Homework 7

Topological Data Analysis with Persistent Homology

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Exercise 37. *Let X, Y be two closed and bounded subsets of $\mathbb{R}^n$. Show that, for every $t \geq 0$, the thickenings satisfy*

$$d_H(X^t, Y^t) \leq d_H(X, Y).$$

Give an example for which $d_H(X^t, Y^t) < d_H(X, Y)$.

Suppose, by contradiction, $d_H(X^t, Y^t) > d_H(X, Y)$. That is,

$$\max\{d_H(X^t; Y^t), d_H(Y^t; X^t)\} > \max\{d_H(X; Y), d_H(Y; X)\}$$

We will have

$$\max\{d_H(X^t; Y^t), d_H(Y^t; X^t)\} > d_H(X; Y), d_H(Y; X)$$

So, without loss of generality, we can consider $d_H(X^t; Y^t) > d_H(X; Y)$ is true. This is the same as

$$\begin{aligned} \sup_{z \in Y^t} \inf_{w \in X^t} \|w - z\| &> \sup_{y \in Y} \inf_{x \in X} \|x - y\| \\ &\geq \inf_{x \in X} \|x - y\|, \quad \forall y \in Y \end{aligned}$$

Because the sets are closed, there must exist $z \in Y^t$ such that

$$\begin{aligned} \inf_{w \in X^t} \|w - z\| &= \sup_{z \in Y^t} \inf_{w \in X^t} \|w - z\| \\ &> \inf_{x \in X} \|x - y\|, \quad \forall y \in Y \end{aligned}$$

Consider this $z \in Y^t$. So there must exist $y_0 \in Y$ such that $\|z - y_0\| \leq t$. This way, there exists $x_0 \in X$ which attains the infimum $\inf_{x \in X} \|x - y_0\| = \|x_0 - y_0\|$. Now, consider the convex combination between $x_0 \in X$ and $y_0 \in Y$ such that we end up with the point $x_1 \in X^t$ with $\|x_0 - x_1\| = t$, that is, $\|x_1 - y_0\| = \|x_0 - y_0\| - t$. This way,

$$\begin{aligned}\|x_1 - z\| &\leq \|x_1 - y_0\| + \|y_0 - z\| \\ &\leq \|x_1 - y_0\| + t \\ &= \|x_0 - y_0\|\end{aligned}$$

We now can conclude that

$$\begin{aligned}\inf_{w \in X^t} \|w - z\| > \inf_{x \in X} &\|x - y_0\| \\ &= \|x_0 - y_0\| \\ &\geq \|x_1 - z\| \\ &\geq \inf_{w \in X_t} \|w - z\|\end{aligned}$$

This is a contradiction. So

$$d_H(X^t, Y^t) \leq d_H(X, Y).$$

As an example of $d_H(X^t, Y^t) < d_H(X, Y)$, let's consider the closed and bounded sets in $\mathbb{R}$ $A = \{0\} \cup [4, 5]$ and $B = [1, 2] \cup \{6\}$. The Hausdorff distance between A, B can be calculated as 2. If $t = 1$, the sets will be $A^1 = [-1, 1] \cup [3, 6]$ and $B^1 = [0, 3] \cup [5, 7]$, and the Hausdorff distance will be 1. So $d_H(A^1, B^1) < d_H(A, B)$.