

# solutions to transportation problems

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## 1 L1

1. scale 1 处于分子尺度，粒子非常稀疏，体微元包含的粒子数的变化 $dN$ 与体微元有关,也就是 $dN = dN(dV)$ ,因而 $\frac{dN}{dV} \neq \text{constant}$ ，也就是密度 $\rho$ 在这个尺度会随体微元的不同而不同.scale 2 处于连续尺度，体元宏观无穷小，而微观无穷大，在此尺度定义密度有：1)定义于一点处，因为体元宏观无穷小，在宏观上就是一点，2)在定义点处连续，因为微观无穷大使体元包含足够多的粒子，这使得附近点之间的粒子数平均不会发生突变，因而密度在空间上是连续分布的.scale 3 处于宏观尺度,这一个尺度上的体元已经感知到了密度的宏观变化

2.

$$\frac{d\alpha}{dt} = \frac{dx/dy}{dt} = \frac{\partial u}{\partial y} \quad (1)$$

$$\frac{d\beta}{dt} = \frac{dy/dx}{dt} = \frac{\partial v}{\partial x} \quad (2)$$

so

$$\frac{d\alpha + d\beta}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \quad (3)$$

$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (4)$$

3.

$$\tau_x = \frac{\partial u}{\partial y} = -\frac{2U_0 y}{b^2} \quad (5)$$

$$\tau_x|_{y=\frac{b}{2}} = -\frac{U_0}{b} \quad (6)$$

4.

$$u = cy \quad (7)$$

$$v = cx \quad (8)$$

$$\tau_x = \mu \frac{\partial \mathbf{v}}{\partial y} = \mu \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} c \\ 0 \end{pmatrix} \quad (9)$$

$$\tau_y = \mu \frac{\partial \mathbf{v}}{\partial x} = \mu \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} 0 \\ c \end{pmatrix} \quad (10)$$

let  $\mathbf{e} = \frac{\mathbf{j} + \mathbf{i}}{\sqrt{2}}$ ,

$$\tau_{xy} = \mu \frac{\partial \mathbf{v}}{\partial e} \quad (11)$$

$$= \mu \left( \frac{\partial \mathbf{v}}{\partial x} \cos(\mathbf{e}, \mathbf{i}) + \frac{\partial \mathbf{v}}{\partial y} \cos(\mathbf{e}, \mathbf{j}) \right) \quad (12)$$

$$= \mu \left( \frac{\partial \mathbf{v}}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial \mathbf{v}}{\partial y} \frac{1}{\sqrt{2}} \right) \quad (13)$$

$$= \frac{\mu}{\sqrt{2}} \left( \begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} \right) \quad (14)$$

$$= \frac{c\mu}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (15)$$

5.

$$2\sigma l = \rho g h l w \quad (16)$$

$$\Rightarrow h = \frac{2\sigma}{\rho g w} \quad (17)$$

$$\sigma = 0.123(1 - 0.00139T) \text{ so}$$

$$h = 9.37 \text{ mm}$$

6.

$$\sigma = 0.025 N/m \quad (18)$$

$$2 \cdot 2\pi r \sigma = \Delta p \pi r^2 \quad (19)$$

$$\Rightarrow \Delta p = \frac{4\sigma}{r} \quad (20)$$

$$\Rightarrow \Delta p = 50 Pa \quad (21)$$

$$7. \rho = 996 kg/m^3, h = 1mm$$

$$\pi \frac{d^2}{4} \rho g h = \sigma \pi d \quad (22)$$

$$\Rightarrow d = \frac{4\sigma}{\rho g h} = 2.989cm \quad (23)$$

$$(24)$$

$$8.$$

$$t = \frac{D_c - D_r}{2} = 0.01cm \quad (25)$$

$$F = \tau A \quad (26)$$

$$= \tau \pi D_r L \quad (27)$$

$$= \mu \frac{v}{t} \pi D_r L \quad (28)$$

$$= \frac{0.85 \times 1000 \times 3.7 \times 10^{-4} \times 0.15 \times 0.3602 \times 3.14 \times \pi}{1 \times 10^{-4}} \quad (29)$$

$$= 1676N \quad (30)$$

$$9.$$

$$G = F \quad (31)$$

$$mg = \tau A \quad (32)$$

$$= \mu \frac{v}{t} A \quad (33)$$

$$= \mu \frac{v}{t} \pi D_r L \quad (34)$$

$$10.$$

$$dM = r dF \quad (35)$$

$$= r \tau dA \quad (36)$$

$$\tau = \mu \frac{v}{t} = \mu \frac{\omega r}{t} \quad (37)$$

$$dA = 2\pi r \cdot r dx \quad (38)$$

$$r = x \sin \alpha \quad (39)$$

$\Rightarrow$

$$M = \int_0^{D/2} r \mu \frac{r\omega}{h} 2\pi r \frac{dr}{\sin \alpha} \quad (40)$$

$$= \frac{\pi \mu \omega D^4}{32h \sin \alpha} \quad (41)$$

## 2 L2

1.

$$\begin{cases} \frac{dp}{dr} = f = \rho a \\ a = kr \\ kR = g \end{cases} \quad (42)$$

$\Rightarrow$

$$\frac{dp}{dr} = \frac{\rho gr}{R} \quad (43)$$

$\Rightarrow$

$$p = \frac{\rho gr^2}{2R} + p_{atm} \quad (44)$$

so

$$p|_{r=R} = \frac{\rho g R}{2} + p_{atm} \quad (45)$$

$$\approx \frac{\rho g R}{2} \quad (46)$$

$$= 176 kPa \quad (47)$$

2.

$$\rho g \Delta h = \Delta p \quad (48)$$

$$\Rightarrow \Delta h = \frac{\Delta p}{\rho g} \quad (49)$$

$\Rightarrow$

$$\Delta h_{water} = 10.33m \quad (50)$$

$$\Delta h_{sea} = \Delta h_{water} / 1.025 = 10.08m \quad (51)$$

$$\Delta h_{Hg} = \Delta h_{water} / 13.6 = 0.76m \quad (52)$$

3. let Hg,water,oil denoted by "H, w, o" respectively

$$p_A + \rho_o g h_o + \rho_w g h_w = p_{atm} + \rho_H g h_H \quad (53)$$

$$\Rightarrow p_A - p_{atm} = g(\rho_H h_H - \rho_o h_o - \rho_w h_w) \quad (54)$$

$\Rightarrow$

$$p_A = 588.6 \text{ pa} \quad (55)$$

4. the resultant acceleration is gravity and inertial

$$\mathbf{r} = \mathbf{g} - \mathbf{a} \quad (56)$$

now the isobars and the direction of the pressure gradient is depict as follow

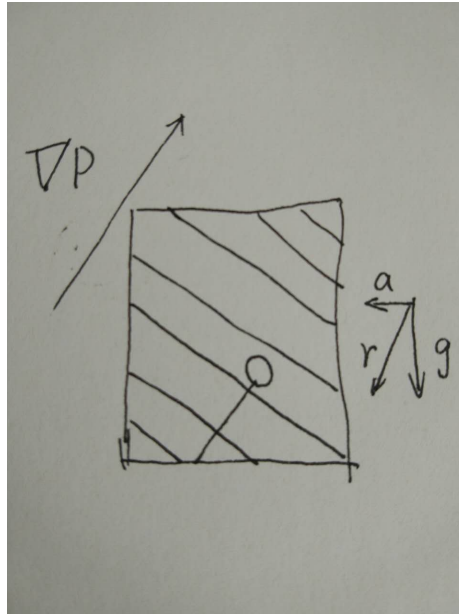


图 1: the balloon

5.

$$dF_y = p_y dA \quad (57)$$

$$\begin{cases} dA = 2\pi r' r d\theta \\ r' = r \sin \theta \\ p_y = p \cos \theta \\ p = \rho g h' \\ h' = h - r \cos \alpha + r \cos \theta \end{cases} \quad (58)$$

$\Rightarrow$

$$dF_y = \rho g (h - r \cos \alpha + r \cos \theta) \cos \theta 2\pi r \sin \theta r d\theta \quad (59)$$

$$= 2\pi \rho g r^2 (h - r \cos \alpha + r \cos \theta) \sin \theta \cos \theta d\theta \quad (60)$$

$$F_y = 2\pi \rho g r^2 \int_{\alpha}^{\pi} (h - r \cos \alpha + r \cos \theta) \sin \theta \cos \theta d\theta \quad (61)$$

$$(62)$$

altanitive solution:

$$F_y + F_D = \rho g V \quad (63)$$

$$F_D = \rho g S_D h = \rho g \cdot \pi \left(\frac{D}{2}\right)^2 \cdot h \quad (64)$$

$$V = V_0 - \frac{\pi}{3} (3r - l) l^2 \quad (65)$$

$$l = r - \sqrt{r^2 - \left(\frac{D}{2}\right)^2} \quad (66)$$

let  $F_y(h) = 0$ , comes the requiring

$$h = \frac{2r}{3 \sin^2 \alpha} + \frac{2r \cos^3 \alpha}{3 \sin^2 \alpha} + r \cos \alpha \quad (67)$$

6.

$$\begin{cases} F = \rho g h A \\ A = \pi r^2 \end{cases} \quad (68)$$

and the acting point is

$$\Delta y_a = \frac{I_{xx}}{hA} = \frac{\pi r^2}{4hA} \quad (69)$$

the momentum equilibrium

$$F\Delta y_a = Pr \quad (70)$$

$\Rightarrow$

$$P = \frac{F\Delta y_a}{r} \quad (71)$$

$$= \frac{\rho g \pi r}{4} \quad (72)$$

$$= 7.7kN \quad (73)$$

7.

$$F_p = \rho g \bar{h} A \quad (74)$$

$$A = lw \quad (75)$$

$$\Delta y_a = \frac{I_{xx}}{y_c A} \quad (76)$$

$$I_{xx} = \frac{l^3 w}{12} \quad (77)$$

$$\tau w l_c = F_p \Delta y_a \quad (78)$$

$\Rightarrow$

$$\tau = \frac{F_p \Delta y_a}{w l_c} \quad (79)$$

$$= 145.2kN \quad (80)$$

8.

$$F = \rho g \bar{h} A = 10^{10} N \quad (81)$$

$$y_a = \frac{2}{3} h = 85.3m \quad (82)$$

9.

$$\begin{aligned}
F_{up} &= F_{down} \\
F + \rho_w g x \left( \frac{4}{3} \pi R^3 \right) &= r g \left( \frac{4}{3} \pi R^3 \right) \\
x &= \frac{r g \left( \frac{4}{3} \pi R^3 \right) - F}{\rho_w g \left( \frac{4}{3} \pi R^3 \right)}
\end{aligned}$$

10.

$$\rho_s v_s = \rho_l v_l \quad (83)$$

$$\Rightarrow \quad \frac{v_l}{v_s} = \frac{\rho_s}{\rho_l} \quad (84)$$

$$(85)$$

$$S = v_s - v_l \quad (86)$$

$$\Rightarrow \quad \frac{1}{2} L L \tan \theta = v_s - v_l = 0.1 L^2 \quad (87)$$

$$\Rightarrow \quad \tan \theta = 0.2 \quad (88)$$

### 3 L3-L4

1. since

$$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) = \mathbf{g}(\mathbf{c}) h(t) \quad (89)$$

 $\Rightarrow$ 

$$\mathbf{g}(\mathbf{c}) = \frac{\mathbf{r}}{h(t)} \quad (90)$$

 $\Rightarrow$ 

$$\mathbf{c} = \mathbf{g}^{-1} \left( \frac{\mathbf{r}}{h(t)} \right) \quad (91)$$

so

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} \quad (92)$$

$$= \mathbf{g}(\mathbf{c}) \dot{h}(t) \quad (93)$$

$$= \mathbf{g}(\mathbf{g}^{-1}(\frac{\mathbf{r}}{\mathbf{h}(\mathbf{t})})) \dot{h}(t) \quad (94)$$



2.

$$g(c) = c \quad (95)$$

$$h(t) = t^2 \quad (96)$$

and it is easy to find

$$g^{-1}(c) = c \quad (97)$$

$$v = g(g^{-1}(\frac{x}{h(t)}))\dot{h}(t) \quad (98)$$

$$= g^{-1}(\frac{x}{h(t)})\dot{h}(t) \quad (99)$$

$$= \frac{x}{h(t)}\dot{h}(t) \quad (100)$$

$$= 2t\frac{x}{t^2} \quad (101)$$

$$= 2\frac{x}{t} \quad (102)$$

$$(103)$$

3. at time  $t$  the  $c$ th element is at  $f(c, t)$  , so the temperature of the  $c$ th element is

$$T = g(f(c, t), t) \quad (104)$$

so the variation rate is

$$\frac{dT}{dt} = \frac{\partial g}{\partial x} \frac{df}{dt} + \frac{\partial g}{\partial t} \quad (105)$$

4.

$$\begin{cases} \frac{dx}{dt} = \frac{x}{1+t} \\ \frac{dy}{dt} = \frac{2y}{2+t} \end{cases} \quad (106)$$

$\Rightarrow$

$$\begin{cases} x = c_x(1+t) \\ y = c_y(2+t)^2 \end{cases} \quad (107)$$

with the boundary condition

$$\mathbf{x}(0) = \mathbf{x}_0 \quad (108)$$

the path line through  $\mathbf{x}_0$  is

$$\begin{cases} x = x_0(1+t) \\ y = \frac{y_0}{4}(2+t)^2 \end{cases} \quad (109)$$

and the streamlines at  $t = 0$  is

$$\begin{cases} x = c_x \\ y = 2c_y \end{cases} \quad (110)$$

$\Rightarrow$

$$y = 2 \frac{c_y}{c_x} x \quad (111)$$

5.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (112)$$

since

$$\rho = \rho_0(2 - \cos \omega t) \quad (113)$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = -\frac{\partial \rho}{\rho \partial t} \quad (114)$$

$$= \frac{\omega \sin \omega t}{\cos \omega t - 2} \quad (115)$$

$$:= f(t) \quad (116)$$

$\Rightarrow$

$$u = \int f(t) dx \quad (117)$$

$$= f(t)x + C \quad (118)$$

apply the boundary condition  $u(0, t) = U$

$$u = f(t)x + U \quad (119)$$

6. df

$$\begin{aligned}
\text{(a)} \quad \int_{A=\partial V} \rho \mathbf{u} \cdot d\mathbf{A} &= \int_0^1 dy \int_0^1 dz 4x^2 y \Big|_{x=1} - \int_0^1 dy \int_0^1 dz 4x^2 y \Big|_{x=0} \\
&\quad + \int_0^1 dz \int_0^1 dx xyz \Big|_{y=1} - \int_0^1 dz \int_0^1 dx xyz \Big|_{y=0} \\
&\quad + \int_0^1 dx \int_0^1 dy yz^2 \Big|_{z=1} - \int_0^1 dx \int_0^1 dy yz^2 \Big|_{z=0} \\
&= 2 + 0 + \frac{1}{4} + 0 + \frac{1}{2} + 0 = \frac{11}{4}.
\end{aligned}$$

$$\text{(b)} \quad \nabla \cdot \mathbf{u} = \partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 8xy + xz + 2yz$$

$$\begin{aligned}
\int_V \nabla \cdot \mathbf{u} dV &= \int_0^1 dx \int_0^1 dy \int_0^1 dz (8xy + xz + 2yz) \\
&= 8 \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{11}{4}.
\end{aligned}$$

7. mass inlet=mass outlet

$$v_1 \cdot 2h = v_2 \cdot 2h - \frac{h \cdot v_2}{2} \quad (120)$$

$$\Rightarrow v_1 = 0.75v_2 \quad (121)$$

8. mass conservation

$$\frac{\partial(\rho \delta v)}{\partial t} + \frac{\partial(\rho u h)}{\partial x} \delta x = 0 \quad (122)$$

$$\Rightarrow \frac{\partial(\rho h \delta x)}{\partial t} + \frac{\partial(\rho u h)}{\partial x} \delta x = 0 \quad (123)$$

$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \quad (124)$$

momentum conservation

$$\frac{\partial(\rho u \delta v)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} \delta x = 0 \quad (125)$$

$$\Rightarrow \frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} = 0 \quad (126)$$

the total force  $p$  due to pressure is

$$p = \int_0^h \rho g y dy = \frac{1}{2} \rho g h^2 \quad (127)$$

so the momentum equation is

$$\frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + \frac{1}{2} \rho g h)}{\partial x} = 0 \quad (128)$$

$$\Rightarrow = \frac{\partial(u h)}{\partial t} + \frac{\partial(u^2 h + \frac{1}{2} g h)}{\partial x} = 0 \quad (129)$$

9. for stream line

$$\frac{dx}{u} = \frac{dy}{v} \quad (130)$$

$$\frac{dx}{Kx} = \frac{dy}{-Ky} \quad (131)$$

$$\ln x = -\ln y + \ln c \quad (132)$$

$$(133)$$

$$\Rightarrow xy = c$$

10. it is a incompressible flow,so

$$\nabla \cdot \mathbf{V} = 0 \quad (134)$$

$$a_1 + b_2 + c_3 = 0 \quad (135)$$