Lecture 7-8

Laminar & Turbulence Flow

Learning Objectives

- To understand
  - How to obtained the exact solution for simple flow problems
  - How to simplify the governing equations for creeping flow
  - Stokes drag for creeping flow past a sphere
  - How to obtain the governing equation for boundary layer by applying scaling analysis
  - Solutions for boundary layer flow past a plate
  - Flow past a circular cylinder
  - Pipe flow

Governing Equation

- Governing equation
  - The governing equations of a mathematical model describes how the unknown variables (i.e. the dependent variables) will change.
  - The governing equation for the flow of incompressible Newtonian fluid are:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \vec{V} = 0$$

$$\frac{D\vec{V}}{Dt} = \vec{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V}$$

$$g = -\nabla \phi$$

Without free surface

$$\frac{D\vec{V}}{Dt} = -\frac{1}{\rho} \nabla p^* + \nu \nabla^2 \vec{V}$$

$$p^* = p + \phi$$

Governing Equation

- Governing equation

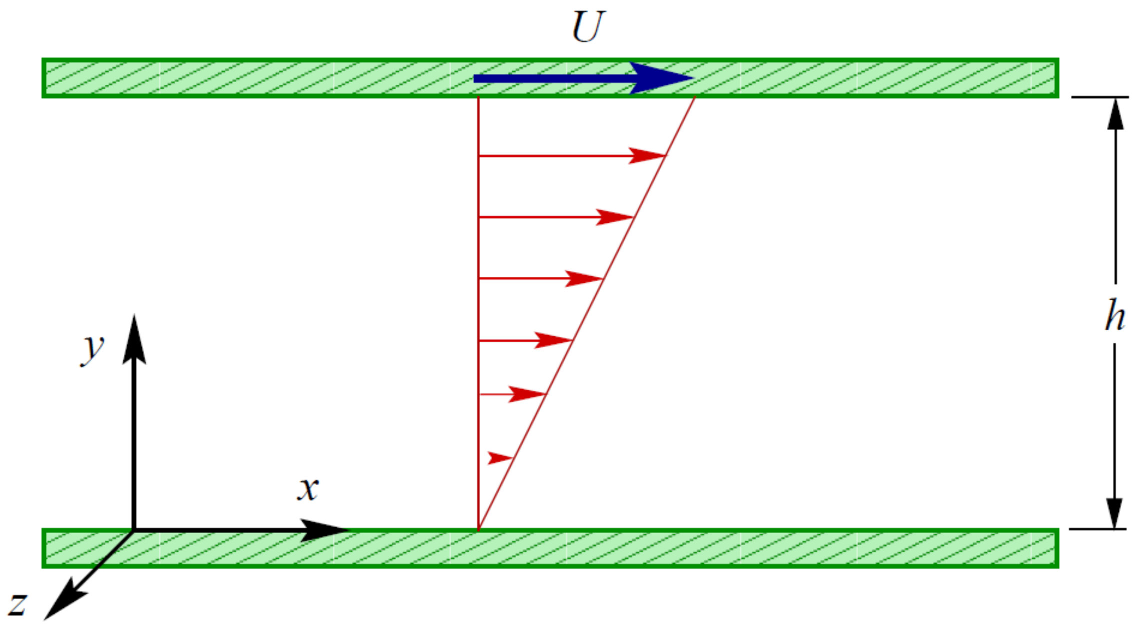
$$\frac{\partial \rho}{\partial t} + \rho \nabla \vec{V} = 0$$

$$\frac{D\vec{V}}{Dt} = \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{V}$$

  - The governing equation for fluid flow is very complicate.
  - Only few very simple flows have analytical solution.
  - The approximated solution is available for some problems with small Reynolds number Re < 1

Plane Couette Flow

Incompressible, steady flow of Newtonian fluid



✓$u = 0,\ v = 0,\ w = 0$ at $y = 0$
✓$u = U,\ v = 0,\ w = 0$ at $y = h$
✓Plate are infinitely long in the x and z directions

Plane Couette Flow

Plate is infinitely long implies there is no reason to expect x and z dependence in any flow variables since there is no way to introduce boundary conditions that can lead to such dependences

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad \frac{\partial p}{\partial x} = 0 \qquad w = 0$$

From continuity equation:

$$\frac{\partial v}{\partial y} = 0$$

$$v = \text{constant},$$

By applying boundary condition

$$v = 0$$

Plane Couette Flow

In the x direction:

$$\frac{\partial^2 u}{\partial y^2} = 0$$

Integrating:

$$\frac{\partial u}{\partial y} = C_1$$

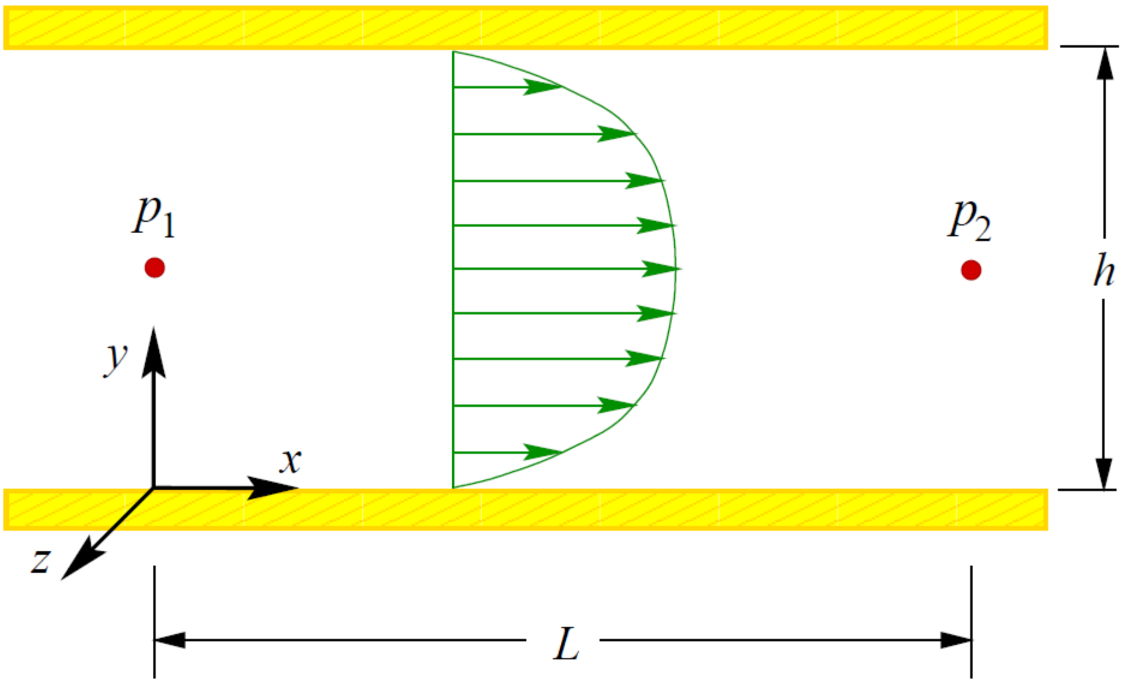
$$u = C_1 y + C_2$$

By applying boundary condition

$$u = \frac{U}{h} y$$

Plane Poiseuille Flow

Incompressible, steady flow of Newtonian fluid



A pressure-driven flow in a duct over a finite length L, but of infinite extent in the z direction. For the flow as shown we assume $p_1 > p_2$ with p_1 and p_2 given, and that pressure is constant in the z direction at each x location.

Plane Poiseuille Flow

Flow varies only in the y direction

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} = 0 \qquad w = 0$$

From continuity equation:

$$\frac{\partial v}{\partial y} = 0$$

$$v = \text{constant},$$

By applying boundary condition

$$v = 0$$

Plane Poiseuille Flow

Momentum equation in the x direction

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{1}{\mu}\frac{p_2 - p_1}{L} = \frac{\Delta p}{\mu L}$$

Integrating:

$$\frac{\partial u}{\partial y} = \frac{\Delta p}{\mu L} y + C_1$$

$$u = \frac{\Delta p}{2\mu L} y^2 + C_1 y + C_2$$

By applying boundary condition

$$u = \frac{\Delta p}{2\mu L} y\left(y - h\right)$$

Plane Poiseuille Flow

Maximum velocity:

$$u_{\max} = -\frac{\Delta p}{8\mu L} h^2 \qquad \text{at } y = \frac{h}{2}$$

$$u_{avg} = \frac{1}{h}\int_0^h u\,dy = \frac{1}{h}\int_0^h \frac{\Delta p}{2\mu L} y\left(y-h\right)dy$$

$$u_{avg} = -\frac{\Delta p}{12\mu L} h^2$$

$$\tau = \mu \frac{\partial u}{\partial y} \qquad \tau\Big|_{y=0} = -\Delta p \frac{h}{2L} \qquad \tau\Big|_{y=h} = \Delta p \frac{h}{2L}$$

Plane Poiseuille Flow

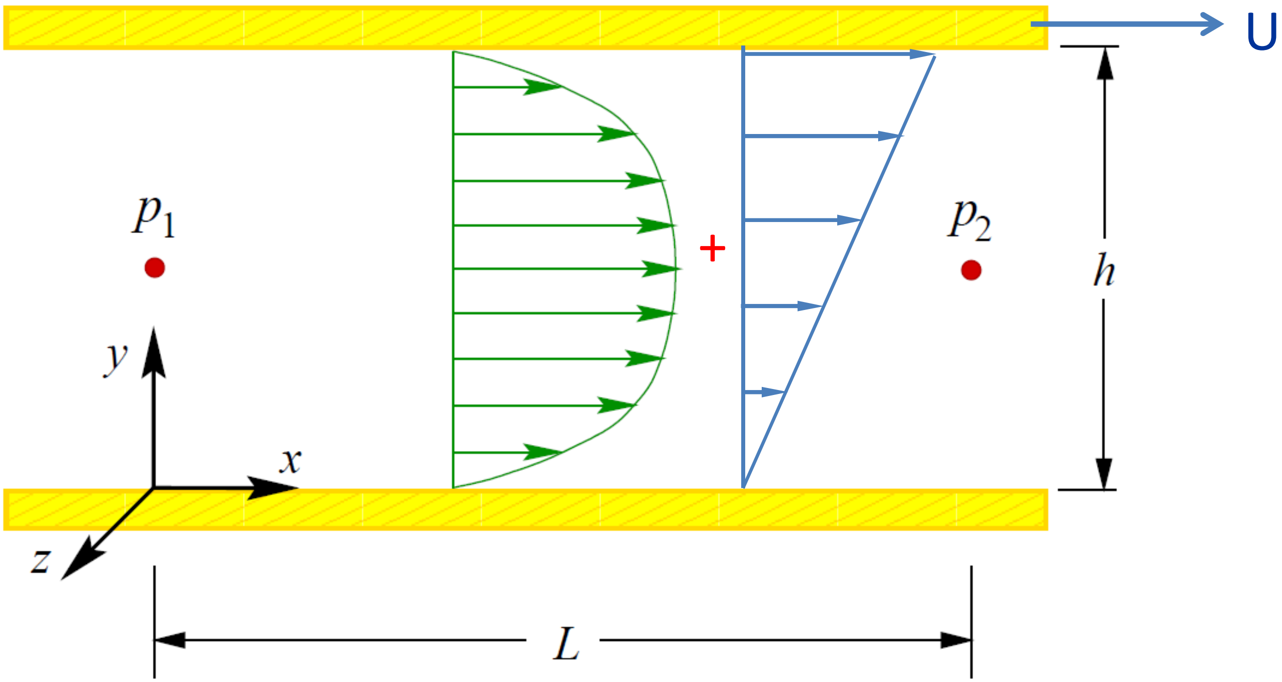
Nondimensionalized equation

$$\bar{u} = \frac{u}{u_{\max}} \quad \text{and} \quad \bar{y} = \frac{y}{h}$$

$$\bar{u} = 4\bar{y} - 4\bar{y}^2$$

Rectilinear Flow Between Parallel Plates

Incompressible, steady flow of Newtonian fluid



A pressure-driven and moving-plate driven flow in a duct over a finite length L, but of infinite extent in the z direction. We assume $p_1 > p_2$ with p_1 and p_2 given, and that pressure is constant in the z direction at each x location.

Rectilinear Flow Between Parallel Plates

Momentum equation in the *x* direction

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} = \frac{1}{\mu}\frac{p_2 - p_1}{L} = \frac{\Delta p}{\mu L}$$

Integrating:

$$\frac{\partial u}{\partial y} = \frac{\Delta p}{\mu L} y + C_1$$

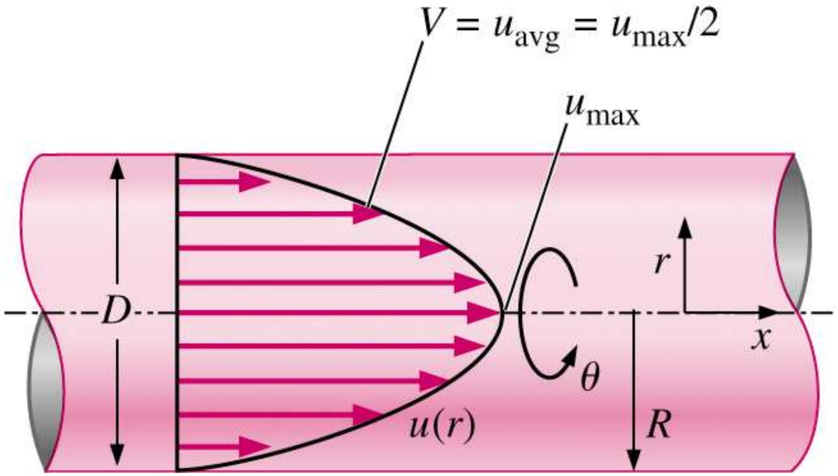
$$u = \frac{\Delta p}{2\mu L} y^2 + C_1 y + C_2$$

By applying boundary condition

$$u = \frac{U}{h} y + \frac{\Delta p}{2\mu L} y\left(y - h\right)$$

Pipe Poiseuille Flow

Incompressible, steady flow of Newtonian fluid



$$r = R, \quad u_x = 0$$

$$r = 0, \quad du_x/dr = 0$$

Steady laminar flow in a long round pipe with an applied constant pressure gradient

$$\frac{\partial p}{\partial x} = \frac{P_2 - P_1}{x_2 - x_1}$$

Pipe Poiseuille Flow

Assumptions

1. The pipe is infinitely long in the x direction
2. The flow is steady
3. This is a parallel flow $u_r = 0$
4. The fluid is incompressible and Newtonian
5. A constant pressure gradient
6. $u_\theta = 0$ and $\partial u_\theta / \partial \theta = 0$
7. Fully developed $\partial u_x / \partial x = 0$

Pipe Poiseuille Flow

Momentum equation in the *x* direction

$$\frac{\partial u_x}{\partial t} + u_r \frac{\partial u_x}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_x}{\partial \theta} + u_x \frac{\partial u_x}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_x}{\partial \theta^2} + \frac{\partial^2 u_x}{\partial x^2}\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating:

$$r\frac{\partial u_x}{\partial r} = \frac{r^2}{2\mu}\frac{\partial p}{\partial x} + C_1$$

$$u_x = \frac{r^2}{4\mu}\frac{\partial p}{\partial x} + C_1 \ln r + C_2$$

Pipe Poiseuille Flow

Applying boundary condition

$$\frac{\partial u_x}{\partial r} = 0 \text{ at } r = 0 \quad \Rightarrow \quad C_1 = 0$$

$$r = R, \quad u_x = 0 \quad \Rightarrow \quad C_2 = -\frac{R^2}{4\mu}\frac{\partial P}{\partial x}$$

The final solution is

$$u_x = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - R^2\right) \qquad u_{\max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 \text{ at } r = 0$$

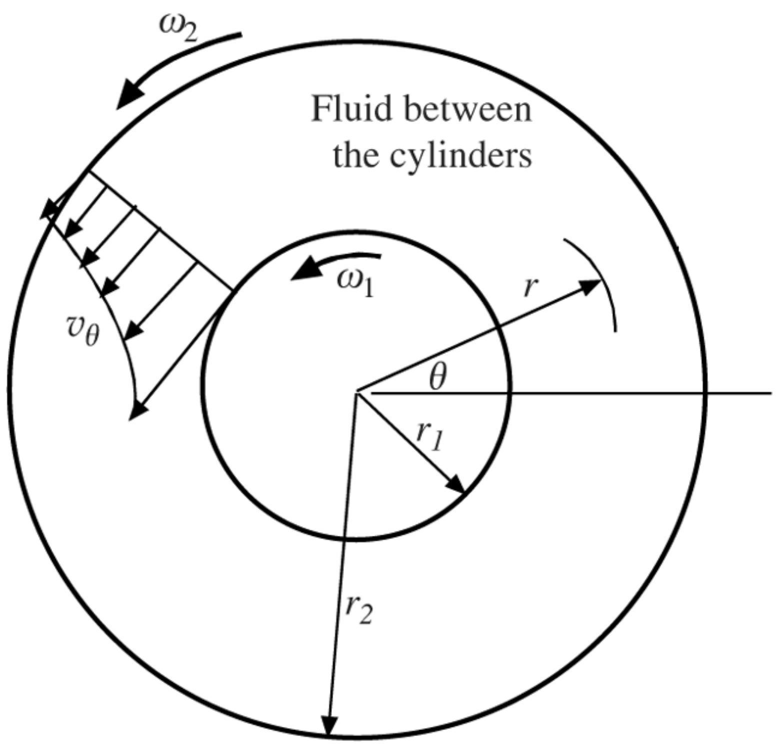
$$Q = \int_0^R u_x 2\pi r dr = \int_0^R \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - R^2\right)2\pi r dr = \frac{\pi}{8\mu}\frac{\partial p}{\partial x}R^4$$

$$u_{avg} = \frac{Q}{\pi R^2} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}R^2$$

$$\tau = \mu\frac{\partial u_x}{\partial r} = \frac{1}{2}\frac{\partial p}{\partial x}r$$

Couette Flow between two Concentric Cylinders

Incompressible, steady flow of Newtonian fluid



2 Dimensional Flow

Couette Flow between two Concentric Cylinders

Momentum equation in the θ direction

$$\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$+\nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}=0$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\left(r v_\theta\right)}{\partial r}\right)=0$$

Couette Flow between two Concentric Cylinders

Integrating

$$v_\theta = Ar + \frac{B}{r}$$

Applying boundary condition

$$A = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} \qquad B = \frac{(\omega_1 - \omega_2) r_1^2 r_2^2}{r_2^2 - r_1^2}$$

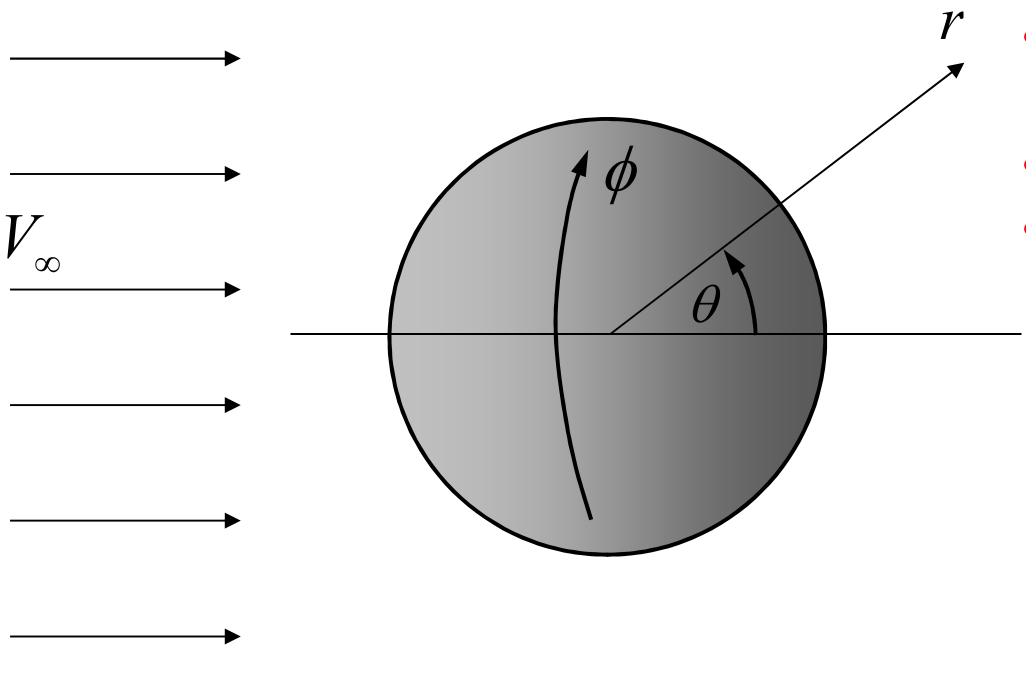
$$v_\theta = \frac{1}{r_2^2 - r_1^2}\left[r\left(\omega_2 r_2^2 - \omega_1 r_1^2\right) - \frac{r_1^2 r_2^2}{r}(\omega_1 - \omega_2) \right]$$

Couette Flow between two Concentric Cylinders

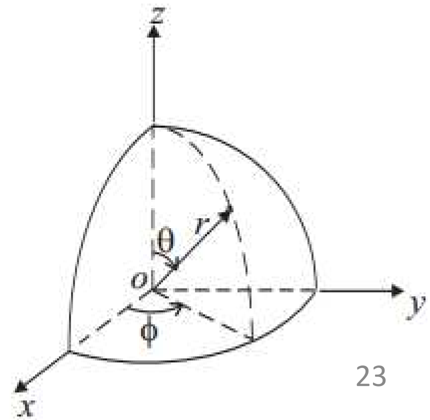
- Special case
  - $\omega_1 \to 0$, $r_1 \to 0$, $v_\theta = \omega_2 r$, steady rotation of cylinder filled with fluid under rigid body rotation
  - $\omega_2 = 0$, $r_2 \to \infty$, $v_\theta = \frac{r_1^2 \omega_2}{r}$, potential vortex driven by rotating cylinder with no-slip boundary condition.
  - Small clearance, $r_2 - r_1 << r_1$, $\omega_2 = 0$, $v_\theta = \omega_1 r_1 \frac{r - r_1}{r_2 - r_1}$, linear Couette flow

Creeping Flow past a Sphere

- Problem Definition
  - Flow past a stationary sphere with radius *R*
  - Creeping flow Re << 1
  - Use Standard Spherical Coordinates, (r, θ, and φ)



- V_∞ is the free stream velocity in Cartesian coordinate
- $v_r = V_\infty \cos\theta$
- $v_\theta = -V_\infty \sin\theta$



Creeping Flow past a Sphere

- Momentum Equation

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \mu \nabla^2 \vec{v}$$

$$\rho \vec{v} \cdot \nabla \vec{v} = -\nabla p + \mu \nabla^2 \vec{v}$$

$$\frac{\rho V_\infty}{R} \qquad \frac{p}{R} \qquad \mu \frac{V_\infty}{R^2}$$

$$\mathrm{Re} = \frac{\frac{\rho V_\infty}{R}}{\mu \frac{V_\infty}{R^2}} = \frac{\rho V_\infty R}{\mu} \ll 1$$

$$\rho \vec{v} \cdot \nabla \vec{v} = -\nabla p$$

Creeping Flow past a Sphere

- Momentum Equation in Spherical Coordinate
  - Further assumptions: axisymmetric flow.
  - Nothing depends on φ.
  - There is no velocity component in the φ direction.

$$\frac{\partial(\bullet)}{\partial\varphi} = 0$$

$$v_{\varphi} = 0$$

Creeping Flow past a Sphere

- Momentum Equation in Spherical Coordinate
  - Radial direction:

$$\frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_r}{r^2} - \frac{2\cot\theta}{r^2}v_\theta\right)$$

  - Azimuthal direction:

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin\theta}\right)$$

Creeping Flow past a Sphere

- Continuity Equation in Spherical Coordinate

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r} = 0$$

- Boundary Condition
  - On the sphere surface $r = R$

$$v_r = 0;\ v_\theta = 0;$$

  - At ∞

$$v_r = V_\infty \cos\theta; \qquad v_\theta = -V_\infty \sin\theta$$

Creeping Flow past a Sphere

- Governing Equations

$$\frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{2v_r}{r} + \frac{v_\theta \cot\theta}{r} = 0$$

$$\frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} - \frac{2v_r}{r^2} - \frac{2\cot\theta}{r^2}v_\theta\right)$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r}\frac{\partial v_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2\sin\theta}\right)$$

- Boundary Condition
  - $r = R$: $v_r = 0$; $v_\theta = 0$;
  - $r = \infty$: $v_r = V_\infty \cos\theta$; $v_\theta = -V_\infty \sin\theta$

Creeping Flow past a Sphere

- Variables

$$v_r(r,\theta),\ v_\theta(r,\theta)\ ),\ p(r,\theta)$$

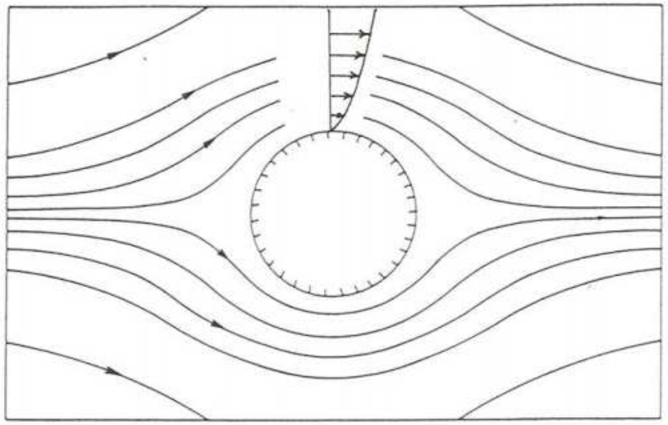
- We can use the method of separation of variables to solve the equations

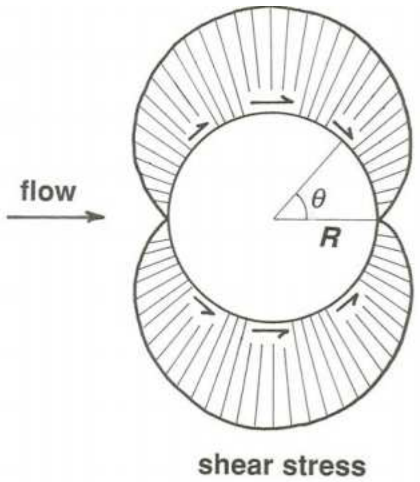
$$v_r(r,\theta) = V_\infty \cos\theta \left(1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\frac{R^3}{r^3}\right)$$

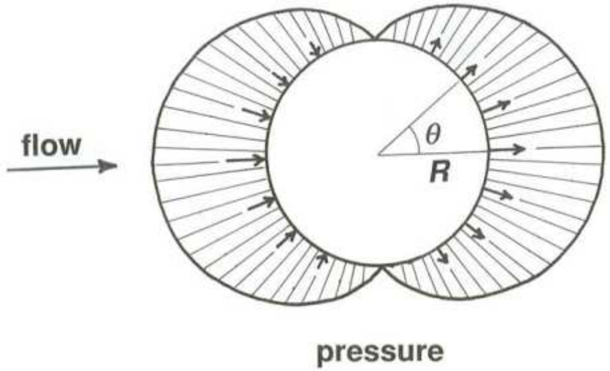
$$v_\theta(r,\theta) = -V_\infty \sin\theta \left(1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\frac{R^3}{r^3}\right)$$

$$p(r,\theta) = -\frac{3}{2}\mu\frac{RV_\infty}{r^2}\cos\theta + p_\infty$$

Creeping Flow past a Sphere







Creeping Flow past a Sphere

- Stress on the surface of sphere

$$\tau_{rr} = -p + 2\mu \frac{\partial v_r}{\partial r}$$

$$\tau_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)$$

$$\tau_{r\varphi} = \mu \left(\frac{\partial v_\varphi}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} \right)$$

  – Simplify the stress expression by applying assumptions

$$\frac{\partial(\bullet)}{\partial \varphi} = 0 \quad \text{and } v_\varphi = 0 \quad \Longrightarrow \quad \tau_{r\varphi} = 0$$

$$\text{at surface } v_r = v_\theta = 0 \quad \Longrightarrow \quad \frac{\partial v_r}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial v_\theta}{\partial \theta} = 0$$

$$\text{continuity equation} \quad \Longrightarrow \quad \frac{\partial v_r}{\partial r} = 0$$

Creeping Flow past a Sphere

- Stress on the surface of sphere

$$\tau_{rr} = -p = \frac{3}{2}\frac{\mu V_\infty}{R}\cos\theta - p_\infty$$

$$\tau_{r\theta} = \mu\frac{\partial v_\theta}{\partial r} = -\frac{3\mu V_\infty}{2R}\sin\theta$$

- Drag on the surface

$$F_D = \int_S \left(\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta\right) dS$$

$$F_D = \int_0^\pi \left(\tau_{rr}\cos\theta - \tau_{r\theta}\sin\theta\right) 2\pi R^2 \sin\theta d\theta$$

$$F_D = 2\pi R^2 \int_0^\pi \frac{3\mu V_\infty}{2}\left(\cos^2\theta - \sin^2\theta\right)\sin\theta d\theta - 2\pi R^2 p_\infty \int_0^\pi \cos\theta\sin\theta d\theta$$

$$F_D = 3\pi\mu V_\infty R\int_0^\pi \sin\theta d\theta = 6\pi\mu V_\infty R$$

Creeping Flow past a Sphere

- Drag coefficient

$$F_D = 6\pi\mu V_\infty R$$

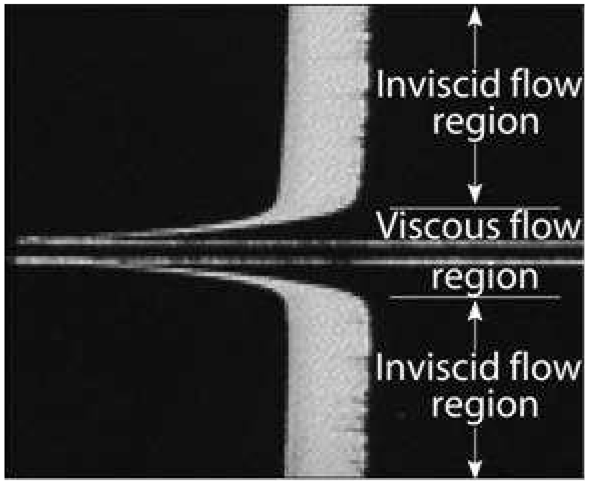
$$C_D = \frac{F_D}{\frac{1}{2}\rho V_\infty^2 \pi R^2} = \frac{12\mu}{\mu V_\infty R} = \frac{24}{\mathrm{Re}}$$

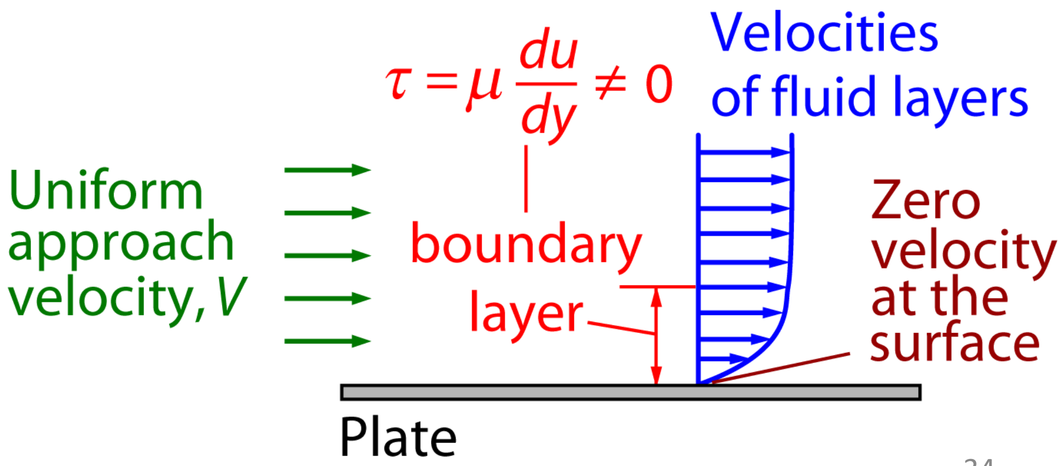
$$\mathrm{Re} == \frac{\mu V_\infty D}{\mu} = \frac{\mu V_\infty 2R}{\mu}$$

Where D is the diameter of sphere

Boundary Layer Theory

- Flow past a infinitely long plate
  - No-slip condition ⇒ fluid has zero velocity at wall
  - Fluid velocity approaches V far away from wall
  - Fluid velocity increases from zero at wall to V far away from wall ⇒ non-zero velocity gradient in a thin layer adjacent to wall ⇒ boundary layer





Boundary Layer Theory

- Boundary layer
  - The layer of fluid in the immediate vicinity of a bounding surface where the effects of viscosity are significant.
  - We define the thickness of this boundary layer δ as the distance from the wall to the point where the velocity is 99% of the "free stream" velocity
  - In fluid dynamics, there are two different types of boundary layer flow: laminar and turbulent
  - In thermal dynamics, we may encounter thermal boundary layer

Boundary Layer Theory

- Boundary layer governing equation
  - Let's start from a two dimensional steady flow of incompressible Newtonian fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Boundary Layer Theory

- Boundary layer governing equation
  - Applying scaling analysis
    - ✓ In the x direction, the length scale is L, which is the reference length of the plate
    - ✓ In the y direction, the length scale is δ, which is the thickness of boundary layer
    - ✓ The velocity scale in the x direction is U_∞, which is the free stream velocity.
    - ✓ The velocity scale in the y direction can be estimate from the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{U_\infty}{L} \sim \frac{v}{\delta}$$

$$v \sim \frac{\delta}{L} U_\infty$$

Boundary Layer Theory

- Boundary layer governing equation
  - Momentum equation in the x direction

$$
\begin{array}{ccccccccc}
u\dfrac{\partial u}{\partial x} & + & v\dfrac{\partial u}{\partial y} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & + & \dfrac{\mu}{\rho}\left(\dfrac{\partial^2 u}{\partial x^2}\right. & + & \left.\dfrac{\partial^2 u}{\partial y^2}\right) \\
\dfrac{U_\infty^2}{L} & & \dfrac{U_\infty^2}{L} & & \dfrac{1}{\rho}\dfrac{p}{L} & & \dfrac{\mu}{\rho}\dfrac{U_\infty}{L^2} & & \dfrac{\mu}{\rho}\dfrac{U_\infty}{\delta^2}
\end{array}
$$

✓ Both convective terms are equally large ~ $U_\infty{}^2/L$

✓ The viscous term with the x-derivatives is much smaller than that with the y-derivatives

✓ The largest of the viscous terms, the convective terms and the pressure term are about the same order

$$\frac{U_\infty^2}{L} \sim \frac{\mu}{\rho}\frac{U_\infty}{\delta^2} \quad \Longrightarrow \quad \frac{\delta}{L} \sim \sqrt{\frac{\mu}{\rho U_\infty L}} = \mathrm{Re}^{-\frac{1}{2}}$$

$$\frac{U_\infty^2}{L} \sim \frac{1}{\rho}\frac{p}{L} \quad \Longrightarrow \quad p \sim \rho U_\infty^2$$

Boundary Layer Theory

- Boundary layer governing equation
  - Momentum equation in the y direction

$$
\begin{array}{ccccccccc}
u\dfrac{\partial v}{\partial x} & + & v\dfrac{\partial v}{\partial y} & = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & + & \dfrac{\mu}{\rho}\left(\dfrac{\partial^2 v}{\partial x^2}\right. & + & \left.\dfrac{\partial^2 v}{\partial y^2}\right) \\
\dfrac{\delta}{L^2}U_\infty^2 & & \dfrac{\delta}{L^2}U_\infty^2 & & \dfrac{U_\infty{}^2}{\delta} & & \dfrac{\mu}{\rho}\dfrac{\delta}{L^3}U_\infty & & \dfrac{\mu}{\rho}\dfrac{U_\infty}{\delta L}
\end{array}
$$

✓ Both convective terms are equally large ~ $\delta U_\infty{}^2/L$

✓ The viscous term with the x-derivatives is much smaller than that with the y-derivatives

✓ The pressure term is much larger than the viscous term in the y direction and the convective term

$$\frac{\text{pressure term}}{\text{x-viscous term}} \sim \frac{U_\infty^2 / \delta}{\dfrac{\mu}{\rho}\dfrac{U_\infty}{\delta L}} = \mathrm{Re} \qquad \frac{\text{pressure term}}{\text{convective term}} \sim \frac{U_\infty^2 / \delta}{\dfrac{\delta}{L^2}U_\infty^2} = \frac{L^2}{\delta^2}$$

Boundary Layer Theory

- Boundary layer governing equation
  - The final equations after dropping the smaller terms

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary condition:

$u = v = 0$ at the surface of plates

$u = u_e$, $v = 0$, $p = p_e$ at the edge of boundary layer

Boundary Layer Theory

- Boundary layer governing equation
  - In the inviscid region, the Euler equation is valid

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Boundary condition:

$u = u_e$, $v = 0$, $p = p_e$ at the edge of boundary layer

$$u_e\frac{du_e}{dx} = -\frac{1}{\rho}\frac{\partial p_e}{\partial x}$$

Boundary Layer Theory

- Blasius' analytical solutions

$$\frac{\delta}{L} \sim \sqrt{\frac{\mu}{\rho U_\infty L}} = \mathrm{Re}^{-\frac{1}{2}}$$

$$\delta \approx 5.0\sqrt{\frac{\mu x}{\rho U_\infty}}$$

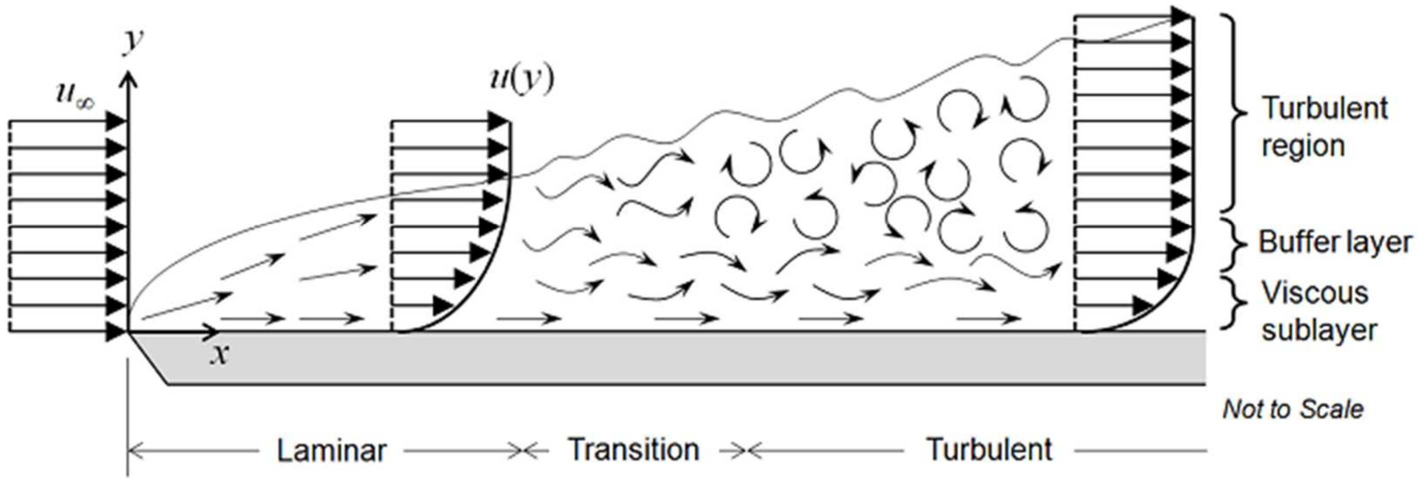
$$\tau = 0.332\mu U_\infty \sqrt{\frac{\rho U_\infty}{\mu x}}$$

$$C_x = \frac{\tau}{\frac{1}{2}\rho U_\infty^{\ 2}} = 0.664\sqrt{\frac{\rho}{U_\infty \mu x}} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

$$C_D = \frac{1.328}{\sqrt{\mathrm{Re}_L}}$$

Boundary Layer Theory

- More on flow past a plate

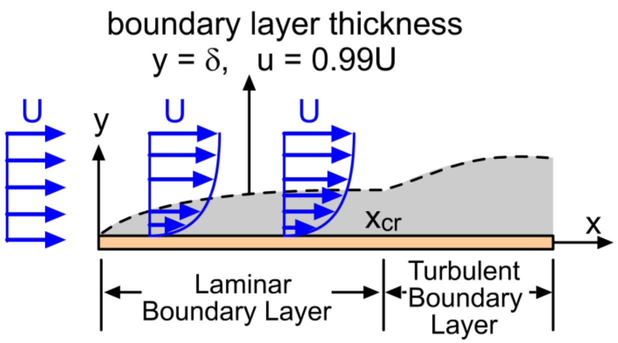


- Boundary layer thickness increase with x
- Flow becomes turbulent with increasing x
- The critical Reynolds number for transition is about 5×10^5

Boundary Layer Theory

- More on flow past a plate

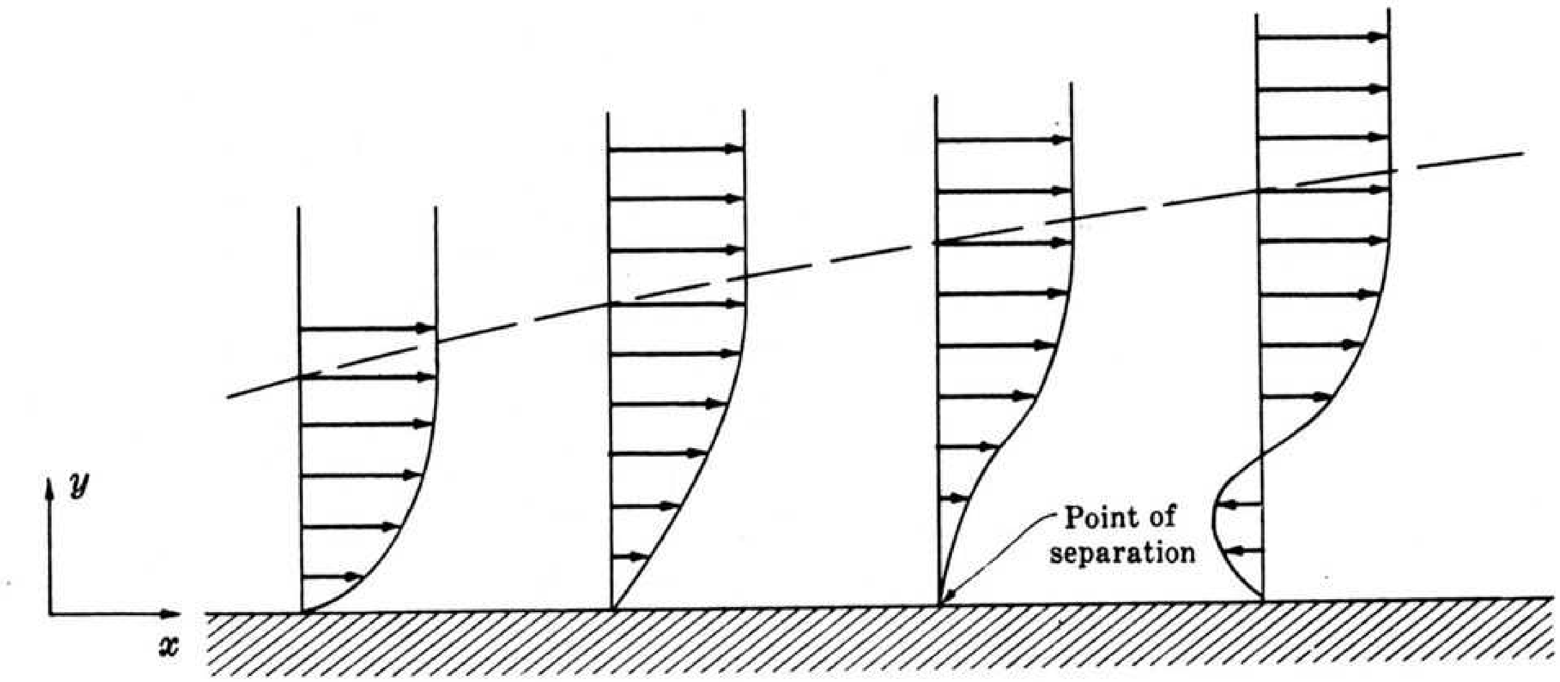
$$\left.\begin{aligned}\frac{\delta}{x} &\approx \frac{5.0}{\sqrt{\mathrm{Re}_x}} \\ C_D &= \frac{1.328}{\sqrt{\mathrm{Re}_L}}\end{aligned}\right\} \quad \mathrm{Re} < 5\times10^5 \quad \text{Laminar flow}$$



$$\left.\begin{aligned}\frac{\delta}{x} &\approx \frac{0.38}{\mathrm{Re}_L^{0.2}} \\ C_D &= \frac{0.074}{\mathrm{Re}_L^{0.2}}\end{aligned}\right\} \quad 5\times10^5 < \mathrm{Re} < 10^7 \quad \text{Turbulent flow}$$

Boundary Layer Theory

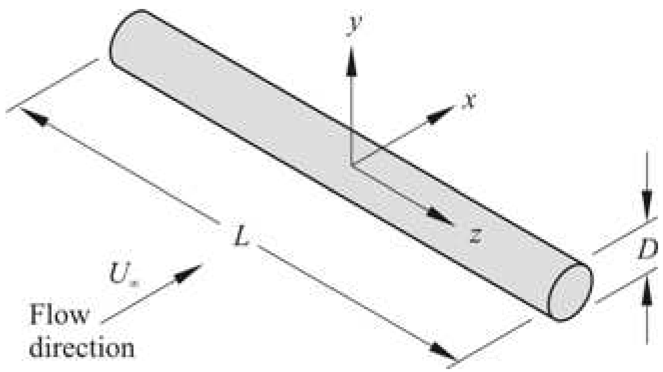
- More on flow past a plate

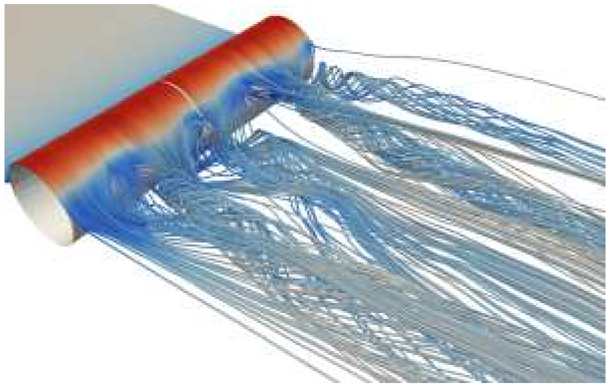


  - $dp/dx > 0$, adverse pressure gradient cause flow separation

Flow Past a Cylinder

- Problem Definition
  - The cylinder is infinitely long $L >> D$
  - Uniform incoming flow velocity U_∞
  - Flow direction is normal to the vertical axis of cylinder
  - Boundary effect on the flow is negligible





Flow Past a Cylinder

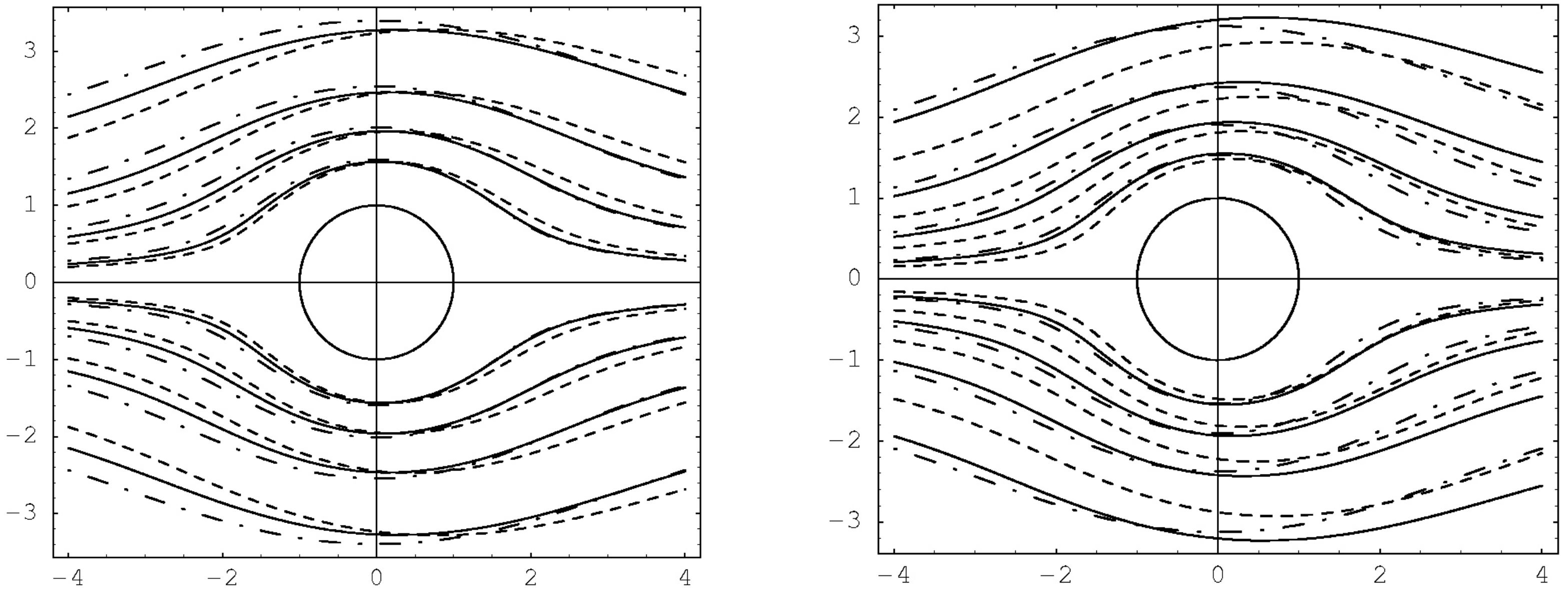
- Introduction
  - Flow past a cylinder is a very classic fluid problem
  - This type of flow occurs in many industrial applications
  - Promote the understanding in the underlying physics behind the flow phenomena
    - ✓ Flow instability
    - ✓ Wake structure
    - ✓ Vortex shedding
    - ✓ Drag, lift

Flow Past a Cylinder

- Flow Feature
  - Creeping Flow Re < 1
    - For flow past sphere, we have Stokes solution
    - For the 2D flow around a circle cylinder, there is no solution of the 2D Navier Stokes flow, this is the "Stokes Paradox".
    - We can not fit the boundary condition far away and the no-slip boundary condition on the cylinder surface at the same time.
    - Lamb obtained the an approximation solution for the 2D flow around a circle cylinder by applying the Oseen's approximation (Lamb, 1911).
    - More accurate results for Re ~ 1 can be obtained by using matched asymptotic expansions (Van Dyke, 1964) or a series truncation method (Mandujano and Peralta-Fabi, 2005).

Flow Past a Cylinder

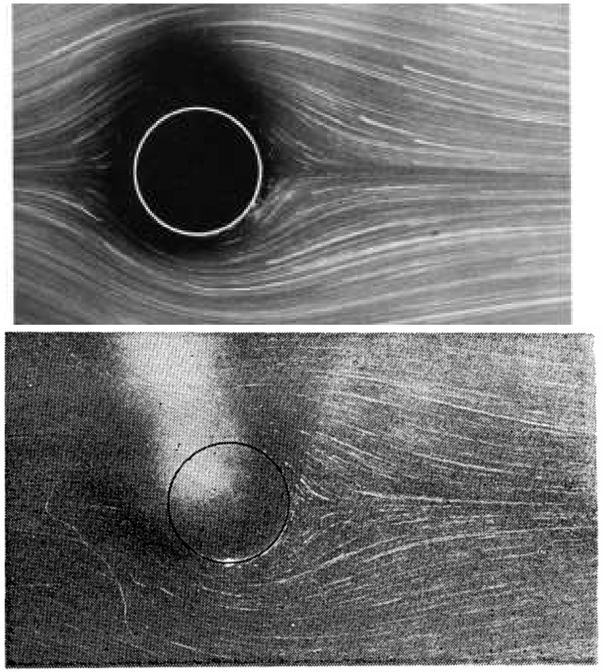
- Flow Feature
  - Creeping Flow Re < 1



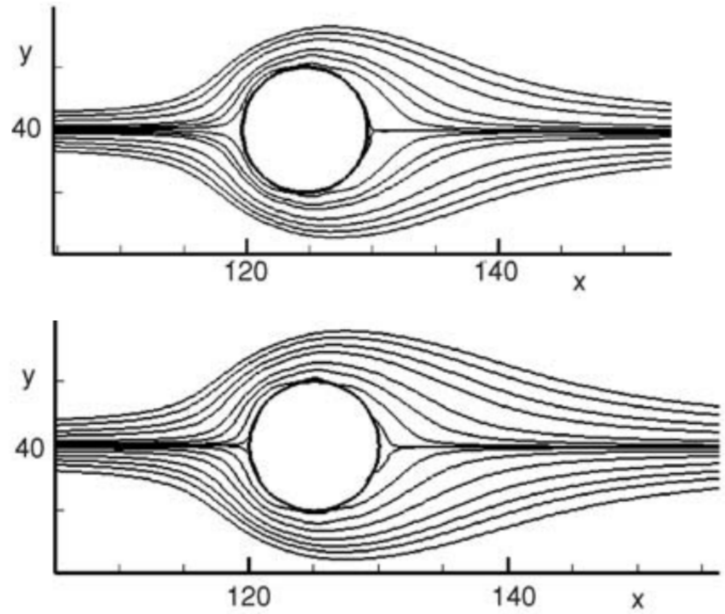
Stream lines for Re = 0.2 (left) and Re = 0.4 (right) for a series truncation method (continuous), Oseen approximation (dashed) and matched asymptotic expansions (dashed-dot).

Flow Past a Cylinder

- Flow Feature
  - Fully Attached Flow $1 < Re < 6$
    - Experimental and numerical solutions
    - Streamlines are fully attached to the surface of the cylinder



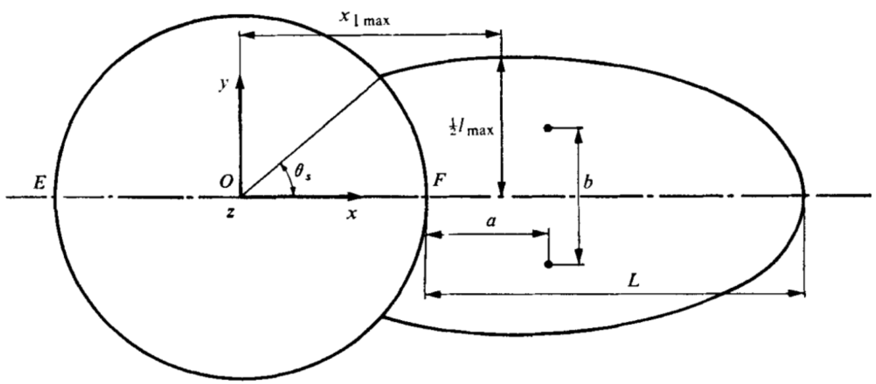
Re = 1.54(top) and Re = 3.64 (below)
Experimental result, Taneda, 1956

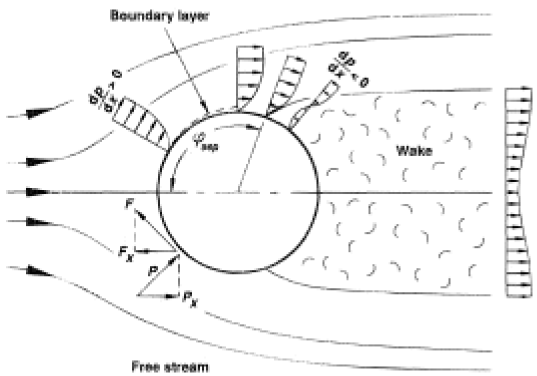


Re = 4 (top) and Re = 5 (below)
Numerical results

Flow Past a Cylinder

- Flow Feature
  - Steady Closed Wake Flow 6 < Re < 49
    - The flow separates on the cylinder surface and the wake behind the cylinder consists of a pair of symmetric contra-rotating vortices on either side of the wake centreline



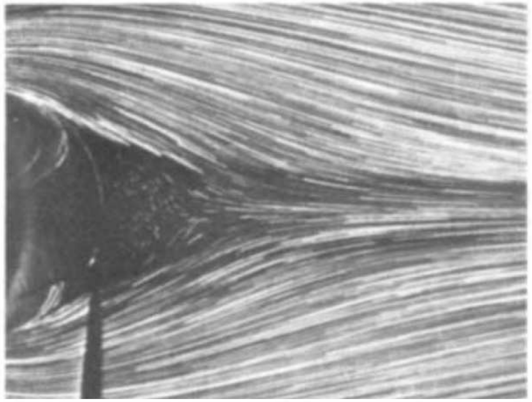


Geometrical parameters of the closed wake.

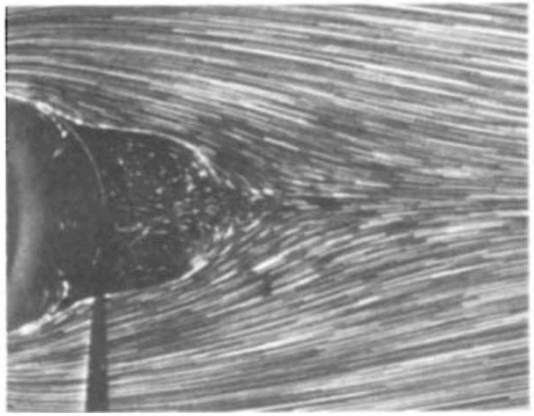
Flow Past a Cylinder

- Flow Feature
  - Steady Closed Wake Flow 6 < Re < 49

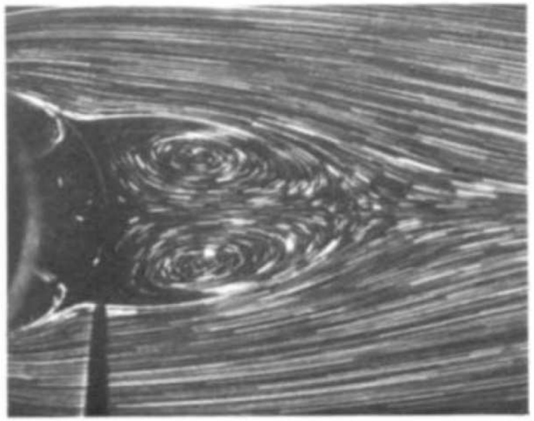
a)



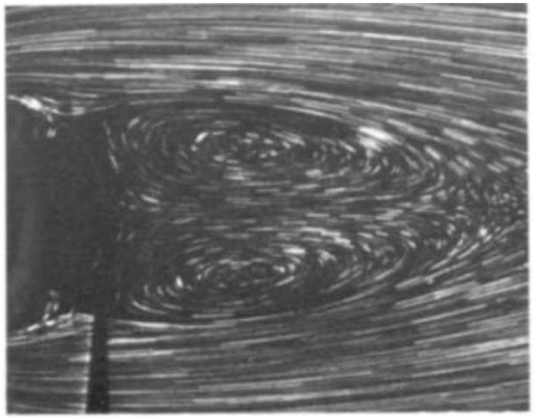
b)



c)



d)



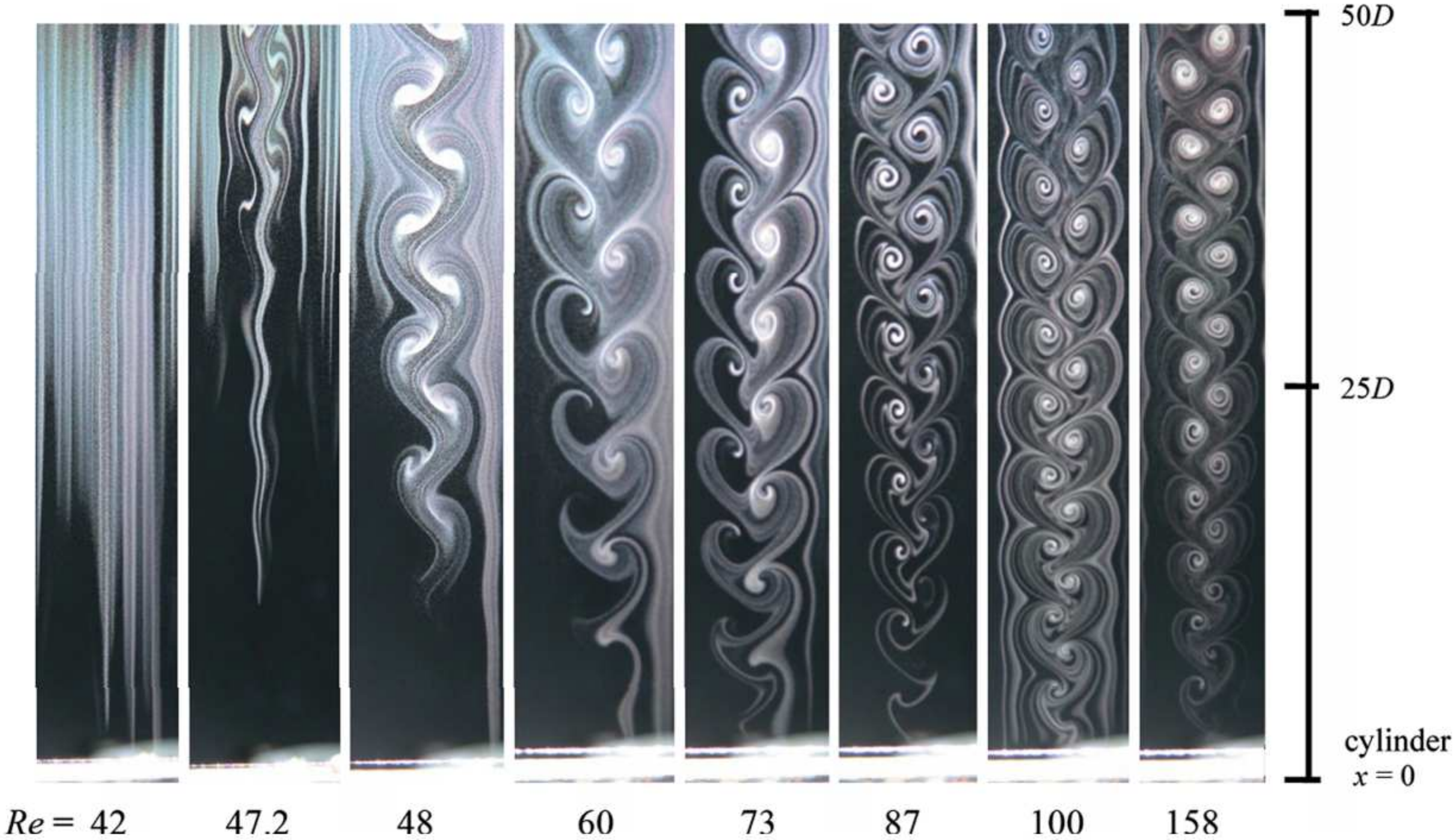
Variation of the closed wake with Re = a) 10.3, b) 16.6, c)25.0, d)35.2; λ=0.07 (blockage). Pictures from Coutanceau and Bouard, 1977, JFM.

Flow Past a Cylinder

- Flow Feature
  - Laminar Vortex Shedding $49 < Re < 180$
    - Flow becomes unsteady,
    - For infinitely long cylinder, the flow is still two-dimensional
    - As the Re is increased further, the vortices are shed alternately from the upper and lower cylinder surface at a definite frequency depending on the Reynolds number (Kármán vortex street).

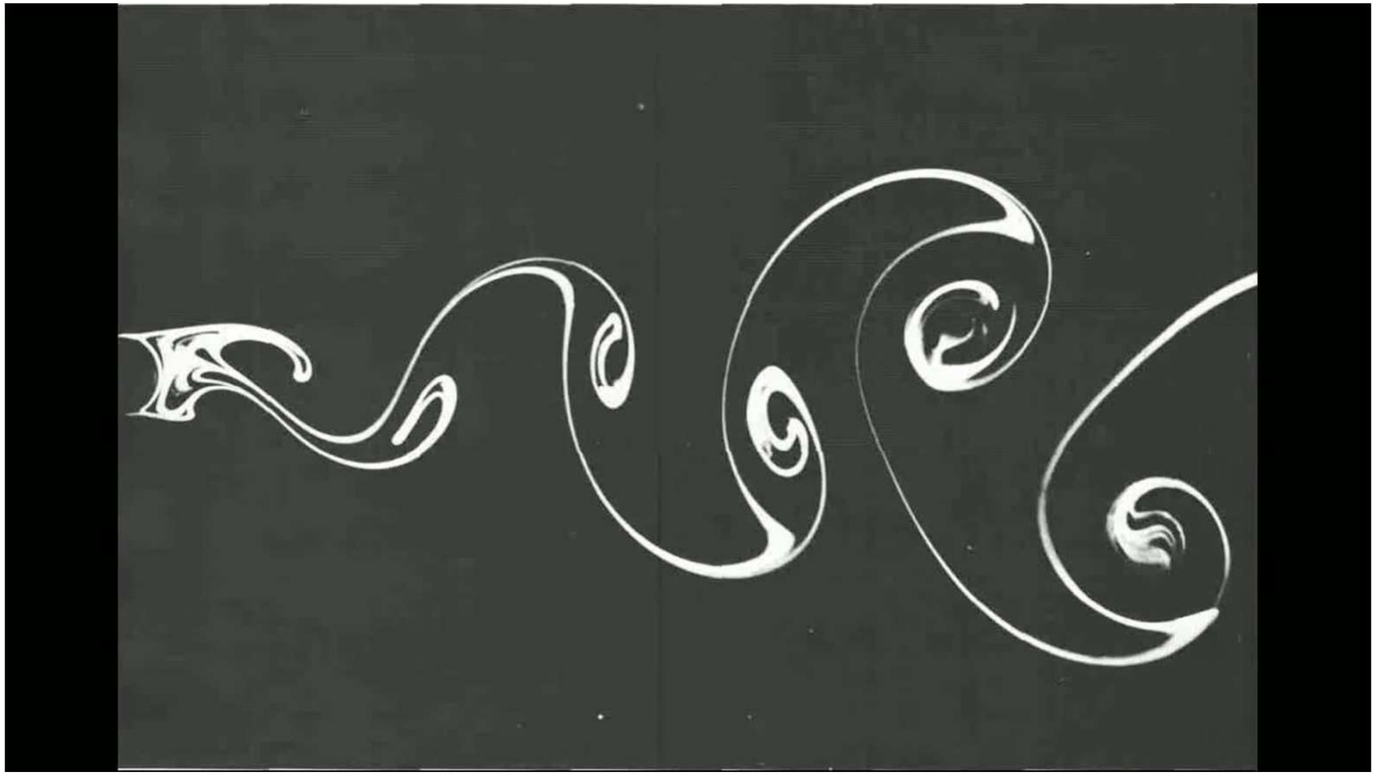
Flow Past a Cylinder

- Flow Feature
  - Laminar Vortex Shedding 49 < Re < 180



Flow Past a Cylinder

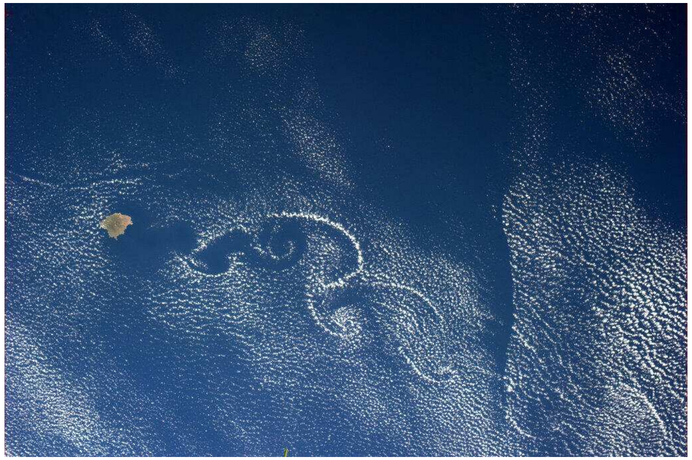
- Flow Feature
  - Laminar Vortex Shedding 49 < Re < 180

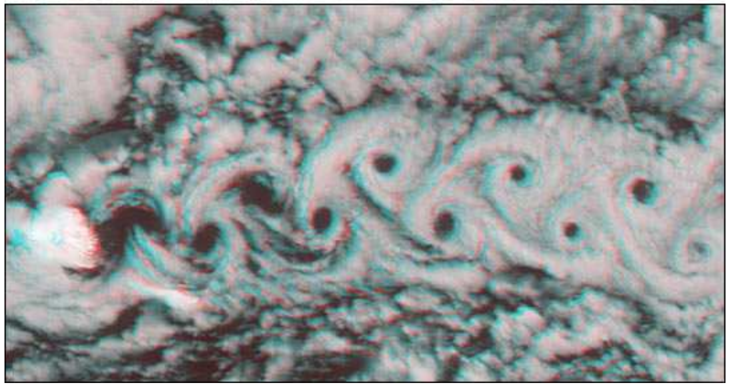


Vortex shedding behind cylinder at Re = 140

Flow Past a Cylinder

- Flow Feature
  - More on Karman Vortex Street

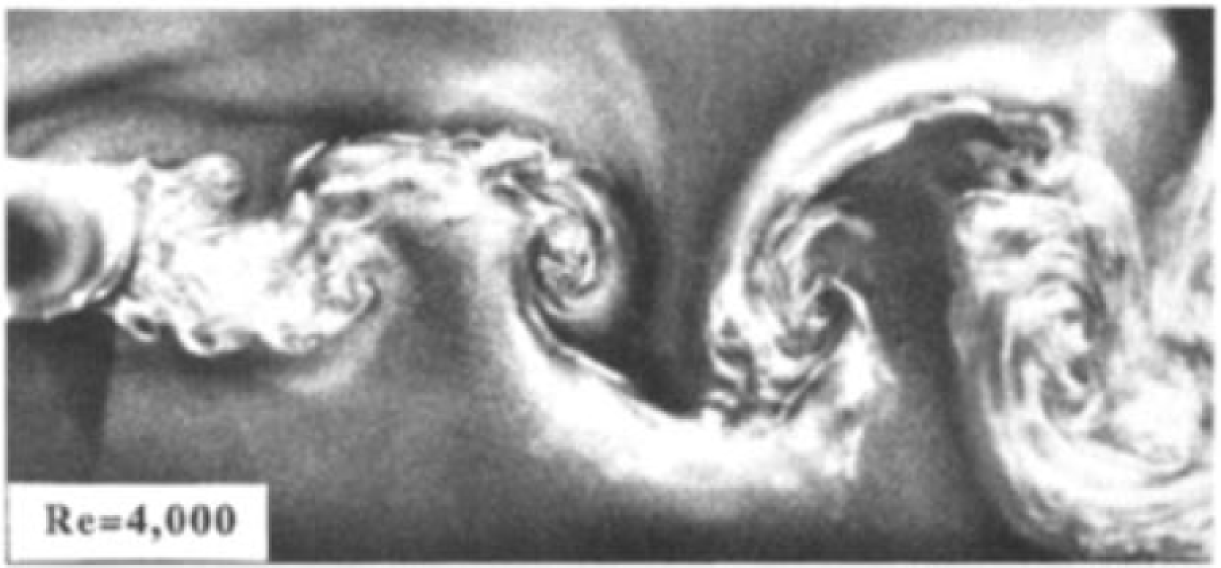




Cloud Karman vortex street. Such wakes often occur downstream of rocky, volcanic islands that rise above the smooth ocean surface and disrupt the atmosphere's boundary layer

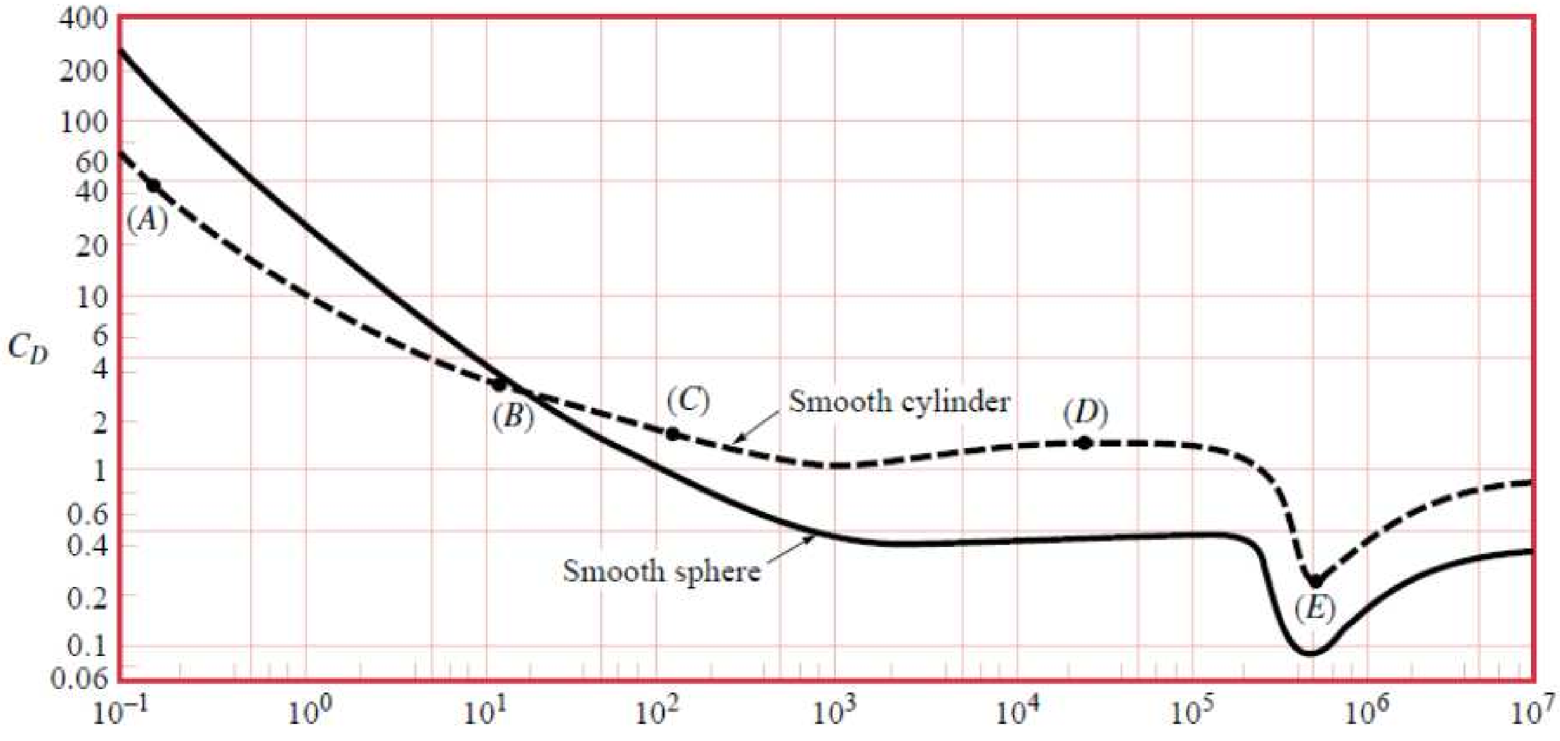
Flow Past a Cylinder

- Flow Feature
  - Re >180
    - Flow become more complicated.
    - Even for an infinite long cylinder, the flow becomes three-dimensional with Re. Wake starts to become turbulent.
    - With a further increase in Re, the flow gradually becomes turbulent.



Flow Past a Cylinder

- Drag coefficient



Sphere

$$C_D = \frac{F_D}{\rho V^2 D^2} = f\left(\text{Re}\right)$$

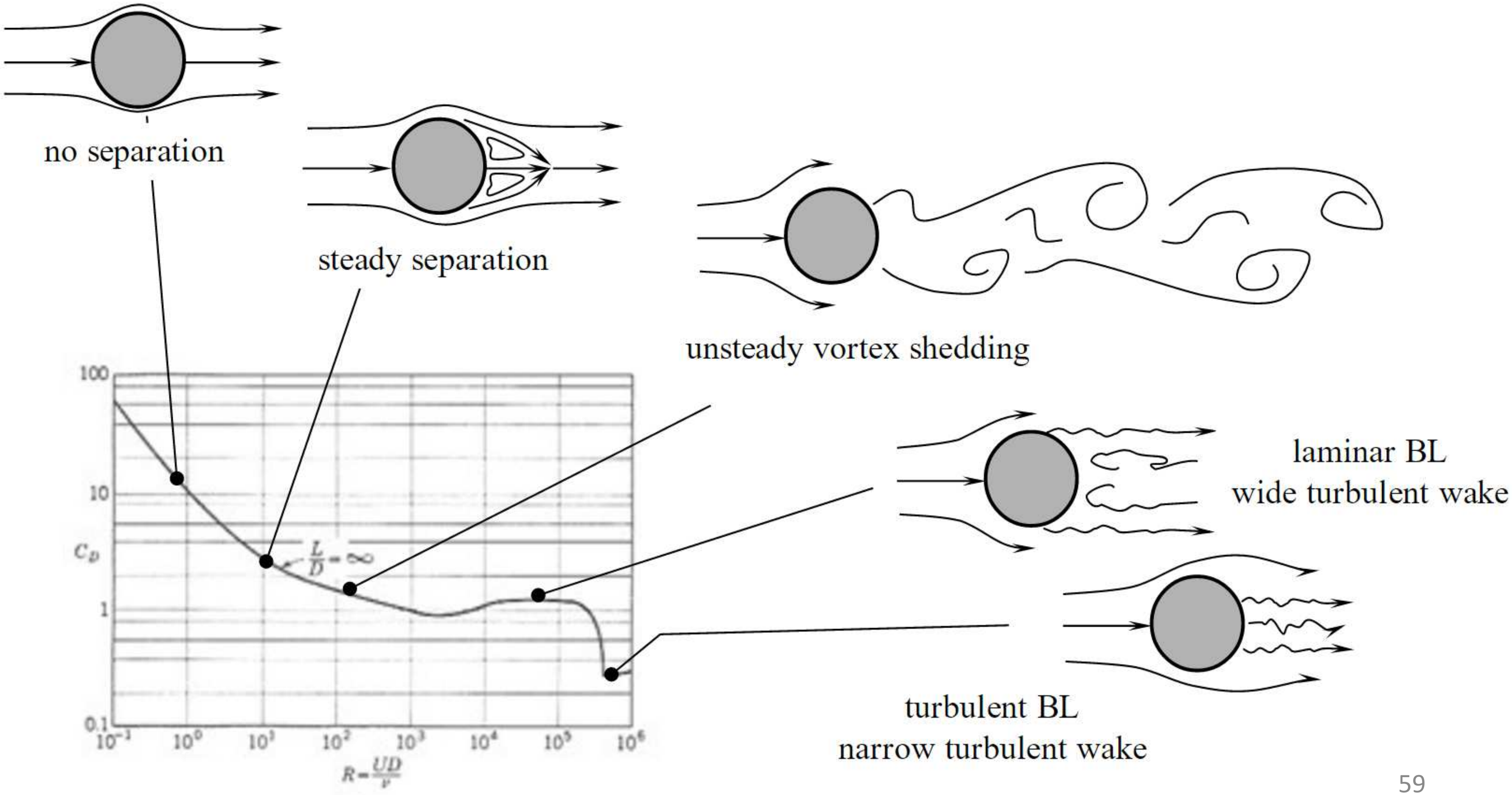
Cylinder

$$C_D = \frac{f_D}{\rho V^2 D} = f\left(\text{Re}\right)$$

f_D is the drag force per unit length

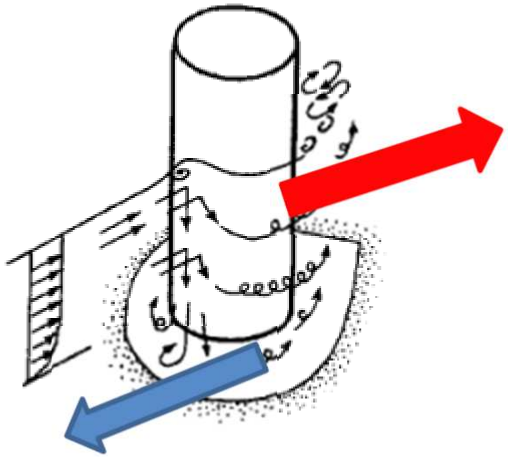
Flow Past a Cylinder

- Flow Feature



Flow Past a Cylinder

- Typhoon and Human Body Weight
  - What's the drag force act on your body



Assumption:
1. Our body has a cylinder shape
2. Ignore the ground effect

Flow Past a Cylinder

- Typhoon and Human Body Weight
  - What's the drag force act on your body
    - ✓ Consider a man with the body weight M_b = 65 kg, body height of H = 1.7 m
    - ✓ The density of the human body is around ρ_b = 1000 kg/m^3, which is almost the same as water.
    - ✓ If he has a cylinder shape, the diameter of the cylinder D is

$$M_b = \rho V = \frac{1}{4}\pi D^2 H \rho$$

$$D = \sqrt{\frac{4M_b}{\pi H \rho}} = 0.221 \text{ m}$$

Flow Past a Cylinder

- Typhoon and Human Body Weight
  - What's the drag force act on your body

✓ Viscosity and density of the air are $\mu = 1.846 \times 10^{-5}$ kg/m/s and ρ_a 1.177 kg/m^3, respectively.

✓ Level 10 Typhoon, velocity is $V = 28$ m/s.

✓ The Reynolds number can be calculated as

$$\mathrm{Re} = \frac{\rho_a V D}{\mu} = \frac{1.177 \times 28 \times 0.221}{1.846 \times 10^{-5}} = 3.394 \times 10^5$$

✓ At this Reynolds number, the drag coefficient for a circular cylinder is about $C_D = 0.8$.

Flow Past a Cylinder

- Typhoon and Human Body Weight
  - What's the drag force act on your body

✓ The drag force acting on the cylinder:

$$C_D = \frac{f_D}{\rho V^2 D}$$

$$f_D = \rho_a V^2 D C_D = 1.177 \times 28^2 \times 0.221 \times 0.8 = 154.83 \text{ N/m}$$

$$F_D = f_D H = 154.83 \times 1.7 = 263.2 \text{ N}$$

✓ Assume the friction coefficient of our feet with the ground is $f_f = 0.26$

✓ The max friction force is:

$$F_f = f_f M_b g = 0.26 \times 65 \times 9.8 = 165.6 \text{ N}$$

$$F_D > F_f$$

The man will be blown away by the wind!!!

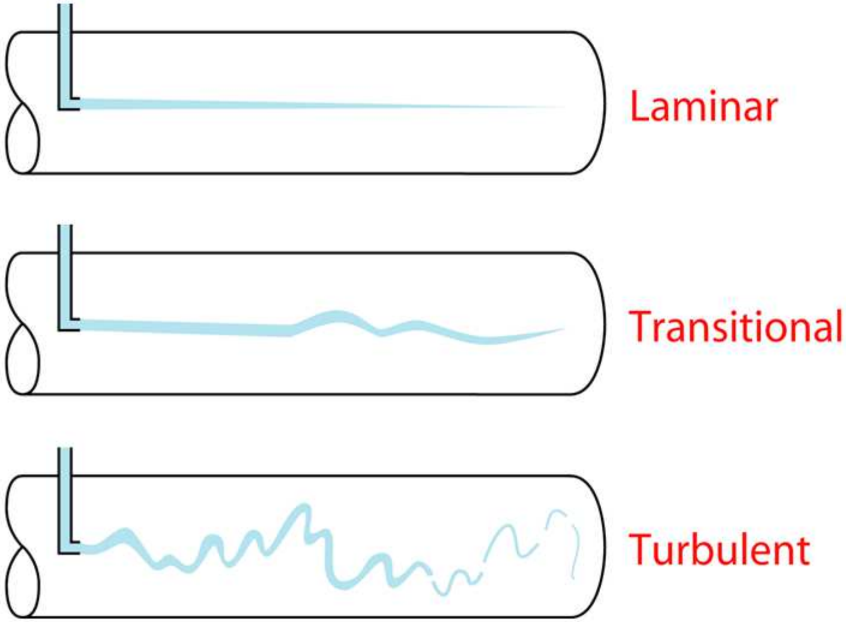
Pipe Flow

- Introduction
  - Pipe flow is very common in industry and everyday life
  - We need estimate the dissipation of energy incurred in maintaining the flow. This requires the knowledge of boundary layer theory.

Pipe Flow

- Three types of flow
  - Laminar flow $Re < 2300$
  - Transitional flow $2300 < Re < 10^5$
  - Turbulent flow $Re > 10^5$

$$Re = \frac{\rho V d}{\mu}$$



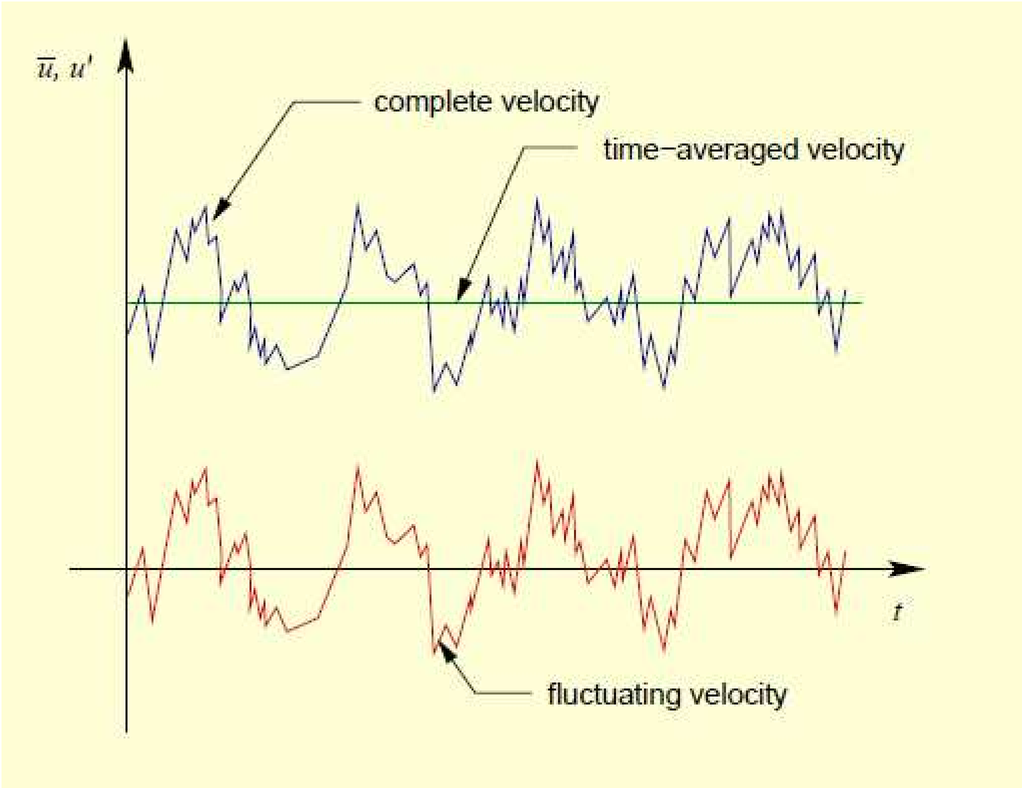
The turbulent flow is characterized by RANDOM, IRREGULAR and UNSTEADY movement of fluid particles, making it impossible to predict the motion of a fluid particle with respect to time and space.

Pipe Flow

- Turbulent flow
  - Turbulent in pipe flow is actually more likely to occur than laminar flow in practical situation.
  - Turbulent flow is a very complex process.
  - In turbulent flow, there are velocity fluctuations which are over a range of time scales and length scales.
  - The fluctuations are relatively small but are of varying amplitude and occur over a range of frequencies.

Pipe Flow

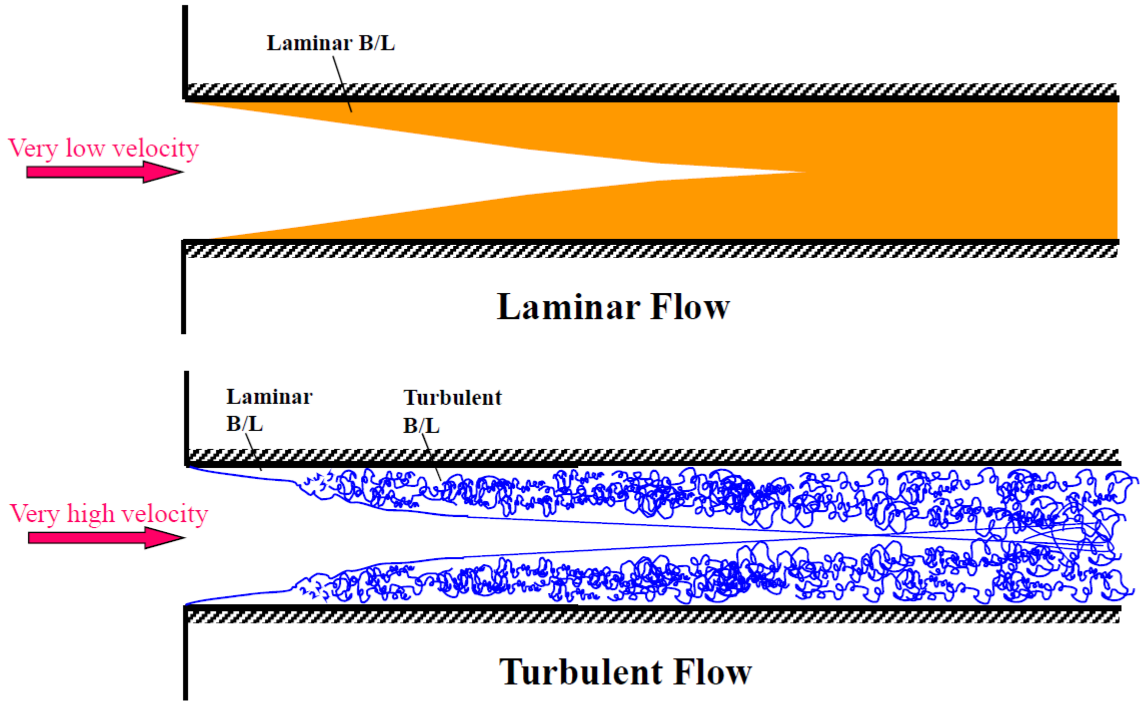
- Turbulent flow



$$u = \overline{u} + u'$$

Pipe Flow

- Transition Reynolds number
  - Initial disturbance of approach flow
  - Shape of pipe entrance
  - Roughness of pipe



Pipe Flow

- Fully Developed Flow
  - The flow in long, straight, constant diameter sections of a pipe become fully developed.
  - Although this is true whether the flow is laminar or turbulent, the details of the velocity profile (and other flow properties) are quite different for these two types of flow.
  - “Boundary layers” from opposite sides of the pipe have merged (and, hence, can no longer continue to grow);
  - The streamwise velocity component satisfies $\partial w/\partial z = 0$;
  - The radial (or in the case of, e.g., square ducts, the wall-normal) component of velocity is zero, i.e., $v = 0$.

Pipe Flow

- Velocity profile
  - Laminar flow (fully developed):
    - ✓ Analytical solution

$$u_x = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - R^2\right)$$

$$u_{avg} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}R^2 = -\frac{1}{32\mu}\frac{\partial p}{\partial x}d^2$$

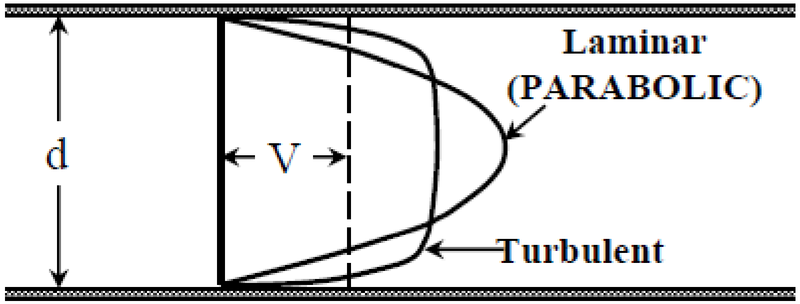
$$u_{\max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x}R^2 = -\frac{1}{16\mu}\frac{\partial p}{\partial x}d^2 \text{at} \quad r = 0$$

Pipe Flow

- Velocity profile
  - Turbulent flow (fully developed):
    - ✓ Experimental measurement

$$\frac{u}{U_c} = \left(1 - \frac{r}{R}\right)^{1/n}$$

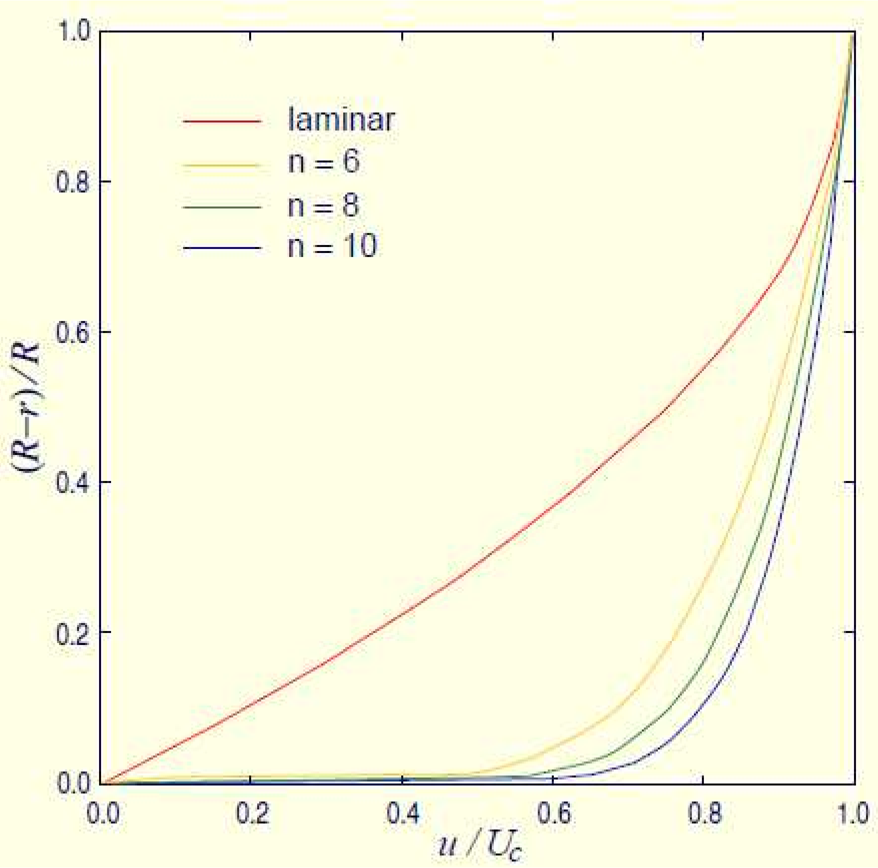
where U_c is pipe centerline velocity.



- ✓ The value of n used in the equation depends on the Reynolds number
- ✓ n increases from $n = 6$ at $\mathrm{Re} \simeq 2 \times 10^4$ to $n = 10$ at $\mathrm{Re} \simeq 3 \times 10^6$ in a nearly linear (on a semi-log plot) fashion.
- ✓ For moderate Re, $n = 7$ is widely used, almost independent of the actual value of Re.

Pipe Flow

- Velocity profile
  - Turbulent flow (fully developed):

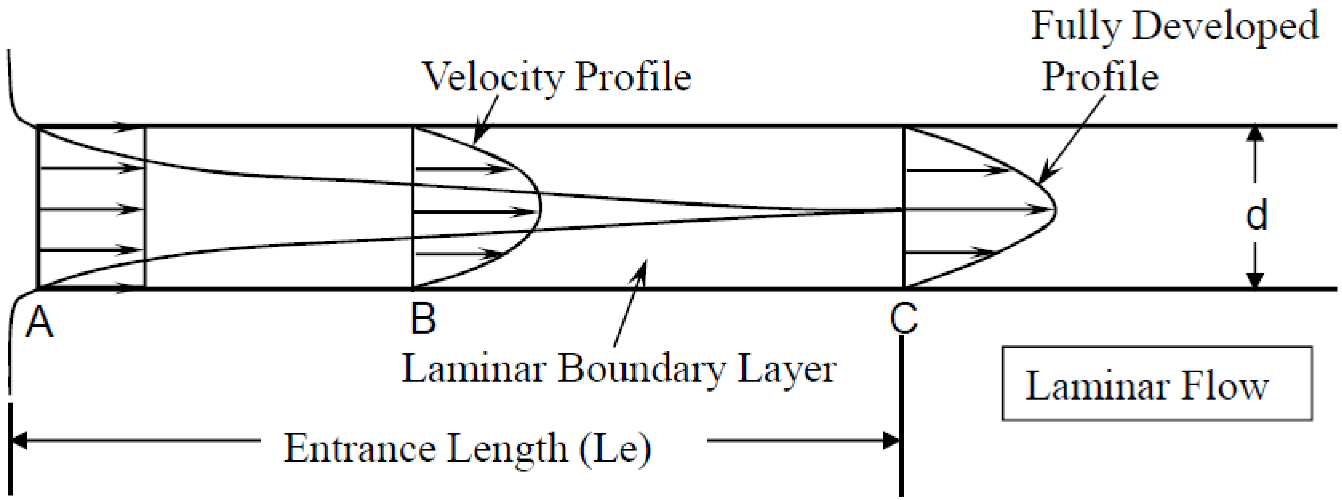


Pipe Flow

- Entrance Length
  - Defined as the distance from the entrance of the pipe that the flow needs to travel before the flow is fully developed (i.e. the velocity profile does not change with distance).
  - As long as the boundary-layer thickness satisfies $\delta \ll R$ the boundary-layer approximations is valid. This, of course, holds only very near the pipe entrance.
  - Development farther from the entrance is still strongly influenced by boundary-layer growth, but now velocity profiles outside the boundary layer are also adjusting.
  - The entrance length in pipe flow is he required distance in the flow direction for the "boundary layers" from opposite sides of the pipe to merge.

Pipe Flow

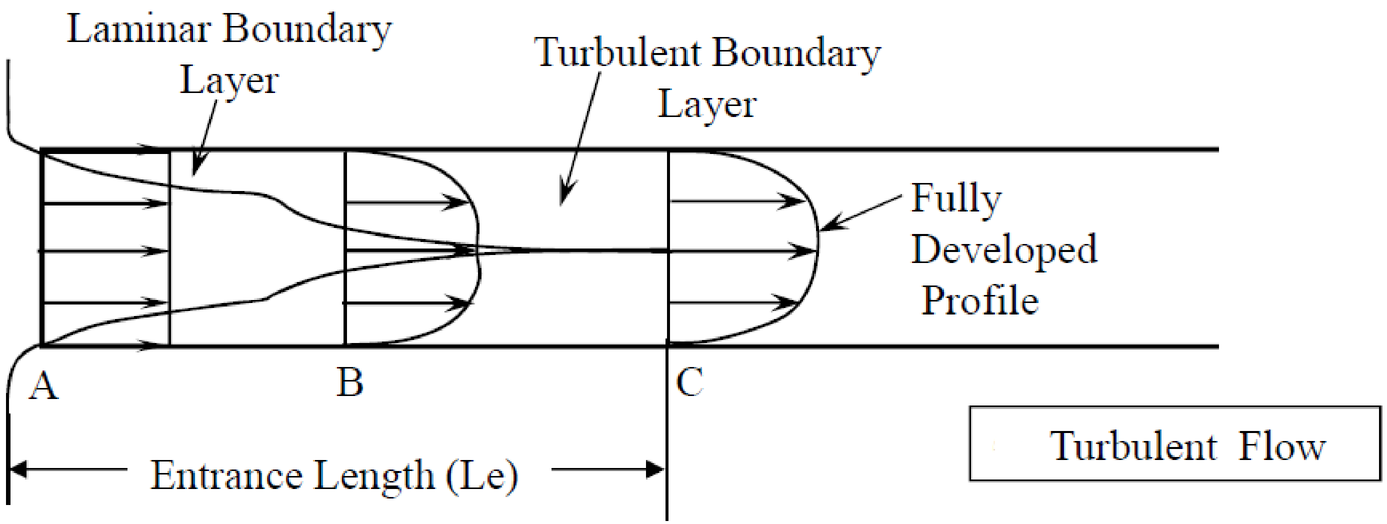
- Entrance Length
  - Laminar flow



$$\frac{L_e}{d} = 0.06Re$$

Pipe Flow

- Entrance Length
  - Laminar flow



$$\frac{L_e}{d} = 4.4 Re^{1/6}$$

✓ In turbulent flow, the boundary layers grow faster, and L_e is relatively short.

Pipe Flow

- Example 1
  - SAE 10 oil at 20^{o}C flows through a 3-cm diameter tube. Estimate the entrance length in cm if the volume flow rate is (a) 0.001 m^3/s and (b) 0.03 m^3/s. The density (μ) and dynamic viscosity (ρ) of SAE 10 oil are 870 kg/m^3 and 0.104 $kg/m{\cdot}s$, respectively. (Assume critical Re is 2300)

Pipe Flow

- Example 1
  - Solution
    - ✓ Before we can determine the entrance length, we need to determine whether the flow is laminar or turbulent

$$Q = \frac{\pi d^2}{4} V_{avg}$$

$$V_{avg} = \frac{4Q}{\pi d^2}$$

$$\mathrm{Re} = \frac{\rho V_{avg} d}{\mu} = \frac{4\rho Q}{\pi \mu d}$$

Pipe Flow

- Example 1
  - Solution

(a)

$$Q = 0.001 \text{ m}^3/\text{s}$$

$$\text{Re} = \frac{4 \times 870 \times 0.001}{\pi \times 0.104 \times 0.03} = 355$$

Flow is laminar since Re < 2300

$$\frac{L_e}{d} = 0.06Re$$

$$L_e = 0.06\,\text{Re}\,d = 0.06 \times 355 \times 0.03 = 0.64 \text{ m}$$

Pipe Flow

- Example 1
  - Solution

(a)

$$Q = 0.03 \text{ m}^3/\text{s}$$

$$\text{Re} = \frac{4 \times 870 \times 0.03}{\pi \times 0.104 \times 0.03} = 10650$$

Flow is turbulent since Re > 2300

$$\frac{L_e}{d} = 4.4\text{Re}^{1/6}$$

$$L_e = 4.4\text{Re}^{1/6} d = 0.06 \times 10650^{1/6} \times 0.03 = 0.62 \text{ m}$$

Pipe Flow

- Friction Factor
  - Laminar flow (fully developed):
    - ✓ The pressure loss due to viscous effects along the length of the system

$$u_x = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - R^2\right) \qquad \tau = \mu\frac{\partial u_x}{\partial r} = \frac{1}{2}\frac{\partial p}{\partial x} r$$

$$u_{avg} = -\frac{1}{8\mu}\frac{\partial p}{\partial x}R^2 = -\frac{1}{32\mu}\frac{\partial p}{\partial x}d^2 \qquad \tau_{r=R} = \frac{1}{2}\frac{\partial p}{\partial x}R = \frac{8\mu u_{avg}}{d}$$

    - ✓ If we consider the pressure along a typical length L

$$u_{avg} = -\frac{1}{32\mu}\frac{\Delta p}{L}d^2$$

$$\Delta p = \frac{32\mu u_{avg} L}{d^2}$$

Pipe Flow

- Friction Factor
  - Laminar flow (fully developed):
    - ✓ Friction factor, f, in terms of the dimensionless pressure difference

$$f = \frac{\Delta p}{\frac{1}{2}\rho u_{avg}^2}\frac{d}{L} == \frac{64\mu}{\rho u_{avg} d} = \frac{64}{\mathrm{Re}}$$

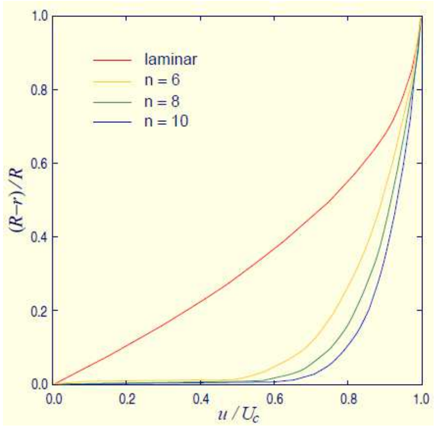
    - ✓ This is also called Darcy friction factor
    - ✓ The head-loss due to friction h_f

$$h_f = \frac{\Delta p}{\rho g} = \frac{32\mu u_{avg} L}{\rho g d^2} = \boxed{f\frac{L}{d}\frac{u_{avg}^2}{2g}}$$

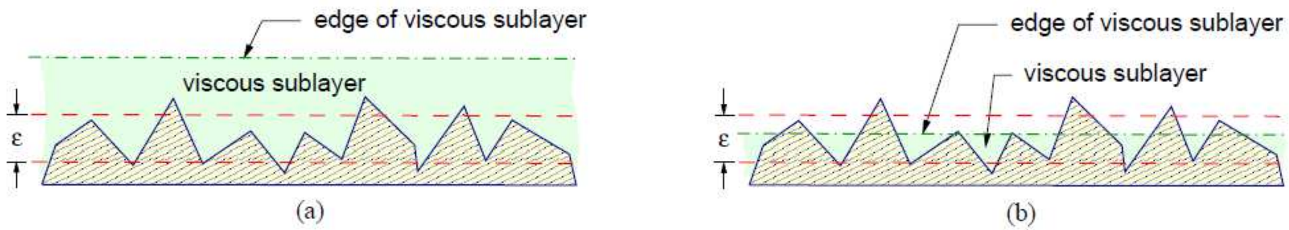
$$\frac{P_1}{\rho g} + \frac{1}{2}\frac{v_1^2}{g} + z_1 = \frac{P_2}{\rho g} + \frac{1}{2}\frac{v_2^2}{g} + z_2 + \frac{32\mu u_{avg} L}{\rho g d^2}$$

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):
    - ✓ Both Reynolds number and surface roughness ε (or ε/d in dimensionless form) affect the friction factor.
    - ✓ For turbulent flow, the boundary layer thins and so also does the thickness of the viscous sublayer.



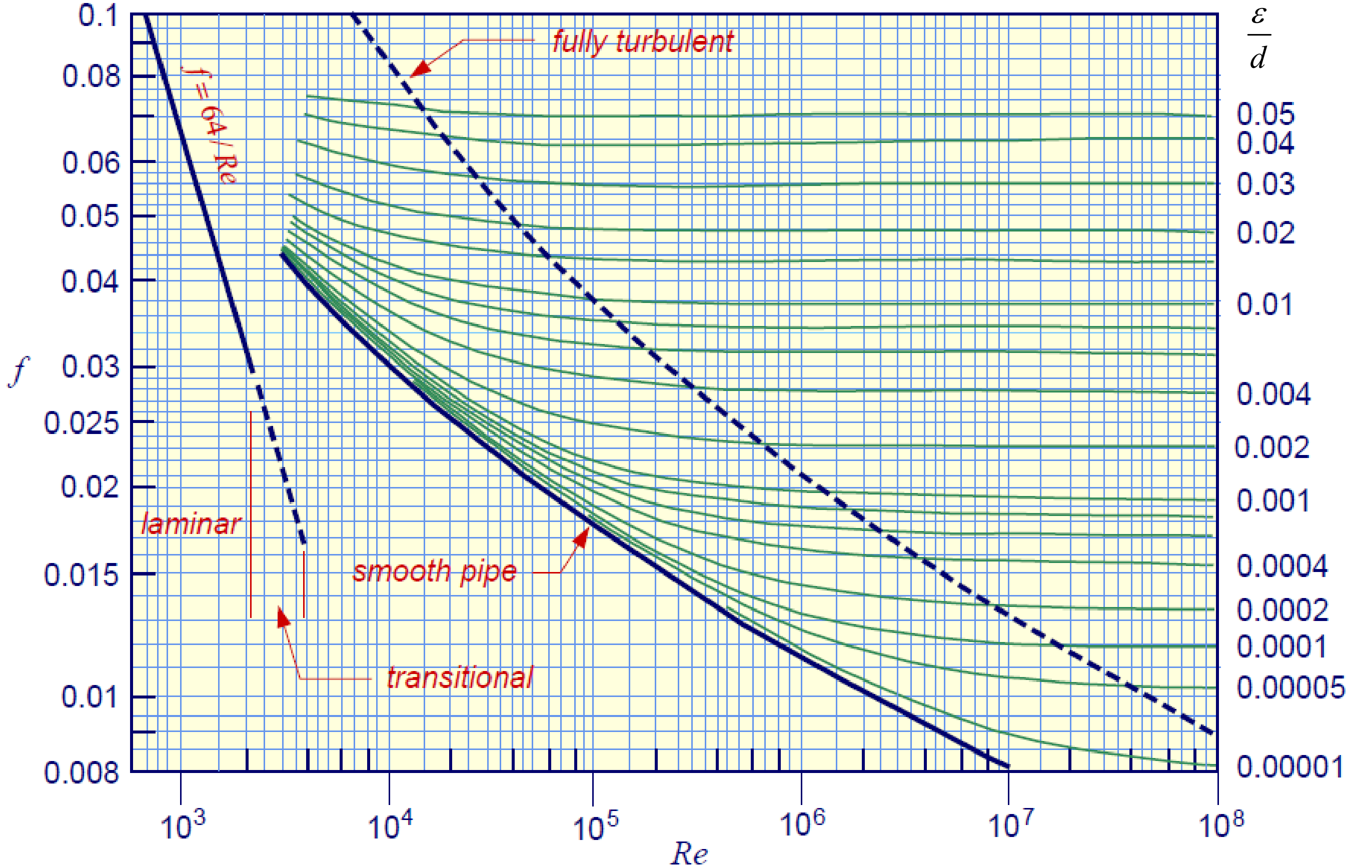
- ✓ As Re increases more of the rough edges of the surface are extending beyond the viscous sublayer and into the buffer and inertial layers.
- ✓ Far more internal friction than does viscosity, so the result is considerable pressure loss and increased friction factor in comparison with laminar flow.



Low Re   High Re

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):



Moody diagram

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):
    - ✓ Moody diagram provides a log-log plot of friction factor over a wide range of Reynolds numbers and for numerous values of dimensionless surface roughness.
    - ✓ "Hydraulically smooth": Within the range of Reynolds numbers of practical importance the normalized surface roughness is so small that it never protrudes through the viscous sublayer. Thus, for any given Reynolds number in the turbulent flow regime a smooth pipe produces the smallest possible friction factor.
    - ✓ "Fully turbulent": provides the locus of Re values beyond which the viscous sublayer is so thin compared with each displayed value of ε/d that it has essentially no effect on the inertial behavior of the turbulent fluctuations; viz., the friction factor shows almost no further change with increasing Reynolds number.

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):
    - ✓ Smooth pipe

$$f = \frac{0.316}{\mathrm{Re}^{1/4}}$$

the Blasius formula at the case of $\varepsilon/d = 0$

- ✓ Colebrook formula

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) \qquad 4\times10^{3} < \mathrm{Re} < 10^{8}$$

within this range calculated values of friction factor differ from experimental results typically by no more than 15%.

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):
    - ✓ Colebrook formula

$$s = \frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}\sqrt{f}}\right) = -2\log_{10}\left(\frac{\varepsilon/d}{3.7} + \frac{2.51}{\mathrm{Re}}s\right) = F(s)$$

1. Guess an initial value of $s = s_0$
2. $s_1 = \mathrm{F}(s_0)$
3. $s_2 = \mathrm{F}(s_1)$
4. …
5. $s_n = \mathrm{F}(s_{n-1})$
6. Until $| s_n - s_{n-1}| < \epsilon$, with ϵ being a specified acceptable level of error

$$f = \frac{1}{s^2} \qquad n = \frac{1}{\sqrt{f}}$$

Pipe Flow

- Friction Factor
  - Turbulence flow (fully developed):
    - ✓ Head loss

$$\frac{P_1}{\rho g}+\frac{1}{2}\frac{v_1^2}{g}+z_1=\frac{P_2}{\rho g}+\frac{1}{2}\frac{v_2^2}{g}+z_2+h_f$$

$$h_f=\frac{\Delta p}{\rho g}=f\frac{L}{d}\frac{u_{avg}^2}{2g}$$

Pipe Flow

- Head loss
  - Example 2:

Oil of density 900 kg/m^3 and kinematic viscosity 330×10^{-6} m^2/s is pumped over a distance of 1.5 km through a 75 mm diameter tube at a rate of 25×10^3 kg/hr. Determine the shear stress at the wall and the head loss through the pipe.

Pipe Flow

- Head loss
  - Example 2:

<u>Solution:</u>

The cross-section $A_w = \pi d^2 / 4 = \pi \times 0.075^2 / 4 = 4.42 \times 10^{-3}$ m

The average velocity of water

$$V = \dot{m} / (\rho A_w) = 25 \times 10^3 / (3600 \times 900 \times 4.42 \times 10^{-3})$$
$$= 1.746 \text{ m/s}$$

The Reynolds number is

$$\mathrm{Re} = V\,d / \nu = 1.746 \times 0.075 / (330 \times 10^{-6}) = 396.94$$

The flow is laminar flow

Pipe Flow

- Head loss
  - Example 2:

<u>Solution:</u>

The wall stress is

$$\tau_w = 8\mu u_{avg} / d = 8 \times 330 \times 10^{-6} \times 900 \times 1.746 / 0.075$$
$$= 55.3 \text{ N/m}^2$$

The friction factor $f = 64 / \text{Re} = 64 / 396.94 = 0.1612$

The head loss is

$$h_f = f \frac{L}{d} \frac{u_{avg}^2}{2g} = 0.1612 \frac{1500 \times 1.746^2}{0.075 \times 2 \times 9.8} = 501.5 \text{ m}$$

Pipe Flow

- Minor loss
  - The head losses are the main contributor to pressure drop, and are often called major losses.
  - The minor losses arise specifically from flow through pipe expansions and contractions, and through tees, bends, branches and various fittings such as valves.
  - The minor losses have usually been obtained empirically
  - The general formula for minor losses takes the form

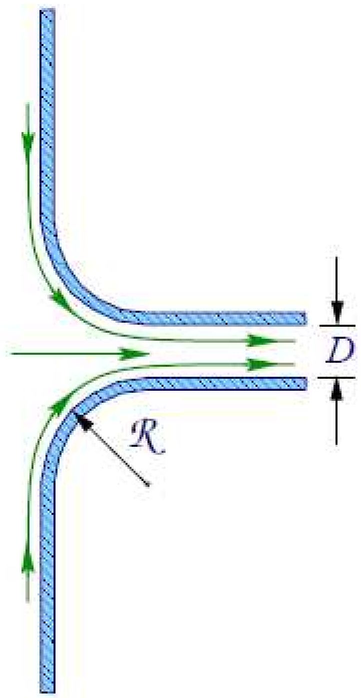
$$h_m = K\frac{u_{avg}^2}{2g}$$ where K is the loss coefficient.

$$\frac{P_1}{\rho g}+\frac{1}{2}\frac{v_1^2}{g}+z_1=\frac{P_2}{\rho g}+\frac{1}{2}\frac{v_2^2}{g}+z_2+h_f+h_m$$

Pipe Flow

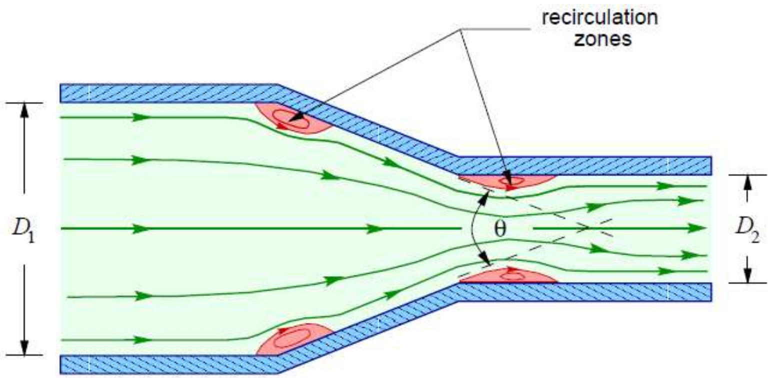
- Minor loss
  - Sharp-edged inlets

$\mathcal{R}/D$	K
0.0	0.5
0.02	0.28
0.06	0.15
≥ 0.15	0.04



Pipe Flow

- Minor loss
  - Contracting pipes



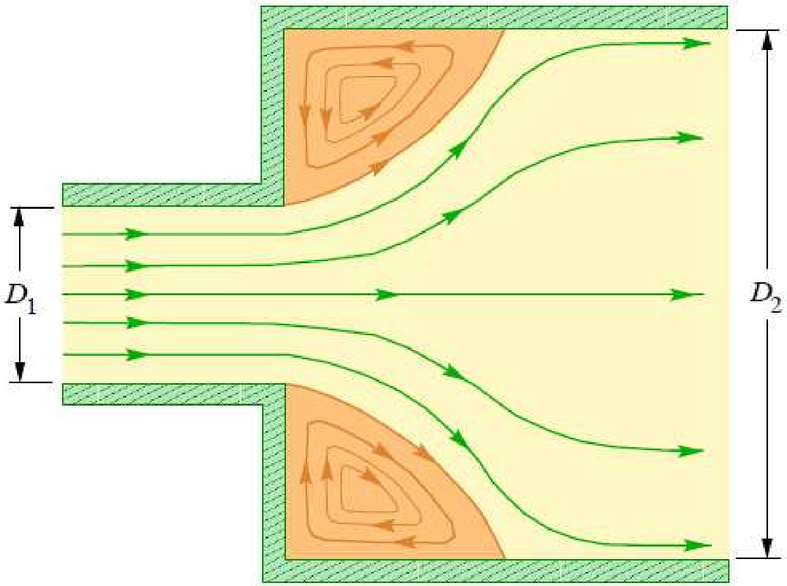
D_2/D_1	K	
	$\theta = 60°$	$\theta = 180°$
0.2	0.08	0.49
0.4	0.07	0.42
0.6	0.06	0.32
0.8	0.05	0.18

$$K \approx \frac{1}{2}\left[1-\left(\frac{D_2}{D_1}\right)^2\right]$$

$$for\ \ \theta = 180^\circ$$

Pipe Flow

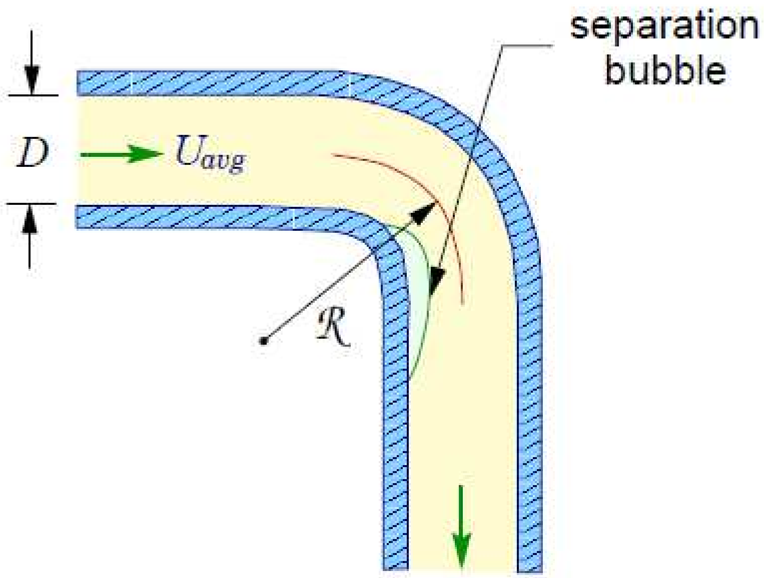
- Minor loss
  - Rapidly-expanding pipes



$$K \approx \left[1 - \left(\frac{D_2}{D_1}\right)^2\right]^2$$

Pipe Flow

- Minor loss
  - Elbows



$\mathcal{R}/D$	K
1.0	0.35
2.0	0.19
4.0	0.16
6.0	0.21
8.0	0.28
10.0	0.32

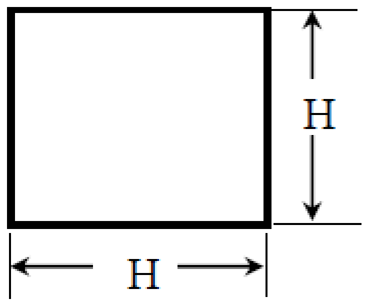
Pipe Flow

- Non-circular pipe
  - Hydraulic diameter:
    - ✓ In some situations, the pipe cross-section is non-circular
    - ✓ We can modify many of the equations that we have derived earlier for circular cross-sections to noncircular sections by using the concept of hydraulic diameter

$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

Pipe Flow

- Hydraulic diameter
  - Square duct:

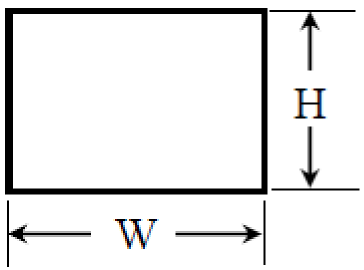


$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

$$D_H = \frac{4 \times H \times H}{4 \times H} = H$$

Pipe Flow

- Hydraulic diameter
  - Rectangular duct:

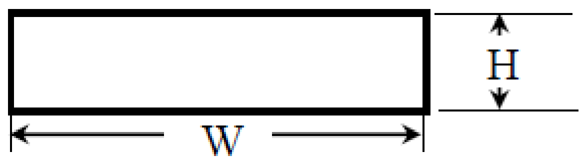


$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

$$D_H = \frac{4 \times H \times W}{2 \times (H + W)} = \frac{2HW}{H + W}$$

Pipe Flow

- Hydraulic diameter
  - Elongated rectangular section ($W >> H$)

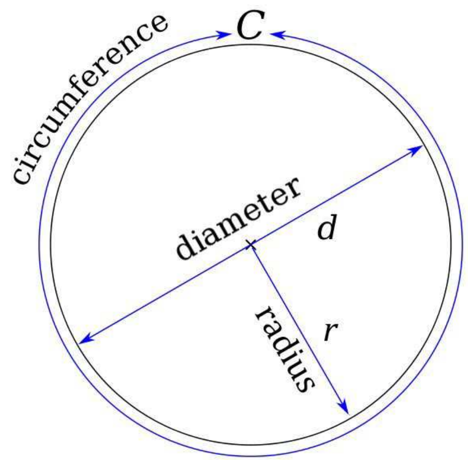


$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

$$D_H = \frac{4 \times H \times W}{2 \times (H + W)} \approx \frac{2HW}{W} = 2H$$

Pipe Flow

- Hydraulic diameter
  - Circular pipe:

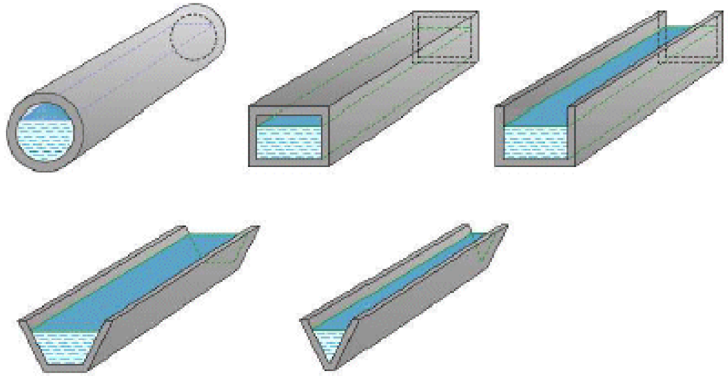


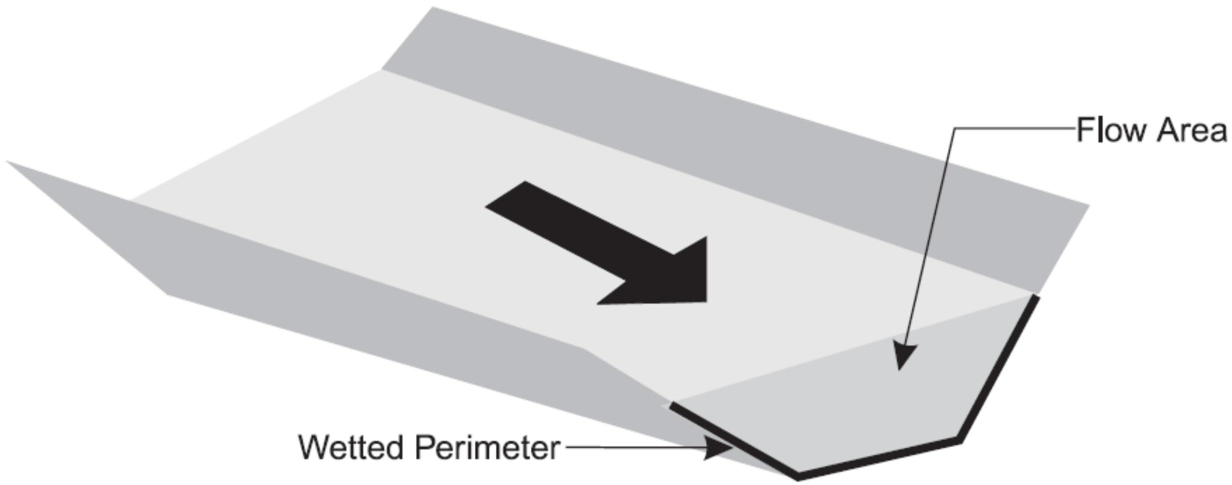
$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

$$D_H = \frac{4 \times \pi \times \left(\frac{d}{2}\right)^2}{\pi \times d} = d$$

Pipe Flow

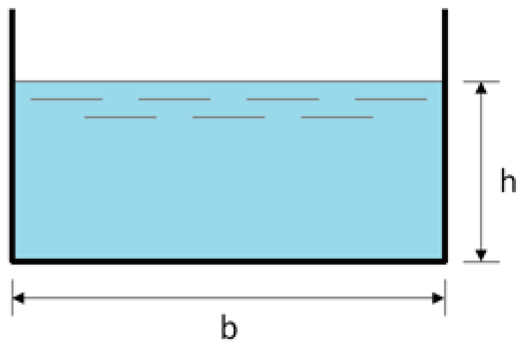
- Hydraulic diameter
  - Open channel and partially filled pipe





Pipe Flow

- Hydraulic diameter
  - Open channel

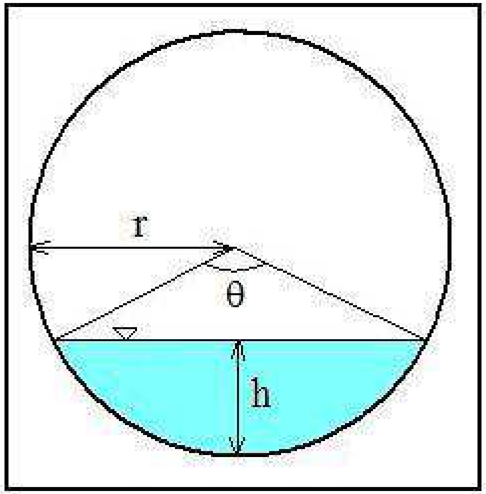


$$D_H = \frac{4 \times \text{Cross-Section Area}}{\text{Wetted Perimeter of the Cross-section}}$$

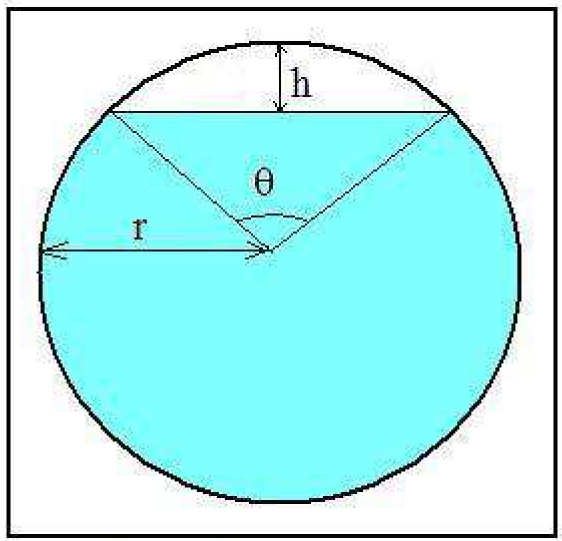
$$D_H = \frac{4 \times h \times b}{2 \times h + b} = \frac{4hb}{2h + b}$$

Pipe Flow

- Hydraulic diameter
  - Partially filled pipe



Partially Full Pipe Flow Parameters
(Less Than Half Full)



Partially Full Pipe Flow Parameters
(More Than Half Full)

Pipe Flow

- Hydraulic diameter
  - Reynolds number

$$\mathrm{Re} = \frac{\rho V D_H}{\mu}$$

  - Head-loss due to friction

$$h_f = f\frac{L}{D_H}\frac{u_{avg}^2}{2g}$$

Pipe Flow

- Hydraulic diameter
  - Example 3:

A rectangular concrete channel is 3 m wide and 2 m high. The water in the channel is 1.5 m deep and is flowing at a rate of 30 m^3/s. Determine the flow area, wetted perimeter, and hydraulic diameter. Is the flow laminar or turbulent? (the kinematic viscosity for water at 20ºC is 1.00×10^{-6} m^2/s)

Pipe Flow

- Hydraulic diameter
  - Example 3:

Solution:

The cross-section area $A_w = 1.5 \times 3 = 4.5\ \mathrm{m}^2$

The wetted perimeter $P_w = 3.0 + 2 \times 1.5 = 6\ \mathrm{m}$

The hydraulic diameter $D_H = 4A_w / P_w = 4 \times 4.5 / 6 = 3\ \mathrm{m}$

The average velocity of water

$$V = Q / A_w = 30 / 4.5 = 6.67\ \mathrm{m}$$

The Reynolds number is

$$\mathrm{Re} = V D_H / v = 3 \times 6.67 / (1.0 \times 10^{-6}) = 2 \times 10^7$$

The flow is turbulent flow

Review

- The governing equation for fluid flow:
  - Continuity equation and Navier Stokes equation.
- Exact solution for simple flow problems
  - Plane Couette flow
  - Plane Poiseuille flow
  - Rectilinear flow between parallel plates
  - Pipe Poiseuille flow
  - Couette flow between two concentric cylinders
- Creeping flow past a sphere
  - Drag coefficient
- Boundary layer theory
  - Flow past a infinitely long plate
  - Boundary thickness and drag coefficient for laminar and turbulent flow

Review

- Flow past a cylinder
  - Different flow regime
  - Drag coefficient

- Pipe flow
  - Laminar and turbluent flow
  - Fully developed flow
  - Velocity profile
  - Entrance effect
  - Friction factor, head loss, and minor loss
  - Hydraulic diameter for non-circular cross-section pipe


Thank You for Your Attention!
Any Questions?
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