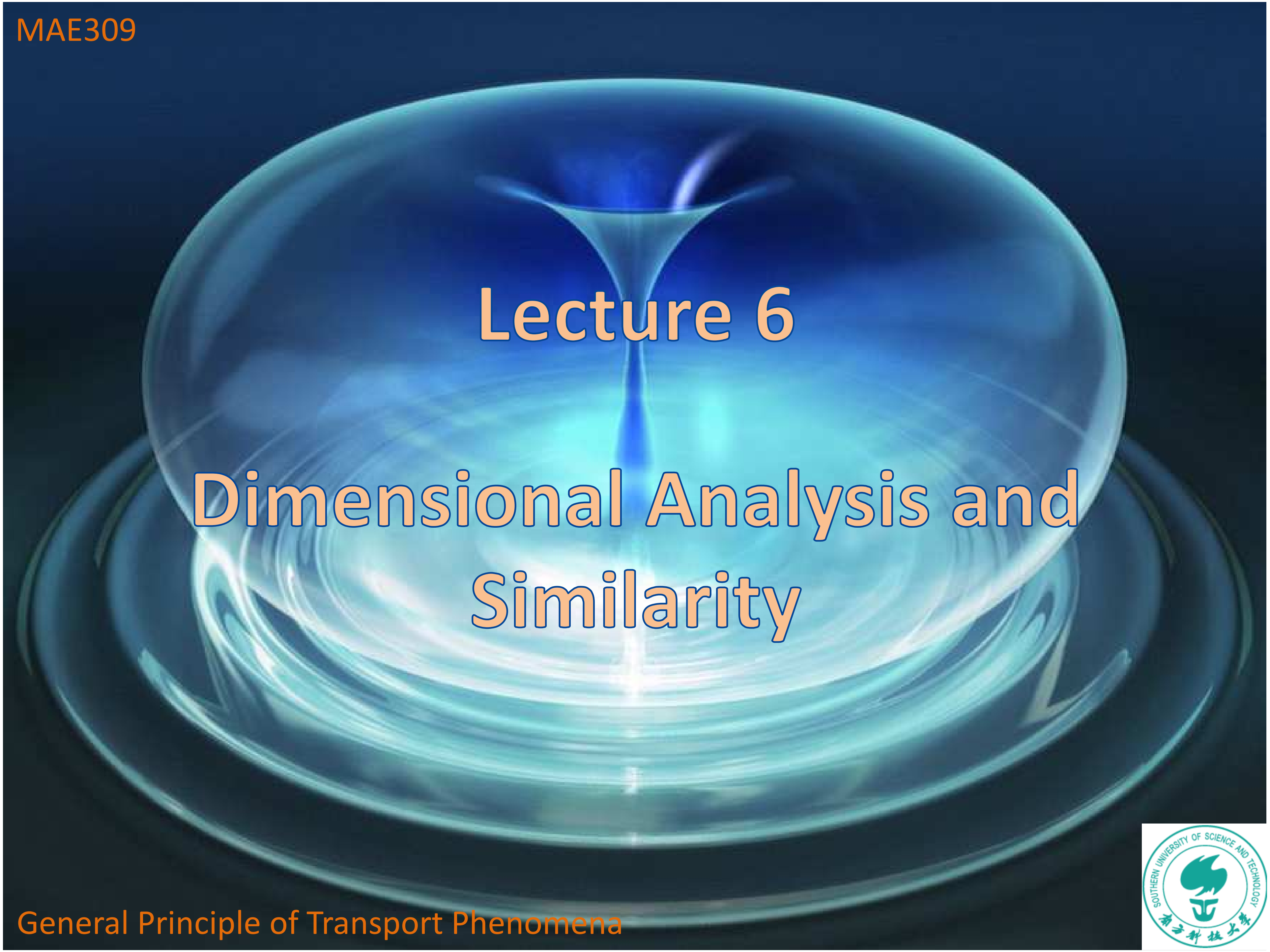


MAE309

Lecture 6

Dimensional Analysis and Similarity

General Principle of Transport Phenomena

Learning Objectives

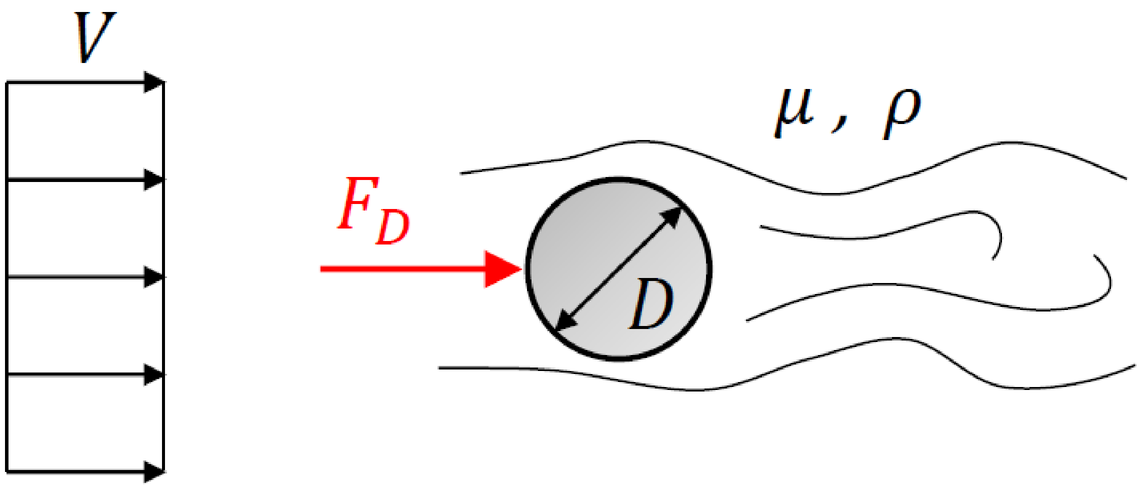
- To understand
  - Dimensions & Unit
  - Be able to determine the dimensions of physical quantities in terms of fundamental dimensions.
  - Understand the Principle of Dimensional Homogeneity and its use in checking equations and reducing physical problems.
  - Be able to carry out a formal dimensional analysis using Buckingham’s Pi Theorem and Rayleigh Method
  - Nondimensionalize governing equations
  - Physical similarity
  - Geometric, kinematic, dynamic similarities

Dimensional Analysis

- Overview
  - Dimensional analysis (量纲分析) is a powerful tool in formulating problems of physical phenomena, which defy analytical solution and must be solved experimentally.
  - This is accomplished by the formation of dimensionless groups containing relevant variables.
  - It is based on the assumption that certain variables, which affect the phenomena are independent variables.
  - It is based on the principle of dimensional homogeneity.

Dimensional Analysis

- Overview
  - Drag force acting on a sphere immersed in a fluid flow

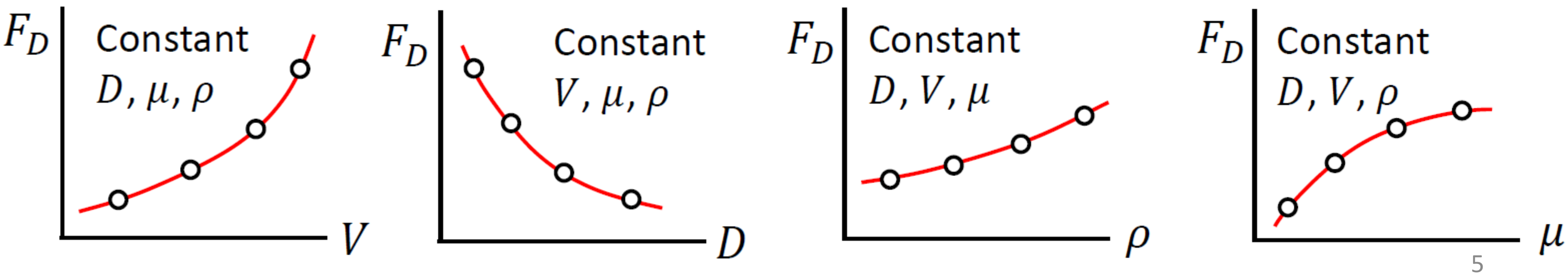


  - Drag force F_D may depend on the following variables

$$F_D = f\left(D, V, \mu, \rho\right)$$

Dimensional Analysis

- Overview
  - In order to find the actual functional relation, the following set of controlled experiments should be done.
    - Fix D, μ and ρ. Change V and measure F_D.
    - Fix V, μ and ρ. Change D and measure F_D.
    - Fix D, V and μ . Change ρ and measure F_D.
    - Fix D, V and ρ. Change μ and measure F_D
  - We need to perform too many experiments



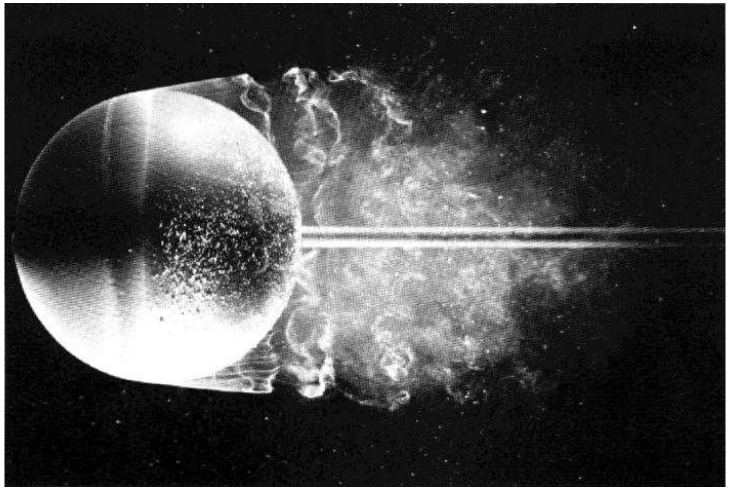
Dimensional Analysis

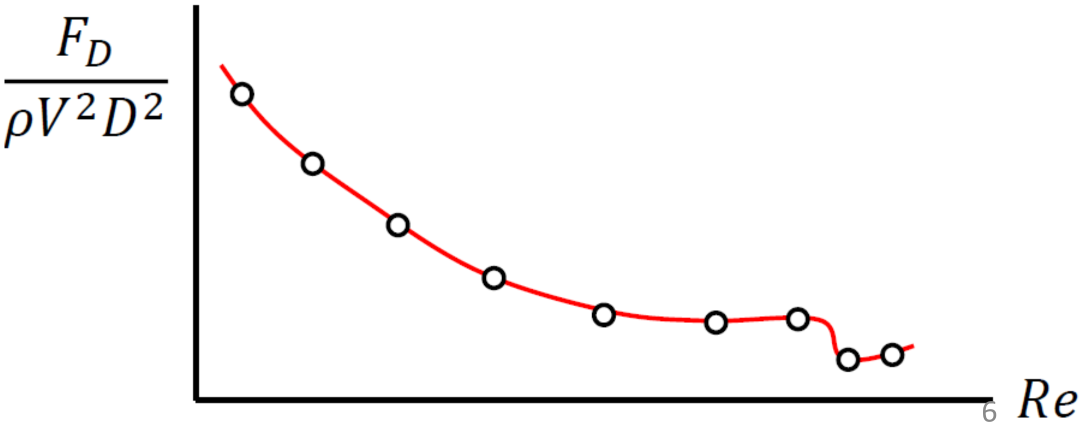
- Overview
  - It is possible to simplify the dependency of drag force on other variables by using nondimensional parameters.

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}\right)$$

Dimensionless drag force
(Drag coefficient)

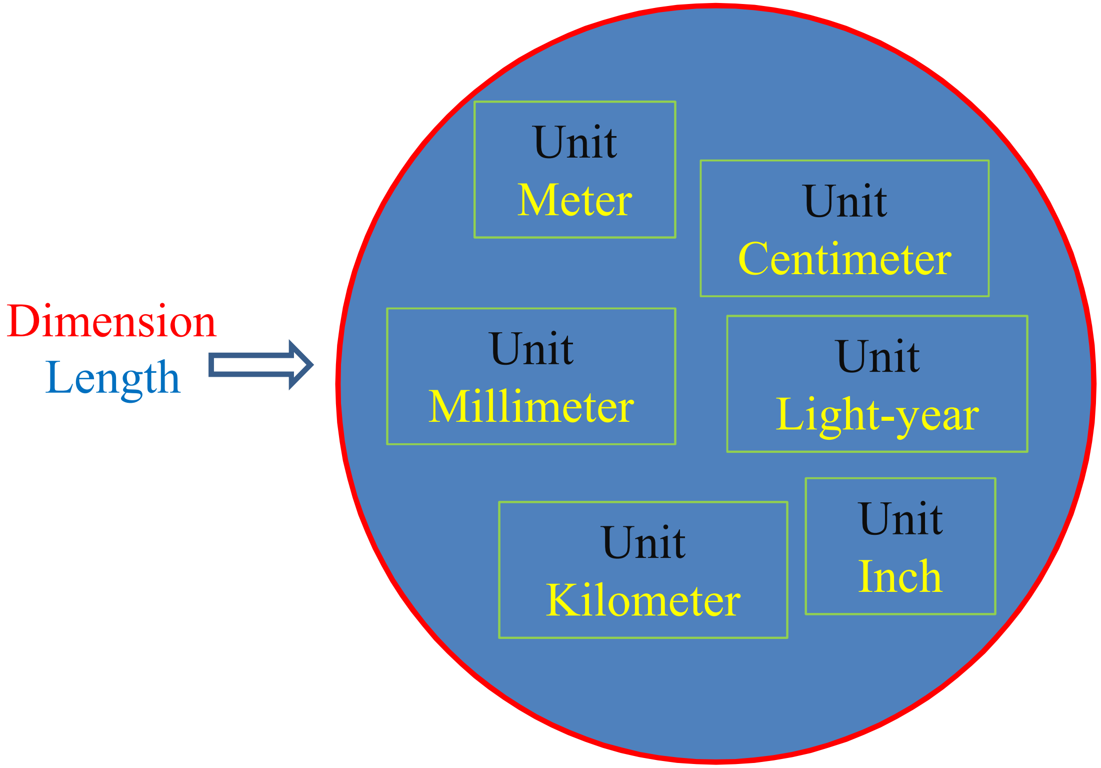
Reynolds number





Dimensional Analysis

- Dimensions (量纲) and Unit (单位)
  - A dimension is the type of physical quantity.
  - A unit is a means of assigning a numerical value to that quantity.



SI units are preferred in scientific work

Dimensional Analysis

- Primary (Basic/Fundamental) Dimensions
  - In <General Principles of Transport Phenomena>, the primary or fundamental dimensions, together with their SI units are:

Dimensions	Symbol	SI Unit
Mass	M	kilogram, kg
Length	L	meter, m
Time	T	second, s
Temperature	Θ	kelvin, K
Amount of Substance	n	mole, mol

Dimensional Analysis

- Dimensions of Derived Quantities

	Quantity	Symbols	Dimensions
Geometry	Area	A	L^2
	Volume	V	L^3
	Second moment of area	I	L^4
Kinematics	Velocity	U	LT^{-1}
	Acceleration	a	LT^{-2}
	Mass flow rate	$\dot{m}$	MT^{-1}
	Volume flow rate	Q	L^3T^{-1}
Dynamics	Force	F	MLT^{-2}
	Energy, work	E, W	ML^2T^{-2}
	Pressure, stress	p, τ	$ML^{-1}T^{-2}$

✓ Variables having only L in their dimension are called geometric variables
✓ Variables having only T or both L and T are called kinematic variables
✓ Variables having M in their dimension are called dynamic variables

Dimensional Analysis

- Dimensions of Derived Quantities

	Quantity	Symbols	Dimensions
Fluid Properties	Density	ρ	ML^{-3}
	Viscosity	μ	$ML^{-1}T^{-1}$
	Kinematic viscosity	ν	$L^{2}T^{-1}$
	Surface tension	σ	MT^{-2}
	Thermal conductivity	k	$MLT^{-3}\Theta^{-1}$
	Specific heat	Cp, Cv	$L^{2}T^{-2}\Theta^{-1}$
	Bulk modulus	K	$ML^{-1}T^{-2}$

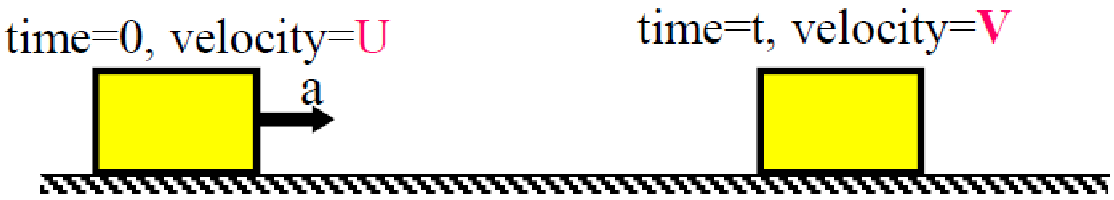
Dimensional Analysis

- Dimensional Homogeneity
  - Two quantities may be added or subtracted if and only if they have the same dimensions.
  - Principle of Dimensional Homogeneity: If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; i.e., each of its additive terms will have the same dimensions.
  - The principle holds for equations of all types.
  - Dimensional homogeneity is a useful tool for checking formulae.

Dimensional Analysis

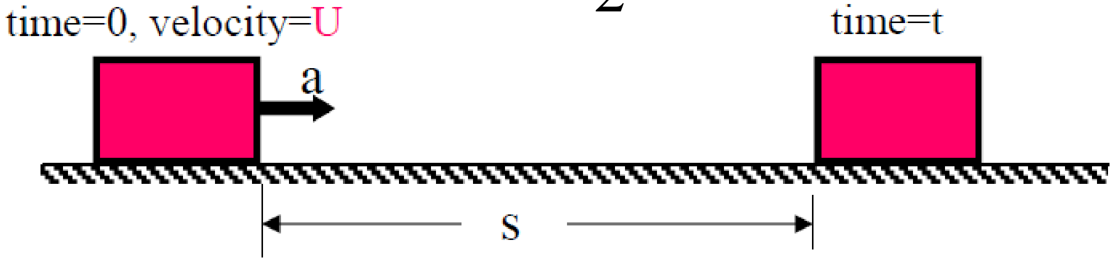
- Dimensional Homogeneity

$$V = U + at$$



The above equation is dimensionally homogeneous because the dimensions of V, U, and at are the same

$$S = Ut + \frac{1}{2}at$$



The above equation is dimensionally incorrect because "*at*" does not have the same dimension as S and Ut.

Dimensional Analysis

- Dimensional Homogeneity
  - The Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant}$$

is dimensionally correct because each term on the left hand side has a dimension of L^2T^{-2} (This implies that the constant also has the same dimension)

  - Steady Euler Equation

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

is dimensionally correct because all the terms have the same dimension of LT^{-2}.

Dimensional Analysis

- Example 1
  - Determine the dimensions of viscosity according to Newton's Law of Viscosity

$$\tau = \mu \frac{du}{dy}$$

  - Solution

From the definition,

$$\mu = \frac{\tau}{du/dy} = \frac{\text{force/area}}{\text{velocity/length}}$$

Hence

$$[\mu] = \frac{[\mathrm{MLT^{-2}}]/[\mathrm{L^2}]}{[\mathrm{LT^{-1}}]/[\mathrm{L}]} = [\mathrm{ML^{-1}T^{-1}}]$$

Dimensional Analysis

- Example 2
  - Determine the dimensions of viscosity according to definition of Reynolds number

$$\mathrm{Re} = \frac{\rho U L}{\mu}$$

  - Solution

    Since the Reynolds number is dimensionless, the dimensions of μ must be the same as those of ρUL; i.e.

$$[\mu] = [\rho UL] = [\mathrm{ML}^{-3}][\mathrm{LT}^{-1}][\mathrm{L}] = [\mathrm{ML}^{-1}\mathrm{T}^{-1}]$$

Dimensional Analysis

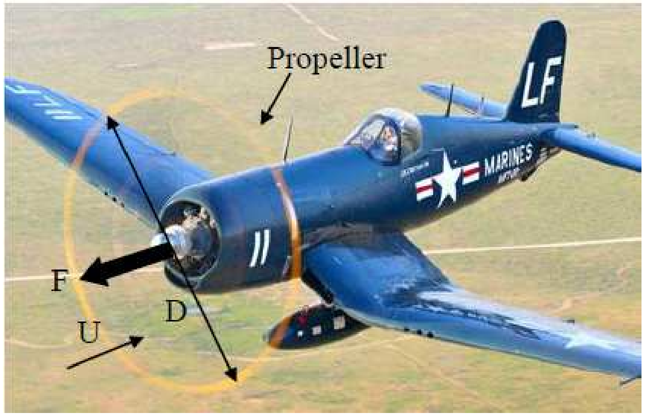
- Determination of Dimensionless Group
  - It is based on the principle of Dimensional Homogeneity
  - There are two methods of finding dimensionless groups, they are:
    - ✓ Rayleigh's Method
    - ✓ Buckingham's Pi Theorem

Dimensional Analysis

- Determination of Dimensionless Group
  - Rayleigh's Method
    - ✓ This method consists of dealing with all variables together.
    - ✓ It involves setting up a set of three indicial equations (M, L, T) and solving them to obtain the dimensionless groups in the desired functional form.

Dimensional Analysis

- Rayleigh's Method
  - Example 3: Force (F, MLT^{-2}) acting on the propeller of an aircraft.



Symbol	Variable	Dimension
U	forward velocity	LT^{-1}
ρ	density	ML^{-3}
μ	dynamic viscosity	$ML^{-1}T^{-1}$
D	Diameter of propeller	L
N	Rotating speed of propeller	T^{-1}

Dimensional Analysis

- Rayleigh's Method
  - Example 3: Force (F, MLT^{-2}) acting on the propeller of an aircraft.
    - From the dimensional consideration, the following equation may be written

$$F = CU^a \rho^b \mu^c D^d N^e$$

where C is a dimensionless constant, and a, b, c, d, e are unknown indices

    - Substitute the dimensions of the variable in the above equation

$$\left[MLT^{-2}\right] = \left[LT^{-1}\right]^a \left[ML^{-3}\right]^b \left[ML^{-1}T^{-1}\right]^c \left[L\right]^d \left[T^{-1}\right]^e$$

$$\left[MLT^{-2}\right] = \left[M^{b+c} L^{a-3b-c+d} T^{-a-c-e}\right]$$

Dimensional Analysis

- Rayleigh's Method
  - Example 3: Force (F, MLT^{-2}) acting on the propeller of an aircraft.
    - Equating the indices of M, L and T gives

$$b + c = 1$$
$$a - 3b - c + d = 1$$
$$-a - c - e = -2$$

    - There are 3 equations with 5 unknowns, the most one can do is to express three of them in terms of the other two unknowns (e.g. c and e)

$$a = 2 - c - e$$
$$b = 1 - c$$
$$d = 2 - c + e$$

Dimensional Analysis

- Rayleigh's Method
  - Example 3: Force (F, MLT^{-2}) acting on the propeller of an aircraft.
    - Substituting the above values in the force equation

$$F = CU^{(2-c-e)}\rho^{(1-c)}\mu^{c}D^{(2-c+e)}N^{e}$$

    - Regroup the variables gives

$$F = C\rho U^{2}D^{2}\left(\frac{\mu}{\rho UD}\right)^{c}\left(\frac{ND}{U}\right)^{e}$$

$$\frac{F}{\rho U^{2}D^{2}} = C\left(\frac{\mu}{\rho UD}\right)^{c}\left(\frac{ND}{U}\right)^{e}$$

Dimensional Analysis

- Rayleigh's Method
  - Example 3: Force (F, MLT^{-2}) acting on the propeller of an aircraft.
    - Instead of express in terms of unknown quantities C, c, and e, the above equation can also be expressed in function form:

$$\frac{F}{\rho U^2 D^2} = f\left(\frac{\mu}{\rho U D}, \frac{ND}{U}\right)$$

Or

$$\underbrace{C_d}_{\text{Drag coefficient}} = \frac{F}{\rho U^2 D^2} = f_1\left(\underbrace{\frac{\rho U D}{\mu}}_{\text{Reynolds number}}, \underbrace{\frac{ND}{U}}_{\text{Speed parameter}}\right)$$

Dimensional Analysis

- Determination of Dimensionless Group
  - Buckingham's Pi Theorem
    - ✓ This method is widely used in the dimensional analysis of a problem.
    - ✓ The name Pi comes from the mathematical notation Π, meaning a product of variables.
    - ✓ The dimensionless groups found from the theorem are power products denoted by Π_1, Π_2, Π_3, etc.
    - ✓ However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

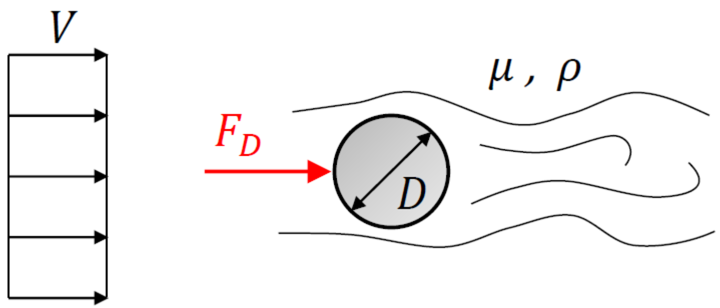
Dimensional Analysis

- Determination of Dimensionless Group
  - Buckingham's Pi Theorem
    - ✓ According to this theorem, we assume a phenomenon involves n variables, say, a_1, a_2, a_3, … a_n.
    - ✓ If a_1 is a dependent variable, and the rest a_2, a_3, a_4, … a_n are independent variables, then the general functional relationship between independent and dependent variables can be expressed as:

$$a_1 = f_1\left(a_2, a_3, a_4, \ldots, a_n\right)$$

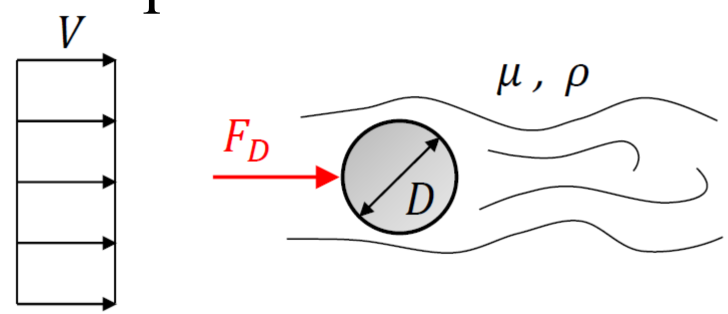
or

$$f_2\left(a_1, a_2, a_3, \ldots, a_n\right) = 0$$



Dimensional Analysis

- Determination of Dimensionless Group
  - Buckingham's Pi Theorem



✓ If a problem involves

n relevant variables

m independent dimensions

Then it can be reduced to a relationship between

$n - m$ non-dimensional parameters $\Pi_1, \Pi_2, \ldots, \Pi_{n\text{-}m}$.

✓ To construct these non-dimensional Π groups:

1. Choose m **dimensionally-distinct** scaling variables (repeating variables).
2. For each of the $n - m$ remaining variables construct a non-dimensional Π of the form

$$\Pi_1 = (\text{variable}_1)(\text{scale}_1)^{a1}(\text{scale}_2)^{b1}(\text{scale}_3)^{c1}\ldots$$

$$\Pi_2 = (\text{variable}_2)(\text{scale}_1)^{a2}(\text{scale}_2)^{b2}(\text{scale}_3)^{c2}\ldots$$

where a, b, c, ... are chosen so as to make each Π dimensionless.

Dimensional Analysis

- Determination of Dimensionless Group
  - Buckingham's Pi Theorem
    - ✓ If each of the basic dimensions (M, L, T) are involved, then m = 3 and the results will be an equation with (n – 3) dimensionless parameters.
    - ✓ In thermodynamics, another basic dimension is involved, namely θ (temperature). Therefore, if the process involves the basic dimensions such as M, L, T, Θ, then m = 4 and the results will be an equation with (n – 4) dimensionless parameters.

Dimensional Analysis

- Determination of Dimensionless Group
  - Buckingham's Pi Theorem

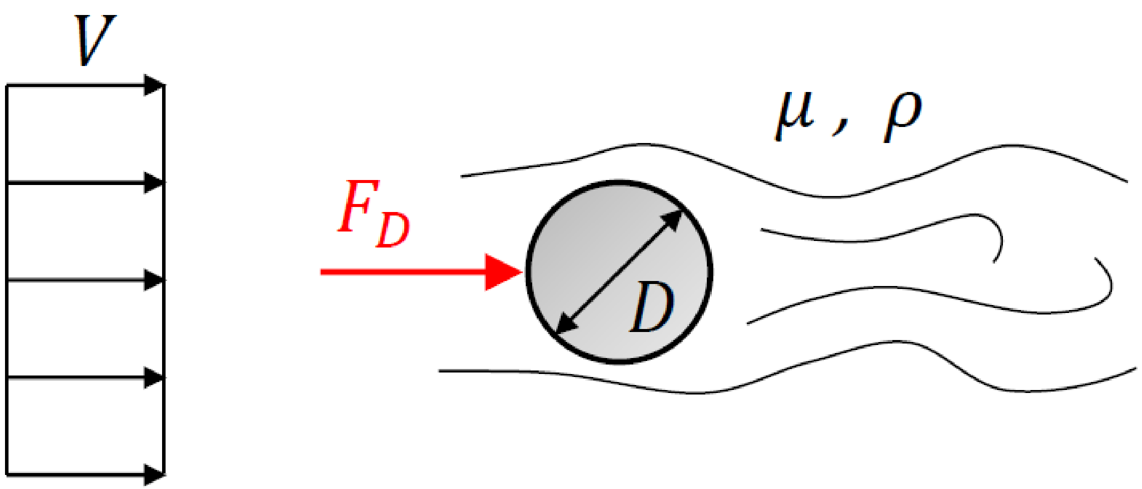
    ✓ In the fluid mechanics problems, it is usual to select repeating variables using the following guidelines:.

    1. Select the first repeating variable from those describing the geometry of the flow (such as size and shape of the fluid passage or for a moving body such as diameter, length, etc).
    2. Select the second repeating variable from those representing fluid properties (such as density, viscosity, surface tension, etc)
    3. Select the third repeating variable from those characterising fluid motion (such as, velocity, acceleration, discharge, etc)

    ✓ It is usually conventional to select a length dimension to represent category 1 (geometric variable), and density for category 2 (dynamic variable), and velocity for category 3 (kinematic variable).

Dimensional Analysis

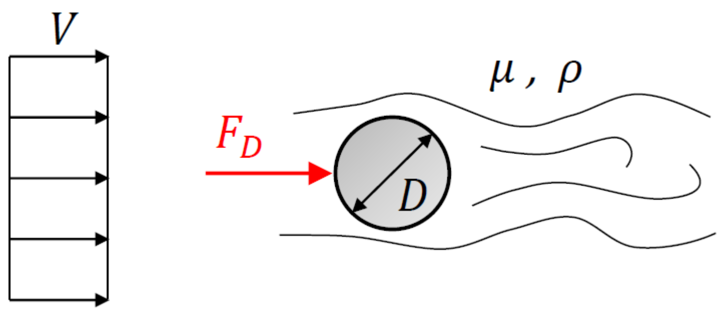
- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow
    - ✓Determine the dimensionless groups of variables.



Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow

✓ Solution:



Step 1:

- List all the dimensional variables involved in the problem F_D, D, V, μ, ρ
- The number of dimensional variables is $n = 5$
- These variables should be independent of each other. For example if the diameter of a sphere is in the list, frontal area of the sphere can not be included.
- If body forces are important in a problem, gravitational acceleration should be in the list, although it is a constant.

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow

    ✓ Solution:

    Step 2:

    - There are $r = 3$ basic dimensions. For most of fluid mechanics problems, r will be 3.

      L: Length, T: time, M: mass

    - Express each of the variables in terms of basic dimensions, which are

      $$[F_D]=\left[\mathrm{MLT^{-2}}\right],\quad [D]=[\mathrm{L}],\quad [V]=\left[\mathrm{LT^{-1}}\right],\quad [\rho]=\left[\mathrm{ML^{-3}}\right],\quad [\mu]=\left[\mathrm{ML^{-1}T^{-1}}\right]$$

    - D is a geometric, V is a kinematic, and F_D , μ and ρ are dynamic variables.

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow

✓ Solution:

<u>Step 3:</u>

- Determine the repeating variables that are allowed to appear in more than one Pi group. There should be $r = 3$ repeating variables.
- Since L, T, M are basic dimensions, we should select one geometric variable, one kinematic variable, and one dynamic variable as the repeating variables.
- In the present example, we can select D , V and ρ as repeating variables
- Note that this selection is not unique and the resulting Pi groups will depend on our selection. Certain selections are "better" than others.
- If there is an obvious dependent variable in the problem, do not select it as a repeating variable. In our example F_D is a dependent variable. We are trying to understand how it depends on other variables.

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow

✓ Solution:

Step 4:

- Determine $(n - r)$ many Pi groups by combining repeating variables with nonrepeating variables and using the fact that Pi groups should be dimensional. For our example we need to find $5 - 3 = 2$ Pi groups. Each Pi group will include only one of the nonrepating variables.

$$\Pi_1 = \underbrace{F_D}_{\text{A nonrepeating parameter}} \underbrace{D^a V^b \rho^c}_{\text{Unknown combination of repeating parameters}}$$

We need to determine a, b, and c

- Π_1 should be unitless: $[-] = \left[\mathrm{MLT^{-2}}\right][\mathrm{L}]^a\left[\mathrm{LT^{-1}}\right]^b\left[\mathrm{ML^{-3}}\right]^c = \left[\mathrm{M}^{1+c}\mathrm{L}^{1+a+b-3c}\mathrm{T}^{-2-b}\right]$

$$\left.\begin{aligned} 1+c&=0 \\ 1+a+b-3c&=0 \\ -2-b&=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a&=-2 \\ b&=-2 \\ c&=-1 \end{aligned}\right\} \Rightarrow \Pi_1 = \frac{F_D}{\rho D^2 V^2}$$

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 4: Drag act on sphere immersed in a fluid flow

    ✓ Solution:

    Step 4:

    - Now determine the second Pi group which has μ as the nonrepeating variable

    $$\Pi_2 = \mu D^a V^b \rho^c$$

    - Π_2 should be unitless: $\left[-\right] = \left[\mathrm{ML^{-1}T^{-1}}\right]\left[\mathrm{L}\right]^a\left[\mathrm{LT^{-1}}\right]^b\left[\mathrm{ML^{-3}}\right]^c = \left[\mathrm{M}^{1+c}\mathrm{L}^{-1+a+b-3c}\mathrm{T}^{-1-b}\right]$

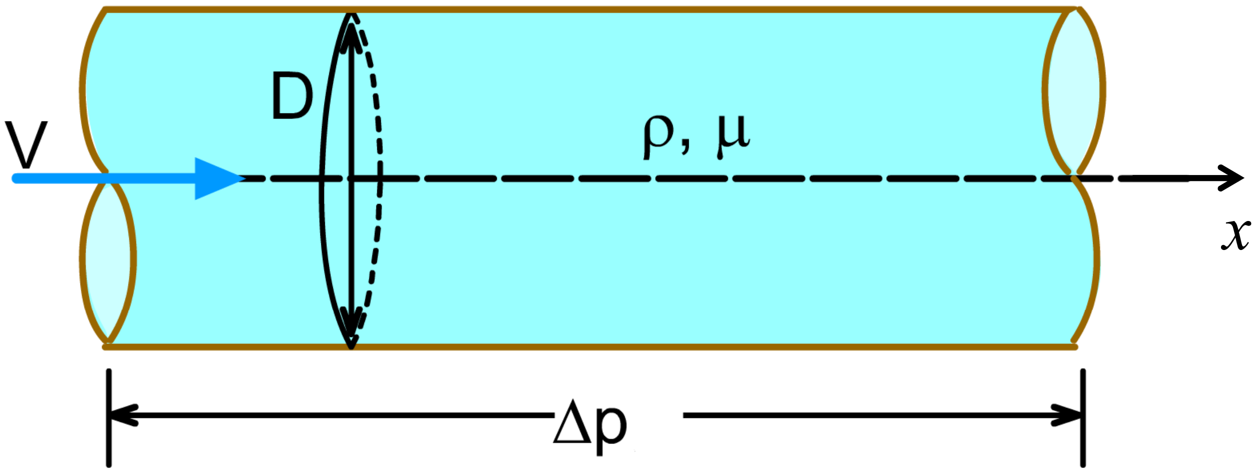
    $$\left.\begin{array}{l} 1+c=0 \\ -1+a+b-3c=0 \\ -1-b=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a=-1 \\ b=-1 \\ c=-1 \end{array}\right\} \Rightarrow \Pi_2 = \frac{\mu}{\rho D V}$$

    - Therefore the relation of nondimensional groups is

    $$\Pi_1 = f_1\left(\Pi_2\right) \Rightarrow \frac{F_D}{\rho V^2 D^2} = f_1\left(\frac{\mu}{\rho V D}\right)$$

Dimensional Analysis

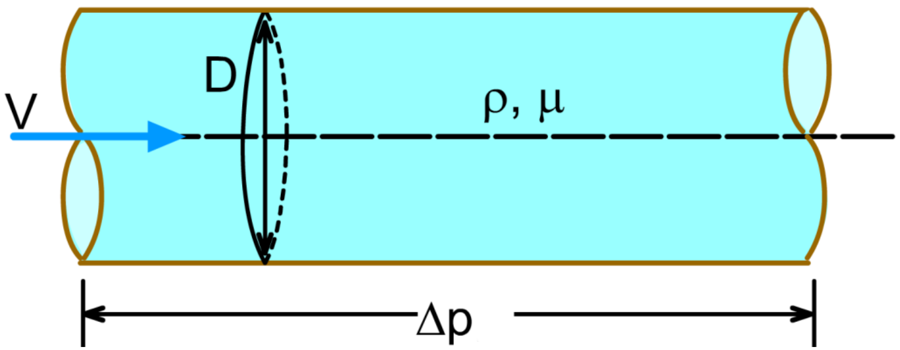
- Buckingham's Pi Theorem
  - Example 5: Obtain an expression in non-dimensional form for the pressure gradient $\Delta p/\Delta x$ in a horizontal pipe of circular.



Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 5: Pressure gradient $\Delta p/\Delta x$ in a circular pipe

    ✓ Solution:



Step 1:

- List all the dimensional variables involved in the problem

$$\Delta p/\Delta x,\ D,\ V,\ \mu,\ \rho$$

- The number of dimensional variables is $n = 5$

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 5: Pressure gradient $\Delta p/\Delta x$ in a circular pipe

    ✓ Solution:

    Step 2:

    - There are $r = 3$ basic dimensions. For most of fluid mechanics problems, r will be 3.

      L: Length, T: time, M: mass

    - Express each of the variables in terms of basic dimensions, which are

$$[\Delta p/\Delta x] = \left[\mathrm{ML^{-2}T^{-2}}\right], \quad [D] = [\mathrm{L}], \quad [V] = \left[\mathrm{LT^{-1}}\right], \quad [\rho] = \left[\mathrm{ML^{-3}}\right], \quad [\mu] = \left[\mathrm{ML^{-1}T^{-1}}\right]$$

    - D is a geometric, V is a kinematic, and $\Delta p/\Delta x$, μ and ρ are dynamic variables.

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 5: Pressure gradient $\Delta p/\Delta x$ in a circular pipe

    ✓ Solution:

    Step 3:

    - There should be $r = 3$ repeating variables.
    - Since L, T, M are basic dimensions, we should select one geometric variable, one kinematic variable, and one dynamic variable as the repeating variables.
    - In the present example, we can select D , V and ρ as repeating variables

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 5: Pressure gradient $\Delta p/\Delta x$ in a circular pipe

✓ Solution:

Step 4:

- Determine $(n-r)$ many Pi groups by combining repeating variables with nonrepeating variables and using the fact that Pi groups should be dimensional. For our example we need to find $5-3=2$ Pi groups. Each Pi group will include only one of the nonrepating variables.

$$\Pi_1 = \frac{\Delta p}{\Delta x} D^a V^b \rho^c$$

- Π_1 should be unitless: $\left[-\right]=\left[\mathrm{ML^{-2}T^{-2}}\right]\left[\mathrm{L}\right]^a\left[\mathrm{LT^{-1}}\right]^b\left[\mathrm{ML^{-3}}\right]^c=\left[\mathrm{M}^{1+c}\mathrm{L}^{-2+a+b-3c}\mathrm{T}^{-2-b}\right]$

$$\left.\begin{aligned} 1+c&=0\\ -2+a+b-3c&=0\\ -2-b&=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} a&=1\\ b&=-2\\ c&=-1 \end{aligned}\right\} \Rightarrow \Pi_1 = \frac{\dfrac{\Delta p}{\Delta x}}{\rho V^2/D}$$

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 5: Pressure gradient $\Delta p/\Delta x$ in a circular pipe

    ✓ Solution:

    Step 4:

    - Now determine the second Pi group which has μ as the nonrepeating variable

$$\Pi_2 = \mu D^a V^b \rho^c$$

    - Π_2 should be unitless: $[-]=\left[\mathrm{ML^{-1}T^{-1}}\right][\mathrm{L}]^a\left[\mathrm{LT^{-1}}\right]^b\left[\mathrm{ML^{-3}}\right]^c=\left[\mathrm{M}^{1+c}\mathrm{L}^{-1+a+b-3c}\mathrm{T}^{-1-b}\right]$

$$\left.\begin{array}{l} 1+c=0 \\ -1+a+b-3c=0 \\ -1-b=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a=-1 \\ b=-1 \\ c=-1 \end{array}\right\} \Rightarrow \Pi_2 = \frac{\mu}{\rho D V}$$

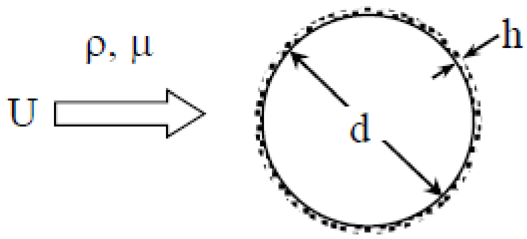
    - Therefore the relation of nondimensional groups is

$$\Pi_1 = f_1(\Pi_2) \Rightarrow \frac{\frac{\Delta p}{\Delta x}}{\rho V^2/D} = f_1\left(\frac{\mu}{\rho V D}\right)$$

Dimensional Analysis

- Buckingham's Pi Theorem
  - Example 6: Flow past a rough sphere generates a drag force (F_D), which is known to depend on the speed of the fluid (U), the density of fluid (ρ), the viscosity of fluid (μ), diameter of the sphere (d) and the roughness height on the sphere (h). Use Buckingham's pi theorem to predict the form of dependence of F_D on the variables.

    1. Use ρ, U, and d as repeating variables.
    2. Use F_D, μ, h as repeating variables



Dimensional Analysis

- Common Dimensionless Groups in Fluid Mechanics
  - Reynolds number (Re):

$$\mathrm{Re} = \frac{\text{Inertia force}}{\text{Viscous Force}} = \frac{\text{Mass}\bullet\text{Accelaration}}{\text{Stress}\bullet\text{Area}} = \frac{ma}{\mu \dfrac{du}{dy} A}$$

$$= \frac{\rho L^3 \left(\dfrac{V}{T}\right)}{\mu \dfrac{V}{L} L^2} = \frac{\rho L^3 \left(\dfrac{V}{T}\dfrac{L}{L}\right)}{\mu V L} = \frac{\rho L^3 \left(\dfrac{V^2}{L}\right)}{\mu V L} = \frac{\rho V^2 L^2}{\mu V L} = \frac{\rho V L}{\mu}$$

Dimensional Analysis

- Common Dimensionless Groups in Fluid Mechanics
  - Froude number (Fr):

$$\mathrm{Fr} = \frac{\text{Inertia force}}{\text{Gravity force}} = \frac{ma}{mg} = \frac{a}{g}$$

$$= \frac{\left(\dfrac{V}{T}\right)}{g} = \frac{\left(\dfrac{V}{T}\dfrac{L}{L}\right)}{g} = \frac{\left(\dfrac{V^2}{L}\right)}{g} = \frac{V^2}{gL}$$

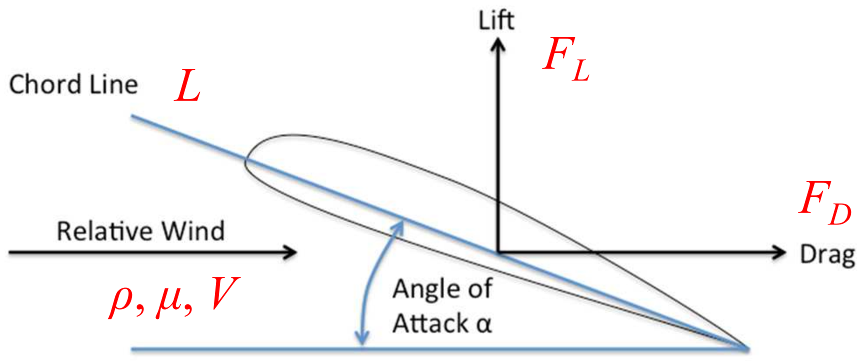
Sometimes, Fr is also defined as

$$\mathrm{Fr} = \frac{V}{\sqrt{gL}}$$

Dimensional Analysis

- Common Dimensionless Groups in Fluid Mechanics
  - Lift and Drag Coefficients (C_D and C_L):

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 L^2} \qquad C_L = \frac{F_L}{\frac{1}{2}\rho V^2 L^2}$$



Dimensional Analysis

- Nondimensionalization of the Basic Equation
  - It is very useful and instructive to nondimensionalize the basic equations and boundary conditions
  - Consider the incompressible-flow continuity and momentum equations of Newtonian fluid

Continuity Equation

$$\nabla \vec{V} = 0$$

Momentum Equation

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p + \mu \nabla^2 \vec{V}$$

Dimensional Analysis

- Nondimensionalization of the Basic Equation
  - The two equations contain three basic dimensions M, L, and T. All variables p, V, x, y, z, and t can be non-dimensionalized by using density ρ and two reference constants that might be characteristic of the particular fluid flow:

    Reference velocity = U

    Reference length (or characteristics length) = L
  - We can define all dimensionless variables:

$$\vec{V}^* = \frac{\vec{V}}{U} \qquad x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L}$$

$$t^* = \frac{tU}{L} \qquad \nabla^* = \frac{\nabla}{L} \qquad p^* = \frac{p + \rho g z}{\rho U^2}$$

Gravity is in the negative z direction

Dimensional Analysis

- Nondimensionalization of the Basic Equation
  - Since U, ρ, and L are constants, the derivatives in the governing equations can all be handled in dimensionless form, for example:

$$\frac{\partial \vec{V}}{\partial x} = \frac{\partial\left(U\vec{V}^*\right)}{\partial\left(Lx^*\right)} = \frac{U\partial \vec{V}^*}{L\partial x^*}$$

  - After substitution and simplifications, we obtain:

$$\nabla^* \vec{V}^* = 0$$

$$\frac{D\vec{V}^*}{Dt^*} = \nabla^* p^* + \frac{\mu}{\rho UL}\nabla^{*2}\vec{V}^*$$

Physical Similarity

- Model and Prototype
  - In experimental fluid mechanics we sometimes can not work with real sized objects, known as prototypes:
  - Instead we use scaled down (or up) versions of them, called models.
  - Also sometimes in experiments we use fluids that are different than actual working fluids, e.g. we use regular tap water instead of salty sea water to test the performance of a marine propeller.

Physical Similarity

- Similarity
  - Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype.
    - ✓ E.g. For a problem described by a functional relationship between non-dimensional parameters

$$\Pi_1 = f\left(\Pi_2, \Pi_3, \ldots \Pi_k\right)$$

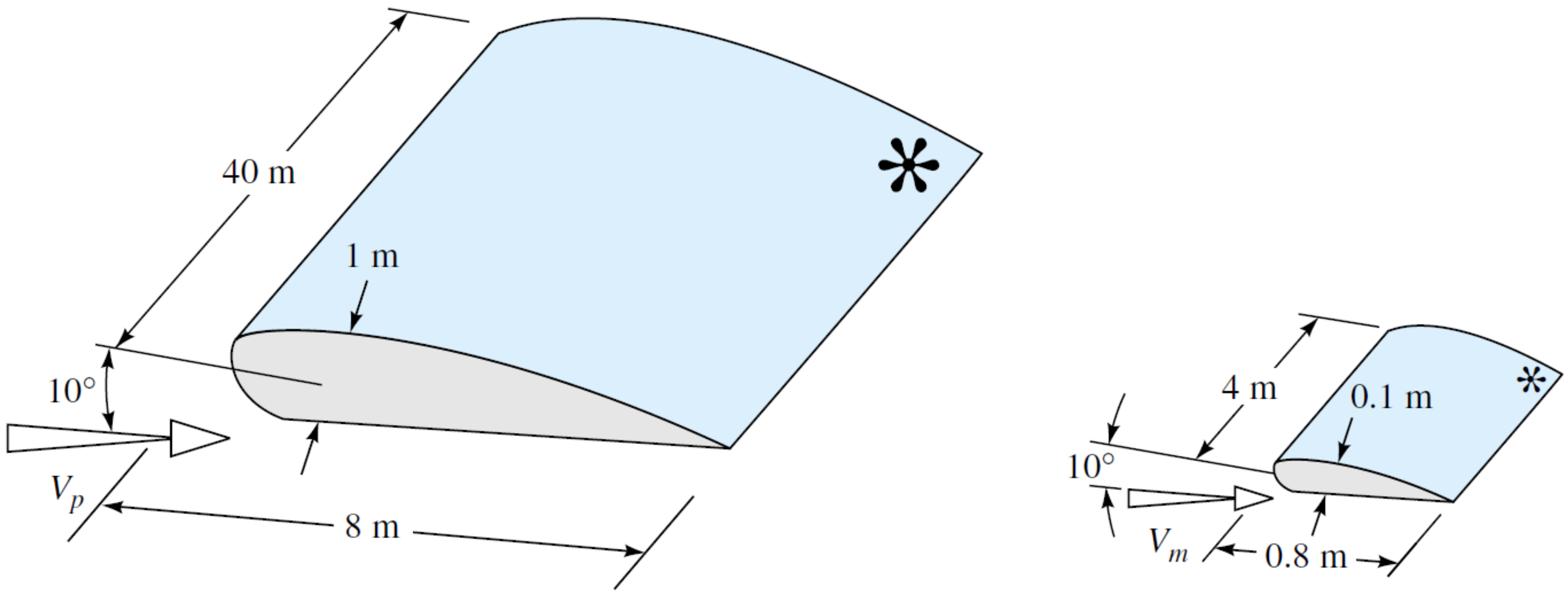
    - ✓ If $\Pi_{2m} = \Pi_{2p}$, $\Pi_{3m} = \Pi_{3p}$, etc., the above relationship guarantees that the desired output Π_{1m} will equal Π_{1p}.
    - ✓ However, this is not easy to achieve, especially for a multi-parameter problem.
    - ✓ Instead of complete similarity, we seek particular types of similarity, the most common being geometric, kinematic, dynamic, and thermal

Physical Similarity

- Geometric Similarity
  - A model and prototype are geometrically similar if and only if all body dimensions in all three coordinates have the same linear-scale ratio.
  - Note that all length scales must be the same.
  - All angles are preserved in geometric similarity. All flow directions are preserved.
  - The orientations of model and prototype with respect to the surroundings must be identical.
  - It is usually impossible to establish 100 % geometric similarity due to very small details that can not be put into the model. Modeling surface roughness exactly is also impossible.

Physical Similarity

- Geometric Similarity



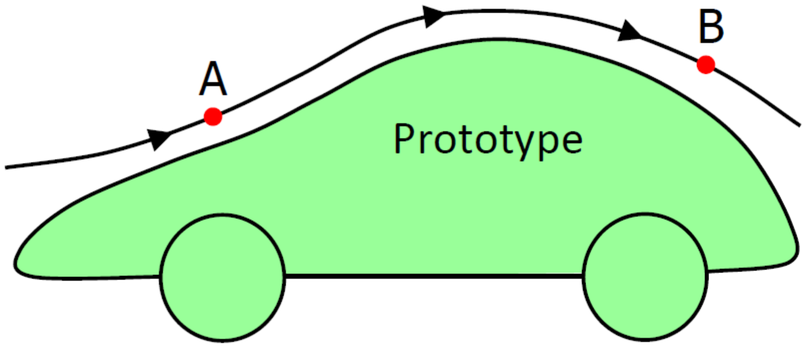
a prototype wing and a one-tenth-scale model.

Physical Similarity

- Kinematic Similarity
  - Kinematic similarity requires that the model and prototype have the same length-scale ratio and the same time-scale ratio. The result is that the velocity-scale ratio will be the same for both.
  - The motions of two systems are kinematically similar if homologous particles (particles that have the same relative location) lie at homologous points at homologous times.
  - It implies that the direction of velocity and acceleration at corresponding points in the two flows should be the same.

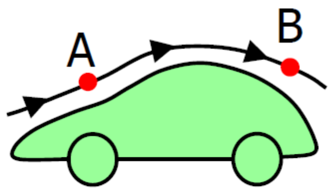
Physical Similarity

- Kinematic Similarity



$$V_r = \frac{V_{p_A}}{V_{m_A}} = \frac{V_{p_B}}{V_{m_B}}$$

Model



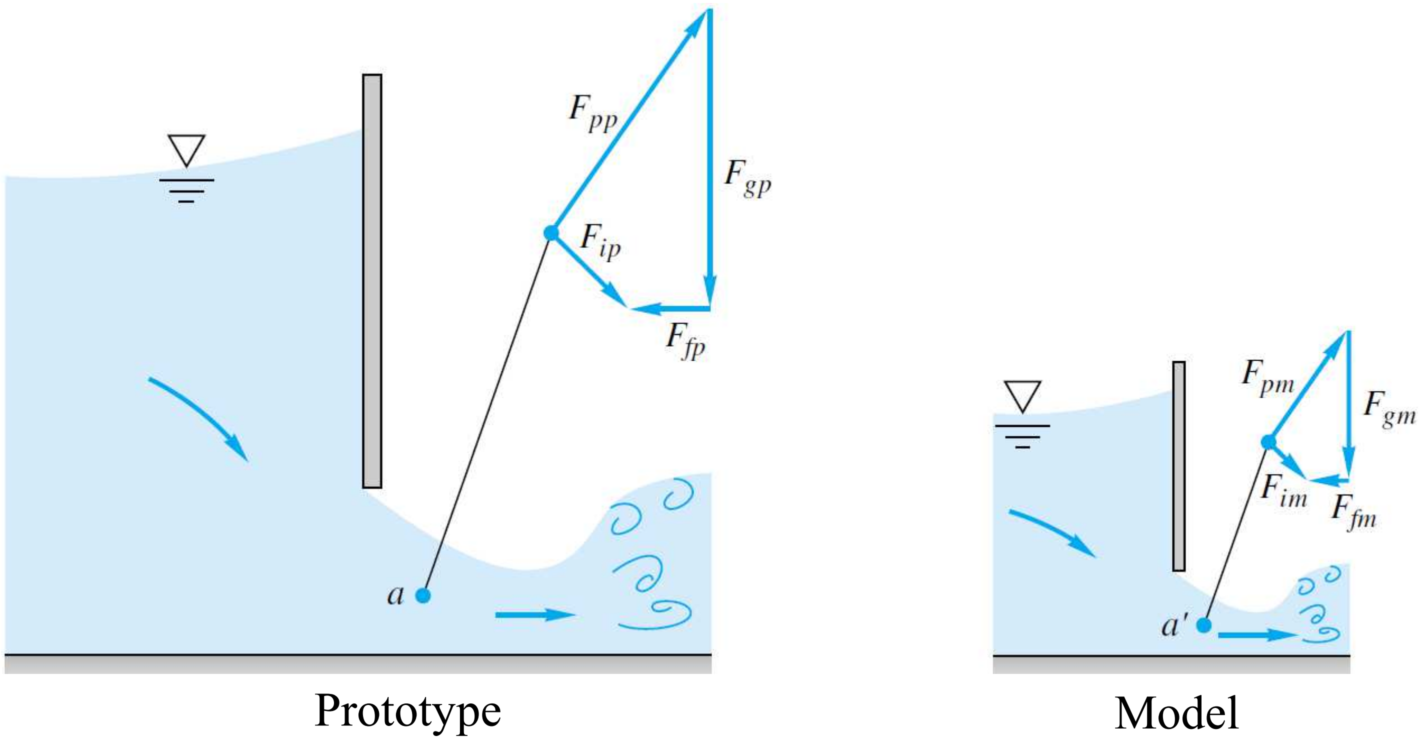
- ➢ the velocities at corresponding points are the same in direction and differ only by a constant factor of velocity ratio, Vr .
- ➢ the streamline patterns of two flow fields also differ only by a constant scale factor.

Physical Similarity

- Dynamic Similarity
  - Dynamic similarity exists when the model and the prototype have the same length-scale ratio, time-scale ratio, and force-scale (or mass-scale) ratio.
  - Dynamic similarity exists, simultaneous with kinematic similarity, if the model and prototype force and pressure coefficients are identical.
    - For compressible flow, the model and prototype Reynolds number and Mach number and specific-heat ratio are correspondingly equal
    - For incompressible flow
      a) With no free surface: model and prototype Reynolds numbers are equal.
      b) With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

Physical Similarity

- Dynamic Similarity



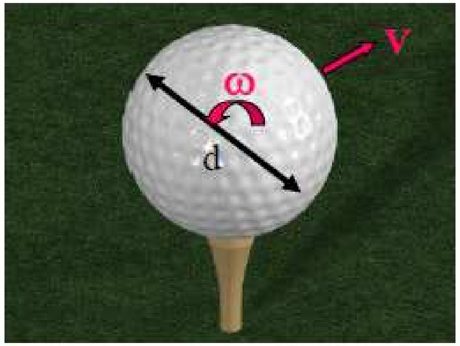
Dynamic similarity in sluice-gate flow. Model and prototype yield identical homologous force polygons if the Reynolds and Froude numbers are the same corresponding values

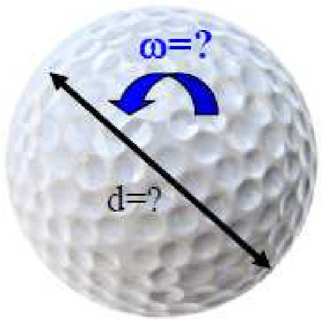
Physical Similarity

- Dynamic Similarity
  - In many situations, the perfect dynamic similarity is more of a dream than a reality.
  - True equivalence of Reynolds and Froude numbers can be achieved only by dramatic changes in fluid properties, whereas in fact most model testing is simply done with water or air, the cheapest fluids available.
  - Matching Reynolds number and Mach number of the prototype and model flows is also very difficult.
  - We need to determine the important forces of the prototype flow and make sure that the nondimensional numbers related to those forces are the same in prototype and model flows.

Physical Similarity

- Example 6
  - The aerodynamic characteristics of a golf ball are to be tested in a wind tunnel. The dependent parameters are the drag force (F_D), and lift force (F_L) that act on the ball. Variables which affects these characteristics include d (diameter of the ball), V (speed of travel), ω (angular speed of spin), d'(depth of dimples), ρ (air density) and μ (air viscosity). Determine the dimensionless parameters that characterise the problem.
  - A golf professional can hit a ball at V = 80m/s and impart a spin of 900 rpm. To model these conditions in a wind tunnel with maximum speed of 27 m/s, what should be the diameter of the model used? How fast must the model rotate if the diameter of the real golf ball is 42.67mm?





Physical Similarity

- Example 6
  - Solution

✓ List all the dimensional variables involved in the problem

$$F_D = f_1\left(d, V, \omega, d', \rho, \mu\right)$$
$$F_L = f_2\left(d, V, \omega, d', \rho, \mu\right)$$

✓ Let us consider the drag first. From Buckingham's Pi theorem

n = 7 (total number of variables)

$$\left[F_D\right] = \left[\mathrm{MLT^{-2}}\right],\ \left[\mu\right] = \left[\mathrm{ML^{-1}T^{-1}}\right],\ \left[V\right] = \left[\mathrm{LT^{-1}}\right],$$
$$\left[\rho\right] = \left[\mathrm{ML^{-3}}\right],\ \left[d\right] = \left[\mathrm{L}\right],\ \left[d'\right] = \left[\mathrm{L}\right],\ \left[\omega\right] = \left[\mathrm{T^{-1}}\right]$$

m = 3 (three repeating variables containing M, L and T)

Physical Similarity

- Example 6
  - Solution

✓ Therefore, there are $n - m = 4$ dimensionless groups or Π parameters.

✓ If we choose ρ, V, d as the repeating variables. Therefore the 4 Π parameters are

$$\Pi_1 = F_D \rho^{a1} V^{b1} d^{c1}$$

$$\Pi_2 = \mu \rho^{a2} V^{b2} d^{c2}$$

$$\Pi_3 = \omega \rho^{a3} V^{b3} d^{c3}$$

$$\Pi_4 = d' \rho^{a4} V^{b4} d^{c4}$$

$$\Pi_1 = f\left(\Pi_2, \Pi_3, \Pi_4\right)$$

Physical Similarity

- Example 6
  - Solution

✓ Consider Π_1 , In term of their dimensions,

$$[-]=\left[\mathrm{MLT}^{-2}\right]\left[\mathrm{ML}^{-3}\right]^{a1}\left[\mathrm{LT}^{-1}\right]^{b1}[\mathrm{L}]^{c1}=\left[\mathrm{M}^{1+a_1}\mathrm{L}^{1-3a_1+b_1+c_1}\mathrm{T}^{-2-b_1}\right]$$

✓ From dimensional homogeneity, the dimensions of M, L and T on both sides should be equated.

$$\left.\begin{array}{l} 1+a_1=0 \\ 1-3a_1+b_1+c_1=0 \\ -2-b_1=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} a_1=-1 \\ b_1=-2 \\ c_1=-2 \end{array}\right\} \Rightarrow \Pi_1=\frac{F_D}{\rho V^2 d^2}$$

Physical Similarity

- Example 6
  - Solution

✓ By using the same technique, we can obtain

$$\Pi_2 = \frac{\mu}{\rho V d} \quad \text{or} \quad \Pi_2 = \frac{\rho V d}{\mu} \qquad \Pi_3 = \frac{\omega d}{V} \quad \Pi_4 = \frac{d'}{d}$$

✓ Thus, the functional form for F_D can be expressed as

$$\frac{F_D}{\rho V^2 d^2} = f_1\left(\frac{\rho V d}{\mu}, \frac{\omega d}{V}, \frac{d'}{d}\right)$$

✓ Similarly, we can obtained

$$\frac{F_L}{\rho V^2 d^2} = f_2\left(\frac{\rho V d}{\mu}, \frac{\omega d}{V}, \frac{d'}{d}\right)$$

Physical Similarity

- Example 6
  - Solution
    - ✓ For the three types of similarity between the model and prototype
    - ✓ Dynamic similarity: $\left(\dfrac{\rho V d}{\mu}\right)_p = \left(\dfrac{\rho V d}{\mu}\right)_m$
    - ✓ Kinematic similarity $\left(\dfrac{\omega d}{V}\right)_p = \left(\dfrac{\omega d}{V}\right)_m$
    - ✓ Geometric similarity $\left(\dfrac{d'}{d}\right)_p = \left(\dfrac{d'}{d}\right)_m$

Physical Similarity

- Example 6
  - Solution
    - ✓ To determine the diameter of the wind-tunnel model, we make us of dynamic similarity criterion

$$\left(\frac{\rho V d}{\mu}\right)_p = \left(\frac{\rho V d}{\mu}\right)_m$$

    - ✓ Since both model and prototype are tested in air

$$\rho_p = \rho_m \qquad \mu_p = \mu_m$$

    - ✓ Thus

$$d_m = \frac{V_p}{V_m} d_p = \frac{80}{27} \times 42.67 = 126.43 \text{ mm}$$

Physical Similarity

- Example 6
  - Solution
    - ✓ To determine ω_m, we make us of kinematic similarity criterion

$$\left(\frac{\omega d}{V}\right)_p = \left(\frac{\omega d}{V}\right)_m$$

    - ✓ Thus

$$\omega_m = \frac{V_m}{V_p}\frac{d_p}{d_m}\omega_p = \frac{27}{80}\times\frac{42.67}{126.43}\times 900 = 102.52 \text{ rpm}$$

Review

- A dimension is the type of physical quantity.
- A unit is a means of assigning a numerical value to that quantity.
- Principle of Dimensional Homogeneity: If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous; i.e., each of its additive terms will have the same dimensions.
- Determination of Dimensionless Group: Rayleigh's Method and Buckingham's Pi Theorem.
- Choose the proper reference scales to nondimensionalize governing equations as well as boundary conditions.
- Use the criterion of the geometric, kinematic, dynamic similarities to determine the relationship between prototypes and models


Thank You for Your Attention!
Any Questions?