Chromatic discrepancy of locally s-colourable graphs

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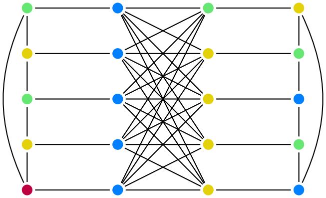
Definition

- **Proper colouring**: colouring of vertices s.t. adjacent vertices receive distinct colours.
- **Chromatic number $\chi(G)$**: minimum number of colours in a proper colouring of G.



$$\chi = 4$$

Chromatic Discrepancy

Definition

- The **discrepancy** of a proper colouring c of G:

$$\varphi_c(G) = \max_{H \subseteq_{\text{ind}} G} \big(|c(V(H))| - \chi(H)\big).$$

- The **chromatic discrepancy** of G:

$$\varphi(G) = \min_{c \in \mathscr{C}(G)} \varphi_c(G).$$

Remark: we may always choose H rainbow in the definition of φ_c.

$\varphi_c = 3 - 1 = 2$

$\varphi_c = 4 - 3 = 1$

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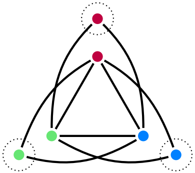

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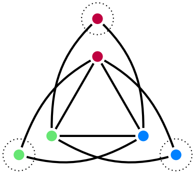

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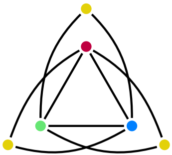

$\varphi_c = 4 - 3 = 1$

First Bounds on the Chromatic Discrepancy

$$\varphi(G) = \min_{c \in \mathscr{C}(G)} \max_{H \subseteq_{\text{ind}} G} (|c(V(H))| - \chi(H))$$

Proposition

Every graph $G \neq \varnothing$ satisfies $0 \leqslant \varphi(G) \leqslant \chi(G) - 1$, and both inequalities are tight.

$\varphi = 0$

$\varphi = \chi(G) - 1$

Question: Are there conditions guaranteeing that $\varphi(G)$ **is close to** $\chi(G) - 1$?

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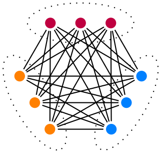

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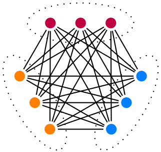

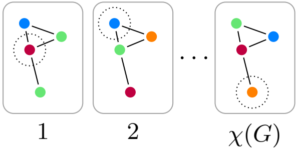

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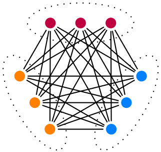

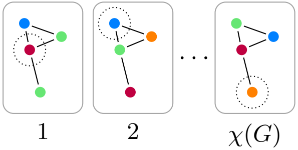

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Chromatic Discrepancy and Clique Number

ω = size of a largest clique 

Conjecture (Aravind, Kalyanasundaram, Sandeep, Sivadasan '15)

Every graph G satisfies $\varphi(G) \geqslant \chi(G) - \omega(G)$.

Theorem (Aravind, Cambie, Cames van Batenburg, Joannis de Verclos, Kang, Patel '21)

For every $r \geqslant 3$ and $k \geqslant 1$, there exists G with $\omega(G) = r$, $\chi(G) \geqslant k$, and

$$\varphi(G) \leqslant \chi(G) - \Omega\left(\chi(G)^{1/4}\right).$$

Theorem (Aravind, Cambie, Cames van Batenburg, Joannis de Verclos, Kang, Patel '21)

Every triangle-free graph G satisfies $\varphi(G) \geqslant \chi(G) - \log(\chi(G)) - 1$.

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Chromatic Discrepancy of Triangle-Free Graphs

Theorem (Corsini, P., Pierron, Pirot, Robinson '25)

Every triangle-free graph G satisfies $\varphi(G) \geqslant \chi(G) - 2$.

Remark: best possible (Micyelski graphs).

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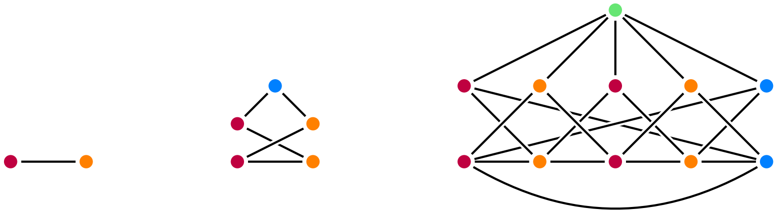

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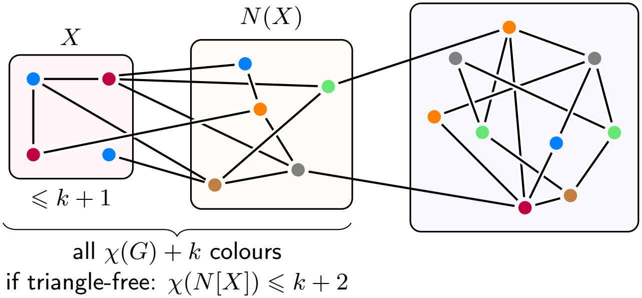
Proof. Follows from the following lemma.

Lemma

For every graph G and every proper $(\chi(G) + k)$-colouring c of G, there exists $X \subseteq V(G)$ s.t. $|X| \leqslant k + 1$ and $|c(N[X])| = \chi(G) + k$.



Spanning $\chi + k$ colours in the neighbourhood of $k + 1$ vertices

Lemma

For every graph G, every proper $(\chi(G) + k)$-colouring c of G, and colour $i \in [\chi(G) + k]$, there exists $X \subseteq V(G)$ s.t. $|X| \leqslant k + 1$, $|c(N[X])| = \chi(G) + k$, and $i \in c(X)$.

Proof. By recurrence on k.

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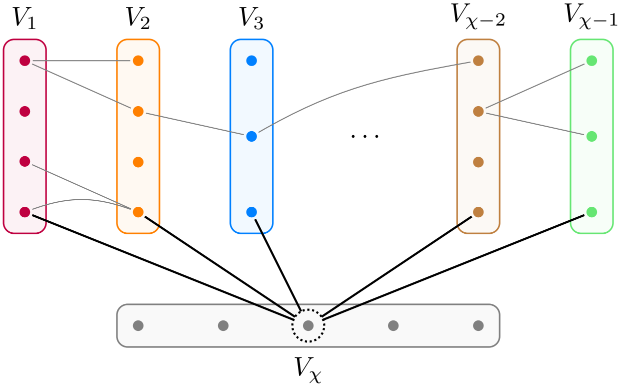

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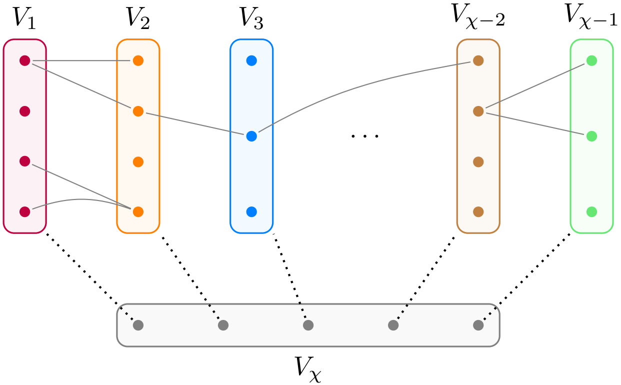

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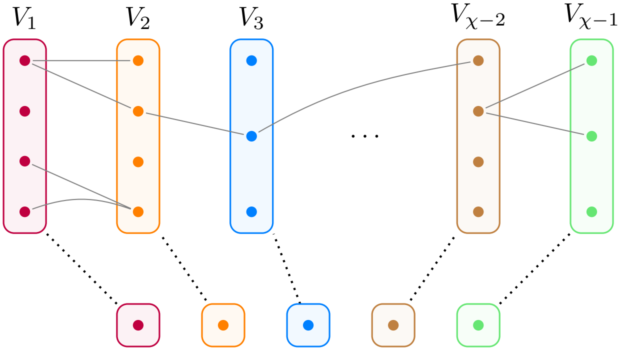

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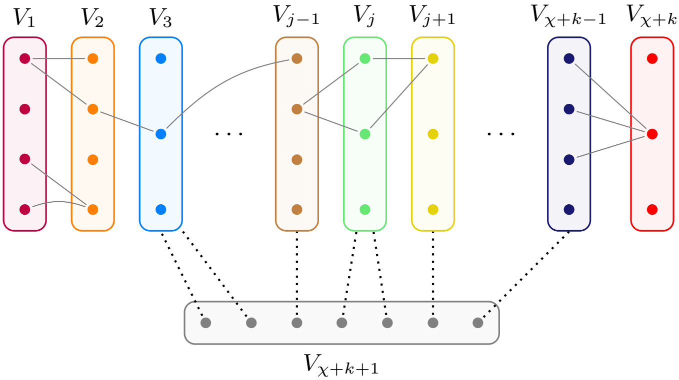

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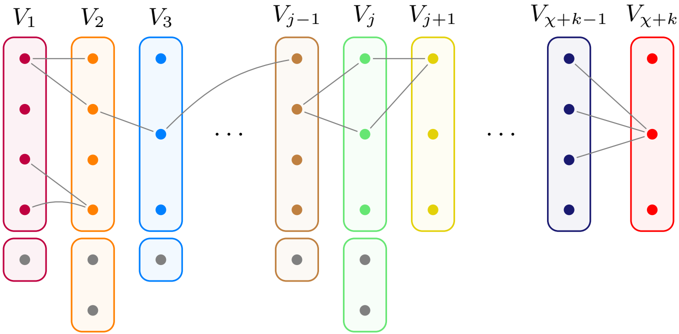

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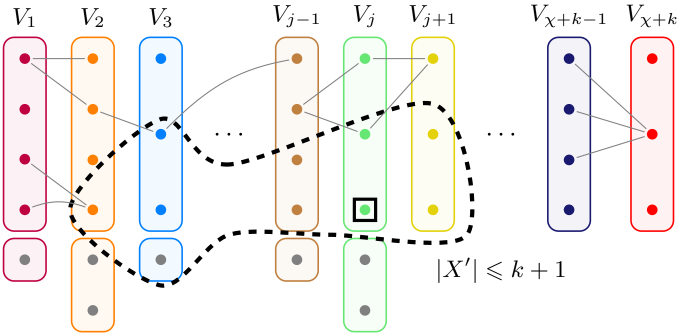

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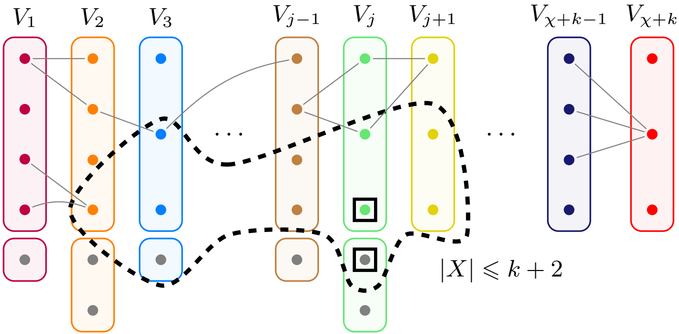

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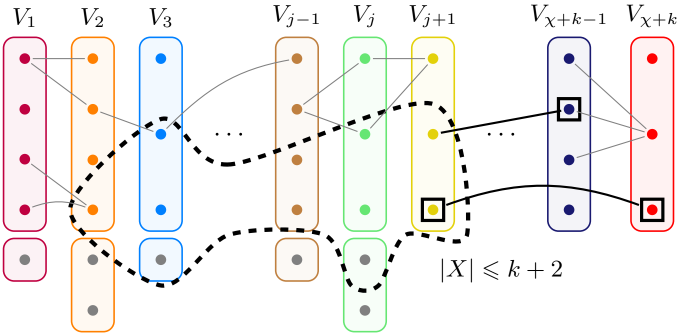

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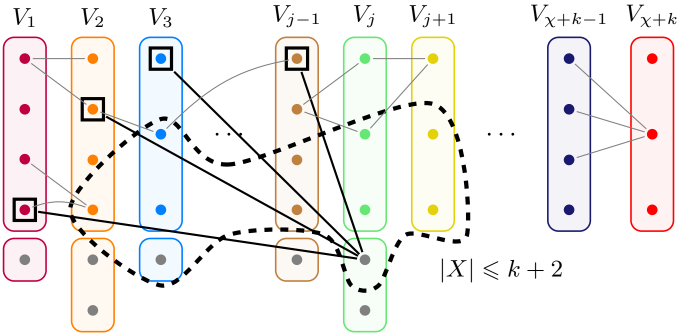

Conjecture

Every graph G satisfies $\varphi(G) \geqslant \chi(G) - s(G)$, where $s(G) = \max_{v \in V(G)} \chi(N[v])$.

Remarks.

- True when $s(G) = 2$ (G is triangle-free iff $s(G) = 2$),
- Weakening of the initial (disproved) conjecture since $s(G) \geqslant \omega(G)$,
- Almost always tight: for all $2 \leqslant s \leqslant \chi$, there exists G with $\chi(G) = \chi$, $s(G) = s$, and

$$\varphi(G) \leqslant \chi - s + 1.$$

Partial results.

- True when $\chi(G) < \frac{11}{6} s(G) + \frac{8}{3}$.
  $\hookrightarrow$ Based on Gallai's decomposition of k-critical graphs of order $\leqslant 2k - 2$.
- Every graph G satisfies $\varphi(G) \geqslant \chi_f(G) - s(G)$.
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A general logarithmic bound

Theorem

Every graph G with $s(G) \geqslant 3$ satisfies $\varphi(G) \geqslant \chi(G) - s(G) \cdot \ln \chi(G)$.

Remark. Not true when $s(G)$ is replaced by $\omega(G)$.

Proof Idea.

1. There exists an independent set I spanning $\geqslant \chi(G)/s(G)$ colours.
   $\hookrightarrow$ take a vertex v, remove $N(v)$ and all vertices coloured $c(v)$, and repeat.

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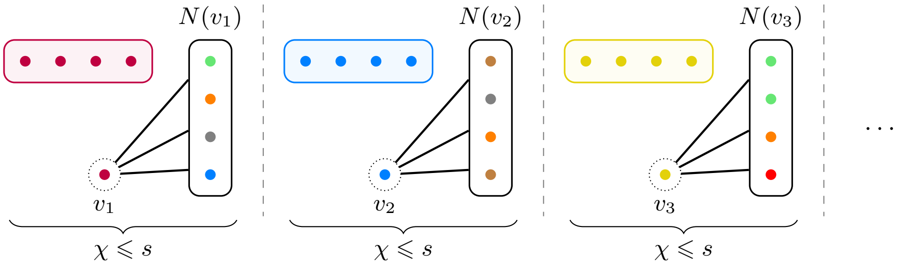

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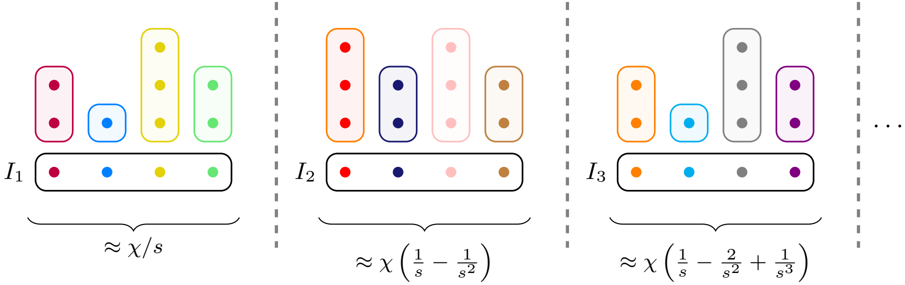
   $\hookrightarrow$ take a vertex v, remove $N(v)$ and all vertices coloured $c(v)$, and repeat.
2. Add I to H, remove all vertices coloured $c(I)$ and repeat. Eventually $|c(V(H))| - \chi(H) \geqslant \chi - s \cdot \ln \chi$.



An improved bound for 2-locally s-colourable graphs

Theorem

Every graph G satisfies $\varphi(G) \geqslant \chi(G) - O(t(G)^2 \cdot \ln\ln\chi(G))$, where $t(G) = \max_{v \in V(G)} \chi(N^2[v])$.

Proof Idea.

1. Every graph H of order p has chromatic number at most $\sqrt{2 \cdot s(H) \cdot p}$.
2. $\Rightarrow$ We can assume that $p \leqslant \chi + 2t\sqrt{\chi}$ colours are used.
3. There exist $v \in V(G)$ and an IS I s.t. $I \subseteq N(v)$ and I spans $\dfrac{p}{t(p-\chi+1)} \geqslant \dfrac{\sqrt{\chi}}{2t^2}$ colours.
4. Remove $N(I)$ and the vertices coloured $c(I)$, and repeat.
   We eventually build $I^\star$ an IS spanning $p - p^{1-1/4s_2^2}$ colours.
5. Add $I^\star$ to H, remove vertices coloured $c(I^\star)$ and repeat.

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The particular case of $C_{\ell+1}$-free graphs

Conjecture

Every $C_{\ell+1}$-free graph G satisfies

$$\varphi(G) \geqslant \chi(G) - \ell.$$

Moreover, equality holds only if $G = K_\ell$ or $\ell = 2$.

Remarks.

- If G is $C_{\ell+1}$-free then $s(G) \leqslant \ell$.

Partial results.

- Every C_4-free graph $G \neq K_3$ satisfies $\varphi(G) \geqslant \chi(G) - 2$.
- Every $C_{\ell+1}$-free graph $G \neq K_\ell$ with $\chi(G) < \frac{5}{3}\ell + \frac{2}{3}$ satisfies $\varphi(G) \geqslant \chi(G) - \ell + 1$.
- Every $C_{\ell+1}$-free graph G satisfies $\varphi(G) \geqslant \chi(G) - O_\ell(\ln \ln \chi(G))$.
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Balls of radius $\ell/2$ have chromatic number $O(\ell)$ in $C_{\ell+1}$-free graphs

Theorem

Let G be a $C_{\ell+1}$-free graph. For every $v \in V(G)$, $\chi\Big(B_{\ell/2}(v)\Big) \leqslant 2\ell$, where $B_{\ell/2}(v)$ is the subgraph of G induced by the vertices at distance at most $\ell/2$ from v.

Lemma

Let G be a $C_{\ell+1}$-free graph. For every $v \in V(G)$ and $1 \leqslant r \leqslant \ell/2$, $L_r(v)$ is $(\ell-1)$-degenerate, where $L_r(v)$ is the subgraph of G induced by the vertices at distance exactly r from v.

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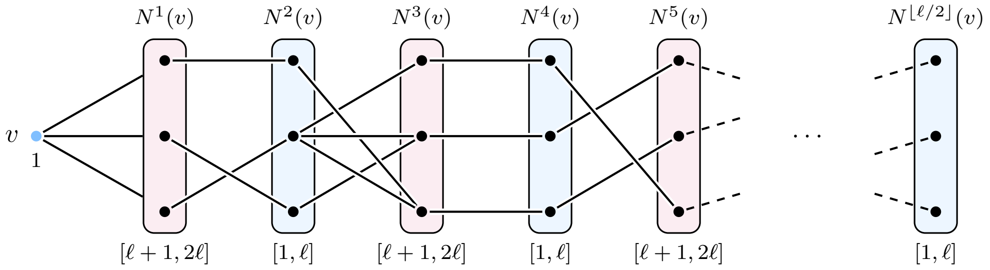

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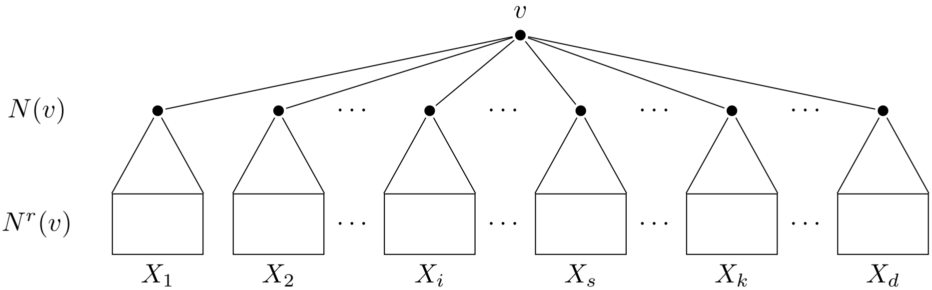

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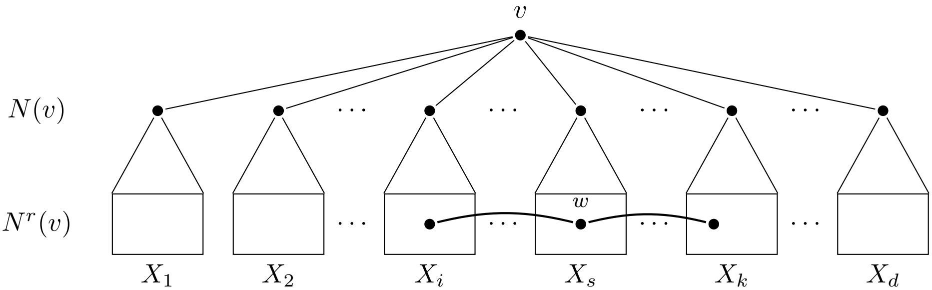

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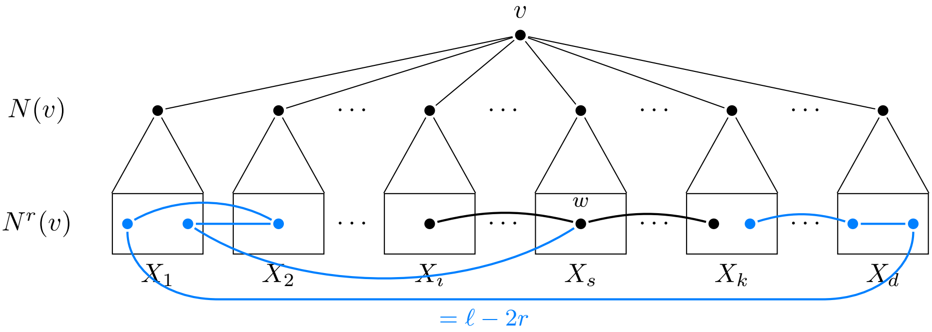

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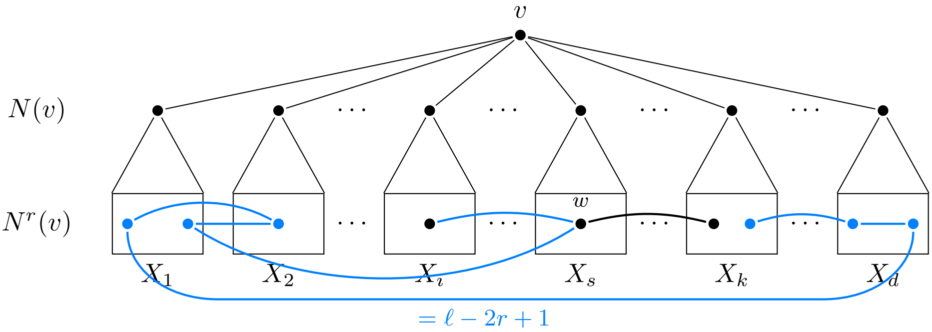

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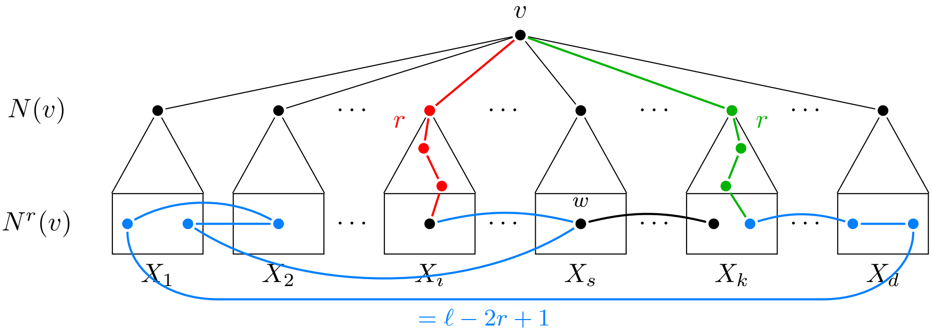

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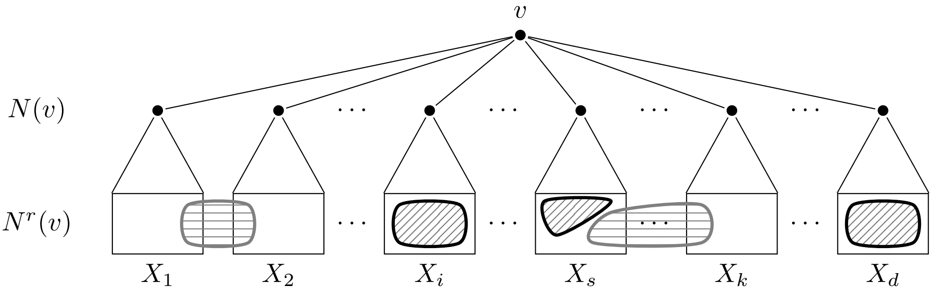

Open problems

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