solutions to transportation problems

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1 L1

1. scalse 1 处于分子尺度，粒子非常稀疏，体微元包含的粒子数的变化dN与体微元有关,也就是$dN = dN(dV)$,因而$\frac{dN}{dV} \neq constant$，也就是密度$\rho$在这个尺度会随体微元的不同而不同.scale 2 处于连续尺度，体元宏观无穷小，而微观无穷大，在此尺度定义密度有：1)定义于一点处，因为体元宏观无穷小，在宏观上就是一点，2)在定义点处连续，因为微观无穷大使体元包含足够多的粒子，这使得附近点之间的粒子数平均不会发生突变，因而密度在空间上是连续分布的.scale 3 处于宏观尺度,这一个尺度上的体元已经感知到了密度的宏观变化

2.

$$\frac{d\alpha}{dt} = \frac{dx/dy}{dt} = \frac{\partial u}{\partial y} \tag{1}$$

$$\frac{d\beta}{dt} = \frac{dy/dx}{dt} = \frac{\partial v}{\partial x} \tag{2}$$

so

$$\frac{d\alpha + d\beta}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \tag{3}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4}$$

3.

$$\tau_x = \frac{\partial u}{\partial y} = -\frac{2U_0 y}{b^2} \tag{5}$$

$$\tau_x\big|_{y=\frac{b}{2}} = -\frac{U_0}{b} \tag{6}$$

4.

$$u = cy \tag{7}$$
$$v = cx \tag{8}$$

$$\tau_x = \mu \frac{\partial \mathbf{v}}{\partial y} = \mu \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{9}$$

$$\tau_y = \mu \frac{\partial \mathbf{v}}{\partial x} = \mu \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} 0 \\ c \end{pmatrix} \tag{10}$$

let $\mathbf{e} = \frac{\mathbf{j}+\mathbf{i}}{\sqrt{\mathbf{2}}}$,

$$\begin{aligned} \tau_{xy} &= \mu \frac{\partial \mathbf{v}}{\partial e} && (11) \\ &= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \cos(\mathbf{e}, \mathbf{i}) + \frac{\partial \mathbf{v}}{\partial y} \cos(\mathbf{e}, \mathbf{j}) \right) && (12) \\ &= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial \mathbf{v}}{\partial y} \frac{1}{\sqrt{2}} \right) && (13) \\ &= \frac{\mu}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} \right) && (14) \\ &= \frac{c\mu}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (15) \end{aligned}$$

5.

$$2\sigma l = \rho g h l w \tag{16}$$
$$\Rightarrow h = \frac{2\sigma}{\rho g w} \tag{17}$$

$\sigma = 0.123(1 - 0.00139T)$ so
$h = 9.37mm$

6.

$$\sigma = 0.025N/m \tag{18}$$

$$2 \cdot 2\pi r \sigma = \Delta p \pi r^2 \tag{19}$$
$$\Rightarrow \Delta p = \frac{4\sigma}{r} \tag{20}$$
$$\Rightarrow \Delta p = 50Pa \tag{21}$$

7. $\rho = 996kg/m^3, h = 1mm$

$$\pi \frac{d^2}{4}\rho g h = \sigma \pi d \tag{22}$$

$$\Rightarrow d = \frac{4\sigma}{\rho g h} = 2.989cm \tag{23}$$

$$\tag{24}$$

8.

$$t = \frac{D_c - D_r}{2} = 0.01cm \tag{25}$$

$$\begin{aligned} F &= \tau A & (26)\\ &= \tau \pi D_r L & (27)\\ &= \mu \frac{v}{t} \pi D_r L & (28)\\ &= \frac{0.85 \times 1000 \times 3.7 \times 10^{-4} \times 0.15 \times 0.3602 \times 3.14 \times \pi}{1 \times 10^{-4}} & (29)\\ &= 1676\mathrm{N} & (30) \end{aligned}$$

9.

$$G = F \tag{31}$$

$$\begin{aligned} mg &= \tau A & (32)\\ &= \mu \frac{v}{t} A & (33)\\ &= \mu \frac{v}{t} \pi D_r L & (34) \end{aligned}$$

10.

$$\begin{aligned} dM &= r dF & (35)\\ &= r \tau dA & (36) \end{aligned}$$

$$\tau = \mu \frac{v}{t} = \mu \frac{\omega r}{t} \tag{37}$$

$$dA = 2\pi r \cdot r dx \tag{38}$$

$$r = x \sin \alpha \tag{39}$$

$\Rightarrow$

$$M = \int_0^{D/2} r\mu \frac{r\omega}{h} 2\pi r \frac{dr}{\sin\alpha} \tag{40}$$

$$= \frac{\pi\mu\omega D^4}{32h\sin\alpha} \tag{41}$$

2 L2

1.

$$\begin{cases} \dfrac{dp}{dr} = f = \rho a \\ a = kr \\ kR = g \end{cases} \tag{42}$$

$\Rightarrow$

$$\frac{dp}{dr} = \frac{\rho g r}{R} \tag{43}$$

$\Rightarrow$

$$p = \frac{\rho g r^2}{2R} + p_{atm} \tag{44}$$

so

$$p|_{r=R} = \frac{\rho g R}{2} + p_{atm} \tag{45}$$

$$\approx \frac{\rho g R}{2} \tag{46}$$

$$= 176Mpa \tag{47}$$

2.

$$\rho g \Delta h = \Delta p \tag{48}$$

$$\Rightarrow \Delta h = \frac{\Delta p}{\rho g} \tag{49}$$

$\Rightarrow$

$$\Delta h_{water} = 10.33m \tag{50}$$

$$\Delta h_{sea} = \Delta h_{water}/1.025 = 10.08m \tag{51}$$

$$\Delta h_{Hg} = \Delta h_{water}/13.6 = 0.8m \tag{52}$$

3. let Hg,water,oil denoted by "H, w, o" respectively

$$p_A + \rho_o g h_o + \rho_w g h_w = p_{atm} + \rho_H g h_H \tag{53}$$
$$\Rightarrow p_A - p_{atm} = g(\rho_H h_H - \rho_o h_o - \rho_w h_w) \tag{54}$$
$$\tag{55}$$

4. the resultant accelaration is gravity and inertial

$$\mathbf{r} = \mathbf{g} - \mathbf{a} \tag{56}$$

now the isobars and the direction of the pressure gradient is depict as follow

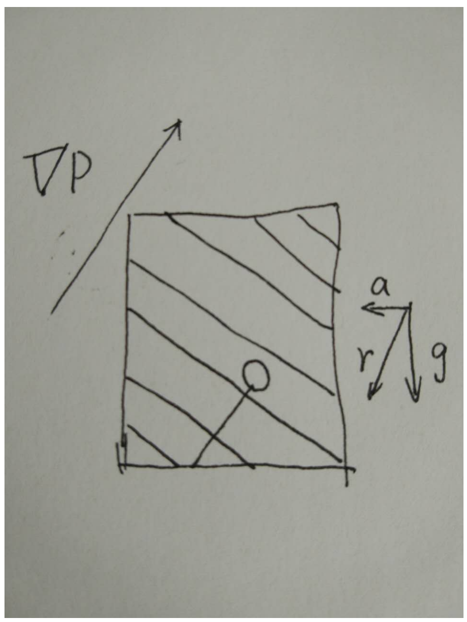

图 1: the balloon

5.

$$dF_y = p_y dA \tag{57}$$

$$
\begin{cases}
dA = 2\pi r' r d\theta \\
r' = r\sin\theta \\
p_y = p\cos\theta \\
p = \rho g h' \\
h' = h - r\cos\alpha + r\cos\theta
\end{cases} \tag{58}
$$

$\Rightarrow$

$$
\begin{aligned}
dF_y &= \rho g(h - r\cos\alpha + r\cos\theta)\cos\theta 2\pi r\sin\theta r d\theta && (59)\\
&= 2\pi\rho g r^2 (h - r\cos\alpha + r\cos\theta)\sin\theta\cos\theta d\theta && (60)
\end{aligned}
$$

$$
\begin{aligned}
F_y =& 2\pi\rho g r^2 \int_\alpha^\pi (h - r\cos\alpha + r\cos\theta)\sin\theta\cos\theta\, d\theta && (61)\\
& && (62)
\end{aligned}
$$

let$F_y(h) = 0$, comes the requiring

$$
h = \frac{2r}{3\sin^2\alpha} + \frac{2r\cos^3\alpha}{3\sin^2\alpha} + r\cos\alpha \tag{63}
$$

6.

$$
\begin{cases}
F = \rho g h A \\
A = \pi r^2
\end{cases} \tag{64}
$$

and the acting point is

$$
\Delta y_a = \frac{I_{yy}}{hA} = \frac{\pi r^2}{4hA} \tag{65}
$$

the momentum equilibrium

$$
F\Delta y_a = Pr \tag{66}
$$

$\Rightarrow$

$$
\begin{aligned}
P &= \frac{F\Delta y_a}{r} && (67)\\
&= \frac{\rho g \pi r}{4} && (68)\\
&= 7.7kN && (69)
\end{aligned}
$$

7.

$$F_p = \rho g \bar{h} A \tag{70}$$
$$A = lw \tag{71}$$

$$\Delta y_a = \frac{I_{yy}}{y_c A} \tag{72}$$
$$I_{yy} = \frac{l^3 w}{12} \tag{73}$$

$$\tau w l_c = F_p \Delta y_a \tag{74}$$

$\Rightarrow$

$$\tau = \frac{F_p \Delta y_a}{w l_c} \tag{75}$$
$$= 145.2kN \tag{76}$$

8.

$$F = \rho g \bar{h} A = 10^{10} N \tag{77}$$
$$y_a = \frac{2}{3} h = 85.3m \tag{78}$$

9.

$$F_{up} = F_{down}$$
$$F + \rho_w g x \left(\frac{4}{3}\pi R^3\right) = rg \left(\frac{4}{3}\pi R^3\right)$$
$$x = \frac{rg\left(\frac{4}{3}\pi R^3\right) - F}{\rho_w g \left(\frac{4}{3}\pi R^3\right)}$$

10.

$$\rho_s v_s = \rho_l v_l \tag{79}$$
$$\Rightarrow \quad \frac{v_l}{v_s} = \frac{\rho_s}{\rho_l} \tag{80}$$
$$\tag{81}$$

$$S = v_s - v_l \tag{82}$$

$$\Rightarrow \quad \frac{1}{2}LL\tan\theta = v_s - v_l = 0.1L^2 \tag{83}$$

$$\Rightarrow \quad \tan\theta = 0.2 \tag{84}$$

3 L3-L4

1. since

   $$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) = \mathbf{g}(\mathbf{c})h(t) \tag{85}$$

   $\Rightarrow$

   $$\mathbf{g}(\mathbf{c}) = \frac{\mathbf{r}}{h(t)} \tag{86}$$

   $\Rightarrow$

   $$\mathbf{c} = \mathbf{g}^{-1}(\frac{\mathbf{r}}{h(t)}) \tag{87}$$

   so

   $$\mathbf{v} = \frac{\partial\mathbf{r}}{\partial t} \tag{88}$$

   $$= \mathbf{g}(\mathbf{c})\dot{h}(t) \tag{89}$$

   $$= \mathbf{g}(\mathbf{g}^{\mathbf{-1}}(\frac{\mathbf{r}}{\mathbf{h(t)}}))\dot{h}(t) \tag{90}$$

2.

   $$g(c) = c \tag{91}$$

   $$h(t) = t^2 \tag{92}$$

   and it is easy to find

   $$g^{-1}(c) = c \tag{93}$$

$$
\begin{aligned}
v &= g(g^{-1}(\frac{x}{h(t)}))\dot{h}(t) && (94)\\
&= g^{-1}(\frac{x}{h(t)})\dot{h}(t) && (95)\\
&= \frac{x}{h(t)}\dot{h}(t) && (96)\\
&= 2t\frac{x}{t^2} && (97)\\
&= 2\frac{x}{t} && (98)\\
& && (99)
\end{aligned}
$$

3. at time t the cth element is at $f(c,t)$, so the temperature of the cth element is

$$T = g(f(c,t),t) \tag{100}$$

so the variation rate is

$$\frac{dT}{dt} = \frac{\partial g}{\partial x}\frac{df}{dt} + \frac{\partial g}{\partial t} \tag{101}$$

4.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{1+t} \\ \dfrac{dy}{dt} = \dfrac{2y}{2+t} \end{cases} \tag{102}$$

$\Rightarrow$

$$\begin{cases} x = c_x(1+t) \\ y = c_y(2+t)^2 \end{cases} \tag{103}$$

with the boundary condition

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{104}$$

the path line through $\mathbf{x}_0$ is

$$\begin{cases} x = x_0(1+t) \\ y = \dfrac{y_0}{4}(2+t)^2 \end{cases} \tag{105}$$

and the streamlines at $t = 0$ is

$$\begin{cases} x = c_x \\ y = 2c_y \end{cases} \tag{106}$$

$\Rightarrow$

$$y = 2\frac{c_y}{c_x}x \tag{107}$$

5.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{108}$$

since

$$\rho = \rho_0(2 - \cos\omega t) \tag{109}$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = -\frac{\partial \rho}{\rho \partial t} \tag{110}$$

$$= \frac{\omega \sin\omega t}{\cos\omega t - 2} \tag{111}$$

$$:= f(t) \tag{112}$$

$\Rightarrow$

$$u = \int f(t)dx \tag{113}$$

$$= f(t)x + C \tag{114}$$

apply the boundary condition$u(0, t) = U$

$$u = f(t)x + U \tag{115}$$

6. df

(a)
$$\int_{A=\partial V} \rho \mathbf{u} \cdot d\mathbf{A} = \int_0^1 dy \int_0^1 dz\, 4x^2 y\Big|_{x=1} - \int_0^1 dy \int_0^1 dz\, 4x^2 y\Big|_{x=0} + \int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=1} - \int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=0} + \int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=1} - \int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=0} = 2 + 0 + \tfrac{1}{4} + 0 + \tfrac{1}{2} + 0 = \tfrac{11}{4}.$$

(b)
$$\nabla \cdot \mathbf{u} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 8xy + xz + 2yz$$

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, (8xy + xz + 2yz) = 8 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{11}{4}.$$

7. df

8. mass conservation

$$\frac{\partial(\rho\delta v)}{\partial t} + \frac{\partial(\rho uh)}{\partial x}\delta x = 0 \tag{116}$$
$$\Rightarrow \frac{\partial(\rho h\delta x)}{\partial t} + \frac{\partial(\rho uh)}{\partial x}\delta x = 0 \tag{117}$$
$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \tag{118}$$

momentum conservation

$$\frac{\partial(\rho u\delta v)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x}\delta x = 0 \tag{119}$$
$$\Rightarrow \frac{\partial(\rho uh)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} = 0 \tag{120}$$

the total force p due to pressure is

$$p = \int_0^h \rho g y dy = \frac{1}{2}\rho g h^2 \tag{121}$$

so the momentum equation is

$$\frac{\partial(\rho uh)}{\partial t} + \frac{\partial(\rho u^2 h + \frac{1}{2}\rho gh)}{\partial x} = 0 \tag{122}$$
$$\Rightarrow = \frac{\partial(uh)}{\partial t} + \frac{\partial(u^2 h + \frac{1}{2}gh)}{\partial x} = 0 \tag{123}$$