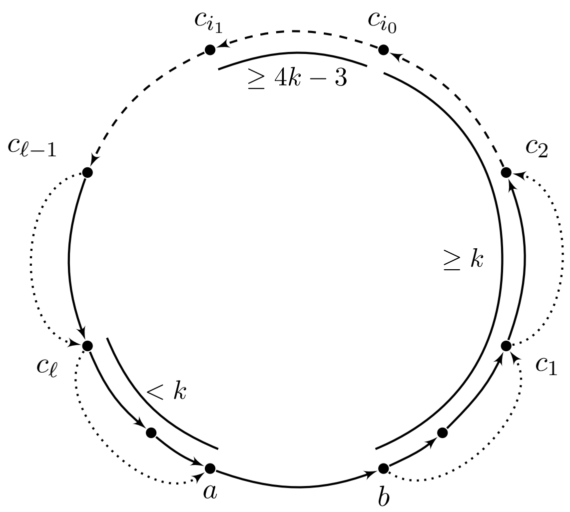


Figure 6.3: The structure of C in D. The solid and dashed arcs represent the arcs of C. A dotted arc from u to v illustrates the existence of two internally-disjoint directed paths from u to v in D.

Let ab be an arc along C. Let $c_1, \dots, c_\ell$ be the (b,a)-vertex-cuts in D. Observe that at least one such vertex-cut exists by Claim 6.6.4. As C contains a (b,a)-path, all these vertices belong to $V(C)$, and we suppose that they appear in this order $c_1, \dots, c_\ell$ along C starting at b. By convention, we also define $c_0 = b$ and $c_{\ell+1} = a$.

Claim 6.6.5. $\mathrm{dist}(c_i, c_{i+1}) \leq k-1$ *for every* $i = 0, \dots, \ell$.

Proof of claim. Suppose for contradiction that $\mathrm{dist}(c_i, c_{i+1}) \geq k$. Assume first that there exists a (c_i, c_{i+1})-vertex-cut x. We claim that x is also a (b,a)-vertex-cut. Consider a (b,a)-path Q. Then Q passes through c_i and c_{i+1} in this order, otherwise the concatenation of $Q[b, c_{i+1}]$ and $C[c_{i+1}, a]$ is a (b,a)-path avoiding c_i, a contradiction. By definition of x, it belongs to $Q[c_i, c_{i+1}]$. Hence x intersects every (b,a)-path, and so $x \in \{c_1, \dots, c_\ell\}$, a contradiction since $c_1, \dots, c_\ell$ are in this order along C.

This shows, by Menger's theorem, that there are two internally vertex-disjoint (c_i, c_{i+1})-paths P_1, P_2. Then P_1 and P_2 have length at least k, and so $P_1 \cup P_2$ is a subdivision of $C(k,k)$ with source c_i and sink c_{i+1}, a contradiction. $\diamondsuit$

Let i_0 be the least index i such that $\mathrm{dist}(b, c_i) \geq k$, and let i_1 be the largest index i such that $\mathrm{dist}(c_i, a) \geq k$. By choice of i_0, we have $\mathrm{dist}(b, c_{i_0-1}) \leq k-1$. By Claim 6.6.5, we have $\mathrm{dist}(c_{i_0-1}, c_{i_0}) \leq k-1$, which implies $\mathrm{dist}(b, c_{i_0}) \leq \mathrm{dist}(b, c_{i_0-1}) + \mathrm{dist}(c_{i_0-1}, c_{i_0}) \leq 2k-2$. Similarly, we have $\mathrm{dist}(c_{i_1}, a) \leq 2k-2$. Therefore, we have

$$\mathrm{dist}(c_{i_0}, c_{i_1}) = n(C) - \mathrm{dist}(c_{i_1}, a) - 1 - \mathrm{dist}(b, c_{i_0}) \geq 4k-3,$$

which implies $i_1 - i_0 \geq 5$ by Claim 6.6.5. See Figure 6.3 for an illustration.

We now define $d_i = c_{i+i_0}$ for $i = 0, \dots, 5$. If i is an index larger than 5, we identify d_i with $d_{i \bmod 6}$. For every $i = 0, \dots, 5$, let X_i be the set of vertices reachable from d_i in $D - d_{i+1}$. Similarly, if i is larger than 5, we identify X_i with $X_{i \bmod 6}$.

Claim 6.6.6. *For every $i = 0, \dots, 5$, $X_i \cap V(C) = V(C[d_i, d_{i+1}[)$.*

Proof of claim. We first consider the case $i \in \{0, \dots, 4\}$. Assume for a contradiction that there exists a directed path P from $V(C[d_i, d_{i+1}[)$ to $V(C) \setminus V(C[d_i, d_{i+1}])$ with internal vertices disjoint from C. Let u be $\mathrm{init}(P)$ and v be $\mathrm{term}(P)$. If $v \in V(C]d_{i+1}, a])$, then $C[b, u] \cup P \cup C[v, a]$ is a (b, a)-path which avoids d_{i+1}, a contradiction. Otherwise $v \in V(C[b, d_i[)$, and then P has length at least k because, since C is isometric,

$$\mathrm{dist}_P(u, v) \geq \mathrm{dist}_C(u, v) \geq \mathrm{dist}_C(d_5, b) \geq \mathrm{dist}_C(c_{i_1}, a) \geq k.$$

But then $P \cup C[u, v]$ is a subdivision of $C(k, k)$ with source u and sink v, a contradiction to D being a counterexample.

Now suppose $i = 5$. Consider first a directed path P from $u \in V(C[d_5, a])$ to $v \in V(C]d_0, d_5[)$ internally disjoint from C. Then P has length at least k because

$$\mathrm{dist}_P(u, v) \geq \mathrm{dist}_C(u, v) \geq \mathrm{dist}_C(b, d_0) \geq k.$$

Thus $P \cup C[u, v]$ is a subdivision of $C(k, k)$ with source u and sink v, a contradiction. Consider finally a directed path P from $u \in V(C[b, d_0[)$ to $v \in V(C]d_0, d_5[)$. Then d_0 is not a (b, a)-cut in D, a contradiction. ♢

Claim 6.6.7. *For every distinct $i, j \in \{0, \dots, 5\}$, we have*

(i) $X_i \cap X_j = \varnothing$ if $i \notin \{j-1, j+1\}$, and

(ii) $X_i \cap X_{i+1} = \varnothing$ if $v_0 \notin X_i \cup X_{i+1}$.

Proof of claim. We first prove (i). Let us fix two distinct integers $i, j \in \{0, \dots, 5\}$ such that $i \notin \{j-1, j+1\}$. Assume for a contradiction that $X_i \cap X_j \neq \varnothing$. Note that $X_i \cap X_j \neq V(D)$ because $d_{i+1} \notin X_i$. Therefore, since D is strongly connected by Claim 6.6.9, there is an arc uv such that $u \in X_i \cap X_j$ and $v \in V(D) \setminus (X_i \cap X_j)$. Assume first that $v \in X_j \setminus X_i$. Since $v \notin X_i$, we must have $v = d_{i+1}$. Hence $d_{i+1} \in X_j$, a contradiction to Claim 6.6.6. Symmetrically, if $v \in X_i \setminus X_j$ then $v = d_{j+1}$ by definition of X_j, implying that $d_{j+1} \in X_i$, a contradiction to Claim 6.6.6. Finally if $v \notin (X_i \cup X_j)$, by definition of X_i and X_j, we must have $v = d_{i+1} = d_{j+1}$, a contradiction. This proves (i).

We now prove (ii). Assume for a contradiction that $v_0 \notin X_i \cup X_{i+1}$ and $X_i \cap X_{i+1} \neq \varnothing$. Recall that $X_i \cap X_{i+1} \neq V(D)$ because $d_{i+1} \notin X_i$. Therefore, since D is strongly connected, there is an arc uv such that $u \in X_i \cap X_{i+1}$ and $v \in V(D) \setminus (X_i \cap X_{i+1})$. First, if $v \in V(D) \setminus (X_i \cup X_{i+1})$ then, by definition of X_i, v must be d_{i+1}, and by definition of X_{i+1}, v must be d_{i+2}, a contradiction. Next if $v \in X_i \setminus X_{i+1}$, then by definition of X_{i+1}, v must be d_{i+2}, but $d_{i+2} \notin X_i$ by Claim 6.6.6, a contradiction. Then we may assume that $v \in X_{i+1} \setminus X_i$, and by definition of X_i, v must be d_{i+1}. As $u \in X_{i+1}$, there is a directed path P from $V(C[d_{i+1}, d_{i+2}[)$ to u in $D - d_{i+2}$ internally disjoint from C. Let x be $\mathrm{init}(P)$. If $x \neq d_{i+1}$, then the union of $P \cup ud_{i+1}$ (which has length at least k because $\mathrm{digirth}(D) \geq 8k - 6 \geq 2k$ and $\mathrm{dist}(d_{i+1}, x) \leq k$ by Claim 6.6.5) and $C[x, d_{i+1}]$ (which has length at least k because $n(C) \geq 8k - 6 \geq 2k$ and $\mathrm{dist}(d_{i+1}, x) \leq k$ by Claim 6.6.5) is a subdivision of $C(k, k)$ with source x and sink d_{i+1}, a contradiction.

So we assume that $x = d_{i+1}$, that is $P \cup ud_{i+1}$ is a cycle C' with $V(C') \cap V(C) = \{d_{i+1}\}$, and $u \in V(C') \cap X_i \cap X_{i+1}$. Let w be the vertex in $V(C') \cap X_i$ such that $\mathrm{dist}_{C'}(w, d_{i+1})$ is maximum (the existence of w is guaranteed because $u \in V(C') \cap X_i$), and let Q be a (d_i, w)-path in $D\langle X_i \rangle$. If $\mathrm{dist}_{C'}(w, d_{i+1}) \leq k-1$, then $C'[d_{i+1}, w]$ has length at least k, and $C[d_{i+1}, d_i] \cup Q$ has length at least k. Moreover, the directed paths $C'[d_{i+1}, w]$ and $C[d_{i+1}, d_i] \cup Q$ are internally vertex-disjoint by the choice of P, w and Q. Hence, their union is a subdivision of $C(k,k)$ with source d_{i+1} and sink w, a contradiction. Henceforth we suppose that $\mathrm{dist}_{C'}(w, d_{i+1}) \geq k$.

We now prove the following statement.

$$D - d_{i+1} \textit{ does not contain any directed path } R \textit{ from } V(C') \textit{ to } V(C). \qquad (\heartsuit)$$

Assume for a contradiction that such a directed path R exists. We assume that R is internally disjoint from $V(C') \cup V(C)$, for otherwise we can extract a subpath of R with this extra property. Let $y = \mathrm{init}(R)$ and $z = \mathrm{term}(R)$. Then by Claim 6.6.6, z belongs to $V(C]d_{i+1}, d_{i+2}])$. Observe that $y \in V(C']d_{i+1}, w[)$, for otherwise y belongs to X_i and so does z, a contradiction to Claim 6.6.6. But then the union of $R \cup C[z, d_{i+1}]$ and $C'[y, d_{i+1}]$ is a subdivision of $C(k,k)$ with source y and sink d_{i+1}, a contradiction to D being a counterexample. This shows ($\heartsuit$).

Let U be the set of vertices reachable from d_i in $D \setminus d_{i+1}t$ where t is the successor of d_{i+1} in C. We claim that $U \subseteq X_i \cup X_{i+1}$. Let u be any vertex in U. By definition, there is a directed path R' from d_i to u in $D \setminus d_{i+1}t$. If $d_{i+1} \notin V(R')$, then $u \in X_i$. Else if $d_{i+1} \in V(R')$ and $d_{i+2} \notin V(R')$, then $u \in X_{i+1}$. Henceforth assume that both d_{i+1} and d_{i+2} belong to R'. Observe that d_{i+1} is before d_{i+2} along R', otherwise $d_{i+2} \in X_i$, a contradiction to Claim 6.6.6. Since $d_D^+(d_{i+1}) = 2$, the successor of d_{i+1} in R' is also its successor in C'. Hence $R'[d_{i+1}, d_{i+2}]$ contains a subpath R from $V(C') \setminus \{d_{i+1}\}$ to $V(C) \setminus \{d_{i+1}\}$ internally disjoint from $V(C') \cup V(C)$, a contradiction to ($\heartsuit$).

This proves that $U \subseteq X_i \cup X_{i+1}$ and in particular, $v_0 \notin U$. Set $v_0' = d_{i+1}$, $D' = D\langle U \rangle$. Then D' equipped with v_0' is such that every vertex in U has out-degree 2 in D' except v_0' which has out-degree at least 1. By minimality of $n(D)$, D' contains a subdivision of $C(k,k)$ and so does D, a contradiction.

$\Diamond$

By (i) of the previous claim, there is an index $j \in \{0, \dots, 5\}$ such that $v_0 \notin X_{j-1}$. Since D is strongly connected, $\bigcup_{i=0}^5 X_i = V(D)$, and so there is an index $i \in \{0, \dots, 5\}$ such that $v_0 \in (X_i \setminus X_{i-1})$. From now on, we fix such an index $i \in \{0, \dots, 5\}$, and we set

$$\begin{aligned} Y_0 &= X_{i-1} \cup X_i \cup X_{i+1} \cup X_{i+2}, \\ Y_1 &= X_{i+3}, \text{ and} \\ Y_2 &= X_{i+4}. \end{aligned}$$

Note that $v_0 \notin X_{i-1} \cup X_{i+2}$, and so, by Claim 6.6.7, Y_0, Y_1, Y_2 are pairwise vertex-disjoint. Moreover, $Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C)$ are pairwise non-adjacent by definition of the X_js (*i.e.* there is no arc of D with head and tail in different parts of $(Y_0 \setminus V(C), Y_1 \setminus V(C), Y_2 \setminus V(C))$). Consider a connected component A of $D - V(C)$ of maximal size, that is with $|A| = \rho(C)$. Then A is included in one of Y_0, Y_1, Y_2. Let $j \in \{1, 2\}$ be such that $A \cap Y_j = \varnothing$. Let q be the predecessor of d_{i+j+3} in C. Let S be the set of vertices reachable from q in $D - d_{i+j+3}$. Observe that S is a subset of X_{i+j+2}. We claim that $D\langle S \rangle$ is not acyclic. Indeed, for every vertex $u \in S$, $N_D^+(u) \subseteq N_{D\langle S \rangle}^+(u) \cup \{d_{i+j+3}\}$. Since $v_0 \notin S$, for every vertex $u \in S$, $d_{D\langle S \rangle}^+(u) \geq d_D^+(u) - 1 = 1$. Therefore $D\langle S \rangle$ has minimum out-degree at least 1. Let C' be an isometric cycle in $D\langle S \rangle$.

Let us show that C' is an isometric cycle in D. Suppose on the contrary that there is a directed path P from $x \in V(C')$ to $y \in V(C')$ internally disjoint from $V(C')$ of length smaller than $\mathrm{dist}_{C'}(x, y)$. As P is not included in S, P must contain d_{i+j+3}. Let d_ι be the last vertex along P in $\{d_\ell \mid \ell = 0, \dots, 5\}$. We have $y \in X_\iota$ by definition of ι and $y \in X_{i+j+2}$ because $y \in S \subseteq X_{i+j+2}$. Therefore $y \in X_\iota \cap X_{i+j+2}$. Since $v_0 \notin X_{i+j+1} \cup X_{i+j+2} \cup X_{i+j+3}$, by Claim 6.6.7, we deduce that $\iota = i + j + 2$. Hence P contains a directed path from d_{i+j+3} to d_{i+j+2}. This implies:

$$\begin{aligned}
dist_{C'}(x, y) &\geq \mathrm{length}(P) && \text{by definition of } P\\
&\geq \mathrm{dist}_D(d_{i+j+3}, d_{i+j+2}) && \text{because } P \text{ contains } d_{i+j+3} \text{ and } d_{i+j+2}\\
&= \mathrm{dist}_C(d_{i+j+3}, d_{i+j+2}) && \text{because } C \text{ is isometric}\\
&= n(C) - \mathrm{dist}_C(d_{i+j+2}, d_{i+j+3}) &&\\
&\geq (8k - 6) - (k - 1) \geq k && \text{by Claim 6.6.5.}
\end{aligned}$$

Therefore both P and $C'[x, y]$ have length at least k, implying that $P \cup C'[x, y]$ is a subdivision of $C(k, k)$ with source x and sink y, a contradiction to D being a counterexample. This proves that C' is isometric in D.

By definition, $N(A) \subseteq V(C) \cup A$. Since D is strongly connected, A has an in-neighbour in $V(C)$. Since $A \cap Y_j = \varnothing$ by choice of j, A has an in-neighbour in $C[d_{i+j+3}, d_{i+j+2}[$. Hence, the connected component in $D - V(C')$ which contains A is strictly larger than A, which contradicts the maximality of $\rho(C)$, and concludes the proof of the theorem. □

6.6.2 Subdivisions of $C(2, 2)$ in oriented graphs

This section is devoted to the proof of Theorem 6.1.12.

Theorem 6.1.12. *Every oriented graph D with $\delta^+(D) \geq 2$ contains a subdivision of $C(2, 2)$.*

Proof. Suppose for contradiction that there exists an oriented graph D with $\delta^+(D) \geq 2$ that contains no subdivision of $C(2, 2)$. Assume that $n(D)$ is minimum, and that among such minimum counterexamples, $m(D)$ is minimum.

Claim 6.6.8. *For every vertex $v \in V(D)$, $d^+(v) = 2$.*

Proof of claim. If v is a vertex with at least 3 out-neighbours w_1, w_2, w_3, then $D \setminus vw_3$ is a smaller counterexample. ♢

Claim 6.6.9. *D is strongly connected. In particular, $d^-(v) \geq 1$ for every vertex v.*

Proof of claim. Let C be a terminal strongly connected component of D. Then C is also a counterexample, so by minimality of D we have $D = C$, and D is strongly connected. ♢

Claim 6.6.10. *For every vertex $v \in V(D)$, $d^-(v) \geq 2$.*

Proof of claim. Suppose that v is a vertex which has at most one in-neighbour. By Claim 6.6.9, it must have a unique in-neighbour u, and let w_1, w_2 be its two out-neighbours. If w_1 is non adjacent to u, then consider $D' = (D - v) \cup uw_1$. By minimality of D, D' contains a subdivision F of $C(2, 2)$, and as $F \not\subseteq D$, we have $uw_1 \in A(F)$. But then $(F \setminus uw_1) \cup uv \cup vw_1 \subseteq D$ is a subdivision of $C(2, 2)$. Hence there is an arc between u and w_1. Similarly, there is an arc between u and w_2.

If $w_1u, w_2u \in A(D)$, then the union of the directed paths vw_1u and vw_2u yields a copy of $C(2,2)$ in D. Moreover, if $uw_1, uw_2 \in A(D)$, then $d^+(u) \geq 3$, a contradiction to Claim 6.6.8. Hence, without loss of generality, $w_1u, uw_2 \in A(D)$.

As D is strongly connected, there is a directed path P from w_2 to $\{u, v, w_1\}$ with internal vertices disjoint from $\{u, v, w_1, w_2\}$. The terminal vertex of P is not v, as the only in-neighbour of v is u. So the terminal vertex of P is either u or w_1. If it is w_1, then the union of the directed paths uvw_1 and uP yields a subdivision of $C(2,2)$. If u is the end-vertex of P, then the union of the directed paths vw_1u and vP yields a subdivision of $C(2,2)$. In both cases, we find a subdivision of $C(2,2)$ in D, a contradiction. ◊

Claim 6.6.11. *D is 2-diregular.*

Proof of claim. By Claim 6.6.8, we know that, for each $v \in V(D)$, $d^+(v) = 2$. It implies that $m(D) = \sum_{v \in V(D)} d^+(v) = 2n(D)$. Since $m(D)$ is also equal to $\sum_{v \in V(D)} d^-(v)$, we get by Claim 6.6.10 that for every vertex v of D, $d^-(v) = d^+(v) = 2$, which implies that D is 2-diregular. ◊

Claim 6.6.12. *For every arc vw, w has a neighbour in $N^-(v)$.*

Proof of claim. Let u_1, u_2 be the in-neighbours of v. If w has no neighbour in $\{u_1, u_2\}$, consider $D' = D - v \cup u_1w \cup u_2w$. By minimality of D, there exists a subdivision F of $C(2,2)$ in D'. If neither u_1w nor u_2w belongs to $A(F)$, then $F \subseteq D$, a contradiction. If both u_1w and u_2w belong to $A(F)$, then w is the sink of F and $F \setminus w_1 \cup v$ is a subdivision of $C(2,2)$ in D, a contradiction. If exactly one of u_1w and u_2w belongs to $A(F)$, say u_1w, then $F \setminus u_1w \cup \{u_1v, vw\}$ is a subdivision of $C(2,2)$ in D, a contradiction. Hence w_1 has a neighbour in $\{u_1, u_2\}$. ◊

Claim 6.6.13. *For every vertex v with in-neighbourhood u_1, u_2 and out-neighbourhood w_1, w_2, either $\{w_1u_1, w_2u_2\} \subseteq A(D)$ or $\{w_1u_2, w_2u_1\} \subseteq A(D)$. In particular, every vertex belongs to two different directed triangles.*

Proof of claim. By Claim 6.6.12, w_1 has a neighbour in $\{u_1, u_2\}$. Without loss of generality, suppose that it is u_1. We now show that $w_1u_1 \in A(D)$, so assume for a contradiction that $u_1w_1 \in A(D)$. By Claim 6.6.12 u_1 has an in-neighbour x which is also a neighbour of w_1.

If $x = u_2$, then $\{u_1, u_2, v, w_1\}$ contains a copy of $C(2,2)$ with source u_2 and sink w_1. If $x = w_2$, then either $w_2w_1 \in A(D)$ and w_1 has in-degree 3, a contradiction to Claim 6.6.11, or $w_1w_2 \in A(D)$ and $\{u_1, v, w_1, w_2\}$ contains a copy of $C(2,2)$ with source u_1 and sink w_2. Hence x, u_1, u_2, v, w_1, w_2 are pairwise distinct.

Moreover, $xw_1 \notin A(D)$ for otherwise w_1 has in-degree 3 contradicting Claim 6.6.11. Hence $w_1x \in A(D)$. Consider the out-neighbour y of w_1 distinct from x. By Claim 6.6.12, y has a neighbour in $N^-(w_1) = \{u_1, v\}$. If v is a neighbour of y, then $y \in \{u_2, w_2\}$. If $y = w_2$, then $\{u_1, v, w_1, w_2\}$ contains a copy of $C(2,2)$ with source u_1 and sink w_2. If $y = u_2$, then the union of the directed paths w_1u_2v and w_1xu_1v yields a subdivision of $C(2,2)$ with source w_1 and sink v.

Hence y is not a neighbour of v, and so y is a neighbour of u_1. If $u_1y \in A(D)$, then u_1 has out-degree at least 3, contradicting Claim 6.6.8. If $yu_1 \in A(D)$, then $\{w_1, x, u_1, y\}$ contains a copy of $C(2,2)$ with source w_1 and sink u_1. In both cases, we reach a contradiction.

This proves that $w_1u_1 \in A$. Similarly, w_2 has an out-neighbour in $N^-(v) = \{u_1, u_2\}$. If $w_2u_1 \in A(D)$, then $\{v, w_1, w_2, u_1\}$ contains a copy of $C(2,2)$ with source v and sink u_1, a contradiction. Hence $w_2u_1 \notin A(D)$, and so $w_2u_2 \in A(D)$ as claimed. $\lozenge$

Claim 6.6.14. *Let t_1 and t_2 be two different directed triangles of D. Then $|V(t_1) \cap V(t_2)| \leq 1$.*

Proof of claim. Since D is an oriented graph, then it is clear that $|V(t_1) \cap V(t_2)| \leq 2$. Assume now that $V(t_1) = \{x, y, z\}$ and $V(t_2) = \{x, y, w\}$, where $z \neq w$. Assume without loss of generality that $xy \in A(t_1)$, then $xy \in A(t_2)$ because x and y must be adjacent in t_2, and D does not contain any digon. Now $t_1 \cup t_2$ contains a copy of $C(2,2)$ with source y and sink x, a contradiction. $\lozenge$

Consider the undirected graph H whose vertices are the directed triangles in D, and such that two triangles t and t' are adjacent if and only if they share a common vertex.

By Claim 6.6.14 and Claim 6.6.11, H is a subcubic graph. Moreover, by Claim 6.6.13, H must be a cubic graph. In particular H is not a forest and so it contains an induced cycle $C = (t_1, \dots, t_k, t_1)$. Let $t_1 = (x, y, z, x)$ and suppose (by possibly relabelling t_1 and C) that $V(t_1) \cap V(t_k) = \{x\}$ and $V(t_1) \cap V(t_2) = \{y\}$. Let P be a directed path in D with vertices in $V(t_2) \cup \dots V(t_k)$ from y to x. Observe that $z \notin V(P)$ because C is an induced cycle of H. Then the union of P and the path yzx is a subdivision of $C(2,2)$ in D, a contradiction. This proves the theorem. $\square$

6.6.3 Subdivisions of out-stars

This section is devoted to the proofs of Theorems 6.1.13 and 6.1.14.

Theorem 6.1.13. *Let $k \geq 2$ and $\ell \geq 1$ be two integers. Every digraph D with $\delta^+(D) \geq k$ and $\mathrm{digirth}(D) \geq \frac{k^\ell - 1}{k-1} + 1$ contains a copy of $S_k^{+(\ell)}$ with centre u for every chosen vertex u.*

Proof. Let D be such a digraph. By taking $m(D)$ minimal, we can suppose that $d^+(v) = k$ for every vertex $v \in V(D)$. Let W be the set of vertices at distance at least ℓ from u. If there are k vertex-disjoint (u, W)-paths then these directed paths have length at least ℓ and so they form a copy of $S_k^{+(\ell)}$. Otherwise, by Menger's Theorem, there is a set $S \subseteq V(D) \setminus \{u\}$ of $k-1$ vertices such that there is no (u, W)-path in $D - S$. Let R be the set of vertices reachable from u in $D - S$. Then every vertex in R is at distance at most $\ell - 1$ from u, so $|R| \leq \frac{k^\ell - 1}{k-1}$. As D has digirth at least $\frac{k^\ell - 1}{k-1} + 1$, this implies that $D\langle R \rangle$ is acyclic. Let $r \in R$ be a sink in $D\langle R \rangle$. Then all the out-neighbours of r in D are in S, and so $d^+(r) \leq k - 1$, a contradiction. $\square$

Theorem 6.1.14. *Every digraph D with $\delta^+(D) \geq 2$ and $\mathrm{digirth}(D) \geq 2\ell$ contains a copy of $S_2^{+(\ell)}$.*

Proof. Suppose for contradiction that there is such a digraph D containing no copy of $S_2^{+(\ell)}$. We assume $\ell \geq 2$, the result being trivial when $\ell = 1$. Without loss of generality, we may also assume that $d^+(v) = 2$ for every vertex v in D. By considering only one terminal strongly connected component of D, we can also assume that D is strong. Let u be a vertex in D, and let w be a vertex at distance exactly ℓ from u. Such a vertex exists because, as D is strong, u has an in-neighbour, which is at distance at least $2\ell - 1 \geq \ell + 1$ from u.

Let us fix P a shortest directed path from u to w. A *P-tricot* is a sequence of pairwise vertex-disjoint directed paths $Q_1, \dots, Q_r$ (where r is the size of the tricot) such that for every $i \in [r]$:

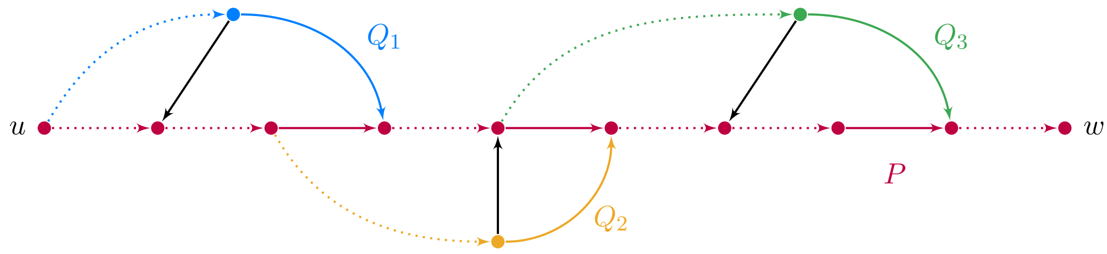


Figure 6.4: An example of a P-tricot of size 3. Dotted arcs represent directed paths.

- $V(Q_i) \cap V(P) = \{\text{term}(Q_i), \text{init}(Q_i)\}$,
- $\text{init}(Q_1) = u$,
- $\text{init}(Q_{i+1}) = \text{pred}_P(\text{term}(Q_i))$ if $i < r$,
- there is an arc from $\text{pred}_{Q_i}(\text{term}(Q_i))$ to $V(P]\text{init}(Q_i), \text{term}(Q_i)[)$.

Let us first prove that D admits a P-tricot. Let Q be a maximum directed path in D, starting on u, that is disjoint from $V(P) \setminus \{u\}$. Since Q is maximum, the two out-neighbours of $\text{term}(Q)$ belong to $V(P) \cup V(Q)$. We have that the length of Q is at most $\ell - 1$, for otherwise the union of P and Q contains a copy of $S_2^{+(\ell)}$, a contradiction to the choice of D. This implies that the out-neighbourhood of $\text{term}(Q)$ is in $V(P) \setminus V(Q)$, for otherwise $D\langle V(Q)\rangle$ contains a directed cycle of length at most $\ell - 1$, a contradiction. Let x be the out-neighbour of $\text{term}(Q)$ which is the furthest from u and y be its other out-neighbour. Let Q' be the extension of Q with x, then (Q') is a P-tricot of size one since Q' starts at u, intersects P exactly on $\{u, x\}$ and y belongs to $P[u, x[$.

Among all P-tricots of D, we choose one with maximum size r and denote it by $\mathcal{T}$. Let P_1 be the directed path corresponding to the concatenation of Q_1 and all $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$ for odd $i \in \{3, \dots, r\}$. Let P_2 be the directed path corresponding to the concatenation of all $P[\text{term}(Q_{i-2}), \text{init}(Q_i)] \cdot Q_i$ for even $i \in \{2, \dots, r\}$ (we identify $\text{term}(Q_0)$ with u).

Observe that P_1 and P_2 are two directed paths starting from u, and that they intersect exactly on $\{u\}$. Note that P_1 can be completed into a (u, w)-path $\widetilde{P}_1 = P_1 \cdot P[\text{term}(P_1), w]$ disjoint from $P_2 - \text{term}(P_2)$. Since it is a (u, w)-path, $\widetilde{P}_1$ has length at least ℓ. Therefore P_2 has length at most ℓ, for otherwise $\widetilde{P}_1 \cup (P_2 - \text{term}(P_2))$ is a copy of $S_2^{+(\ell)}$ in D. Analogously, P_2 can be completed into a (u, w)-path, which implies that P_1 has length at most ℓ.

Let $i \in \{1, 2\}$ be such that P_i does not contain Q_r and let $j \in \{1, 2\}$ be different from i. Let v be $\text{pred}_P(\text{term}(Q_r))$. We consider Q' a maximal directed path starting from v in $D - (V(P) \cup V(P_1) \cup V(P_2) \setminus \{v\})$. Let t be $\text{term}(Q')$, let P'_i be the concatenation $P_i \cdot P[\text{term}(P_i), v] \cdot Q'$ and P'_j be the concatenation $P_j \cdot P[\text{term}(Q_r), w]$. See Figure 6.5 for an illustration.

Since P'_j is a (u, w)-path, P'_j has length at least ℓ. If also P'_i has length at least ℓ, then P'_i and P'_j are two directed paths of length at least ℓ, sharing their source u and no other vertices, a contradiction since D is a counterexample. We know that t has two out-neighbours x, y in D. Since Q' is a maximal directed path, we know that both x and y belong to $V(P'_i) \cup V(P'_j) \cup V(P)$.

- First, if one of x, y, say x, belongs to $V(P'_i)$, $P'_i[x, t] \cup \text{term}(P'_i)x$ is a directed cycle of length at most
$$\text{length}(P'_i[x, t]) + 1 \leq \text{length}(P'_i) + 1 \leq \ell,$$

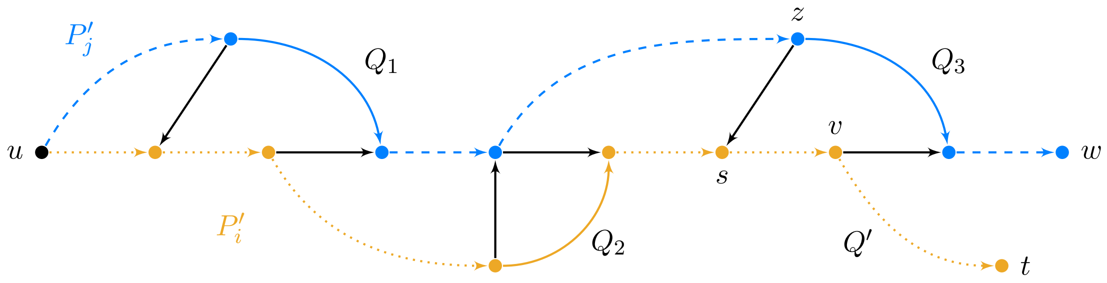


Figure 6.5: An illustration of the paths P'_i and P'_j.

where in the last inequality we used that P'_i has length at most $\ell - 1$. This is a contradiction since $\mathrm{digirth}(D) \geq 2\ell > \ell$.

- Else, if one of x, y, say x, belongs to $V(P[u,v])$, then $P[x,v] \cdot Q' \cup tx$ is a directed cycle of length at most:

$$\begin{aligned}
&\mathrm{length}(P[x,v]) + \mathrm{length}(Q'[v,t]) + 1 & \\
\leq\, &\mathrm{length}(P[u,v]) + \mathrm{length}(Q'[v,t]) + 1 && \text{because } u \text{ is before } x \text{ in } P \\
\leq\, &\mathrm{length}(P'_i[u,v]) + \mathrm{length}(Q'[v,t]) + 1 && \text{because } P \text{ is a shortest path} \\
=\, &\mathrm{length}(P'_i) + 1 \leq \ell, &
\end{aligned}$$

a contradiction since $\mathrm{digirth}(D) \geq 2\ell > \ell$.

- Else if one of x, y, say x, belongs to $V(P_j) \setminus V(P[u,v])$, let z be $\mathrm{pred}_{P_j}(\mathrm{term}(P_j))$, which is also $\mathrm{pred}_{Q_r}(\mathrm{term}(Q_r))$. By definition of $\mathcal{T}$, z has an out-neighbour s in $V(P]\mathrm{init}(Q_r), \mathrm{term}(Q_r)[)$. Then $P_j[x,z] \cdot zs \cdot P[s,v] \cdot Q' \cup tx$ is a directed cycle with length at most

$$\begin{aligned}
&\mathrm{length}(P_j[x,z]) + \mathrm{length}(P[s,v]) + \mathrm{length}(Q') + 2 & \\
\leq\, &\mathrm{length}(P_j) + \mathrm{length}(P[s,v]) + \mathrm{length}(Q') && \text{as } x \neq u \text{ and } z \neq \mathrm{term}(P_j) \\
\leq\, &\mathrm{length}(P_j) + \mathrm{length}(P'_i[u,v]) + \mathrm{length}(Q') && \text{as } P \text{ is a shortest path} \\
\leq\, &\mathrm{length}(P_j) + \mathrm{length}(P'_i) \leq 2\ell - 1, &
\end{aligned}$$

a contradiction since $\mathrm{digirth}(D) \geq 2\ell$.

- Finally if both x and y belong to $P[\mathrm{term}(Q_r), w]$, we can assume that x is before y on the path P. But then the P-tricot $(Q_1, \ldots, Q_r, Q' \cdot ty)$ contradicts the maximality of $\mathcal{T}$. □

6.7 Further research directions

In this chapter, for a fixed digraph F, we gave some sufficient conditions on a digraph D to ensure that D contains F as a subdivision. Many open questions arise. In particular, the exact value of $\mathrm{mader}_{\vec{\chi}}(F)$ is known only for very few digraphs F. The smallest digraph F for which it is unknown is $\overleftrightarrow{K_3}$.

Conjecture 6.7.1 (Gishboliner, Steiner, and Szabó [80])**.**

$$\mathrm{mader}_{\vec{\chi}}(\overleftrightarrow{K_3}) = 4$$

In the first part of this chapter, we looked for paths and cycles in large dicritical digraphs. In particular, we proved in Theorem 6.1.5 that for every integer $k \geq 3$, there are infinitely many k-dicritical digraphs without any directed path on $3k+1$ vertices. Conversely, Bermond et al. [24] proved that every connected digraph with $\delta^+(D) \geq k$ and $\delta^-(D) \geq \ell$ contains a directed path of order at least $\min\{n, k+\ell+1\}$. As every vertex in a k-dicritical digraph has in- and out-degree at least $k-1$, we obtain that there are finitely many k-dicritical digraphs with no directed path on $2k-1$ vertices. The following problem then naturally comes.

Problem 6.7.2. *For every integer $k \geq 3$, find the largest integer $f(k) \in [2k-1, 3k]$ such that the set of k-dicritical $\vec{P}_{f(k)}$-free digraphs is finite.*

Given a digraph F, we say that F is δ^+*-maderian* if there is an integer k such that every digraph D with $\delta^+(D) \geq k$ contains a subdivision of F. The smallest such integer k is then denoted by $\mathrm{mader}_{\delta^+}(F)$. The problem of characterising δ^+-maderian digraphs is widely open. In particular, Mader [125] conjectured that every acyclic digraph is δ^+-maderian, but this remains unproven albeit many efforts to prove or disprove it (see [122] for a partial answering to the conjecture).

In the remaining of the chapter, we focused on digraphs of large digirth. Given a digraph F, and an integer g, we can define $\mathrm{mader}_{\delta^+}^{(g)}(F)$ to be the smallest integer k, if it exists, such that every digraph D with $\delta^+(D) \geq k$ and $\mathrm{digirth}(D) \geq g$ contains a subdivision of F.

It is interesting to note that there are digraphs which are not δ^+-maderian even when restricted to digraphs of large digirth. Indeed, for every integers g, d there is a digraph D with $\mathrm{digirth}(D) \geq g$ and $\delta^+(D) \geq d$ such that D does not contain any subdivision of $\overleftrightarrow{K_3}$. Such a digraph D can be easily obtained from a construction by DeVos et al. [56] of digraphs with arbitrarily large out-degree in which every directed cycle has odd length, by removing a few arcs in order to increase the digirth. Since every subdivision of $\overleftrightarrow{K_3}$ has an even directed cycle, such a digraph does not contain $\overleftrightarrow{K_3}$ as a subdivision.

In Theorem 6.1.11, we proved that $\mathrm{mader}_{\delta^+}^{(8k-4)}(C(k,k)) \leq 2$. On the other hand, the value $8k-4$ cannot be replaced by $k-1$. To see this, consider the digraph D with vertex-set $\mathbb{Z}/(2k-1)\mathbb{Z}$ and arc-set $\{(i,i+1),(i,i+2) \mid i \in V(D)\}$. Since $n(D) < 2k$, D has no subdivision of $C(k,k)$. Since $\delta^+(D) = 2$ and $\mathrm{digirth}(D) = k-1$, we deduce that $\mathrm{mader}_{\delta^+}^{(k-1)}(C(k,k)) > 2$. Thus, the following problem arises.

Problem 6.7.3. *Find the minimum $g \in [k, 8k-4]$ such that $\mathrm{mader}_{\delta^+}^{(g)}(C(k,k)) \leq 2$.*

In this chapter, we studied the value of $\min\{\mathrm{mader}_{\vec{\chi}}^{(g)}(X) \mid g \geq 0\}$ given a digraph X. We believe that this value is upper bounded by a function of the maximum degree.

Conjecture 6.7.4. *There is a function f such that for every digraph F with maximum degree Δ, there is an integer g such that $\mathrm{mader}_{\vec{\chi}}^{(g)}(F) \leq f(\Delta)$.*

This is motivated by the following result by Mader [126], which is somehow the undirected analogue of Conjecture 6.7.4.

Theorem 6.7.5 (Mader [126]). *There is a function f such that for every graph F, for every graph G with $\delta(G) \geq \max\{\Delta(F), 3\}$, if $\mathrm{girth}(G) \geq f(F)$ then G contains a subdivision of F.*

It was later proved by Kühn and Osthus [118] that one can take $f(H) = 166 \frac{\log n(H)}{\log \Delta(H)}$, which is optimal up to the constant factor.

Harutyunyan and Mohar [94] proved that there is a positive constant c such that for every large enough Δ, g, there is a digraph D with $\mathrm{girth}(D) \geq g$, $\Delta(D) \leq \Delta$ and $\vec{\chi}(D) \geq c \cdot \frac{\Delta}{\log \Delta}$. This is a generalisation to the directed case of a classical result by Bollobás [29]. This implies that any function f satisfying Conjecture 6.7.4 is such that $f(\Delta) \geq c \cdot \frac{\Delta}{\log \Delta}$. We are inclined to believe that this is optimal and that there is such a function f in $O\left(\frac{\Delta}{\log \Delta}\right)$.

CHAPTER 7

Redicolouring digraphs

This chapter contains joint work with Nicolas Bousquet, Frédéric Havet, Nicolas Nisse, Amadeus Reinald, and Ignasi Sau and is based on [41, 139, 136].

7.1 Introduction

7.1.1 Graph recolouring

Given a graph G, for any $k \geq \chi(G)$, recall that the *k-colouring graph* of G, denoted by $\mathcal{C}_k(G)$, is the graph whose vertices are the proper k-colourings of G and in which two k-colourings are adjacent if they differ on the colour of exactly one vertex. A walk between two given colourings in $\mathcal{C}_k(G)$ corresponds to a *recolouring sequence*, that is, a sequence of pairs composed of a vertex of G, which is going to receive a new colour, and a new colour for this vertex. If $\mathcal{C}_k(G)$ is connected, we say that G is *k-mixing*. An isolated vertex in $\mathcal{C}_k(G)$ is called a *k-frozen* proper colouring. One of the main problems in recolouring is to decide whether a given graph is k-mixing.

IS k-MIXING
Input: A graph G satisfying $\chi(G) \leq k$
Output: Is G k-mixing?

Note that IS 2-MIXING is trivial since only edgeless graphs are 2-mixing. On the other hand, Cereceda [48] proved that IS 3-MIXING is coNP-complete. For higher values of k, Bousquet [40] proved that IS k-MIXING is coNP-hard, but we do not know whether the problem belongs to coNP. Cereceda, van den Heuvel, and Johnson [48] conjectured that it is PSPACE-complete.

A related problem is that of deciding whether two given proper k-colourings of a graph G are in the same connected component of $\mathcal{C}_k(G)$.

k-COLOURING PATH
Input: A graph G along with two k-colourings α and β of G.
Output: Is there a recolouring sequence between α and β?

2-COLOURING PATH is trivial since only isolated vertices can be recoloured in a bipartite graph. Cereceda, van den Heuvel, and Johnson [49] proved that 3-COLOURING PATH is polynomial-time solvable. Moreover the authors proved that the diameter of each component of $\mathcal{C}_3(G)$ is at most $O(n(G)^2)$.

In contrast, for every $k \geq 4$, Bonsma and Cereceda [36] showed the existence of a family $\mathcal{G}_k$ of graphs such that for every $G \in \mathcal{G}_k$ of order n, there exist two proper k-colourings whose distance in $\mathcal{C}_k(G)$ is finite and superpolynomial in n. They also proved that k-COLOURING PATH is PSPACE-complete for all $k \geq 4$ even restricted to bipartite graphs. However, the situation is different for degenerate graphs. Bonsma and Cereceda [36] and Dyer et al. [67] independently proved the following.

Theorem 7.1.1 (Bonsma and Cereceda [36] ; Dyer et al. [67])**.** *Let $k \in \mathbb{N}$ and G be a graph. If $k \geq \delta^*(G) + 2$, then G is k-mixing.*

The original proof of Theorem 7.1.1 also implies that $\mathcal{C}_k$ has diameter at most $2^{n(G)}$. Cereceda's conjecture states that the diameter of $\mathcal{C}_k$ is actually quadratic in n.

Conjecture 7.1.2 (Cereceda [47])**.** *Let $k \in \mathbb{N}$ and G be a graph. If $k \geq \delta^*(G) + 2$, then the diameter of $\mathcal{C}_k(G)$ is at most $O(n^2)$.*

In the remainder of this section, we recall several results approaching this conjecture, which can be seen as evidences for the general conjecture. The following are the best existing bounds approaching Conjecture 7.1.2 in the general case*.

Theorem 7.1.3 (Bousquet and Heinrich [42])**.** *Let $k \in \mathbb{N}$ and G be a graph. Then $\mathcal{C}_k(G)$ has diameter at most:*

(i) $O(n^2)$ *if* $k \geq \frac{3}{2}(\delta^*(G) + 1)$,

(ii) $O_\varepsilon(n^{\lceil \frac{1}{\varepsilon} \rceil})$ *if* $k \geq (1 + \varepsilon)(\delta^*(G) + 1)$, *and*

(iii) $O_d(n^{d+1})$ *if* $k \geq \delta^*(G) + 2 = d + 2$.

*Given two computable functions f, g and a parameter Γ, $f(n) = O_\Gamma(g(n))$ means that there exists a computable function h such that $f(n) = O(h(\Gamma) \cdot g(n))$. Also Γ can be a sequence of parameters $\Gamma_1, \dots, \Gamma_r$ in which case $f(n) = O_\Gamma(g(n))$ means that $f(n) = O(h(\Gamma_1, \dots, \Gamma_r) \cdot g(n))$.

Bousquet and Perarnau [43] also proved the following.

Theorem 7.1.4 (Bousquet and Perarnau [43]). *Let $k \in \mathbb{N}$ and G be a graph. If $k \geq 2\delta^*(G) + 2$, then $\mathrm{diam}(\mathcal{C}_k(G)) \leq (\delta^*(G) + 1)n$.*

In order to obtain possibly simpler versions of Conjecture 7.1.2, one can restrict to graphs with bounded maximum average degree. Recall that the *maximum average degree* of a graph G is $\mathrm{Mad}(G) = \max\left\{\frac{2m(H)}{n(H)} \mid H \text{ subgraph of } G\right\}$. It is easy to see that every graph G satisfies $\lfloor \mathrm{Mad}(G) \rfloor \geq \delta^*(G)$. Hence, if true, Conjecture 7.1.2 would imply that every graph G with $\mathrm{Mad}(G) \leq d - \varepsilon$, where $d \in \mathbb{N}$ and $\varepsilon > 0$, satisfies $\mathrm{diam}(\mathcal{C}_k(G)) = O(n^2)$ for every $k \geq d + 1$. Feghali showed the following analogue result.

Theorem 7.1.5 (Feghali [73]). *Let d, k be integers such that $k \geq d+1$. For every $\varepsilon > 0$ and every graph G with n vertices and maximum average degree at most $d - \varepsilon$,*

$$\mathrm{diam}(\mathcal{C}_k(G)) = O_{d,\varepsilon}(n(\log n)^{d-1}).$$

Another approach toward Conjecture 7.1.2 consists of considering the maximum degree of a graph instead of its degeneracy. Cereceda has shown (see [47, Proposition 5.23]) that, for every graph G on n vertices and integer $k \geq \Delta(G) + 2$, $\mathcal{C}_k(G)$ has diameter at most $(\Delta(G) + 1)n$, where $\Delta(G)$ denotes the maximum degree of G. In order to get a more precise bound, Bonamy and Bousquet considered the grundy number. Let G be a graph and $\mathcal{O} = (x_1, \ldots, x_n)$ be an ordering of $V(G)$. The *greedy colouring* $\alpha_g(\mathcal{O}, G)$ is the proper colouring in which every vertex x_i receives the smallest colour that does not appear in $N(x_i) \cap \{x_1, \ldots, x_{i-1}\}$. The *grundy number* of G, denoted by $\chi_g(G)$, is the maximum, over all orderings $\mathcal{O}$, of the number of colours used in $\alpha_g(\mathcal{O}, G)$.

Theorem 7.1.6 (Bonamy and Bousquet [31]). *For any graph G on n vertices, if $k \geq \chi_g(G) + 1$, then G is k-mixing and $\mathrm{diam}(\mathcal{C}_k(G)) \leq 4 \cdot \chi(G) \cdot n$.*

Considering graphs of bounded maximum degree, Feghali, Johnson, and Paulusma [74] proved the following analogue of Brooks Theorem for graph recolouring. Note that it is of interest only when $k = \Delta + 1$, because Theorem 7.1.6 already gives a better bound when $k \geq \Delta + 2$.

Theorem 7.1.7 (Feghali, Johnson, and Paulusma [74]). *Let $G = (V, E)$ be a connected graph with $\Delta(G) = \Delta \geq 3$, $k \geq \Delta + 1$, and α, β two proper k-colourings of G. Then at least one of the following holds:*

- *α is k-frozen, or*
- *β is k-frozen, or*
- *there is a recolouring sequence of length at most $c_\Delta |V|^2$ between α and β, where $c_\Delta = O(\Delta)$ is a constant linear on Δ.*

A last result approaching Conjecture 7.1.2 is due to Bonamy and Bousquet, and makes a connection between the treewidth of a graph and its recolourability. It is easy to see that, for any graph G, $\mathrm{tw}(G) \geq \delta^*(G)$. Hence, the following result is a weaker version of Conjecture 7.1.2.

Theorem 7.1.8 (Bonamy and Bousquet [31]). *Let $k \in \mathbb{N}$ and G be a graph of order n. If $k \geq \mathrm{tw}(G) + 2$, then $\mathrm{diam}(\mathcal{C}_k(G)) \leq 2(n^2 + n)$.*

Bonamy et al. [33] have shown that the bound of Theorem 7.1.8 is asymptotically sharp (up to a constant factor).

7.1.2 Analogues for digraphs

In this chapter, we study the notion of digraph redicolouring, which is a generalisation of graph recolouring. We first give and recall some definitions specific to this chapter. For any digraph D and integer $k \geq \vec{\chi}(D)$, the *k-dicolouring graph* of D, denoted by $\mathcal{D}_k(D)$, is the undirected graph whose vertices are the k-dicolourings of D and in which two k-dicolourings are adjacent if they differ on exactly one vertex. Observe that $\mathcal{C}_k(G) = \mathcal{D}_k(\overleftrightarrow{G})$ for any bidirected graph $\overleftrightarrow{G}$. A *redicolouring sequence* between two k-dicolourings is a walk between these dicolourings in $\mathcal{D}_k(D)$. The digraph D is *k-mixing* if $\mathcal{D}_k(D)$ is connected, and *k-freezable* if $\mathcal{D}_k(D)$ contains an isolated vertex, and such an isolated vertex is called a *k-frozen* dicolouring. A vertex v is *blocked* to its colour in a k-dicolouring α if, for every colour $c \in [k]$ different from $\alpha(v)$, recolouring v to c in α creates a monochromatic directed cycle. We say that v is *frozen* in α if $\beta(v) = \alpha(v)$ for any k-dicolouring β in the same connected component of α in $\mathcal{D}_k(D)$.

In Section 7.2, we consider the directed analogues of IS k-MIXING and k-COLOURING PATH.

DIRECTED IS k-MIXING
Input: A digraph D.
Output: Is D k-mixing?

k-DICOLOURING PATH
Input: A digraph D along with two k-dicolourings α and β of D.
Output: Is there a path between α and β in $\mathcal{D}_k(D)$?

Note that IS k-MIXING and k-DICOLOURING PATH may be seen as the restrictions of DIRECTED IS k-MIXING and k-DICOLOURING PATH to bidirected graphs. Therefore hardness results transfer to those problems. It follows that DIRECTED IS k-MIXING is coNP-hard for all $k \geq 3$ and k-DICOLOURING PATH is PSPACE-complete for all $k \geq 4$. We strengthen this second result in Section 7.2 by proving that 2-DICOLOURING PATH is PSPACE-complete, and that k-DICOLOURING PATH remains PSPACE-complete when restricted to some digraph classes.

Given a digraph $D = (V, A)$ and a vertex $v \in V$, recall that the *cycle-degree* of v, denoted by $d_c(v)$, is the minimum size of a set $S \subseteq (V \setminus \{v\})$ such that S intersects every directed cycle of D containing v. The *minimum cycle-degree* of D, denoted by $\delta_c(D)$, corresponds to $\min\{d_c(v) \mid v \in V\}$. The *c-degeneracy* of D is defined as $\delta_c^*(D) = \max\{\delta_c(H) \mid H \subseteq D\}$. As we will see in Section 7.5, the notion of c-degeneracy appears to be a natural generalisation of the undirected degeneracy when dealing with directed treewidth.

We define the *average cycle-degree* of D, denoted by $\mathrm{Ad}_c(D)$, as $\frac{1}{n(D)} \sum_{v \in V} d_c(v)$. The *maximum average cycle-degree* of D, denoted by $\mathrm{Mad}_c(D)$, is defined as $\max_{H \subseteq D}(\mathrm{Ad}_c(H))$. It follows from the definitions that $\delta_c^*(D) \leq \lfloor \mathrm{Mad}_c(D) \rfloor$ holds for every digraph D.

Let us show that $\vec{\chi}(D) \leq \delta_c^*(D) + 1$. By definition of c-degeneracy, we can find an ordering $v_1, \dots, v_n$ of V such that, for every $i \in [n]$, there exists $S_i \subseteq \{v_{i+1}, \dots, v_n\}$ of size at most $\delta_c^*(D)$, and such that $S_i \cup \{v_1, \dots, v_{i-1}\}$ intersects every directed cycle of D containing v_i. From now on, such an ordering will be called a *c-degeneracy ordering*. Hence, considering the vertices from v_n to v_1, one can greedily find a $(\delta_c^*(D) + 1)$-dicolouring of D by colouring each v_i with a colour that has not been chosen in S_i. Indeed, suppose for contradiction that, for some $i \in [\delta_c^*(D) + 1]$, the subdigraph of D induced by the set of vertices assigned colour i contains a directed cycle C. Let v_j be the leftmost vertex in C according to the considered ordering. Then, by definition of c-

degeneracy we have that $(V(C) \setminus \{v_j\}) \cap S_j \neq \varnothing$, but this contradicts the fact that, by construction of the colouring, the colour of v_j is different from all the colours of the vertices in S_j.

The following is a stronger version of Conjecture 7.1.2.

Conjecture 7.1.9. *Let $k \in \mathbb{N}$ and D be a digraph. If $k \geq \delta_c^*(D) + 2$, then the diameter of $\mathcal{D}_k(D)$ is at most $O(n^2)$.*

In Section 7.3 we show that Theorems 7.1.1, 7.1.3, 7.1.4 and 7.1.5 can be generalised to digraphs using the c-degeneracy.

In Section 7.4 we look at digraphs of bounded maximum degree. We show that both Theorems 7.1.6 and 7.1.7 can be generalised to digraphs. When restricted to oriented graphs, we improve the quadratic bound of Theorem 7.1.7 into a linear bound.

In Section 7.5 we look for extensions of Theorem 7.1.8 to digraphs. We first give a general result which makes a connection between the redicolourability of a digraph and the recolourability of its underlying graph. In particular, it extends Theorem 7.1.8 when taking the treewidth of the underlying graph. This is not completely satisfying, since this notion does not take under consideration the orientations in the digraph. Using a generalisation of treewidth to digraphs, namely the $\mathscr{D}$-width, we give a more precise generalisation of Theorem 7.1.8.

In Section 7.6, we turn our focus to the density of non-mixing graphs and digraphs. We first provide a construction witnessing that there exist $(k-1)$-regular graphs of arbitrarily large girth that are not k-mixing, which was first shown by Bonamy, Bousquet, and Perarnau [32] using probabilistic arguments. Therefore, the upper bound on the minimum density of non-mixing graphs derived from Theorem 7.1.1 is best possible even on graphs of arbitrarily large girth. However, this is not the case for digraphs with arbitrary large digirth. In fact, this is not even the case for oriented graphs, which are exactly digraphs with digirth at least 3. We pose a conjecture on the minimum density of non-mixing oriented graphs and provide some support for it.

We conclude in Section 7.7 by discussing the consequences of our results, especially on planar digraph redicolouring, and detail a few related open questions.

7.2 Complexity of k-DICOLOURING PATH

In this section, we establish some hardness results for k-DICOLOURING PATH. We will mainly prove the following result.

Theorem 7.2.1. *Each of the following holds.*

(i) *For every $k \geq 2$, k-*DICOLOURING PATH *is PSPACE-complete on digraphs with maximum degree $2k+1$.*

(ii) *For every $k \geq 2$, k-*DICOLOURING PATH *is PSPACE-complete on oriented graphs.*

(iii) *For every $2 \leq k \leq 4$, k-*DICOLOURING PATH *is PSPACE-complete on planar digraphs with maximum degree $2k+2$.*

(iv) 2-DICOLOURING PATH *is PSPACE-complete on oriented planar graphs of maximum degree* 6.

We first need some definitions. Given a graph $G = (V, E)$ together with a mapping $\phi\colon V \to \{1,2\}$, an orientation $\vec{G}$ of G is *proper* if for any $v \in V$, $d^-_{\vec{G}}(v) \geq \phi(v)$. A *reorienting sequence* from $\vec{G}_1$ to $\vec{G}_2$ is a sequence of proper orientations $\vec{\Gamma}_1, \dots, \vec{\Gamma}_p$ of G such that $\vec{\Gamma}_1 = \vec{G}_1$, $\vec{\Gamma}_p = \vec{G}_2$, every $\vec{\Gamma}_i$ is proper and every $\vec{\Gamma}_{i+1}$ can be obtained from $\vec{\Gamma}_i$ by reversing exactly one arc. The following problem has been shown to be PSPACE-complete in [98] by a reduction from Quantified Boolean Formulas.

PLANAR-CUBIC-NCL **Input:** A cubic planar graph G, a mapping $\phi\colon V \to \{1,2\}$, two proper orientations $\vec{G}_1$ and $\vec{G}_2$ of G. **Output:** Is there a reorienting sequence from $\vec{G}_1$ to $\vec{G}_2$?

We will derive the hardness results for DICOLOURING PATH from a hardness result on its list dicolouring version. Let D be a digraph. Recall that a *list assignment* L is a function which associates a list of colours to every vertex v of D. An *L-dicolouring* of D is a dicolouring α of D such that $\alpha(v) \in L(v)$ for all vertex v. A *k-list assignment* is a list assignment L such that $L(v) \subseteq [k]$ for all vertex v. We denote by $\mathcal{D}(D, L)$ the reconfiguration graph of the L-dicolourings of D, *i.e.* the undirected graph in which the vertices are the L-dicolourings of D and two colourings are adjacent if they differ on the colour of exactly one vertex. We call vertices v such that $|L(v)| = 1$ *forced vertices*. We will consider the following problem.

k-LIST DICOLOURING PATH **Input:** A digraph D, a k-list assignment L, and two L-dicolourings α and β of D. **Output:** Is there a path between α and β in $\mathcal{D}(D, L)$?

Let us start by proving the following result.

Theorem 7.2.2. 2-LIST DICOLOURING PATH *is PSPACE-complete on digraphs D even when:*

- *forced vertices have degree at most* 3 *and,*
- *either all the vertices have degree at most* 5 *or, the digraph D is planar and all the vertices have in and out-degree at most* 3.

Proof. First, note that 2-LIST DICOLOURING PATH is indeed in NPSPACE. Given a digraph D and two dicolourings α and β of D together with a redicolouring sequence from α to β, we can easily check with a polynomial amount of space that each dicolouring is valid. Then, we get that k-LIST DICOLOURING PATH belongs to PSPACE thanks to Savitch's Theorem [151], which asserts that PSPACE = NPSPACE.

We shall now give a polynomial reduction from PLANAR-CUBIC-NCL.

Let G be a planar cubic graph on n vertices $x_1, \dots, x_n$ with a mapping $\phi\colon V(G) \to \{1,2\}$. Let $\vec{G}_1$ and $\vec{G}_2$ be two proper orientations of G. From (G, ϕ) we construct the digraph D and the function L as follows (see Figure 7.1 for an illustration).

- For each vertex $x_i \in V(G)$, we create a *vertex-gadget* as follows.

  We associate three vertices $x_{i,1}, x_{i,2}$ and $x_{i,3}$ in V_D so that each of these vertices is associated to exactly one edge of G incident to x_i, and each edge of G is associated to exactly two vertices of D. The function L assigns to each of these vertices the list $\{1,2\}$.

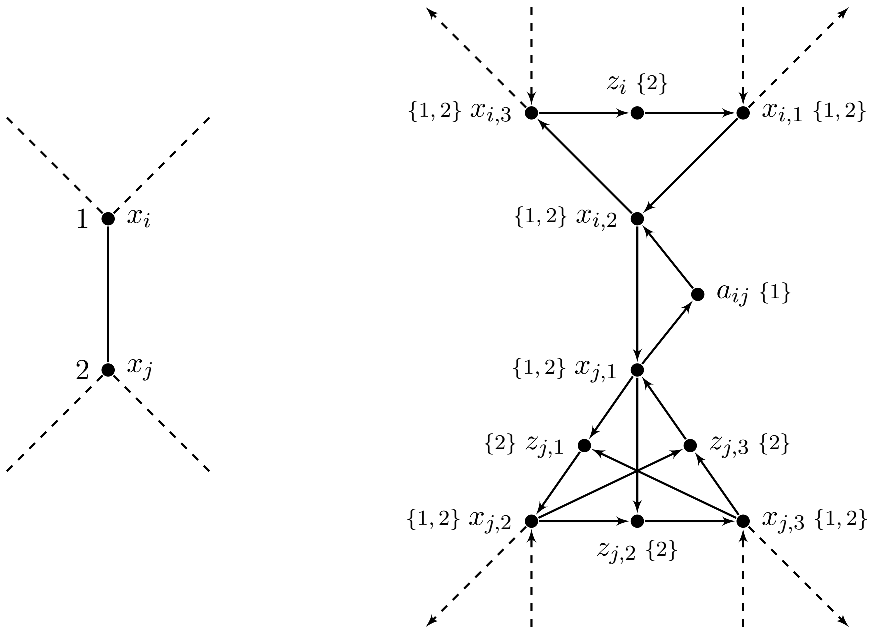


Figure 7.1: An example of building (D, L) from (G, ϕ), where $\phi(x_i) = 1$, $\phi(x_j) = 2$, and $i < j$.

We complete the vertex-gadget in two ways, depending on the value of $\phi(x_i)$. If $\phi(x_i) = 1$, then we create a new vertex z_i in such a way that $(z_i, x_{i,1}, x_{i,2}, x_{i,3}, z_i)$ is a directed 4-cycle. We set $L(z_i) = \{2\}$.

If $\phi(x_i) = 2$, then we create three new vertices $z_{i,1}, z_{i,2}$, and $z_{i,3}$ in such a way that $(x_{i,1}, z_{i,1}, x_{i,2}, z_{i,2}, x_{i,3}, z_{i,3}, x_{i,1})$ is a directed 6-cycle, and we add the arcs $x_{i,1}z_{i,2}$, $x_{i,2}z_{i,3}$ and $x_{i,3}z_{i,1}$. The function L assigns the list $\{2\}$ to $z_{i,1}$, $z_{i,2}$, and $z_{i,3}$.

- For each edge $x_ix_j \in E(G)$, where $i < j$, we create a vertex a_{ij} and create the directed 3-cycle $(a_{ij}, x_{i,r}, x_{j,r'}, a_{ij})$, where $x_{i,r}$ and $x_{j,r'}$ are the vertices of D corresponding to the edge x_ix_j in G. We set $L(a_{i,j}) = \{1\}$. This directed 3-cycle is the *edge-gadget* of x_ix_j.

Note that the digraph D has maximum degree at most 5 and that its forced vertices have degree at most 3.

For every proper orientation $\vec{G}$ of G, we define the *dicolouring associated to* $\vec{G}$, a particular dicolouring of D that we denote by $\alpha_{\vec{G}}$, as follows.

- Forced vertices are assigned the colour of their list: vertices $z_i, z_{i,1}, z_{i,2}, z_{i,3}$ are coloured 2, and vertices a_{ij} are coloured 1.
- For each arc $x_ix_j \in A(\vec{G})$, we set $x_{i,r}$ to colour 2 and $x_{j,r'}$ to colour 1, where $x_{i,r}$ and $x_{j,r'}$ are the vertices of D corresponding to the edge x_ix_j in G.

Claim 7.2.3. *For every proper orientation $\vec{G}$ of G and corresponding digraph D, the following hold.*

(i) $\alpha_{\vec{G}}$ is an L-dicolouring of D.

(ii) Unless $d^-_{\vec{G}}(x_i) = \phi(x_i)$ in $\vec{G}$, changing the colour of $x_{i,r}$ from 1 to 2 still yields a valid L-dicolouring of D.

Proof of claim. Let us only show the first item, the second follows by similar arguments noting that $d^-_{\vec{G}}(x_i) > \phi(x_i)$ in this case. Note that, by definition, $\alpha_{\vec{G}}$ satisfies the colouring constraints imposed by the list assignment L. Let us then show that $\alpha_{\vec{G}}$ is indeed an L-dicolouring. Assume, for a contradiction, that there is a monochromatic directed cycle C in $\alpha_{\vec{G}}$. For every edge-gadget, say corresponding to x_ix_j, the vertices $x_{i,r}, x_{j,r'}$ are coloured differently. Therefore, all vertices of C must be contained in a single vertex-gadget of D. Let x_i be the vertex such that C is in the vertex-gadget of x_i. If $\phi(x_i) = 1$, then C must be $(z_i, x_{i,1}, x_{i,2}, x_{i,3}, z_i)$ and all its vertices should be coloured 2. This is a contradiction since $d^-_{\vec{G}}(x_i) \geq 1$ implies, by construction, that at least one of $x_{i,1}, x_{i,2}, x_{i,3}$ is coloured 1. If $\phi(x_i) = 2$, then C contains at least two vertices in $\{x_{i,1}, x_{i,2}, x_{i,3}\}$ and two vertices in $\{z_{i,1}, z_{i,2}, z_{i,3}\}$. Then, at least two vertices of $\{x_{i,1}, x_{i,2}, x_{i,3}\}$ are coloured 2 because vertices $z_{i,j}$ are. This is a contradiction since $d^-_{\vec{G}}(x_i) \geq 2$ implies, by construction, that at least two vertices of $x_{i,1}, x_{i,2}, x_{i,3}$ are coloured 1. $\Diamond$

Let us take $\alpha_1 = \alpha_{\vec{G}_1}$ and $\alpha_2 = \alpha_{\vec{G}_2}$ to be the dicolourings obtained from the two proper orientations $\vec{G}_1, \vec{G}_2$. We will now show that there exists a reorienting sequence in G from $\vec{G}_1$ to $\vec{G}_2$ if and only if there exists a redicolouring sequence in D from α_1 to α_2.

Assume first that there is a reorienting sequence $\vec{\Gamma}_1, \dots, \vec{\Gamma}_p$ from $\vec{G}_1$ to $\vec{G}_2$, and let us show how to build a corresponding redicolouring sequence. Consider any step s of the reorienting sequence, say when $\vec{\Gamma}_s$ is transformed into $\vec{\Gamma}_{s+1}$ by reversing an arc x_ix_j into x_jx_i. We will exhibit a path from $\alpha_{\vec{\Gamma}_s}$ to $\alpha_{\vec{\Gamma}_{s+1}}$ in $\mathcal{D}(D, L)$. Consider vertices $x_{i,r}, x_{j,r'}$ in D, corresponding to the edge x_ix_j, and coloured 2 and 1 respectively in $\alpha_{\vec{\Gamma}_s}$. We first set the colour of $x_{j,r'}$ from 1 to 2. Since $a_{i,j}$ is forced to colour 1, the edge-gadget is not monochromatic at this point. Moreover, since step s reorients arc x_ix_j and still yields a proper orientation, $d^-_{\vec{\Gamma}_s}(x_j) = d^-_{\vec{\Gamma}_{s+1}}(x_j) + 1 \geq \phi(x_j) + 1$. The resulting colouring is an L-dicolouring by Claim 7.2.3 (ii). We then set the colour of $x_{i,r}$ from 2 to 1, yielding dicolouring $\alpha_{\vec{\Gamma}_{s+1}}$. Concatenating the redicolouring sequences obtained through the process above from steps $s = 1$ to $s = p$ yields a redicolouring sequence from α_1 to α_2.

Conversely, assume that there is a redicolouring sequence $\gamma_1, \dots, \gamma_p$ from α_1 to α_2. Observe that the only vertices of D that are possibly recoloured in a step of our sequence are those defined as $x_{i,k}$ for $i \in [n]$ and $k \in [3]$, since all others are forced. Now, at any step s of the redicolouring, for each edge x_ix_j of G, at most one of the two corresponding vertices is coloured 1 because a_{ij} is forced to colour 1. This allows us to define an orientation $\vec{\Gamma}_s$ of G as follows. If the vertices $x_{i,r}, x_{j,r'} \in V(D)$, corresponding to the gadget of edge x_ix_j, are not coloured the same in γ_s, orientation $\vec{\Gamma}_s$ sets x_ix_j to be directed from the vertex coloured 2 towards the vertex coloured 1. Otherwise, both vertices are coloured 2 in γ_s and we preserve the orientation of the corresponding edge given by $\vec{\Gamma}_{s-1}$. In the first and last dicolourings, α_1 and α_2, for each edge x_ix_j the corresponding vertices $x_{i,k}$ and $x_{j,k'}$ are coloured differently. Thus $\vec{\Gamma}_1 = \vec{G}_1$ and $\vec{\Gamma}_p = \vec{G}_2$. Therefore, $\vec{\Gamma}_1, \dots, \vec{\Gamma}_p$ is a sequence of orientations of G from $\vec{G}_1$ to $\vec{G}_2$ such that $\vec{\Gamma}_{s+1}$ is either obtained by reversing an arc of $\vec{\Gamma}_s$ (when one of the $x_{i,k}$ is recoloured to 1, and the edge x_ix_j whose edge-gadget contains $x_{i,k}$ was not oriented towards x_i), or equal to $\vec{\Gamma}_s$ otherwise (and in particular when one of the $x_{i,k}$ is recoloured to 2). Moreover, at each step s, $\vec{\Gamma}_s$ is a proper orientation of G. Indeed, if $\phi(x_i) = 1$ (resp. $\phi(x_i) = 2$), then at least one vertex (resp. two vertices) of $\{x_{i,1}, x_{i,2}, x_{i,3}\}$ is coloured 1, and so x_i has in-degree at least 1 (resp. at least 2) in $\vec{\Gamma}_s$. Hence,

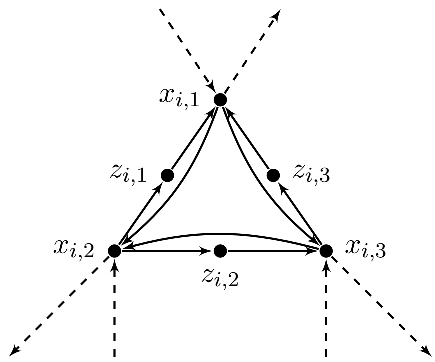


Figure 7.2: A planar vertex-gadget when $\phi(x_i) = 2$

taking the subsequence of $\vec{\Gamma}_1, \dots, \vec{\Gamma}_p$ that omits constant steps yields a reorienting sequence from $\vec{G}_1$ to $\vec{G}_2$.

Since PLANAR-CUBIC-NCL is PSPACE-complete, at this point our reduction already yields the PSPACE-completeness of 2-LIST DICOLOURING PATH. By construction, forced vertices $z_{j,k}$ and a_{ij} have degree at most 3, and all other vertices have degree at most 5. This achieves the proof of the first case of the result.

Now, since the input instances are already planar, to get the PSPACE-completeness of 2-LIST DICOLOURING PATH on planar digraphs, it suffices to use planar vertex and edge gadgets. In our current reduction, the only gadget which is not planar is the vertex-gadget corresponding to $x_i \in V(G)$ such that $\phi(x_i) = 2$. We now consider the same reduction, replacing the vertex-gadget for vertices such that $\phi(x_i) = 2$ with a planar one. For these vertices, the planar vertex-gadget is defined on the same set of vertices, i.e. $\{x_{i,1}, x_{i,2}, x_{i,3}, z_{i,1}, z_{i,2}, z_{i,3}\}$, but with the arcs of the directed 3-cycles $(z_{i,1}, x_{i,1}, x_{i,2}, z_{i,1})$, $(z_{i,3}, x_{i,1}, x_{i,3}, z_{i,3})$ and $(z_{i,2}, x_{i,3}, x_{i,2}, z_{i,2})$, as depicted in Figure 7.2. This replacement produces a planar digraph in which all forced vertices still have degree at most 3, and all vertices have maximum in- and out-degree at most 3. This completes the proof. □

The problem k-COLOURING PATH is known to be PSPACE-complete for every $k \geq 4$ in the undirected case [36]. Leveraging Theorem 7.2.2, we prove that this also holds for its dicolouring analogue for $k \geq 2$ colours, in both directed and oriented graphs.

Theorem 7.2.1. *Each of the following holds.*

(i) *For every $k \geq 2$, k-*DICOLOURING PATH *is PSPACE-complete on digraphs with maximum degree $2k + 1$.*

(ii) *For every $k \geq 2$, k-*DICOLOURING PATH *is PSPACE-complete on oriented graphs.*

(iii) *For every $2 \leq k \leq 4$, k-*DICOLOURING PATH *is PSPACE-complete on planar digraphs with maximum degree $2k + 2$.*

(iv) 2-DICOLOURING PATH *is PSPACE-complete on oriented planar graphs of maximum degree* 6.

Proof. (i) We give a reduction from 2-LIST DICOLOURING PATH on instances where forced vertices have degree at most 3 and the graph has maximum degree 5. The problem is PSPACE-complete by Theorem 7.2.2. Let $(D, L, \alpha_1, \alpha_2)$ be an instance of the problem, we construct an instance $(D', \alpha_1', \alpha_2')$ for k-DICOLOURING PATH as follows.

We build D' starting with $D' = D$. Then, for every vertex $v \in V(D)$, we let $\overleftrightarrow{K_k^v}$ be a bidirected complete graph on vertex-set $\{z_i^v \mid i \in [k]\}$. We then add a digon between v and each z_i^v such that $i \notin L(v)$. We define dicolourings α_1' and α_2' on D' by extending dicolourings α_1 and α_2 as follows. All vertices of D' that were vertices of D are coloured the same, and we set z_i^v to colour i for all $v \in V(D)$ and all $i \in [k]$. Note that all the vertices of gadget $\overleftrightarrow{K_k^v}$ are then frozen in any k-dicolouring, letting us simulate in D' the list dicolouring constraints on D. An L-dicolouring path from α_1 to α_2 in D is then exactly a dicolouring path from α_1' to α_2' in D' restricted to vertices of D, achieving equivalence between the instances.

We will now show that the maximal degree of a vertex u in D' is $2k+1$. If u belongs to some gadget $\overleftrightarrow{K_k^v}$, then its degree is at most $2(k-1)+2 = 2k$. Note that when $u \in V(D)$, u is of degree $2, 3$ or 5 in D, and our reduction adds exactly $2(k - |L(u)|)$ arcs incident to u. If $|L(u)| = 2$, this yields $d_{D'}(u) \leq 5 + 2k - 4 = 2k + 1$. If $|L(u)| = 1$, we know by construction that $d_D(u) \leq 3$, yielding $d_{D'}(u) \leq 3 + 2k - 2 = 2k + 1$. This achieves the proof that D' has maximum degree at most $2k + 1$, concluding (i).

(ii) We give a reduction from k-DICOLOURING PATH to k-DICOLOURING PATH restricted to oriented graphs. Let (D, α_1, α_2) be an instance of k-DICOLOURING PATH, we will build an equivalent instance $(\vec{G}, \alpha_1', \alpha_2')$ where $\vec{G}$ is an oriented graph. Take $\vec{H}$ to be an arbitrary oriented graph with dichromatic number exactly k. We construct $\vec{G}$ from D by replacing digons of D as follows. For each digon $[u, v]$ of D, create a copy $\vec{H}_{uv}$ of $\vec{H}$, then replace $[u, v]$ by a single arc from u to v, and add all arcs from v to $\vec{H}_{uv}$ and all arcs from $\vec{H}_{uv}$ to u. By construction, $\vec{G}$ is an oriented graph.

In the following, we let ξ be a fixed k-dicolouring of $\vec{H}$. We show how to transform any k-dicolouring α of D to a k-dicolouring α' of $\vec{G}$, and vice versa. Given a k-dicolouring α for D, we define α' for $\vec{G}$ by colouring each copy $\vec{H}_{uv}$ of $\vec{H}$ with ξ, and keeping the same colours as α on $V(D)$. Any monochromatic directed cycle in $(\vec{G}, \alpha')$ must contain a vertex of some $\vec{H}_{uv}$, as otherwise it would be a subdigraph of D and would already be monochromatic in (D, α). Since ξ is a dicolouring of $\vec{H}$, the cycle must contain both u, v, but then u and v being coloured the same would yield a monochromatic digon in (D, α), so α' is indeed a k-dicolouring of $\vec{G}$. Conversely, given any k-dicolouring α' of $\vec{G}$, we define α for D as the restriction of α' on $V(D)$. Similarly, if (D, α) were to contain a monochromatic directed cycle, any arc (u, v) of the cycle that is not present in $\vec{G}$ may be replaced with (u, w, v), taking $w \in \vec{H}_{uv}$ to be a vertex of the same colour as u and v (since $\vec{\chi}(\vec{H}) = k$). This would yield a monochromatic directed cycle in $(\vec{G}, \alpha')$, so α must be a k-dicolouring of D.

Now, we define the k-dicolourings α_1', α_2' on $\vec{G}$ obtained from α_1, α_2 by the transformation above, and let our output instance be $(\vec{G}, \alpha_1', \alpha_2')$. If there is a redicolouring sequence from α_1 to α_2 in D, we perform the same recolouring steps in $\vec{G}$ starting from α_1' and yielding α_2'. Since we only recolour vertices of $V(D)$, the last paragraph yields that this sequence is valid. Conversely, if there is a redicolouring sequence from α_1' to α_2', its restriction to $V(D)$ (omitting recolourings of vertices in subgraphs $\vec{H}_{uv}$) yields a valid sequence from α_1 to α_2 in D. This achieves the proof of the equivalence of the instances and proves (ii).

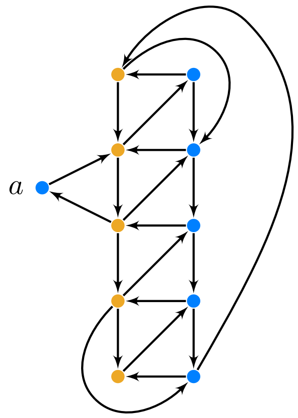


Figure 7.3: How to freeze the vertex a in an oriented planar graph with two colours.

(iii) We give a reduction from 2-LIST DICOLOURING PATH where D is planar, forced vertices have degree at most 3 and D has in- and out-degree at most 3. The problem is PSPACE-complete by Theorem 7.2.2. Let $(D, L, \alpha_1, \alpha_2)$ be an instance of the problem. We make the same reduction as in the proof of (i) by ensuring that $\overleftrightarrow{K_k^b}$ is embedded and coloured in such a way that (at most 3) forbidden colours of v lie on its external face. This allows us to keep a planar representation of D' which has maximum degree $2k + 2$. This proves (iii).

(iv) As in (iii), we give a reduction from 2-LIST DICOLOURING PATH where D is planar, forced vertices have degree at most 3 and every vertex has in- and out-degree at most 3. Since we are considering 2-dicolourings, vertices with a list of size 2 do not require a gadget to simulate forbidden colours. The main difference with case (iii) is that we cannot use a bidirected complete graph $\overleftrightarrow{K_2^b}$ to freeze a vertex v with list of size 1. To overcome this, we use the gadget depicted in Figure 7.3, where the colour of all vertices is frozen. Therefore, we simply have to attach such a gadget on each vertex with list of size one, that is, those of the form $z_{i,r}$ or $a_{i,j}$. This can be done by creating a directed triangle including the vertex and two vertices of the opposite colour in the gadget, as depicted in the figure. $\square$

7.3 On a directed version of Cereceda's conjecture

In this section, we establish some connections between the redicolourability of a digraph and its c-degeneracy. Note that, together with Bousquet, Havet, Nisse, and Reinald, we first proved some of these results in [41] with the min-degeneracy, before strengthening them together with Nisse and Sau when we introduced c-degeneracy in [136]. The min-*degeneracy* $\delta^*_{\min}(D)$ of a digraph D is defined as the least integer ℓ such that every subdigraph H of D contains a vertex v satisfying $d_{\min}(v) \leq \ell$. Observe that for every digraph D and every vertex $v \in V(D)$, we have $d_c(v) \leq d_{\min}(v)$ because both the in-neighbourhood and the out-neighbourhood of D intersect all the directed cycles of D containing v. Therefore, we have $\delta^*_c(D) \leq \delta^*_{\min}(D)$ for every digraph D.

The results in this section provide support to Conjecture 7.1.9 that we recall here for convenience.

$k \geq$	$d+2$	$\frac{3}{2}(d+1)$	$2(d+1)$	$\lfloor \mathrm{Mad}_c \rfloor + 2$
$\mathrm{diam}(\mathcal{D}_k(D))$	$O_d\left(n^{d+1}\right)$	$O\left(n^2\right)$	$\leq (d+1)n$	$O_{\mathrm{Mad}_c,\varepsilon}\left(n(\log n)^{\lfloor \mathrm{Mad}_c \rfloor}\right)$
Theorem	7.3.2	7.3.9	7.3.13	7.3.3

Table 7.1: Bounds on the diameter of $\mathcal{D}_k(D)$ where D is a digraph on n vertices with c-degeneracy d, maximum average cycle-degree Mad_c and where $\varepsilon = \lfloor \mathrm{Mad}_c \rfloor + 1 - \mathrm{Mad}_c$.

Conjecture 7.1.9. *Let $k \in \mathbb{N}$ and D be a digraph. If $k \geq \delta_c^*(D) + 2$, then the diameter of $\mathcal{D}_k(D)$ is at most $O(n^2)$.*

Recall that Theorem 7.1.1 states that every graph G is k-mixing when $k \geq \delta^*(G) + 2$. We first generalise this result by proving the following.

Theorem 7.3.1. *Let D be a digraph and $k \in \mathbb{N}$ be such that $k \geq \delta_c^*(D) + 2$. Then D is k-mixing.*

Proof. The proof is by induction on $n = n(D)$. The result clearly holds for $n \leq 1$. Let us assume that $n > 1$ and that the result holds for $n - 1$. Let α, β be any two k-dicolourings of D and let $v \in V$ be a vertex satisfying $d_c(v) \leq \delta_c^*(D)$. Let α', β' be the two k-dicolourings induced, respectively, by α and β on $D - \{v\}$. By induction, there exists a redicolouring sequence $\alpha' = \alpha'_1, \dots, \alpha'_q = \beta'$ where α'_i and α'_{i-1} differ by the colour of exactly one vertex $v_i \in V \setminus \{v\}$, for every $1 < i \leq q$.

Now, we build the following redicolouring sequence from α to β. At step i, if v_i can be recoloured as from α'_{i-1} to α'_i, then recolour v_i accordingly. Otherwise, this implies that there exists a directed cycle containing v and v_i whose all vertices (but v_i) have colour $\alpha'_i(v_i)$. By definition of cycle-degree, there exists a transversal X of the directed cycles containing v, with $|X| \leq \delta_c^*(D)$ and $v \notin X$. Let $c \neq \alpha'_i(v_i)$ be a colour that does not appear in X (it exists since $k \geq \delta_c^*(D) + 2$). Colour v with c and then v_i with $\alpha'_i(v_i)$.

Finally (after step q), recolour v with its final colour $\beta(v)$ to obtain a redicolouring sequence from α to β. ☐

Let D be a digraph on n vertices and $k \geq \delta_c^*(D) + 2$ be an integer. Note that the proof of Theorem 7.3.1 above, inspired from the original proof of Theorem 7.1.1, also shows that $\mathcal{D}_k(D)$ has diameter at most 2^n. In the reminder of this section, we will give different polynomial bounds on the diameter of $\mathcal{D}_k(D)$, depending on how large k is compare to the c-degeneracy of D. In particular, we will generalise Theorems 7.1.3(i), 7.1.3(iii), 7.1.4 and 7.1.5. These bounds are summarised in Table 7.1.

In Subsection 7.3.1 we use some ideas introduced in [73] to generalise both Theorems 7.1.3(iii) and 7.1.5. Then we will generalise Theorem 7.1.3(i) in Subsection 7.3.2. The proof will use the result of Bousquet and Heinrich as a black box. Finally in Subsection 7.3.3 we give a generalisation of Theorem 7.1.4, the proof of which is based on the original proof of Bousquet and Perarnau.

7.3.1 Bounds on the diameter of $\mathcal{D}_k(D)$ when $k \geq \delta_c^*(D) + 2$

This section is devoted to the proofs of the two following generalisations of Theorems 7.1.3(iii) and 7.1.5.

Theorem 7.3.2. *Let D be a digraph and $k \geq \delta_c^*(D) + 2 = d + 2$, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O_d(n^{d+1}).$$

Theorem 7.3.3. *Let $d \geq 1$ and $k \geq d + 1$ be two integers, and let $\varepsilon > 0$. If D is a digraph satisfying $\operatorname{Mad}_c(D) \leq d - \varepsilon$, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O_{d,\varepsilon}(n(\log n)^{d-1}).$$

Observe that every digraph D satisfies $\operatorname{Mad}_c(D) \leq \frac{1}{2}\operatorname{Mad}(D)$. This holds because, for every vertex $v \in V(D)$, $d_c(v) \leq \frac{1}{2}(d^+(v) + d^-(v))$. Hence, for every subdigraph H of D, we have:

$$2m(H) = \sum_{v \in V(H)} (d^+(v) + d^-(v)) \geq 2 \sum_{v \in V(H)} d_c(v) = 2 \cdot \operatorname{Ad}_c(H) \cdot n(H).$$

Thus the following is a direct consequence of Theorem 7.3.3.

Corollary 7.3.4. *Let $d \geq 1, k \geq \lfloor \frac{d+3}{2} \rfloor$ be two integers, and let $\varepsilon > 0$. If D is a digraph satisfying $\operatorname{Mad}(D) \leq d - \varepsilon$, then $\mathcal{D}_k(D)$ has diameter at most:*

(i) $O_d\left(n(\log n)^{\frac{d-1}{2}}\right)$ *if d is odd, and*

(ii) $O_{d,\varepsilon}\left(n(\log n)^{\frac{d-2}{2}}\right)$ *otherwise.*

In the remaining of this section, $f, g \colon \mathbb{N}^2 \to \mathbb{N}$ are the functions defined as $f(s,t) = (s+1)!(2t)^s$ and $g(s,t) = 2sf(s,t) + 2s + 1$ respectively. One can show the following using elementary calculus.

Proposition 7.3.5. *For every $s, t \in \mathbb{N}$, $s \neq 0$, the following inequalities hold:*

$$f(s,t) \geq \sum_{q=1}^{t} (2(s+1)f(s-1,q)). \tag{7.1}$$

$$g(s,t) \geq 2f(s,t) + 2 + g(s-1,t). \tag{7.2}$$

$$g(s,t) = O_s(t^s). \tag{7.3}$$

We now prove the following main lemma[†].

Lemma 7.3.6. *Let $D = (V, A)$ be a digraph, $(V_1, \ldots, V_t)$ be a partition of V, and $s \geq 0$, $k \geq s + 2$ be two integers. Let $h \in [t]$ be such that, for every $p \leq h$ and every $u \in V_p$, there exists $X_u \subseteq \bigcup_{i=p+1}^{t} V_i$ such that $|X_u| \leq s$ and X_u intersects every directed cycle containing u in $D - \bigcup_{i=1}^{p-1} V_i$.*

Then, for every k-dicolouring α of D and for any colour $c \in [s+2]$, there exists a redicolouring sequence between α and some k-dicolouring β such that:

[†]In his original proof, Feghali claims to obtain, in the statement corresponding to the third item of Lemma 7.3.6 for undirected graphs, a multiplicative factor $(s + 1)$ instead of $(s + 1)!$ (in the function f). Since we are not able to understand how the smaller factor is obtained in the original proof, we state our result with the larger factor of which we are sure of the correctness, which anyway is hidden in the asymptotic notation.

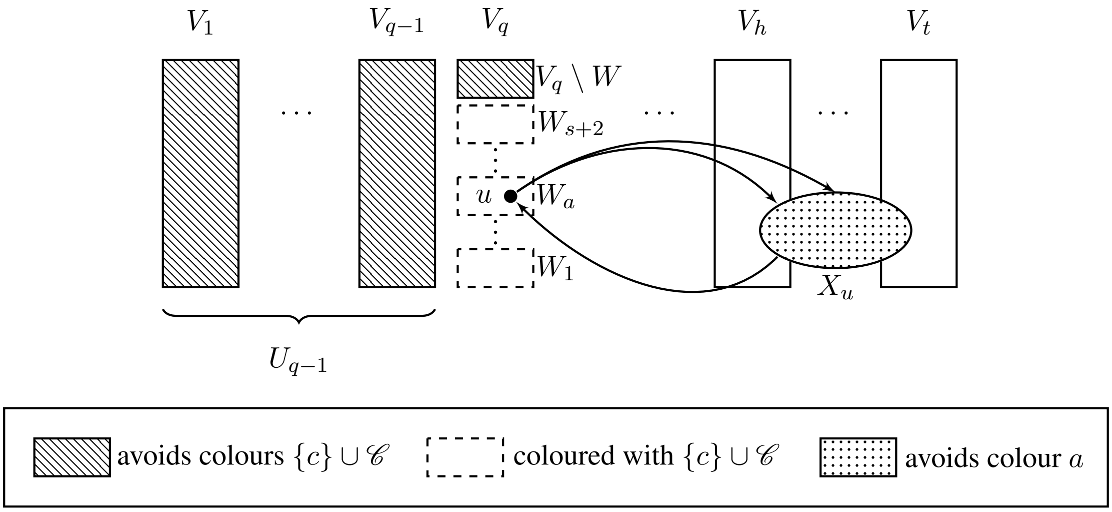


Figure 7.4: The structure of the digraph D dicoloured with α, which we assume to be an $(s+2)$-dicolouring for clarity. Note that W_c does not exist and X_u may intersect $V_{q+1} \cup \cdots \cup V_{h-1}$.

- *for every $v \in \bigcup_{i=1}^{h} V_i$, $\beta(v) \neq c$ and $\beta(v) \leq s+2$,*
- *no vertex of $\bigcup_{i=h+1}^{t} V_i$ is recoloured, and*
- *each vertex in $\bigcup_{i=1}^{h} V_i$ is recoloured at most $f(s,h)$ times.*

Proof. We proceed by induction on s. Assume first that $s = 0$ and let H the subdigraph of D induced by $\bigcup_{i=1}^{h} V_i$. We claim that D does not contain any directed cycle which intersects $V(H)$. Indeed, if D contains such a directed cycle C, let $q \in [h]$ be the smallest index such that $V(C) \cap V_q \neq \varnothing$, and let $u \in V(C) \cap V_q$. Then C is a directed cycle containing u in $D - \bigcup_{i=1}^{q-1} V_i$, so X_u must intersect C. This yields a contradiction because $X_u = \varnothing$ (since $|X_u| = 0$). Thus, since no directed cycle of D intersects $V(H)$, in α we can recolour each vertex of H with the colour $c' \in [2]$ different from c. Since $f(0,h) = 1$, we get the result.

Assume now that $s > 0$. Let $\mathscr{C}$ be the set of colours greater than $s+2$ and let W be the set of vertices with colour c or any colour $c' \in \mathscr{C}$ in α. Formally, $\mathscr{C} = [s+3, k]$ and $W = \{v \in V(D) \mid \alpha(v) = c \vee \alpha(v) \in \mathscr{C}\}$. If $k = s+2$, $\mathscr{C}$ is empty and W is the set of vertices coloured c.

Let $q \in [h]$ be the smallest index such that $V_q \cap W \neq \varnothing$. If such an index does not exist, then we take $\alpha = \beta$ and we are done. Let $U_{q-1} = \bigcup_{i=1}^{q-1} V_i$ (when $q = 1$ we let U_0 be the empty set). For each colour $a \in [s+2]$ different from c, we define W_a as follows:

$$W_a = \{u \in W \cap V_q \mid \forall v \in X_u, \alpha(v) \neq a\}$$

Observe that every vertex in W_a is recolourable to colour a in $D - U_{q-1}$ since X_u intersects every directed cycle containing u in $D - U_{q-1}$. Note also that every vertex $u \in W$ belongs to some W_a (maybe to several) because $|X_u| \leq s$. Whenever a vertex belongs to several sets W_a, we remove it from one, so at the end the collection $(W_a)_{a \in [s+2], a \neq c}$ is a partition of W. Figure 7.4 illustrates the structure of D dicoloured with α.

Claim 7.3.7. *Let ϕ be a k-dicolouring of D such that:*

- *ϕ and α agree on $\bigcup_{i=q+1}^{t} V_i$ and $V_q \setminus W$,*
- *$(\{c\} \cup \mathscr{C}) \cap \phi(U_{q-1}) = \varnothing$, and*
- *$\forall a \in [s+2], a \neq c$, either $\phi(W_a) = \{a\}$ or $\phi(W_a) \subseteq (\{c\} \cup \mathscr{C})$.*

Then for every $a \in [s+2], a \neq c$ such that $\phi(W_a) \subseteq (\{c\} \cup \mathscr{C})$, there exists a redicolouring sequence between ϕ and a k-dicolouring ψ such that:

- *each vertex in U_{q-1} is recoloured at most $2f(s-1, q-1)$ times,*
- *each vertex in W_a is recoloured exactly once (to colour a),*
- *no vertex of $D - (U_{q-1} \cup W_a)$ is recoloured, and*
- *$(\{c\} \cup \mathscr{C}) \cap \psi(U_{q-1}) = \varnothing$.*

Proof of claim. By definition of W_a and because ϕ and α agree on $\bigcup_{i=q+1}^{t} V_i$, note that every vertex $u \in W_a$ is recolourable to colour a if $a \notin \phi(U_{q-1})$. Hence the key idea is to remove colour a from U_{q-1}, then recolour every vertex in W_a with a, and finally remove the colour c from U_{q-1} that we may have introduced. Along this process, we will never introduce any colour of $\mathscr{C}$. When $q = 1$, note that we can just recolour every vertex in W_a with a.

Let $u_1, \ldots, u_r$ be an ordering of U_{q-1} such that the vertices in V_p appear before the vertices in $V_{p'}$ for every $1 \leq p' < p \leq q-1$. Whenever it is possible, in ϕ, we recolour every vertex $u_1, \ldots, u_r$ (in this order) with colour c. Let η be the obtained dicolouring of D, and let S be the set of vertices coloured c in η. We define $D' = D - S$, $h' = q - 1$ and $s' = s - 1$. Also for every $p \in [t]$ we define $V'_p = V_p \setminus S$. Finally, we define η' as the induced dicolouring η on D'.

Let us prove that, for every $p \leq h'$ and every $u \in V'_p$, the set of vertices $X'_u = X_u \setminus S$ satisfies $|X'_u| \leq |X_u| - 1 \leq s'$ and intersects every directed cycle containing u in $D' - \bigcup_{i=1}^{p-1} V'_i$.

First, since $u \in V'_p$, we know that u has not been recoloured to c in the previous process. It means that recolouring u with c creates a monochromatic directed cycle C. Moreover, since $c \notin \phi(U_{q-1})$ and by choice of the ordering $u_1, \ldots, u_r$, we know that such a directed cycle C is included in $\bigcup_{i=p}^{t} V_i$. By assumption on X_u, we have $X_u \cap V(C) \neq \varnothing$. Since $X_u \subseteq \bigcup_{i=p+1}^{t} V_i$ and $\phi(V(C) \setminus \{u\}) = \{c\}$ we deduce that $X_u \cap S \neq \varnothing$, which shows $|X'_u| \leq |X_u| - 1 \leq s'$.

We now prove that X'_u intersects every directed cycle containing u in $D' - \bigcup_{i=1}^{p-1} V'_i$. Let C be such a directed cycle. Since C is also a directed cycle in $D - \bigcup_{i=1}^{p-1} V_i$, we know that C intersects X_u. We also know that $V(C) \cap S = \varnothing$ because C is a directed cycle in D'. Hence, C intersects $X_u \setminus S = X'_u$ as desired.

By the remark above, we can apply the induction on Lemma 7.3.6 with D', $(V'_1, \ldots, V'_t)$, h', s', η' and a playing the roles of D, $(V_1, \ldots, V_t)$, h, s, α and c respectively. Hence, by induction, there exists a redicolouring sequence (which does not use colour c) in D' from η' to some dicolouring ζ' such that:

- for every $v \in \bigcup_{i=1}^{q-1} V'_i, \zeta'(v) \notin (\{a\} \cup \mathscr{C})$,
- no vertex of $\bigcup_{i=q}^{t} V'_i$ is recoloured, and
- each vertex of $\bigcup_{i=1}^{q-1} V'_i$ is recoloured at most $f(s-1, q-1)$ times.

Since this redicolouring sequence does not use colour c, and because $\eta(S) = \{c\}$, it extends into a redicolouring sequence in D between η and ζ, where $\zeta(u) = \zeta'(u)$ when $u \in (U_{q-1} \setminus S)$, $\zeta(u) = c$ when $u \in S$ and $\zeta(u) = \phi(u)$ otherwise. Since $\zeta(v) \neq a$ for every vertex $v \in U_{q-1}$ and by choice of W_a, in ζ we can recolour every vertex in W_a to colour a. Note that the vertices in $S \cap U_{q-1}$ have been recoloured exactly once (to colour c), which is less than $f(s-1, q-1)$.

We will now remove the colour c we introduced in U_{q-1}. We use the same process as before, swapping the roles of c and a. So, whenever it is possible, starting with u_1 and moving forwards towards u_r, we recolour each vertex of U_{q-1} with colour a. Let ξ be the obtained dicolouring of D. We define $R = \{v \in V(D) \mid \xi(v) = a\}$, $\widetilde{D} = D - R$, and $\widetilde{V}_i = V_i \setminus R$ for every $i \in [t]$. Finally, let $\widetilde{\xi}$ be the induced dicolouring ξ on $\widetilde{D}$. By induction, there exists a redicolouring sequence (which does not use colour a) in $\widetilde{D}$ from $\widetilde{\xi}$ to some dicolouring $\widetilde{\psi}$ such that:

- for every $v \in \bigcup_{i=1}^{q-1} \widetilde{V}_i$, $\widetilde{\psi}(v) \neq c$,
- no vertex of $\bigcup_{i=q}^{t} \widetilde{V}_i$ is recoloured, and
- each vertex of $\bigcup_{i=1}^{q-1} \widetilde{V}_i$ is recoloured at most $f(s-1, q-1)$ times.

This gives, in D, a redicolouring sequence from ξ to some dicolouring ψ which does not use colour c on U_{q-1}.

Concatenating the redicolouring sequences we built, we conclude the existence of the desired redicolouring sequence from ϕ to ψ in which every vertex in U_{q-1} is recoloured at most $2f(s-1, q-1)$ times, vertices in W_a are recoloured exactly once to colour a, and the other vertices of D are not recoloured. $\Diamond$

Now we may apply Claim 7.3.7 on α (playing the role of ϕ) to obtain a redicolouring sequence from α to a dicolouring α' (corresponding to ψ) in which W_a has been recoloured to a (for some fixed $a \in [s+2]$, $a \neq c$), and colours $\{c\} \cup \mathscr{C}$ do not appear in $\alpha'(U_{q-1})$. Note that, in Claim 7.3.7, the obtained dicolouring ψ satisfies the assumptions on ϕ. Thus, we may repeat this argument on α' to recolour $W_{a'}$ for some $a' \in [s+2]$, $a' \notin \{c, a\}$.

Repeating this process for each colour $a \in [s+2]$ different from c, we obtain a redicolouring sequence between α and some dicolouring in which colours $\{c\} \cup \mathscr{C}$ do not appear in $U_{q-1} \cup V_q$ and such that each vertex of U_{q-1} is recoloured at most $(s+1) \cdot 2f(s-1, q-1)$ times and every vertex in V_q is recoloured at most once. Since $f(s-1, q) \geq 1$ and $f(x, y)$ is non-decreasing in y, a fortiori every vertex of $\bigcup_{i=1}^{q} V_i$ is recoloured at most $2(s+1)f(s-1, q)$ times.

We have shown above that, if colours $\{c\} \cup \mathscr{C}$ are not appearing in $\bigcup_{i=1}^{q-1} V_i$, then in at most $2(s+1)f(s-1, q)$ recolourings per vertex, we can also remove them from V_q (and we do not recolour vertices in $\bigcup_{i=q+1}^{t} V_i$). Thus we can repeat this argument at most h times to find a redicolouring sequence between α and a dicolouring β in which colours $\{c\} \cup \mathscr{C}$ do not appear in $\bigcup_{i=1}^{h} V_i$. In this redicolouring sequence, by (7.1), the number of recolourings per vertex of $\bigcup_{i=1}^{h} V_i$ is at most

$$\sum_{q=1}^{h} \left(2(s+1)f(s-1, q)\right) \leq f(s, h),$$

which concludes the proof. $\square$

Lemma 7.3.8. *Let $D = (V, A)$ be a digraph on n vertices and let $(V_1, \dots, V_t)$ be a partition of V such that for every $p \in [t]$ and $u \in V_p$, there exists $X_u \subseteq \cup_{i=p+1}^{t} V_i$ such that $|X_u| \leq s$ and X_u intersects every directed cycle containing u in $D - \bigcup_{i=1}^{p-1} V_i$. Then, for any $k \geq s+2$, $\mathcal{D}_k(D)$ has diameter at most $g(s,t) \cdot n$.*

Proof. We will show that, for any two k-dicolourings α, β of D, there exists a redicolouring sequence between them where each vertex is recoloured at most $g(s,t)$ times, showing the result. We proceed by induction on s. When $s = 0$, D is acyclic, so we can directly recolour every vertex v from $\alpha(v)$ to $\beta(v)$. Since $g(0,t) = 1$, we get the result.

Assume now that $s > 0$. By Lemma 7.3.6, there is a redicolouring sequence from α to an $(s+1)$-dicolouring $\widetilde{\alpha}$ in which each vertex of D is recoloured at most $f(s,t)$ times (by taking $h = t$ and $c = s+2$). Symmetrically, we have a redicolouring sequence from β to an $(s+1)$-dicolouring $\widetilde{\beta}$ in which each vertex of D is recoloured at most $f(s,t)$ times. We will now find a redicolouring sequence between $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Let $v_1, \dots, v_n$ be an ordering of V such that the vertices in V_p appear before the vertices in $V_{p'}$ for every $1 \leq p' < p \leq t$. In both $\widetilde{\alpha}$ and $\widetilde{\beta}$, starting with v_1 and moving forwards towards v_n, we recolour, whenever it is possible, each vertex of V with colour $s+2$. This is done in at most two recolourings per vertex (one in both dicolourings). Let $\widehat{\alpha}$ and $\widehat{\beta}$ be the two obtained dicolourings. Observe that the vertices coloured $s+2$ in $\widehat{\alpha}$ are exactly the vertices coloured $s+2$ in $\widehat{\beta}$. We define $S = \{v \in V \mid \widehat{\alpha}(v) = s+2\}$ and $H = D - S$. Let $\widehat{\alpha}_{|H}$ and $\widehat{\beta}_{|H}$ be the dicolourings induced by $\widehat{\alpha}$ and $\widehat{\beta}$ on H, respectively. For each $p \in [t]$, let $V'_p = V_p \setminus S$. Observe that $(V'_1, \dots, V'_t)$ is a partition of $V(H)$ such that for every $p \in [t]$ and $u \in V'_p$, $X'_u = X_u \setminus S$ has size at most $s-1$ and intersects every directed cycle containing u in $D - \bigcup_{i=1}^{p-1} V_i$ (the arguments are the same as in the proof of Lemma 7.3.6). Thus, by induction, there exists in H a redicolouring sequence between $\widehat{\alpha}_{|H}$ and $\widehat{\beta}_{|H}$, using only colours in $[s+1]$ in which every vertex is recoloured at most $g(s-1,t)$ times. Since the vertices in S are coloured $s+2$, this redicolouring sequence extends to D and gives a redicolouring sequence between $\widehat{\alpha}$ and $\widehat{\beta}$. Thus, we have obtained a redicolouring sequence between α and β in which the number of recolourings per vertex is at most $2f(s,t) + 2 + g(s-1,t) \leq g(s,t)$ by Inequality (7.2). □

We will now prove Theorems 7.3.2 and 7.3.3 with Lemma 7.3.8.

Theorem 7.3.2. *Let D be a digraph and $k \geq \delta_c^*(D) + 2 = d+2$, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O_d(n^{d+1}).$$

Proof. Take any c-degeneracy ordering $v_1, \dots, v_n$ of D, and set $V_i = \{v_i\}$ for every $i \in [n]$. Set $s = d$ and $t = n$, and the result follows directly from Lemma 7.3.8 and Inequality (7.3). □

Theorem 7.3.3. *Let $d \geq 1$ and $k \geq d+1$ be two integers, and let $\varepsilon > 0$. If D is a digraph satisfying $\operatorname{Mad}_c(D) \leq d - \varepsilon$, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O_{d,\varepsilon}(n (\log n)^{d-1}).$$

Proof. Our goal is to find a partition $(V_1, \dots, V_{t(n)})$ of $V(D)$ such that $t(n) = O_{d,\varepsilon}(\log n)$. Moreover, we need for every $p \in [t(n)]$ and every $u \in V_p$, that there exists $X_u \subseteq \bigcup_{i=p+1}^{t(n)} V_i$, $|X_u| \leq d-1$, that intersects every directed cycle containing u in $D - \bigcup_{i=1}^{p-1} V_i$. If we find such a partition, then by Lemma 7.3.8, applied for $s = d-1$ and $t = t(n)$, we get that $\mathrm{diam}(\mathcal{D}_k(D)) \leq g(d-1, t(n)) \cdot n$, implying that $\mathrm{diam}(\mathcal{D}_k(D)) = O_{d,\varepsilon}(n(\log n)^{d-1})$ since $t(n) = O_{d,\varepsilon}(\log n)$ and by Inequality 7.3.

Let us guarantee the existence of such a partition. Let H be any subdigraph of D on n_H vertices. For every vertex $u \in V(H)$, we let $X_u \subseteq V(H) \setminus \{u\}$ be a set of $d_c^H(u)$ vertices intersecting every directed cycle containing u in H (where $d_c^H(u)$ denotes the cycle-degree of u in H). The existence of X_u is guaranteed by definition of the cycle-degree.

Then let $J = (V(H), F)$ be an auxiliary digraph, built from H, where $F = \{uv \mid v \in X_u\}$. Let $S \subseteq V(H)$ be the set of all vertices v with $d_J^+(v) \leq d-1$ (where $d_J^+(v)$ denotes the out-degree of v in J). Then $|S| \geq \frac{\varepsilon}{d} n_H$, for otherwise we have the following contradiction:

$$\begin{aligned}
\mathrm{Mad}_c(D) \geq \mathrm{Mad}_c(H) &\geq \frac{1}{n_H} \sum_{v \in V(H)} d_c^H(v) = \frac{1}{n_H} \sum_{v \in V(H)} d_J^+(v) \\
&\geq \frac{1}{n_H} \sum_{v \in V(H) \setminus S} d_J^+(v) \\
&\geq \frac{1}{n_H} (n_H - |S|) d > \left(1 - \frac{\varepsilon}{d}\right) d = d - \varepsilon,
\end{aligned}$$

where in the last inequality we have used that $|S| < \frac{\varepsilon n_H}{d}$. Now let us prove that $J\langle S\rangle$ has an independent set I of size at least $\frac{|S|}{2d-1}$. By choice of S, every subdigraph J' of $J\langle S\rangle$ satisfies $\Delta^+(J') \leq d-1$. Hence, for every such J', we have

$$\sum_{v \in V(J')} (d_{J'}^+(v) + d_{J'}^-(v)) = 2m(J') \leq 2(d-1)n(J').$$

In particular, this implies that $\mathrm{UG}(J\langle S\rangle)$ is $(2d-2)$-degenerate, and $\chi(\mathrm{UG}(J\langle S\rangle)) \leq 2d-1$. Take any proper $(2d-1)$-colouring of $\mathrm{UG}(J\langle S\rangle)$, its largest colour class is the desired I.

Hence, we have shown that H admits a set of vertices $I \subseteq V(H)$, of size at least $\frac{\varepsilon}{(2d-1)d} n_H$, such that for every vertex $u \in I$ there exists $X_u \subseteq (V(H) \setminus I)$, $|X_u| \leq d-1$, that intersects every directed cycle of H containing u.

Since the remark above holds for every subdigraph H of D, we can greedily construct the desired partition $(V_1, \ldots, V_{t(n)})$ by picking successively such a set I in the digraph induced by the non-picked vertices. By construction, we get that t satisfies the following recurrence:

$$t(i) \leq t\left(i - \frac{\varepsilon i}{(2d-1)d}\right) + 1.$$

We thus have $t(n) \leq \log_b(n)$ where $b = \frac{1}{1 - \frac{\varepsilon}{(2d-1)d}}$, implying that

$$t(n) \leq \frac{1}{-\log(1 - \frac{\varepsilon}{(2d-1)d})} \cdot \log(n) = O_{d,\varepsilon}(\log(n)),$$

which concludes the proof. □

7.3.2 Bounds on the diameter of $\mathcal{D}_k(D)$ when $k \geq \frac{3}{2}(\delta_c^*(D) + 1)$

This section is devoted to the proof of the following theorem, which generalises Theorem 7.1.3(i).

Theorem 7.3.9. *Let D be a digraph on n vertices and $k \geq \frac{3}{2}(\delta_c^*(D)+1)$ be an integer, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O(n^2).$$

Before we prove Theorem 7.3.9, we need a preliminary result. Let L be a list assignment of a graph G. We denote by $\mathcal{C}(G, L)$ the graph whose vertices are the L-colourings of G and in which two colourings are adjacent if they differ by the colour of exactly one vertex. An *L-recolouring sequence* is a walk in $\mathcal{C}(G, L)$. We say that L is *a-feasible* if, for some ordering $v_1, \dots, v_n$ of V, $|L(v_i)| \geq |N(v) \cap \{v_{i+1}, \dots, v_n\}| + 1 + a$ for every $i \in [n]$. Bousquet and Heinrich proved the following [42, Theorem 6].

Theorem 7.3.10 (Bousquet and Heinrich [42])**.** *Let G be a graph and $a \in \mathbb{N}$. Let L be an a-feasible list assignment and k be the total number of colours. Then $\mathcal{C}(G, L)$ has diameter at most:*

(i) *kn if $k \leq 2a$,*

(ii) *Cn^2 if $k \leq 3a$ (where C a constant independent of k, a).*

For a graph G and a list assignment L of G, we say that an L-colouring α *avoids a set of colours S* if for every vertex $v \in V(G)$, $\alpha(v)$ does not belong to S. We need the following consequence of Theorem 7.3.10.

Lemma 7.3.11. *Let $G = (V, E)$ be an undirected graph on n vertices, L be a k-list assignment of G that is $\left\lceil \frac{k}{3} \right\rceil$-feasible and α an L-colouring of G that avoids a set S of $\left\lceil \frac{k}{3} \right\rceil$ colours. Then for any set of $\left\lceil \frac{k}{3} \right\rceil$ colours S', there is an L-colouring β of G that avoids S' and such that there is an L-recolouring sequence from α to β of length at most $\frac{4k+12}{3}n$.*

Let us mention that Lemma 7.3.11 was indirectly proved in the proof of [42, Lemma 8]. We give it for the sake of completeness.

Proof. Let S' be any set of $\left\lceil \frac{k}{3} \right\rceil$ colours.

We start with G coloured by α. Let $(v_1, \dots, v_n)$ be an ordering of $V(G)$ such that for every $i \in [n]$, we have

$$|L(v_i)| \geq |N(v) \cap \{v_{i+1}, \dots, v_n\}| + 1 + \left\lceil \frac{k}{3} \right\rceil,$$

the existence of which is guaranteed by L being $\left\lceil \frac{k}{3} \right\rceil$-feasible. We consider each vertex from v_n to v_1. For each vertex, if it is possible, we recolour it with a colour of S. We denote by η the obtained L-colouring. This is done in less than n steps. Observe that, for each colour $c \in S$ and each vertex v_i of G, at least one of the following holds:

- $\eta(v_i) \in S$,
- v_i has a neighbour in $\{v_{i+1}, \dots, v_n\}$ coloured c, or
- $c \notin L(v_i)$.

Let H be the subgraph of G induced by the vertices whose colour in η is not in S. We define L_H by $L_H(v) = L(v) \setminus S$ for every $v \in V(H)$. Using the previous observation, we get that for every vertex v_i of H, v_i has at least $|L(v_i) \cap S|$ neighbours in $\{v_{i+1}, \dots, v_n\} \setminus V(H)$. This implies that L_H is a $\left\lceil \frac{k}{3} \right\rceil$-feasible list assignment of H with a total number of colours bounded by $k - |S| \leq \frac{2k}{3}$. By Theorem 7.3.10 (i), the diameter of $\mathcal{C}(H, L_H)$ is at most $\frac{2k}{3}n$. Note that every L_H-recolouring sequence in H starting from η_H (the colouring η induced on H) gives an L-recolouring of G starting from η.

Consider the following preference ordering on the colours: an arbitrary ordering of $[k] \setminus (S \cup S')$, followed by an ordering of $S' \setminus S$, and finally the colours from S. Let γ be the L-colouring of G obtained by colouring G greedily from v_n to v_1 with this preference ordering. Since L is $\left\lceil \frac{k}{3} \right\rceil$-feasible, and $|S| = \left\lceil \frac{k}{3} \right\rceil$, no vertex is coloured with a colour in S in γ. This implies that γ_H, the colouring γ induced on H, is an L_H-colouring of H. Thus, there is an L_H-recolouring sequence from η_H to γ_H of length at most $\frac{2k}{3}n$ steps. This gives a recolouring sequence in G. We can then recolour the vertices of $G - H$ to their target colour in γ in at most n steps. This shows that, in G, there is an L-recolouring sequence from α to γ of length at most $n + \frac{2k}{3}n + n = \frac{2k+6}{3}n$.

Now observe that, for each colour $c \in R = [k] \setminus (S \cup S')$ and each vertex v_i of G, at least one of the following must hold:

- $\gamma(v_i) \in R$,
- v_i has a neighbour in $\{v_{i+1}, \dots, v_n\}$ coloured c, or
- $c \notin L(v_i)$

Let Γ be the subgraph of G induced by all vertices coloured with a colour in S' by γ. Note that Γ is also the subgraph induced by all vertices coloured with a colour in $(S \cup S')$ by γ because no vertex is coloured in S by γ. Let L_Γ be the list assignment defined by $L_\Gamma(v) = L(v) \cap (S \cup S')$ for all $v \in V(\Gamma)$. By the previous observation, L_Γ is $\left\lceil \frac{k}{3} \right\rceil$-feasible, and the total number of colours is $|S \cup S'| \leq 2\left\lceil \frac{k}{3} \right\rceil$. Thus, by Theorem 7.3.10 (i), $\mathcal{C}(\Gamma, L_\Gamma)$ has diameter at most $2\left\lceil \frac{k}{3} \right\rceil n$. Let β_Γ be an L_Γ-colouring of Γ that avoids the colours of S' (such a colouring exists because $|S'| = \left\lceil \frac{k}{3} \right\rceil$ and L_Γ is $\left\lceil \frac{k}{3} \right\rceil$-feasible) and γ_Γ the colouring γ induced on Γ. There is an L_Γ-recolouring sequence of length at most $2\left\lceil \frac{k}{3} \right\rceil n$ from γ_Γ to β_Γ.

This extends to an L-recolouring sequence in G from γ to β where β does not use any colour of S'. The total number of steps to reach β from α is then at most $\frac{2k+6}{3}n + 2\left\lceil \frac{k}{3} \right\rceil n$ which is bounded by $\frac{4k+12}{3}n$. This shows the result. □

We are now ready to prove Theorem 7.3.9.

Theorem 7.3.9. *Let D be a digraph on n vertices and $k \geq \frac{3}{2}(\delta_c^*(D) + 1)$ be an integer, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) = O(n^2).$$

Proof. Let $D = (V, A)$ be a digraph on n vertices and $k \geq \frac{3}{2}(\delta_c^*(D) + 1)$. Let $(v_1, \dots, v_n)$ be a c-degeneracy ordering of D, that is, an ordering such that for each $i \in [n]$, there exists $X_i \subseteq \{v_{i+1}, \dots, v_n\}$, $|X_i| \leq \delta_c^*(D)$, such that every directed cycle of D containing v_i must intersect $\{v_1, \dots, v_{i-1}\} \cup X_i$.

Let $G = (V, E)$ be the undirected graph where $E = \{v_i v_j \mid v_j \in X_i, i \in [n]\}$. We first prove that each proper colouring of G is a dicolouring of D. Assume that this is not the case, and there exists a proper colouring α of G such that D, coloured with α, contains a monochromatic directed cycle C. Let v_i be the least vertex of C in the ordering $(v_1, \dots, v_n)$. Then C must contain a vertex v_j in X_i. This is a contradiction, since α is a proper colouring of G and $v_i v_j \in E$.

By construction, G has degeneracy at most $\delta_c^*(D)$. Using Theorem 7.1.3(i), we get that $\mathcal{C}_k(G)$ has diameter at most $C_0 n^2$ for some constant C_0.

Let α be any k-dicolouring of D. We will now show that there exists a dicolouring α' of D that is also a proper colouring of G, and such that there exists a redicolouring sequence between α and α' of length at most $C_1 n^2$ for some constant C_1. Set $\delta^* = \delta_c^*(D) \geq \delta^*(G)$, $Y_i = \{v_{i+1}, \dots, v_n\}$, and $H_i = G - Y_i$ for all $i \in [n]$.

Let L_i be the k-list assignment of H_i defined by

$$L_i(v_j) = [k] \setminus \{\alpha(v) \mid v \in X_j \cap Y_i\} \text{ for all } j \in [i].$$

Since k, the total number of colours, is at least $\frac{3}{2}(\delta^* + 1)$, for every $j \in [i]$ we have:

$$\begin{aligned} |L_i(v_j)| &\geq k - |X_j \cap Y_i| \\ &\geq \frac{k}{3} + \frac{2}{3}\frac{3}{2}(\delta^* + 1) - |X_j \cap Y_i| \\ &\geq |X_j \cap \{v_{j+1}, \dots, v_i\}| + 1 + \frac{k}{3}. \end{aligned}$$

Hence, since $|L_i(v_j)|$ is an integer, L_i is a $\left\lceil \frac{k}{3} \right\rceil$-feasible k-list assignment of H_i.

Remark 7.3.1 – Let γ be a dicolouring of D such that for some i, γ agrees with α on $\{v_{i+1}, \dots, v_n\}$ and $\gamma_{|H_i}$ (the restriction of γ to H_i) is an L_i-colouring of H_i. Then any L_i-recolouring sequence starting from $\gamma_{|H_i}$ on H_i is a redicolouring sequence in D. Indeed, assume this is not the case and at one step, we get to an L_i-colouring ζ of H_i but ζ_D contains a monochromatic cycle C, where $\zeta_D(v) = \zeta(v)$ when v belongs to H_i and $\zeta_D(v) = \gamma(v)$ otherwise. Let v_j be the vertex of C such that j is minimum in the c-degeneracy ordering of D. Then C must intersect X_j in some vertex v_q. Thus either $q \leq i$ and then $v_q v_j$ is a monochromatic edge in H_i or $q \geq i + 1$ but then $\zeta(v_q) = \gamma(v_q) = \alpha(v_q)$ does not belong to $L_i(v_j)$. In both cases, we get a contradiction.

Claim 7.3.12. *Let γ_i be a k-dicolouring of D, agreeing with α on Y_i, which induces an L_i-colouring of H_i avoiding at least $\left\lceil \frac{k}{3} \right\rceil$ colours in H_i. Then there is a redicolouring sequence of length at most $\frac{8k+24}{3} n + \left\lceil \frac{k}{3} \right\rceil$ from γ_i to a dicolouring $\gamma_{i+\lceil \frac{k}{3} \rceil}$ which induces an $L_{i+\lceil \frac{k}{3} \rceil}$-colouring of $H_{i+\lceil \frac{k}{3} \rceil}$ avoiding at least $\left\lceil \frac{k}{3} \right\rceil$ colours in $H_{i+\lceil \frac{k}{3} \rceil}$. Moreover, $\gamma_{i+\lceil \frac{k}{3} \rceil}$ agrees with α on $Y_{i+\lceil \frac{k}{3} \rceil}$.*

Proof of claim. Figure 7.5 illustrates the different steps of the proof of the claim. The main steps are first to remove the colours of a set S' in H_i which then allows us to remove the colours of a set S'' for vertices v_{i+1} to $v_{i+\lceil \frac{k}{3} \rceil}$ and finally reach a colouring where no colour of S'' appears in $H_{i+\lceil \frac{k}{3} \rceil}$ (see the definitions of S' and S'' below).

Let S be a set of colours of size exactly $\left\lceil \frac{k}{3} \right\rceil$ avoided by γ_i on H_i. For each vertex v_j in $\{v_{i+1}, \dots, v_{i+\lceil \frac{k}{3} \rceil}\}$, we choose a colour c_j so that each of the following holds:

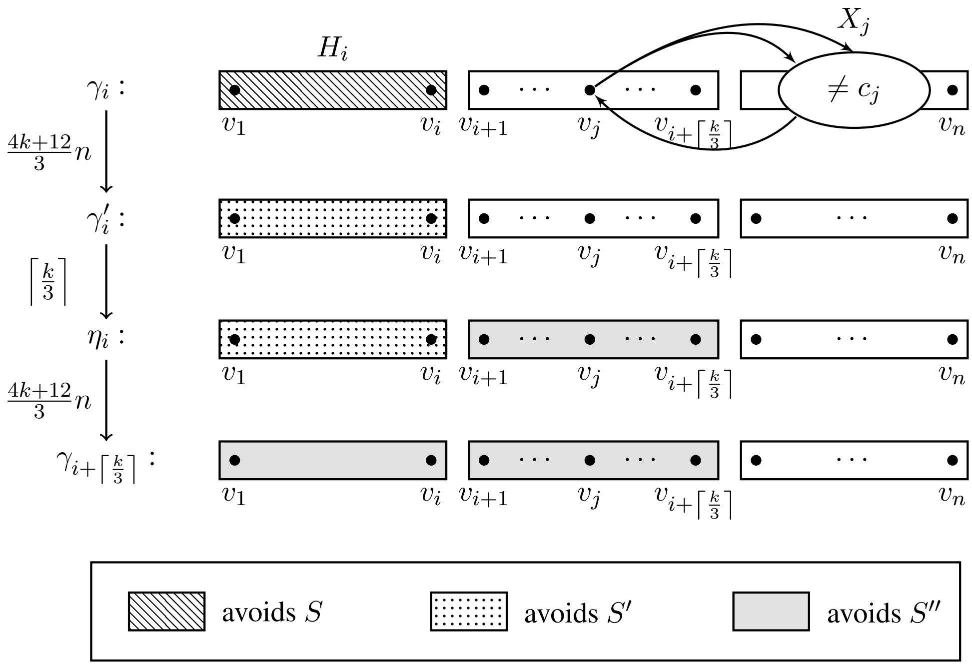


Figure 7.5: The redicolouring sequence between γ_i and $\gamma_{i+\lceil \frac{k}{3} \rceil}$.

- c_j belongs to $L_j(v_j)$, and
- for each $\ell \in \{i+1, \dots, j-1\}$, c_ℓ is different from c_j.

Note that this is possible because L_j is $\left\lceil \frac{k}{3} \right\rceil$-feasible. Now let S' be the set $\{c_{i+1}, \dots, c_{i+\lceil \frac{k}{3} \rceil}\}$. Observe that $|S'| = \left\lceil \frac{k}{3} \right\rceil$. By Lemma 7.3.11, there is, in H_i, an L_i-recolouring sequence of length at most $\frac{4k+12}{3}n$ from γ_i to some γ'_i that avoids S'. This recolouring sequence extends to a redicolouring sequence in D by Remark 7.3.1. In the obtained dicolouring, since γ'_i avoids S' on H_i, we can recolour successively v_j with c_j for all $i+1 \leq j \leq i + \left\lceil \frac{k}{3} \right\rceil$ (starting from v_{i+1} and moving forward to $v_{i+\lceil \frac{k}{3} \rceil}$). This does not create any monochromatic directed cycle by choice of c_j. Let η_i be the resulting dicolouring of D. Now, we define a list assignment $\widetilde{L}_i$ of H_i as follows:

$$\widetilde{L}_i(v_j) = [k] \setminus \{\eta_i(v) \mid v \in N(v_j) \cap \{v_{i+1}, \dots, v_n\}\} \text{ for all } j \in [i].$$

Using the same arguments as we did for L_i, we get that $\widetilde{L}_i$ is $\left\lceil \frac{k}{3} \right\rceil$-feasible for H_i. Note that η_i is an $\widetilde{L}_i$-colouring of H_i that avoids S'. Let S'' be any set of $\left\lceil \frac{k}{3} \right\rceil$ colours disjoint from S'. By Lemma 7.3.11, there is, in H_i, an $\widetilde{L}_i$-recolouring sequence of length at most $\frac{4k+12}{3}n$ from η_i to some η'_i that avoids S''. This recolouring sequence extends directly to a redicolouring sequence in D. Since S' is disjoint from S'', the obtained dicolouring is an $L_{i+\lceil \frac{k}{3} \rceil}$-colouring of $H_{i+\lceil \frac{k}{3} \rceil}$ that avoids at least $\left\lceil \frac{k}{3} \right\rceil$ colours in $H_{i+\lceil \frac{k}{3} \rceil}$. Hence, we get a redicolouring sequence from γ_i to the desired $\gamma_{i+\lceil \frac{k}{3} \rceil}$, in at most $\frac{8k+24}{3}n + \left\lceil \frac{k}{3} \right\rceil$ steps. $\Diamond$

Note that $\gamma_{\lceil \frac{k}{3} \rceil}$ (a dicolouring satisfying the assumptions of Claim 7.3.12 for $i = \left\lceil \frac{k}{3} \right\rceil$) can be reached from α in less than n steps: for all $j \in [\left\lceil \frac{k}{3} \right\rceil]$, choose a colour c_j so that each of the following holds:

- c_j belongs to $L_j(v_j)$, and
- for each $\ell \in [j-1]$, c_ℓ is different from c_j.

Now we can recolour successively $v_1, \dots, v_{\lceil \frac{k}{3} \rceil}$ (in this order) to their corresponding colour in $\{c_1, \dots, c_{\lceil \frac{k}{3} \rceil}\}$. Then applying Claim 7.3.12 iteratively at most $\left\lfloor \frac{n}{\lceil \frac{k}{3} \rceil} \right\rfloor \leq \frac{3n}{k}$ times, we get that there is a redicolouring sequence of length at most $n + \frac{3n}{k}\left(\frac{8k+24}{3}n + \frac{k}{3}\right)$ from α to a dicolouring α' of D that is also a proper colouring of G. Note that there exists a constant C_1, independent of k, such that $n + \frac{3n}{k}\left(\frac{8k+24}{3}n + \frac{k}{3}\right) \leq C_1 n^2$.

Let α and β be two k-dicolourings of D. As proved above, there is a redicolouring sequence of length at most $C_1 n^2$ from α (resp. β) to a dicolouring α' (resp. β') of D that is also a proper colouring of G. Since $\mathcal{C}_k(G)$ has diameter at most $C_0 n^2$, there is a recolouring sequence of G of length at most $C_0 n^2$ from α' to β', which is also a redicolouring sequence of D (since every proper colouring of G is a dicolouring of D). The union of those three sequences yields a redicolouring sequence from α to β of length at most $(2C_1 + C_0)n^2$. □

7.3.3 Bounds on the diameter of $\mathcal{D}_k(D)$ when $k \geq 2\delta_c^*(D) + 2$

This section is devoted to the proof of the following theorem which generalises Theorem 7.1.4.

Theorem 7.3.13. *Let D be a digraph on n vertices and $k \geq 2(\delta_c^*(D) + 1)$ be an integer, then*

$$\operatorname{diam}(\mathcal{D}_k(D)) \leq (\delta_c^*(D) + 1)n.$$

Proof. Let α and β be two k-dicolourings of D. Let us show by induction on the number of vertices that there exists a redicolouring sequence from α to β where every vertex is recoloured at most $\delta_c^*(D) + 1$ times.

If $n = 1$ the result is obviously true. Let D be a digraph on at least two vertices, let u be a vertex such that $d_c(u) \leq \delta_c^*(D)$ and let $D' = D - u$. We denote by α' and β' the dicolourings of D' induced, respectively, by α and β. By induction and since $\delta_c^*(D') \leq \delta_c^*(D)$, there exists a redicolouring sequence from α' to β' such that each vertex is recoloured at most $\delta_c^*(D) + 1$ times. Now we consider the same recolouring steps to recolour D, starting from α. If for some step i, it is not possible to recolour v_i to c_i, this must be because u is currently coloured c_i and recolouring v_i to c_i would create a monochromatic directed cycle. By definition of cycle-degree, there exists a transversal X of the directed cycles containing u, with $|X| \leq \delta_c^*(D)$ and $u \notin X$. Since $k \geq 2\delta_c^*(D) + 2$, there are at least $\delta_c^*(D) + 2$ colours that do not appear in X. We choose c among these colours so that c does not appear in the next $\delta_c^*(D) + 1$ recolourings of X, and we recolour u with c.

Since $|X| \leq \delta_c^*(D)$ and since each vertex in D' is recoloured at most $\delta_c^*(D) + 1$ times, the total number of recolourings in X is at most $\delta_c^*(D)(\delta_c^*(D) + 1)$ in the redicolouring sequence obtained by induction. Hence, in this new redicolouring sequence, u is recoloured at most $\delta_c^*(D)$ times. We finally have to set u to its colour in β. Doing so u is recoloured at most $\delta_c^*(D) + 1$ times. This concludes the proof. □

7.4 Digraphs having bounded maximum degree

This section is devoted to the study of digraphs with bounded maximum degree. We first show that Theorem 7.1.6 generalises to digraphs via the digrundy number.

The *digrundy number* of a digraph $D = (V, A)$, introduced in [15], is the natural analogue of the grundy number for digraphs. If ϕ is a dicolouring of D, then ϕ is a *greedy dicolouring* if there is an ordering $v_1, \dots, v_n$ of V such that, for each vertex v_i and each colour c smaller than $\phi(v_i)$, the set of vertices $(\{v_1, \dots, v_{i-1}\} \cap \phi^{-1}(c)) \cup \{v_i\}$ contains a directed cycle. The digrundy number of D, denoted by $\vec{\chi}_g(D)$, corresponds to the maximum number of colours used in a greedy dicolouring of D.

Theorem 7.4.1. *For any digraph D, if $k \geq \vec{\chi}_g(D) + 1$, then $\mathrm{diam}(\mathcal{D}_k(D)) \leq 4 \cdot \vec{\chi}(D) \cdot n$.*

Analogously to the undirected case, we always have $\vec{\chi}(D) \leq \vec{\chi}_g(D) \leq \Delta_c(D) + 1$, where $\Delta_c(D) = \max\{d_c(v) \mid v \in V(D)\}$. Thus, the following is a direct consequence of Theorem 7.4.1. In particular, it shows that Conjecture 7.1.9 holds when considering the maximum cycle-degree instead of the c-degeneracy.

Corollary 7.4.2. *For any digraph D, if $k \geq \Delta_c(D) + 2$, then $\mathrm{diam}(\mathcal{D}_k(D)) \leq 4 \cdot \vec{\chi}(D) \cdot n$.*

Proof of Theorem 7.4.1. Let α be any k-dicolouring of D and β be any $\vec{\chi}(D)$-dicolouring of D. We will show by induction on $\vec{\chi}(D)$ that there exists a redicolouring sequence of length at most $2 \cdot \vec{\chi}(D) \cdot n$ between α and β. The claimed result will then follow. If $\vec{\chi}(D) = 1$, the result is clear since D is acyclic.

Starting from α, whenever a vertex can be recoloured to colour k, we recolour it. Then we try to recolour the remaining vertices with colour $k - 1$, and we repeat this process for every colour $k - 1, \dots, 2$. At the end, the obtained dicolouring γ is greedy (with colours ordered from k to 1). Actually, γ is exactly the greedy dicolouring obtained from any ordering $v_1, \dots, v_n$ of $V(D)$ where $i < j$ whenever $\gamma(v_i) > \gamma(v_j)$.

Since γ is a greedy dicolouring, and because $k \geq \vec{\chi}_g(D) + 1$, colour 1 is not used in γ. This allows us to recolour every vertex of $V_1 = \{v \in V(D) \mid \beta(v) = 1\}$ to colour 1 ($V_1 \neq \varnothing$ since β uses colours $[\vec{\chi}(D)]$). If η is the obtained dicolouring, then η and β agree on colour 1. Note also that, starting from α, we reached η by recolouring each vertex at most twice. Thus, the distance between α and η in $\mathcal{D}_k(D)$ is at most $2n$.

Consider $H = D - V_1$. Since β is an optimal dicolouring of D, $\vec{\chi}(H) = \vec{\chi}(D) - 1$. Thus, by induction, there exists a redicolouring sequence between $\eta_{|H}$ and $\beta_{|H}$ (that is, the restrictions of η and β, respectively, to H) of length at most $2(\vec{\chi}(D) - 1)n$, that does not use colour 1. This directly extends to a redicolouring sequence between η and β in D, which together with the redicolouring sequence between α and η gives a redicolouring sequence between α and β of length at most $2 \cdot \vec{\chi}(D) \cdot n$. □

We now show that Theorem 7.1.7 extends to digraphs as follows.

Theorem 7.4.3. *Let $D = (V, A)$ be a connected digraph with $\Delta_{\max}(D) = \Delta \geq 3$, $k \geq \Delta + 1$, and α, β two k-dicolourings of D. Then at least one of the following holds:*

- *α is k-frozen, or*
- *β is k-frozen, or*

- *there is a redicolouring sequence of length at most $c_\Delta|V|^2$ between α and β, where $c_\Delta = O(\Delta^2)$ is a constant depending only on Δ.*

Furthermore, we prove that a digraph D is k-freezable only if D is bidirected and its underlying graph is k-freezable. Thus, an obstruction in Theorem 7.4.3 is exactly the bidirected graph of an obstruction in Theorem 7.1.7.

Lemma 7.4.4. *Let $D = (V, A)$ be a digraph and L be a list assignment of D such that, for every vertex $v \in V$, $|L(v)| \geq d_{\max}(v) + 1$. Let α be an L-dicolouring of D. If $u \in V$ is blocked in α, then for each colour $c \in L(u)$ different from $\alpha(u)$, u has exactly one out-neighbour u_c^+ and one in-neighbour u_c^- coloured c. Moreover, if $u_c^+ \neq u_c^-$, there must be a monochromatic directed path from u_c^+ to u_c^-. In particular, u is not incident to a monochromatic arc.*

Proof. Since u is blocked to its colour in α, for each colour $c \in L(u)$ different from $\alpha(u)$, recolouring u to c must create a monochromatic directed cycle C. Let v be the out-neighbour of u in C and w be the in-neighbour of u in C. Then $\alpha(v) = \alpha(w) = c$, and there is a monochromatic directed path (in C) from v to w.

This implies that, for each colour $c \in L(u)$ different from $\alpha(u)$, u has at least one out-neighbour and at least one in-neighbour coloured c. Since $|L(u)| \geq d_{\max}(u) + 1$, then $|L(u)| = d_{\max}(u) + 1$, and u must have exactly one out-neighbour and exactly one in-neighbour coloured c. In particular, u cannot be incident to a monochromatic arc. □

Lemma 7.4.5. *Let $D = (V, A)$ be a digraph such that for every vertex $v \in V$, $N^+(v) \setminus N^-(v) \neq \varnothing$ and $N^-(v) \setminus N^+(v) \neq \varnothing$. Let L be a list assignment of D, such that for every vertex $v \in V$, $|L(v)| \geq d_{\max}(v) + 1$. Then for any pair of L-dicolourings α, β of D, there is an L-redicolouring sequence of length at most $(|V| + 3)|V|$.*

Proof. Let $x = \text{diff}(\alpha, \beta) = |\{v \in V \mid \alpha(v) \neq \beta(v)\}|$. We will show by induction on x that there is an L-redicolouring sequence from α to β of length at most $(|V| + 3)x$. The result clearly holds for $x = 0$ (i.e. $\alpha = \beta$). Let $v \in V$ be such that $\alpha(v) \neq \beta(v)$. We denote $\alpha(v)$ by c and $\beta(v)$ by c'. If v can be recoloured to c', then we recolour it and we get the result by induction.

Assume now that v cannot be recoloured to c'. Whenever v is contained in a directed cycle C of length at least 3, such that every vertex of C but v is coloured c', we do the following: we choose a vertex w of C different from v, such that $\beta(w) \neq c'$. We know that such a vertex w exists, for otherwise C would be a monochromatic directed cycle in β. Now, since w is incident to a monochromatic arc in C, and because $|L(w)| \geq d_{\max}(w) + 1$, by Lemma 7.4.4, we know that w can be recoloured to some colour different from c'. We thus recolour w to this colour. Observe that it does not increase x.

After repeating this process, maybe v cannot be recoloured to c' because it is adjacent by a digon to some vertices coloured c'. We know that these vertices are not coloured c' in β. Thus, whenever such a vertex can be recoloured, we recolour it. After this, let η be the obtained dicolouring. If v can be recoloured to c' in η, we are done. Otherwise, there must be some vertices, blocked to colour c' in η, adjacent to v by a digon. Let S be the set of such vertices. Observe that, by Lemma 7.4.4, for every vertex $s \in S$, c belongs to $L(s)$, for otherwise s would not be blocked in η. We distinguish two cases, depending on the size of S.

- If $|S| \geq 2$, then by Lemma 7.4.4, v can be recoloured to a colour c'', different from both c and c' because v is adjacent by a digon with two neighbours coloured c'. Hence, we can

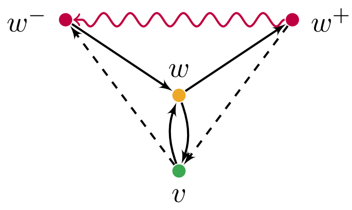


Figure 7.6: The vertices v, w, w^+ and w^-.

successively recolour v to c'', and every vertex of S to c. This does not create any monochromatic directed cycle because for each $s \in S$, since s is blocked in η, by Lemma 7.4.4 v must be the only neighbour of s coloured c in η.

We can finally recolour v to c'.

- If $|S| = 1$, let w be the only vertex in S. If v can be recoloured to any colour (different from c' since w is coloured c'), then we first recolour v, allowing us to recolour w to c because v is the single neighbour of w coloured c in η by Lemma 7.4.4. We finally can recolour v to c'.

  Assume then that v is blocked to colour c in η. Let us fix $w^+ \in N^+(w) \setminus N^-(w)$. Since w is blocked to c' in η, by Lemma 7.4.4, there exists exactly one vertex $w^- \in N^-(w) \setminus N^+(w)$ such that $\eta(w^+) = \eta(w^-) = c''$ and there must be a monochromatic directed path from w^+ to w^-.

  Since v is blocked to colour c in η, either $vw^- \notin A$ or $w^+v \notin A$, otherwise, by Lemma 7.4.4, there must be a monochromatic directed path from w^- to w^+, which is blocking v to its colour. But since there is also a monochromatic directed path from w^+ to w^- (blocking w) there would be a monochromatic directed cycle, a contradiction (see Figure 7.6). We distinguish the two possible cases:

  – if $vw^- \notin A$, then we start by recolouring w^- with a colour that does not appear in its in-neighbourhood. This is possible because w^- has a monochromatic entering arc, and because $|L(w^-)| \geq d_{\max}(w^-) + 1$. We first recolour w with c'', since c'' does not appear in its in-neighbourhood anymore (w^- was the only one by Lemma 7.4.4). Next we recolour v with c': this is possible because v does not have any out-neighbour coloured c' since w was the only one by Lemma 7.4.4 and w^- is not an out-neighbour of v. We can finally recolour w to colour c and w^- to c''. After all these operations, we exchanged the colours of v and w.

  – if $w^+v \notin A$, then we use a symmetric argument.

Observe that we found an L-redicolouring sequence from α to a α', in at most $|V| + 3$ steps, such that $\text{diff}(\alpha', \beta) < \text{diff}(\alpha, \beta)$. Thus, by induction, we get an L-redicolouring sequence of length at most $(|V| + 3)x$ between α and β. □

We are now able to prove Theorem 7.4.3. The idea of the proof is to divide the digraph D into two parts. One of them is bidirected and we will use Theorem 7.1.7 as a black box on it. In the other part, we know that each vertex is incident to at least two simple arcs, one leaving and one entering, and we will use Lemma 7.4.5 on it.

Proof of Theorem 7.4.3. Let $D = (V, A)$ be a connected digraph with $\Delta_{\max}(D) = \Delta$, $k \geq \Delta+1$. Let α and β be two k-dicolourings of D. Assume that neither α nor β is k-frozen.

We first make a simple observation. For any simple arc $xy \in A$, we may assume that $N^+(y) \setminus N^-(y) \neq \varnothing$ and $N^-(x) \setminus N^+(x) \neq \varnothing$. If this is not the case, then every directed cycle containing xy must contain a digon, implying that the k-dicolouring graph of D is also the k-dicolouring graph of $D \setminus \{xy\}$. Then we may look for a redicolouring sequence in $D \setminus \{xy\}$.

Let $X = \{v \in V \mid N^+(v) = N^-(v)\}$ and $Y = V \setminus X$. Observe that $D\langle X \rangle$ is bidirected, and thus the dicolourings of $D\langle X \rangle$ are exactly the colourings of $UG(D\langle X \rangle)$. We first show that $\alpha_{|D\langle X \rangle}$ and $\beta_{|D\langle X \rangle}$ are not frozen k-colourings of $D\langle X \rangle$. If Y is empty, then $D\langle X \rangle = D$ and $\alpha_{|D\langle X \rangle}$ and $\beta_{|D\langle X \rangle}$ are not k-frozen by assumption. Otherwise, since D is connected, there exists $x \in X$ such that, in $D\langle X \rangle$, $d^+(x) = d^-(x) \leq \Delta - 1$, implying that x is not blocked in any dicolouring of $D\langle X \rangle$. Thus, by Theorem 7.1.7, there is a redicolouring sequence $\gamma'_1, \dots, \gamma'_r$ in $D\langle X \rangle$ from $\alpha_{|D\langle X \rangle}$ to $\beta_{|D\langle X \rangle}$, where $r \leq c_\Delta |X|^2$, and $c_\Delta = O(\Delta)$ is a constant depending on Δ.

We will show that, for each $i \in [r-1]$, if γ_i is a k-dicolouring of D which agrees with γ'_i on X, then there exist a k-dicolouring γ_{i+1} of D that agrees with γ'_{i+1} on X and a redicolouring sequence from γ_i to γ_{i+1} of length at most $\Delta + 2$.

Observe that α agrees with γ'_1 on X. Now assume that there is such a γ_i, which agrees with γ'_i on X, and let $v_i \in X$ be the vertex for which $\gamma'_i(v_i) \neq \gamma'_{i+1}(v_i)$. We denote by c (resp. c') the colour of v_i in γ'_i (resp. γ'_{i+1}). If recolouring v_i to c' in γ_i is valid, then we have the desired γ_{i+1}. Otherwise, we know that v_i is adjacent with a digon (since v_i is only adjacent to digons) to some vertices (at most Δ) coloured c' in Y. Whenever such a vertex can be recoloured to a colour different from c', we recolour it. Let η_i be the reached k-dicolouring after these operations. If v_i can be recoloured to c' in η_i we are done. If not, then the neighbours of v_i coloured c' in Y are blocked to colour c' in η_i. We denote by S the set of these neighbours. We distinguish two cases:

- If $|S| \geq 2$, then by Lemma 7.4.4, v_i can be recoloured to a colour c'', different from both c and c', because v_i has two neighbours with the same colour. Then we successively recolour v_i to c'', and every vertex of S to c. This does not create any monochromatic directed cycle because, by Lemma 7.4.4, for each $s \in S$, v_i is the only neighbour of s coloured c in η_i. We can finally recolour v_i to c' to reach the desired γ_{i+1}.

- If $|S| = 1$, let y be the only vertex in S. Since y belongs to Y and is blocked to its colour in η_i, by Lemma 7.4.4, we know that y has an out-neighbour $y^+ \in N^+(y) \setminus N^-(y)$ and an in-neighbour $y^- \in N^-(y) \setminus N^+(y)$ such that there is a monochromatic directed path from y^+ to y^-. Observe that both y^+ and y^- are recolourable in η_i by Lemma 7.4.4 because they are incident to a monochromatic arc.

  - If v_i is not adjacent to y^+, then we recolour y^+ to any possible colour, and we recolour y to $\eta_i(y^+)$. We can finally recolour v_i to c' to reach the desired γ_{i+1}.
  - If v_i is not adjacent to y^-, then we recolour y^- to any possible colour, and we recolour y to $\eta_i(y^-)$. We can finally recolour v_i to c' to reach the desired γ_{i+1}.
  - Finally, if v_i is adjacent to both y^+ and y^-, since $\eta_i(y^+) = \eta_i(y^-)$, then v_i can be recoloured to a colour c'' different from c and c'. This allows us to recolour y to c, and we finally can recolour v_i to c' to reach the desired γ_{i+1}.

We have shown that there is a redicolouring sequence of length at most $(\Delta+2)c_\Delta n^2$ from α to some α' that agrees with β on X. Now we define the list assignment: for each $y \in Y$,

$$L(y) = [k] \setminus \{\beta(x) \mid x \in N(y) \cap X\}.$$

Observe that, for every $y \in Y$,

$$|L(y)| \geq k - |N^+(y) \cap X| \geq \Delta + 1 - (\Delta - d_Y^+(y)) \geq d_Y^+(y) + 1.$$

Symmetrically, we get $|L(y)| \geq d_Y^-(y) + 1$. This implies, in $D\langle Y\rangle$, $|L(y)| \geq d_{\max}(y) + 1$. Note also that both $\alpha'_{|D\langle Y\rangle}$ and $\beta_{|D\langle Y\rangle}$ are L-dicolourings of $D\langle Y\rangle$. Note finally that, for each $y \in Y$, $N^+(y) \setminus N^-(y) \neq \varnothing$ and $N^+(y) \setminus N^-(y) \neq \varnothing$ by choice of X and Y and by the initial observation. By Lemma 7.4.5, there is an L-redicolouring sequence in $D\langle Y\rangle$ between $\alpha'_{|D\langle Y\rangle}$ and $\beta_{|D\langle Y\rangle}$, with length at most $(|Y|+3)|Y|$. By choice of L, this extends directly to a redicolouring sequence from α' to β on D of the same length.

The concatenation of the redicolouring sequence from α to α' and the one from α' to β leads to a redicolouring sequence from α to β of length at most $c'_\Delta|V|^2$, where $c'_\Delta = O(\Delta^2)$ is a constant depending on Δ. □

Remark 7.4.1 – If α is a k-frozen dicolouring of a digraph D, with $k \geq \Delta_{\max}(D)+1$, then D must be bidirected. If D is not bidirected, then we choose v a vertex incident to a simple arc. If v cannot be recoloured in α, by Lemma 7.4.4, since v is incident to a simple arc, there exists a colour c for which v has an out-neighbour w and an in-neighbour u both coloured c, such that $u \neq w$ and there is a monochromatic directed path from w to u. But then, every vertex on this path is incident to a monochromatic arc, and it can be recoloured by Lemma 7.4.4. Thus, α is not k-frozen. This shows that an obstruction of Theorem 7.4.3 is exactly the bidirected graph of an obstruction of Theorem 7.1.7.

In the reminder of this section, we restrict our focus to oriented graphs. We first show the following result.

Theorem 7.4.6. *Let $\vec{G}$ be an oriented graph of order n such that $\Delta_{\min}(\vec{G}) \leq 1$. Then $\mathcal{D}_2(\vec{G})$ is connected and has diameter exactly n.*

Then, we will prove the following as a consequence of both Theorem 7.4.6 and Corollary 3.3.6 which states that every oriented graph $\vec{G}$ satisfies $\vec{\chi}(\vec{G}) \leq \max(2, \Delta_{\min}(\vec{G}))$.

Corollary 7.4.7. *Let $\vec{G}$ be an oriented graph of order n with $\Delta_{\min}(\vec{G}) = \Delta \geq 1$, and let $k \geq \Delta + 1$. Then $\mathcal{D}_k(\vec{G})$ is connected and has diameter at most $2\Delta n$.*

This is a significant improvement on Theorem 7.4.3 when restricted to oriented graphs. The interested reader may note that Corollary 7.4.7 does not hold for digraphs in general: indeed, $\overleftrightarrow{P_n}$, the bidirected path on n vertices, satisfies $\Delta_{\min}(\overleftrightarrow{P_n}) = 2$ and $\mathcal{D}_3(\overleftrightarrow{P_n}) = \mathcal{C}_3(P_n)$ has diameter $\Omega(n^2)$, as proved in [33].

Observe that, if $\mathcal{D}_2(\vec{G})$ is connected, then its diameter must be at least n: for any 2-dicolouring α, we can define its mirror $\overline{\alpha}$ where, for every vertex $v \in V(\vec{G})$, $\alpha(v) \neq \overline{\alpha}(v)$; then every redicolouring sequence between α and $\overline{\alpha}$ has length at least n. Hence, to prove Theorem 7.4.6 it is indeed sufficient to prove that the diameter of $\mathcal{D}_2(\vec{G})$ has diameter at most n. The following proves Theorem 7.4.6 for directed cycles of length at least 3.

Lemma 7.4.8. *Let C be a directed cycle of length at least 3. Then $\mathcal{D}_2(C)$ is connected and has diameter exactly n.*

Proof. Let α and β be any two 2-dicolourings of C. Let $x = \text{diff}(\alpha, \beta) = |\{v \in V(C) \mid \alpha(v) \neq \beta(v)\}|$. By induction on $x \geq 0$, let us show that there exists a path of length at most x from α to β in $\mathcal{D}_2(C)$. This clearly holds for $x = 0$ (i.e., $\alpha = \beta$). Assume $x > 0$ and the result holds for $x - 1$. Let $v \in V(C)$ be such that $\alpha(v) \neq \beta(v)$.

If v can be recoloured in $\beta(v)$, then we recolour it and reach a new 2-dicolouring α' such that $\text{diff}(\alpha', \beta) = x - 1$ and the result holds by induction. Else if v cannot be recoloured, then recolouring v must create a monochromatic directed cycle, which must be C. Then there must be a vertex v', different from v, such that $\beta(v) = \alpha(v') \neq \beta(v')$, and v' can be recoloured. We recolour it and reach a new 2-dicolouring α' such that $\text{diff}(\alpha', \beta) = x - 1$. The result then holds by induction. □

We are now ready to prove Theorem 7.4.6.

Proof of Theorem 7.4.6. Let α and β be any two 2-dicolourings of $\vec{G}$. We will show that there exists a redicolouring sequence of length at most n between α and β. We may assume that $\vec{G}$ is strongly connected, for otherwise we consider each strongly connected component independently. This implies in particular that $\vec{G}$ does not contain any sink nor source. Let (X, Y) be a partition of $V(\vec{G})$ such that, for every $x \in X$, $d^+(x) = 1$, and for every $y \in Y$, $d^-(y) = 1$.

Assume first that $\vec{G}\langle X\rangle$ contains a directed cycle C. Since every vertex in X has exactly one out-neighbour, there is no arc leaving C. Thus, since $\vec{G}$ is strongly connected, $\vec{G}$ must be exactly C, and the result holds by Lemma 7.4.8. Using a symmetric argument, we get the result when $\vec{G}\langle Y\rangle$ contains a directed cycle.

Assume now that both $\vec{G}\langle X\rangle$ and $\vec{G}\langle Y\rangle$ are acyclic. An *out-arborescence* is a rooted tree in which every edge is oriented away from the root. An *in-arborescence* is obtained from an out-arborescence by reversing every arc. Since every vertex in X has exactly one out-neighbour, $\vec{G}\langle X\rangle$ is the union of disjoint and independent in-arborescences. We denote by X_r the set of roots of these in-arborescences. Symmetrically, $\vec{G}\langle Y\rangle$ is the union of disjoint and independent out-arborescences, and we denote by Y_r the set of roots of these out-arborescences. Set $X_\ell = X \setminus X_r$ and $Y_\ell = Y \setminus Y_r$. Observe that the arcs from X to Y form a perfect matching directed from X_r to Y_r. We denote by M_r this perfect matching. Observe also that there can be any arc from Y to X. Now we define X_r^1 and Y_r^1 two subsets of X_r and Y_r respectively, depending on the two 2-dicolourings α and β, as follows:

$$X_r^1 = \{x \mid xy \in M_r, \alpha(x) = \beta(y) \neq \alpha(y) = \beta(x)\}$$
$$Y_r^1 = \{y \mid xy \in M_r, \alpha(x) = \beta(y) \neq \alpha(y) = \beta(x)\}$$

Set $X_r^2 = X_r \setminus X_r^1$ and $Y_r^2 = Y_r \setminus Y_r^1$. We denote by M_r^1 (resp. M_r^2) the perfect matching from X_r^1 to Y_r^1 (resp. from X_r^2 to Y_r^2). Figure 7.7 shows a partitioning of $V(\vec{G})$ into $X_r^1, X_r^2, X_\ell, Y_r^1, Y_r^2, Y_\ell$.

Claim 7.4.9. *There exists a redicolouring sequence of length s_α from α to some 2-dicolouring α' and a redicolouring sequence of length s_β from β to some 2-dicolouring β' such that each of the following holds:*

(i) For any arc $xy \in M_r$, $\alpha'(x) \neq \alpha'(y)$ and $\beta'(x) \neq \beta'(y)$,

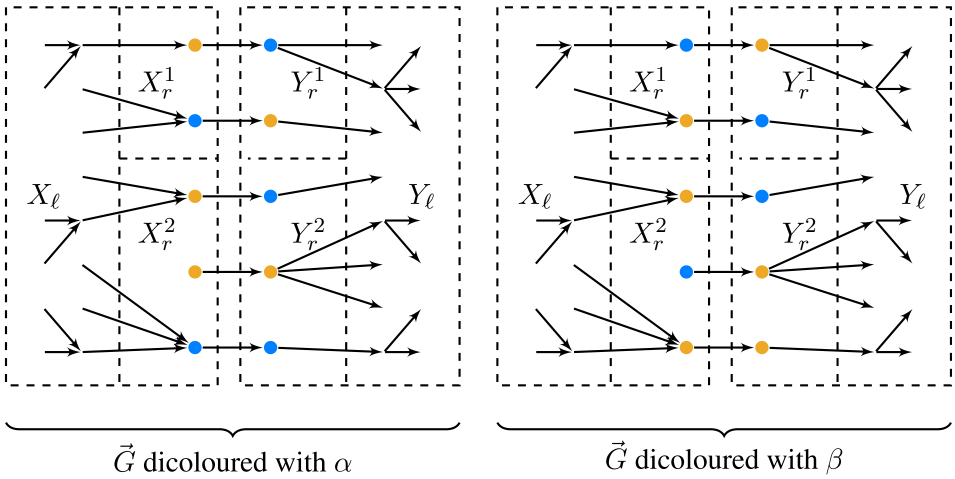


Figure 7.7: The partitioning of $V(\vec{G})$ into $X_r^1, X_r^2, X_\ell, Y_r^1, Y_r^2, Y_\ell$.

(ii) For any arc $xy \in M_r^2$, $\alpha'(x) = \beta'(x)$ *(and so* $\alpha'(y) = \beta'(y)$ *by (i)), and*

(iii) $s_\alpha + s_\beta \leq |X_r^2| + |Y_r^2|$.

Proof of claim. We consider the arcs xy of M_r^2 one after another and do the following recolourings depending on the colours of x and y in both α and β to get α' and β'.

- If $\alpha(x) = \alpha(y) = \beta(x) = \beta(y)$, then we recolour x in both α and β;
- Else if $\alpha(x) = \alpha(y) \neq \beta(x) = \beta(y)$, then we recolour x in α and we recolour y in β;
- Else if $\alpha(x) = \beta(x) \neq \alpha(y) = \beta(y)$, then we do nothing;
- Else if $\alpha(x) \neq \alpha(y) = \beta(x) = \beta(y)$, then we recolour x in β;
- Finally, if $\alpha(y) \neq \alpha(x) = \beta(x) = \beta(y)$, then we recolour y in β.

Each of these recolourings is valid because, when a vertex in X_r^2 (resp. Y_r^2) is recoloured, it gets a colour different from its only out-neighbour (resp. in-neighbour). Let α' and β' be the two resulting 2-dicolourings. By construction, α' and β' agree on $X_r^2 \cup Y_r^2$. For each arc $xy \in M_r^2$, either $\alpha(x) = \alpha'(x)$ or $\alpha(y) = \alpha'(y)$, and the same holds for β and β'. This implies that $s_\alpha + s_\beta \leq 2|M_r^2| = |X_r^2| + |Y_r^2|$. ◊

Claim 7.4.10. *There exists a redicolouring sequence from* α' *to some 2-dicolouring* $\widetilde{\alpha}$ *of length* s'_α *and a redicolouring sequence from* β' *to some 2-dicolouring* $\widetilde{\beta}$ *of length* s'_β *such that each of the following holds:*

(i) $\widetilde{\alpha}$ *and* $\widetilde{\beta}$ *agree on* $V(\vec{G}) \setminus (X_r^1 \cup Y_r^1)$*,*

(ii) α' *and* $\widetilde{\alpha}$ *agree on* $X_r \cup Y_r$*,*

(iii) β' *and* $\widetilde{\beta}$ *agree on* $X_r \cup Y_r$*,*

(iv) $X_\ell \cup Y_\ell$ *is monochromatic in* $\widetilde{\alpha}$ *(and in* $\widetilde{\beta}$ *by (i)), and*

(v) $s'_\alpha + s'_\beta \leq |X_\ell| + |Y_\ell|$.

Proof of claim. Observe that in both 2-dicolourings α' and β', we are free to recolour any vertex of $X_\ell \cup Y_\ell$ since there is no monochromatic arc from X to Y and both $\vec{G}\langle X\rangle$ and $\vec{G}\langle Y\rangle$ are acyclic. Let n_1 (resp. n_2) be the number of vertices in $X_\ell \cup Y_\ell$ that are coloured 1 (resp. 2) in both α' and β'. Without loss of generality, assume that $n_1 \leq n_2$. Then we set each vertex of $X_\ell \cup Y_\ell$ to colour 2 in both α' and β'. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ the resulting 2-dicolouring. Then $s'_\alpha + s'_\beta$ is exactly $|X_\ell| + |Y_\ell| + n_1 - n_2 \leq |X_\ell| + |Y_\ell|$. ♢

Claim 7.4.11. *There is a redicolouring sequence between $\widetilde{\alpha}$ and $\widetilde{\beta}$ of length $|X_r^1| + |Y_r^1|$.*

Proof of claim. By construction of $\widetilde{\alpha}$ and $\widetilde{\beta}$, we only have to exchange the colours of x and y for each arc $xy \in M_r^1$. Without loss of generality, we may assume that the colour of all vertices in $X_\ell \cup Y_\ell$ by $\widetilde{\alpha}$ and $\widetilde{\beta}$ is 1.

We first prove that, by construction, we can recolour any vertex of $X_r^1 \cup Y_r^1$ from 1 to 2. Assume not, then there is such a vertex $x \in X_r^1 \cup Y_r^1$ such that recolouring x from 1 to 2 creates a monochromatic directed cycle C. Since both $\vec{G}\langle X\rangle$ and $\vec{G}\langle Y\rangle$ are acyclic, C must contain an arc of M_r. Since M_r does not contain any monochromatic arc in $\widetilde{\alpha}$, then this arc must be incident to x. Now observe that colour 2, in $\widetilde{\alpha}$, induces an independent set on both $\vec{G}\langle X\rangle$ and $\vec{G}\langle Y\rangle$. This implies that C must contain at least 2 arcs in M_r. This is a contradiction, since recolouring x creates exactly one monochromatic arc in M_r.

Then, for each arc $xy \in M_r^1$, we can first recolour the vertex coloured 1 and then the vertex coloured 2. Note that we maintain the invariant that colour 2 induces an independent set on both $\vec{G}\langle X\rangle$ and $\vec{G}\langle Y\rangle$. We get a redicolouring sequence from $\widetilde{\alpha}$ to $\widetilde{\beta}$ in exactly $2|M_r^1| = |X_r^1| + |Y_r^1|$ steps. ♢

Combining the three claims, we finally proved that there exists a redicolouring sequence between α and β of length at most n. □

We will now prove Corollary 7.4.7, let us restate it first for convenience.

Corollary 7.4.7. *Let $\vec{G}$ be an oriented graph of order n with $\Delta_{\min}(\vec{G}) = \Delta \geq 1$, and let $k \geq \Delta + 1$. Then $\mathcal{D}_k(\vec{G})$ is connected and has diameter at most $2\Delta n$.*

Proof. We will show the result by induction on Δ.

Assume first that $\Delta = 1$, let $k \geq 2$. Let α be any k-dicolouring of $\vec{G}$ and γ be any 2-dicolouring of $\vec{G}$. To ensure that $\mathcal{D}_k(\vec{G})$ is connected and has diameter at most $2n$, it is sufficient to prove that there is a redicolouring sequence between α and γ of length at most n. Let H be the digraph induced by the set of vertices coloured 1 or 2 in α, and let J be $V(\vec{G}) \setminus V(H)$. By Theorem 7.4.6, since $\Delta_{\min}(H) \leq \Delta_{\min}(\vec{G}) \leq 1$, we know that there exists a redicolouring sequence, in H, from $\alpha_{|H}$ to $\gamma_{|H}$ of length at most $n(H)$. This redicolouring sequence extends in $\vec{G}$ because it only uses colours 1 and 2. Let α' be the obtained dicolouring of $\vec{G}$. Since $\alpha'(v) = \gamma(v)$ for every $v \in H$, we can recolour each vertex in J to its colour in γ. This proves that there is a redicolouring sequence between α and γ of length at most $n(H) + |J| = n(\vec{G})$. This ends the case $\Delta = 1$.

Assume now that $\Delta \geq 2$ and let $k \geq \Delta + 1$. Let α and β be two k-dicolourings of $\vec{G}$. By Corollary 3.3.6, we know that $\vec{\chi}(\vec{G}) \leq \Delta \leq k - 1$. We first show that there is a redicolouring sequence of length at most $2n$ from α to some $(k-1)$-dicolouring γ of $\vec{G}$. From α, whenever it is possible, we recolour each vertex coloured $1, 2$ or k with a colour of $\{3, \dots, k-1\}$ (when $k = 3$ we do nothing). Let $\widetilde{\alpha}$ be the obtained dicolouring, and let M be the set of vertices coloured in $\{3, \dots, k-1\}$ by $\widetilde{\alpha}$ (when $k = 3$, M is empty). We get that $H = \vec{G} - M$ satisfies $\Delta_{\min}(H) \leq 2$, since every vertex in H has at least one in-neighbour and one out-neighbour coloured c for every $c \in \{3, \dots, k-1\}$. By Corollary 3.3.6, there exists a 2-dicolouring $\gamma_{|H}$ of H. From $\widetilde{\alpha}_{|H}$, whenever it is possible, we recolour a vertex coloured 1 or 2 to colour k. Let $\widehat{\alpha}$ be the resulting dicolouring, and $\widehat{H}$ be the subdigraph of H induced by the vertices coloured 1 or 2 in $\widehat{\alpha}$. We get that $\Delta_{\min}(\widehat{H}) \leq 1$ since every vertex in $\widehat{H}$ has, in $\vec{G}$, at least one in-neighbour and one out-neighbour coloured c for every $c \in \{3, \dots, k\}$. In at most $n(\widehat{H})$ steps, using Theorem 7.4.6, we can recolour the vertices of $V(\widehat{H})$ to their colour in $\gamma_{|H}$ (using only colours 1 and 2). Then we can recolour each vertex coloured k to its colour in $\gamma_{|H}$. This results in a redicolouring sequence of length at most $2n$ from α to some $(k-1)$-dicolouring γ of $\vec{G}$, since colour k is not used in the resulting dicolouring (recall that M is coloured with $\{3, \dots, k-1\}$).

Now, from β, whenever it is possible, we recolour each vertex to colour k. Let $\widetilde{\beta}$ be the obtained k-dicolouring, and let N be the set of vertices coloured k in $\widetilde{\beta}$. We get that $J = \vec{G} - N$ satisfies $\Delta_{\min}(J) \leq \Delta - 1$. Thus, by induction, there exists a redicolouring sequence from $\widetilde{\beta}_{|J}$ to $\gamma_{|J}$, in at most $2(\Delta - 1)n(J)$ steps (using only colours $\{1, \dots, k-1\}$). Since N is coloured k in $\widetilde{\beta}$, this extends to a redicolouring sequence in $\vec{G}$. Now, since γ does not use colour k, we can recolour each vertex in N to its colour in γ. We finally get a redicolouring sequence from β to γ of length at most $2(\Delta - 1)n$. Concatenating the redicolouring sequence from α to γ and the one from γ to β, we get a redicolouring sequence from α to β in at most $2\Delta n$ steps. □

7.5 Digraphs having bounded treewidth

This section is devoted to generalisations of Theorem 7.1.8. In Subsection 7.5.1 we show the following general result which makes a connection between the recolourability of a digraph and the recolourability of its underlying graph.

Theorem 7.5.1. *Let $\mathcal{G}$ be a family of undirected graphs, closed under edge-deletion and with bounded chromatic number, and let $k \geq \chi(\mathcal{G})$ (i.e. $k \geq \chi(G)$ for every $G \in \mathcal{G}$) be such that, for every graph $G \in \mathcal{G}$, the diameter of $\mathcal{C}_k(G)$ is bounded by $f(n(G))$ for some function f. Then for every digraph D such that $\mathrm{UG}(D) \in \mathcal{G}$, the diameter of $\mathcal{D}_k(D)$ is bounded by $2f(n(D))$.*

Theorem 7.5.1 directly extends to digraphs numerous known results about planar graphs recolouring. We discuss further these applications in Section 7.7.

On the other hand, since removing edges does not increase the treewidth of a graph, the following is a consequence of Theorems 7.1.8 and 7.5.1 (by taking $\mathcal{G} = \{G \mid \mathrm{tw}(G) \leq \ell\}$ for some constant ℓ).

Corollary 7.5.2. *Let $k \in \mathbb{N}$ and D be a digraph. If $k \geq \mathrm{tw}(\mathrm{UG}(D)) + 2$, then $\mathrm{diam}(\mathcal{D}_k(D)) = O(n^2)$.*

However, the treewidth of the underlying graph of D is not a satisfying extension of treewidth to digraphs, since it does not take under consideration the orientations in D. There exist, at least, four well-known generalisations of treewidth to digraphs: the directed treewidth (introduced in [103], see also [146]), the $\mathscr{D}$-width (introduced in [149], see also [150]), the DAG-width (introduced in [25]) and the Kelly-width (introduced in [100]).

Recall that an *out-arborescence* is a rooted tree in which every edge is oriented away from the root. A *directed tree-decomposition* $(T, \mathcal{W}, \mathcal{X})$ of a digraph $D = (V, A)$ consists of an out-arborescence $T = (I, F)$ rooted in $r \in I$, a partition $\mathcal{W} = (W_t)_{t \in I}$ of V into non-empty parts, and a family $\mathcal{X} = (X_e)_{e \in F}$ of subsets of vertices of D such that, for every $tt' \in F$ we have:

1. $X_{tt'} \cap (\bigcup_{t'' \in T_{t'}} W_{t''}) = \varnothing$ (where $T_{t'}$ denotes the subtree of T rooted in t'), and

2. for every directed walk P with both ends in $\bigcup_{t'' \in T_{t'}} W_{t''}$ and some internal vertex not in $\bigcup_{t'' \in T_{t'}} W_{t''}$, it holds that $V(P) \cap X_{tt'} \neq \varnothing$.

The *width* of $(T, \mathcal{W}, \mathcal{X})$ equals $\max_{t \in I} |H_t| - 1$, where $H_t = W_t \cup \bigcup_{e \in F, t \in e} X_e$, and the *directed treewidth* of D, denoted by $\operatorname{dtw}(D)$, is the minimum width of its directed tree-decompositions. Recall that the treewidth of an undirected graph is always at least its degeneracy. However, it is well-known that there exist digraphs with arbitrary large $\min$-degeneracy and directed tree-width exactly one. We include a proof for completeness.

Proposition 7.5.3. *For every integer d, there exists a digraph $D = (V, A)$ such that every vertex $v \in V$ satisfies $d^+(v) \geq d$, $d^-(v) \geq d$, and* $\operatorname{dtw}(D) = 1$.

Proof. Let T be a tree rooted in $r \in V(T)$ with depth at least d (that is, all leaves are at distance at least d from the root), such that every non-leaf vertex has at least d children. We orient each edge uv of T from the parent to its child. Then we add every arc uv such that v is an ancestor of u. In the obtained digraph D, every vertex has out-degree at least d.

Then we add a disjoint copy $\overleftarrow{D}$ of D in which we reverse every arc, so in $\overleftarrow{D}$ every vertex has in-degree at least d. We finally add every arc from $\overleftarrow{D}$ to D.

In the resulting digraph, every vertex has out-degree and in-degree at least d. Moreover, the directed treewidth of a digraph is equal to the maximum directed treewidth of its strongly connected components. For each edge uv of T, such that u is the parent of v, we label uv with u. Then T, together with this labelling, is a directed tree-decomposition of both D and $\overleftarrow{D}$ and has width 1. This follows from the fact that every directed cycle of D containing a vertex must also contain its father in T. □

The following proposition shows that, dealing with directed treewidth, c-degeneracy, compared to $\min$-degeneracy, appears to be a better generalisation of the undirected one.

Proposition 7.5.4. *For every digraph D, it holds that* $\operatorname{dtw}(D) \geq \delta_c^*(D)$.

Proof. Consider an optimal directed tree-decomposition $(T, \mathcal{W}, \mathcal{X})$ of D. If t is a leaf of T and v is a vertex in W_t, then $H_t \setminus \{v\}$ intersects every directed cycle of D containing v. Thus, $d_c(v) \leq \operatorname{dtw}(D)$. Moreover, since t is a leaf of T, it is easy to verify that removing v from D does not increase its directed treewidth, and we can repeat the same argument in $D \setminus \{v\}$. □

Proposition 7.5.4 implies that $\vec{\chi}(D) \leq \operatorname{dtw}(D) + 1$ for every digraph D and, together with Theorems 7.3.2, 7.3.9 and 7.3.13, implies the following.

Corollary 7.5.5. *Let $k \in \mathbb{N}$ and D be a digraph on n vertices.*

(i) if $k \geq \operatorname{dtw}(D) + 2$ then D is k-mixing and $\operatorname{diam}(\mathcal{D}_k(D)) = O_{\operatorname{dtw}}(n^{\operatorname{dtw}+1})$,

(ii) if $k \geq \frac{3}{2}(\operatorname{dtw}(D) + 1)$ then $\operatorname{diam}(\mathcal{D}_k(D)) = O(n^2)$*, and*

(iii) if $k \geq 2(\mathrm{dtw}(D)+1)$ *then* $\mathrm{diam}(\mathcal{D}_k(D)) \leq (\mathrm{dtw}(D)+1)n$*.*

The interested reader may have a look at [25] (resp. [100]) and see that DAG-width (resp. Kelly-width) is bounded below by min-degeneracy (resp. by min-degeneracy plus one). Thus, Corollary 7.5.5 also holds for DAG-width (resp. Kelly-width minus one).

In Subsection 7.5.2, we show that the proof of Theorem 7.1.8 extends to digraphs using $\mathscr{D}$-width. A *$\mathscr{D}$-decomposition* of a digraph $D=(V,A)$ is a pair $(T,\mathcal{X})$ such that $T=(I,F)$ is an undirected tree and $\mathcal{X}=(X_v)_{v\in I}$ is a family of subsets (called *bags*) of V indexed by the nodes of T, which satisfies Property ($\star$) stated below, for which we first need a definition. For a vertex subset $S\subseteq V$, let the *support* of S in $(T,\mathcal{X})$, denoted by T_S, be the subgraph of T with vertex-set $\{t\in I \mid X_t\cap S\neq\varnothing\}$ and edge-set $\{tt'\in F \mid \exists u\in S\cap X_t\cap X_{t'}\}$. A $\mathscr{D}$-decomposition must ensure the following property:

$$\forall S\subseteq V \textit{ such that } D\langle S\rangle \textit{ is strongly connected, } T_S \textit{ is a non-empty subtree of } T. \tag{$\star$}$$

Similarly to undirected tree-decompositions, the *width* of $(T,\mathcal{X})$ is the maximum size of its bags minus one, and the *$\mathscr{D}$-width* of D, denoted by $\mathscr{D}\mathrm{w}(D)$, is the minimum width of its $\mathscr{D}$-decompositions. Note that, for every $v\in V$, $D\langle\{v\}\rangle$ is strongly connected, and therefore Property ($\star$) can be seen as a generalisation of the basic properties of undirected tree-decompositions that state that the set of bags containing some vertex v must induce a (connected) subtree and that every vertex must belong to at least one bag. Note also that, if $\{u,v\}$ is a digon of D, Property ($\star$) implies that u and v must belong to a common bag of $(T,\mathcal{X})$. Hence, every bidirected graph G satisfies $\mathrm{tw}(G)=\mathscr{D}\mathrm{w}(\overleftrightarrow{G})$, and the following is actually a generalisation of Theorem 7.1.8. Our proof is strongly based on the proof of Theorem 7.1.8.

Theorem 7.5.6. *If $D=(V,A)$ is a digraph of order n with $\mathscr{D}\mathrm{w}(D)\leq k-1$, then*

$$\mathrm{diam}(\mathcal{D}_{k+1}(D)) \leq 2(n^2+n).$$

Note that the bound of Theorem 7.5.6 is asymptotically sharp (up to a constant factor) since Theorem 7.1.8 is already known to be sharp. Finally, observe that the digraph D built in the proof of Proposition 7.5.3 also satisfies $\mathscr{D}\mathrm{w}(D)=1$ but $\delta^*_{\min}(D)\geq d$. Again, the following easy proposition shows that, dealing with the $\mathscr{D}$-width, c-degeneracy, compared to min-degeneracy, appears to be a better generalisation of the undirected one.

Proposition 7.5.7. *For every digraph D, it holds that $\mathscr{D}\mathrm{w}(D)\geq\delta^*_c(D)$.*

Proof. Consider an optimal directed $\mathscr{D}$-decomposition $(T,\mathcal{X}=(X_t)_{t\in V(T)})$ of D. Let t be a leaf of T and v be a vertex in X_t that belongs to no other bag $X_{t'}$ (this is possible unless $X_t\subseteq X_{t'}$, $tt'\in E(T)$, in which case we just remove the bag t from the decomposition). We claim that $X_t\setminus\{v\}$ intersects every directed cycle of D containing v (which directly implies $d_c(v)\leq\mathscr{D}\mathrm{w}(D)$). Assume not, and let C be a directed cycle such that $X_t\cap V(C)=\{v\}$. Then, since $D\langle V(C)\rangle$ is strongly connected, $T_{V(C)}$ must be connected. This is a contradiction since t is an isolated vertex in $T_{V(C)}$. □

Analogously to the directed treewidth, Proposition 7.5.7, together with Theorems 7.3.2, 7.3.9 and 7.3.13, implies the following (note that the two first items are also implied by Theorem 7.5.6, but the third one is not).

Corollary 7.5.8. *Let $k \in \mathbb{N}$ and D be a digraph on n vertices.*

(i) if $k \geq \mathscr{D}\mathrm{w}(D) + 2$ then D is k-mixing and $\mathrm{diam}(\mathcal{D}_k(D)) = O_{\mathscr{D}\mathrm{w}}(n^{\mathscr{D}\mathrm{w}+1})$,

(ii) if $k \geq \frac{3}{2}(\mathscr{D}\mathrm{w}(D) + 1)$ then $\mathrm{diam}(\mathcal{D}_k(D)) = O(n^2)$, and

(iii) if $k \geq 2(\mathscr{D}\mathrm{w}(D) + 1)$ then $\mathrm{diam}(\mathcal{D}_k(D)) \leq (\mathscr{D}\mathrm{w}(D) + 1)n$.

7.5.1 Using the underlying graph to bound the diameter of $\mathcal{D}_k(D)$

This section is devoted to the proof of Theorem 7.5.1.

Theorem 7.5.1. *Let $\mathcal{G}$ be a family of undirected graphs, closed under edge-deletion and with bounded chromatic number, and let $k \geq \chi(\mathcal{G})$ (i.e. $k \geq \chi(G)$ for every $G \in \mathcal{G}$) be such that, for every graph $G \in \mathcal{G}$, the diameter of $\mathcal{C}_k(G)$ is bounded by $f(n(G))$ for some function f. Then for every digraph D such that $\mathrm{UG}(D) \in \mathcal{G}$, the diameter of $\mathcal{D}_k(D)$ is bounded by $2f(n(D))$.*

Proof. Let $D = (V, A)$ be such a digraph, and let γ be any proper k-colouring of $\mathrm{UG}(D)$. We will show, for any k-dicolouring α of D, that there is a redicolouring sequence between α and γ of length at most $f(n)$, showing the result. Let $G_\alpha = (V, E)$ be the undirected graph where $E = \{uv \mid uv \in A, \alpha(u) \neq \alpha(v)\}$.

First, observe that α is a proper k-colouring of G_α by construction of G_α. Note also that γ is a proper k-colouring of G_α because G_α is a subgraph of $\mathrm{UG}(D)$. Moreover, since G_α is a subgraph of $\mathrm{UG}(D)$, by assumption on $\mathcal{G}$, we know that there exists a recolouring sequence between α and γ in G_α of length at most $f(n)$.

Next, we show that every proper k-colouring of G_α is a dicolouring of D, implying that the recolouring sequence between α and γ in G_α is also a redicolouring sequence between α and γ in D. For purpose of contradiction, let us assume that β is a proper k-colouring of G_α but D, coloured with β, contains a monochromatic directed cycle C. Then, by construction of G_α, for each arc xy of C, we must have $\alpha(x) = \alpha(y)$, for otherwise xy would be a monochromatic edge in G_α. This shows that C is monochromatic in D coloured with α, a contradiction. □

7.5.2 Case of digraphs with bounded $\mathscr{D}$-width

This section is devoted to the proof of Theorem 7.5.6.

Theorem 7.5.6. *If $D = (V, A)$ is a digraph of order n with $\mathscr{D}\mathrm{w}(D) \leq k - 1$, then*

$$\mathrm{diam}(\mathcal{D}_{k+1}(D)) \leq 2(n^2 + n).$$

The following claim can be easily deduced from the definition of a $\mathscr{D}$-decomposition.

Claim 7.5.9. *Let $(T, \mathcal{X} = (X_v)_{v \in V(T)})$ be a $\mathscr{D}$-decomposition of a digraph $D = (V, A)$ and $tt' \in E(T)$ such that $v \in X_{t'} \setminus X_t$. Then, $(T, \mathcal{X}' = (X'_v)_{v \in V(T)})$ such that $X'_u = X_u$ for all $u \neq t$ and $X'_t = X_t \cup \{v\}$ is a $\mathscr{D}$-decomposition of $D = (V, A)$. Moreover, if $|X_t| < |X_{t'}|$, $(T, \mathcal{X}')$ has the same width as $(T, \mathcal{X})$.*

A $\mathscr{D}$-decomposition $(T, \mathcal{X})$ is *reduced* if, for every $tt' \in E(T)$, $X_t \setminus X_{t'}$ and $X_{t'} \setminus X_t$ are non-empty. It is easy to see that any digraph D admits an optimal (i.e., of width $\mathscr{D}\mathrm{w}(D)$) $\mathscr{D}$-decomposition which is reduced (indeed, if $X_t \subseteq X_{t'}$ for some edge $tt' \in E(T)$, then contract this edge and remove X_t from $\mathcal{X}$).

A $\mathscr{D}$-decomposition $(T, \mathcal{X})$ of $\mathscr{D}$-width $k \geq 0$ is *full* if every bag has size exactly $k+1$. A $\mathscr{D}$-decomposition $(T, \mathcal{X})$ is *valid* if $|X_t \setminus X_{t'}| = |X_{t'} \setminus X_t| = 1$ for every $tt' \in E(T)$. Note that any valid $\mathscr{D}$-decomposition is full and reduced. Note also that, if $(T, \mathcal{X})$ is valid and $t \in V(T)$ is a leaf of T, then there exists a (unique) vertex $v \in V$ that belongs only to the bag X_t. Such a vertex v is called a *baby*.

Lemma 7.5.10. *Every digraph $D = (V, A)$ admits a valid $\mathscr{D}$-decomposition of width $\mathscr{D}\mathrm{w}(D)$.*

Proof. Let $(T, \mathcal{X})$ be an optimal reduced $\mathscr{D}$-decomposition of $D = (V, A)$, which exists by the remark above the lemma. We will progressively modify $(T, \mathcal{X})$ in order to make it first full and then valid.

While the current decomposition is not full, let $tt' \in E(T)$ such that $|X_t| < |X_{t'}| = \mathscr{D}\mathrm{w}(D) + 1$ and let $v \in X_{t'} \setminus X_t$. Add v to X_t. The obtained decomposition is still a $\mathscr{D}$-decomposition of width $\mathscr{D}\mathrm{w}(D)$ by Claim 7.5.9. Moreover, the updated decomposition remains reduced all along the process, as since $|X_t| < |X_{t'}|$ and the initial decomposition is reduced, $X_{t'}$ must contain another vertex $u \neq v$ with $u \notin X_t$. Eventually, the obtained decomposition $(T, \mathcal{X})$ becomes an optimal full $\mathscr{D}$-decomposition.

Now, while $(T, \mathcal{X})$ is not valid, let $tt' \in E(T)$, $x, y \in X_t \setminus X_{t'}$ and $u, v \in X_{t'} \setminus X_t$ (such an edge of T and four distinct vertices of V must exist since $(T, \mathcal{X})$ is full and reduced but not valid). Then, add a new node t'' to T, with corresponding bag $X_{t''} = (X_{t'} \setminus \{u\}) \cup \{x\}$ and replace the edge tt' in T by the two edges tt'' and $t''t'$. Clearly, subdividing the edge tt' by adding a bag $X_{t''} = X_{t'}$ still leads to an optimal full (but not reduced) $\mathscr{D}$-decomposition of the same width. Then, adding x to $X_{t''}$ makes that $(T, \mathcal{X})$ remains a $\mathscr{D}$-decomposition (by the first statement of Claim 7.5.9). Finally, we must prove that removing u from $X_{t''}$ preserves the fact that we still have a $\mathscr{D}$-decomposition. Indeed, let S be a strong subset whose support T_S (before the subdivision) contains tt' (clearly, the other strong subsets are not affected by the change in the decomposition). It must be because of some vertex in $z \in X_t \cap X_{t'}$ and so $z \in X_{t''}$. Therefore, the support T_S, obtained after the subdivision and the modifications to $X_{t''}$, contains both edges tt'' and $t''t'$, and therefore it remains connected. Note that, after the modifications, $(T, \mathcal{X})$ is still full and reduced.

Note that, after the application of each step as described above, either the maximum of $|X_t \setminus X_{t'}|$ over all edges $tt' \in E(T)$, or the number of edges $tt' \in E(T)$ that maximise $|X_t \setminus X_{t'}|$, strictly decreases, and none of these two quantities increases. Therefore, the process terminates, and eventually $(T, \mathcal{X})$ becomes an optimal valid $\mathscr{D}$-decomposition. $\square$

Given a valid $\mathscr{D}$-decomposition $(T, \mathcal{X})$ of a digraph $D = (V, A)$, two vertices $u, v \in V$ are *parents*, denoted by $u \sim_p v$, if their supports T_u and T_v (we use T_v instead of $T_{\{v\}}$ for denoting the support of a single vertex $\{v\}$) are vertex-disjoint and there is an edge $tt' \in E(T)$ with $t \in V(T_v)$ and $t' \in V(T_u)$. Let $\sim_{(T,\mathcal{X})}$ be the transitive closure of $\sim_p$.

Lemma 7.5.11. *Let $(T, \mathcal{X})$ be an optimal valid $\mathscr{D}$-decomposition of a digraph $D = (V, A)$. Then, $\sim_{(T,\mathcal{X})}$ defines an equivalence relation on V which has exactly $\mathscr{D}\mathrm{w}(D) + 1$ classes. Moreover, the vertices of each class induce an acyclic subdigraph of D.*

Proof. The facts that $\sim_{(T,\mathcal{X})}$ is well-defined and that there are $\mathscr{D}\mathrm{w}(D)+1$ classes follow from the fact that $(T, \mathcal{X})$ is valid. Now, let C be any equivalence class of $\sim_{(T,\mathcal{X})}$. For the purpose of contradiction, let us assume that $D\langle C \rangle$ contains a directed cycle Q. By definition of a $\mathscr{D}$-decomposition, the support T_Q must induce a subtree of T. Since, by definition of $\sim_{(T,\mathcal{X})}$, the supports of the vertices of Q are pairwise vertex-disjoint, the support T_Q must consist precisely of the disjoint union of the supports of the vertices of Q, and hence T_Q is not connected, a contradiction. $\square$

Given a valid $\mathscr{D}$-decomposition $(T, \mathcal{X})$ of $D = (V, A)$, the corresponding equivalence relation $\sim_{(T,\mathcal{X})}$, and a subset $X \subseteq V$, a k-colouring α of D (which is not necessarily a dicolouring) is *X-coherent with respect to* $(T, \mathcal{X})$ if, for every $u, v \in X$ such that $u \sim_p v$, and for every $t \in V(T_v)$, v is the unique vertex coloured $\alpha(v)$ in X_t, and $\alpha(u) = \alpha(v)$. In what follows, we will just say *X-coherent*, as the $\mathscr{D}$-decomposition will always be clear from the context.

Claim 7.5.12. *Let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of a digraph $D = (V, A)$ of width $k-1 \geq 0$. Then, D admits a k-colouring that is V-coherent.*

Proof of claim. Consider any ordering $C_1, \dots, C_{k+1}$ of the classes of $\sim_{(T,\mathcal{X})}$. Let α be the colouring associating with each vertex v the index i such that $v \in C_i$. Then α is V-coherent. $\Diamond$

The following claim is straightforward, so we skip its proof.

Claim 7.5.13. *Let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of a digraph $D = (V, A)$ of width $k-1 \geq 0$ and let α be a k-colouring of D that is V-coherent. Then, the equivalence classes of $\sim_{(T,\mathcal{X})}$ are precisely $\alpha_1, \dots, \alpha_k$ (the colours classes of α). In particular, by Lemma 7.5.11, α is a k-dicolouring.*

Lemma 7.5.14. *Let $D = (V, A)$ be a digraph of order n and let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1 = \mathscr{D}\mathrm{w}(D)$. Let α and β be two $(k+1)$-dicolourings of D that are V-coherent. Then, α and β are at distance at most $2n$ in $\mathcal{D}_{k+1}(D)$.*

Proof. We prove the existence of a redicolouring sequence from α to β in $\mathcal{D}_{k+1}(D)$ that recolours each vertex at most twice.

Let $S_1, \dots, S_k$ be the equivalence classes of $\sim_{(T,\mathcal{X})}$. By Claim 7.5.13, each S_i corresponds exactly to one colour class of α and exactly one colour class of β. In particular, both α and β use indeed k colours (not necessarily the same). Consider the undirected complete graph H on k vertices $x_1, \dots, x_k$, and the two colourings α_H, β_H of H defined as $\{\alpha_H(x_i)\} = \alpha(S_i)$ and $\{\beta_H(x_i)\} = \beta(S_i)$. It is known (see [31, Lemma 5]) that there is a redicolouring sequence in H between α_H and β_H in which every vertex is recoloured at most twice. This directly extends to a redicolouring sequence between α and β (when x_i is recoloured with colour c in H, we recolour every vertex in S_i with c in D). Note that this is indeed a redicolouring sequence because at each step of the sequence, in the corresponding colouring, every colour class is a subset of some S_i, which induces an acyclic subdigraph by Lemma 7.5.11. $\square$

Given a tree T rooted in $r \in V(T)$ and two vertices u, v of T, we say that v is a *descendant* of u if u belongs to the path between r and v in T.

Lemma 7.5.15. *Let $D = (V, A)$ be a digraph of order n and $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1 \geq 0$. Let T be rooted in $r \in V(T)$. Let α be a $(k+1)$-colouring of D that is $(V \setminus X_r)$-coherent and let $c \in [k+1]$ such that $\alpha(v) \neq c$ for all $v \in X_r$. Then, for every $t, t' \in V(T)$ with t' being a descendant of t, if there exists $v \in X_t$ with $\alpha(v) = c$, then v is the unique vertex of X_t coloured with c and there exists a unique $u \in X_{t'}$ with $\alpha(u) = c$.*

Proof. For purpose of contradiction, let us assume that there exist $t, t' \in V(T)$ such that t' is a descendant of t, a vertex in X_t is coloured with c, and no vertex in $X_{t'}$ is coloured with c. Over all possible such pairs $\{t, t'\}$, we choose one such that the distance between t and t' in T is minimum. Then t' is a child of t.

Let $v \in X_t$ such that $\alpha(v) = c$ and let $\{u\} = X_{t'} \setminus X_t$, where we have used that $(T, \mathcal{X})$ is valid. Note that $u \sim_{(T,\mathcal{X})} v$ and that, since T_u is connected, $u \notin X_r$. Since α is $(V \setminus X_r)$-coherent, we must have $\alpha(u) = \alpha(v) = c$, a contradiction.

The uniqueness of u and v comes from the fact that $u, v \notin X_r$ since $\alpha(v) = \alpha(u) = c$, and because by hypothesis α is $(V \setminus X_r)$-coherent. □

Lemma 7.5.16. *Let $D = (V, A)$ be a digraph of order n and let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1$, and let T be rooted in $r \in V(T)$. Let $y \in X_r$ and let α be a $(k + 1)$-dicolouring of D that is $(V \setminus (X_r \setminus \{y\}))$-coherent. Let D' be the digraph obtained by identifying y and all vertices v such that $y \sim_{(T,\mathcal{X})} v$ and let α' be the dicolouring of D arising from α. If there exists a $(k + 1)$-dicolouring β' of D' that can be reached from α' by recolouring each vertex at most once, then the $(k + 1)$-dicolouring β that naturally extends β' to D (i.e., $\beta(v) = \beta'(y)$ for all $v \sim_{(T,\mathcal{X})} y$) can be reached from α by recolouring each vertex at most once.*

Proof. This follows from the fact that α is $(V \setminus (X_r \setminus \{y\}))$-coherent, and thus every bag X_t has exactly one vertex of the colour of y, which is the colour of the unique vertex $v \sim_{(T,\mathcal{X})} y$ that belongs to X_t. Hence, to go from α to β, we follow the same redicolouring sequence for every vertex $u \nsim_{(T,\mathcal{X})} y$, and when we recolour y in D', we simply recolour every vertex $v \sim_{(T,\mathcal{X})} y$ with the same colour as y. □

Claim 7.5.17. *Let $D = (V, A)$ be a digraph of order n and let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1$. Let α be a $(k + 1)$-dicolouring of D and $x \in V(D)$ be any vertex. Let $c \in [k + 1]$ be a colour such that, for every vertex $v \in \bigcup_{x \in X_t} X_t$, $\alpha(v) \neq c$. Then the $(k + 1)$-colouring obtained from α by recolouring x with c is a dicolouring.*

Proof of claim. Assume this is not the case, and recolouring x with c creates a monochromatic directed cycle C. Then since $D\langle V(C)\rangle$ is strongly connected, the support $T_{V(C)}$ of $V(C)$ is a non-empty subtree. This implies that there exists $y \in V(C) \setminus \{x\}$ such that y and x belong to one same bag X_t. This contradicts the choice of c. ♢

Lemma 7.5.18. *Let $D = (V, A)$ be a digraph of order n and let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1$, and let T be rooted in $r \in V(T)$. Let α be a $(k + 1)$-dicolouring of D that is $(V \setminus X_r)$-coherent and such that the colour c does not appear in X_r, i.e., there exists $c \in [k + 1]$ such that $\alpha(v) \neq c$ for all $v \in X_r$. Then, there exists a $(k + 1)$-dicolouring β of D and a redicolouring sequence $\alpha = \gamma_1, \dots, \gamma_\ell = \beta$ such that:*

- *β is $(V \setminus X_r)$-coherent,*
- *$\beta(v) \neq c$ for all $v \in V$,*
- *every vertex of $D \setminus X_r$ is recoloured at most once,*
- *no vertex of X_r is recoloured, and*
- *if x_i is the vertex recoloured between γ_i and γ_{i+1}, then, for every vertex $v \in \bigcup_{x_i \in X_t} X_t$, $\gamma_i(v) \neq \gamma_{i+1}(x_i)$.*

Proof. The proof is by induction on $k-1+n(T)$, where $k-1$ is the width of $(T, \mathcal{X})$. If $k-1=0$, then D is acyclic and colour c can be eliminated by recolouring every vertex at most once with a same colour distinct from c. Note that the vertices in X_r are not recoloured and the last condition holds trivially since every bag has size one.

If $n(T) = 1$, the result holds trivially since the colour c does not appear in X_r, so we may take $\beta = \alpha$. Hence, r must have at least one child. Let us fix one child v of r, and let $\{y\} = X_v \setminus X_r$. Let T_v be the subtree of T rooted in v and let D_v be the subdigraph of D induced by $\bigcup_{t \in V(T_v)} X_t$.

- If $\alpha(y) \neq c$, then c does not appear in X_v. Let $(T_v, \mathcal{Y}) = (T_v, \{X_t \mid t \in V(T_v)\})$ be the decomposition of D_v obtained from T. Let D'_v be the digraph obtained from D_v by identifying y with all vertices of its class with respect to $(T_v, \mathcal{Y})$. Note that $(T_v, \mathcal{Y})$ is a full decomposition of D'_v. By contracting each edge $tt' \in E(T_v)$ such that $Y_t = Y_{t'}$ (in D'), we obtain $(T'_v, \mathcal{Y}')$ a valid decomposition of D'_v.

  Note that $n(T'_v) < n(T)$ and the width $(T'_v, \mathcal{Y}')$ equals the width of $(T_v, \mathcal{X})$. Hence, by induction, there exists a $(k+1)$-dicolouring β'_v of D'_v that is $(V(D'_v) \setminus X_v)$-coherent and such that $\beta'_v(w) \neq c$ for all $w \in V(D'_v)$. Moreover, there is a redicolouring sequence from α'_v, the dicolouring α restricted to D'_v, to β'_v such that every vertex of $D'_v \setminus X_v$ is recoloured at most once, and vertices in X_v are not recoloured. Note finally that, whenever a vertex x is recoloured, it is recoloured with a colour that is not appearing in $\bigcup_{x \in X_t} X_t$.

  By Lemma 7.5.16, there exists a $(k+1)$-dicolouring β_v of D_v that is $(V(D_v) \setminus X_v)$-coherent and such that $\beta_v(w) \neq c$ for every vertex $w \in V(D_v)$. Moreover, there is a redicolouring sequence $\gamma_v = (\gamma_1, \cdots, \gamma_\ell)$ from $\gamma_1 = \alpha_v$, the dicolouring α restricted to D_v, to $\gamma_\ell = \beta_v$ such that every vertex of $D_v \setminus X_v$ is recoloured at most once. Furthermore, note that β_v is indeed $(V(D_v) \setminus (X_v \setminus \{y\}))$-coherent because we identified y with all vertices of its class in D'_v.

  Along this redicolouring sequence γ_v, when a vertex x is recoloured between γ_i and γ_{i+1}, let us show that, for every vertex $z \in \bigcup_{x \in X_t} X_t$, $\gamma_i(z) \neq \gamma_{i+1}(x)$. Let us first assume that x does not belong to the class of y with respect to $(T_v, \mathcal{X})$. If $z \sim_{(T_v, \mathcal{X})} y$, then by induction, $\gamma_{i+1}(x) \neq \gamma_i(y) = \gamma_i(z)$. Otherwise ($z$ not in the class of y), by induction, we directly have that $\gamma_{i+1}(x) \neq \gamma_i(z)$. Second, let us assume that x belongs to the class of y. Hence, $z \nsim_{(T_v, \mathcal{X})} y$ since x and z belong to a same bag. By induction,

  Finally, this redicolouring sequence in D_v is indeed a redicolouring sequence in D because of the property above and by Claim 7.5.17.

- If $\alpha(y) = c$, by Lemma 7.5.15 and because $(T, \mathcal{X})$ is $(V \setminus X_r)$-coherent, every bag in T_v contains exactly one vertex coloured c and the set of vertices coloured with c is precisely $Y = \{w \in X_t \mid t \in V(T_v), w \sim_{(T,\mathcal{X})} y\}$. The reduced decomposition obtained from $(T_v, \mathcal{Y}) = (T_v, \{X_t \setminus Y \mid t \in V(T_v)\})$ is a valid decomposition of $D'_v = D_v - Y$ of width $k-2$. Observe that it is full because we remove exactly one vertex from each bag. Let c' be a colour that does not appear in $X_v \setminus \{y\}$. Observe that the width of $(T_v, \mathcal{Y})$ is at most $k-2$. Thus, by induction, there exists a k-dicolouring β' of D'_v that is $(V(D'_v) \setminus X_v)$-coherent and such that $\beta'(w) \notin \{c, c'\}$ for every vertex $w \in V(D'_v)$ and β' can be obtained from α'_v, the restriction of α to D'_v, by recolouring each vertex of $V(D'_v) \setminus X_v$ at most once. By then recolouring all vertices of Y to c', we obtain a $(k+1)$-dicolouring β of D_v that is $(V(D_v) \setminus (X_v \setminus \{y\}))$-coherent and such that $\beta(w) \neq c$ for every vertex $w \in V(D_v)$

and β can be obtained from α_v, the restriction of α to D_v, by recolouring each vertex of $D_v \setminus (X_v \setminus \{y\})$ at most once, as we wanted to prove.

Along the obtained redicolouring sequence, by induction, when a vertex x that is not in the class y is recoloured, it is recoloured with a colour different from the colours of $\bigcup_{x\in X_t}(X_t \setminus Y)$. Since Y is coloured with c, and no vertex is recoloured with c, it is recoloured with a colour different from the colours of $\bigcup_{x\in X_t}(X_t\setminus Y)$. Finally, when $y' \sim_{(T,\mathcal{X})} y$ is recoloured, it is recoloured with c' which is a colour that is not appearing in D_v. This shows the last property.

Finally, this redicolouring sequence in D_v is indeed a redicolouring sequence in D because of the property above and by Claim 7.5.17.

Repeating the process above for each child v of r, we finally obtain a redicolouring sequence from α to some $(k+1)$-dicolouring β such that $\beta(w) \neq c$ for all $w \in V$. Moreover, at each step, in both cases, no vertex of X_r is recoloured, so no vertex of X_r is recoloured all along the redicolouring sequence. Furthermore, if v and v' are two children of r, and x is a vertex of D that belongs to $V(D_v) \cap V(D_{v'})$, then x also belongs to X_r, implying that x is not recoloured. Thus, every vertex that is recoloured belongs to $V(D_v)$ for exactly one child v of r, implying that, in the obtained redicolouring sequence, every vertex of $D \setminus X_r$ is recoloured at most once.

Finally, note that β is $(V \setminus X_r)$-coherent because, in both cases, the obtained dicolouring β_v is $(V \setminus \{X_v \setminus \{y\}\})$-coherent. □

Lemma 7.5.19. *Let $D = (V, A)$ be a digraph of order n and $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of $D = (V, A)$ of width $k - 1 \geq \mathscr{D}\mathrm{w}(D)$. For every $(k+1)$-dicolouring α of D there exists a V-coherent $(k+1)$-colouring β of D such that α and β are at distance at most n^2 in $\mathcal{D}_{k+1}(D)$.*

Proof. Let us root T at $r \in V(T)$ arbitrarily. For a vertex $t \in V(T)$, let T_t be the subtree of T rooted at t and let $D_t = D\langle\bigcup_{v\in V(T_t)} X_v\rangle$.

We will define inductively an ordering $(v_1,\dots,v_n)$ on V and a sequence $(\gamma_0,\gamma_1,\dots,\gamma_n)$ of $(k+1)$-dicolourings of D such that $\gamma_0 = \alpha$, $\gamma_n = \beta$ and γ_i is X_i-coherent with $X_i = \{v_1,\dots,v_i\}$ (set $X_0 = \varnothing$) for every $0 \leq i \leq n$ and such that it is possible to go from γ_i to γ_{i+1} by recolouring every vertex of X_i at most twice and v_{i+1} at most once. Note that $\gamma_n = \beta$ is V-coherent.

First, note that γ_0 is trivially X_0-coherent since $X_0 = \varnothing$.

Let $i \geq 0$ and assume $(v_1,\dots,v_i)$ and $(\gamma_1,\dots,\gamma_i)$ that satisfy the above properties have already been defined. Let $v_{i+1} \in V \setminus X_i$ be any vertex that appears in some bag X_t such that $V(D_t)\setminus X_t \subseteq X_i$ (if $i = 0$ then t is a leaf and v_1 is a baby). Note that $\gamma_{i|D_t}$, the restriction of γ_i to D_t, is a $(V(D_t) \setminus X_t)$-coherent dicolouring of D_t. Let c be any colour that does not appear in X_t coloured with γ_i. By Lemma 7.5.18, there exists a $(k+1)$-dicolouring ξ of D_t that is $(V(D_t)\setminus X_t)$-coherent and such that $\xi(v) \neq c$ for every vertex $v \in V(D_t)$ and ξ can be reached from $\gamma_{i|D_t}$ by recolouring each vertex of $V(D_t) \setminus X_t \subseteq X_i$ at most once (when $i = 0$, this sequence is empty). Note that the same recolouring sequence allows to go from γ_i to the $(k+1)$-colouring γ'_i (whose restriction to D_t is ξ) by recolouring each vertex of X_i at most once and so that $\gamma'_i(v) \neq c$ for all $v \in V(D_t)$. Then, we can go from γ'_i to γ_{i+1} by recolouring each vertex $w \in V(D_t)$ such that $w \sim_{(T,\mathcal{X})} v_{i+1}$ with colour c (note that all the vertices that are recoloured at this phase are in X_i except v_{i+1}). Then, γ_{i+1} is X_{i+1}-coherent. Therefore, the induction properties hold for $i + 1$.

At the end, we find the desired redicolouring sequence between α and β in which the total number of recolourings is at most:

$$\sum_{0 \leq i \leq n-1} (2|X_i| + 1) = \sum_{0 \leq i \leq n-1} (2i + 1) = n^2,$$

which concludes the proof. □

The proof of Theorem 7.5.6 follows from Lemma 7.5.14 and Lemma 7.5.19.

Theorem 7.5.6. *If $D = (V, A)$ is a digraph of order n with $\mathscr{D}\mathrm{w}(D) \leq k - 1$, then*

$$\mathrm{diam}(\mathcal{D}_{k+1}(D)) \leq 2(n^2 + n).$$

Proof. Take α and β two k-dicolourings. Let $(T, \mathcal{X})$ be a valid $\mathscr{D}$-decomposition of D of width $k - 1 \geq \mathscr{D}\mathrm{w}(D)$. By Lemma 7.5.14, there is a redicolouring sequence from α (resp. β) to a dicolouring α' (resp. β') that is V-coherent. Moreover, there is such a redicolouring sequence of length at most n^2. Then by Lemma 7.5.19, there is a redicolouring sequence between α' and β' of length at most $2n$. Altogether, this gives a redicolouring sequence between α and β of length at most $2(n^2 + n)$.

Note that you can always build a valid $\mathscr{D}$-decomposition of width $k - 1 \geq \mathscr{D}\mathrm{w}(D)$ unless $k \geq n(D) + 1$, in which case follows from Corollary 7.4.2. □

7.6 Density of non-mixing graphs and digraphs

In this section, we turn our focus to the density of non-mixing graphs and digraphs. We first consider undirected graphs. In the undirected case, one can easily deduce from Theorem 7.1.1 that any non k-mixing graph G contains a subgraph H with minimum degree at least $k - 1$. This bound is tight because the complete graph on k vertices is $(k - 1)$-regular and is not k-mixing. Using probabilistic arguments, Bonamy, Bousquet, and Perarnau [32] showed that this bound is tight even on graphs of arbitrary large girth. We provide a construction witnessing this fact in Subsection 7.6.1.

Next, we show that this result on undirected graphs cannot be generalised to digraphs. Given a digraph D, recall that the *maximum average degree* of D is defined as $\mathrm{Mad}(D) = \max\{\frac{2m(H)}{n(H)} \mid H$ a non-empty subdigraph of $D\}$. It follows from Theorem 7.4.2 that every non k-mixing digraph D contains a subdigraph H with minimum out-degree and minimum in-degree at least $k - 1$. This shows that such a digraph D has maximum average degree at least $2k - 2$. This bound is tight because the bidirected complete graph on k vertices is $k - 1$-diregular and is not k-mixing. However, unlike the undirected case, this is not the case for digraphs with arbitrary large digirth. In fact, this is not even the case for oriented graphs, which are exactly the digraphs with digirth at least 3. We conjecture that if an oriented graph is not k-mixing, then it has maximum average degree at least $2k$. Even the case $k = 2$ remains open.

Conjecture 7.6.1. *Any non 2-mixing oriented graph has maximum average degree at least 4.*

Remark 7.6.1 – If true, this conjecture would be tight since there exist 2-freezable oriented graphs with maximum average degree 4. Consider for example the oriented graph $\vec{F}_n$ obtained from the disjoint union of two disjoints directed paths $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ by adding the set of

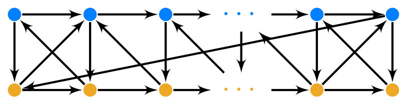


Figure 7.8: The 2-freezable oriented graph $\vec{F}_n$ and a frozen 2-dicolouring.

arcs $\{u_iv_i \mid i \in [n]\} \cup \{v_{i+1}u_i \mid i \in [n-1]\} \cup \{v_1u_2, u_nv_1, v_{n-1}u_n\}$ (see Figure 7.8 for an illustration). Let α be the 2-dicolouring of $\vec{F}_n$ in which all the u_i are coloured 1 and all the v_i are coloured 2. One can easily check that $\mathrm{Mad}(\vec{F}_n) = 4$ and α is a 2-frozen dicolouring of $\vec{F}_n$.

We prove two results providing some support for this conjecture in Subsection 7.6.2. Firstly, using the Discharging Method, we prove the conjecture in the special case of freezable oriented graphs.

Theorem 7.6.2. *Let $\vec{G} = (V, A)$ be an oriented graph. If $\vec{G}$ is 2-freezable, then $|A| \geq 2|V|$.*

From this result, we derive the following lower bound on the density of k-freezable oriented graphs.

Corollary 7.6.3. *Let $\vec{G} = (V, A)$ be an oriented graph. If $\vec{G}$ is k-freezable, then $|A| \geq k|V| + k(k-2)$.*

We give a family of oriented graphs for which this bound is reached. Secondly, again with the Discharging Method, we show a statement weaker than Conjecture 7.6.1 with $7/2$ instead of 4.

Theorem 7.6.4. *Let $\vec{G}$ be an oriented graph. If $\vec{G}$ is not 2-mixing, then $\mathrm{Mad}(\vec{G}) \geq \frac{7}{2}$.*

7.6.1 Density of non k-mixing undirected graphs

This section is devoted to a constructive proof of the following result, based on an explicit construction of regular bipartite graphs from Lazebnik and Ustimenko in [120].

Theorem 7.6.5 (Bonamy, Bousquet, and Perarnau [32])**.** *For any $k, \ell \in \mathbb{N}^*$, there exists a $(k-1)$-regular k-freezable graph $G_{k,\ell}$ with girth at least ℓ.*

We first make the following remark that we will use in the proof of Theorem 7.6.5:

Remark 7.6.2 – Let $k \in \mathbb{N}^*$, G be a $(k-1)$-regular k-freezable graph and α be a frozen k-colouring of G, then all colour classes of α have the same size. This follows from the fact that, for every vertex v of G, $N[v]$ uses all colours in α. Thus, given two colours $i, j \in [k]$, there must be a perfect matching in G between the vertices coloured i and the vertices coloured j. In particular, this implies that the number of vertices coloured i is the same as the number of vertices coloured j.

Proof of Theorem 7.6.5. Let us fix $\ell \in \mathbb{N}$. We prove the statement by induction on k, the result holding trivially for $k = 1$. Let $k > 1$ and assume that there exists a $(k-2)$-regular $(k-1)$-freezable graph $G_{k-1,\ell}$ with girth at least ℓ. Let α be a frozen k-colouring of $G_{k-1,\ell}$.

We denote by n the number of vertices of $G_{k-1,\ell}$. Consider H an n-regular bipartite graph with girth at least ℓ (such a graph exists by a construction from Lazebnik and Ustimenko [120]). Since H is bipartite, it is well-known that we can colour the edges of H with exactly n colours such that two adjacent edges receive different colours (see for instance [34, Theorem 17.2]). By Remark 7.6.2, all colour classes of α have the same size. Thus there is an ordering $(v_1, \ldots, v_n)$ of $V(G_{k-1,\ell})$ such that for each $i \in [n-k+1]$, the vertices $v_i, \ldots, v_{i+k-1}$ have different colours in α.

We denote by (A, B) the bipartition of H. Let $G_{k,\ell}$ be the graph obtained from H as follows.

- For each $a \in A$, replace a by a copy G^a of $G_{k-1,\ell}$, and connect v_i^a (the vertex corresponding to v_i in G^a) to the edge coloured i that was incident to a.
- For each $b \in B$, replace b by an independent set $I^b = \{x_1^b, \ldots, x_{\frac{n}{k}}^b\}$ of size $\frac{n}{k}$ (by Remark 7.6.2, $\frac{n}{k}$ is an integer). Connect x_i^b to the edges coloured $\{k(i-1)+1, \ldots, ki\}$ that were incident to b.

Observe that $G_{k,\ell}$ is k-regular: every vertex in a G^a is adjacent to its $k-1$ neighbours in G^a and exactly one neighbour in one of the I^bs; every vertex in an I^b has exactly k neighbours by construction. Moreover, $G_{k,\ell}$ has girth at least ℓ. Indeed, assume, for a contradiction, that it contains a cycle C of length at most $\ell - 1$. Then C cannot contain an edge of H, otherwise, contracting each copy of G^a would transform C into a cycle of length at most $\ell - 1$ in H. Thus C must be contained in some G^a, which is a copy of $G_{k-1,\ell}$, which is impossible since $G_{k-1,\ell}$ has girth at least ℓ.

Let β be the $(k+1)$-colouring of $G_{k,\ell}$ such that the restriction of β to each G^a corresponds to α, and $\beta(x_i^b) = k+1$ for all $b \in B$ and $i \in [n/k]$. Let v be a vertex of G. If v belongs to some G^a, then since α is frozen in $G_{k-1,\ell}$, $N_{G^a}[v]$ contains all colours of $[k]$. Moreover, by construction, v has a neighbour in some I^b which is coloured $k+1$. If v is in some I^b, then v is coloured $k+1$ and by construction it has exactly one neighbour in each colour class. In both cases, $N[v] = [k+1]$. Thus no vertex can be recoloured and so β is a frozen colouring of $G_{k,\ell}$. □

7.6.2 Density of non 2-mixing and 2-freezable oriented graphs

This section is devoted to the proof of two results supporting Conjecture 7.6.1. First we prove that it holds with the stronger assumption that G is 2-freezable.

Theorem 7.6.2. *Let $\vec{G} = (V, A)$ be an oriented graph. If $\vec{G}$ is 2-freezable, then $|A| \geq 2|V|$.*

Proof. Let $\vec{G} = (V, A)$ be a 2-freezable oriented graph, and α a frozen 2-dicolouring of $\vec{G}$. For a vertex $v \in V$, we say that a vertex $u \in V$ is *blocking* for v (in dicolouring α), if one of the following holds:

- u is an out-neighbour of v, $\alpha(u) \neq \alpha(v)$, and there exists a directed path $(u, ..., x, v)$, such that $(u, ..., x)$ is monochromatic, or
- u is an in-neighbour of v, $\alpha(u) \neq \alpha(v)$, and there exists a directed path $(x, ..., u, v)$ such that $(x, ..., u)$ is monochromatic.

We shall use a discharging argument. We set the initial charge of every vertex v to be $d(v)$. Observe that $d(v) \geq 2$ otherwise v can be recoloured in α. We then use the following discharging rule.

(R) Every vertex receives 1 from each of its blocking neighbours.

Let $f(v)$ be the final charge of every vertex v. Let us show that $f(v) \geq 4$ for every $v \in V$.

Let $v \in V$. We assume without loss of generality that $\alpha(v) = 1$. Since α is frozen, v admits at least one out-neighbour v^+ and one in-neighbour v^- coloured 2 that are blocking, and thus sending 1 to v by (R). Let us now examine the charge that v sends to others vertices. Let w be a vertex to which v sends charge. The vertex v is blocking for w, so $\alpha(w) = 2$. Moreover, if w is an out-neighbour (resp. in-neighbour) of v, then v has an in-neighbour (resp. out-neighbour) coloured 1. We are in one of the following cases.

- If v sends no charge, then $f(v) \geq d(v) + 2 \geq 4$.
- If v sends charge only to some out-neighbours, then it does not send to its in-neighbours. Since v has at least two in-neighbours (one blocking v and one coloured 1), $f(v) \geq d(v) + 2 - (d(v) - 2) \geq 4$.
- If v sends charge only to some in-neighbours, symmetrically to above, $f(v) \geq 4$.
- If v sends charge only to both out-neighbours and in-neighbours, then its has both an in-neighbour and an out-neighbour coloured 1 to which it sends no charge. Hence $f(v) \geq d(v) + 2 - (d(v) - 2) \geq 4$.

In all cases, we have $f(v) \geq 4$. Consequently, $2|A| = \sum_{v\in V} d(v) = \sum_{v \in V} f(v) \geq 4|V|$. □

We can deduce from Theorem 7.6.2 the following lower bound on the density of a k-freezable oriented graph.

Corollary 7.6.3. *Let $\vec{G} = (V, A)$ be an oriented graph. If $\vec{G}$ is k-freezable, then $|A| \geq k|V| + k(k-2)$.*

Proof. Suppose for a contradiction that there is a k-freezable oriented graph $\vec{G} = (V, A)$ such that $|A| < k|V| + k(k-2)$. Without loss of generality, we may take $\vec{G}$ having a minimum number of arcs among all such graphs. Let α be a frozen k-dicolouring of $\vec{G}$. For each $i, j \in [k]$, let $\vec{G}_i$ be the subdigraph of $\vec{G}$ induced by the vertices coloured i in α, and let $\vec{G}_{i,j}$ be the subdigraph of $\vec{G}$ induced by the vertices coloured i or j in α. We set $n_i = n(\vec{G}_i)$, $m_i = m(\vec{G}_i)$ and $m_{i,j} = m(\vec{G}_{i,j})$.

We first show that, for any $i \in [k]$, $m_i \leq n_i - 1$. Suppose not. Then, since $\vec{G}_i$ is acyclic, it admits an acyclic ordering $(x_1, \dots, x_{n_i})$. Now consider

$$\vec{G}' = (\vec{G} \setminus A(\vec{G}_i)) \cup \{x_j x_{j+1} \mid j \in [n_i - 1]\}$$

with the same dicolouring α. Clearly $m(\vec{G}') < m(\vec{G})$. Let v be a vertex of $\vec{G}'$. If $v \in V(\vec{G}_i)$, then x is still blocked in $(\vec{G}', \alpha)$ because it is blocked in $(\vec{G}, \alpha)$. Now, suppose $v \notin V(\vec{G}_i)$. For any colour j distinct from i and $\alpha(v)$, it is impossible to recolour v with j because it was already impossible in $\vec{G}$. Now, in $\vec{G}$, it was impossible to recolour v to i, so there is a directed path in $\vec{G}_i$ whose initial vertex x_k is an out-neighbour of v and whose terminal vertex x_ℓ is an in-neighbour of v. Since $(x_1, \dots, x_{n_i})$ is an acyclic ordering of $\vec{G}_i$, we have $k \leq \ell$. Thus $(x_k, \dots, x_\ell)$ is a directed path in $\vec{G}'$. Hence, v cannot be recoloured to i in $(\vec{G}', \alpha)$, meaning that it is also blocked in $(\vec{G}', \alpha)$. Since all vertices of $(\vec{G}', \alpha)$ are blocked, α is a frozen k-dicolouring of $\vec{G}'$, contradicting the minimality of $\vec{G}$.

We will now prove the result by bounding $S = \sum_{1 \le i < j \le k} m_{i,j}$. First, since α induces a 2-frozen dicolouring on $\vec{G}_{i,j}$, Theorem 7.6.2 yields $m_{i,j} \ge 2(n_i + n_j)$ for any $1 \le i < j \le k$. Thus,

$$S = \sum_{1 \le i < j \le k} m(\vec{G}_{i,j}) \ge 2 \sum_{1 \le i < j \le k} (n_i + n_j) = 2n(k-1).$$

Next, observe that $S = m(\vec{G}) + \sum_{i=1}^{k}(k-2)m_i$ because every monochromatic arc of D is counted exactly $(k-1)$ times in S, and every other arc only once. Thus

$$\begin{aligned} S &= m(\vec{G}) + (k-2)\sum_{i=1}^{k} m_i \\ &< kn + k(k-2) + (k-2)\sum_{i=1}^{k}(n_i - 1) \\ &= 2n(k-1)\ . \end{aligned}$$

Putting the two inequalities together, we get $2n(k-1) \le S < 2n(k-1)$, which is a contradiction. □

Remark 7.6.3 – The bound of Corollary 7.6.3 is tight: we can extend the construction of $\vec{F}_n$ (defined in Remark 7.6.1) to k-freezable oriented graphs $\vec{F}^k_n$ with exactly $k|V(\vec{F}^k_n)| + k(k-2)$ arcs. The oriented graph $\vec{F}^k_n$ is constructed as follows. We start from k disjoint directed paths $P_1, \ldots, P_k$ of length n. For each $j \in [k]$, let $P_s = (v^j_1, \ldots, v^j_n)$. For each pair $1 \le j < \ell \le k$, we add the set of arcs $\{v^j_i v^\ell_i \mid i \in [n]\} \cup \{v^\ell_{i+1} v^j_i \mid i \in [n-1]\} \cup \{v^\ell_1 v^j_2, v^j_n v^\ell_1, v^\ell_{n-1} v^j_n\}$, so that the subdigraph induced by $V(P_j) \cup V(P_\ell)$ is isomorphic to $\vec{F}_n$. By construction, $|A(\vec{F}^k_n)| = k|V(\vec{F}^k_n)| + k(k-2)$.

Let α_k be the dicolouring of $\vec{F}^k_n$ assigning colour j to the vertices of P_j for all $j \in [k]$. Since $V(P_j) \cup V(P_\ell)$ is isomorphic to $\vec{F}_n$, then every vertex of P_j cannot be recoloured with ℓ (and vice versa). Therefore, α_k is a frozen k-dicolouring of $\vec{F}^k_n$.

Theorem 7.6.4. *Let $\vec{G}$ be an oriented graph. If $\vec{G}$ is not 2-mixing, then* $\mathrm{Mad}(\vec{G}) \ge \frac{7}{2}$.

Proof. Let $\vec{G} = (V, A)$ be an oriented graph which is not 2-mixing, and take $\vec{G}$ to be minimal for this property (every proper induced subdigraph $\vec{H}$ of $\vec{G}$ is 2-mixing). In order to get a contradiction, assume that $|A| < \frac{7}{4}|V|$. Let α and β be two 2-dicolourings of $\vec{G}$ such that there is no redicolouring sequence from α to β. We will first prove some structural properties on $\vec{G}$, which we then leverage through a discharging strategy to show that $|A| \ge \frac{7}{4}|V|$.

Claim 7.6.6. *$\vec{G}$ has no source nor sink.*

Proof of claim. Assume that $\vec{G}$ contains a vertex s which is either a source or a sink. By minimality of $\vec{G}$, we know that $\vec{G}' = \vec{G} - s$ is 2-mixing. In particular, there is a redicolouring sequence $\gamma'_0, \ldots, \gamma'_r$ where γ'_0 and γ'_r are the restrictions of α and β to $\vec{G}'$ respectively.

For all $i \in [r]$, let γ_i be defined by $\gamma_i(s) = \alpha(s)$ and $\gamma_i(v) = \gamma'_i(v)$ for all $v \ne s$. Since s is either a source or a sink, γ_i is a 2-dicolouring of $\vec{G}$. If $\beta(s) = \alpha(s)$, then $\beta = \gamma_r$ and so $\gamma_0, \ldots, \gamma_r$ is a redicolouring sequence from α to β, a contradiction. Thus $\beta(s) \ne \alpha(s)$. Then setting $\gamma_{r+1} = \beta$, we have that $\gamma_0, \ldots, \gamma_r, \gamma_{r+1}$ is a redicolouring sequence from α to β, a contradiction. ◇

Claim 7.6.7. $\delta(\vec{G}) = 3$.

Proof of claim. First note that if $\delta(\vec{G}) \geq 4$ then $|A| \geq 2|V|$, contradicting our assumption that $|A| < \frac{7}{4}|V|$. Therefore $\delta(\vec{G}) \leq 3$.

Assume now that $\vec{G}$ contains a vertex u such that $d(u) \leq 2$. By Claim 7.6.6 we know that $d^+(u) = d^-(u) = 1$. Let $\vec{G}'$ be $\vec{G} - u$. By minimality of $\vec{G}$, we know that $\vec{G}'$ is 2-mixing. Let $\gamma'_0, \dots, \gamma'_r$ be a redicolouring sequence in $\vec{G}'$ where γ'_0 and γ'_r are the restrictions of α and β to $\vec{G}'$. Towards the contradiction, we exhibit a redicolouring sequence from α to β. To do so, we show, for any $i \in \{0, 1, \dots, r\}$, the existence of a 2-dicolouring γ_i of $\vec{G}$ such that γ'_i is the restriction of γ_i on $\vec{G}'$, and there is a redicolouring sequence from α to γ_i.

For $i = 0$, the result holds trivially with $\gamma_0 = \alpha$. Assume now that γ_{i-1} exists, and that there exists a redicolouring sequence reaching γ_{i-1} from α. Let x_i be the vertex such that $\gamma'_{i-1}(x_i) \neq \gamma'_i(x_i)$. Consider the operation of recolouring x_i by its opposite colour in γ_{i-1}. If this creates no monochromatic directed cycle, we let γ_i be the output dicolouring. Otherwise, this creates a monochromatic directed cycle containing u and its two neighbours u^- and u^+. Since in γ'_{i-1}, the colour of x_i is different from at least one of $\{u^-, u^+\}$, we may first recolour u and then x_i to obtain the desired γ_i. At the end of this process, we obtain a redicolouring sequence from α to a 2-dicolouring γ_r agreeing with β on $V(\vec{G}')$. If necessary, we may then recolour u to yield a redicolouring sequence from α to β, achieving the contradiction and yielding $\delta(\vec{G}) = 3$. $\diamondsuit$

For every positive integer i, an *i-vertex* (resp. *$(\geq i)$-vertex*) is a vertex of degree i (resp. at least i) in $\vec{G}$, and the set of i-vertices in $\vec{G}$ is denoted V_i. Let u be a 3-vertex. By Claim 7.6.6, either $d^-(u) = 1$ or $d^+(u) = 1$. Then, the *lonely neighbour* of u is its unique in-neighbour if $d^-(u) = 1$ and its unique out-neighbour if $d^+(u) = 1$. Observe that, in any dicolouring, recolouring u with a colour different from the one of its lonely neighbour yields another dicolouring because every directed cycle containing u must contain its lonely neighbour. We will use this argument several times along the remaining of this proof.

Claim 7.6.8. *Let u be a 3-vertex and v its lonely neighbour. There exist two 2-dicolourings ϕ and ψ that agree on $V(\vec{G}) \setminus \{u, v\}$ and such that there is no redicolouring sequence from ϕ to ψ.*

Proof of claim. Assume for a contradiction that for any pair ϕ, ψ of 2-dicolourings that agree on $V(\vec{G}) \setminus \{u, v\}$, there is a redicolouring sequence from ϕ to ψ. Let $\vec{G}' = \vec{G} - u$, and α', β' be the restrictions of our initial dicolourings α, β to $\vec{G}'$. By minimality of $\vec{G}$, there is a redicolouring sequence $\alpha' = \gamma'_0, \dots, \gamma'_r = \beta'$ in $\vec{G}'$. In order to get a contradiction, we will extend the latter into a redicolouring sequence from α to β. As in Claim 7.6.7, for $i = 0$ to r, we show inductively that there exists a 2-dicolouring γ_i of $\vec{G}$ such that γ'_i is the restriction of γ_i on $\vec{G}'$, and there is a redicolouring sequence from α to γ_i.

For $i = 0$ the result holds trivially with $\gamma_0 = \alpha$. Assume now that γ_{i-1} exists, and let $x_i \in V(\vec{G}')$ be the vertex recoloured at step i, such that $\gamma'_{i-1}(x_i) \neq \gamma'_i(x_i)$. We assume without loss of generality that $\gamma'_i(x_i) = 1$. We try to recolour x_i to colour 1 in γ_{i-1}. If this creates no monochromatic directed cycle, then we have the desired γ_i. If it does, the fact that γ'_i is a dicolouring of $\vec{G}'$ yields that any resulting monochromatic directed cycle must contain both u and its lonely neighbour v. This implies that $\gamma_{i-1}(u) = 1$, and we distinguish two cases, depending on the colour of v in γ_{i-1}.

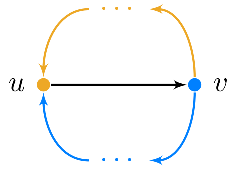


Figure 7.9: A 3-vertex u and its lonely neighbour v in a dicolouring ϕ from which we cannot reach the dicolouring ψ swapping the colours of u and v via a redicolouring sequence. Vertices coloured 1 are shown in orange, while vertices coloured 2 are shown in blue.

- If $\gamma_{i-1}(v) = 1$, then $\gamma_{i-1}(v) \neq \gamma_{i-1}(x_i) = 2$, which implies that x_i and v are different. In this case, we can first recolour u and then x_i to obtain the desired γ_i from γ_{i-1}, which combined with the sequence from α to γ_{i-1} obtained by induction yields the redicolouring sequence from α to γ_i.

- Else, $\gamma_{i-1}(v) = 2$ and since recolouring x_i creates a monochromatic directed cycle containing both u and v, it must be because $x_i = v$. Now, let γ_i be the 2-dicolouring of $\vec{G}$ obtained from γ_{i-1} by swapping the colours of u and v (*i.e.* $\gamma_i(u) = 2$ and $\gamma_i(v) = 1$). Observe that γ_i is a 2-dicolouring since γ_i' is, v is the lonely neighbour of u, and $\gamma_i(u) \neq \gamma_i(v)$. Since γ_i and γ_{i-1} agree on $V(\vec{G}) \setminus \{u, v\}$, our assumption yields a redicolouring sequence from γ_{i-1} to γ_i. Then, our induction hypothesis yields a sequence from α to γ_{i-1}, which, combined with the one above, yields a sequence from α to γ_i.

Therefore, by induction, we obtain a redicolouring sequence from α to a 2-dicolouring γ_r which agrees with β on $V(\vec{G}')$. Then, either $\gamma_r = \beta$, or we may recolour u in γ_r to obtain β, which in any case yields a redicolouring sequence from α to β, a contradiction. $\Diamond$

Claim 7.6.9. *Let u be a 3-vertex and v its lonely neighbour. Then $d^+(v) \geq 2$ and $d^-(v) \geq 2$.*

Proof of claim. By directional duality, we may assume $d^+(u) = 1$, such that uv is an arc of $\vec{G}$.

Let ϕ and ψ be two 2-dicolourings of $\vec{G}$ given by Claim 7.6.8. That is, both dicolourings agree on $V(\vec{G}) \setminus \{u, v\}$, and there is no redicolouring sequence from ϕ to ψ. Without loss of generality, we assume that $\phi(u) = 1$. Note that ϕ and ψ must differ on both vertices u, v, for otherwise ϕ, ψ would yield a trivial sequence from ϕ to ψ. Moreover, u cannot be recoloured in ϕ, for otherwise we can successively recolour u and v to obtain a redicolouring sequence between ϕ and ψ. Altogether, this shows that $\psi(u) = \phi(v) = 2$ and $\phi(u) = \psi(v) = 1$. Finally, there must be two paths from v to u such that one is internally coloured 1, and the other is internally coloured 2, respectively, in both ϕ and ψ (see Figure 7.9). Otherwise, either v, respectively u, could be recoloured in ϕ or ψ to get a 2-dicolouring in which u and v have the same colour allowing us to get a contradiction as above. This shows $d^+(v) \geq 2$.

Now, assume for the sake of contradiction that $d^-(v) = 1$, and recall that α, β are two 2-dicolourings witnessing that $\vec{G}$ is non-mixing. Up to recolouring u or v in α or β if both vertices are coloured the same, we may assume $\alpha(u) \neq \alpha(v)$ and $\beta(u) \neq \beta(v)$. We assume without loss of generality $\beta(u) = 1$ (implying $\beta(v) = 2$). We first show that we can further assume $\alpha(u) = 1$ (and therefore $\alpha(v) = 2$). To do so, if $\alpha(u) = 2$, we build a redicolouring sequence from α to another 2-dicolouring $\tilde{\alpha}$ such that $\tilde{\alpha}(u) = 1$. Let t_1 and t_2 be the two in-neighbours of u. If t_1

and t_2 are coloured the same in α, then we can either recolour u then v if $\alpha(t_1) = \alpha(t_2) = 2$, or recolour v then u if $\alpha(t_1) = \alpha(t_2) = 1$, yielding the desired $\widetilde{\alpha}$. Hence we may assume that $\alpha(t_1) = 1$ and $\alpha(t_2) = 2$. Let $\vec{G}' = \vec{G} - \{u, v\}$ and α' be the restriction of α to $\vec{G}'$. Given a 2-dicolouring ζ of $\vec{G}$, we define its *mirror* $\overline{\zeta}$ as the 2-dicolouring of $\vec{G}$ such that $\zeta(x) \neq \overline{\zeta}(x)$ for all $x \in V(\vec{G})$. By minimality of $\vec{G}$, there is a redicolouring sequence from α' to its mirror $\overline{\alpha'}$ in $\vec{G}'$. Since the colours of all vertices are swapped, there must be a 2-dicolouring γ' at some point in the sequence where $\gamma'(t_1) = \gamma'(t_2)$ (for instance the first time t_1 or t_2 is recoloured). We extend the sequence from α' to γ' into a redicolouring sequence from α to γ in $\vec{G}$, where u and v are constantly coloured 2 and 1 in both dicolourings. This is possible because v (resp. u) is the lonely neighbour of u (resp. v). Now, since $\gamma(t_1) = \gamma(t_2)$, in γ we may exchange the colours of u and v as above to obtain the desired $\widetilde{\alpha}$.

Since α can be obtained from $\widetilde{\alpha}$ through a redicolouring sequence, we can redefine α to be $\widetilde{\alpha}$. Now, $\alpha(u) = \beta(u) = 1$ and $\alpha(v) = \beta(v) = 2$, and we claim there exists a redicolouring sequence from α to β. Indeed, let α' and β' be the restrictions of α and β to $\vec{G}'$. By minimality of $\vec{G}$, there is a redicolouring sequence from α' to β'. This sequence extends to $\vec{G}$ by fixing the colour of u and v to 1 and 2 respectively. This gives a contradiction, and yields $d^-(v) \geq 2$. $\Diamond$

Claim 7.6.10. *Let u be a 3-vertex such that $|N(u) \cap V_3| = 2$ and v its lonely neighbour, then:*

- (i) *$d(v) \geq 5$, and*
- (ii) *$d(v) = 5$ only if v is adjacent to two (≥ 4)-vertices.*

Proof of claim. By directional duality, we may assume $d^+(u) = 1$. Let t_1 and t_2 be the in-neighbours of u. By Claim 7.6.9, we have $d(v) \geq 4$ so $N(u) \cap V_3 = \{t_1, t_2\}$. Let ϕ and ψ be two 2-dicolourings given by Claim 7.6.8.

Since t_1 and t_2 are 3-vertices, Claim 7.6.9 yields $d^-(t_1) = d^-(t_2) = 1$ because u cannot be their lonely neighbour. By Claim 7.6.8, neither u nor v can be recoloured in ϕ. Thus, for each $j \in [2]$, there is a (v, u)-path P_j whose internal vertices are all coloured j. In particular, we have $\phi(t_1) \neq \phi(t_2)$ because one of $\{t_1, t_2\}$ belongs to P_1 and the other one belongs to P_2. Without loss of generality, we may assume that both u and t_1 are coloured 1, and both v and t_2 are coloured 2.

(i). Let us first show that t_1 is blocked in ϕ. Towards this, consider the sequence successively recolouring t_1, v, u and t_1 again. For this not to be a redicolouring sequence from ϕ to ψ, which would contradict our assumption, it must create a monochromatic directed cycle at one of the steps. Assume that t_1 is recolourable, that is, the first step of the sequence does not create a monochromatic directed cycle. Now, since both t_1, t_2 are coloured 2 at this point, v can also be recoloured. Indeed, any monochromatic directed cycle resulting from setting v to colour 1 would already be a monochromatic directed cycle in ψ. Then, u is coloured the same as its lonely neighbour v, allowing us to set the colour of u to 2, then to recolour t_1 and obtain ψ. Therefore, t_1 is blocked in ϕ. Then, we must have $P_1 = (v, t_1, u)$, otherwise the only in-neighbour of t_1 would be coloured 1, and there is a (t_1, v)-path P_3 with internal vertices coloured 2. In particular, the out-neighbour of t_1 distinct from u is coloured 2.

We now consider t_2. The existence of P_2 ensures that t_2 has its unique in-neighbour coloured 2, therefore t_2 is not blocked to colour 2 in ϕ. Thus we recolour t_2 with colour 1, which allows us to then recolour u with colour 2 since its in-neighbourhood is coloured 1. In the resulting dicolouring ξ, if v is recolourable, then we can successively recolour v to colour 1 and t_2 to colour

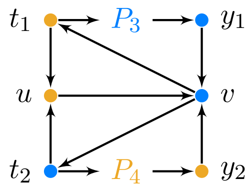


Figure 7.10: A 3-vertex u and its lonely neighbour v in a dicolouring ϕ from which we cannot reach the dicolouring ψ swapping the colours of u and v via a redicolouring sequence. Vertices coloured 1 are shown in orange, while vertices coloured 2 are shown in blue. Vertex t_1 must be blocked, yielding the existence of monochromatic path P_3 and the arc vt_1. Vertex v must be blocked after recolouring t_2, yielding P_4 and the arc vt_2.

2. This gives a redicolouring sequence between ϕ and ψ, a contradiction. Thus, recolouring v in ξ must create a monochromatic directed cycle. Since ξ agrees with ψ on $V(G) \setminus \{v, t_2\}$, this implies that $P_2 = (v, t_2, u)$ and there is a (t_2, v)-path P_4 in $\vec{G} - u$ with internal vertices coloured 1 (in both ψ and ψ).

The existence of P_1, P_2, P_3, and P_4 ensures that $d(v) \geq 5$ (see Figure 7.10), and completes the proof of (i).

(ii) Assume $d(v) = 5$. Let y_1 (resp. y_2) be the in-neighbour of v in P_3 (resp. P_4). Recall also that, without loss of generality, we can take $\phi(u) = 1$ and $\phi(v) = 2$. We shall prove that y_1 and y_2 both have degree at least 4 in $\vec{G}$.

- Assume for a contradiction that $d(y_1) = 3$. If the in-neighbour of y_1 on P_3 is coloured 2, we can recolour y_1 to 1 without creating a monochromatic directed cycle coloured 1. If not, the in-neighbour must be t_1, and since its lonely neighbour v is coloured 2, we can again recolour y_1 without creating a monochromatic directed cycle. Having recoloured y_1, the whole in-neighbourhood of v is now coloured 1. Since v is the lonely neighbour of t_1, we can set t_1 to colour 2 without creating any monochromatic directed cycle. Next we can successively recolour v to 1 since its out-neighbours are coloured 2, then u to 2 since its lonely out-neighbour v is now coloured 1. Now if recolouring y_1 yields a monochromatic directed cycle C, it does not contain neither u nor t_1 because they are both coloured 2 and v is coloured 1. Then C must have already been a monochromatic directed cycle in ϕ, which is a contradiction. Hence we can recolour y_1, and finally recolour t_1 to obtain a redicolouring sequence from ϕ to ψ, a contradiction.

- Assume for a contradiction that $d(y_2) = 3$. We start by recolouring t_2 to colour 1, and u to colour 2. Now we can recolour y_2 to colour 2 since $d(y_2) = 3$, the in-neighbour of y_2 in P_4 is coloured 1 and the out-neighbourhood of v is coloured 1. Then, we can recolour v to colour 1 because its in-neighbourhood is coloured 2. Next we can recolour t_2 to colour 2 since its in-neighbour v is coloured 1, and we can finally recolour y_2 to obtain a redicolouring sequence from ϕ to ψ, a contradiction.

$\Diamond$

From now on, for each (≥ 4)-vertex v, we let S_v be the set of 3-vertices x such that v is the lonely neighbour of x.

Claim 7.6.11. *Let v be a (≥ 4)-vertex, then $|S_v| \leq d(v) - 2$.*

Proof of claim. Assume for a contradiction that $|S_v| \geq d(v) - 1$. If $|S_v| = d(v) - 1$, we let w be the only neighbour of v which does not belong to S_v. By directional duality, we may assume that w is an out-neighbour of v. If $|S_v| = d(v)$, let w be any out-neighbour of v.

We shall find a redicolouring sequence from α to β. Without loss of generality, we may assume $\beta(v) = 1$. We first find a redicolouring sequence from α to a 2-dicolouring $\widetilde{\alpha}$ such that $\widetilde{\alpha}(v) = 1$. If $\alpha(v) = 1$, there is nothing to do. Assume now that $\alpha(v) = 2$. We distinguish two cases, depending on the colour of w.

- If $\alpha(w) = 1 \neq \alpha(v)$, then we can set every vertex in $N^+(v)$ to colour 1 without creating any monochromatic directed cycle. Next we can set every vertex in $N^-(v)$ to colour 2. Finally we can recolour v to 1.
- If $\alpha(w) = 2 = \alpha(v)$, then we set every vertex in $N^-(v)$ to colour 1. Next we set every vertex in $N^+(v)$ to colour 2. Finally we can recolour v to 1.

We now have a 2-dicolouring $\widetilde{\alpha}$ such that $\widetilde{\alpha}(v) = \beta(v) = 1$. First we set each vertex in S_v to colour 2. Then we consider $\vec{G}' = \vec{G} - (\{v\} \cup S_v)$. By the minimality of $\vec{G}$, there is a redicolouring sequence from $\widetilde{\alpha}'$ to β', the restrictions of $\widetilde{\alpha}$ and β to $\vec{G}'$. This sequence extends directly to $\vec{G}$, since v is coloured 1 and every vertex in S_v is coloured 2. Finally we only have to set each vertex $x \in S_v$ to colour $\beta(x)$. This operation does not create any monochromatic directed cycle, because such a cycle C should be coloured 1 (since we only recolour some vertices of S_v to colour 1). But then, every vertex in C is also coloured 1 in β, a contradiction.

We finally get a redicolouring sequence from $\widetilde{\alpha}$ to β. Since we described above a redicolouring sequence from α to $\widetilde{\alpha}$, there is a redicolouring sequence from α to β, a contradiction. $\Diamond$

Claim 7.6.12. *Let v be a 4-vertex. If $|S_v| = 0$, then $|N(v) \cap V_3| \leq 2$.*

Proof of claim. Assume for a contradiction that $|S_v| = 0$ and $|N(v) \cap V_3| \geq 3$. We consider $\vec{G}' = \vec{G} - (\{v\} \cup (N(v) \cap V_3))$. By minimality of $\vec{G}$, there is a redicolouring sequence $\gamma'_0, \dots, \gamma'_r$ where γ'_0 and γ'_r are the restrictions of α and β to $\vec{G}'$. In order to get a contradiction, we exhibit a redicolouring sequence from $\alpha = \gamma_0$ to $\beta = \gamma_r$ in $\vec{G}$ as follows. For $i = 1$ to r, we show that there exists a 2-dicolouring γ_i of $\vec{G}$ such that γ'_i is the restriction of γ_i on $\vec{G}'$, and there is a redicolouring sequence from γ_{i-1} to γ_i. The concatenation of these sequences will then yield the desired sequence between α and β.

Assume that for any $i \geq 1$, we have defined dicolouring γ_{i-1} such that γ'_{i-1} is the restriction of γ_{i-1} to $\vec{G}'$, let us and exhibit a redicolouring sequence from γ_{i-1} to some γ_i such that γ'_i is a restriction of γ_i to $\vec{G}'$. In γ_{i-1}, we first recolour each vertex of $N(v) \cap V_3$ with the colour different from the one of its lonely neighbour to get a new 2-dicolouring $\widetilde{\gamma}_{i-1}$ (possibly equal to γ_{i-1}). Note that γ'_{i-1} is still the restriction of $\widetilde{\gamma}_{i-1}$ on $\vec{G}'$. Let x_i be the vertex recoloured between γ'_{i-1} and γ'_i.

If recolouring x_i in our current dicolouring $\widetilde{\gamma}_{i-1}$ of $\vec{G}$ creates no monochromatic directed cycle, we let γ_i be the resulting dicolouring. Then, we have defined a sequence from γ_{i-1} to $\widetilde{\gamma}_{i-1}$ to γ_i, yielding the desired property. If the recolouring of x_i in $\widetilde{\gamma}_{i-1}$ does yield a monochromatic directed cycle, the cycle must contain at least one vertex in $\{v\} \cup (N(v) \cap V_3)$. Moreover, it must always contain a 3-vertex x that is a neighbour of v, because when it contains v it must also contain two of its neighbours, and by assumption at most one is not a 3-vertex. Since in $\widetilde{\gamma}_{i-1}$, x is coloured differently from its lonely neighbour, x_i must be the lonely neighbour of x. We may assume without loss of generality that x is an out-neighbour of v. Since all 3-vertices in $N(v)$, that is, all but at most one vertex in $N(v)$, are now coloured differently from their lonely neighbour, we can recolour v without creating any monochromatic directed cycle. Hence, up to this recolouring, we may assume that the two in-neighbours of x have the same colour. If this colour is the same as x, then we can recolour x and then recolour x_i, otherwise we can directly recolour x_i. In both cases, we let γ_i be the resulting dicolouring, which has been obtained by a redicolouring sequence from γ_{i-1}.

Concatenating the sequences $\gamma_{i-1}, ..., \gamma_i$ for $i \in [r]$ yields a sequence from α to a dicolouring γ_r which agrees with β on $V(\vec{G}')$. To extend this into a redicolouring sequence to β, in γ_r we start by recolouring each vertex of $N(v) \cap V_3$ with the colour different from the one of its lonely neighbour. Then, if necessary, we can recolour v to $\beta(v)$ without creating any monochromatic directed cycle. Finally, we recolour in any order the neighbours of v that need to be recoloured with their colour in β. If recolouring one of these vertices, say y, were to create a monochromatic directed cycle C, since β is a 2-dicolouring, C would have to contain both y and a neighbour of v coloured differently than in β. But these neighbours can only be other 3-vertices that we have not recoloured at this point, which are therefore coloured differently from their lonely neighbour, thus cannot be part of a monochromatic directed cycle. Hence, we have a redicolouring sequence from α to β, yielding the contradiction. $\lozenge$

Claim 7.6.13. *Let v be a 4-vertex. If $|S_v| = 1$, then $|N(v) \cap V_3| \leq 2$.*

Proof of claim. Assume for a contradiction that $|S_v| = 1$ and $|N(v) \cap V_3| \geq 3$, and let u be the only vertex in S_v. By directional duality, we may assume that u is an in-neighbour of v. Note that by Claim 7.6.9, $d^-(v) = d^+(v) = 2$. This implies that v has at least one out-neighbour x in V_3, and $d^+(x) = 1$ since $S_v = \{u\}$.

By Claim 7.6.8, there are two 2-dicolourings ϕ and ψ of $\vec{G}$ that agree on $V(\vec{G}) \setminus \{u, v\}$ and such that there is no redicolouring sequence from ϕ to ψ. In this case, $\phi(u) \neq \phi(v)$ and both u and v are blocked in ϕ, for otherwise we could recolour them one after the other to yield ψ. Without loss of generality, let $\phi(u) = 1$ and $\phi(v) = 2$. For each $j \in [2]$ there is a (v, u)-path P_j whose internal vertices are all coloured j in ϕ. We now distinguish two cases, depending on whether x belongs to P_1 or P_2.

- Assume first that x is the out-neighbour of v in P_1. Then x can be recoloured since its only out-neighbour is coloured 1. After recolouring x, every out-neighbour of v is coloured 2; hence we can successively recolour v to 1 and u to 2. We finally can recolour x to 1, since this directly gives dicolouring ψ, yielding a redicolouring sequence from ϕ to ψ, a contradiction.

- Assume now that x is the out-neighbour of v in P_2. Then, x can be recoloured since its only out-neighbour is either coloured 2 or it is u (and the only out-neighbour of u is coloured 2). After recolouring x, every out-neighbour of v has colour 1, and we can recolour u to colour 2. Now, recolouring v to colour 1 does not create any monochromatic directed cycle. Indeed, such a cycle C would necessarily contain x, otherwise C would already be monochromatic in ψ, and be coloured 1. Note that x has only one out-neighbour, which is necessarily coloured 2, because it is either internal to P_2, or it is u, which is now coloured 2. We can finally recolour x, yielding a redicolouring sequence from ϕ to ψ. This is a contradiction.

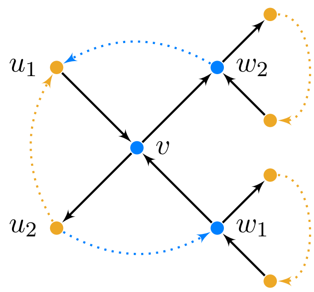


Figure 7.11: The neighbourhood of a 4-vertex v with $|S_v| = 2$ coloured with the particular dicolouring γ defined in Claim 7.6.14. Vertices coloured 1 are shown in orange, while vertices coloured 2 are shown in blue. The existence of the coloured dotted paths is guaranteed because no vertex of $N[v]$ is recolourable in γ.

$\lozenge$

Claim 7.6.14. *Let v be a 4-vertex with $|S_v| = 2$. Set $S_v = \{u_1, u_2\}$ and $N(v) \setminus S_v = \{w_1, w_2\}$. Then each of the following holds:*

- (i) *v has exactly one in-neighbour and one out-neighbour in S_v;*
- (ii) *v has exactly one in-neighbour and one out-neighbour in $\{w_1, w_2\}$;*
- (iii) *$\vec{G}$ has a 2-dicolouring γ such that $\gamma(u_1) = \gamma(u_2) = 1$, $\gamma(v) = \gamma(w_1) = \gamma(w_2) = 2$ and there exists no redicolouring sequence from γ that recolours at least one vertex in $N[v]$;*
- (iv) *$d(w_1) \geq 4$ and $d(w_2) \geq 4$.*

Proof of claim. Figure 7.11 illustrates the neighbourhood of such a 4-vertex v coloured with γ.

(i) By Claim 7.6.9, we have $d^+(v) = d^-(v) = 2$. Suppose (i) does not hold, and assume without loss of generality that both u_1 and u_2 are in-neighbours of v, so both w_1 and w_2 are out-neighbours of v. Towards a contradiction, we exhibit a redicolouring sequence from α to β. Without loss of generality, we assume $\beta(v) = 1$.

Starting from α, we first show how to obtain a dicolouring where v is coloured 1. If $\alpha(v) = 1$, there is nothing to do. If $\alpha(v) = 2$, we distinguish three cases depending on the colours of $\alpha(w_1)$ and $\alpha(w_2)$:

- If $\alpha(w_1) = \alpha(w_2) = 2 = \alpha(v)$, then we can directly recolour v.
- If $\alpha(w_1) = \alpha(w_2) = 1 \neq \alpha(v)$, then we can set both u_1 and u_2 to 2, and then recolour v with 1.
- Else, $\alpha(w_1) \neq \alpha(w_2)$. Since $\alpha(v) = 2$, we first set u_1 and u_2 to colour 1, naming the resulting 2-dicolouring $\widehat{\alpha}$. Then, we consider $\vec{G}' = \vec{G} - \{v, u_1, u_2\}$ and let $\widehat{\alpha}'$ be the restriction of $\widehat{\alpha}$ to $\vec{G}'$. By minimality of $\vec{G}$, $\vec{G}'$ is 2-mixing. In particular, there is a redicolouring sequence from $\widehat{\alpha}'$ to its mirror. Along this sequence, since we initially set $\widehat{\alpha}'(w_1) \neq \widehat{\alpha}'(w_2)$, there is a 2-dicolouring $\breve{\alpha}'$ of $\vec{G}'$ such that $\breve{\alpha}'(w_1) = \breve{\alpha}'(w_2)$. The

redicolouring sequence from $\widehat{\alpha}'$ to $\breve{\alpha}'$ in $\vec{G}'$ directly extends to a redicolouring sequence from $\widehat{\alpha}$ to a 2-dicolouring $\breve{\alpha}$ in $\vec{G}$. Indeed, in $\widehat{\alpha}$, both u_1 and u_2 are coloured differently from their lonely neighbour v, and v has no other in-neighbours, thus applying the same sequence in $\widehat{\alpha}$ does not create any monochromatic directed cycle. After this recolouring, we obtain dicolouring $\breve{\alpha}$ such that $\breve{\alpha}(w_1) = \breve{\alpha}(w_2)$, allowing us to apply one of the first two items to recolour v with 1.

We have shown that there is a (possibly empty) redicolouring sequence from α to a 2-dicolouring α^* such that $\alpha^*(v) = \beta(v) = 1$. We now start from α^*, set both u_1 and u_2 to 2, and name the resulting 2-dicolouring $\widetilde{\alpha}$. Consider $\vec{G}' = \vec{G} - \{v, u_1, u_2\}$ and $\widetilde{\alpha}'$, β' the restrictions of $\widetilde{\alpha}$ and β to $\vec{G}'$. By minimality of $\vec{G}$, there is a redicolouring sequence from $\widetilde{\alpha}'$ to β' in $\vec{G}'$. This redicolouring sequence immediately extends to $\vec{G}$, by keeping $\widetilde{\alpha}$ on v, u_1, u_2, and we may only need to alter the colour of u_1 and u_2 at the end. Therefore, there is a redicolouring sequence from α to β, a contradiction. This proves (i).

(ii) Follows directly from (i) by Claim 7.6.9.

(iii) By (i) and (ii), we may assume that u_1 and w_1 are the in-neighbours of v and u_2 and w_2 are the out-neighbours of v. Assume for the sake of contradiction that (iii) does not hold, that is:

For any 2-dicolouring ξ of $\vec{G}$, such that $\xi(u_1) = \xi(u_2)$ and $\xi(v) = \xi(w_1) = \xi(w_2)$ with $\xi(v) \neq \xi(u_1)$, there is a redicolouring sequence from ξ to a 2-dicolouring ξ' such that for at least one vertex x in $N[v]$, $\xi(x) \neq \xi'(x)$. (♠)

Assuming this, we will prove that there is a redicolouring sequence from α to β, which is a contradiction, showing the existence of γ.

First we show that there is a redicolouring sequence from α to some dicolouring $\widetilde{\alpha}$, where $\widetilde{\alpha}(w_1) \neq \widetilde{\alpha}(w_2)$. We assume without loss of generality that $\alpha(v) = 1$. If $\alpha(w_1) \neq \alpha(w_2)$, we let $\widetilde{\alpha} = \alpha$, otherwise, $\alpha(w_1) = \alpha(w_2)$ and we distinguish two cases according to the colour of $\alpha(w_1)$.

- Assume $\alpha(w_1) = \alpha(w_2) = 2 \neq \alpha(v)$. Starting from α, we first set both u_1 and u_2 to 2. Then, we consider $\vec{G}' = \vec{G} - \{u_1, u_2, v\}$, which must be 2-mixing by minimality of $\vec{G}$. In particular, there is a redicolouring sequence in $\vec{G}'$ from α', the restriction of α to $\vec{G}'$, to its mirror $\overline{\alpha'}$. Along this sequence, we let $\widetilde{\alpha}'$ be the first dicolouring where w_1 and w_2 are coloured differently. The redicolouring sequence from α' to $\widetilde{\alpha}'$ in $\vec{G}'$ directly extends to a redicolouring sequence from α to $\widetilde{\alpha}$ in $\vec{G}$. Indeed, we keep v coloured 1 and u_1 and u_2 coloured 2 throughout the sequence, which does not create any monochromatic directed cycle since v does not have any neighbour coloured 1 until reaching $\widetilde{\alpha}$, when it has exactly one.
- Assume $\alpha(w_1) = \alpha(w_2) = 1 = \alpha(v)$. Starting from α, we first set both u_1 and u_2 to 2. Now by (♠) there is a redicolouring sequence from α to a 2-dicolouring $\widehat{\alpha}$ where one vertex x in $N[v]$ has a different colour. If x is w_1 or w_2, then we are done. If x is v, then we are done by the previous case. Finally if x is u_1 or u_2, then v has three neighbours coloured 1, hence v can be recoloured 2, and we are done by the previous case (swapping the roles of colours 1 and 2).

The same proof applies to β to yield a redicolouring sequence from β to some 2-dicolouring $\widetilde{\beta}$ such that $\widetilde{\beta}(w_1) \neq \widetilde{\beta}(w_2)$. It now suffices to exhibit a redicolouring sequence from $\widetilde{\alpha}$ to $\widetilde{\beta}$.

Consider $\vec{G}' = \vec{G} - \{v, u_1, u_2\}$, which is 2-mixing by minimality of $\vec{G}$, giving us a redicolouring sequence $\widetilde{\alpha}' = \eta'_0, \dots, \eta'_r = \widetilde{\beta}'$, where $\widetilde{\alpha}'$ and $\widetilde{\beta}'$ are the restrictions of $\widetilde{\alpha}$ and $\widetilde{\beta}$ to $\vec{G}'$. We will extend this redicolouring sequence to $\vec{G}$. To do so, for $i = 1$ to r, we show that there exists a 2-dicolouring η_i of $\vec{G}$ such that η'_i is the restriction of η_i on $\vec{G}'$, and there is a redicolouring sequence from η_{i-1} to η_i. To ensure that extending this sequence does not create any monochromatic directed cycle, we will maintain the property that $\{v, w_1, w_2\}$ is not monochromatic in each η_i.

We first set η_0 to $\widetilde{\alpha}$. Since $\widetilde{\alpha}(w_1) \neq \widetilde{\alpha}(w_2)$, $\{v, w_1, w_2\}$ is not monochromatic in η_0. Assume now that η_{i-1} exists, and let us define η_i. Let x_i be the vertex that is recoloured from η'_{i-1} to η'_i. Without loss of generality, say v is coloured 1 in η_{i-1}. We may assume that $\eta_{i-1}(u_1) = \eta_{i-1}(u_2) = 2$, otherwise we may recolour them. Then, if recolouring x_i does not break the invariant that $\{v, w_1, w_2\}$ is not monochromatic, we can recolour x_i without creating any monochromatic directed cycle and obtain the desired η_i. Indeed, a resulting monochromatic directed cycle cannot contain u_1 or u_2, so it must contain v and therefore both w_1 and w_2, which are coloured differently. If recolouring x_i breaks the invariant, x_i must belong to $\{w_1, w_2\}$. Assume then that $x_i = w_1$, and recolouring w_1 breaks the invariant. This implies that w_2 is currently coloured 1 and w_1 will be recoloured from colour 2 to colour 1. In this case, all in-neighbours of v are coloured 2, allowing us to first recolour u_2 without creating any monochromatic directed cycle. Now, all out-neighbours of v are coloured 1, letting us set v to colour 2. We finally can recolour w_1 to colour 1 and we get the desired η_i extending η'_i and preserving the invariant. The arguments hold symmetrically when the recoloured vertex x_i is w_2.

Concatenating the resulting sequences $\eta_{i-1}, ..., \eta_i$ for $i \in [1, r]$, we obtain a redicolouring sequence from $\widetilde{\alpha}$ to a 2-dicolouring β^* which agrees with $\widetilde{\beta}$ on $V(\vec{G}')$. Moreover, $\{v, w_1, w_2\}$ is not monochromatic in β^*. We now turn to exhibiting a redicolouring sequence from β^* to $\widetilde{\beta}$. Let us assume, without loss of generality, that $\beta^*(v) = 1$. We may first set u_1 and u_2 to colour 2, as v is their lonely neighbour. We show how to recolour v to colour 2, when required, without affecting the colours of $V(\vec{G}')$. Note that $\widetilde{\beta}$ is a dicolouring, so any monochromatic directed cycle resulting from recolouring v must use one of u_1 or u_2. We distinguish two cases depending on the colour of $\widetilde{\beta}(w_1)$.

- If $\widetilde{\beta}(w_1) = 1$, we know by assumption on $\widetilde{\beta}$ that $\widetilde{\beta}(w_2) = 2$. Now, all out-neighbours of v are coloured 2, letting us first recolour u_1 to 1. Then, all in-neighbours of v are coloured 1, letting us set v to colour 2.
- If $\widetilde{\beta}(w_1) = 2$, we know by assumption that $\widetilde{\beta}(w_2) = 1$. Symmetrically, all in-neighbours of v are coloured 2, letting us recolour u_2 to 1. Then, all out-neighbours of v are coloured 1, letting us recolour v to 2.

Finally, if necessary, we may recolour u_1 and u_2. To do so, we may initially set u_1 and u_2 to the colour different from $\widetilde{\beta}(v)$. Then, if only one of u_1, u_2 needs to be recoloured, this can be done since $\widetilde{\beta}$ is a dicolouring. If both need to be recoloured, this can also be done in any order, as any resulting monochromatic directed cycle would also exist in $\widetilde{\beta}$. This yields redicolouring sequence from $\widetilde{\alpha}$ to $\widetilde{\beta}$, which together with the redicolouring sequences from α to $\widetilde{\alpha}$ and β to $\widetilde{\beta}$ yields a redicolouring sequence from α to β, a contradiction. This achieves to prove (iii).

(iv) This is a consequence of (iii), let us consider the dicolouring γ given by (iii) with $\gamma(u_1) = \gamma(u_2) = 1$ and $\gamma(v) = \gamma(w_1) = \gamma(w_2) = 2$. In γ, u_2 cannot be recoloured to 2, therefore there is a (u_2, w_1)-path with internal vertices coloured 2. If $d(w_1) = 3$, then w_1 could be recoloured 1 because its out-neighbour v is coloured 2 and one of its in-neighbours is either coloured 2 or is u_2, whose unique in-neighbour v is coloured 2. This would contradict (iii), hence $d(w_1) \geq 4$. We can show symmetrically that $d(w_2) \geq 4$, achieving to prove (iv).

$\Diamond$

Claim 7.6.15. *If v is a 4-vertex, then $|N(v) \cap V_3| \leq 2$.*

Proof of claim. This is a direct consequence of Claims 7.6.11, 7.6.12, 7.6.13 and 7.6.14. $\Diamond$

We shall now use the Discharging Method. The initial charge of each vertex x is $d(x)$, and we apply the following rules:

(R1) Each vertex x such that $d(x) = 3$ and $|N(x) \cap V_3| \geq 2$ receives $\frac{1}{2}$ charge from its lonely neighbour.

(R2) Each vertex x such that $d(x) = 3$ and $|N(x) \cap V_3| \leq 1$ receives $\frac{1}{4}$ from each of its (≥ 4)-neighbours.

Let us now show that the final charge $w^*(x)$ of a vertex x in $\vec{G}$ is at least $\frac{7}{2}$, showing that $|A| \geq \frac{7}{4}|V|$, which contradicts the assumption $|A| < \frac{7}{4}|V|$.

- Assume $d(x) = 3$ and $|N(u) \cap V_3| \geq 2$, then x receives $\frac{1}{2}$ by (R1). Hence $w^*(x) = d(x) + \frac{1}{2} = \frac{7}{2}$.
- Assume $d(x) = 3$ and $|N(u) \cap V_3| \leq 1$, then x receives $\frac{1}{4}$ by (R2) at least twice. Hence $w^*(x) \geq d(x) + \frac{1}{2} = \frac{7}{2}$.
- Assume $d(x) = 4$. By Claim 7.6.10, x does not give any charge through (R1), and by Claim 7.6.15 it gives at most twice $\frac{1}{4}$ by (R2). Hence $w^*(x) \geq d(x) - 2 \cdot \frac{1}{4} = \frac{7}{2}$.
- Assume $d(x) = 5$. If x gives $\frac{1}{2}$ charge at least once by (R1), then by Claim 7.6.10, x has at least two neighbours that did not take any charge from x. Therefore $w^*(x) \geq d(x) - 3 \times \frac{1}{2} = \frac{7}{2}$.
  If x does not give by (R1), then it gives at most $\frac{1}{4}$ to each of its neighbours. Thus $w^*(x) \geq d(x) - d(x) \times \frac{1}{4} = \frac{15}{4} > \frac{7}{2}$.
- Finally, assume $d(x) \geq 6$. By Claim 7.6.11, x gives $\frac{1}{2}$ by (R1) to at most $d(x) - 2$ of its neighbours. Thus $w^*(x) \geq d(x) - \frac{1}{2}(d(x) - 2) - 2 \times \frac{1}{4} = \frac{d(x)}{2} + \frac{1}{2} \geq \frac{7}{2}$.

This completes the proof of Theorem 7.6.4. $\square$

7.7 Further research directions

This chapter gives some results on digraph redicolouring. This is obviously just the tip of the iceberg and many open questions arise. Forthwith, we detail a few of them.

In Section 7.2, we proved that k-DICOLOURING PATH is PSPACE-complete for all $k \geq 2$. But we did not prove any complexity result about DIRECTED IS k-MIXING.

Problem 7.7.1. *What is the complexity of* DIRECTED IS k-MIXING*?*

We believe that this is PSPACE-hard for all $k \geq 2$. To settle the complexity of DIRECTED IS k-MIXING, it could be helpful to settle the complexity of the following particular case of 2-DICOLOURING PATH. (Recall that the mirror of a 2-dicolouring α of D, is the 2-dicolouring $\overline{\alpha}$ of D such that $\alpha(v) \neq \overline{\alpha}(v)$ for all $v \in V(D)$.)

> MIRROR-REACHABILITY
> **Input:** A 2-dicolouring α of a digraph D.
> **Output:** Is there a path between α and its mirror $\overline{\alpha}$ in $\mathcal{D}_2(D)$?

Problem 7.7.2. *What is the complexity of* MIRROR-REACHABILITY*?*

A particular case of non k-mixing digraphs are k-freezable digraphs. It would then be interesting to consider the complexity of the following problem.

> DIRECTED IS k-FREEZABLE
> **Input:** A k-dicolourable digraph D.
> **Output:** Is D k-freezable?

Note that deciding whether a digraph is k-freezable is NP-complete in general, since we can reduce easily from k-dicolourability for all $k \geq 2$. Indeed, for a digraph D, let D' be the digraph obtained from D by adding to each vertex $v \in V(D)$ a complete bidirected graph K_k^v which contains v and $k-1$ new vertices. Trivially, D' is k-dicolourable if and only if D is k-dicolourable, and every k-dicolouring of D' (if one exists) is necessarily frozen because of the complete bidirected graphs. Therefore, D' is k-freezable if and only if D is k-dicolourable.

A related problem is the one of deciding whether a vertex is frozen in a given k-dicolouring α of a digraph D. Recall that a vertex v is frozen in α if $\beta(v) = \alpha(v)$ for any k-dicolouring β in the same connected component of α in $\mathcal{D}_k(D)$.

> k-FROZEN VERTEX
> **Input:** A digraph D, a k-dicolouring α of D, and a vertex v of D.
> **Output:** Is v frozen in α?

2-FROZEN VERTEX is PSPACE-complete. This comes from the following result from [98]: given a cubic graph G, a mapping $\phi\colon V(G) \to \{1,2\}$, a proper orientation $\vec{G}$ of G, and an edge xy of G, deciding whether there is a reorienting sequence from $\vec{G}$ that reverse xy is PSPACE-complete. Hence, the same reduction as the one used for Theorem 7.2.1 also yields PSPACE-completeness of 2-FROZEN VERTEX.

One can then easily derive that k-FROZEN VERTEX is PSPACE-complete for any $k \geq 2$. Indeed, consider a (non-acyclic) digraph D_2 and a 2-dicolouring α_2 of D_2. Let D_k be the digraph obtained the disjoint union of D_2 and bidirected complete graph $\overleftrightarrow{K}_{k-2}$ on $k-2$ vertices

g \ k	4	5	6	7	9	≥ 10
3	$+\infty$	$+\infty$	$+\infty$	$O(n^6)$ [42]	$O(n^2)$ [42]	$O(n)$ [65]
4	$+\infty$	$O(n^4)$ [42]	$O(n \log^3 n)$ [73]	$O(n)$ [66]	-	-
5	$< +\infty$ [22]	$O(n \log^2 n)$ [73]	-	-	-	-
6	$O(n^3)$ [42]	$O(n)$ [22]	-	-	-	-
≥ 7	$O(n \log n)$ [73]	-	-	-	-	-

Table 7.2: Bound on the diameter of $\mathcal{C}_k(G)$ when G is a planar graph with girth at least g. The value '$+\infty$' means that there exists a graph for which $\mathcal{C}_k(G)$ is not connected, and the value '$< +\infty$' means that $\mathcal{C}_k(G)$ is connected but no reasonable upper bound is known.

$y_1, \dots, y_{k-2}$ and adding a digon between any vertex of D_2 and any vertex of $\overleftrightarrow{K}_{k-2}$. Let α_k be the k-dicolouring of D_k defined by $\alpha_k(v) = \alpha_2(v)$ for all $v \in V(D_2)$ and $\alpha_k(y_i) = i + 2$ for every $i \in [k-2]$. One can easily check that a vertex v in $V(D_2)$ is frozen in α_2 if and only if it is frozen in α_k.

In this chapter, we generalised several evidence for Cereceda's conjecture to digraphs. In particular, our results give more support to Conjecture 7.1.9. Using Proposition 7.5.4, an analogue question is the following.

Question 7.7.3. *Let $k \in \mathbb{N}$ and D be a digraph such that $k \geq \mathrm{dtw}(D) + 2$. Is it true that $\mathrm{diam}(\mathcal{D}_k(D)) = O(n(D)^2)$?*

The same question remains open when we replace directed treewidth by DAG-width or Kelly-width. In every case, if true, it would give another generalisation of Theorem 7.1.8.

Note that Conjecture 7.1.9 implies Conjecture 7.1.2. We ask if the converse is true.

Question 7.7.4. *Does proving Cereceda's conjecture for undirected graphs imply its analogue in digraphs?*

Finally, we pose a few questions on redicolouring planar digraphs. Using Theorem 7.5.1, we know that every result on recolouring planar graphs extends to digraphs (up to a factor two). In particular, numerous results from [22, 42, 66, 73], that we recap in Table 7.2, remain true on digraphs by taking the girth of the underlying graph.

We ask if these results can be improved for oriented planar graphs. Using results of this chapter, we obtain bounds on the diameter of $\mathcal{D}_k(\vec{G})$ when $\vec{G}$ is an oriented planar graph. We recap them in Table 7.3 and pose the following conjecture about the missing values in it.

Conjecture 7.7.5. *Every oriented planar graph is 3-mixing.*

Conjecture 7.7.6. *Every oriented planar graph $\vec{G}$ with $\mathrm{girth}(\vec{G}) \geq 4$ is 2-mixing.*

Recall the celebrated conjecture of Neumann-Lara (Conjecture 1.2.4) stating that every oriented planar graph has dichromatic number at most 2. As a partial result, one may explore the following question which, if true, implies that Question 7.7.5 is a consequence of Neumann-Lara's conjecture.

Conjecture 7.7.7. *Every oriented planar graph $\vec{G}$ with $\vec{\chi}(G) \leq 2$ is 3-mixing.*

g \ k	2	3	4	5	6
3	$+\infty$ Rmk. 7.6.1	?	$O(n^3)$ Th. 7.3.2	$O(n\log^3(n))$ Cor. 7.3.4	$O(n)$ Th. 7.3.13
4	?	$O(n^2)$ Th. 7.3.9	$O(n)$ Th. 7.3.13	-	-
5	$< +\infty$ Th. 7.6.4	$O(n\log(n))$ Cor. 7.3.4	-	-	-

Table 7.3: Bound on the diameter of $\mathcal{D}_k(\vec{G})$ when $\vec{G}$ is an oriented planar graph with girth at least g. The value '?' means that we do not know whether $\mathcal{D}_k(\vec{G})$ is connected.

Chapter 8

Conclusion and perspectives

In this thesis, we established a number of results on digraph dicolouring, aiming to gain deeper insight into the distinctions and the parallels between graph colouring and its directed counterpart. We detail in this chapter a few questions related to this topic that remain open, including both well-known classical problems and inquiries emerging from recent advances.

8.1 Original problems due to Erdős and Neumann-Lara

We start with three old problems posed by Neumann-Lara when he introduced the notions of dicolouring and dichromatic number. The first one asks for the dichromatic number of oriented planar graphs. As mentioned at the beginning of this thesis (Proposition 1.2.3), oriented planar graphs are 3-dicolourable, but Neumann-Lara conjectured that they are indeed 2-dicolourable. This conjecture can be seen as the analogue of the Four Colour Theorem (Theorem 1.1.1) for digraphs.

Conjecture 8.1.1 (Neumann-Lara [134])**.** *Every oriented planar graph is* 2*-dicolourable.*

In Chapter 3 we presented some upper bounds on the dichromatic number of oriented graphs with bounded maximum degree Δ. All these bounds are linear in Δ, and it is still open whether the dichromatic number of an oriented graph $\vec{G}$ is bounded above by a sublinear function of $\Delta(\mathrm{UG}(\vec{G}))$. Erdős and Neumann-Lara conjectured that it is indeed the case.

Conjecture 8.1.2 (Erdős and Neumann-Lara (see [69]))**.** *Let $\vec{G}$ be an orientation of a graph G, then*

$$\vec{\chi}(\vec{G}) = O\left(\frac{\Delta(G)}{\log \Delta(G)}\right).$$

Note that, if true, Conjecture 8.1.2 can be seen as the directed counterpart of Johansson's seminal result (Theorem 1.1.8). It is also natural to ask for a generalisation of Conjecture 1.1.9 stating that every H-free graph G satisfies $\chi(G) = O\left(\frac{\Delta(G)}{\ln \Delta(G)}\right)$.

Conjecture 8.1.3. *For every fixed digraph H, there exists a positive constant c_H such that every H-free digraph D satisfies $\vec{\chi}(G) \leq c_H \frac{\Delta(D)}{\ln \Delta(D)}$.*

For our last problem, let us define the dichromatic number $\vec{\chi}(G)$ of an undirected graph G as the maximum dichromatic number over all its orientations. Clearly, every graph G satisfies $\vec{\chi}(G) \leq \chi(G)$. Erdős and Neumann-Lara conjectured that indeed these two parameters are functionally equivalent.

Conjecture 8.1.4 (Erdős and Neumann-Lara (see [69])). *There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every graph G satisfies*

$$\chi(G) \leq f(\vec{\chi}(G)).$$

This conjecture is widely open, as it is not even known whether graphs of arbitrarily large chromatic number have dichromatic number at least 3. Mohar and Wu [130] proved the fractional version of the conjecture.

8.2 Digraphs with bounded maximum degree

As mentioned in Chapter 6, Harutyunyan and Mohar [94] proved the existence of digraphs D with arbitrarily large girth and dichromatic number at least $c \cdot \frac{\Delta(D)}{\log(\Delta(D))}$ for some absolute constant c. This generalises a result of Bollobas [29] and implies that Conjecture 8.1.2, if true, would be best possible. In the light of this result, together with Rambaud we remarked that even the following weakening of Conjecture 8.1.2 is open.

Conjecture 8.2.1. *There is a function g and a constant $c > 0$ such that every digraph D satisfies at least one of the following:*

- $\mathrm{digirth}(D) < g(\Delta(D))$, *or*
- $\vec{\chi}(D) < c \cdot \frac{\Delta(D)}{\log(\Delta(D))}$.

It motivated us to pose the following stronger conjecture (Conjecture 6.7.4 in Chapter 6).

Conjecture 8.2.2. *There is a function g and a constant $c > 0$ such that, for every pair of digraphs F, D, at least one of the following holds:*

- $\mathrm{digirth}(D) < g(F)$, *or*
- $\vec{\chi}(D) < c \cdot \frac{\Delta(F)}{\log \Delta(F)}$, *or*
- *D contains F as a subdivision.*

Observe that Conjecture 8.2.2 implies Conjecture 8.2.1 when F is set to an oriented star on $\Delta(D) + 2$ vertices.

8.3 An analogue of a conjecture of Tarsi for digraphs

Let $G = (V, E = E_1 \cup E_2)$ be a graph such that $G_1 = (V, E_1)$ is a forest and $G_2 = (V, E_2)$ is 2-degenerate. Such a graph is called a $(1, 2)$-composed graph. It is easy to show that every $(1, 2)$-composed graph is 6-colourable, and that K_5 is a $(1, 2)$-composed graph. Tarsi conjectured the following.

Conjecture 8.3.1 (Tarsi (see [101, Problem 4.2]))**.** *If G is a $(1, 2)$-composed graph, then $\chi(G) \leq 5$.*

Tarsi's conjecture is still widely open. It has been generalised to $(m_1, \dots, m_s)$-composed graphs, which are unions of s graphs $G_1, \dots, G_s$ such that G_i is m_i-degenerate, by Klein and Schönheim in [107] (see [108]).

It is natural to investigate whether there are generalisations of Conjecture 8.3.1 for dicolouring depending on the various possible definitions of degeneracy for a digraph. If we use the c-degeneracy, the $\min$-degeneracy there is no hope for an analogue of Conjecture 8.3.1 because every digraph $D = (V, A)$ admits a partition of its arc-set $A = A_1 \cup A_2$ in such a way that both $D_1 = (V, A_1)$ and $D = (V, A_2)$ are acyclic (*i.e.* have $\min$-degeneracy 0).

However, an analogue of the conjecture can be considered when we use the $\max$-*degeneracy*. Analogously to the $\min$-degeneracy, the $\max$-degeneracy of a digraph D is the least integer d such that every subdigraph H of D contains a vertex v satisfying $d_{\max}(v) \leq d$. We say that a digraph $D = (V, A)$ is $(1, 2)$-composed if there exists a partition $A = A_1 \cup A_2$ such that $D_1 = (V, A_1)$ is 1-$\max$-degenerate and $D_2 = (V, A_2)$ is 2-$\max$-degenerate. Again, every $(1, 2)$-composed digraph has dichromatic number at most 6, and $\overleftrightarrow{K_5}$ is a $(1, 2)$-composed digraph. We believe that the following generalisation of Tarsi's conjecture holds.

Conjecture 8.3.2. *If D is a $(1, 2)$-composed digraph, then $\vec{\chi}(D) \leq 5$.*

It is clear that Conjecture 8.3.2 implies Conjecture 8.3.1, but we do not know if the converse is true, that is if the two conjectures are indeed equivalent. One way to prove it would be to answer the following by the affirmative.

Question 8.3.3. *Is it true that, for every $(1, 2)$-composed digraph $D = (V, A)$, there exists $F \subseteq Q$ such that $D' = (V, F)$ is acyclic and $\mathrm{UG}(D \setminus F)$ is a $(1, 2)$-composed graph?*

Motivated by the question above, we conjecture the following.

Conjecture 8.3.4. *There exists a constant $\varepsilon > 0$ such that for every digraph $D = (V, A)$, there exists $F \subseteq A$ such that $D_1 = (V, F)$ is acyclic and $D_2 = (V, A \setminus F)$ satisfies*

$$\mathrm{Mad}(D_2) \leq (1 - \varepsilon) \cdot \mathrm{Mad}(D).$$

Observe that this is false when $\varepsilon > \frac{1}{2}$ because of bidirected graphs. We are inclined to believe that it turns out to be true even for $\varepsilon = \frac{1}{2}$.

8.4 Dichromatic number of the product of two digraphs

There exist different definitions of the product of graphs or digraphs. For each of these definitions, given two digraphs D_1, D_2, it is natural to ask for an expression of the dichromatic number of the product of D_1 and D_2 depending only on $\vec{\chi}(D_1)$ and $\vec{\chi}(D_2)$. For the undirected case, the interested reader is referred to the survey of Klavžar [106]. In this section, we study these questions for the tensor product and the Cartesian product.

Tensor product

The *tensor product* $G_1 \times G_2$ of two graphs G_1 and G_2 is the graph with vertex-set $V(G_1) \times V(G_2)$ in which (u_1, u_2) is adjacent to (v_1, v_2) if and only if u_1v_1 is an edge of G_1 and u_2v_2 is an edge of G_2. It is straightforward that $\chi(G_1 \times G_2) \leq \min\{\chi(G_1), \chi(G_2)\}$. Hedetniemi made a celebrated conjecture in 1966, stating that indeed equality holds for all G_1, G_2. This conjecture remained open for a long time, until Shitov [157] refuted it in 2019.

Theorem 8.4.1 (Shitov [157]). *There exist graphs G_1, G_2 such that*

$$\chi(G_1 \times G_2) < \min\{\chi(G_1), \chi(G_2)\}.$$

Analogously to the undirected case, we define the *tensor product* $D_1 \times D_2$ of digraphs D_1, D_2 as the digraph with vertex-set $V = V(D_1) \times V(D_2)$ in which there is an arc from (u_1, u_2) to (v_1, v_2) if and only if u_1v_1 is an arc of D_1 and u_2v_2 is an arc of D_2. Again, it is straightforward that $\vec{\chi}(D_1 \times D_2) \leq \min\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}$. Given two graphs G_1, G_2, observe that $\overleftrightarrow{G_1 \times G_2}$ is exactly $\overleftrightarrow{G_1} \times \overleftrightarrow{G_2}$. Therefore, the analogue of Hedetniemi's conjecture does not hold neither.

Harutyunyan and Puig i Surroca [95] asked whether Hedetniemi's conjecture holds for oriented graphs. We answer their question by the negative, using the result of Shitov and the following trick introduced by Aboulker, Havet, Pirot, and Schabanel [9] for building oriented graphs with large dichromatic number. For a graph G and integer k, we let $G^{(k)}$ be the blow-up of G with power k, that is the graph with vertex-set $V(G) \times [k]$ in which (u, i) is adjacent to (v, j) whenever $i \neq j$ and uv is an edge of G.

Lemma 8.4.2 (Aboulker et al. [9]). *For every graph G, there exists an integer k such that, for every $\ell \geq k$,*

$$\vec{\chi}(G^{(\ell)}) = \chi(G).$$

Proposition 8.4.3. *There exist oriented graphs $\vec{G}_1$, $\vec{G}_2$ with*

$$\vec{\chi}(\vec{G}_1 \times \vec{G}_2) < \min\{\vec{\chi}(\vec{G}_1), \vec{\chi}(\vec{G}_2)\}.$$

Proof. Let G_1 and G_2 be two graphs such that $\chi(G_1 \times G_2) < \min\{\chi(G_1), \chi(G_2)\} = s$, the existence of which is guaranteed by Theorem 8.4.1. Let k be an integer such that $\vec{\chi}(G_1^{(k)}) = \chi(G_1)$ and $\vec{\chi}(G_2^{(k)}) = \chi(G_2)$, the existence of which is guaranteed by Lemma 8.4.2. We fix two orientations $\vec{G}_1^{(k)}$ and $\vec{G}_2^{(k)}$ of $G_1^{(k)}$ and $G_2^{(k)}$ respectively, such that $\vec{\chi}(\vec{G}_1^{(k)}) = \chi(G_1)$ and $\vec{\chi}(\vec{G}_2^{(k)}) = \chi(G_2)$. We now justify that

$$\vec{\chi}(\vec{G}_1^{(k)} \times \vec{G}_2^{(k)}) < \min\left\{\vec{\chi}(\vec{G}_1^{(k)}), \vec{\chi}(\vec{G}_2^{(k)})\right\} = s,$$

which implies the result. It is sufficient to prove that $\chi(G_1^{(k)} \times G_2^{(k)}) < s$, as the dichromatic number is bounded above by the chromatic number. By choice of G_1 and G_2, there exists a proper $(s-1)$-colouring α of $G_1 \times G_2$. We let β be the $(s-1)$-colouring of $G_1^{(k)} \times G_2^{(k)}$ defined as $\beta\Big(((u_1, i), (u_2, j))\Big) = \alpha\Big((u_1, u_2)\Big)$ for every $((u_1, i), (u_2, j)) \in V(G_1^{(k)}) \times V(G_2^{(k)})$.

Assume for a contradiction that $G_1^{(k)} \times G_2^{(k)}$, coloured with β, contains a monochromatic edge $\{((u_1, i), (u_2, j)), ((v_1, i'), (u_2, j'))\}$. By definition, $\{(u_1, i), (v_1, i')\}$ is an edge of $G_1^{(k)}$ and

$\{(u_2, j), (v_2, j')\}$ is an edge of $G_2^{(k)}$. Hence $\{u_1, v_1\}$ is an edge of G_1 and $\{u_2, v_2\}$ is an edge of G_2, implying that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge of $G_1 \times G_2$. Hence $G_1 \times G_2$, coloured with α, contains a monochromatic edge, a contradiction. $\square$

When they studied Hedetniemi's conjecture, Poljak and Rödl [142] defined the function $f(n)$ as follows:

$$f(n) = \min\{\chi(G_1 \times G_2) \mid \chi(G_1) = \chi(G_2) = n\}.$$

The following is a weakening of Hedetniemi's conjecture which is still open.

Conjecture 8.4.4. *The function $f(n)$ tends to infinity.*

It is known that either $f(n)$ tends to infinity or $f(n) \leq 9$ for all $n \in \mathbb{N}$ (see [170]). In the same vein, we define $g(n)$ as follows:

$$g(n) = \min\{\vec{\chi}(D_1 \times D_2) \mid \vec{\chi}(D_1) = \vec{\chi}(D_2) = n\}.$$

The following is the directed counterpart of Conjecture 8.4.4.

Conjecture 8.4.5. *The function $g(n)$ tends to infinity.*

Clearly, Conjecture 8.4.5 implies Conjecture 8.4.4. We ask for the equivalence.

Question 8.4.6. *Does Conjecture 8.4.4 imply Conjecture 8.4.5?*

Cartesian product

The *Cartesian product* $D_1 \,\square\, D_2$ of two digraphs D_1, D_2 is the digraph with vertex-set $V(D_1) \times V(D_2)$ which contains an arc from (u_1, u_2) to (v_1, v_2) if and only if $u_1 = v_1$ and $u_2v_2 \in A(D_2)$ or $u_2 = v_2$ and $u_1v_1 \in A(D_1)$. We clearly have $\vec{\chi}(D_1 \,\square\, D_2) \geq \max\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}$ (except if one of D_1, D_2 is empty) since $D_1 \square D_2$ contains both D_1 and D_2 as subdigraphs. Harutyunyan and Puig i Surroca [95] proved that indeed equality holds, so if D_1, D_2 are two non-empty digraphs, then

$$\vec{\chi}(D_1 \,\square\, D_2) = \max\{\vec{\chi}(D_1), \vec{\chi}(D_2)\}.$$

5he dichromatic number of Cartesian products of digraphs is therefore well-understood. However, it is not clear what is happening for the dichromatic number of Cartesian products of undirected graphs. The *Cartesian product* $G_1 \,\square\, G_2$ of two graphs is the underlying graph of $\overleftrightarrow{G_1} \,\square\, \overleftrightarrow{G_2}$.

If G_1, G_2 are two non-empty graphs, we clearly have $\vec{\chi}(G_1 \,\square\, G_2) \geq \max\{\vec{\chi}(G_1), \vec{\chi}(G_2)\}$. However, equality does not occur in general, for instance $K_2 \,\square\, K_2$ is C_4, and we have $\vec{\chi}(C_4) = 2$ while $\vec{\chi}(K_2) = 1$. The following bound is straightforward:

$$\vec{\chi}(G_1 \,\square\, G_2) \leq \chi(G_1) \cdot \vec{\chi}(G_2).$$

It is natural to ask for the existence of a function $f(k)$ such that $\vec{\chi}(G_1 \,\square\, G_2) \leq f(\max\{\vec{\chi}(G_1), \vec{\chi}(G_2)\})$ for all graphs G_1, G_2. Observe that Conjecture 8.1.4, together with the inequality above, implies the existence of such a function. We conjecture that not only such a function exists, but even a linear one does.

Conjecture 8.4.7. *There exists a constant c such that, for all graphs G_1, G_2,*

$$\vec{\chi}(G_1 \,\square\, G_2) \leq c \cdot \max\{\vec{\chi}(G_1), \vec{\chi}(G_2)\}.$$

The very first interesting case of the conjecture is for G_1, G_2 being the complete graph on n vertices. In this case, we have $\vec{\chi}(K_n) = O\left(\frac{n}{\log n}\right)$, as shown by Erdős, Gimbel and Kratsch in [70]. If true, the conjecture above implies $\vec{\chi}(K_n \,\square\, K_n) = O\left(\frac{n}{\log n}\right)$. Observe that this bound is also a consequence of Conjecture 8.1.2, since $\Delta(K_n \square K_n) = 2(n-1)$. We thus ask the following.

Question 8.4.8. *Is it true that* $\vec{\chi}(K_n \,\square\, K_n) = O\left(\frac{n}{\log n}\right)$*?*

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Appendix A

Appendix to Chapter 5

In this appendix we present the complete code used in the proof of Theorem 5.1.6. As mentioned in Chapter 5, this is a joint work with Frédéric Havet and Florian Hörsch. The code is implemented using SageMath and is accessible on the author's GitHub webpage.

We first give the adjacency matrices of T^1, T^2, T^3, and T^4 in Section A.1. In Section A.2, we give a collection of useful subroutines we use in the main part of the code. In Sections A.3, A.4, A.5, and A.6, we give the code use in the proofs of Lemmas 5.4.5, 5.4.6, 5.4.8, and 5.4.10, respectively.

A.1 The tournaments T^1, T^2, T^3, and T^4

We give the adjacency matrices of T^1, T^2, T^3, and T^4.

$$T^1 : \quad \begin{array}{c c} & \begin{array}{ccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$T^2: \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$T^3: \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccccccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$
T^4 : \quad \begin{array}{c|ccccccccc}
 & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & x_1 & x_2 & x_3 \\
\hline
u_1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
u_2 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
u_3 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
u_4 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
u_5 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
u_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
x_1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
x_2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
x_3 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}
$$

A.2 Preliminaries for the code

In the following code, we give a collection of subroutines we use in our code.

```python
# The following function displays a progress bar
def printProgressBar (iteration, total):
    percent = ("{0:.1f}").format(100 * (iteration / float(total)))
    filledLength = int(50 * iteration // total)
    bar = "#" * filledLength + "-" * (50 - filledLength)
    print(f"\rProcess: |{bar}| {percent}% Complete", end = "\r")
    # Print New Line on Complete
    if iteration == total:
        print()

# k,n : integers such that k < 3**n
# Returns: the decomposition of k in base 3 of length n
def ternary(k,n):
    b = 3**(n-1)
    res=""
    for i in range(n):
        if(k >= 2*b):
            k -= 2*b
            res = res + "2"
        elif(k >= b):
            k -= b
            res = res + "1"
        else:
            res = res + "0"
        b /= 3
    return res

# d: DiGraph
```

```
# u: vertex
# v: vertex
# Returns: True if and only if d contains a directed path from u to v
def contains_directed_path(d,u,v):
    to_be_treated = [u]
    i=0
    while(len(to_be_treated) != i):
        x = to_be_treated[i]
        if (x==v):
            return True
        for y in d.neighbors_out(x):
            if (not y in to_be_treated):
                to_be_treated.append(y)
        i+=1
    return False

# d: DiGraph
# u: vertex of d
# v: vertex of d
# current_colouring: partial 2-dicolouring with colours {0,1} of d such that current_colouring[u] = current_colouring[v] = 0
# Returns: True if and only if current_colouring can be extended into a 2-dicolouring of d with no monochromatic directed path from u to v.
def can_be_subgraph_of_3_dicritical_aux(d,u,v, current_colouring):
    #build the colour classes
    colours = {}
    colours[0] = []
    colours[1] = []
    for (x,i) in current_colouring.items():
        colours[i].append(x)
    #check whether both colour classes are acyclic
    for i in range(2):
        d_i = d.subgraph(colours[i])
        if(not d_i.is_directed_acyclic()):
            return False
    #check whether there is a monochromatic directed path from u to v
    d_0 = d.subgraph(colours[0])
    if(contains_directed_path(d_0,u,v)):
        return False
    #check whether current_colouring is partial
    if(len(current_colouring) == d.order()):
        return True
    else:
        #find a vertex x that is not coloured yet
        x = len(current_colouring)
        while(x in current_colouring):
            x-=1
        #check recursively whether current_colouring can be extended to x
        for i in range(2):
            current_colouring[x] = i
```

```
            if(can_be_subgraph_of_3_dicritical_aux(d,u,v,current_colouring)):
                return True
            current_colouring.pop(x, None)
        return False

# d: DiGraph
# forbidden_subtournaments: list of DiGraphs
# Returns: True if and only if d is {forbidden_subdigraphs}-free, {forbidden_induced_subdigraphs}-free  and, for every arc (u,v) of d, d admits a uv-colouring.
def can_be_subgraph_of_3_dicritical(d, forbidden_subdigraphs, forbidden_induced_subdigraphs):
    #check whether d contains a forbidden subgraph
    for T in forbidden_subdigraphs:
        if(d.subgraph_search(T, False) != None):
            return False
    #check whether d contains a forbidden induced subgraph
    for T in forbidden_induced_subdigraphs:
        if(d.subgraph_search(T, True) != None):
            return False
    #check for every arc uv if d admits a uv-colouring.
    for e in d.edges():
        d_aux = DiGraph(len(d.vertices()))
        d_aux.add_edges(d.edges())
        d_aux.delete_edge(e)
        current_colouring = {}
        current_colouring[e[0]] = 0
        current_colouring[e[1]] = 0
        if(not can_be_subgraph_of_3_dicritical_aux(d_aux,e[0],e[1],current_colouring)):
            return False
    return True

# d: DiGraph
# Returns: True if and only if d is 2-dicolourable
def is_two_dicolourable(d):
    n = d.order()
    for bipartition in range(2**n):
        #build the binary word corresponding to the bipartition
        binary = bin(bipartition)[2:]
        while(len(binary)<(n)):
            binary = "0" + binary
        #build the bipartition
        V1 = []
        V2 = []
        for v in range(n):
            if(binary[v] == '0'):
                V1.append(v)
            else:
                V2.append(v)
        #check whether (V1,V2) is actually a dicolouring
```

```
        d1 = d.subgraph(V1)
        d2 = d.subgraph(V2)
        if(d1.is_directed_acyclic() and d2.is_directed_acyclic()):
            return True
    return False

#C3_C3 is the digraph made of two disjoint directed triangles, the
   vertices of one dominating the vertices of the other
C3_C3 = DiGraph(6)
for i in range(3):
    C3_C3.add_edge(i,(i+1)%3)
    C3_C3.add_edge(i+3,((i+1)%3)+3)
    for j in range(3,6):
        C3_C3.add_edge(i,j)

#F is the digraph on nine vertices made of a TT6 u1,...,u6 and the arcs
    of the directed triangles u1u2x1u1, u3u4x2u3, u5u6x3u5.
F = DiGraph(9)
for i in range(6):
    for j in range(i):
        F.add_edge(j,i)
F.add_edge(6,0)
F.add_edge(1,6)
F.add_edge(7,2)
F.add_edge(3,7)
F.add_edge(8,4)
F.add_edge(5,8)

#TT8 is the transitive tournament on 8 vertices
TT8 = DiGraph(8)
for i in range(8):
    for j in range(i):
        TT8.add_edge(j,i)

#reversed_TT8 is the set of tournaments, up to isomorphism, obtained
   from TT8 by reversing exactly one arc
reversed_TT8 = []
for e in TT8.edges():
    rev = DiGraph(8)
    rev.add_edges(TT8.edges())
    rev.delete_edge(e)
    rev.add_edge(e[1],e[0])
    check = True
    for T in reversed_TT8:
        check = check and (not T.is_isomorphic(rev))
    if(check):
        reversed_TT8.append(rev)

#K2 is the complete digraph on 2 vertices
K2 = DiGraph(2)
K2.add_edge(0,1)
K2.add_edge(1,0)
```

```
#S4 is the bidirected star on 4 vertices
S4 = DiGraph(4)
for i in range(1,4):
    S4.add_edge(i,0)
    S4.add_edge(0,i)

#C3_K2 is the digraph with a directed triangle dominating a digon.
C3_K2 = DiGraph(5)
for i in range(3):
    C3_K2.add_edge(i,(i+1)%3)
    for j in range(3,5):
        C3_K2.add_edge(i,j)
C3_K2.add_edge(3,4)
C3_K2.add_edge(4,3)

#K2_C3 is the digraph with a digon dominating a directed triangle
K2_C3 = DiGraph(5)
for i in range(3):
    K2_C3.add_edge(i,(i+1)%3)
    for j in range(3,5):
        K2_C3.add_edge(j,i)
K2_C3.add_edge(3,4)
K2_C3.add_edge(4,3)

#K2_K2 is the digraph with a digon dominating a digon
K2_K2 = DiGraph(4)
for i in range(2):
    K2_K2.add_edge(i,(i+1)%2)
    K2_K2.add_edge(i+2,((i+1)%2)+2)
    for j in range(2,4):
        K2_K2.add_edge(i,j)

#O4 and O5 are the obstructions described in the paper.
O4 = DiGraph(4)
for i in range(1,4):
    O4.add_edge(0,i)
for i in range(1,3):
    O4.add_edge(i,3)
O4.add_edge(1,2)
O4.add_edge(2,1)

O5 = DiGraph(5)
for i in range(1,5):
    O5.add_edge(0,i)
for i in range(1,4):
    O5.add_edge(i,4)
O5.add_edge(1,2)
O5.add_edge(2,3)
O5.add_edge(3,1)
```

A.3 Code used in the proof of Lemma 5.4.5

We here give the code used in the proof of Lemma 5.4.5.

```
load("tools.sage")

# binary_code: a string of fifteen characters '0' and '1'
# Returns: a tournament of \mathcal{F}. The orientations of the fifteen non-forced arcs correspond to the characters of binary_code.
def digraph_blowup_TT3(binary_code):
    iterator_binary_code = iter(binary_code)
    d = DiGraph(9)
    #the vertices 0,...,8 correspond respectively to u_1,...,u_6,x_1,x_2,x_3

    #add the arcs of the TT_6
    for i in range(6):
        for j in range(i):
            d.add_edge(j,i)

    #add the arcs of the directed triangles
    for i in range(3):
        d.add_edge(6+i, 2*i)
        d.add_edge(2*i+1, 6+i)

    missing_edges=[(6,2),(6,3),(6,4),(6,5),(6,7),(6,8),(7,0),(7,1),(7,4),(7,5),(7,8),(8,0),(8,1),(8,2),(8,3)]
    #we orient the missing_edges according to binary_code
    for e in missing_edges:
        if(next(iterator_binary_code) == '0'):
            d.add_edge(e[0],e[1])
        else:
            d.add_edge(e[1],e[0])
    return d

print("Computing all possible candidates of \mathcal{F} for being a subtournament of a 3-dicritical semi-complete digraph...")
list_candidates = []
list_forbidden_induced_subdigraphs = [C3_C3]

#print progress bar
printProgressBar(0, 2**15)
for i in range(2**15):
    binary_value = bin(i)[2:]
    while(len(binary_value)<15):
        binary_value = '0' + binary_value
    d = digraph_blowup_TT3(binary_value)
    if(can_be_subgraph_of_3_dicritical(d,[],list_forbidden_induced_subdigraphs)):
        list_candidates.append(d)
    #update progress bar
    printProgressBar(i + 1, 2**15)
```

```

print("Number of candidates: ",len(list_candidates),".")
for i in range(len(list_candidates)):
    print("Candidate ",i+1,": ")
    list_candidates[i].export_to_file("T"+str(i+1)+".pajek")
    print(list_candidates[i].adjacency_matrix())
```

Running this code produces the following output after roughly 2 minutes of execution on a standard desktop computer:

```
Computing all possible candidates of \mathcal{F} for being a
    subtournament of a 3-dicritical semi-complete digraph...
Process: |##################################################| 100.0%
    Complete
Number of candidates:  4 .
Candidate  1 :
[0 1 1 1 1 1 0 1 1]
[0 0 1 1 1 1 1 1 1]
[0 0 0 1 1 1 0 0 0]
[0 0 0 0 1 1 0 1 0]
[0 0 0 0 0 1 0 1 0]
[0 0 0 0 0 0 1 1 1]
[1 0 1 1 1 0 0 0 1]
[0 0 1 0 0 0 1 0 1]
[0 0 1 1 1 0 0 0 0]
Candidate  2 :
[0 1 1 1 1 1 0 1 1]
[0 0 1 1 1 1 1 1 1]
[0 0 0 1 1 1 1 0 0]
[0 0 0 0 1 1 0 1 0]
[0 0 0 0 0 1 0 1 0]
[0 0 0 0 0 0 0 1 1]
[1 0 0 1 1 1 0 1 0]
[0 0 1 0 0 0 0 0 1]
[0 0 1 1 1 0 1 0 0]
Candidate  3 :
[0 1 1 1 1 1 0 0 0]
[0 0 1 1 1 1 1 0 1]
[0 0 0 1 1 1 1 0 1]
[0 0 0 0 1 1 1 1 1]
[0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 1]
[1 0 0 0 1 1 0 1 1]
[1 1 1 0 1 1 0 0 0]
[1 0 0 0 1 0 0 1 0]
Candidate  4 :
[0 1 1 1 1 1 0 0 1]
[0 0 1 1 1 1 1 0 1]
[0 0 0 1 1 1 1 0 1]
[0 0 0 0 1 1 1 1 0]
[0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 1]
[1 0 0 0 1 1 0 1 0]
```

```
[1 1 1 0 1 1 0 0 1]
[0 0 0 1 1 0 1 0 0]
```

The graphs in the output are exactly T_1, T_2, T_3, and T_4.

A.4 Code used in the proof of Lemma 5.4.6

We here give the code used in the proof of Lemma 5.4.6.

```
import networkx
load("tools.sage")

list_candidates = []
list_forbidden_induced_subdigraphs = [C3_C3]

#import T1, T2, T3 and T4
for i in range(1,5):
    candidate = DiGraph(9)
    nx = networkx.read_pajek("T"+str(i)+".pajek")
    for e in nx.edges():
        candidate.add_edge(int(e[0]),int(e[1]))
    list_candidates.append(candidate)

print("We start from the ", len(list_candidates), " candidates on 9
    vertices.")

#We want to prove that a 3-dicritical semi-complete digraph does not
    contain a digraph in {F+,F-}. By directional duality, it is
    sufficient to prove that it does not contain F+.
#For each candidate computed above, we try to add a new vertex that
    dominates the transitive tournament, and then we build every
    possible orientation between this vertex and the three other
    vertices.
print("Computing for F+...")
list_Fp = []
printProgressBar(0, 8)
for orientation in range(2**3):
    binary = bin(orientation)[2:]
    while(len(binary)<3):
        binary = '0' + binary
    for T9 in list_candidates:
        iterator = iter(binary)
        T10 = DiGraph(10)
        T10.add_edges(T9.edges())
        for v in range(6):
            T10.add_edge(9,v)
        for v in range(6,9):
            if(next(iterator) == '0'):
                T10.add_edge(v,9)
            else:
                T10.add_edge(9,v)
```

```
        check = can_be_subgraph_of_3_dicritical(T10,[], list_forbidden_induced_subdigraphs)
        if(check):
            list_Fm.append(T2)
    printProgressBar(orientation+1, 8)

print("Number of 1-extensions of {T1,T2,T3,T4} containing F+: ",len(list_Fp))
```

Running this code produces the following output after roughly 1 second of execution on a standard desktop computer:

```
We start from the  4  candidates on 9 vertices.
Computing for F+...
Process: |##################################################| 100.0% Complete
Number of 1-extensions of {T1,T2,T3,T4} containing F+:  0
```

A.5 Code used in the proof of Lemma 5.4.8

We here give the code used in the proof of Lemma 5.4.8.

```
import networkx
load("tools.sage")

all_candidates = []
current_candidates = []
next_candidates = []

list_forbidden_subdigraphs = [S4, K2_K2, O4, O5, K2_C3, C3_K2, C3_C3]
list_forbidden_induced_subdigraphs = [TT8]

#import the candidates T1, ..., T4 on 9 vertices:
for i in range(1,5):
    candidate = DiGraph(9)
    nx = networkx.read_pajek("T"+str(i)+".pajek")
    for e in nx.edges():
        candidate.add_edge(int(e[0]),int(e[1]))
    current_candidates.append(candidate)
print("We start from the tournaments {T^1,T^2,T^3,T^4} on 9 vertices, and look for every possible completion of them that is potentially a subdigraph of a larger 3-dicritical semi-complete digraphs.\n")


all_candidates.extend(current_candidates)
#completions of T1, ..., T4
while(len(current_candidates)>0):
    for old_D in current_candidates:
        for e in old_D.edges():
            #we try to complete old_D by replacing e by a digon. It actually makes sense only if e is not already in a digon.
            if(not (e[1],e[0],None) in old_D.edges()):
```

```
                new_D = DiGraph(9)
                new_D.add_edges(old_D.edges())
                new_D.add_edge(e[1],e[0])
                #we check whether this completion of old_D is potentially a subdigraph of a larger 3-dicritical semi-complete digraph.
                check = can_be_subgraph_of_3_dicritical(new_D, list_forbidden_subdigraphs, list_forbidden_induced_subdigraphs)
                for D in next_candidates:
                    check = check and (not D.is_isomorphic(new_D))
                if(check):
                    next_candidates.append(new_D)
    all_candidates.extend(next_candidates)
    current_candidates = next_candidates
    next_candidates=[]

print("-----------------------------------------")
print("There are",len(all_candidates),"possible completions (up to isomorphism) of {T^1,T^2,T^3,T^4} that are potentially subdigraphs of a larger 3-dicritical semi-complete digraphs.\n")


count_dic_3 = 0
for D in all_candidates:
    if(not is_two_dicolourable(D)):
        count_dic_3 += 1
print(count_dic_3, " of them have dichromatic number at least 3. In particular,",count_dic_3,"of them are 3-dicritical.\n")

current_candidates = all_candidates
next_candidates = []
for n in range(10,12):
    #computes the extensions on n vertices of {T1,T2,T3,T4}
    print("-----------------------------------------")
    print("Computing "+str(n-9)+"-extensions of the candidates on 9 vertices that are potentially subtournaments of 3-dicritical tournaments (up to isomorphism).")
    printProgressBar(0, 3**(n-1))

    for orientation in range(3**(n-1)):
        ternary_code = ternary(orientation,n-1)
        #build every 1-extension of current_candidates
        for old_D in current_candidates:
            new_D = DiGraph(n)
            new_D.add_edges(old_D.edges())
            for v in range(n-1):
                if(ternary_code[v] == '0'):
                    new_D.add_edge(v,n-1)
                elif(ternary_code[v]=='1'):
                    new_D.add_edge(n-1,v)
                else:
                    new_D.add_edge(v,n-1)
```

```
                    new_D.add_edge(n-1,v)
            check = can_be_subgraph_of_3_dicritical(new_D, list_forbidden_subdigraphs,list_forbidden_induced_subdigraphs)
            for D in next_candidates:
                check = check and (not new_D.is_isomorphic(D))
            if(check):
                next_candidates.append(new_D)
        printProgressBar(orientation+1, 3**(n-1))

    print("Number of ", n-9,"-extensions up to isomorphism: ",len(next_candidates))
    #check if one of the candidates has dichromatic number at least 3.
    count_dic_3 = 0
    for D in next_candidates:
        if(not is_two_dicolourable(D)):
            count_dic_3 += 1
    print(count_dic_3, " of them have dichromatic number at least 3. In particular,", count_dic_3,"of them are 3-dicritical.\n")
    current_candidates = next_candidates
    next_candidates = []
```

Running this code produces the following output after roughly 12 minutes of execution on a standard desktop computer:

```
We start from the tournaments {T^1,T^2,T^3,T^4} on 9 vertices, and look for every possible completion of them that is potentially a subdigraph of a larger 3-dicritical semi-complete digraphs.

-----------------------------------------
There are 14 possible completions (up to isomorphism) of {T^1,T^2,T^3,T^4} that are potentially subdigraphs of a larger 3-dicritical semi-complete digraphs.

0  of them have dichromatic number at least 3. In particular, 0 of them are 3-dicritical.

-----------------------------------------
Computing 1-extensions of the candidates on 9 vertices that are potentially subtournaments of 3-dicritical tournaments (up to isomorphism).
Process: |##################################################| 100.0% Complete
Number of  1 -extensions up to isomorphism:  34
0  of them have dichromatic number at least 3. In particular, 0 of them are 3-dicritical.

-----------------------------------------
Computing 2-extensions of the candidates on 9 vertices that are potentially subtournaments of 3-dicritical tournaments (up to isomorphism).
Process: |##################################################| 100.0% Complete
Number of  2 -extensions up to isomorphism:  0
```

```
0  of them have dichromatic number at least 3. In particular, 0 of them
   are 3-dicritical.
```

A.6 Code used in the proof of Lemma 5.4.10

We here give the code used in the proof of Lemma 5.4.10.

```
load("tools.sage")

def possible_completions(graph_to_complete, nb_vertices,
    list_forbidden_subdigraphs, list_forbidden_induced_subdigraphs,
    progress=0):
    if(progress == nb_vertices-1):
        return [graph_to_complete]
    else:
        result = []
        for i in range(3):
            #we make a copy of the graph_to_complete
            new_D = DiGraph(nb_vertices)
            new_D.add_edges(graph_to_complete.edges())

            #we consider every possible orientation between the
    vertices (nb_vertices-1) and (progress)
            if(i==0):
                new_D.add_edge(nb_vertices-1, progress)
            elif(i==1):
                new_D.add_edge(progress, nb_vertices-1)
            else:
                new_D.add_edge(nb_vertices-1, progress)
                new_D.add_edge(progress, nb_vertices-1)

            #for each of the 3 possible orientations, we check whether
    the obtained digraph is already an obstruction. If it is not, we
    compute all possible completions recursively
            if(can_be_subgraph_of_3_dicritical(new_D,
    list_forbidden_subdigraphs, list_forbidden_induced_subdigraphs)):
                result.extend(possible_completions(new_D, nb_vertices,
    list_forbidden_subdigraphs, list_forbidden_induced_subdigraphs,
    progress+1))
        return result

for tt in range(1,8):
    transitive_tournament = DiGraph(tt)
    for i in range(tt):
        for j in range(i):
            transitive_tournament.add_edge(j,i)

    next_transitive_tournament = DiGraph(tt+1)
    for i in range(tt+1):
        for j in range(i):
            next_transitive_tournament.add_edge(j,i)
```

```
    list_forbidden_subdigraphs = [S4, K2_K2, O4, O5, K2_C3, C3_K2, C3_C3, F]
    list_forbidden_induced_subdigraphs = []
    if(tt<7):
        list_forbidden_induced_subdigraphs = [next_transitive_tournament]
    else:
        list_forbidden_induced_subdigraphs = reversed_TT8

    print("\n---------------------------------------------------------\n")
    print("Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly " + str(tt) + ".")

    n=tt+1
    candidates = [transitive_tournament]
    next_candidates = []
    while(len(candidates)>0):
        print("\nComputing candidates on "+str(n)+" vertices.")
        printProgressBar(0, len(candidates))
        for i in range(len(candidates)):
            old_D = candidates[i]
            new_D = DiGraph(n)
            new_D.add_edges(old_D.edges())
            all_possible_completions_new_D = possible_completions(new_D, n, list_forbidden_subdigraphs, list_forbidden_induced_subdigraphs)
            for candidate in all_possible_completions_new_D:
                check = True
                for D in next_candidates:
                    check = not D.is_isomorphic(candidate)
                    if(not check):
                        break
                if(check):
                    next_candidates.append(candidate)
            printProgressBar(i + 1, len(candidates))

        #check the candidates that are actually 3-dicritical.
        print("We found",len(next_candidates),"candidates on "+str(n)+" vertices.")
        dicriticals = []
        for D in next_candidates:
            if(not is_two_dicolourable(D)):
                dicriticals.append(D)
        print(len(dicriticals), " of them are actually 3-dicritical.\n")
        for D in dicriticals:
            print("adjacency matrix of a 3-dicritical digraph that we found:")
            print(D.adjacency_matrix())
```

```

        candidates = next_candidates
        next_candidates = []
        n+=1
```

Running this code produces the following output after roughly 2 hours of execution on a standard desktop computer:

```
--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 1.

Computing candidates on 2 vertices.
Process: |##################################################| 100.0% Complete
We found 1 candidates on 2 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 3 vertices.
Process: |##################################################| 100.0% Complete
We found 1 candidates on 3 vertices.
1  of them are actually 3-dicritical.

adjacency matrix of a 3-dicritical digraph that we found:
[0 1 1]
[1 0 1]
[1 1 0]

Computing candidates on 4 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 4 vertices.
0  of them are actually 3-dicritical.


--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 2.

Computing candidates on 3 vertices.
Process: |##################################################| 100.0% Complete
We found 5 candidates on 3 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 4 vertices.
Process: |##################################################| 100.0% Complete
```

```
We found 5 candidates on 4 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 5 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 5 vertices.
0  of them are actually 3-dicritical.


--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 3.

Computing candidates on 4 vertices.
Process: |##################################################| 100.0% Complete
We found 13 candidates on 4 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 5 vertices.
Process: |##################################################| 100.0% Complete
We found 37 candidates on 5 vertices.
1  of them are actually 3-dicritical.

adjacency matrix of a 3-dicritical digraph that we found:
[0 1 1 0 0]
[0 0 1 0 1]
[0 0 0 1 1]
[1 1 0 0 1]
[1 0 1 1 0]

Computing candidates on 6 vertices.
Process: |##################################################| 100.0% Complete
We found 8 candidates on 6 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 7 vertices.
Process: |##################################################| 100.0% Complete
We found 1 candidates on 7 vertices.
1  of them are actually 3-dicritical.

adjacency matrix of a 3-dicritical digraph that we found:
[0 1 1 0 0 0 1]
[0 0 1 0 1 1 0]
```

```
[0 0 0 1 0 1 1]
[1 1 0 0 0 1 0]
[1 0 1 1 0 0 0]
[1 0 0 0 1 0 1]
[0 1 0 1 1 0 0]

Computing candidates on 8 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 8 vertices.
0  of them are actually 3-dicritical.


--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 4.

Computing candidates on 5 vertices.
Process: |##################################################| 100.0% Complete
We found 27 candidates on 5 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 6 vertices.
Process: |##################################################| 100.0% Complete
We found 116 candidates on 6 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 7 vertices.
Process: |##################################################| 100.0% Complete
We found 10 candidates on 7 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 8 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 8 vertices.
0  of them are actually 3-dicritical.


--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 5.

Computing candidates on 6 vertices.
```

```
Process: |##################################################| 100.0% Complete
We found 49 candidates on 6 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 7 vertices.
Process: |##################################################| 100.0% Complete
We found 266 candidates on 7 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 8 vertices.
Process: |##################################################| 100.0% Complete
We found 20 candidates on 8 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 9 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 9 vertices.
0  of them are actually 3-dicritical.


-----------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 6.

Computing candidates on 7 vertices.
Process: |##################################################| 100.0% Complete
We found 80 candidates on 7 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 8 vertices.
Process: |##################################################| 100.0% Complete
We found 500 candidates on 8 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 9 vertices.
Process: |##################################################| 100.0% Complete
We found 39 candidates on 9 vertices.
0  of them are actually 3-dicritical.

```

```

Computing candidates on 10 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 10 vertices.
0  of them are actually 3-dicritical.


--------------------------------------------------------

Generating all 3-dicritical semi-complete digraphs with maximum acyclic induced subdigraph of size exactly 7.

Computing candidates on 8 vertices.
Process: |##################################################| 100.0% Complete
We found 110 candidates on 8 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 9 vertices.
Process: |##################################################| 100.0% Complete
We found 459 candidates on 9 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 10 vertices.
Process: |##################################################| 100.0% Complete
We found 16 candidates on 10 vertices.
0  of them are actually 3-dicritical.


Computing candidates on 11 vertices.
Process: |##################################################| 100.0% Complete
We found 0 candidates on 11 vertices.
0  of them are actually 3-dicritical.
```

The adjacency matrices in the output are exactly those of the digraphs $\overleftrightarrow{K_3}$, $\mathcal{H}_5$ and $\mathcal{P}_7$.

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Symbol Index

Coloration de Graphes Dirigés

Lucas PICASARRI-ARRIETA

Résumé

Cette thèse est dédiée à l'étude de la dicoloration, une notion de coloration pour les digraphes introduite par Erdős et Neumann-Lara à la fin des années 1970, ainsi que le paramètre qui lui est associé, à savoir le nombre dichromatique. Lors des dernières décennies, ces deux notions ont permis de généraliser de nombreux résultats classiques de coloration de graphes.
Nous commençons par donner différentes bornes sur le nombre dichromatique des digraphes dont le graphe sous-jacent est un graphe cordal. Ensuite, nous améliorons la borne donnée par le théorème de Brooks pour les digraphes sans arcs antiparallèles et introduisons une notion de dégénérescence variable pour les digraphes, ce qui nous permet de prouver une version plus générale du théorème de Brooks.
Nous étudions ensuite les digraphes k-dicritiques, c'est-à-dire les obstructions minimales à la $(k-1)$-dicolorabilité. En particulier, nous généralisons un résultat de Gallai au cas dirigé, et nous prouvons une conjecture de Kostochka et Stiebitz dans le cas particulier $k = 4$. Nous discutons également la densité maximum de tels digraphes, et prouvons qu'il n'y a qu'un nombre fini de digraphes semi-complets 3-dicritiques. On donne par la suite certains résultats structurels sur les digraphes dicritiques de grand ordre.
Enfin, nous étudions la notion de redicoloration pour les digraphes. En particulier, nous prouvons que de nombreux résultats soutenant la conjecture de Cereceda se généralisent au cas dirigé.

Mots-clés : Digraphes, dicoloration, nombre dichromatique, reconfiguration, digraphes dicritiques.

Abstract

This thesis focuses on a notion of colouring of digraphs introduced by Erdős and Neumann-Lara in the late 1970s, namely the dicolouring, and its associated digraph parameter: the dichromatic number. It appears in the last decades that many classical results on graph colouring have directed counterparts using these notions.
We first give a collection of bounds on the dichromatic number of digraphs for which the underlying graph is chordal. We then introduce a notion of variable degeneracy for digraphs which leads to a more general version of Brooks Theorem. We also strengthen this theorem on a large class of digraphs which contains digraphs without antiparallel arcs.
Next we prove a collection of results on k-dicritical digraphs, the digraphs that are minimal obstructions for the $(k-1)$-dicolourability. We first generalise a result of Gallai to the directed case, and then prove a conjecture of Kostochka and Stiebitz in the particular case $k = 4$. We also discuss the maximum density of such digraphs and prove that the number of 3-dicritical semi-complete digraphs is finite. We then give a collection of results on the substructures in large dicritical digraphs.
We finally study the notion of redicolouring for digraphs. In particular, we prove that a large collection of evidences for Cereceda's conjecture admit a directed counterpart.

Keywords: Digraphs, dicolouring, dichromatic number, reconfiguration, dicritical digraphs.