PHY324 Pendulum Project

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Abstract

In this experiment, I refuted the idea that a pendulum's period is independent from its amplitude and determined a functional form for this relationship for fixed length. I found a relationship between the pendulum's period and length for small angles and verified that the period is independent of mass. The pendulum's amplitude decays exponentially for angles around 0.6 radians, but I demonstrated that decay was not exponential for larger angles. I also determined that the pendulum's decay constant is dependent on its length and mass, and suggested functional forms for these two relationships.

1 Introduction

Consider a mass m attached to a length L string of negligible mass, hanging from some fixed pivot. Suppose the distance between the point on the mass where the string is attached and the center of mass is D. If we assume that this mass is released at some angle θ_0 from the vertical at $t = 0$ and is constrained to move in a plane, the theory of damped harmonic oscillators predicts that the pendulum's trajectory will follow

$$\theta(t) = \theta_0 e^{-t/\tau} \cos\left(\frac{2\pi}{T}t + \phi_0\right) \tag{1}$$

where T is the period of the oscillation, τ is the decay constant, and ϕ_0 is the phase constant. Further, this model predicts that T is independent of θ_0, m, and ϕ_0 but

$$T = 2(L + D)^{1/2} \tag{2}$$

provided that $D^2 \ll L^2$. It makes no predictions on the relationship of τ with these parameters [1].

In this experiment, I put several of these relationships to the test. I refuted the fact that T is independent of θ_0, and demonstrated instead that T is quadratic in θ_0 for $0 \leq \theta_0 \leq \frac{\pi}{2}$. This also refutes eq. (2) as a general formula for the period. However, for small angles ($\theta_0 \ll 0.5$ rad) the linear and quadratic terms are negligible and T obeys a relationship very close to eq. (2). I did, however, verify that T is independent of the pendulum's mass.

I also studied the pendulum's decay. I verified that it decays exponentially for $\theta_0 \leq 0.6$ rad, but for angles close to $\frac{\pi}{2}$ it is clearly not the case. Additionally, I successfully demonstrated that τ is dependent on both L and m, although my apparatus failed to discriminate between the direct influence of m on τ versus indirect influence through changes in D. Although I was able to find an adequate functional form for this relationship, the accuracy of the model could be largely improved.

2 Materials and Methods

2.1 Materials

- Crayola Model magic molding clay
- Nylon String

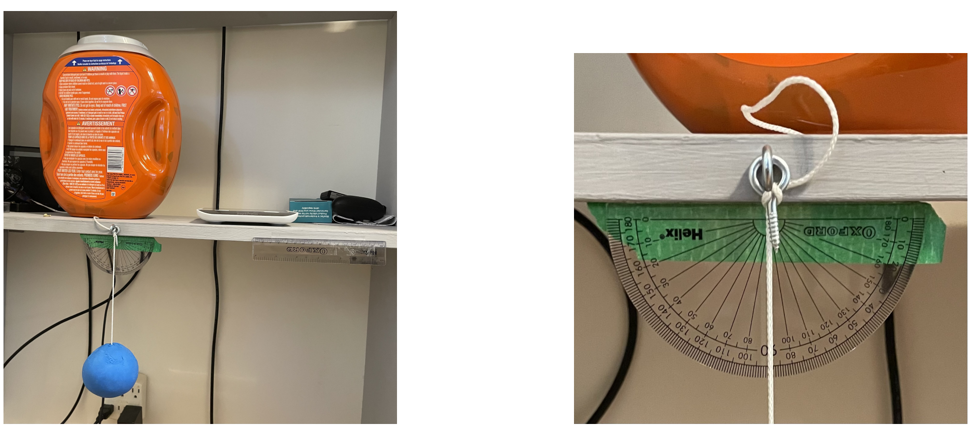
- Tape
- Phone camera (30 FPS)
- Tracking Software [2]
- Ruler (error of 0.1 cm)
- Protractor
- Ring Screws
- Food scale (error of 1 g)

2.2 Methods

2.2.1 Setup

A ball of clay was molded around the end of a nylon string, which was tied to a ring screw for extra support. Together, the screw and clay weighed 105 ± 1 g, with the screw weighing about 2 grams so its affect on the mass of the ball was minuscule. Since the clay has roughly uniform density, I took D to be the ball's radius, which was 3.5 ± 0.1 cm. For analyses which did not require varying the mass, I used the same ball of clay each time. The clay's dryness prevented changes in D.

The other end of the Nylon string was tied to a ring screw, while a second screw was screwed into a desk through the ring of the first. I tightened the screw so that the ring experienced some pressure between the side of the desk and the screw. This prevented oscillations in the direction normal to the side of the desk, which forced the pendulum to swing in a plane. This pressure also kept the ring from slipping and the pivot position from moving. However, the additional friction between the pivot and the desk may have had an adverse effect on my decay analysis. I placed a large mass on the remaining string beyond the knot at the pivot. This was done to prevent string slippage and increases in L during oscillation. A ruler was taped near the pendulum and a protractor directly behind it to provide length scales and a reference angle for later. The lab setup is pictured below.



(a) (b)

Figure 1: (a) displays the full lab setup for a particularly short length while (b) takes a closer look at the pivot point.

2.2.2 Collecting Data

Once the pendulum was setup, I propped up my phone as far as I could from the pendulum and began filming. I did the best I could to place my phone camera parallel to the pendulum's swinging plane to avoid error due to perspective. Since my phone was relatively far from the pendulum (much farther away than the relevant length scales), the error due to camera angle should be minuscule. For each trial I began filming and then released the pendulum from its leftmost position. For trials where I was interested in measuring the period, I filmed for 5-10 oscillations. For trials where I was interested in measuring the decay constant, I needed most of the energy of the pendulum to dissipate. I decided to film until the pendulum's maximum angle was well within 10 degrees on the protractor. This usually took 1-2 minutes. Once I was done taking data, I sent my videos to the motion tracker.

In the motion tracker, I used the ruler I had previously taped to the desk in order to set a length scale for the tracker. This is how I measured all my lengths, including D. I used the pendulum itself to line up my axes with the vertical; I did this for a section of the video where the pendulum was still and pointing straight down. I also placed the pivot at the origin. However, my videos had to be low quality for the computer to track the pendulum at a reasonable timescale. To account for this, I added an uncertainty of 1 degree to all angle measurements.

For each trial where I was interested in the period, I computed it using

$$T = \frac{t_2 - t_1}{N} \tag{3}$$

where t_1 and t_2 are the times of two different peaks in the data set and N is the number of oscillations between my two peaks. Since my camera rate has an FPS of 30, the peak can actually occur at any time within 1/30 seconds of each data point with equal probability. Then, I took the uncertainty in each peak time to be

$$\sigma_{t_i} = \frac{1}{2\sqrt{3}} \frac{1}{30} \text{ sec} \tag{4}$$

The factor at the front comes from the fact that this can occur at any time within the uncertainty range with equal probability [3].

In general, I chose $t_1 = 0$ since I started taking data in each trial when the pendulum was at its leftmost position, and I chose $N = 5$. I chose these values since my typical oscillation time was on the order of 1 sec while my typical decay time was greater than 20 seconds. This choice of N was large enough that the period could be calculated to a high degree of certainty, but small enough so that possible decay effects are negligible.

In total, I took data for 31 trials, including my test data set.

2.2.3 Reducing Uncertainties

Throughout the course of this experiment, I had to vary the parameters θ_0, L, and m one at a time. I had to take precautions to ensure no other parameters changed where possible. Here I document my efforts towards achieving this.

For trials where I varied θ_0, I fixed the length by simply taking all my data without removing the pendulum from the pivot.

For trials where I varied L, I picked some reference angle on the protractor taped under the desk at the beginning of taking data, and tried to stick to that angle for the entire data set. Of course, since the base of the protractor was not directly behind the pivot, this angle did not represent the true release angle of my pendulum. However, it provided a marker which I knew was at a constant angle from the vertical. Generally, I was able to keep this angle within a total range of 0.06 radians.

Trials where I varied m were the most complicated. I took two different approaches depending on whether I was measuring T or τ. For T, I determined a functional form depending only on the value of $L + D$ for small angles in section 3.2.2. Then, all that matters is that $L + D$ does not change. To accomplish this, I simply molded more clay to the bob without removing it from the pivot between trials. For this section, I managed to keep all lengths within 0.6 cm of an average length of 42.3 cm, with the exception of one value which differed by 1.2 cm. This was later accounted for in the uncertainty.

When measuring τ vs mass, I did not have the luxury of a functional form which depended on $L + D$. Thus, I needed L to actually stay constant. To achieve this, I removed the ruler from the desk and lined it up with the top of the bob and the point which would make contact with the ring when I tied. I did this while the bob was hanging to account for stretching. I made the length of string about 15.5 cm, which is the top to bottom length of the ruler. However, since the pivot is actually at the top of the ring screw, I added the 1 cm diameter of the ring for a total length of 17 ± 1 cm to account for error in my procedure. I repeated this process each time I added more mass to the string. Since it was impossible to change the mass without changing D, my model does not discriminate between the direct affect of m on τ versus the indirect affect of m through variations in D as larger masses had larger D, since my clay was of uniform density. For both T and τ, I fixed the release angle the same way I did when varying length.

In the analysis section, I explain any additional uncertainties I included to account for changes in parameters which were meant to stay constant.

3 Data and Analysis

3.1 Verifying Symmetry

One should expect that a perfect pendulum spends just as much time moving left as it does moving right during an integer number of oscillation. As a test of my apparatus symmetry, I computed the time my pendulum spent moving in each direction for the largest possible integer number of oscillations for each of my 31 data sets. I then took the absolute value of the difference of these two numbers. The mean value and standard error were 0.2 ± 0.2 seconds. Since this value is within uncertainty from 0, I concluded that my pendulum is symmetric.

3.2 Analysis of T

3.2.1 T vs θ_0

I attempted to do a χ^2_{red} fit my T vs θ_0 data with a power series, using the first one, two, and three terms respectively. The result was as follows

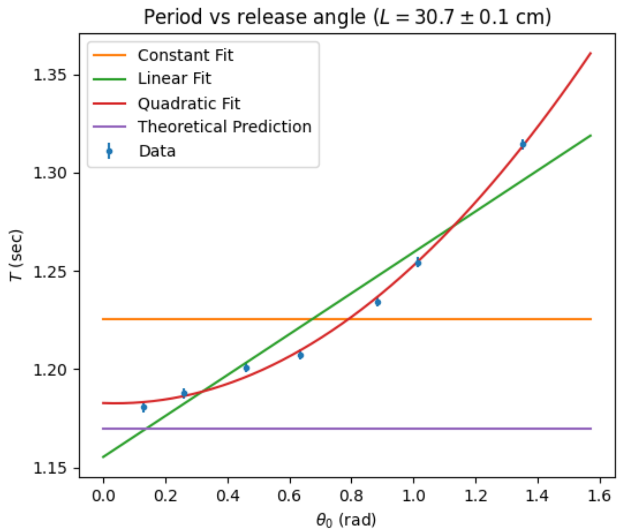


Figure 2: A plot of period vs release angle including three chi squared fits, and a theoretical prediction obtained via eq. (2). This data is well modelled by a quadratic function.

The model parameters for each chi squared fit are included in the table below

Model Type	a (sec)	b (sec)	c (sec)	χ^2_{red}	Fit Probability
$T = a$	1.226 ± 0.001	N/A	N/A	297	0%
$T = a + b\theta_0$	1.155 ± 0.002	0.104 ± 0.003	N/A	24.7	0%
$T = a + b\theta_0 + c\theta_0^2$	1.183 ± 0.003	0.01 ± 0.01	0.076 ± 0.007	1.47	11.7%

Table 1: Table of data for model fits for period vs release angle data.

The quadratic model is the best fit for this regime, and it is clear that T does depend on θ_0. However, for $\theta_0 \ll 0.5$ rad, the quadratic model is approximately constant and comparable to the theoretical fit from eq. (2).

3.2.2 T vs L

In this section, I tested the relationship T vs L from a release angle of 0.17 ± 0.02 rad (the value is the median angle and the uncertainty is half of the range). Using the Quadratic model for $T(\theta_0)$ I obtained in 3.2.1, I got that for this angle, T deviates by 0.3% of its initial value. Then, for each T measurement I added an additional uncertainty of $0.003 \cdot T$ to account for this. To satisfy $D^2 \ll L^2$, I took L no shorter than 14.2 ± 0.1 cm while $D = 3.5 \pm 0.01$ cm.

To test eq. (2) I fit the data with the functional form

$$T = C \cdot (L + D)^k \tag{5}$$

where L and D are measured in meters. The resulting best fit parameters were $C = 2.03 \pm 0.01$ and $k = 0.509 \pm 0.004$. The result is plotted below

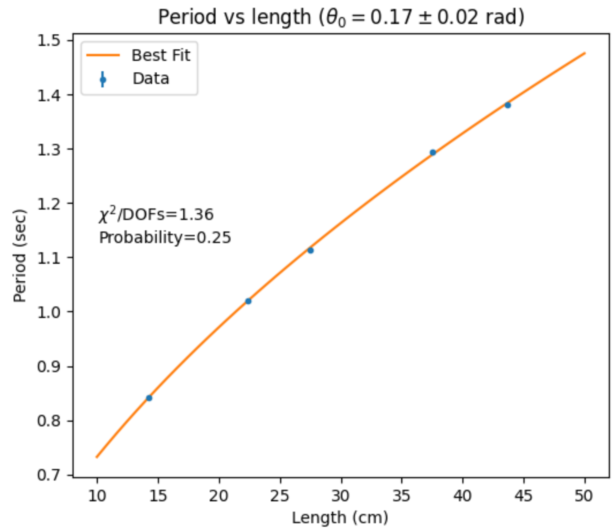


Figure 3: Plot of period versus length for a fixed (small) angle. This data is clearly well represented by the functional form eq. (5).

3.2.3 T vs m

Similarly to the previous section, I released the pendulum from a fixed angle of 0.15 ± 0.01 rad. This accounts for a 0.3% deviation from the initial value of the quadratic model, which was incorporated into uncertainty.

To account for changes in $L + D$ between masses, I computed the period as predicted by eq. (5), using the average value of $L + D$, T_{avg}. I then defined an uncertainty $|T_{\text{avg}} - T_{\text{exp}}|$ where T_{exp} is the expected value of T by eq. (5) for each length measurement to account for any change in T due to change in length.

I attempted to fit the T vs m data using both a constant and linear fit. The results were as follows

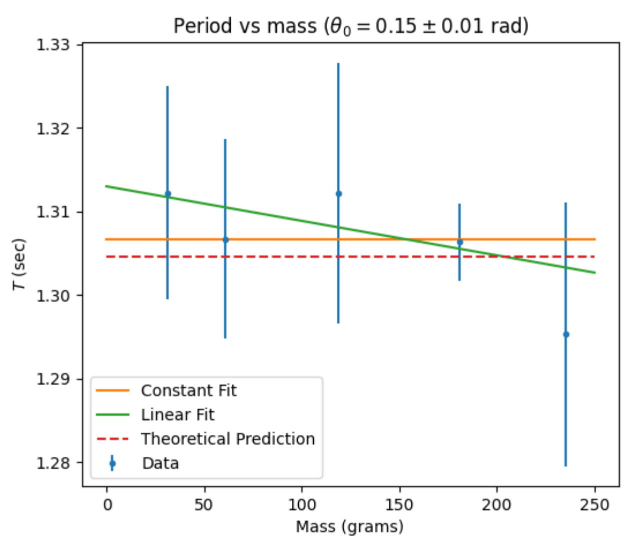


Figure 4: Plot of period vs mass data. The abnormally large uncertainties on points near the theoretical prediction may suggest that my length measurements were not accurate enough for these data points.

Model Type	a (sec)	b (sec/g)	χ^2_{red}	Fit Probability
$T = a$	1.307 ± 0.004	N/A	0.21	36.2%
$T = a + bm$	1.31 ± 0.01	$(-4 \pm 6) \cdot 10^{-5}$	0.15	79.6%

Table 2: Table of data for model fits for period vs mass.

According to the data T is independent of m, but the extremely large uncertainties suggest that this result is pushing the limits of what my experirental apparatus can measure.

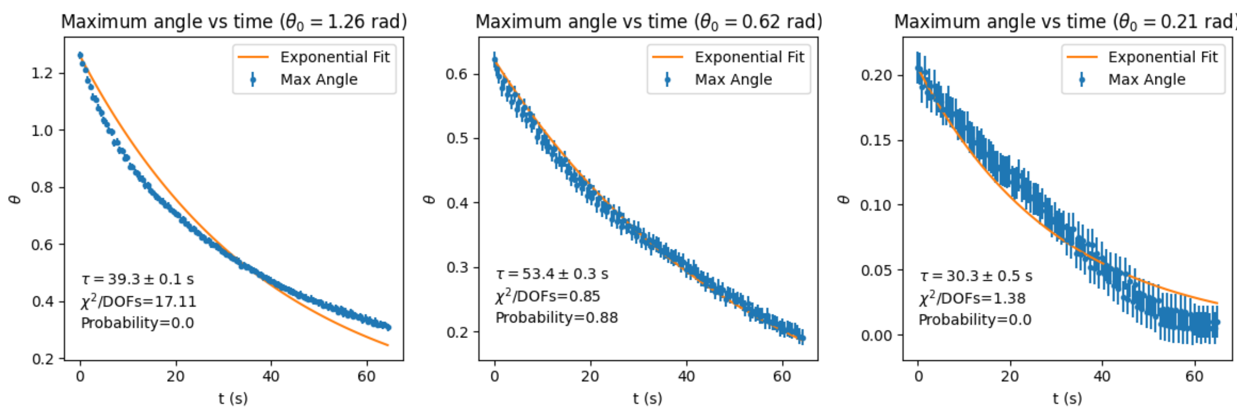
3.3 Analysis of τ

3.3.1 Functional Form of Decay

To model exponential decay, I considered only peaks and troughs of the data set. In particular, according to eq. (1), if the pendulum is at a turning point in its swing,

$$\theta(t) = \theta_0 e^{-t/\tau} \tag{6}$$

In this section, I performed a χ^2_{red} fit on three different peak / trough data sets. The results are included below:



(a) **(b)** **(c)**

Figure 5: Plot of the data peaks for three different scales of initial angles. For the largest angle, τ is clearly changing. For the smallest, the data appears very messy. There is a large discrepancy between the magnitude of "left" peaks versus "right" peaks which messes with the exponential fit. For the middle regime, the data appears to be well modelled by an exponential fit.

The decay is clearly exponential for angles on the order of 0.6 radians, so I will stick to this regime.

3.3.2 τ vs L

For this section, I attempted to model the τ vs L data with constant, linear, and quadratic relationships.

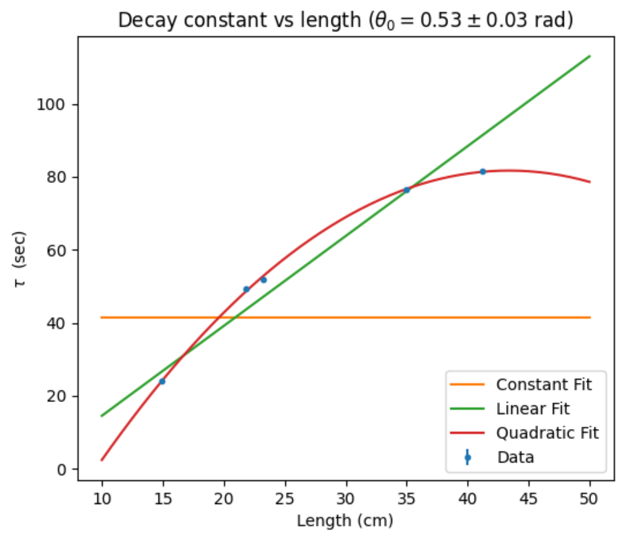


Figure 6: A plot of τ vs length including three possible models. The quadratic model is clearly superior at modellin the data.

Model Type	a (sec)	b (sec/cm)	c (sec/cm^2)	χ^2_{red}	Fit Probability
$\tau = a$	41.3 ± 0.2	N/A	N/A	3070	0%
$\tau = a + bL$	10.1 ± 0.5	2.46 ± 0.02	N/A	202	0%
$\tau = a + bL + cL^2$	-52 ± 2	6.2 ± 0.2	-0.071 ± 0.003	3.70	6.0%

Table 3: Table of data for model fits for tau vs length data. Note lengths are measured in centimeters for the purpose of this table

3.3.3 τ vs m

Based on the shape of this curve, I attempted to model the relationship with

$$\tau = C(m - m_0)^{1/3} + \tau_0 \tag{7}$$

where C, m_0, and τ_0 are some unknown parameters. The best fit values were $C = 4.45 \pm 0.04$, $m_0 = 76 \pm 2$, and $\tau_0 = 31.1 \pm 3$.

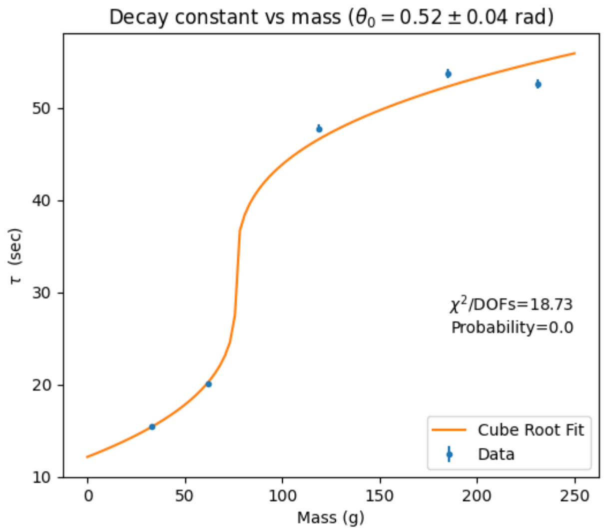


Figure 7: A plot of τ vs m with an attempt at a model. The model captures the main features of the curve but fails to accurately predict τ.

4 Discussion and Conclusion

Section 3.2.1 demonstrates a clear quadratic dependence of T on θ_0 for $0 \leq T \leq \frac{\pi}{2}$, as this was by far the best fit with $\chi^2_{\text{red}} = 1.47$ and a best fit probability of 11.7%, refuting the theory's prediction that T is independent of θ_0. However, since quadratic functions are approximately constant for small inputs, this does not necessarily refute eq. (2) for all angles. In fact, for angles around 0.17 radians, I found that T obeys eq. (5) for $C = 2.03 \pm 0.01$ and $k = 0.509 \pm 0.004$ with $\chi^2_{\text{red}} = 1.36$ and a best fit probability of 25%. Since my model accounts for deviation from a constant value of $T(\theta_0)$, these uncertainties show that the combined values of 2 and 0.5 for the coefficient and power cannot quite be right.

The result from Section 3.2.2 of a constant fit with a χ^2_{red} of 0.21 strongly suggests that something is wrong with my uncertainty estimations. I believe that my length measurements were much less accurate in this section than I had hoped. If eq. (5) accurately predicts the expected period of the pendulum given $L + D$, which I have shown it does, then data points near the red line should have error bars much smaller than those farther away, if my length measurements are accurate. However, this does not seem to be the case for the first few data points. A possible source of this large error in L is the blurriness of the video that I tracked. Unfortunately, I am limited by computing power so I cannot use high resolution video for motion tracking. To fix this issue in the future I could use an opaque ruler for scale reference instead of a transparent one, or clearly mark the beginning and end of the ruler with colours that contrast against my desk. Since this issue does not seem to be present in my other analyses, it may also stem from the fact that I was using new clay to form the pendulum for the mass varying sections. Since this clay ball was not dry like the main one I used, it was much more prone to variations in D. I could fix this by pre-preparing my masses, and making sure they all have very similar $L + D$ values, but this would require a lot more clay. I could also use traditional weights instead of clay, but this method would be quite costly. All that being said, I think my data still clearly demonstrates the independence of T from m, as even the slope of the linear fit is within uncertainty from 0.

My apparatus refutes exponential decay for large angles. Figure (5a) clearly displays a varying value of τ for an initial angle of 1.26 radians. For the middle range of angles an exponential model seems appropriate, as my data set had a $\chi^2_{\text{red}} = 0.85$ and a goodness of fit probability of 85%. However, as I mentioned earlier, in general there was an angle between my phone camera and the pendulum's swinging plane. This can affect the actual registered value of the peak and cause a discrepency between peaks to the left and peaks to the right. This effect is most obvious for small release angles since the angle difference due to the phone's tilt is proportionally much larger for small peaks as compared to large peaks. The effect is clearly visible in Figure (5c) as it displays peaks alternating from low and high values. However, the χ^2_{red} of 1.38 still leads me to believe that the decay is exponential.

My data also demonstrates a dependence of τ on both L and m. In Section 3.3.2 I found a quadratic fit for τ vs L with a χ^2_{red} of 3.70 and a best fit probability of 6.0%. However, I was not as successful in modelling the relationship between τ and m. The best fit I could find was eq. (7) which still has a χ^2_{red} of 18.7. My lack of success in this regard may be more due to the fact that I was unable to account for the slight variations in release angle and string length while taking these data sets like I did for the period, as it is not clear how τ depends on θ_0.

5 Appendix

References

[1] Brian Wilson. "PHY324 Pendulum Project - 2022". In: (2022). Accessed March 6th, 2022.

[2] *Try tracker online. Tracker Video Analysis and Modeling Tool for Physics Education. (n.d.).* `https://physlets.org/tracker/`. Retrieved January 10, 2022.

[3] Chris E. Kuyatt Barry N. Taylor. "Guidelines for Evaluating and Expressing the Uncertainty of NIST Measurement Results". In: *NIST Technical Note 1297* (1994). Accessed March 6th, 2022.