Digraph redicolouring

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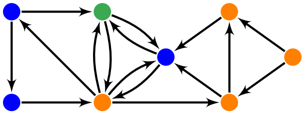
Digraph dicolouring

- **k-dicolouring** of D : partition of $V(D)$ in k parts inducing an acyclic subdigraph.



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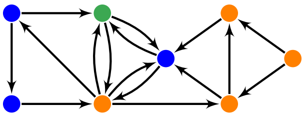

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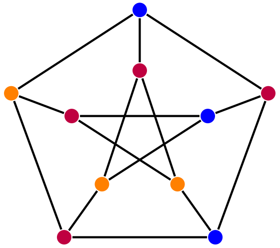

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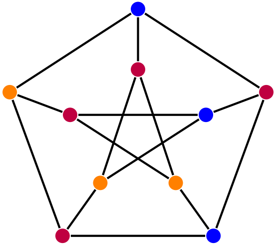

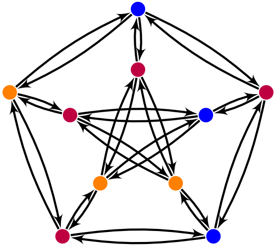

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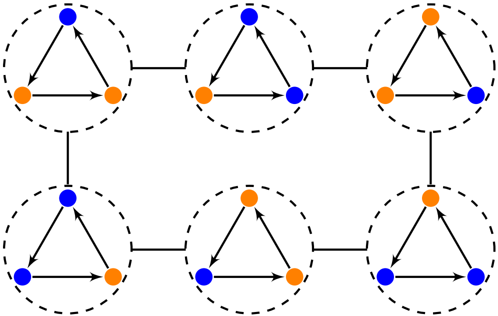
$\mathcal{D}_k(D)$: the ***k*-dicolouring graph** of D :

- $V(\mathcal{D}_k(D))$ are the k-dicolourings of D,
- $\gamma_i\gamma_j \in E(\mathcal{D}_k(D))$ if $\gamma_i = \gamma_j$ except on one vertex.



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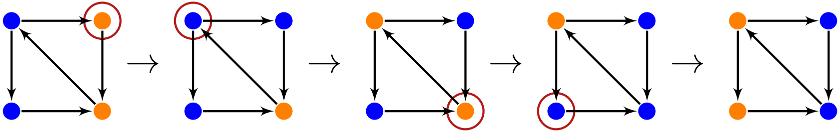
$\mathcal{C}_k(G)$: the **k-colouring graph** of G is similar.

- **recolouring sequence** : a path (or a walk) in $\mathcal{D}_k(D)$.



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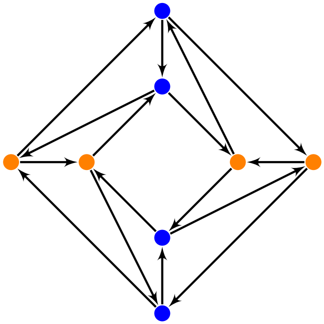
- D is ***k*-mixing** : $\mathcal{D}_k(D)$ is connected.



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$\longrightarrow$ **Is D k-mixing ?**

$\longrightarrow$ **Can we bound the diameter of $\mathcal{D}_k(D)$?**

Undirected graphs

Theorem (Bonsma et al. ; Dyer et al.)

If $k \geq \delta^*(G) + 2$, then G is k-mixing, and $diam(\mathcal{C}_k(G)) \leq 2^n - 1$.

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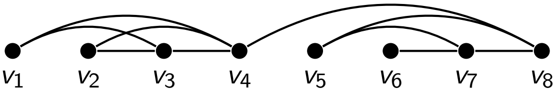

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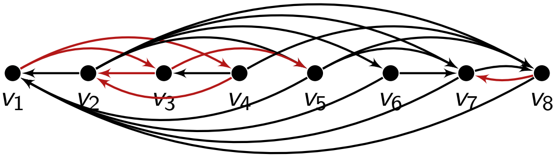

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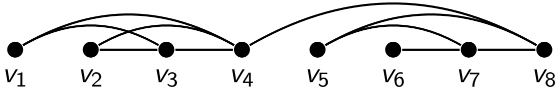

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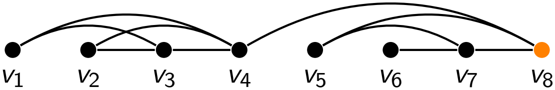

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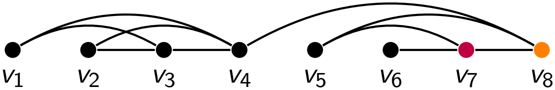

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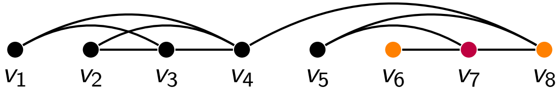

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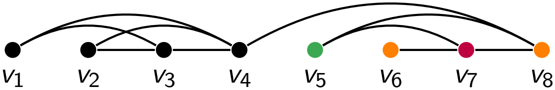

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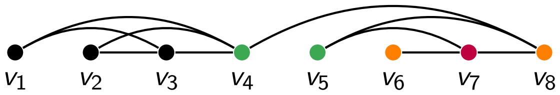

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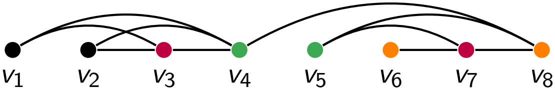

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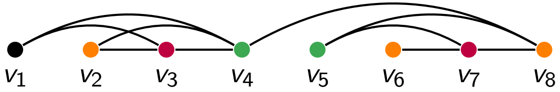

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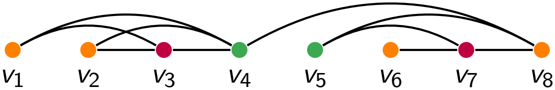

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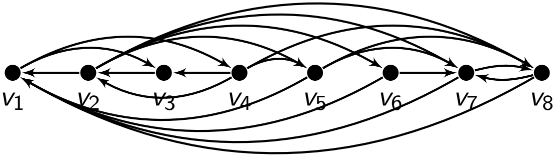

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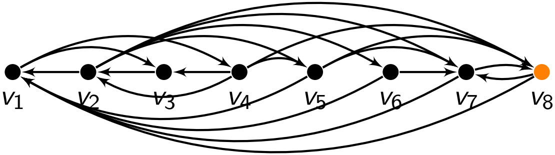

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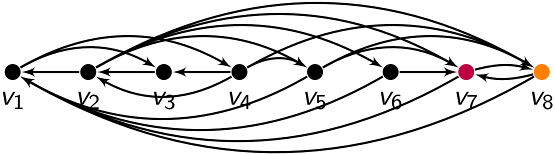

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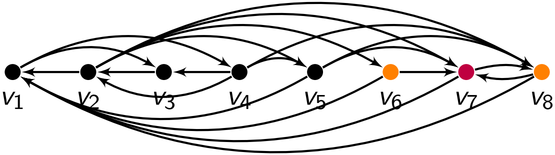

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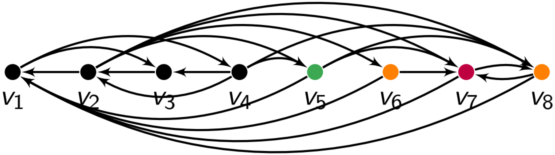

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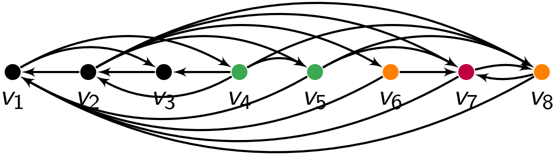

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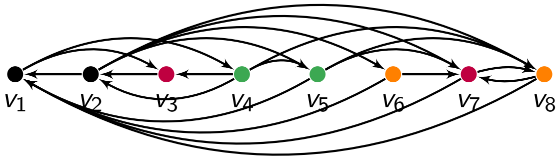

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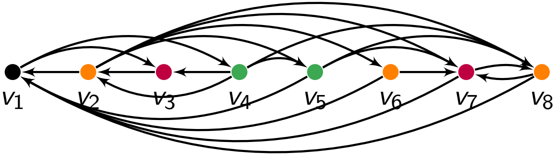

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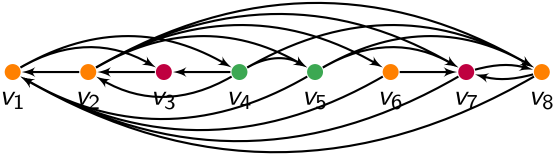

A generalization of a result from Bonsma, Cereceda, Dyer, Flaxman, Frieze and Vigoda.

Theorem (Bonsma et al. ; Dyer et al.)

If $k \geq \delta^*(G) + 2$, then G is k-mixing, and $diam(\mathcal{C}_k(G)) \leq 2^n - 1$.

This **generalizes** to the following :

Theorem

If $k \geq \delta^*_{\text{min}}(D) + 2$, then D is k-mixing, and $diam(\mathcal{D}_k(D)) \leq 2^n - 1$.

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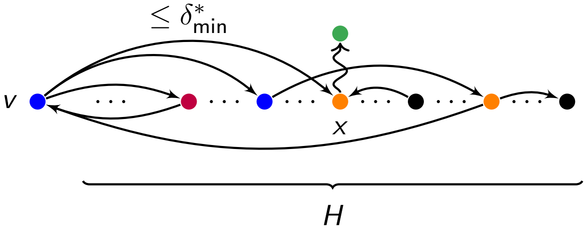

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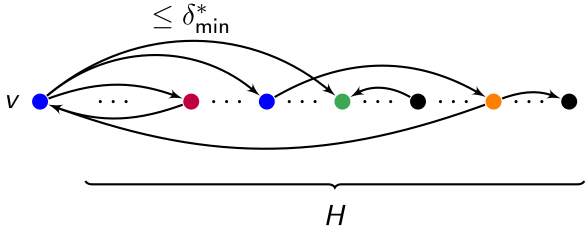

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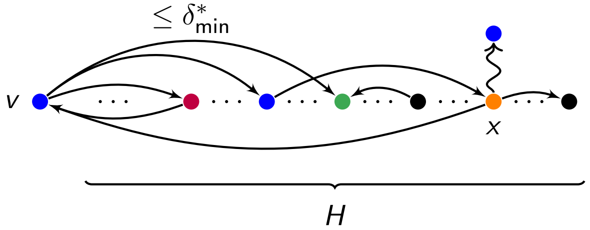

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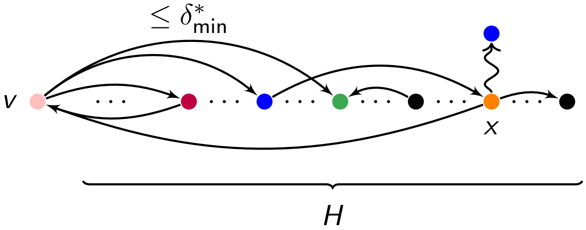

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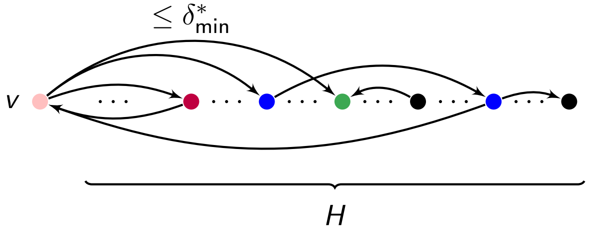

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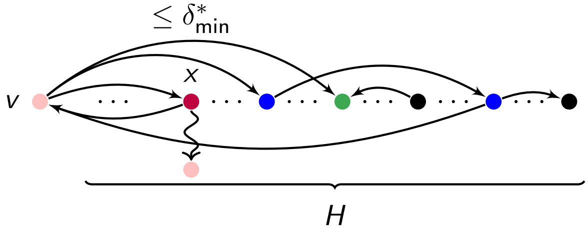
- $n = 1$: Trivial.
- $n \geq 2$: choose $v \in V$ s.t. $d^+(v) \leq k-2$ (or $d^-(v) \leq k-2$), $H = D - v$.

  By **induction** $\exists \alpha_{|H} \xrightarrow{2^{n-1}-1} \beta_{|H}$.

  When x **is recoloured** in H, either we can recolour it in D, or we can first recolour v and then recolour x :



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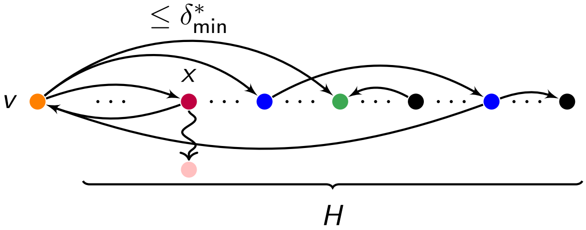

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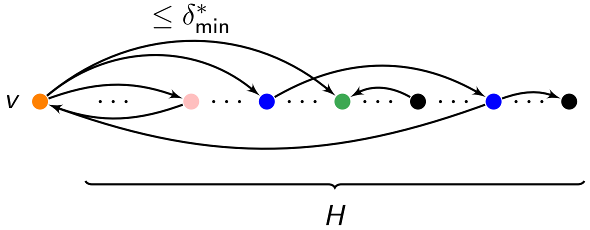

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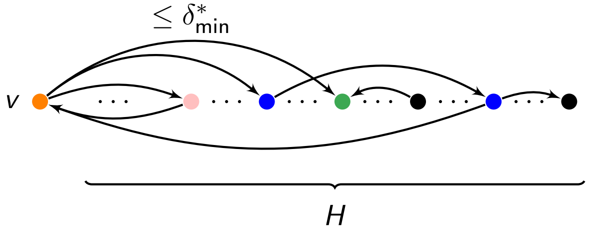

At the end we find $\alpha \longrightarrow \beta$ of length $\leq 2(2^{n-1}-1)+1 = 2^n - 1$

An analogue of Cereceda's conjecture.

Conjecture (Cereceda, 2007)

If $k \geq \delta^*(G) + 2$, then $diam(\mathcal{C}_k(G)) = O(n^2)$.

We posed the analogue for digraphs :

Conjecture

If $k \geq \delta^*_{\min}(D) + 2$, then $diam(\mathcal{D}_k(D)) = O(n^2)$.

A partial result

Theorem (Bousquet, Heinrich)

If $k \geq \frac{3}{2}(\delta^*(G) + 1)$, then $diam(\mathcal{C}_k(G)) = O(n^2)$.

Theorem

If $k \geq \frac{3}{2}(\delta^*_{\min}(D) + 1)$, then $diam(\mathcal{D}_k(D)) = O(n^2)$.

Using the Maximum Average Degree

$$MAD(D) = \max\left\{\frac{2|A(H)|}{|V(H)|} \middle| H \text{ subdigraph of } D\right\}$$

Theorem

If an oriented graph D satisfies $MAD(D) < \frac{7}{2}$ then it is 2-mixing.

Conjecture

It is also true when $MAD(D) < 4$.

Using the planarity

Conjecture (Neumann-Lara)

Every oriented planar graph D has dichromatic number at most 2.

It is known that $\vec{\chi}(D) \leq 3$.

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Problem

Is every oriented planar graph D 3-mixing ?

About complexity

Theorem

For every $k \geq 2$, given a digraph D together with two k-dicolourings α,β of D, deciding if there is a recolouring sequence (with k colours) between α and β is PSPACE-complete.

Problem

What is the complexity of deciding if D is k-mixing for any fixed $k \geq 2$?

Thanks for your attention.