Homework 2

Topological Data Analysis with Persistent Homology

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Exercise 8. *Show that the topological spaces $\mathbb{R}^n$ and $\mathcal{B}(0,1) \subset \mathbb{R}^n$ are homeomorphic.*

Let f be the map

$$\begin{aligned} f : \mathcal{B}(0,1) &\to \mathbb{R}^n \\ x &\mapsto \begin{cases} \dfrac{x}{\|x\|} \tan\left(\dfrac{\pi}{2}\|x\|\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \end{aligned}$$

If we take two different $x_1 \neq x_2$:

- if one is multiple of the other: they will have different modules, because $\tan\left(\frac{\pi}{2}\|x_1\|\right) \neq \tan\left(\frac{\pi}{2}\|x_2\|\right)$
- if one is not multiple of the other: they will continue not being multiple of the other.

So the function is injective.

For each point $y \neq 0 \in \mathbb{R}^n$, we can reach it by taking $x = \dfrac{y}{\|y\|}\dfrac{2}{\pi}\arctan\|y\|$. So the function is surjective. We conclude f is bijective.

Considering the above, we arrive with the following inverse function:

$$\begin{aligned} f^{-1} : \mathbb{R}^n &\to \mathcal{B}(0,1) \\ y &\mapsto \begin{cases} \dfrac{y}{\|y\|}\dfrac{2}{\pi}\arctan\|y\|, & y \neq 0 \\ 0, & y = 0 \end{cases} \end{aligned}$$

The two functions are clearly continuous out of zero. In order to check if they are continuous at zero, we will consider the module of the functions.

$$
\begin{aligned}
\|f(x)\| &= \begin{cases} \left\| \dfrac{x}{\|x\|} \tan\left(\dfrac{\pi}{2}\|x\|\right) \right\|, & x \neq 0 \\ 0, & x = 0 \end{cases} \\
&= \begin{cases} \tan\left(\dfrac{\pi}{2}\|x\|\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \\
&\to_{x\to 0} 0
\end{aligned}
$$

The same for f^{-1}. So the functions are continuous at zero, therefore continuous. This way, f is a homeomorphism.

Exercise 11. *Show that* $[0,1)$ *and* $(0,1)$ *are not homeomorphic.*

Suppose there is a homeomorphism f between $[0,1)$ and $(0,1)$. So

$$
g : [0,1)\backslash\{0\} = (0,1) \to (0,1)\backslash\{f(0)\}
$$

must be a homeomorphism too. $(0,1)$ has one connected component, and $(0,1)\backslash\{f(0)\}$, because of $f(0) \in (0,1)$, has two connected components. This is a contradiction.