solutions to transportation problems

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1 L1

1. scalse 1 处于分子尺度，粒子非常稀疏，体微元包含的粒子数的变化dN与体微元有关,也就是$dN = dN(dV)$,因而$\frac{dN}{dV} \neq constant$，也就是密度$\rho$在这个尺度会随体微元的不同而不同.scale 2 处于连续尺度，体元宏观无穷小，而微观无穷大，在此尺度定义密度有：1)定义于一点处，因为体元宏观无穷小，在宏观上就是一点，2)在定义点处连续，因为微观无穷大使体元包含足够多的粒子，这使得附近点之间的粒子数平均不会发生突变，因而密度在空间上是连续分布的.scale 3 处于宏观尺度,这一个尺度上的体元已经感知到了密度的宏观变化

2.

$$\frac{d\alpha}{dt} = \frac{dx/dy}{dt} = \frac{\partial u}{\partial y} \tag{1}$$

$$\frac{d\beta}{dt} = \frac{dy/dx}{dt} = \frac{\partial v}{\partial x} \tag{2}$$

so

$$\frac{d\alpha + d\beta}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \tag{3}$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4}$$

3.

$$\tau_x = \frac{\partial u}{\partial y} = -\frac{2U_0 y}{b^2} \tag{5}$$

$$\tau_x\big|_{y=\frac{b}{2}} = -\frac{U_0}{b} \tag{6}$$

4.

$$u = cy \tag{7}$$

$$v = cx \tag{8}$$

$$\tau_x = \mu \frac{\partial \mathbf{v}}{\partial y} = \mu \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{9}$$

$$\tau_y = \mu \frac{\partial \mathbf{v}}{\partial x} = \mu \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} 0 \\ c \end{pmatrix} \tag{10}$$

let $\mathbf{e} = \frac{\mathbf{j}+\mathbf{i}}{\sqrt{\mathbf{2}}}$,

$$\begin{aligned} \tau_{xy} &= \mu \frac{\partial \mathbf{v}}{\partial e} && (11) \\ &= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \cos(\mathbf{e}, \mathbf{i}) + \frac{\partial \mathbf{v}}{\partial y} \cos(\mathbf{e}, \mathbf{j}) \right) && (12) \\ &= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial \mathbf{v}}{\partial y} \frac{1}{\sqrt{2}} \right) && (13) \\ &= \frac{\mu}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} \right) && (14) \\ &= \frac{c\mu}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (15) \end{aligned}$$

5.

$$\begin{aligned} 2\sigma l &= \rho g h l w && (16) \\ \Rightarrow h &= \frac{2\sigma}{\rho g w} && (17) \end{aligned}$$

$\sigma = 0.123(1 - 0.00139T)$ so
$h = 9.37mm$

6.

$$\sigma = 0.025N/m \tag{18}$$

$$\begin{aligned} 2\pi r \sigma &= \Delta p \pi r^2 && (19) \\ \Rightarrow \Delta p &= \frac{2\sigma}{r} && (20) \\ \Rightarrow \Delta p &= 25Pa && (21) \end{aligned}$$

2 L2

1. (1)

$$\begin{cases} \dfrac{dp}{dr} = f = \rho a \\ a = kr \\ kR = g \end{cases} \tag{22}$$

$\Rightarrow$

$$\frac{dp}{dr} = \frac{\rho g r}{R} \tag{23}$$

$\Rightarrow$

$$p = \frac{\rho g r^2}{2R} + p_{atm} \tag{24}$$

so

$$p|_{r=R} = \frac{\rho g R}{2} + p_{atm} \tag{25}$$

$$\approx \frac{\rho g R}{2} \tag{26}$$

$$= 176 Mpa \tag{27}$$

2. (2)

$$\rho g \Delta h = \Delta p \tag{28}$$

$$\Rightarrow \Delta h = \frac{\Delta p}{\rho g} \tag{29}$$

$\Rightarrow$

$$\Delta h_{water} = 10.33m \tag{30}$$

$$\Delta h_{sea} = \Delta h_{water}/1.025 = 10.08m \tag{31}$$

$$\Delta h_{Hg} = \Delta h_{water}/13.6 = 0.8m \tag{32}$$

3. (3) let Hg,water,oil denoted by "H, w, o" respectively

$$p_A + \rho_o g h_o + \rho_w g h_w = p_{atm} + \rho_H g h_H \tag{33}$$

$$\Rightarrow p_A - p_{atm} = g(\rho_H h_H - \rho_o h_o - \rho_w h_w) \tag{34}$$

$$\tag{35}$$

4. (4) the resultant accelaration is gravity and inertial

$$\mathbf{r} = \mathbf{g} - \mathbf{a} \tag{36}$$

now the isobars and the direction of the pressure gradient is depict as follow

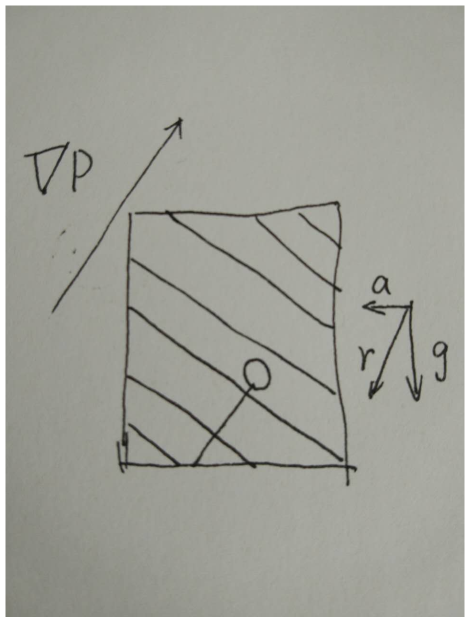

图 1: the balloon

5. (8)

$$F = \rho g \bar{y} A = 369.8kN \tag{37}$$

3 L3-L4

1. since

$$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) = \mathbf{g}(\mathbf{c})h(t) \tag{38}$$

$\Rightarrow$

$$\mathbf{g}(\mathbf{c}) = \frac{\mathbf{r}}{h(t)} \tag{39}$$

$\Rightarrow$

$$\mathbf{c} = \mathbf{g}^{-1}(\frac{\mathbf{r}}{h(t)}) \tag{40}$$

so

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} \tag{41}$$
$$= \mathbf{g}(\mathbf{c})\dot{h}(t) \tag{42}$$
$$= \mathbf{g}(\mathbf{g^{-1}}(\frac{\mathbf{r}}{\mathbf{h(t)}}))\dot{h}(t) \tag{43}$$

2.

$$g(c) = c \tag{44}$$
$$h(t) = t^2 \tag{45}$$

and it is easy to find

$$g^{-1}(c) = c \tag{46}$$

$$v = g(g^{-1}(\frac{x}{h(t)}))\dot{h}(t) \tag{47}$$
$$= g^{-1}(\frac{x}{h(t)})\dot{h}(t) \tag{48}$$
$$= \frac{x}{h(t)}\dot{h}(t) \tag{49}$$
$$= 2t\frac{x}{t^2} \tag{50}$$
$$= 2\frac{x}{t} \tag{51}$$
$$\tag{52}$$

3. at time t the cth element is at $f(c,t)$, so the temperature of the cth element is

$$T = g(f(c,t),t) \tag{53}$$

so the variation rate is

$$\frac{dT}{dt} = \frac{\partial g}{\partial x}\frac{df}{dt} + \frac{\partial g}{\partial t} \tag{54}$$

4.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{1+t} \\ \dfrac{dy}{dt} = \dfrac{2y}{2+t} \end{cases} \tag{55}$$

$\Rightarrow$

$$\begin{cases} x = c_x(1+t) \\ y = c_y(2+t)^2 \end{cases} \tag{56}$$

with the boundary condition

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{57}$$

the path line through $\mathbf{x}_0$ is

$$\begin{cases} x = x_0(1+t) \\ y = \dfrac{y_0}{4}(2+t)^2 \end{cases} \tag{58}$$

and the streamlines at $t = 0$ is

$$\begin{cases} x = c_x \\ y = 2c_y \end{cases} \tag{59}$$

$\Rightarrow$

$$y = 2\frac{c_y}{c_x}x \tag{60}$$

5.

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{61}$$

since

$$\rho = \rho_0(2 - \cos\omega t) \tag{62}$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = -\frac{\partial\rho}{\rho\partial t} \tag{63}$$

$$= \frac{\omega\sin\omega t}{\cos\omega t - 2} \tag{64}$$

$$:= f(t) \tag{65}$$

$\Rightarrow$

$$u = \int f(t)dx \tag{66}$$

$$= f(t)x + C \tag{67}$$

apply the boundary condition$u(0,t) = U$

$$u = f(t)x + U \tag{68}$$

6. df

(a) $$\int_{A=\partial V} \rho \mathbf{u} \cdot d\mathbf{A} = \int_0^1 dy \int_0^1 dz\, 4x^2 y\Big|_{x=1} - \int_0^1 dy \int_0^1 dz\, 4x^2 y\Big|_{x=0} + \int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=1} - \int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=0} + \int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=1} - \int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=0} = 2 + 0 + \tfrac{1}{4} + 0 + \tfrac{1}{2} + 0 = \tfrac{11}{4}.$$

(b) $\nabla \cdot \mathbf{u} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 8xy + xz + 2yz$

$$\int_V \nabla \cdot \mathbf{u}\, dV = \int_0^1 dx \int_0^1 dy \int_0^1 dz\, (8xy + xz + 2yz) = 8 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{11}{4}.$$

7. df

8. mass conservation

$$\frac{\partial(\rho\delta v)}{\partial t} + \frac{\partial(\rho u h)}{\partial x}\delta x = 0 \tag{69}$$

$$\Rightarrow \frac{\partial(\rho h \delta x)}{\partial t} + \frac{\partial(\rho u h)}{\partial x}\delta x = 0 \tag{70}$$

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \tag{71}$$

momentum conservation

$$\frac{\partial(\rho u \delta v)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x}\delta x = 0 \tag{72}$$

$$\Rightarrow \frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} = 0 \tag{73}$$

the total force p due to pressure is

$$p = \int_0^h \rho g y dy = \frac{1}{2}\rho g h^2 \tag{74}$$

so the momentum equation is

$$\frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + \frac{1}{2}\rho g h)}{\partial x} = 0 \tag{75}$$

$$\Rightarrow = \frac{\partial(u h)}{\partial t} + \frac{\partial(u^2 h + \frac{1}{2} g h)}{\partial x} = 0 \tag{76}$$