SSY156 - Modelling and Control of Mechatronic Systems
Peer-2-Peer Homework 03

Question 1 (2 points)

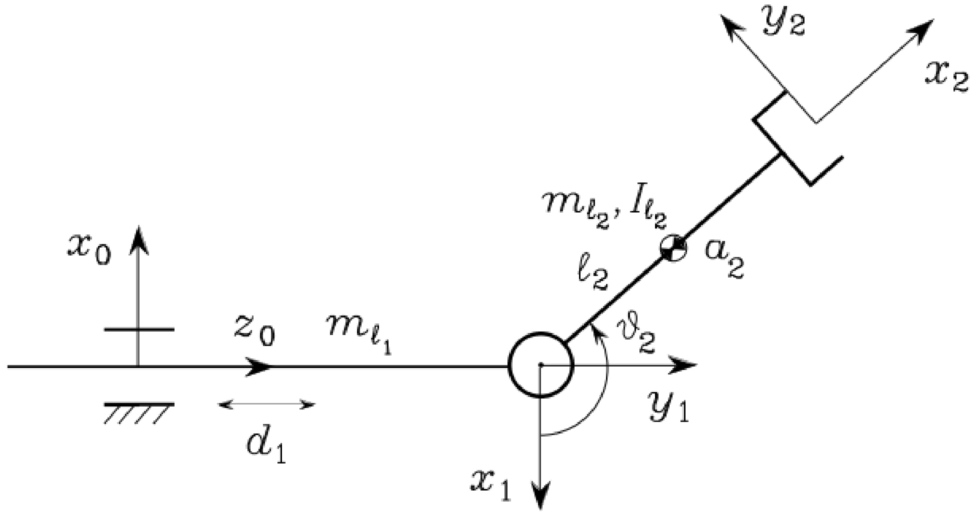

Figure 1: *Two planar manipulator*

First, we define the main coordinates

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ \theta_2 \end{bmatrix} \tag{1}$$

Then, we define:
Masses of the links: m_{l_1}, m_{l_2}.
Length of the links: a_2. The link 1 is length-variable defined by d_1.
Distance from the center of mass to the respective joint: l_1, l_2
We find the DH parameters for this manipulator:

Table 1: *DH parameters Table*

$Link(i)$	a_i	α_i	d_i	$\theta_i(q_i)$
1	0	$\frac{\pi}{2}$	d_1	π
2	a_2	0	0	θ_2

With this DH table we calculate the HT matrices A_1^0 and A_2^0.

Position of each link-center of mass We got this vectors using the homogeneous transformation (HT) matrices and extracting the points P_0 and P_1, and the rotation matrices R_0^0 and R_0^1 and add the section missing to reach the center of mass of each link which is assume in its center. Nevertheless, as been a prismatic joint, the center of mass it is assume in the end of the link.

$$p_{l_1} = P_0 + \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} \tag{2}$$

$$p_{l_2} = P_1 + R_0^1 \begin{bmatrix} l_2\, c_2 \\ l_2\, s_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_2\, c_2 \\ 0 \\ d_1 + l_2\, s_2 \end{bmatrix} \tag{3}$$

We have also to define the Inertia of link i w.r.t. frame i: I_{l_2}. There is no need to define the inertia of link 1 since it is not rotating.

Now we are ready to find the energies and build the lagrange formulation.

Potential energy for the 3 links $i = 1, 2$

$$\mathcal{U} = -\sum_{i=1}^{3} \mathcal{U}_i = -\sum_{i=1}^{3} (m_{l_i} \mathbf{g}_0^T \mathbf{p}_{l_i}) \tag{4}$$

Link 1:

$$\mathcal{U}_1 = m_{l_1} \mathbf{g}_0^T \mathbf{p}_{l_1} = m_{l_1} \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix} = 0 \tag{5}$$

Link 2:

$$\mathcal{U}_2 = m_{l_2} \mathbf{g}_0^T \mathbf{p}_{l_2} = m_2 \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}^T \mathbf{p}_{l_2} = g\, l_2\, m_{l_2}\, c_2 \tag{6}$$

Kinetic Energy:

$$\mathcal{T} = \sum_{i=1}^{2} \mathcal{T}_{l_i} = \sum_{i=1}^{2} \frac{1}{2} m_{l_i} \dot{\mathbf{p}}_{l_i}^T \dot{\mathbf{p}}_{l_i} + \frac{1}{2} \omega_i^T \mathbf{I}_{l_i} \omega_i \tag{7}$$

$$\mathcal{T} = \sum_{i=1}^{2} \frac{1}{2} m_{l_i} \dot{\mathbf{q}}^T \mathbf{J}_P^{(l_i)T} \mathbf{J}_P^{(l_i)} \dot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{J}_O^{(l_i)T} \mathbf{R}_i \mathbf{I}_{l_i}^i \mathbf{R}_i^T \mathbf{J}_O^{(l_i)} \dot{\mathbf{q}} \tag{8}$$

$$\mathcal{T} = \frac{1}{2} m_{l_1} \dot{\mathbf{q}}^T \mathbf{J}_P^{(l_1)T} \mathbf{J}_P^{(l_1)} \dot{\mathbf{q}} + \frac{1}{2} m_{l_2} \dot{\mathbf{q}}^T \mathbf{J}_P^{(l_2)T} \mathbf{J}_P^{(l_2)} \dot{\mathbf{q}} + \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{J}_O^{(l_2)T} \mathbf{R}_2 \mathbf{I}_{l_i}^2 \mathbf{R}_2^T \mathbf{J}_O^{(l_2)} \dot{\mathbf{q}} \tag{9}$$

Jacobian of each link wrt frame 0:

$$\mathbf{J}_P^{(l_1)} = \begin{bmatrix} z_0 & \mathbf{0}_{3\times 1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\mathbf{J}_O^{(l_1)} = \begin{bmatrix} \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{3\times 1} & \mathbf{0}_{3\times 1} \end{bmatrix}$$

$$\mathbf{J}_P^{(l_2)} = \begin{bmatrix} z_0 & z_1 \times (P_{l_2} - P_1) \end{bmatrix} = \begin{bmatrix} 0 & l_2\, s_2 \\ 0 & 0 \\ 1 & l_2\, c_2 \end{bmatrix}$$

$$\mathbf{J}_O^{(l_2)} = \begin{bmatrix} \mathbf{0}_{3\times 1} & z_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Rotational matrix each link w.r.t. frame $\{0\}$:

$$\mathbf{R}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathbf{R}_2 = \begin{bmatrix} -c_2 & s_2 & 0 \\ 0 & 0 & 1 \\ s_2 & c_2 & 0 \end{bmatrix}$$

Inertia tensor relative to the center of mass of Link i w.r.t. link frame:

$$\mathbf{I}_{l_2}^2 = \begin{bmatrix} I_{l_2xx} & 0 & 0 \\ 0 & I_{l_2yy} & 0 \\ 0 & 0 & I_{l_2zz} \end{bmatrix}$$

Angular velocity of link i w.r.t frame i:

$\omega_2 = \begin{bmatrix} \omega_{2x} \\ \omega_{2y} \\ \omega_{2z} \end{bmatrix} = \mathbf{J}_O^{(l_2)} \dot{\mathbf{q}}$

Lagrangian

$\mathcal{L} = \mathcal{T} - \mathcal{U}$

$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \mathcal{T}(\mathbf{q}, \dot{\mathbf{q}}) - \mathcal{U}(\mathbf{q})$

Equations of Motion

$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g} = \tau$

$$\mathbf{B}(\mathbf{q}) = \begin{bmatrix} m_1 + m_2 & l_2\, m_2\, c_2 \\ l_2\, m_2\, c_2 & I_{2,3} + m_2\, {l_2}^2 \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -\dot{q}_2\, l_2\, m_2\, s_2 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} 0 \\ g\, l_2\, s_2\, (m_1 + m_2) \end{bmatrix}$$

$$\dot{\mathbf{B}}(\mathbf{q}) = \begin{bmatrix} 0 & -\dot{q}_2\, l_2\, m_2\, s_2 \\ -\dot{q}_2\, l_2\, m_2\, s_2 & 0 \end{bmatrix}$$

$$\dot{\mathbf{B}} - 2\mathbf{C} = \begin{bmatrix} 0 & \dot{q}_2\, l_2\, m_2\, s_2 \\ -\dot{q}_2\, l_2\, m_2\, s_2 & 0 \end{bmatrix}$$

The matrix is skew-symmetric.

Question 2 (1 point)

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{g}(\mathbf{q}) = \tau \tag{10}$$

$$\begin{bmatrix} m_1 + m_2 & l_2\, m_2\, c_2 \\ l_2\, m_2\, c_2 & I_{2,3} + m_2\, {l_2}^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 & -\dot{q}_2\, l_2\, m_2\, s_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g\, l_2\, s_2\, (m_1 + m_2) \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

The condition needed for the robot be considered decoupled is when $q_2 = \pi/2, -\pi/2$ and $\dot{q}_2 = 0$.