REPORT ON THE THESIS OF LUCAS PICASARRI-ARRIETA

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The thesis is devoted to the study of digraph coloring, where one seeks to partition the vertex set of a directed graph (abbreviated *digraph* in the remainder), in such a way that no set in the partition contains a directed cycle. The *dichromatic number* of a digraph is the minimum number of sets in such a partition. Taking any graph G, it is easy to see that the chromatic number of G is equal to the dichromatic number of the digraph obtained from G by replacing every edge by a digon (i.e., two edges in both directions). Therefore, results on the dichromatic number of digraphs generalize results on the chromatic number of graphs and it is natural to seek directed extension of classical results on graph coloring.

The contributions of the candidate are divided in four main parts. The first part is devoted to the study of the coloring properties of digraphs whose underlying graph G is *chordal* (i.e., does not contain an induced cycle of length at least 4). The second part is devoted to extensions of Brooks' theorem (a classical result relating the chromatic number of a graph and its maximum degree). The third part is devoted to the average degree of *critical* digraphs (digraphs whose dichromatic number drops by one if we remove any edge). The final part studies the complexity of reconfiguring digraph colorings, and the maximum length of a reconfiguration sequence.

Chordal graphs are an important subclass of *perfect* graphs, graphs whose chromatic and clique numbers coincide (and the same holds for all induced subgraphs). The chromatic number of chordal graphs is well understood and thus it makes sense to study digraphs whose underlying graphs are chordal. If all edges of a chordal graph G are replaced by digons, then the dichromatic number is equal to $\omega(G)$, the clique number of G. In joint work with Frédéric Havet and Stéphane Bessy, the candidate studies the dichromatic number when only a subset S of edges is replaced by digons (the remaining edges being replaced by single arcs). If the subgraph induced by the edges of S is sparse or does not contain a 4-cycle, then the dicromatic number is close to $(1/2 + o(1))\omega(G)$. The connection between treewidth and chordal graphs directly implies bounds on the dichromatic number of digraphs whose underlying graph has bounded treewidth. Constructions are then given to prove that the bounds are close to best possible.

A classical theorem of Brooks states that every connected graph of maximum degree Δ and chromatic number $\Delta + 1$ is either an odd cycle or a complete graph on $\Delta + 1$ vertices. This theorem has been generalized to directed graphs earlier. In joint work with Daniel Gonçalves and Amadeus Reinald, the candidate proves an even more general version, in a setting where each color class is strictly f-degenerate for some function f taking arbitrary values on the vertex set of the digraph (and where the functions f can be different for the different color classes). The proof is algorithmic, and yields a linear time algorithm producing either the desired coloring, or an obstruction to the existence of such a coloring. The proof uses a nice variant of ear-decompositions involving stars and paths, instead of just paths. The chapter on Brook's theorem is concluded with several results relating the dichromatic number of a digraph G and $\Delta_{\min}(G)$, which is the maximum for all vertices v of G, of the minimum between the in-degree and out-degree of v. In particular, local list-coloring versions are given where each vertex v has a list of colors of size at least the minimum of the in-degree and the out-degree of v, and every vertex mush be colored with a color from its list. As corollaries, every digraph G with no digon has dichromatic number at most $\max(2, \Delta_{\min}(G))$, and digraphs G without the complete bidirected graph on $\lceil \frac{1}{2}(\Delta_{\min}(G)+1)\rceil$ vertices have dichromatic number at most $\Delta_{\min}(G)$. These results are complemented by a nice NP-hardness result showing that when G only excludes the complete bidirected graph on $\lceil \frac{1}{2}(\Delta_{\min}(G)+1)\rceil + 1$ vertices, no such finite characterization of tight examples is expected.

A digraph is *k-dicritical* if its dichromatic number is k but every proper subgraph has dichromatic number at most $k-1$. Any digraph of dichromatic number k has a k-dicritical subdigraph, and thus understanding the structure and density of k-dicritical digraphs can lead to interesting results in digraph coloring. In joint work with Frédéric Havet, Clément Rambaud and Michael Stiebitz, the candidate proves bounds on the minimum number of arcs in a k-critical digraph G when k is close ot the number of vertices of G. It is also proved that if G has no digon and is 4-dicritical, then it contains at least $(10/3+1/51)|V(G)|-1$ arcs. The proof is highly non-trvial. It uses the potential method of Kostochka and Yancey, together with some discharging arguments (along the lines of a proof of Postle in the undirected setting). In joint work with Frédéric Havet and Florian Hörsch, the candidate proves upper bounds on the size of 3-dicritical digraphs whose underlying graphs are complete (so there are only finitely many such graphs). The proof of finiteness is a short Ramsey-theoretic argument while the precise description of all such graphs is computer-assisted. Bounds on the number of arcs (as a function of the number of vertices) in 3-dicritical general digraphs are then given. In joint work with Clément Rambaud, the candidate proves that for any $k \geqslant 3$ there are infinitely many k-dicritical digraphs without directed paths of length $3k$. On the other hand, it is proved that for any k and ℓ, there are only finitely many k-dicritical digraphs whose underlying graph does not contain a cycle of length at least ℓ. Other results include the existence of subdivisions of some trees or other specific graphs if some condition on the dichromatic number, the minimum out-degree, or the digirth is given.

The final part considers the reconfiguration problem for digraph colorings. Given two colorings of a digraph G, is it possible to go from one to the other in a sequence of colorings of G, where we only recolor one vertex at a time? If so, how many recolorings are needed? Is it possible to go from one coloring to the other for any pair of colorings? Again, how many recolorings are needed? These are the natural directed analogues of the classical undirected recoloring questions. In joint work with Nicolas Bousquet, Frédéric Havet, Nicolas Nisse, Amadeus Reinald and Ignasi Sau, the candidate explores these questions, and in particular directed versions of a conjecture of Cereceda about recoloring graphs of bounded degeneracy. Several partial results on this conjecture are extended to the directed setting (identifying a natural notion of degeneracy in the directed setting). Moreover, several PSPACE-hardness results are proved for the problem where the two colorings are given as part of the input, and results are obtained on the density of graphs and digraphs with no recoloring sequence.

The candidate concludes the thesis by mentioning future research directions and partial results on the dichromatic number of graph products.

The author has obtained significant results on many different problems and the thesis contains a lot of interesting material on digraph coloring. In addition to the large number of very nice ideas, the thesis shows that the candidate clearly masters a number of tools from extremal and structural graph theory (discharging and the potential technique), algorithmic graph theory (with a non-trivial linear time algorithm requiring some care on the date structures used), complexity theory (with NP-hardness and PSPACE-hardness reductions) and does not hesitate to write programs to produce computer-assisted proofs.

I have found the manuscript very well written and illustrated, and reading it was a real pleasure.

I am therefore in favor of allowing Lucas Picasarri-Arrieta to defend his thesis.

May 27, 2024. Louis Esperet

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