SSY156 - Modelling and Control of Mechatronic systems
Assignment A02

Lucas Rath

Question 1

The Omni bundle robot consists of a robot with 3 degrees of freedom. Therefore we need to define three generalized coordinates q. It is known that on each revolute joint, there is a motor creating a torque. It is then most naturally to write the generalized coordinates and generalized forces as:

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \tag{1}$$

where ξ_i is the torque actuating on the joint corresponding to joint angle q_i.

Question 2

The Lagrange function is given by:

$$\mathcal{L}(\dot{q}, q) = \mathcal{T}(\dot{q}, q) - \mathcal{U}(q) \tag{2}$$

First, the total kinetic energy of the system $\mathcal{T}(\dot{q}, q)$ is the sum of the kinetic energy for each of the 3 links of the Omni bundle, which consists of an energy due to the translation velocity and due to rotational velocity:

$$\begin{aligned} \mathcal{T}(\dot{q}, q) &= \frac{1}{2}\dot{q}^T B(q)\dot{q} \\ B(q) &= \sum_{i=1}^{3} \left(J_P^{(li)T}\, m_{li}\, J_P^{(li)} + J_O^{(li)T}\, R_0^{i\,T}\, I_{li}^i\, R_0^i\, J_O^{(li)} \right) \end{aligned} \tag{3}$$

where the inertial matrix of each link I_{li}^i is a diagonal matrix of the form:

$$I_{li}^i = \begin{bmatrix} I_{li}\_1 & 0 & 0 \\ 0 & I_{li}\_2 & 0 \\ 0 & 0 & I_{li}\_3 \end{bmatrix} \tag{4}$$

and the Jacobian $J^{(li)}$ that maps the joint velocities to the velocity of the center of mass of link i is given by the following matrices for the case of revolute joints:

$$J^{(li)} = \left[\frac{J_P^{(li)}}{J_O^{(li)}}\right] \qquad J^{(l1)} = \begin{bmatrix} z_0 \times (p_{l1} - p_0) & 0 & 0 \\ z_0 & 0 & 0 \end{bmatrix} \tag{5}$$

$$J^{(l2)} = \begin{bmatrix} z_0 \times (p_{l2} - p_0) & z_1 \times (p_{l2} - p_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix} \quad J^{(l3)} = \begin{bmatrix} z_0 \times (p_{l3} - p_0) & z_1 \times (p_{l3} - p_1) & z_2 \times (p_{l3} - p_2) \\ z_0 & z_1 & z_2 \end{bmatrix} \tag{6}$$

where the position of the center of mass of each link p_{li}, the position of each frame p_i and the vector z_i are defined as follows:

$$\begin{aligned} p_0 &= [0\ 0\ 0\ 1]^T \\ p_1 &= A_1^0 \cdot p_0; \\ p_2 &= A_1^0 \cdot A_2^1 \cdot p_0; \end{aligned} \quad \textit{and} \quad \begin{aligned} z0 &= [0\ 0\ 1]^T \\ z1 &= R_1^0 \cdot z0 \\ z2 &= R_1^0 \cdot R_2^1 \cdot z0 \end{aligned} \quad \textit{and} \quad \begin{aligned} p_{l1} &= A_1^0 \cdot p_1 \\ p_{l2} &= A_1^0\, A_2^1\, [-L1/2\ \ 0\ \ 0\ \ 1]^T \\ p_{l3} &= A_1^0\, A_2^1\, A_3^2\, [-L2/2\ \ 0\ \ 0\ \ 1]^T \end{aligned}$$

Furthermore, the potential energy of the robot is given by:

$$\mathcal{U}(q) = \sum_{i=1}^{3} -m_i \, g_0^T \, p_{li} \qquad \text{where:} \quad g_0 = [0\;0\;-g]^T, g = 9.8 m/s^2 \tag{7}$$

Leading to the following results:

$$p_{l1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix} \; p_{l2} = \begin{bmatrix} 0.5\,L_1\,\cos(q_1)\,\cos(q_2) \\ 0.5\,L_1\,\cos(q_2)\,\sin(q_1) \\ -0.5\,L_1\,\sin(q_2) \\ 1.0 \end{bmatrix} \; p_{l3} = \begin{bmatrix} 0.5\,\cos(q_1)\,(L_2\,\sin(q_2+q_3) + 2.0\,L_1\,\cos(q_2)) \\ 0.5\,\sin(q_1)\,(L_2\,\sin(q_2+q_3) + 2.0\,L_1\,\cos(q_2)) \\ 0.5\,L_2\,\cos(q_2+q_3) - 1.0\,L_1\,\sin(q_2) \\ 1.0 \end{bmatrix}$$

$$J_{l1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.0 & 0 & 0 \end{bmatrix} \; J_{l2} = \begin{bmatrix} -0.5\,L_1\,\cos(q_2)\,\sin(q_1) & -0.5\,L_1\,\cos(q_1)\,\sin(q_2) & 0 \\ 0.5\,L_1\,\cos(q_1)\,\cos(q_2) & -0.5\,L_1\,\sin(q_1)\,\sin(q_2) & 0 \\ 0 & -0.5\,L_1\,\cos(q_2) & 0 \\ 0 & -1.0\,\sin(q_1) & 0 \\ 0 & \cos(q_1) & 0 \\ 1.0 & 0 & 0 \end{bmatrix}$$

$$J_{l3} = \begin{bmatrix} -0.5\,\sin(q_1)\,(L_2\,\sin(q_2+q_3) + 2.0\,L_1\,\cos(q_2)) & 0.5\,\cos(q_1)\,(L_2\,\cos(q_2+q_3) - 2.0\,L_1\,\sin(q_2)) & 0.5\,L_2\,\cos(q_2+q_3)\,\cos(q_1) \\ 0.5\,\cos(q_1)\,(L_2\,\sin(q_2+q_3) + 2.0\,L_1\,\cos(q_2)) & 0.5\,\sin(q_1)\,(L_2\,\cos(q_2+q_3) - 2.0\,L_1\,\sin(q_2)) & 0.5\,L_2\,\cos(q_2+q_3)\,\sin(q_1) \\ 0 & -0.5\,L_2\,\sin(q_2+q_3) - 1.0\,L_1\,\cos(q_2) & -0.5\,L_2\,\sin(q_2+q_3) \\ 0 & -1.0\,\sin(q_1) & -1.0\,\sin(q_1) \\ 0 & \cos(q_1) & \cos(q_1) \\ 1.0 & 0 & 0 \end{bmatrix}$$

$$A_1^0 = \begin{bmatrix} \cos(q_1) & 0 & -1.0\,\sin(q_1) & 0 \\ \sin(q_1) & 0 & \cos(q_1) & 0 \\ 0 & -1.0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \; A_2^0 = \begin{bmatrix} \cos(q_1)\,\cos(q_2) & -1.0\,\cos(q_1)\,\sin(q_2) & -1.0\,\sin(q_1) & L_1\,\cos(q_1)\,\cos(q_2) \\ \cos(q_2)\,\sin(q_1) & -1.0\,\sin(q_1)\,\sin(q_2) & \cos(q_1) & L_1\,\cos(q_2)\,\sin(q_1) \\ -1.0\,\sin(q_2) & -1.0\,\cos(q_2) & 0 & -1.0\,L_1\,\sin(q_2) \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \tag{8}$$

$$A_3^0 = \begin{bmatrix} \sin(q_2+q_3)\,\cos(q_1) & \cos(q_2+q_3)\,\cos(q_1) & -1.0\,\sin(q_1) & \cos(q_1)\,(L_2\,\sin(q_2+q_3) + L_1\,\cos(q_2)) \\ \sin(q_2+q_3)\,\sin(q_1) & \cos(q_2+q_3)\,\sin(q_1) & \cos(q_1) & \sin(q_1)\,(L_2\,\sin(q_2+q_3) + L_1\,\cos(q_2)) \\ \cos(q_2+q_3) & -1.0\,\sin(q_2+q_3) & 0 & L_2\,\cos(q_2+q_3) - 1.0\,L_1\,\sin(q_2) \\ 0 & 0 & 0 & 1.0 \end{bmatrix}$$

$$\mathcal{U}(q) = -0.5\,g\,(L_1\,m_2\,\sin(q_2) - 1.0\,L_2\,m_3\,\cos(q_2+q_3) + 2.0\,L_1\,m_3\,\sin(q_2))$$

The kinetic energy equation is too large to be displayed in this report.

Question 3

Given the Lagrangian $\mathcal{L}$ of the system, the equations of motion can be derived by applying the following equation:

$$\frac{d}{dt}\left(\nabla_{\dot{q}}\mathcal{L}(\dot{q},q)\right) - \nabla_q \mathcal{L}(\dot{q},q) = \xi - F\dot{q} - F_s \text{sign}(\dot{q}) \tag{9}$$

which after some manipulations, can be given by:

$$B(q)\ddot{q} + \underbrace{\dot{B}(q)\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}\left(\dot{q}^T B(q)\dot{q}\right)}_{\dot{q}^T H(q)\dot{q}} + g(q) = \xi - F\dot{q} - F_s \text{sign}(\dot{q}) \tag{10}$$

a) where the gravity function is given by:

$$g(q) = \frac{\partial}{\partial q}\mathcal{U}(q) = \begin{bmatrix} 0 \\ -0.5\,g\,(L_2\,m_3\,\sin(q_2+q_3) + L_1\,m_2\,\cos(q_2) + 2.0\,L_1\,m_3\,\cos(q_2)) \\ -0.5\,L_2\,g\,m_3\,\sin(q_2+q_3) \end{bmatrix} \tag{11}$$

b) The diagonal element (3,3) of inertia matrix $B(q)$ is:

$$B(q)\,(3,3) = 0.25\,m_3\,{L_2}^2 + I_{3,3} \tag{12}$$

c) The Coriolis effect induced on Joint J2 by velocities of Joints J2 and J3 is the (2,3) element of the hessian matrix $H(q)$ in equation 10 with respect to q_2:

$$H_{q_2}(2,3)\cdot \dot{q}_2\,\dot{q}_3 = 0.5\Big(L_1\,L_2\,m_3\,\cos\left(q_3\right)\Big)\dot{q}_2\,\dot{q}_3 \tag{13}$$

Note that $H_{q_2}(2,3) = H_{q_2}(3,2)$ because the Hessian is symmetric, and therefore the total Coriolis effect is twice the value presented above.

d) Centrifugal effect induced on Joint J3 by the velocity of Joint J1 is the (1,1) element of the hessian matrix $H(q)$ in equation 10 with respect to q_3:

$$\begin{aligned} H_{q_3}(1,1)\cdot \dot{q}_1^2 = 0.5\Big(&I_{3,1}\,\sin\left(2\,q_2+2.0\,q_3\right) - I_{3,2}\,\sin\left(2\,q_2+2\,q_3\right) - 0.25\,{L_2}^2\,m_3\,\sin\left(2\,q_2+2\,q_3\right) \\ &- 0.5\,L_1\,L_2\,m_3\,\cos\left(q_3\right) - 0.5\,L_1\,L_2\,m_3\,\cos\left(2\,q_2+q_3\right)\Big)\dot{q}_1^2 \end{aligned} \tag{14}$$

e) Coriolis effect induced on Joint J3 by velocities of Joints J1 and J2 is the (1,2) element of the hessian matrix $H(q)$ in equation 10 with respect to q_3:

$$H\{3\}(1,2) = 0 \tag{15}$$

Question 4

a) Unknown parameters π:

$$\begin{bmatrix} I_{1,2} & I_{2,1} & I_{2,2} & I_{2,3} & I_{3,1} & I_{3,2} & I_{3,3} \\ {L_2}^2\,m_3 & L_1\,L_2\,m_3 & L_2\,m_3 & {L_1}^2\,m_2+4\,{L_1}^2\,m_3 & \frac{L_1\,m_2}{2}+L_1\,m_3 & & \\ f_1 & f_2 & f_3 & \mathrm{fs}_1 & \mathrm{fs}_2 & \mathrm{fs}_3 & \end{bmatrix} \tag{16}$$

Known parameters: $q_1, q_2, q_3, \quad \dot{q}_1, \dot{q}_2, \dot{q}_3, \quad \ddot{q}_1, \ddot{q}_2, \ddot{q}_3 \quad g.$

b) The sub-array with dimension 31 of matrix Y that contain the gravity acceleration parameter g is:

$$\underbrace{\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}}_{\tau} = \underbrace{\begin{bmatrix} 0 & 0 & \dots \\ -0.5\,g\,\sin\left(q_2+q_3\right) & -g\,\cos\left(q_2\right) & \dots \\ -0.5\,g\,\sin\left(q_2+q_3\right) & 0 & \dots \end{bmatrix}}_{\mathbf{Y}(q,\dot{q},\ddot{q})} \underbrace{\begin{bmatrix} L2*m3 \\ L1*m2/2+L1*m3 \\ \vdots \end{bmatrix}}_{\pi} \tag{17}$$

c) The data obtained in the lab can be seen in figure 1.

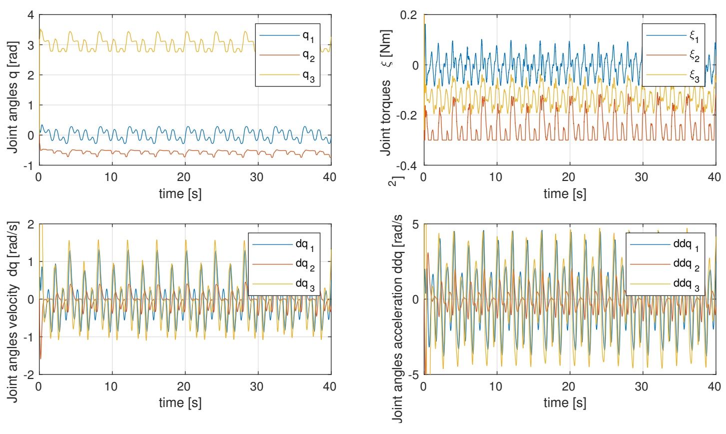

Figure 1: *Joint angles and torques for the Omnibundle lab experiment*

d) Unfortunately, it was not possible to perform system identification using the data collected from the lab. The main reason for that might be the lower-bound saturation observed in the torque sensor of the link 2, as can be seen in 1, that is removing all the torques below -0.3 Nm. This strong non-linearity resulted in a very bad fitting and therefore it was not possible to obtain a meaningful parameters representation from this data.

Nevertheless, using the data provided by the TA, a very good fitting could be done. Below on the left, follow the values of the π vector found with the left pseudo inverse procedure. On the right, the value for each parameters, obtained by solving the equations through the Newton method.

```
        Pi =
                I1_2:    0.0688
                I2_1:    0.0330
                I2_2:    0.0358
                I2_3:    0.1714
                I3_1:    0.0250
                I3_2:    0.0438
                I3_3:    0.0766
             L2^2*m3:    0.0301
            L1*L2*m3:    0.0348
               L2*m3:    0.2700
 L1^2*m2 + 4*L1^2*m3:    0.0518
   (L1*m2)/2 + L1*m3:    0.1261
                  f1:    0.5023
                  f2:    0.5001
                  f3:    0.4990
                 fs1:   -0.0002
                 fs2:   -0.0001
                 fs3:    0.0001
```

```
Unknown_parameters =
      I1_2:    0.0688
      I2_1:    0.0330
      I2_2:    0.0358
      I2_3:    0.1714
      I3_1:    0.0250
      I3_2:    0.0438
      I3_3:    0.0766
        m2:    0.7931
        m3:    0.5818
        L1:    0.1288
        L2:    0.4640
        f1:    0.5023
        f2:    0.5001
        f3:    0.4990
       fs1:   -0.0002
       fs2:   -0.0001
       fs3:    0.0001
```

where, f are the viscous friction forces and fs are the Coulomb friction forces according to equation 10. The root mean square error RMS obtained with this fitting was **7e-4**, i.e. almost perfect.

Question 5

Since there is no torque being applied to the joints, the robot should behave like a 2-link pendulum and oscillate in the vertical position until the friction forces brings the system to the steady-state. This can be confirmed by the simulation below, which was done using the identified parameters and the following initial conditions $q_0 = [0, 0, 0]$ and $\dot{q} = [1, 0, 0]$

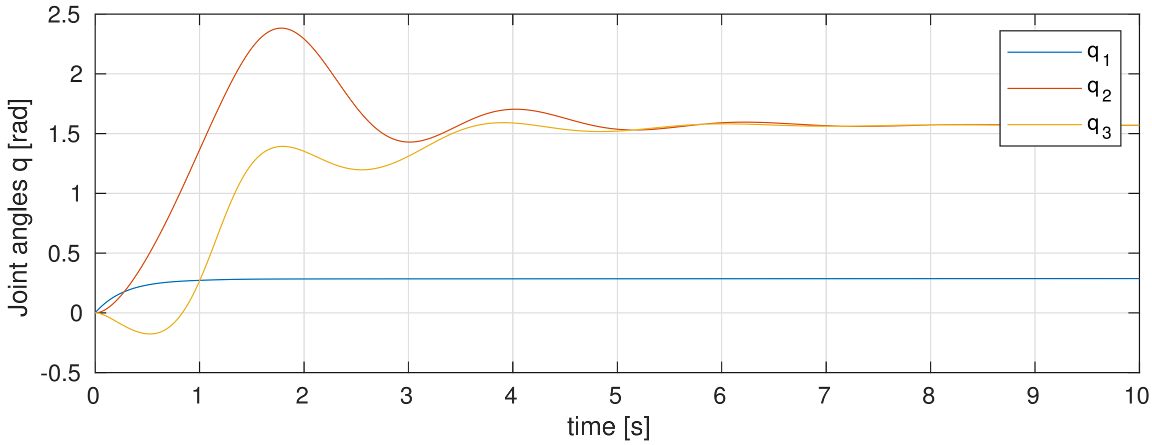

Figure 2: *Simulation using identified parameters*