solutions to transportation problems

2016.9.27

1 L1

1. scalse 1 处于分子尺度，粒子非常稀疏，体微元包含的粒子数的变化dN与体微元有关,也就是$dN = dN(dV)$,因而$\frac{dN}{dV} \neq constant$，也就是密度$\rho$在这个尺度会随体微元的不同而不同.scale 2 处于连续尺度，体元宏观无穷小，而微观无穷大，在此尺度定义密度有：1)定义于一点处，因为体元宏观无穷小，在宏观上就是一点，2)在定义点处连续，因为微观无穷大使体元包含足够多的粒子，这使得附近点之间的粒子数平均不会发生突变，因而密度在空间上是连续分布的.scale 3 处于宏观尺度,这一个尺度上的体元已经感知到了密度的宏观变化

2.
$$\frac{d\alpha}{dt} = \frac{dx/dy}{dt} = \frac{\partial u}{\partial y} \tag{1}$$
$$\frac{d\beta}{dt} = \frac{dy/dx}{dt} = \frac{\partial v}{\partial x} \tag{2}$$

so

$$\frac{d\alpha + d\beta}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} \tag{3}$$
$$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4}$$

3.
$$\tau_x = \frac{\partial u}{\partial y} = -\frac{2U_0 y}{b^2} \tag{5}$$
$$\tau_x\big|_{y=\frac{b}{2}} = -\frac{U_0}{b} \tag{6}$$

4.

$$u = cy \tag{7}$$

$$v = cx \tag{8}$$

$$\tau_x = \mu \frac{\partial \mathbf{v}}{\partial y} = \mu \frac{\partial}{\partial y} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{9}$$

$$\tau_y = \mu \frac{\partial \mathbf{v}}{\partial x} = \mu \frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} = \mu \begin{pmatrix} 0 \\ c \end{pmatrix} \tag{10}$$

let $\mathbf{e} = \frac{\mathbf{j}+\mathbf{i}}{\sqrt{\mathbf{2}}}$,

$$\begin{aligned}
\tau_{xy} &= \mu \frac{\partial \mathbf{v}}{\partial e} && (11)\\
&= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \cos(\mathbf{e}, \mathbf{i}) + \frac{\partial \mathbf{v}}{\partial y} \cos(\mathbf{e}, \mathbf{j}) \right) && (12)\\
&= \mu \left(\frac{\partial \mathbf{v}}{\partial x} \frac{1}{\sqrt{2}} + \frac{\partial \mathbf{v}}{\partial y} \frac{1}{\sqrt{2}} \right) && (13)\\
&= \frac{\mu}{\sqrt{2}} \left(\begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} \right) && (14)\\
&= \frac{c\mu}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} && (15)
\end{aligned}$$

5.

$$\begin{aligned}
2\sigma l &= \rho g h l w && (16)\\
\Rightarrow h &= \frac{2\sigma}{\rho g w} && (17)
\end{aligned}$$

$\sigma = 0.123(1 - 0.00139T)$ so

$h = 9.37mm$

6.

$$\sigma = 0.025N/m \tag{18}$$

$$\begin{aligned}
2 \cdot 2\pi r \sigma &= \Delta p \pi r^2 && (19)\\
\Rightarrow \Delta p &= \frac{4\sigma}{r} && (20)\\
\Rightarrow \Delta p &= 50Pa && (21)
\end{aligned}$$

7. $\rho = 996kg/m^3, h = 1mm$

$$\pi\frac{d^2}{4}\rho gh = \sigma\pi d \tag{22}$$

$$\Rightarrow d = \frac{4\sigma}{\rho gh} = 2.989cm \tag{23}$$

$$\tag{24}$$

8.

$$t = \frac{D_c - D_r}{2} = 0.01cm \tag{25}$$

$$\begin{aligned} F &= \tau A & (26)\\ &= \tau\pi D_r L & (27)\\ &= \mu\frac{v}{t}\pi D_r L & (28)\\ &= \frac{0.85 \times 1000 \times 3.7 \times 10^{-4} \times 0.15 \times 0.3602 \times 3.14 \times \pi}{1 \times 10^{-4}} & (29)\\ &= 1676\text{N} & (30) \end{aligned}$$

9.

$$G = F \tag{31}$$

$$\begin{aligned} mg &= \tau A & (32)\\ &= \mu\frac{v}{t}A & (33)\\ &= \mu\frac{v}{t}\pi D_r L & (34) \end{aligned}$$

10.

$$\begin{aligned} dM &= rdF & (35)\\ &= r\tau dA & (36) \end{aligned}$$

$$\tau = \mu\frac{v}{t} = \mu\frac{\omega r}{t} \tag{37}$$

$$dA = 2\pi r \cdot rdx \tag{38}$$

$$r = x\sin\alpha \tag{39}$$

$\Rightarrow$

$$M = \int_0^{D/2} r\mu \frac{r\omega}{h} 2\pi r \frac{dr}{\sin\alpha} \tag{40}$$

$$= \frac{\pi\mu\omega D^4}{32h\sin\alpha} \tag{41}$$

2 L2

1.

$$\begin{cases} \frac{dp}{dr} = f = \rho a \\ a = kr \\ kR = g \end{cases} \tag{42}$$

$\Rightarrow$

$$\frac{dp}{dr} = \frac{\rho g r}{R} \tag{43}$$

$\Rightarrow$

$$p = \frac{\rho g r^2}{2R} + p_{atm} \tag{44}$$

so

$$p|_{r=R} = \frac{\rho g R}{2} + p_{atm} \tag{45}$$

$$\approx \frac{\rho g R}{2} \tag{46}$$

$$= 176kMpa \tag{47}$$

2.

$$\rho g \Delta h = \Delta p \tag{48}$$

$$\Rightarrow \Delta h = \frac{\Delta p}{\rho g} \tag{49}$$

$\Rightarrow$

$$\Delta h_{water} = 10.33m \tag{50}$$

$$\Delta h_{sea} = \Delta h_{water}/1.025 = 10.08m \tag{51}$$

$$\Delta h_{Hg} = \Delta h_{water}/13.6 = 0.76m \tag{52}$$

3. let Hg,water,oil denoted by "H, w, o" respectively

$$p_A + \rho_o g h_o + \rho_w g h_w = p_{atm} + \rho_H g h_H \tag{53}$$
$$\Rightarrow p_A - p_{atm} = g(\rho_H h_H - \rho_o h_o - \rho_w h_w) \tag{54}$$

$\Rightarrow$

$$p_A = 588.6pa \tag{55}$$

4. the resultant accelaration is gravity and inertial

$$\mathbf{r} = \mathbf{g} - \mathbf{a} \tag{56}$$

now the isobars and the direction of the pressure gradient is depict as follow

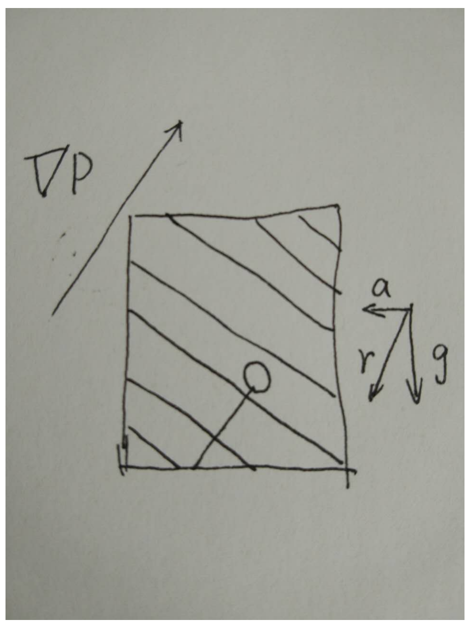


图 1: the balloon

5.

$$dF_y = p_y dA \tag{57}$$

$$\begin{cases} dA = 2\pi r' r d\theta \\ r' = r\sin\theta \\ p_y = p\cos\theta \\ p = \rho g h' \\ h' = h - r\cos\alpha + r\cos\theta \end{cases} \tag{58}$$

$\Rightarrow$

$$dF_y = \rho g(h - r\cos\alpha + r\cos\theta)\cos\theta 2\pi r\sin\theta r d\theta \tag{59}$$

$$= 2\pi\rho g r^2 (h - r\cos\alpha + r\cos\theta)\sin\theta\cos\theta d\theta \tag{60}$$

$$F_y = 2\pi\rho g r^2 \int_\alpha^\pi (h - r\cos\alpha + r\cos\theta)\sin\theta\cos\theta\, d\theta \tag{61}$$

$$\tag{62}$$

$$h = \frac{2r}{3\sin^2\alpha} + \frac{2r\cos^3\alpha}{3\sin^2\alpha} + r\cos\alpha \tag{63}$$

where $\cos\alpha = \frac{D}{2r}$

altanitive solution:

$$F_y + F_D = \rho g V \tag{64}$$

$$F_D = \rho g S_D h = \rho g \cdot \pi r'^2 \cdot h \tag{65}$$

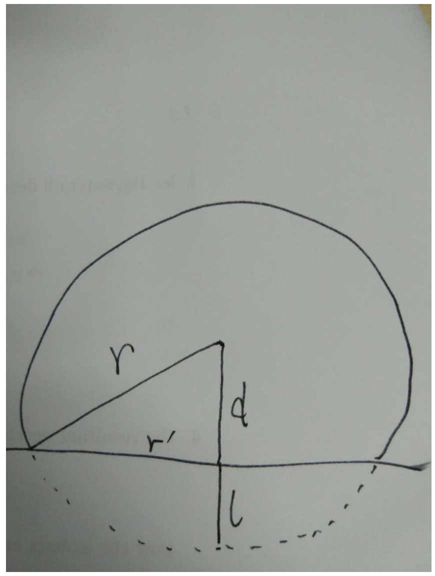
$$r' = \frac{D}{2} \tag{66}$$

$$V = V_a - V_b \tag{67}$$

$$V_b = \frac{\pi}{3}(3r - l)l^2 \tag{68}$$

$$l = r - d \tag{69}$$

$$d = \sqrt{r^2 - r'^2} \tag{70}$$



let$F_y(h) = 0$, comes the requiring

$$\begin{aligned}
h &= V/S_D && (71)\\
&= \frac{V_a - V_b}{\pi r'^2} && (72)\\
&= \frac{\frac{4}{3}\pi r^3 - \frac{\pi}{3}(3r-l)l^2}{\pi r'^2} && (73)\\
&= \frac{4r^3 - (2r+d)(r-d)^2}{3r'^2} && (74)\\
&= \frac{4}{3}\frac{4r^3 - (2r-d)(r-d)^2}{D^2} && (75)
\end{aligned}$$

where $d = \sqrt{r^2 - (\frac{D}{2})^2}$

6.

$$\begin{cases} F = \rho g h A \\ A = \pi r^2 \end{cases} \tag{76}$$

and the acting point is

$$\Delta y_a = \frac{I_{xx}}{hA} = \frac{\pi r^2}{4hA} \tag{77}$$

the momentum equilibrium

$$F\Delta y_a = Pr \tag{78}$$

$\Rightarrow$

$$\begin{aligned} P &= \frac{F\Delta y_a}{r} && (79)\\ &= \frac{\rho g \pi r}{4} && (80)\\ &= 7.7kN && (81) \end{aligned}$$

7.

$$\begin{aligned} F_p &= \rho g \bar{h} A && (82)\\ A &= lw && (83) \end{aligned}$$

$$\begin{aligned} \Delta y_a &= \frac{I_{xx}}{y_c A} && (84)\\ I_{xx} &= \frac{l^3 w}{12} && (85) \end{aligned}$$

$$\tau w l_c = F_p \Delta y_a \tag{86}$$

$\Rightarrow$

$$\begin{aligned} \tau &= \frac{F_p \Delta y_a}{w l_c} && (87)\\ &= 145.2kN && (88) \end{aligned}$$

8.

$$\begin{aligned} F &= \rho g \bar{h} A = 10^{10} N && (89)\\ y_a &= \frac{2}{3} h = 85.3m && (90) \end{aligned}$$

9.

$$F_{up} = F_{down}$$
$$F + \rho_w g x \left(\frac{4}{3}\pi R^3\right) = rg\left(\frac{4}{3}\pi R^3\right)$$
$$x = \frac{rg\left(\frac{4}{3}\pi R^3\right) - F}{\rho_w g\left(\frac{4}{3}\pi R^3\right)}$$

10.

$$\rho_s v_s = \rho_l v_l \tag{91}$$
$$\Rightarrow \quad \frac{v_l}{v_s} = \frac{\rho_s}{\rho_l} \tag{92}$$
$$\tag{93}$$

$$S = v_s - v_l \tag{94}$$
$$\Rightarrow \quad \frac{1}{2}LL\tan\theta = v_s - v_l = 0.1L^2 \tag{95}$$
$$\Rightarrow \quad \tan\theta = 0.2 \tag{96}$$

3 L3-L4

1. since

$$\mathbf{r} = \mathbf{f}(\mathbf{c}, t) = \mathbf{g}(\mathbf{c})h(t) \tag{97}$$

$\Rightarrow$

$$\mathbf{g}(\mathbf{c}) = \frac{\mathbf{r}}{h(t)} \tag{98}$$

$\Rightarrow$

$$\mathbf{c} = \mathbf{g}^{-1}(\frac{\mathbf{r}}{h(t)}) \tag{99}$$

so

$$\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t} \tag{100}$$
$$= \mathbf{g}(\mathbf{c})\dot{h}(t) \tag{101}$$
$$= \mathbf{g}(\mathbf{g^{-1}}(\frac{\mathbf{r}}{\mathbf{h(t)}}))\dot{h}(t) \tag{102}$$

2.

$$g(c) = c \tag{103}$$
$$h(t) = t^2 \tag{104}$$

and it is easy to find

$$g^{-1}(c) = c \tag{105}$$

$$\begin{aligned} v &= g(g^{-1}(\frac{x}{h(t)}))\dot{h}(t) && (106)\\ &= g^{-1}(\frac{x}{h(t)})\dot{h}(t) && (107)\\ &= \frac{x}{h(t)}\dot{h}(t) && (108)\\ &= 2t\frac{x}{t^2} && (109)\\ &= 2\frac{x}{t} && (110)\\ & && (111) \end{aligned}$$

3. at time t the cth element is at $f(c,t)$, so the temperature of the cth element is

$$T = g(f(c,t),t) \tag{112}$$

so the variation rate is

$$\frac{dT}{dt} = \frac{\partial g}{\partial x}\frac{df}{dt} + \frac{\partial g}{\partial t} \tag{113}$$

4.

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{x}{1+t} \\ \dfrac{dy}{dt} = \dfrac{2y}{2+t} \end{cases} \tag{114}$$

$\Rightarrow$

$$\begin{cases} x = c_x(1+t) \\ y = c_y(2+t)^2 \end{cases} \tag{115}$$

with the boundary condition

$$\mathbf{x}(0) = \mathbf{x}_0 \tag{116}$$

the path line through $\mathbf{x}_0$ is

$$\begin{cases} x = x_0(1+t) \\ y = \dfrac{y_0}{4}(2+t)^2 \end{cases} \tag{117}$$

and the streamlines at $t = 0$ can be obtained by let $t = 0$

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \end{cases} \tag{118}$$

$\Rightarrow$

$$y = cx \tag{119}$$

5.

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{120}$$

since

$$\rho = \rho_0(2 - \cos\omega t) \tag{121}$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = -\frac{\partial \rho}{\rho \partial t} \tag{122}$$

$$= \frac{\omega \sin\omega t}{\cos\omega t - 2} \tag{123}$$

$$:= f(t) \tag{124}$$

$\Rightarrow$

$$u = \int f(t) dx \tag{125}$$

$$= f(t)x + C \tag{126}$$

apply the boundary condition$u(0,t) = U$

$$u = f(t)x + U \tag{127}$$

6. df

(a)
$$\int_{A=\partial V} \rho\mathbf{u}\cdot d\mathbf{A} = \int_0^1 dy \int_0^1 dz\, 4x^2y\Big|_{x=1} - \int_0^1 dy \int_0^1 dz\, 4x^2y\Big|_{x=0}$$
$$+\int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=1} - \int_0^1 dz \int_0^1 dx\, xyz\Big|_{y=0}$$
$$+\int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=1} - \int_0^1 dx \int_0^1 dy\, yz^2\Big|_{z=0}$$
$$= 2+0+\tfrac{1}{4}+0+\tfrac{1}{2}+0 = \tfrac{11}{4}.$$

(b)
$$\nabla\cdot\mathbf{u} = \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 8xy + xz + 2yz$$

$$\int_V \nabla\cdot\mathbf{u}\, dV = \int_0^1 dx \int_0^1 dy \int_0^1 dz\,(8xy + xz + 2yz)$$
$$= 8\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2} + 2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{11}{4}.$$

7. mass inlet=mass outlet

$$v_1 \cdot 2h = v_2 \cdot 2h - \frac{h\cdot v_2}{2} \tag{128}$$
$$\Rightarrow v_1 = 0.75 v_2 \tag{129}$$

8. mass conservation

$$\frac{\partial(\rho\delta v)}{\partial t} + \frac{\partial(\rho u h)}{\partial x}\delta x = 0 \tag{130}$$
$$\Rightarrow \frac{\partial(\rho h\delta x)}{\partial t} + \frac{\partial(\rho u h)}{\partial x}\delta x = 0 \tag{131}$$
$$\Rightarrow \frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0 \tag{132}$$

momentum conservation

$$\frac{\partial(\rho u\delta v)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x}\delta x = 0 \tag{133}$$
$$\Rightarrow \frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + p)}{\partial x} = 0 \tag{134}$$

the total force p due to pressure is

$$p = \int_0^h \rho g y dy = \frac{1}{2}\rho g h^2 \tag{135}$$

so the momentum equation is

$$\frac{\partial(\rho u h)}{\partial t} + \frac{\partial(\rho u^2 h + \frac{1}{2}\rho g h)}{\partial x} = 0 \tag{136}$$

$$\Rightarrow = \frac{\partial(uh)}{\partial t} + \frac{\partial(u^2 h + \frac{1}{2} g h)}{\partial x} = 0 \tag{137}$$

9. for stream line

$$\frac{dx}{u} = \frac{dy}{v} \tag{138}$$

$$\frac{dx}{Kx} = \frac{dy}{-Ky} \tag{139}$$

$$\ln x = -\ln y + \ln c \tag{140}$$

$$\tag{141}$$

$$\Rightarrow xy = c$$

10. it is a incompressible flow,so

$$\nabla \cdot \mathbf{V} = 0 \tag{142}$$

$$a_1 + b_2 + c_3 = 0 \tag{143}$$