Homework 3

Topological Data Analysis with Persistent Homology

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Proposition 3.8. *Let $f\colon \mathbb{R}^n \to X$ be a continuous map. Then f is homotopic to a constant map.*

Exercise 12. *Prove the previous proposition.*

Consider the continuous application:

$$\begin{aligned} F\colon \mathbb{R}^n \times [0,1] &\longrightarrow X \\ (x,t) &\longmapsto f(xt) \end{aligned}$$

We have $F(\cdot,0)\colon x \to f(0)$ constant and $F(\cdot,1) = f$.

Exercise 13. *Let $f\colon \mathbb{S}_1 \to \mathbb{S}_2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map.*

Take x_0 outside of the image of f but on $\mathbb{S}_2$, that is, $x_0 \in \mathbb{S}_2$ such as $x_0 \notin f(\mathbb{S}_1)$. Our constant map will be $g\colon x \mapsto -x_0$.

Consider the following continuous application:

$$\begin{aligned} F\colon \mathbb{S}_1 \times [0,1] &\longrightarrow \mathbb{S}_2 \\ (x,t) &\longmapsto \frac{(1-t)f(x) - tx_0}{\|(1-t)f(x) - tx_0\|} \end{aligned}$$

We are doing a convex combination between the points of the image of f and our point $-x_0$. Because x_0 is not on the image, the convex combination never passes through the origin, hence the denominator is never zero, therefore F is continuous.

Note that $F(\cdot, 0) = f$ and $F(\cdot, 1)\colon x \mapsto -x_0$.

Exercise 14. *Show that being homotopic is a transitive relation between maps: for every triplet of maps $f, g, h\colon X \to Y$, if f, g are homotopic and g, h are homotopic, than f, h are homotopic.*

Because f, g are homotopic, we know there must exist a homotopy. Let's call it $F\colon X \times [0,1] \to Y$. The same for g, h: $G\colon X \times [0,1] \to Y$. Consider now the following continuous application:

$$\begin{aligned} H\colon X \times [0,1] &\longrightarrow Y \\ (x,t) &\mapsto \begin{cases} F(x, 2t), & t \in [0, 0.5) \\ G(x, 2t-1), & t \in [0.5, 1] \end{cases} \end{aligned}$$

For every $t \in [0, 0.5) \cup (0.5, 1]$, because F, G are both continuous. When $t = 0.5$, we will end up with $F(x, 2t) = F(x, 1) = G(x, 0) = G(x, 2t-1)$, for every x, so the continuity also holds for $t = 0.5$. We conclude H is continuous. We can deduce now that $H(\cdot, 0) = F(\cdot, 0) = f$ and $H(\cdot, 1) = G(\cdot, 1) = h$. Therefore, f, h are homotopic.

Exercise 16. *Classify the letters of the alphabet into homotopy equivalence classes.*

- The class of circles:

$$\{\text{A}, \text{D}, \text{O}, \text{P}, \text{R}\}$$

- The class of "two holes":

$$\{\text{B}, \text{Q}\}$$

- The class of points:

$$\{\mathrm{C}, \mathrm{E}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{I}, \mathrm{J}, \mathrm{K}, \mathrm{L}, \mathrm{M},$$
$$\mathrm{N}, \mathrm{S}, \mathrm{T}, \mathrm{U}, \mathrm{V}, \mathrm{W}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$$