Homework 4

Topological Data Analysis with Persistent Homology

Lucas Emanuel Resck Domingues
Professor: Raphaël Tinarrage

Escola de Matemática Aplicada
Fundação Getulio Vargas

January 31, 2021

Exercise 20. *What is the Euler characteristic of a sphere of dimension 1? 2? 3?*

- A sphere of dimension 1 is a circunference, so its triangulation can be writen with 3 vertices and simplicial complex

$$K = \{[0], [1], [2], [0, 1], [0, 2], [1, 2]\}$$

  The Euler characteristic is $3 - 3 = 0$.

- A sphere of dimension 2 is a conventional sphere in $\mathbb{R}^3$. We will create a simplicial complex that looks like a tetrahedron, and its topological realization will be homeomorphic to the sphere. $V = \{0, 1, 2, 3\}$,

$$\begin{aligned} K = \{&[0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3], [0, 1], [0, 2], \\ &[0, 3], [1, 2], [1, 3], [2, 3], [0], [1], [2], [3]\} \end{aligned}$$

  The Euler characteristic is $4 - 6 + 4 = 2$.

- A sphere of dimension 3 is more difficult to assimilate. We will add one vertex and write down all possibilities of simplices of dimension 3: $V = \{0, 1, 2, 3, 4\}$,

$$[1, 2, 3, 4], [0, 2, 3, 4], [0, 1, 3, 4], [0, 1, 2, 4], [0, 1, 2, 3]$$

  If we construct the resulting simplicial complex of it (consisting of subsets), we will have

  - 5 simplices with dimension 3
  - $\dfrac{5 \cdot 4 \cdot 3}{3!}$ simplices with dimension 2 (combinations)

- $\dfrac{5 \cdot 4}{2!}$ simplices with dimension 1
- 5 simplices with dimension 0

Therefore, the Euler characteristic is $5 - 10 + 10 - 5 = 0$.

Exercise 21. *Using the previous exercise, show that $\mathbb{R}^3$ and $\mathbb{R}^4$ are not homeomorphic.*

Suppose, by contradiction, that $\mathbb{R}^3 \simeq \mathbb{R}^4$, that is, there exists a homeomorphism f between these two sets. So

$$g\colon \mathbb{R}^3\backslash\{0\} \longrightarrow \mathbb{R}^4\backslash\{f(0)\},\ x \longmapsto f(x)$$

is a homeomorphism too. We can deduce there exist a homeomorphism

$$h\colon \mathbb{R}^3\backslash\{0\} \longrightarrow \mathbb{R}^4\backslash\{0\},\ x \longmapsto f(x) - f(0)$$

so we are good to go.

We already know that the unity sphere of dimension n is homotopic equivalent to $\mathbb{R}^{n+1}\backslash\{0\}$. Hence, $\mathbb{S}_2 \approx \mathbb{R}^3\backslash\{0\}$ and $\mathbb{S}_3 \approx \mathbb{R}^4\backslash\{0\}$. Because $\mathbb{R}^3\backslash\{0\} \simeq \mathbb{R}^4\backslash\{0\}$ (by initial assumption), we have $\mathbb{R}^3\backslash\{0\} \approx \mathbb{R}^4\backslash\{0\}$ and, therefore, $\mathbb{S}_2 \approx \mathbb{S}_3$.

Two homotopic spaces must have the same Euler characteristic, but we already know $\chi(\mathbb{S}_2) = 2$ and $\chi(\mathbb{S}_3) = 0$. This contradiction leads us to the fact that $\mathbb{R}^3 \not\simeq \mathbb{R}^4$.