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Report on the Thèse de Doctorat of Lucas Picasarri-Arrieta

This is an excellent thesis on the theory of directed graphs.

Graph colouring is an extensively studied field at the heart of combinatorics, with a huge literature and a deep theoretical picture. The study of colourings of *directed graphs* is by contrast far less developed. There is a well-known notion of colouring for digraphs, namely the *dichromatic number*, which was introduced by Erdős and Neumann-Lara in the 1970s. This has been studied from various perspectives, but over the last fifteen years researchers in the field have started to build a more systematic theory of digraph colouring that parallels the theory of graph colouring (for example, Ararat Harutyunyan and more recently Guillaume Aubian both worked in this direction in their PhD theses, among many other excellent mathematicians with interests in this area).

The thesis of Lucas Picasarri-Arrieta makes very significant contributions to this broad project. The thesis develops the theory in several interesting directions – I will mention two of these.

One of the cornerstones of graph colouring is the theorem of Brooks, which says that (with two specified exceptions) the chromatic number of a graph is at most its maximal degree. There is an attractive analogue of Brooks theorem for directed graphs, proved in 2011 by Harutyunyan and Mohar. However, more recently it has been noticed that the digraph structure allows for more sophisticated versions of the theorem. Picasarri-Arrieta proves just such a theorem, using clever structural tools he has proved a theorem that is both stronger than previous results and, as a bonus, comes with an efficient algorithm to find the colouring. A further (quite surprising) strengthening is presented for oriented graphs. This is a nice piece of work.

Another strong piece of work concerns subdigraphs in dicritical graphs with large order or directed girth. A digraph is *dicritical* if deleting any edge or vertex gives a subdigraph with smaller dichromatic number. A breakthrough result of Aboulker, Cohen, Havet, Lochet, Moura and Thomassé from 2019 shows that every digraph with suitably large dichromatic number contains a subdivision of any fixed digraph D. It is natural here to focus on the dicritical digraphs, and ask whether for large digraphs the conditions of Aboulker et are sufficient to guarantee long subdivisions of F. Picasarri-Arrieta shows that this is not the case: he shows that for every k>2 there are infinitely many k-dicritical digraphs with no directed path of length 3k+1. This is a sharp contrast to the case of graph colouring, where Kelly and Kelly showed in the 1950s that no analogous result holds for graphs.

The thesis contains a substantial body of other interesting results, including work on the dichromatic number of orientations of chordal graphs, the density of diacritical graphs, and a substantial and interesting chapter on recolouring digraphs.

Overall, this is a very strong thesis building a substantial body of theory. The thesis is well-written throughout and is also thoughtful, with interesting suggestions on the next steps that the field should take.

The thesis is well above the standard expected of a doctoral thesis, and I have no hesitation at all in giving a very positive recommendation.

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