Colouring Digraphs with Bounded Maximum Degree

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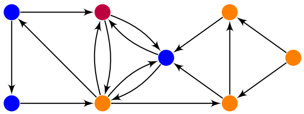
Digraph dicolouring

- ***k*-dicolouring** of D: partition of $V(D)$ into k acyclic subdigraphs (*i.e.* no monochromatic directed cycle).



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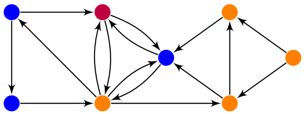

$$\vec{\chi}(D) = 3$$

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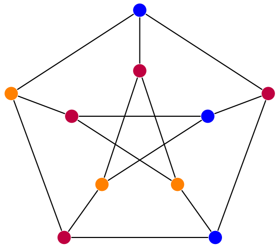

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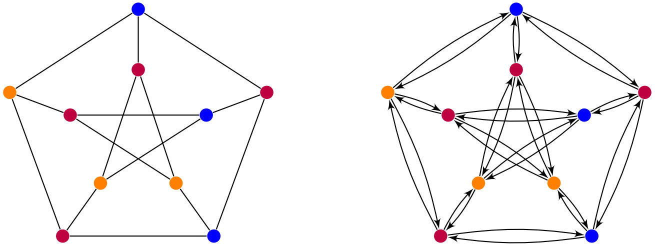
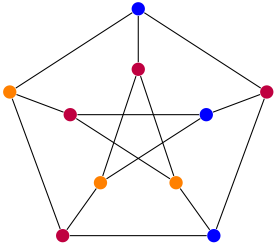


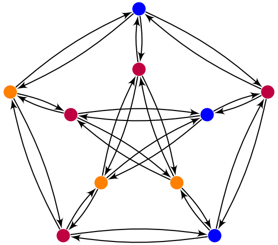
$$\chi(G) = \vec{\chi}(\overleftrightarrow{G})$$

From graphs to digraphs: main questions

Given any result on graph colouring, two questions arise:

- **Question 1:** Does it generalize to all digraphs?
- **Question 2:** If it does, can we strengthen it on oriented graphs?





Maximum Degrees of a digraph

Max-max-degree: $\Delta_{\max}(D) = \max\left\{\max(d^-(v), d^+(v)) \mid v \in V(D)\right\}.$

Max-min-degree: $\Delta_{\min}(D) = \max\left\{\min(d^-(v), d^+(v)) \mid v \in V(D)\right\}.$

Max-geometric-degree: $\tilde{\Delta}(D) = \max\left\{\sqrt{d^-(v) \cdot d^+(v)} \mid v \in V(D)\right\}.$

$\Delta(G) = \Delta_{\min}\left(\overleftrightarrow{G}\right) = \tilde{\Delta}\left(\overleftrightarrow{G}\right) = \Delta_{\max}\left(\overleftrightarrow{G}\right)$

First easy bound:

$$\vec{\chi}(D) \leq \Delta_{\min}(D) + 1 \leq \tilde{\Delta}(D) + 1 \leq \Delta_{\max}(D) + 1$$

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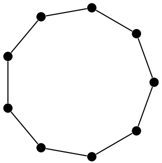
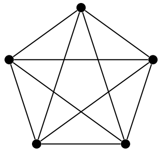
Theorem (Brooks 1941)

Let G be a connected graph, then $\chi(G) \leq \Delta(G)$ unless G is an **odd cycle** *or a* **complete graph**.



$\Delta = 2$, $\chi = 3$



$\Delta = n - 1$, $\chi = n$

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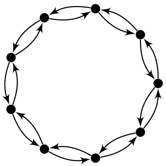
Theorem (Mohar 2010)

Let D be a connected digraph, then $\vec{\chi}(D) \leq \Delta_{\max}(D)$ unless D is a **bidirected odd cycle**, *a* **bidirected complete graph**, *or a* **directed cycle**.



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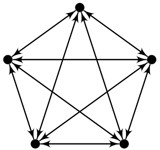

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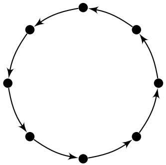

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Analogues for $\Delta_{\min}$

Theorem (Aboulker and Aubian 2022)

*Deciding $\vec{\chi}(D) \leq \Delta_{\min}(D)$ is an **NP-complete** problem.*

$\Longrightarrow$ **No easy characterisation** (unless P=NP) in general.

Theorem (P. 2023)

*Let $\vec{G}$ be an **oriented graph** with $\Delta_{\min}(\vec{G}) \geq 2$, then $\vec{\chi}(\vec{G}) \leq \Delta_{\min}(\vec{G})$.*

Corollary

*Let $\vec{G}$ be an **orientation** of G with $\Delta(G) \leq 5$, then $\vec{\chi}(\vec{G}) \leq 2$.*

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Asymptotic bounds

Conjecture (Erdős and Neumann-Lara 1979)

Let $\vec{G}$ be an oriented graph, then $\vec{\chi}(\vec{G}) = O\left(\frac{\Delta_{\max}(\vec{G})}{\log \Delta_{\max}(\vec{G})}\right)$.

Remark: Best possible for **random tournaments**.

Theorem (Harutyunyan and Mohar 2011)

Every oriented graph $\vec{G}$ with $\tilde{\Delta}(\vec{G})$ **large enough** *satisfies* $\vec{\chi}(\vec{G}) \leq (1 - e^{-13})\tilde{\Delta}(\vec{G})$.

- $\vec{\chi}(\vec{G}) \leq 0.816\tilde{\Delta}(\vec{G}) + O(1)$. [Golowich 2016]
- $\vec{\chi}(\vec{G}) \leq 0.707\tilde{\Delta}(\vec{G}) + O(1)$. (consequence of) [P. 2023]

Question: What about digraphs with bounded **biclique number**?

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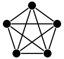
ω = size of a largest clique 

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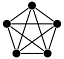

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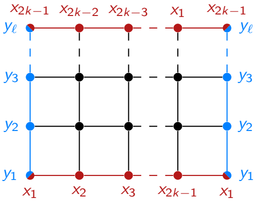

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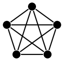
ω = size of a largest clique 

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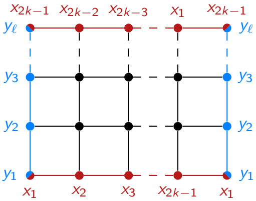

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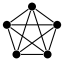
ω = size of a largest clique 

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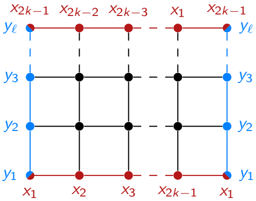
- $\varepsilon = 0.038$ [Bonamy et al. 2015]
- $\varepsilon = 0.077$ [Delcourt and Postle 2017]
- $\varepsilon = 0.119$ [Hurley et al. 2022]



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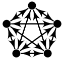
An analogue of Reed's conjecture for digraphs

$\overleftrightarrow{\omega}$ = size of a largest biclique



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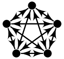

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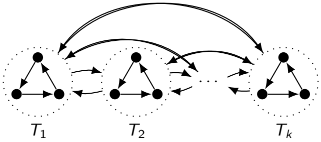

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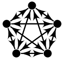

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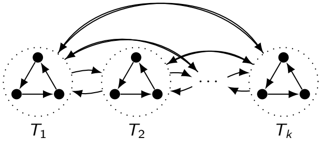

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Sketch of the proof – Lemma 1/3 – Exclude large bicliques

Lemma

Let D be a digraph with $\overleftrightarrow{\omega}(D) > \frac{2}{3}(\Delta_{\max}(D) + 1)$. Then D has an **acyclic set** *of vertices I such that $\overleftrightarrow{\omega}(D - I) = \overleftrightarrow{\omega}(D) - 1$.*

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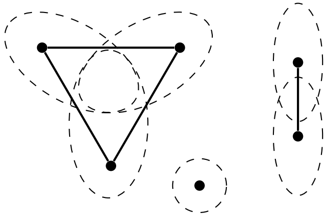

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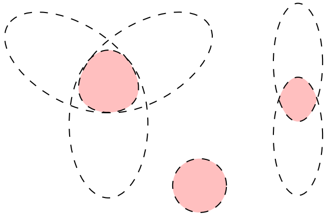

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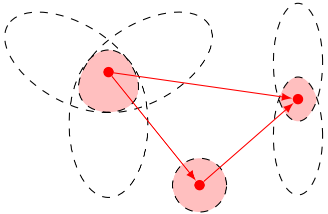

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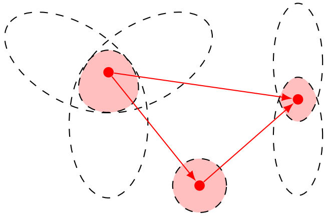

Corollary

If D is a minimal counter example to $\vec{\chi}(D) \leq \left\lceil (1-\varepsilon)(\tilde{\Delta}(D) + 1) + \varepsilon\overleftrightarrow{\omega}(D) \right\rceil$, then $\overleftrightarrow{\omega}(D) \leq \frac{2}{3}(\Delta_{\max}(D) + 1)$.

Sketch of the proof – Lemma 2/3 – Exclude dense out-neighbourhoods

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If D is a minimal counterexample to $\vec{\chi}(D) \leq \left\lceil (1-\varepsilon)(\tilde{\Delta}(D)+1) + \varepsilon \overleftrightarrow{\omega}(D) \right\rceil$, for every $v \in V(D)$, $|A(D[N^+(v)])| \leq (1-a)\Delta_{\max}^2(D)$.

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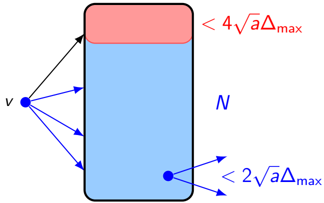

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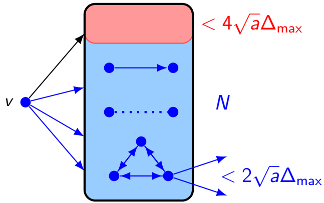

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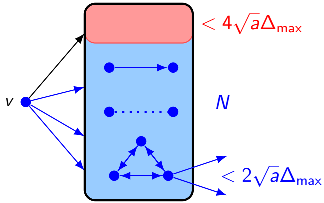
3. Use this matching to extend a dicolouring of $D - N$ to D, and contradict the minimality of D.



Sketch of the proof – Lemma 3/3 – Coloring sparse digraphs

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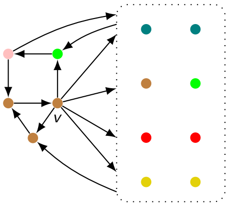

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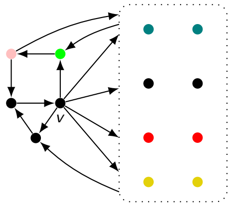

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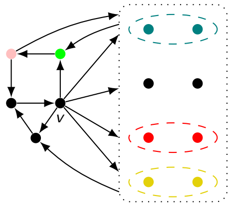

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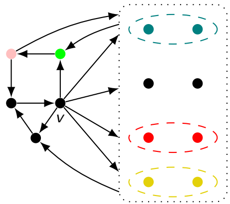

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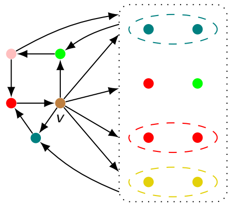
5. Greedily colour the uncoloured vertices.



Further Research

- **Problem:** Prove the existence of $\varepsilon > 0$ such that every oriented graph $\vec{G}$ satisfies $\vec{\chi}(\vec{G}) \leq \lceil (1-\varepsilon)\Delta^+(\vec{G}) \rceil$.
- **Problem:** Find the values of $k \in [1, \Delta + 1]$ for which k-$\textsc{Dicolourability}$ is solvable in polynomial time on digraphs D with $\Delta_{\max}(D) = \Delta$.

  Undirected case (Molloy and Reed 2014): $k \in [\Delta - \sqrt{\Delta}, \Delta + 1]$.
- **Conjecture (Neumann-Lara 1982):** Every oriented planar graph is 2-colourable.

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