Complexity of some arc-partition problems for digraphs

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Arc(edge)-partitioning problems

Given two properties P_1, P_2, **the** (P_1, P_2)**-arc-partitioning problem** consists of deciding whether a digraph $D = (V, A)$ has a partition of its arcs in two subsets A_1 and A_2 such that (V, A_i) **has property** P_i.

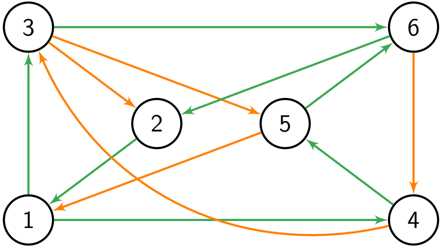
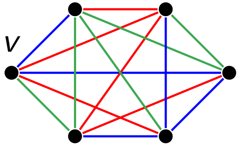


Figure: A digraph with a (*strongly connected, having out-branching*)-arc-partition

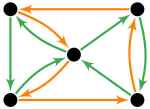
Arc-partitioning or edge-partitioning problems are related to **fault tolerance**.

Undirected case



Using Tutte-Nash-Williams theorem (1961), one can decide in **polynomial** time if $G = (V, E)$ has k **edge-disjoint spanning trees**.

Directed case



It is **NP-complete** to decide if $D = (V, A)$ has 2 arc-disjoint **strongly connected spanning subdigraphs** (Bang-Jensen & Yeo, 2004).

We fixed 15 properties we wanted to study :

- *bipartite*,
- *connected*,
- *strongly connected*,
- *acyclic*,
- *acyclic spanning*,
- *having an out-branching*,
- *having an in-branching*,
- $\delta^+ \geq k$,
- $\delta^- \geq k$,
- *cycle factor*,
- $\geq k$ *arcs*,
- $\leq k$ *arcs*,
- *balanced*,
- *eulerian*,
- *being a cycle*.

$\Longrightarrow$ 120 arc-partitioning problems to study

	Bipartite	Connected	Strongly Connected	Acyclic	Acyclic span.	Out-Branching	In-Branching
Bipartite	NPC	Poly	Poly	NPC	NPC	Poly	Poly
Connected	×	Poly	NPC	Poly	NPC	NPC	NPC
Strongly Connected	×	×	NPC	Poly	NPC	NPC	NPC
Acyclic	×	×	×	Poly	Poly	Poly	Poly
Acyclic span.	×	×	×	×	Poly	NPC	NPC
Out-Branching	×	×	×	×	×	Poly	NPC
In-branching	×	×	×	×	×	×	Poly

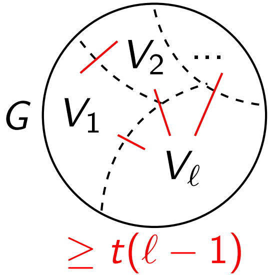
	$\delta^- \geq k$	$\delta^+ \geq k$	Cycle Factor	$\leq k$ arcs	$\geq k$ arcs	Balanced	Eulerian	Cycle
Bipartite	Poly	Poly	NPC	NPC	Poly	NPC	NPC	NPC
Connected	Poly	Poly	NPC	Poly	Poly	Poly	NPC	NPC
Strongly Connected	NPC	NPC	NPC	Poly	NPC	Poly	NPC	NPC
Acyclic	Poly	Poly	NPC	NPC	Poly	Poly	NPC	NPC
Acyclic spanning	Poly	Poly	NPC	NPC	Poly	NPC	NPC	NPC
Out-Branching	Poly	NPC	NPC	Poly	Poly	Poly	NPC	NPC
In-branching	NPC	Poly	NPC	Poly	Poly	Poly	NPC	NPC

	$\delta^- \geq k$	$\delta^+ \geq k$	Cycle Factor	$\leq k$ arcs	$\geq k$ arcs	Balanced	Eulerian	Cycle
$\delta^- \geq k$	Poly	Poly	Poly	Poly	Poly	Poly	NPC	Poly
$\delta^+ \geq k$	×	Poly	Poly	Poly	Poly	Poly	NPC	Poly
Cycle Factor	×	×	Poly	Poly	Poly	Poly	NPC	NPC
$\leq k$ arcs	×	×	×	Poly	Poly	Poly	NPC	NPC
$\geq k$ arcs	×	×	×	×	Poly	Poly	NPC	Poly
Balanced	×	×	×	×	×	Poly	Poly	Poly
Eulerian	×	×	×	×	×	×	NPC	NPC
Cycle	×	×	×	×	×	×	×	NPC

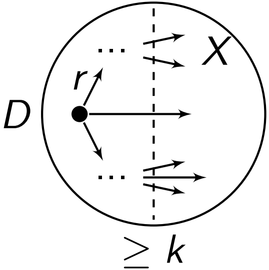
Classification of arc-partitioning problems for digraphs

Some known results

- **(connected, connected)** : Polynomial (Tutte-Nash-Williams' theorem, 1961).
  $G = (V, E)$ has t edge-disjoint spanning trees iff for every partition $V_1, \dots, V_\ell$ of V, there are at least $t(\ell - 1)$ crossing edges.
  One can compute them in polynomial time (Kaiser's algorithmic proof, 2012).



- **(connected, connected)** : Polynomial (Tutte-Nash-Williams' theorem, 1961).
- **(having an out-branching, having an out-branching)** : Polynomial (Edmonds' branching theorem, 1973).
  $D = (V, A)$ has k arc-disjoint out-branchings rooted in r if and only if, $\forall X \subseteq V \setminus \{r\}$, there are k arcs from $V \setminus X$ to X.



- **(connected, connected)** : Polynomial (Tutte-Nash-Williams' theorem, 1961).
- **(out-branching, out-branching)** : Polynomial (Edmonds' branching theorem, 1973).
- **(out-branching, in-branching)** : NP-complete (Thomassen, 1989).

Conjecture (Thomassen)

There is $k \in \mathbb{N}$ such that every k-arc-strong digraph has an (out-branching, in-branching)-arc-partition.

- *solved for digraphs with a universal vertex (Bang-Jensen, Huang, 1995),*
- *solved for digraphs with independence number at most 2 (Bang-Jensen, Bessy, Havet, Yeo, 2020)*

- **(connected, connected)** : Polynomial (Tutte-Nash-Williams' theorem, 1961).
- **(out-branching, out-branching)** : Polynomial (Edmonds' branching theorem, 1973).
- **(out-branching, in-branching)** : NP-complete (Thomassen, 1989).
- **(strongly connected, strongly connected)** : NP-complete (Bang-Jensen, Yeo, 2004).

Conjecture (Bang-Jensen, Yeo)

There is $k \in \mathbb{N}$ such that every k-arc-strong digraph has an (strongly connected, strongly connected)-arc-partition.

solved for locally semi-complete digraphs (Bang-Jensen, Huang, 2012)

- **(connected, connected)** : Polynomial (Tutte-Nash-Williams' theorem, 1961).
- **(out-branching, out-branching)** : Polynomial (Edmonds' branching theorem, 1973).
- **(out-branching, in-branching)** : NP-complete (Thomassen, 1989).
- **(strongly connected, strongly connected)** : NP-complete (Bang-Jensen, Yeo, 2004).
- **(out-branching, connected)** : NP-complete (Bang-Jensen, Yeo, 2012).
- **(strongly connected, connected)** : NP-complete (Bang-Jensen, Yeo, 2012).

An overview on arc-partitioning problems

- **Trivial problems :** The (P_1, P_2)-arc-partitioning problem is trivially polynomial when :
  - P_1 holds for the **arcless digraph**,
    bipartite, acyclic, $\leq k$ arcs, balanced
  - P_2 is **upward closed**,
    connected, strongly connected, having an out(in)-branching, $\delta^+ \geq k, \delta^- \geq k$, $\geq k$ arcs

  A digraph D has such a partition if and only if D has property P_2. If this is the case then $(\emptyset, A(D))$ is a partition.

- **Trivial problems :** polynomial, 28 problems.
- **($\geq k$ arcs, P_2)** : it can be solved in polynomial time when computing the **minimum size** of a subgraph of D having property P_2 can be solved in **polynomial time**.
  $\geq k$ arcs, $\delta^+ \geq k$, $\delta^- \geq k$, cycle, connected, having an out(in)-branching, acyclic spanning, cycle factor.

- **Trivial problems :** polynomial, 28 problems.
- **($\geq k$ arcs, P_2) :** polynomial, 9 problems.
- **Equivalent of being hamiltonian in 2-regular digraphs :**
  - Since the hamiltonian cycle problem is known to be NP-complete on 2-regular digraphs (Bang-Jensen, Gutin, 2009), one can easily show that 16 arc-partitioning problems are NP-complete.
  - For example, a 2-regular digraph has a hamiltonian cycle if and only if it has a (connected, cycle factor)-arc partition.

- **Trivial problems :** polynomial, 28 problems.
- **($\geq k$ arcs, P_2) :** polynomial, 9 problems.
- **Equivalent of being hamiltonian in 2-regular digraphs :** NP-complete, 16 problems.
- **Equivalent of having two arc-disjoint hamiltonian cycles in 2-regular digraphs :**
    - Since deciding if a 2-regular digraph has two arc-disjoint hamiltonian cycles is known to be NP-complete (Bang-Jensen & Yeo, 2012), one can easily show that 12 arc-partitioning problems are NP-complete.
    - For example, a 2-regular digraph has two arc-disjoint hamiltonian cycles if and only if it has a (eulerian, connected)-arc-partition.

- **Trivial problems :** polynomial, 28 problems.
- **($\geq k$ arcs, P_2) :** polynomial, 9 problems.
- **Equivalent of being hamiltonian in 2-regular digraphs :** NP-complete, 16 problems.
- **Equivalent of having two arc-disjoint hamiltonian cycles in 2-regular digraphs :** NP-complete, 12 problems.
- **Already known problems :** 13 problems.

A polynomial-time solvable arc-partitioning problem

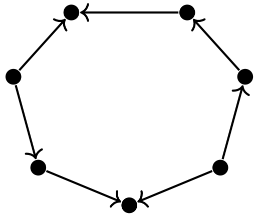
Theorem

a connected digraph D has an (acyclic spanning, acyclic spanning)-arc-partition iff $\delta(D) \geq 2$ and D is not the orientation of an odd cycle.

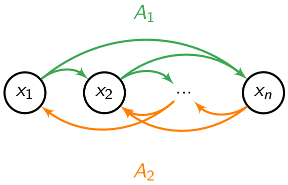
Le D be a connected digraph, then :

- if $\delta(D) < 2$ or if D is the orientation of an odd cycle, clearly D does not have such a partition,
- if D is the orientation of an even cycle, clearly D has such a partition.

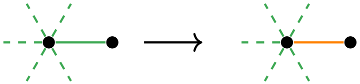
We assume that $\delta(D) \geq 2$ and D **is not the orientation of a cycle**.



- First, note that D has an **(acyclic,acyclic)**-arc-partition.



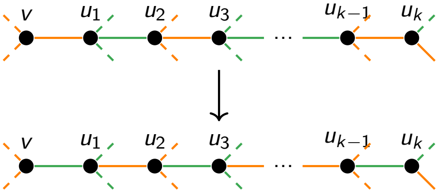
- Since $\delta(D) \geq 2$, it is easy to see that D has an **(acyclic, acyclic spanning)**-arc-partition.



- Let (A_1, A_2) be such a partition which minimize the number of vertices not covered by A_1, and assume there is a vertex v not covered by A_1.

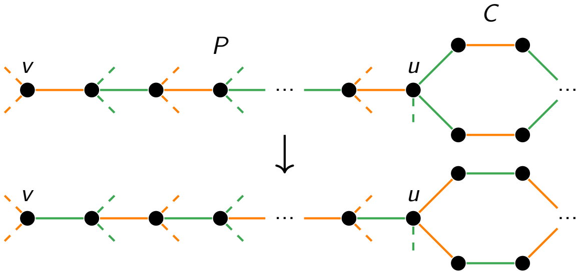
Forgetting the orientation in D,

1. each path from v must be alternating between A_1 and A_2,



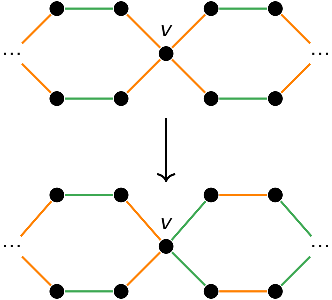
Forgetting the orientation in D,

1. each path from v must be alternating between A_1 and A_2,
2. the vertex v belongs to every cycle in D,



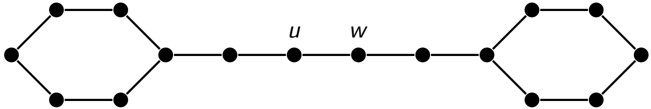
Forgetting the orientation in D,

1. each path from v must be alternating between A_1 and A_2,
2. the vertex v belongs to every cycle in D,
3. there are not two edge-disjoint cycles in D,



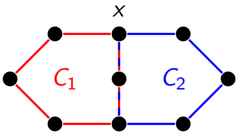
Forgetting the orientation in D,

1. each path from v must be alternating between A_1 and A_2,
2. the vertex v belongs to every cycle in D,
3. there are not two edge-disjoint cycles in D,
4. there are two different cycles in D,



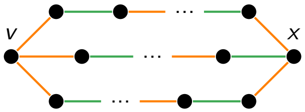
Forgetting the orientation in D,

1. each path from v must be alternating between A_1 and A_2,
2. the vertex v belongs to every cycle in D,
3. there are not two edge-disjoint cycles in D,
4. there are two different cycles in D,
5. there is a vertex x, different from v, which has degree at least 3,



Forgetting the orientation in D,

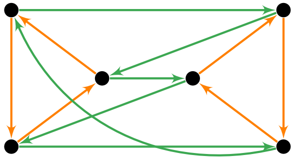
1. each path from v must be alternating between A_1 and A_2,
2. the vertex v belongs to every cycle in D,
3. there are not two edge-disjoint cycles in D,
4. there are two different cycles in D,
5. there is a vertex x, different from v, which has degree at least 3,
6. considering three maximal path from x, one can find three vertex-disjoint path from x to v,



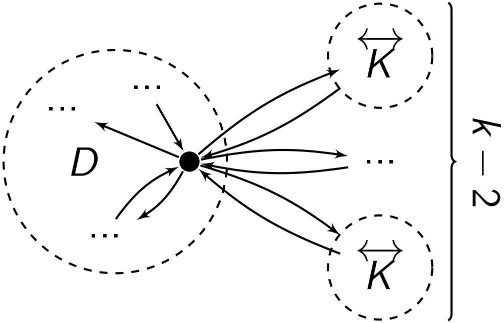
This is a contradiction because of rule 1. This shows that A_1 must cover every vertex, and (A_1, A_2) is an (acyclic spanning, acyclic spanning)-arc-partition of D.

The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem

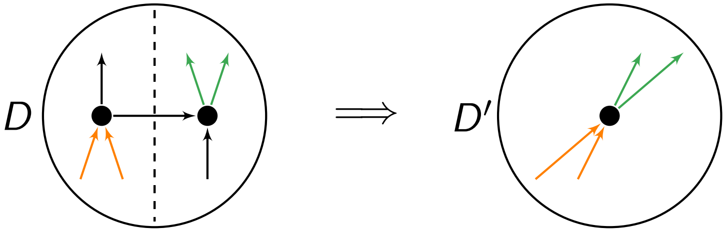
- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-regular digraphs, because it is exactly the hamiltonian cycle problem.



- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-regular digraphs, because it is exactly the hamiltonian cycle problem.
- For every natural number $k \geq 2$, it is NP-complete to decide whether a digraph of minimum out and in-degree at least k has a (strongly connected, $\delta^+ \geq 1$)-arc-partition.



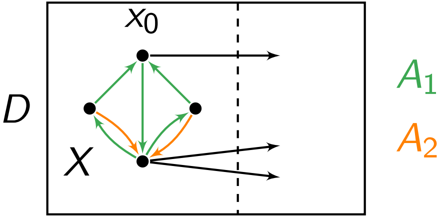
- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-regular digraphs, because it is exactly the hamiltonian cycle problem.
- For every natural number $k \geq 2$, it is NP-complete to decide whether a digraph of minimum out and in-degree at least k has a (strongly connected, $\delta^+ \geq 1$)-arc-partition.
- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-arc-strong 2-regular digraphs, because the hamiltonian cycle problem is NP-complete on this class of graphs :



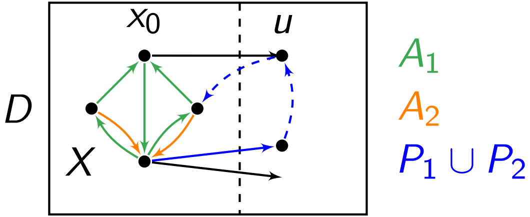
- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-regular digraphs, because it is exactly the hamiltonian cycle problem.
- For every natural number $k \geq 2$, it is NP-complete to decide whether a digraph of minimum out and in-degree at least k has a (strongly connected, $\delta^+ \geq 1$)-arc-partition.
- The (strongly connected, $\delta^+ \geq 1$)-arc-partitioning problem is NP-complete on 2-arc-strong 2-regular digraphs.
- Every 2-arc-strong digraph with minimum out-degree at least 4 has a (strongly connected, $\delta^+ \geq 1$)-arc-partition.

Let $D = (V, A)$ be a **2-arc-strong** digraph with minimum **out-degree at least 4**.
Let $X \subseteq V$ and (A_1, A_2) be a partition of $A(D[X])$.
(X, A_1, A_2) is **good** iff $\exists x_0 \in X$ such that :

- $D_1 = (X, A_1)$ is **strongly connected**,
- $\forall x \in X, x \neq x_0$, either $d^+_{A_2}(x) \geq 1$ or $|N(x) \setminus X| \geq 2$,
- $d^+_{A_2}(x_0) \geq 1$ or $|N(x_0) \setminus X| \geq 1$.

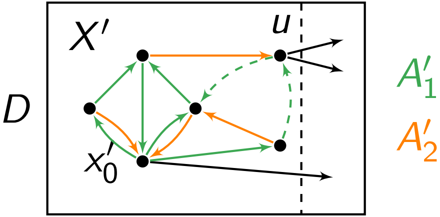


D always has such a good tuple, let (X, A_1, A_2) be such a tuple which **maximize the size of** X, and assume that $|X| < |V|$.
Let u be an out-neighbour of x_0 in X, P_1 a path from X to u in $D - \{x_0u\}$, and P_2 a path from u to X.



One can get a better tuple (X', A'_1, A'_2) where :

- $X' = X \cup V(P_1) \cup V(P_2)$
- $A'_1 = X \cup A(P_1) \cup A(P_2)$
- $A'_2 = A(D[X']) \setminus A'_1$



Then we know that $X = V$ and (A_1, A_2) is a **(strong, $\delta^+ \geq 1$)**-arc-partition of D.

	In general	2-arc-strong	3-arc-strong
$\delta^+ \geq 2$	NP-c	NP-c	$\times$
$\delta^+ \geq 3$	NP-c	?	?
$\delta^+ \geq 4$	NP-c	Always true	Always true

Problem

Does every 2-arc-strong digraph with minimum out-degree at least 3 have a (strongly connected, $\delta^+ \geq 1$)-arc-partition ?

Open problems

Theorem

Every 2-arc-strong outerplanar multi-digraph have a (strong,strong)-arc-partition.

Problem

Does every 3-arc-strong planar digraph have a (out-branching,in-branching)-arc-partition ? a (strong,strong)-arc-partition ?

We know that every 2-arc-strong digraph with a universal vertex have an (out-branching, in-branching)-arc-partition.

Problem

Does every 3-arc-strong digraph with a universal vertex have a (strong,strong)-arc-partition ?

Thanks for your attention.