

solutions to transportation problems

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1 L1

1. scale 1 处于分子尺度, 粒子非常稀疏, 体微元包含的粒子数的变化 dN 与体微元有关, 也就是 $dN = dN(dV)$, 因而 $\frac{dN}{dV} \neq \text{constant}$, 也就是密度 ρ 在这个尺度会随体微元的不同而不同. scale 2 处于连续尺度, 体元宏观无穷小, 而微观无穷大, 在此尺度定义密度有: 1) 定义于一点处, 因为体元宏观无穷小, 在宏观上就是一点, 2) 在定义点处连续, 因为微观无穷大使体元包含足够多的粒子, 这使得附近点之间的粒子数平均不会发生突变, 因而密度在空间上是连续分布的. scale 3 处于宏观尺度, 这一个尺度上的体元已经感知到了密度的宏观变化

2.

$$\frac{d\alpha}{dt} = \frac{dx/dy}{dt} = \frac{\partial u}{\partial y} \quad (1)$$

$$\frac{d\beta}{dt} = \frac{dy/dx}{dt} = \frac{\partial v}{\partial x} \quad (2)$$

so

$$\frac{d\alpha + d\beta}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \quad (3)$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (4)$$

3.

$$\tau_x = \frac{\partial u}{\partial y} = -\frac{2U_0 y}{b^2} \quad (5)$$

$$\tau_x|_{y=\frac{b}{2}} = -\frac{U_0}{b} \quad (6)$$

4.

$$u = cy \quad (7)$$

$$v = cx \quad (8)$$

$$\tau_x = \mu \frac{\partial \mathbf{v}}{\partial y} = \mu \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} c \\ 0 \end{pmatrix} \quad (9)$$

$$\tau_y = \mu \frac{\partial \mathbf{v}}{\partial x} = \mu \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} 0 \\ c \end{pmatrix} \quad (10)$$

$$\text{let } \mathbf{e} = \frac{\mathbf{j} + \mathbf{i}}{\sqrt{2}},$$

$$\tau_{xy} = \mu \frac{\partial \mathbf{v}}{\partial e} \quad (11)$$

$$= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \cos(\mathbf{e}, \mathbf{i}) + \frac{\partial \mathbf{v}}{\partial y} \cos(\mathbf{e}, \mathbf{j}) \right) \quad (12)$$

$$= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial \mathbf{v}}{\partial y} \frac{1}{\sqrt{2}} \right) \quad (13)$$

$$= \frac{\mu}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} \right) \quad (14)$$

$$= \frac{c\mu}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (15)$$

5.

$$2\sigma l = \rho g h l w \quad (16)$$

$$\Rightarrow h = \frac{2\sigma}{\rho g w} \quad (17)$$

$$\sigma = 0.123(1 - 0.00139T) \text{ so}$$

$$h = 9.37 \text{ mm}$$

6.

$$\sigma = 0.025 \text{ N/m} \quad (18)$$

$$2\pi r \sigma = \Delta p \pi r^2 \quad (19)$$

$$\Rightarrow \Delta p = \frac{2\sigma}{r} \quad (20)$$

$$\Rightarrow \Delta p = 25 \text{ Pa} \quad (21)$$

2 L2

1. (1)

$$\begin{cases} \frac{dp}{dr} = f = \rho a \\ a = kr \\ kR = g \end{cases} \quad (22)$$

\Rightarrow

$$\frac{dp}{dr} = \frac{\rho gr}{R} \quad (23)$$

\Rightarrow

$$p = \frac{\rho gr^2}{2R} + p_{atm} \quad (24)$$

so

$$p|_{r=R} = \frac{\rho g R}{2} + p_{atm} \quad (25)$$

$$\approx \frac{\rho g R}{2} \quad (26)$$

$$= 176 \text{ Mpa} \quad (27)$$

2. (2)

$$\rho g \Delta h = \Delta p \quad (28)$$

$$\Rightarrow \Delta h = \frac{\Delta p}{\rho g} \quad (29)$$

\Rightarrow

$$\Delta h_{water} = 10.33m \quad (30)$$

$$\Delta h_{sea} = \Delta h_{water}/1.025 = 10.08m \quad (31)$$

$$\Delta h_{Hg} = \Delta h_{water}/13.6 = 0.8m \quad (32)$$

3. (3) let Hg, water, oil denoted by "H, w, o" respectively

$$p_A + \rho_o g h_o + \rho_w g h_w = p_{atm} + \rho_H g h_H \quad (33)$$

$$\Rightarrow p_A - p_{atm} = g(\rho_H h_H - \rho_o h_o - \rho_w h_w) \quad (34)$$

$$(35)$$

4. (4) the resultant acceleration is gravity and inertial

$$\mathbf{r} = \mathbf{g} - \mathbf{a} \quad (36)$$

now the isobars and the direction of the pressure gradient is depict as follow

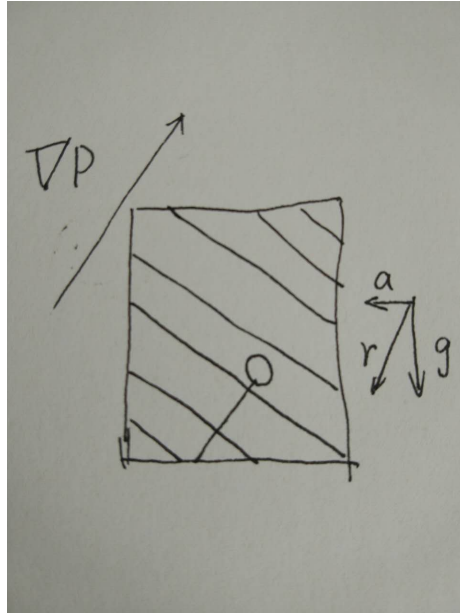


图 1: the balloon

5. (8)

$$F = \rho g \bar{y} A = 369.8 kN \quad (37)$$

3 L3-L4

1. since

$$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) = \mathbf{g}(\mathbf{c})h(t) \quad (38)$$

\Rightarrow

$$\mathbf{g}(\mathbf{c}) = \frac{\mathbf{r}}{h(t)} \quad (39)$$

\Rightarrow

$$\mathbf{c} = \mathbf{g}^{-1}\left(\frac{\mathbf{r}}{h(t)}\right) \quad (40)$$

so

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} \quad (41)$$

$$= \mathbf{g}(\mathbf{c})\dot{h}(t) \quad (42)$$

$$= \mathbf{g}\left(\mathbf{g}^{-1}\left(\frac{\mathbf{r}}{h(t)}\right)\right)\dot{h}(t) \quad (43)$$

2.

$$g(c) = c \quad (44)$$

$$h(t) = t^2 \quad (45)$$

and it is easy to find

$$g^{-1}(c) = c \quad (46)$$

$$v = g\left(g^{-1}\left(\frac{x}{h(t)}\right)\right)\dot{h}(t) \quad (47)$$

$$= g^{-1}\left(\frac{x}{h(t)}\right)\dot{h}(t) \quad (48)$$

$$= \frac{x}{h(t)}\dot{h}(t) \quad (49)$$

$$= 2t\frac{x}{t^2} \quad (50)$$

$$= 2\frac{x}{t} \quad (51)$$

$$(52)$$

3. at time t the c th element is at $f(c, t)$, so the temperature of the c th element is

$$T = g(f(c, t), t) \quad (53)$$

so the variation rate is

$$\frac{dT}{dt} = \frac{\partial g}{\partial x} \frac{df}{dt} + \frac{\partial g}{\partial t} \quad (54)$$

4.

$$\begin{cases} \frac{dx}{dt} = \frac{x}{1+t} \\ \frac{dy}{dt} = \frac{2y}{2+t} \end{cases} \quad (55)$$

 \Rightarrow

$$\begin{cases} x = c_x(1+t) \\ y = c_y(2+t)^2 \end{cases} \quad (56)$$

with the boundary condition

$$\mathbf{x}(0) = \mathbf{x}_0 \quad (57)$$

the path line through \mathbf{x}_0 is

$$\begin{cases} x = x_0(1+t) \\ y = \frac{y_0}{4}(2+t)^2 \end{cases} \quad (58)$$

and the streamlines at $t = 0$ is

$$\begin{cases} x = c_x \\ y = 2c_y \end{cases} \quad (59)$$

 \Rightarrow

$$y = 2 \frac{c_y}{c_x} x \quad (60)$$

5.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (61)$$

since

$$\rho = \rho_0(2 - \cos \omega t) \quad (62)$$

 \Rightarrow

$$\frac{\partial u}{\partial x} = - \frac{\partial \rho}{\rho \partial t} \quad (63)$$

$$= \frac{\omega \sin \omega t}{\cos \omega t - 2} \quad (64)$$

$$:= f(t) \quad (65)$$

\Rightarrow

$$u = \int f(t) dx \quad (66)$$

$$= f(t)x + C \quad (67)$$

apply the boundary condition $u(0, t) = U$

$$u = f(t)x + U \quad (68)$$

6. df

$$\begin{aligned} \text{(a)} \quad \int_{A=\partial V} \rho \mathbf{u} \cdot d\mathbf{A} &= \int_0^1 dy \int_0^1 dz 4x^2 y \Big|_{x=1} - \int_0^1 dy \int_0^1 dz 4x^2 y \Big|_{x=0} \\ &\quad + \int_0^1 dz \int_0^1 dx xyz \Big|_{y=1} - \int_0^1 dz \int_0^1 dx xyz \Big|_{y=0} \\ &\quad + \int_0^1 dx \int_0^1 dy yz^2 \Big|_{z=1} - \int_0^1 dx \int_0^1 dy yz^2 \Big|_{z=0} \\ &= 2 + 0 + \frac{1}{4} + 0 + \frac{1}{2} + 0 = \frac{11}{4}. \end{aligned}$$

$$\text{(b)} \quad \nabla \cdot \mathbf{u} = \partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 8xy + xz + 2yz$$

$$\begin{aligned} \int_V \nabla \cdot \mathbf{u} dV &= \int_0^1 dx \int_0^1 dy \int_0^1 dz (8xy + xz + 2yz) \\ &= 8 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{11}{4}. \end{aligned}$$

7. df

8. mass conservation

$$\frac{\partial(\rho \delta v)}{\partial t} + \frac{\partial(\rho u h)}{\partial x} \delta x = 0 \quad (69)$$

$$\Rightarrow \frac{\partial(\rho h \delta x)}{\partial t} + \frac{\partial(\rho u h)}{\partial x} \delta x = 0 \quad (70)$$

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \quad (71)$$

momentum conservation

$$\frac{\partial(\rho u \delta v)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} \delta x = 0 \quad (72)$$

$$\Rightarrow \frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} = 0 \quad (73)$$

the total force p due to pressure is

$$p = \int_0^h \rho g y dy = \frac{1}{2} \rho g h^2 \quad (74)$$

so the momentum equation is

$$\frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + \frac{1}{2} \rho g h)}{\partial x} = 0 \quad (75)$$

$$\Rightarrow = \frac{\partial(u h)}{\partial t} + \frac{\partial(u^2 h + \frac{1}{2} g h)}{\partial x} = 0 \quad (76)$$