Blow-ups and extensions of trees in tournaments

P. Aboulker, F. Havet, W. Lochet, R. Lopes, L. Picasarri-Arrieta, and C. Rambaud

Generalities on Ramsey Theory

Ramsey Number $R(s,t)$: min. integer n such that all (blue/red)-edge-colourings of K_n contains K_s **in red** or K_t **in blue**.

$R(3,3) = 6$

Ramsey (1930) : $R(s,t)$ exists for all $s,t \in \mathbb{N}$.

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Proof that $R(s,t) \le 2^{s+t}$ by induction on $s+t$:

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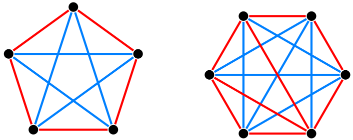

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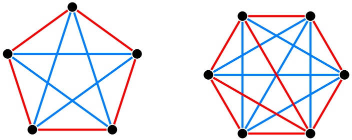

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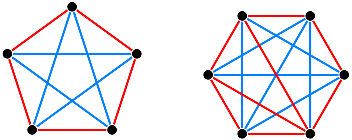

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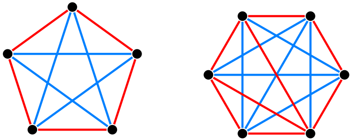

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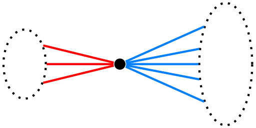

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Diagonal Ramsey Number $R(s,s)$

Erdős (1947) : $R(s,s) \geq 2^{s/2}$ for every $s \in \mathbb{N}$.

Proof via the probabilistic method: $\Pr(\text{Red } K_s \text{ or Blue } K_s) \leq 2 \cdot \binom{n}{s} \cdot \left(\frac{1}{2}\right)^{\binom{s}{2}} < 1.$ $\square$

We thus have $(\sqrt{2})^s \leq R(s,s) \leq 4^{s - c\log s}$.

Campos et al. (2023) : $R(s,s) \leq (4-\varepsilon)^s$ for some $\varepsilon > 0$.

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Tournament: orientation of a complete graph.

For an **acyclic** digraph D, $\vec{R}(D)$ **:** min. integer n such that all tournaments of order n contain H.

Example: $\vec{R}\Big($ $\Big) = 4.$

Transitive tournament on n vertices TT_n:

Theorem (Erdős and Moser, 1963)

For every $k \in \mathbb{N}$, $\sqrt{2}^{n} \leq \vec{R}(TT_{n+1}) \leq 2^{n}$.

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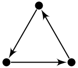

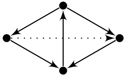

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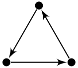

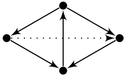

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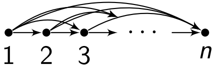

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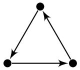

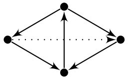

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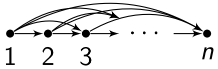

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A class $\mathcal{C}$ is **linearly unavoidable** if, for some c, $\vec{R}(H) \leq c \cdot |V(H)|$ for every $H \in \mathcal{C}$.

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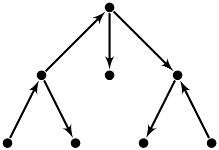
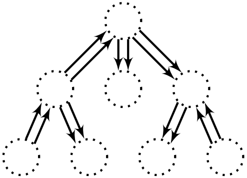
For every fixed $k \in \mathbb{N}$, **k-th blow-ups of oriented trees** *are* **linearly unavoidable**.





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Let D be a k-th blow-up of oriented tree, then $\vec{R}(D) \leq 2^{10+18k} k \cdot |V(D)|$.

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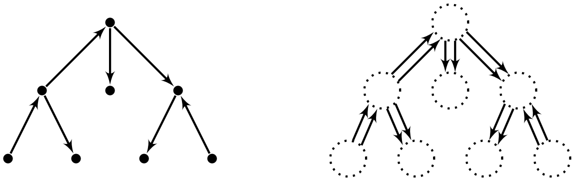

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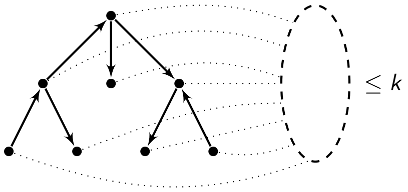

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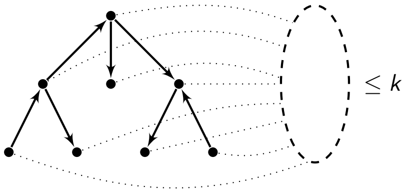

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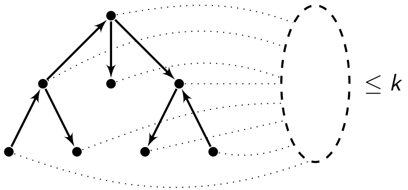

Theorem

Let D be a k-extension of an oriented tree, then $\vec{R}(D) \leq 3^{\binom{2k+2}{2}+1} \cdot |V(D)|$.

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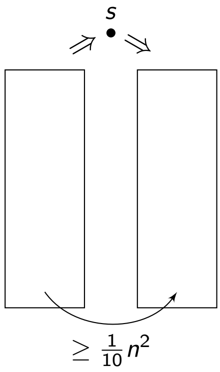

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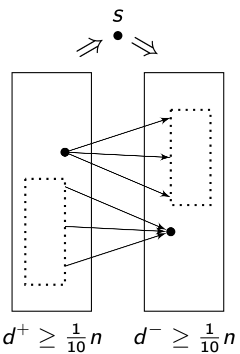

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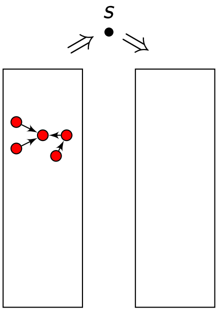

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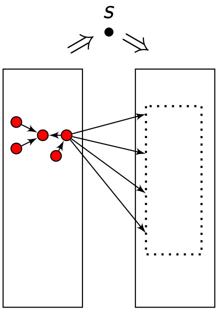

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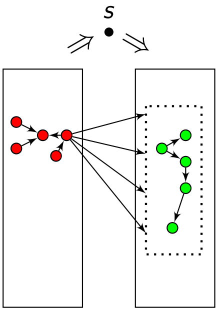

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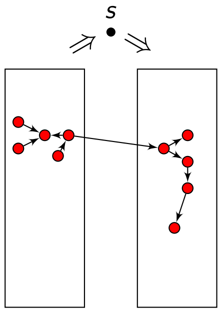

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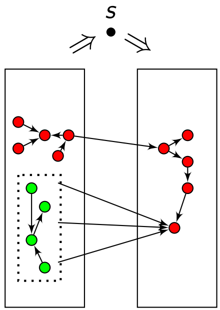

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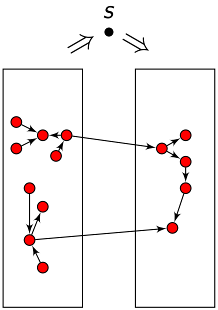

Further Research

Conjecture (Aboulker, Havet, Lochet, Lopes, P-A, Rambaud)

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$\hookrightarrow$ **Question:** Are k-extensions of k-th blow-ups of oriented trees linearly unavoidable?

Problem

For fixed k, is the class of acyclic orientations of $\{G \mid \text{treewidth}(G) \leq k\}$ linearly unavoidable?

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