# Minimum number of arcs in $k$-critical digraphs with order at most $2k-1$

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**Abstract**

The dichromatic number $\vec{\chi}(D)$ of a digraph $D$ is the least integer $k$ for which $D$ has a coloring with $k$ colors such that there is no monochromatic directed cycle in $D$. The digraphs considered here are finite and may have antiparallel arcs, but no parallel arcs. A digraph $D$ is called $k$-critical if each proper subdigraph $D'$ of $D$ satisfies $\vec{\chi}(D') < \vec{\chi}(D) = k$. For integers $k$ and $n$, let $\overrightarrow{\text{ext}}(k,n)$ denote the minimum number of arcs possible in a $k$-critical digraph of order $n$. It is easy to show that $\overrightarrow{\text{ext}}(2,n) = n$ for all $n \geq 2$, and $\overrightarrow{\text{ext}}(3,n) \geq 2n$ for all possible $n$, where equality holds if and only if $n$ is odd and $n \geq 3$. As a main result we prove that if $n, k$ and $p$ are integers with $n = k+p$ and $2 \leq p \leq k-1$, then $\overrightarrow{\text{ext}}(k,n) = 2(\binom{n}{2} - (p^2+1))$, and we give an exact characterisation of $k$-critical digraphs for which equality holds. This generalizes a result about critical graphs obtained in 1963 by Tibor Gallai.

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## 1 Introduction and main result

This paper is concerned with the coloring problem for directed graphs (shortly **digraphs**). Digraphs are always assumed to be finite and simple, that is,

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no loops and no parallel arcs in the same direction are allowed. However, digraphs may have **antiparallel arcs**, that is, two arcs with opposite directions between the same pair of vertices. For a digraph $D$, we denote the **vertex set** of $D$ by $V(D)$, and the **arc set** of $D$ by $A(D)$. The number of vertices of $D$ is called the **order** of $D$, denoted by $|D|$. A digraph $D$ with $|D| = 0$ is said to be **empty**, in this case we also write $D = \varnothing$. A digraph is said to be **acyclic** if it contains no directed cycle as a subdigraph.

Let $D$ be a digraph. An **acyclic $k$-coloring** of $D$ is a map $\varphi : V(D) \to C$ with **color set** $C = \{1, 2, \ldots, k\}$ such that for each color $c \in C$ the subdigraph of $D$ induced by the color class $\varphi^{-1}(c) = \{v \in V(D) \mid \varphi(v) = c\}$ is acyclic. We denote by $\mathcal{AC}_k(D)$ the set of acyclic $k$-colorings of $D$. The **dichromatic number** of $D$, denoted by $\vec{\chi}(D)$, is the least integer $k \geq 0$ for which $\mathcal{AC}_k(D) \neq \varnothing$. This coloring concept for digraphs, introduced in 1982 by Neumann-Lara and Erdős (cf. [10], [24]) generalizes the ordinary coloring concepts for graphs and has been studied in numerous papers until today (cf. [2], [3], [9], [13], [15], [22], [23], [25], [26]).

Let $k$ be a non-negative integer. We call a digraph $D$ **critical** and **$k$-critical** if every proper subdigraph $D'$ of $D$ satisfies $\vec{\chi}(D') < \vec{\chi}(D) = k$. Note that any digraph $D$ has a critical subdigraph with the same dichromatic number as $D$. Hence several problems related to the dichromatic number can be reduced to critical digraphs. Critical digraphs were first introduced and investigated by Neumann-Lara [24]. Let $\overrightarrow{\mathrm{CRI}}(k)$ denote the class of $k$-critical digraphs, and for an integer $n$, let

$$\overrightarrow{\mathrm{CRI}}(k,n) = \{D \in \overrightarrow{\mathrm{CRI}}(k) \mid |D| = n\}.$$

Note that $\vec{\chi}(D) = 0$ if and only if $D = \varnothing$, and $\vec{\chi}(D) \leq 1$ if and only if $D$ is acyclic. Consequently, $\overrightarrow{\mathrm{CRI}}(0) = \{\varnothing\}$, $\overrightarrow{\mathrm{CRI}}(1) = \{K_1\}$, and $\overrightarrow{\mathrm{CRI}}(2)$ is the class of directed cycles. In 2014, Bokal, Fijavž, Juvan, Kayll and Mohar [3] proved that the decision problem whether a given digraph $D$ satisfies $\vec{\chi}(D) \leq 2$ is $\mathsf{NP}$-complete. Hence a characterization of the class $\overrightarrow{\mathrm{CRI}}(k)$ for fixed $k \geq 3$ is unlikely. In this paper, we are interested in the extremal function $\overrightarrow{\mathrm{ext}}(k,n)$ defined by

$$\overrightarrow{\mathrm{ext}}(k,n) = \min\{|A(D)| \mid D \in \overrightarrow{\mathrm{CRI}}(k,n)\}.$$

The following theorem, which is the main result of this paper, implies a classical result for critical graphs that was obtained by Gallai [12] in the 1960s.

**Main Theorem.** *Let $n$, $k$ and $p$ be integers satisfying $n = k + p$ and $2 \leq p \leq k - 1$. Then $\overrightarrow{\mathrm{ext}}(k, n) = 2(\binom{n}{2} - (p^2 + 1))$.*

We also characterise exactly the digraphs for which equality holds.

# 2 Preliminaries

Our notation is standard. For integers $k$ and $\ell$, let $[k, \ell] = \{x \in \mathbb{Z} \mid k \leq x \leq \ell\}$.

## 2.1 Notation for graphs and digraphs

The term **graph** in this paper always refers to a finite undirected graph without multiple edges and without loops. For a graph $G$, we denote by $V(G)$ and $E(G)$ the **vertex set** and the **edge set** of $G$, respectively. The **order** of $G$, denoted by $|G|$, is the number of vertices of $G$. If $|G| = 0$, then $G$ is called the **empty graph**, briefly $G = \varnothing$. As usual, $K_n$ denotes the complete graph of order $n$, and $C_n$ denotes the cycle of order $n$ with $n \geq 3$. A cycle is called **odd** or **even** depending on whether its order is odd or even.

For a digraph $D$ we use the following notation. Let $u$ and $v$ be two distinct vertices of $D$. Then we denote by $a = uv$ the **arc** $a$ whose **initial vertex** is $u$ and whose **terminal vertex** is $v$; $u$ and $v$ are then said to be the **ends** of the arc $a$. The two arcs $uv$ and $vu$ are said to be **opposite arcs**. We say that $u$ and $v$ are **adjacent** in $D$ if $uv$ or $vu$ belongs to $A(D)$. If $u$ and $v$ are adjacent in $D$, we also say that $u$ is a **neighbor** of $v$ in $D$, and vice versa. If $uv \in A(D)$, then we say that $v$ is an **out-neighbor** of $u$ and $u$ is an **in-neighbor** of $v$. By $N_D^+(v)$ we denote the set of out-neighbors of $v$; by $N_D^-(v)$ the set of in-neighbors of $v$. Then $d_D^+(v) = |N_D^+(v)|$ is the **out-degree** of $v$ in $D$, and $d_D^-(v) = |N_D^-(v)|$ is the **in-degree** of $v$ in $D$.

Given $X, Y \subseteq V(D)$, let $A_D(X, Y) = \{a \in A(D) \mid a = uv, u \in X, v \in Y\}$. We denote by $D[X]$ the subdigraph of $D$ that is **induced** by the vertex set $X$, that is, $V(D[X]) = X$ and $A(D[X]) = A_D(X, X)$. A digraph $D'$ is called an induced subdigraph of $D$ if $D' = D[V(D')]$. If $X$ is a subset of $V(D)$, then we define $D - X = D[V(D) \setminus X]$. If $X = \{v\}$ is a singleton, we use $D - v$ rather than $D - X$. For an arc set $F \subseteq A(D)$, let $D - F$ denote the subdigraph of $D$ with vertex set $V(D)$ and arc set $A(D) \setminus F$. If $F = \{a\}$ is a singleton, we write $D - a$ instead of $D - F$. We say that $D$ is a $\vec{C}_n$ if $D$ is a

**directed cycle** of order (length) $n$, respectively a **directed $n$-cycle**, where $n \geq 2$. A digraph $D$ is called **acyclic** if $D$ contains no directed cycle as a subdigraph.

A **complete biorientation** of a graph $G$ is a digraph that can be obtained from $G$ by replacing each edge $e$ of $G$ by two opposite arcs with the same ends as $e$, we denote the resulting digraph by $D^{\pm}(G)$. So $D^{\pm}(K_1) = K_1$, and $D^{\pm}(K_2) = \vec{C}_2$ is a directed cycle of order 2, such a digraph is also refereed to as a **digon**. An **orientation** of a graph $G$ is a digraph $D$ obtained from $G$ by directing every edge of $G$ from one of its ends to the other one. An orientation of a graph is also called an **oriented graph**. Note that a digraph is an oriented graph if and only if it is digon-free.

## 2.2 Coloring of graphs and digraphs

Let $G$ be a graph. An **independent $k$-coloring** of $G$ is a map $\varphi : V(G) \to [1,k]$ such that for each color $c \in [1,k]$ the color class $\varphi^{-1}(c) = \{v \in V(G) \mid \varphi(v) = c\}$ is an independent set of $G$, that is, the subgraph of $G$ induced by $\varphi^{-1}(c)$ is edgeless. Let $\mathcal{IC}_k(G)$ denote the set of independent $k$-colorings of $G$. Then the **chromatic number** of $G$, denoted by $\chi(G)$, is the least integer $k$ for which $\mathcal{IC}_k(G) \neq \varnothing$. Clearly, $\chi(G) = 0$ if and only if $G = \varnothing$; and $\chi(G) \leq 1$ if and only if $G$ is edgeless. A graph $G$ with $\chi(G) \leq 2$ is also referred to as a **bipartite graph**. By a result of König it is known that a graph $G$ is bipartite if and only if $G$ contains no odd cycle. However, the decision problem whether a given graph $G$ satisfies $\chi(G) \leq 3$ is $\mathsf{NP}$-complete. If $G$ is a graph and $D = D^{\pm}(G)$, then $V(G) = V(D) = V$ and a map $\varphi : V \to [1,k]$ is an independent coloring of $G$ if and only if $\varphi$ is an acyclic coloring of $D$. Consequently, every graph $G$ satisfies $\chi(G) = \vec{\chi}(D^{\pm}(G))$.

## 2.3 Minimum number of edges in critical graphs

Deleting a vertex or edge of a graph decreases its chromatic number by at most one. Let $k$ be a non-negative integer. A graph $G$ is called **critical** and **$k$-critical** if every proper subgraph $G'$ of $G$ satisfies $\chi(G') < \chi(G) = k$. Let $\mathrm{CRI}(k)$ denote the class of $k$-critical graphs, and for $n \geq 0$, let

$$\mathrm{CRI}(k,n) = \{G \in \mathrm{CRI}(k) \mid |G| = n\}.$$

Clearly, if $G \in \mathrm{CRI}(k)$, then $|G| \geq k$ and equality holds if and only if $G = K_k$. For $k \leq 2$, $K_k$ is the only $k$-critical graph. In particular, $\mathrm{CRI}(0) = \{\varnothing\}$,

$\mathrm{CRI}(1) = \{K_1\}$, and $\mathrm{CRI}(2) = \{K_2\}$. Furthermore, König's theorem about bipartite graphs implies that

$$\mathrm{CRI}(3) = \{C_n \mid n \equiv 1 \pmod 2\}.$$

A good characterization of $\mathrm{CRI}(k)$ with fixed $k \geq 4$ is very unlikely. It is known that if $k \geq 4$, then $\mathrm{CRI}(k,n) \neq \varnothing$ if and only if $n \geq k$ and $n \neq k+1$. The concept of critical graphs is due to G. A. Dirac. In the 1950s he established the basic properties of critical graphs in a series of papers (see [5], [6] and [7]), and the study of critical graphs has attracted a lot of attention since then. Dirac also started to investigate the function $\mathrm{ext}(k,n)$ defined by

$$\mathrm{ext}(k,n) = \min\{|E(G)| \mid G \in \mathrm{CRI}(k,n)\}$$

as well as the corresponding class

$$\mathrm{EXT}(k,n) = \{G \in \mathrm{CRI}(k,n) \mid |E(G)| = \mathrm{ext}(k,n)\}$$

of extremal graphs. For a detailed discussion of the many partial results related to this extremal function the reader is referred to the survey by Kostochka [18] (see also [21]).

It is easy to show that every graph in $\mathrm{CRI}(k)$ with $k \geq 1$ has minimum degree at least $k-1$, which gives the trivial lower bound $2\mathrm{ext}(k,n) \geq (k-1)n$. By Brooks' famous theorem [4] it follows that $2\mathrm{ext}(k,n) = (k-1)n$ if and only if $n = k$ or $k = 3$ and $n$ is odd. In 1957 Dirac [7] (see also [8]) proved that $2\mathrm{ext}(k,n) \geq (k-1)n + k - 3$ for every $k \geq 4$ and $n \geq k+2$. For $k \geq 3$, we denote by $\mathcal{DG}(k)$ the class of graphs that can be obtained from two disjoints $K_k$'s by splitting a vertex of one $K_k$ into an edge of the other $K_k$. So a graph $G$ belongs to $\mathcal{DG}(k)$ if and only if the vertex set of $G$ consists of three non-empty pairwise disjoint sets $X, Y_1$ and $Y_2$ with

$$|Y_1| + |Y_2| = |X| + 1 = k - 1$$

and two additional vertices $v_1$ and $v_2$ such that $G[X]$ and $G[Y_1 \cup Y_2]$ are complete graphs not joined by any edge in $G$, and $X \cup Y_i$ is the neighborhood of $v_i$ in $G$ for $i \in \{1,2\}$. Then it is easy to show that $\mathcal{DG}(k) \subseteq \mathrm{CRI}(k, 2k-1)$. The class $\mathcal{DG}(k)$ was introduced and investigated by Dirac [8] and by Gallai [11]. Dirac [8] proved that $\mathrm{EXT}(k, 2k-1) = \mathcal{DG}(k)$ for every $k \geq 3$. In 1963, Gallai published two fundamental papers [11] and [12] about the structure of critical graphs. In particular, he proved the following two remarkable results.

**Theorem 2.1** (Gallai 1963)**.** *Let $G$ be a $k$-critical graph of order $n$. If $n \leq 2k-2$, then the complement of $G$ is disconnected.*

**Theorem 2.2** (Gallai 1963)**.** *Let $n = k + p$ be an integer, where $k, p \in \mathbb{N}$ and $2 \leq p \leq k-1$. Then*

$$\mathrm{ext}(k,n) = \binom{n}{2} - (p^2+1) = \frac{1}{2}((k-1)n + p(k-p) - 2)$$

*and* $\mathrm{EXT}(k,n)$ *is the class of graphs $G$ that can be obtained from two disjoint subgraphs $G_1$ and $G_2$ with $G_1 = K_{k-p-1}$ and $G_2 \in \mathcal{DG}(p+1)$ by joining each vertex of $G_1$ to each vertex of $G_2$ by exactly one edge.*

In 2014 Kostochka and Yancey [20] established a lower bound for $\mathrm{ext}(k,n)$ that is sharp when $k \geq 4$ and $n \equiv 1 \pmod{(k-1)}$. In particular, they proved that if $k \geq 4$, then

$$\lim_{n\to\infty} \frac{\mathrm{ext}(k,n)}{n} = \frac{1}{2}(k - \frac{2}{k-1}). \tag{2.1}$$

Furthermore, Kostochka and Yancey [21] proved that

$$\mathrm{ext}(4,n) = \left\lceil \frac{5n-2}{3} \right\rceil \tag{2.2}$$

As pointed out in [21] the above result and its short proof leads to a short proof of Grötzsch' famous Dreifarbensatz [14] saying that any planar triangle free graph $G$ satisfies $\chi(G) \leq 3$.

## 2.4 Minimum number of arcs in critical digraphs

While the study of critical graphs has attracted a lot of attention, not so much is known about the structure of critical digraphs. Neumann-Lara [24] established some basic properties of critical digraphs. The proof of the next result is straightforward and left to the reader.

**Proposition 2.3** (Neumann-Lara)**.** *Let $D \in \overrightarrow{\mathrm{CRI}}(k)$ with $k \geq 1$. Then the following statements hold:*

(a) *If $v \in V(D)$ and $\varphi \in \mathcal{AC}_{k-1}(D-v)$, then for each color $c \in [1,k]$ the color class $\varphi^{-1}(c)$ contains an out-neighbor of $v$ and an in-neighbor of $v$. As a consequence $|\varphi(N_D^+(v))| = |\varphi(N_D^-(v))| = k-1$.*

(b) *Each vertex* $v \in V(D)$ *satisfies* $d_D^+(v) \geq k-1$ *and* $d_D^-(v) \geq k-1$.

(c) *If* $a = uv \in A(D)$ *and* $\varphi \in \mathcal{AC}_{k-1}(D-a)$*, then there is a color* $c \in C$ *such that* $D[\varphi^{-1}(c)]$ *contains a directed path from* $v$ *to* $u$.

(d) $|D| \geq k$ *and equality holds if and only if* $D = D^{\pm}(K_k)$.

(e) *If* $k=1$*, then* $|D|=1$*; and if* $k=2$*, then* $D$ *is a directed cycle.*

Recall that $\overrightarrow{\mathrm{CRI}}(0) = \{\varnothing\}$, $\overrightarrow{\mathrm{CRI}}(1) = \{K_1\}$, and $\overrightarrow{\mathrm{CRI}}(2) = \{\vec{C}_n \mid n \geq 2\}$. Given $k, n \in \mathbb{N}$, let

$$\overrightarrow{\mathrm{ext}}(k,n) = \min\{|A(D)| \mid D \in \overrightarrow{\mathrm{CRI}}(k,n)\}$$

and

$$\overrightarrow{\mathrm{EXT}}(k,n) = \{D \in \overrightarrow{\mathrm{CRI}}(k,n) \mid |A(D)| = \overrightarrow{\mathrm{ext}}(k,n)\}.$$

Mohar [23] established a Brooks-type result for digraphs. Together with Proposition 2.3(b) this leads to some basic information about $\overrightarrow{\mathrm{ext}}(k,n)$ and $\overrightarrow{\mathrm{EXT}}(k,n)$.

**Theorem 2.4** (Mohar)**.** *If* $D \in \overrightarrow{\mathrm{CRI}}(k)$ *with* $k \geq 2$ *and* $d_D^+(v) = d_D^-(v) = k-1$ *for all* $v \in V(D)$*, then* $k=2$ *and* $D$ *is a directed cycle, or* $k=3$ *and* $D = D^{\pm}(C_n)$ *with* $n$ *odd and* $n \geq 3$*, or* $k \geq 4$ *and* $D = D^{\pm}(K_k)$.

**Corollary 2.5.** *Let* $k, n \in \mathbb{N}$ *with* $n \geq k \geq 2$*. Then* $\overrightarrow{\mathrm{ext}}(k,n) \geq (k-1)n$ *and equality holds if and only if* $k=2$ *and* $n \geq 2$*, or* $k=3$ *and* $n \geq 3$ *is odd, or* $n = k \geq 4$*. Furthermore,* $\overrightarrow{\mathrm{EXT}}(2,n) = \{\vec{C}_n\}$ *for* $n \geq 2$*,* $\overrightarrow{\mathrm{EXT}}(3,n) = \{D^{\pm}(C_n)\}$ *for odd* $n$ *and* $n \geq 3$*, and* $\overrightarrow{\mathrm{EXT}}(k,k) = \{D^{\pm}(K_k)\}$ *for* $k \geq 4$.

Let $D = D^{\pm}(G)$ be a complete biorientation of the graph $G$. Then it is easy to see that $G \in \mathrm{CRI}(k)$ if and only if $D \in \overrightarrow{\mathrm{CRI}}(k)$. As a consequence, we obtain that $\overrightarrow{\mathrm{ext}}(k,n) \leq 2\mathrm{ext}(k,n)$ for $n \geq k \geq 4$ and $n \neq k+1$. Note that $\mathrm{CRI}(k,k+1) = \varnothing$, but $\overrightarrow{\mathrm{CRI}}(k,k+1) \neq \varnothing$ (see Proposition 3.2). Kostochka and Stiebitz [19] proposed the following conjecture.

**Conjecture 1.** *Let* $k, n \in \mathbb{N}$ *with* $n \geq k \geq 4$ *and* $n \neq k+1$*. Then* $\overrightarrow{\mathrm{ext}}(k,n) = 2\mathrm{ext}(k,n)$ *and hence*

$$\lim_{n \to \infty} \frac{\overrightarrow{\mathrm{ext}}(k,n)}{n} = (k - \frac{2}{k-1}).$$

*Furthermore,* $\overrightarrow{\mathrm{EXT}}(k,n) = \{D^{\pm}(G) \mid G \in \mathrm{EXT}(k,n)\}$.

A first result supporting this conjecture was obtained by Kostochka and Stiebitz [19], they proved that if $n \geq 4$, then

$$\overrightarrow{\mathrm{ext}}(4, n) \geq \frac{10n - 4}{3}.$$

Havet, Picasarri-Arrieta and Rambaud [16] later characterised the equality case. If $D$ is a 4-critical digraph on $n$ vertices with $\frac{10n-4}{3}$ arcs, they show that $D \in \{D^{\pm}(G) \mid G \in \mathrm{EXT}(4, n)\}$. The best result towards Conjecture 1 is due to Aboulker and Vermande (Theorem 1.10 in [1]), who showed

$$\lim_{n\to\infty} \frac{\overrightarrow{\mathrm{ext}}(k, n)}{n} = (k - \frac{1}{2} - \frac{1}{k-1}).$$

In the proof of the main theorem we will use two recent results. The first one, due to Aboulker and Vermande (Theorems 1.3 and 1.4 in [1]), is the following generalisation of Dirac's result to digraphs.

**Theorem 2.6** (Aboulker and Vermande 2022)**.** *Let $n > k \geq 4$ and $D \in \overrightarrow{\mathrm{CRI}}(k, n)$, then*

$$|A(D)| \geq (k - 1)n + k - 3.$$

*Moreover, equality holds if and only if $D \in \overrightarrow{\mathcal{DG}}(k)$.*

Observe that the result above implies our main theorem when $p = k - 1$. The second result we need is due to Stehlík [28] and deals with decomposable critical digraphs. We introduce it in the next section.

## 2.5 Decomposable critical digraphs

As pointed out by Bang-Jensen, Bellitto, Schweser, and Stiebitz [2], two well known constructions for critical graphs have counterparts for digraphs, namely the Hajós join and the Dirac join.

Let $D_1$ and $D_2$ be two disjoint digraphs. First we construct a digraph $D$ as follows. For $i \in \{1, 2\}$, we choose an arc $u_i v_i \in A(D_i)$. Then let $D$ be the digraph obtained from the union $D_1 \cup D_2$ by deleting both arcs $u_1 v_1$ and $v_2 u_2$, identifying the vertices $v_1$ and $v_2$ to a new vertex, and adding the new arc $u_1 u_2$. We say that $D$ is the **Hajós join** of $D_1$ and $D_2$ and write $D = D_1 \nabla D_2$. Furthermore, let $D'$ be the digraph obtained from the union $D_1 \cup D_2$ by adding all possible arcs in both directions between $D_1$ and $D_2$, that is, $V(D') = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2) \cup \{uv, vu \mid u \in V(D_1) \text{ and } v \in V(D_2)\}$. We call $D'$ the **Dirac join** of $D_1$ and $D_2$ and write $D' = D_1 \boxplus D_2$. The following result was proved in [2]; a proof of the second result (also mentioned in [2]) is straightforward.

**Theorem 2.7** (Hajos Construction)**.** *Let $D = D_1 \nabla D_2$ be the Hajós join of two disjoint non-empty digraphs $D_1$ and $D_2$, and let $k \geq 3$ be an integer. Then $D$ belongs to $\overrightarrow{\mathrm{CRI}}(k)$ if and only if both $D_1$ and $D_2$ belong to $\overrightarrow{\mathrm{CRI}}(k)$.*

**Theorem 2.8** (Dirac Construction)**.** *Let $D = D_1 \boxplus D_2$ be the Dirac join of two disjoint non-empty digraphs $D_1$ and $D_2$. Then, $\chi(D) = \chi(D_1) + \chi(D_2)$ and $D$ is critical if and only if both $D_1$ and $D_2$ are critical.*

The Hajós construction can be used to establish an upper bound for $\overrightarrow{\mathrm{ext}}(k,n)$ for $n \geq k \geq 3$. If $D_1 \in \overrightarrow{\mathrm{EXT}}(k,n)$ and $D_2 \in \overrightarrow{\mathrm{EXT}}(k,m)$ with $n, m \geq k$, then $D = D_1 \nabla D_2 \in \overrightarrow{\mathrm{CRI}}(k, n+m-1)$ and $|A(D)| = |A(D_1)| + |A(D_2)| - 1$, which implies that

$$\overrightarrow{\mathrm{ext}}(k, n+m-1) \leq \overrightarrow{\mathrm{ext}}(k,n) + \overrightarrow{\mathrm{ext}}(k,m) - 1.$$

By Fekete's lemma (see also Jensen and Toft [17, Problem 5.3.]), this implies that the limit $\lim_{n\to\infty}(\overrightarrow{\mathrm{ext}}(k,n)/n)$ exists.

A digraph is called **decomposable** if it is the Dirac join of two non-empty subdigraphs; otherwise the digraph is called **indecomposable**. Theorem 2.8 implies that a decomposable critical digraph is the Dirac join of its indecomposable critical subdigraphs. In this sense, the indecomposable critical digraphs are the building elements of critical digraphs. In 2019, Stěhlík [28] proved the following result, thereby answering a question proposed in [2]. Note that this result applied to bidirected graphs implies the decomposition result for critical graphs due to Gallai (see Theorem 2.1). Stěhlík's proof uses matching theory. However, one can also use the hypergraph version of Theorem 2.1 obtained by Stiebitz, Storch and Toft [27].

**Theorem 2.9** (Stehlík 2019)**.** *If $D$ is an indecomposable critical digraph, then $|D| \geq 2\vec{\chi}(D) - 1$.*

*Sketch of Proof:* Let $D \in \overrightarrow{\mathrm{CRI}}(k)$ with $|D| \leq 2k-2$. Our aim is to show that $D$ is decomposable. Let $H$ be a hypergraph with $V(H) = V(D)$ such that $e \subseteq V(H)$ is an edge of $H$ if and only if $D[e]$ is a directed cycle.

Note that no edge of $H$ contains another edge of $H$. Then $\chi(H) = k$ and $\chi(H - v) < k$ for every vertex $v$ of $H$. Then $H$ has a subhypergraph $H'$ such that $\chi(H'') < \chi(H') = \chi(H) = k$ for every proper subhypergraph $H''$ of $H'$. Furthermore, we have that $V(H') = V(H)$. Then, by [27, Main Theorem 1], $H'$ is obtained from the disjoint union of two nonempty subhypergraphs by adding all 2-edges between these two subhypergraphs. Since $e$ is a 2-edge of $H$ if and only if $D[e]$ is a digon, we obtain that $D$ is decomposable. $\square$

The complement of a digraph $D$, denoted by $\overline{D}$, is the digraph with $V(\overline{D}) = V(D)$ and $A(\overline{D}) = \{uv \mid uv \notin A(D)\}$. So if $D$ has order $n$, then $D \cup \overline{D} = D^{\pm}(K_n)$. Clearly, a non-empty digraph $D$ is indecomposable if and only if $\overline{D}$ is connected.

## 3 Critical digraphs whose order is near to $\vec{\chi}$

Let $D$ be a digraph. A non-empty subdigraph $D'$ of $D$ is called a **dominating subdigraph** of $D$ if there exists a non-empty subdigraph $D''$ of $D$ such that $D = D' \boxplus D''$. A vertex $v$ of $D$ is called a **dominating vertex** of $D$ if the subdigraph $D[\{v\}]$ is a dominating subdigraph of $D$. Note that if $X \subseteq V(D)$ is a set of $p \geq 0$ vertices such that any vertex of $X$ is a dominating vertex of $D$, then $D = D^{\pm}(K_p) \boxplus (D - X)$.

**Theorem 3.1.** *Let $D$ be a $k$-critical digraph with $k \geq 1$, let $p$ be the number of dominating vertices of $D$, and $q$ be the number of dominating directed cycles of $D$ having order at least three. Then the following statements hold:*

(a) *$0 \leq p \leq k$ and there exists a digraph $D' \in \overrightarrow{\mathrm{CRI}}(k-p)$ such that $D = D^{\pm}(K_p) \boxplus D'$, $D'$ has no dominating vertex, and $|D'| \geq \frac{3}{2}(k-p)$. Furthermore, $p \geq 3k - 2|D|$ and equality holds if and only if $D'$ is the Dirac join of $\frac{1}{2}(k-p)$ disjoint directed 3-cycles of $D$.*

(b) *$0 \leq p + 2q \leq k$ and there exist digraphs $D_1 \in \overrightarrow{\mathrm{CRI}}(2q)$ and $D_2 \in \overrightarrow{\mathrm{CRI}}(k-p-2q)$ such that $D = D^{\pm}(K_p) \boxplus D_1 \boxplus D_2$, $D_1$ is the Dirac-join of $q$ directed cycles of $D$ each of which has order at least three, $D_2$ has no dominating vertex and no dominating directed cycle, and $|D_2| \geq \frac{5}{3}(k-p-2q)$. Furthermore, $2p + q \geq 5k - 3|D|$ and equality holds if and only if $D_1$ is the Dirac-join of $q$ directed 3-cycles of $D$, and $D_2$ is the Dirac-join of $\frac{1}{3}(k-p-2q)$ disjoint subdigraphs of $D$ each of which belong to $\overrightarrow{\mathrm{CRI}}(3,5)$.*

*Proof.* In what follows, let $D \in \overrightarrow{\mathrm{CRI}}(k)$ with $k \geq 1$. Then $D$ is a connected digraph and it follows from Theorem 2.9 that

$$D = D_1 \boxplus D_2 \boxplus \cdots \boxplus D_s,$$

where $\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_s$ are the components of $\overline{D}$. Obviously, $s \geq 1$. For $i \in [1, s]$, let $k_i = \vec{\chi}(D_i)$ and $n_i = |D_i|$. By Theorem 2.8, we obtain that

(1) $k = k_1 + k_2 + \cdots + k_t$ and $D_i \in \overrightarrow{\mathrm{CRI}}(k_i, n_i)$ for $i \in [1, s]$.

Since $\overline{D}_i$ is connected, Theorem 2.9 implies that

(2) $|D_i| \geq 2k_i - 1$ for $i \in [1, s]$.

Since $\overrightarrow{\mathrm{CRI}}(1) = \{K_1\}$, and $\overrightarrow{\mathrm{CRI}}(2) = \{\vec{C}_n \mid n \geq 2\}$, we obtain that $k_i = 1$ and $D_i = K_1$, or $k_i = 2$ and $|D_i| \geq 3$ (where equality holds if and only if $D_i = \vec{C}_3$), or $k_i \geq 3$ and $|D_i| \geq 5$. For a subset $I$ of $[1, s]$, let $G_I = \boxplus_{i \in I} G_i$ be the Dirac join of the digraphs $D_i$ with $i \in I$, and let $k_I = \sum_{i \in I} k_i$, where $D_\varnothing = \varnothing$ and $k_\varnothing = 0$. Then it follows from Theorem 2.8 that $D_I \in \overrightarrow{\mathrm{CRI}}(k_I)$. Let $P = \{i \in [1, s] \mid k_i = 1\}$, $Q = \{i \in [1, s] \mid k_i = 2\}$, $R = [1, s] \setminus (P \cup Q)$, $p = |P|$, $q = |Q|$, and $r = |R|$. Then $P, Q$ and $R$ are pairwise disjoint sets whose union is $[1, s]$. Thus we obtain that

(3) $D = D_P \boxplus D_Q \boxplus D_R$, where $D_P = D^{\pm}(K_p)$ and $D_Q \in \overrightarrow{\mathrm{CRI}}(2q)$.

Note that $p$ is the number of dominating vertices of $D$, and $q$ is the number of dominating directed cycles of $D$.

To establish a lower bound for $p$, let $\overline{P} = [1, s] \setminus P$. Then $\overline{P} = R \cup Q$ and $D = D_P \boxplus D_{\overline{P}}$. For $i \in \overline{P}$, we have that $k_i \geq 2$ and so, by (2), $|D_i| \geq 2k_i - 1 \geq \frac{3}{2}k_i$, where equality holds if and only if $D_i = \vec{C}_3$. From Theorem 2.8 and (1) it follows that $k_P = p$ and $k_{\overline{P}} = k - p$. For the order of $D$, it then follows from (1) and (2) that

$$|D| = p + \sum_{i \in \overline{P}} |D_i| \geq p + \tfrac{3}{2} \sum_{i \in \overline{P}} k_i = p + \tfrac{3}{2}(k - p),$$

which is equivalent to $p \geq 3k - 2|D|$. Clearly, $p = 3k - 2|D|$ if and only if $D_{\overline{P}}$ is the Dirac join of $\frac{1}{2}(k - p)$ disjoint $\vec{C}_3$'s. This proves (a).

For $i \in R$, we have $k_i \geq 3$ and so, by (2), $|D_i| \geq 2k_i - 1 \geq \frac{5}{3}k_i$, where equality holds if and only if $D_i \in \overrightarrow{\mathrm{CRI}}(3, 5)$. From Theorem 2.8 we conclude that $k_P = p$, $k_Q = 2q$, and $k_R = k - p - 2q$. For the order of $D$ we then obtain that

$$|D| = p + \sum_{i \in Q} |D_i| + \sum_{i \in R} |D_i| \geq p + 3q + \tfrac{5}{3} \sum_{i \in R} k_i = p + 3q + \tfrac{5}{3}(k - p - 2q),$$

which is equivalent to $2p + q \geq 5k - 3|D|$. Clearly, $2p + q = 5k - 3|D|$ if and only if $D_i = \vec{C}_3$ for all $i \in Q$ and $D_i \in \overrightarrow{\mathrm{CRI}}(3,5)$ for all $i \in R$. Thus (b) is proved. □

In the remaining of this section, we discuss some consequences of Theorem 3.1.

For a digraph $K$ and a class of digraphs $\mathcal{D}$, define $K \boxplus \mathcal{D} = \{K \boxplus D \mid D \in \mathcal{D}\}$ if $\mathcal{D} \neq \varnothing$, and $K \boxplus \mathcal{D} = \varnothing$ otherwise. If $\mathcal{D}$ is a digraph property (i.e., a class of digraphs closed under taking isomorphic copies), then we do not distinguish between isomorphic digraphs, so we are only interested in the number of isomorphism types of $\mathcal{D}$, that is, the number of equivalence classes of $\mathcal{D}$ with respect to the isomorphism relation for digraphs. Clearly, the number of isomorphisms types of the class $\overrightarrow{\mathrm{CRI}}(k,n)$ is finite.

From Theorem 3.1(a) it follows that $\overrightarrow{\mathrm{CRI}}(k, k+1) = K_1 \boxplus \overrightarrow{\mathrm{CRI}}(k-1, k)$ if $k \geq 3$. Since $\overrightarrow{\mathrm{CRI}}(2,3) = \{\vec{C}_3\}$, we have the following result by induction on $k$.

**Proposition 3.2.** *For every integer $k \geq 2$,*

$$\overrightarrow{\mathrm{CRI}}(k, k+1) = \{D^{\pm}(K_{k-2}) \boxplus \vec{C}_3\}.$$

Theorem 3.1(b) implies that a 4-critical digraph on 6 vertices either contains a dominating vertex or two disjoint dominating directed triangles. Hence we have $\overrightarrow{\mathrm{CRI}}(4,6) = (K_1 \boxplus \overrightarrow{\mathrm{CRI}}(3,5)) \cup \{\vec{C}_3 \boxplus \vec{C}_3\}$. If $k \geq 5$, Theorem 3.1(a) implies that every $k$-critical digraph on $k+2$ vertices contains at least one dominating vertex. $\overrightarrow{\mathrm{CRI}}(k, k+2) = K_1 \boxplus \overrightarrow{\mathrm{CRI}}(k-1, k+1)$ if $k \geq 5$. Therefore we obtain the following by induction on $k$.

**Corollary 3.3.** *For every integer $k \geq 4$,*

$$\overrightarrow{\mathrm{CRI}}(k, k+2) = \Big(D^{\pm}(K_{k-4}) \boxplus \vec{C}_3 \boxplus \vec{C}_3\Big) \cup \Big(D^{\pm}(K_{k-3}) \boxplus \overrightarrow{\mathrm{CRI}}(3,5)\Big).$$

Let $\overrightarrow{\mathrm{CRI}}^*(k,n)$ be the class consisting of all digraphs in $\overrightarrow{\mathrm{CRI}}(k,n)$ with no dominating vertex. For $k$-critical digraphs on $k+p$ vertices, the following is a nice consequence of Theorem 3.1. In particular, it implies that the number of $k$-critical digraphs on $k+p$ vertices (up to isomorphism) is bounded by a function depending only on $p$.

**Corollary 3.4.** *For every integer $k,p$ such that $k > 2p \geq 4$,*

$$\overrightarrow{\mathrm{CRI}}(k,k+p) = \bigcup_{\ell=2}^{2p} \left\{ D^{\pm}(K_{k-\ell}) \boxplus \overrightarrow{\mathrm{CRI}}^*(\ell,\ell+p) \right\}$$

*Proof.* Let $D$ be a $k$-critical on $k+p$ vertices. Then $|D| = k+p < \frac{3}{2}k$. Let $s$ be the number of dominating vertices of $D$. By Theorem 3.1(a), $D = D^{\pm}(K_s) \boxplus D'$ where $D'$ belongs to $\overrightarrow{\mathrm{CRI}}^*(\ell,\ell+p)$. It remains to show that $\ell \in [2,2p]$.

Since $p \geq 2$, we obviously have $\ell \geq 2$. On the other hand, $D'$ contains no dominating vertex, so $|D'| \geq \frac{3}{2}\ell$ (by applying Theorem 3.1(a) on $D'$). We deduce

$$|D'| = \ell + p \geq \frac{3}{2}\ell.$$

This implies $\ell \leq 2p$ as desired. $\square$

# 4 Proof of the main theorem

For every $k \geq 2$, let us define $\overrightarrow{\mathcal{DG}}(k)$ as follows: $\overrightarrow{\mathcal{DG}}(2) = \{\vec{C}_3\}$ and $\overrightarrow{\mathcal{DG}}(k) = \{D^{\pm}(G) \mid G \in \mathcal{DG}(k)\}$ for every $k \geq 3$.

The following result clearly implies the main Theorem.

**Theorem 4.1.** *Let $p,k,n \in \mathbb{N}$ such that $1 \leq p \leq k-1$ and $n = k+p$. Then*

$$\overrightarrow{\mathrm{EXT}}(k,n) = D^{\pm}(K_{k-p-1}) \boxplus \overrightarrow{\mathcal{DG}}(p+1).$$

*As a consequence, we have*

$$\overrightarrow{\mathrm{ext}}(k,n) = \begin{cases} 2\binom{n}{2} - 3 & \text{if } p = 1 \\ 2\left(\binom{n}{2} - (p^2+1)\right) & \text{otherwise.} \end{cases}$$

*Proof.* We proceed by induction on $k$. If $k = 2$, then $p = 1$ and we have $\overrightarrow{\mathrm{EXT}}(2,3) = \{\vec{C}_3\} = \overrightarrow{\mathcal{DG}}(2)$. If $k = 3$, then $p \in \{1,2\}$ and we have $\overrightarrow{\mathrm{CRI}}(3,4) = \{D^{\pm}(K_1) \boxplus \vec{C}_3\} = \overrightarrow{\mathrm{EXT}}(3,4)$ (by Proposition 3.2) and $\overrightarrow{\mathrm{EXT}}(3,5) = \overrightarrow{\mathcal{DG}}(3)$ (by Theorem 2.6). So Theorem 4.1 holds for $k = 2$ and $k = 3$.

We now assume $k \geq 4$. We also assume that $2 \leq p \leq k-2$, for otherwise the result is a consequence of Proposition 3.2 or Theorem 2.6. Let $n = k+p$ and let $D \in \overrightarrow{\mathrm{EXT}}(k,n)$. We will show that $D$ belongs to digraph class $D^{\pm}(K_{k-p-1}) \boxplus \overrightarrow{\mathcal{DG}}(p+1)$, which implies the result since every digraph in this class contains exactly $2\left(\binom{n}{2} - (p^2+1)\right)$ arcs.

**Case 1:** *$D$ contains a dominating vertex $v$.*

Then $D' = D - v$ is a $(k-1)$-critical digraph on $n-1$ vertices. Moreover, $D'$ must contain exactly $\overrightarrow{\mathrm{ext}}(k-1, n-1)$ arcs. If this is not the case, there exists a $(k-1)$-critical digraph $\tilde{D}$ on $n-1$ vertices such that $|A(\tilde{D})| < |A(D')|$, and $K_1 \boxplus \tilde{D}$ contradicts the minimality of $D$.

Since $n \leq 2k-2$, we have $(k-1)+1 \leq n-1 \leq 2(k-1)-1$, which allow us to apply the induction hypothesis on $D'$. Hence $D'$ belongs to $D^{\pm}(K_{k-p-2}) \boxplus \overrightarrow{\mathcal{DG}}(p+1)$. Since $D$ is obtained from $D'$ by adding a dominating vertex, we deduce that $D$ belongs to $D^{\pm}(K_{k-p-1}) \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ as desired.

**Case 2:** *$D$ contains a dominating $\vec{C}_3$ and no dominating vertex.*

Let us denote $D = \vec{C}_3 \boxplus D'$. Then $D'$ is a $(k-2)$-critical digraph on $n-3$ vertices. Observe that $n-3 \leq 2(k-2)-1$. Note also that $n-3 \geq k-1$, for otherwise $D'$ is the complete digraph on $k-2$ vertices, which implies that $D$ contains a dominating vertex. Hence we may apply the induction on $D'$, which implies that $D'$ contains at least $2\left(\binom{n-3}{2} - ((p-1)^2+1)\right)$ arcs. We deduce the following.

$$\begin{aligned}
|A(D)| &= |A(D')| + 3 + 6(n-3) \\
&\geq 2\left(\binom{n-3}{2} - ((p-1)^2+1)\right) + 3 + 6(n-3) \\
&= 2\left(\binom{n}{2} - (p^2+1)\right) - 5 + 4p \\
&\geq 2\left(\binom{n}{2} - (p^2+1)\right) + 3.
\end{aligned}$$

where in the last inequality we used $p \geq 2$. This is a contradiction to the minimality of $D$, since every digraph in $D^{\pm}(K_{k-p-1}) \boxplus \overrightarrow{\mathcal{DG}}(p+1)$ contains exactly $2\left(\binom{n}{2} - (p^2+1)\right)$ arcs.

**Case 3:** *$D$ does not contain any dominating vertex nor dominating $\vec{C}_3$.*

Let $t$ be the number of connected components of $\bar{D}$. Since $n \leq 2k-2$, by Theorem 2.9 we know that $t \geq 2$. Then

$$D = D_1 \boxplus D_2 \boxplus \ldots \boxplus D_t$$

where $D_1, \ldots, D_t$ are the connected components of $\bar{D}$. For every $i \in [t]$, $D_i$ is a $k_i$-critical digraph on $n_i$ vertices. Since it is not decomposable, Theorem 2.9 implies $n_i \geq 2k_i - 1$.

Let $s$ be the number of indices $i$ for which $n_i = 2k_i - 1$. Assume first that $s \leq 1$, then we have

$$n = \sum_{i=1}^{t} n_i \geq \sum_{i=1}^{t} 2k_i - 1 = 2k - 1.$$

This is a contradiction since $n \leq 2k-2$. Assume then that $s \geq 2$. We assume without loss of generality that $n_1 = 2k_1 - 1$ and $n_2 = 2k_2 - 1$. Since $D$ does not contain any dominating vertex nor dominating $\vec{C}_3$, we have $k_1, k_2 \geq 3$. Hence we have $n_1 \geq k_1 + 2$ and $n_2 \geq k_2 + 2$.

The minimality of $D$ implies that $D_1$ uses exactly $\overrightarrow{\mathrm{ext}}(k_1, n_1)$ arcs. By induction, $D_1$ then belongs to $\overrightarrow{\mathcal{DG}}(k_1)$. Analogously, $D_2$ belongs to $\overrightarrow{\mathcal{DG}}(k_2)$. This implies the following.

$$\begin{aligned}|A(D_1 \boxplus D_2)| &= 2\left(\binom{2k_1-1}{2} - ((k_1-1)^2+1)\right) \\ &+ 2\left(\binom{2k_2-1}{2} - ((k_2-1)^2+1)\right) \\ &+ 2(2k_1-1)(2k_2-1).\end{aligned}$$

On the other hand, the minimality of $D$ implies that $D_1 \boxplus D_2$ contains exactly $\overrightarrow{\mathrm{ext}}(k_1+k_2, 2(k_1+k_2)-2)$ arcs. We know that every digraph in $K_1 \boxplus \overrightarrow{\mathcal{DG}}(k_1+k_2-1)$ contains exactly $2\left(\binom{2k_1+2k_2-2}{2} - ((k_1+k_2-2)^2+1)\right)$. We finally reach the following contradiction to the minimality of $|A(D_1 \boxplus D_2)|$.

$$\begin{aligned}&|A(D_1 \boxplus D_2)| - 2\left(\binom{2k_1+2k_2-2}{2} - ((k_1+k_2-2)^2+1)\right) \\ &= 4k_1k_2 - 4k_1 - 4k_2 + 2 \\ &\geq 2.\end{aligned}$$

This conclues the proof. $\square$

Theorem 4.1 applied to bidirected graphs gives Gallai's result (Theorem 2.2). Gallai's original proof was much longer, since he did not use Dirac's result from 1974. On the other hand, Gallai's result combined with the Kotochka–Yancey bound for $\text{ext}(k, n)$ implies Dirac's result, since the Kostochka–Yancey bound is better that the Dirac bound if $n \geq 2k$.

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