Exercise 1

Problem 1:

Consider the following two optimization problems:

$$\begin{aligned} \min \quad & \frac{1}{2}(x_1^2 + x_2^2) \\ \text{s.t.} \quad & x_1 + x_2 + 2 \leq 0, \end{aligned} \tag{P1}$$

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & (x_1 - 1)^2 + x_2^2 - 1 = 0 \\ & (x_1 - 2)^2 + x_2^2 - 4 = 0. \end{aligned} \tag{P2}$$

Solve each of the problems

a) graphically,

b) analytically using the KKT-conditions,

c) numerically using the command `fmincon` in MATLAB.

Problem 2: Linear least square fit

Suppose we have a set of M measurement points $\mathcal{M} = \{(x_1, y_1), (x_2, y_2), \dots, (x_M, y_M)\}$. We want to find a function $h(x)$ such that the quadratic error

$$\sum_{i=1}^{M} \big(h(x_i) - y_i\big)^2$$

is minimized.

a) Assume that $h(x)$ is of the form

$$h(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

where $\beta_{1,2,3,4} \in \mathbb{R}$ are the unknown fitting parameters. Write the least square fitting problem in the form

$$\min_{\beta} \|X\beta - y\|^2$$

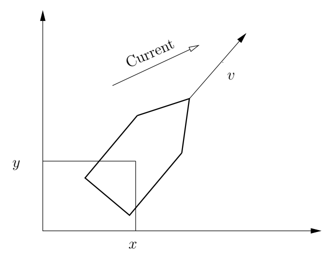
where $\beta = \begin{bmatrix}\beta_1 & \beta_2 & \beta_3 & \beta_4\end{bmatrix}^\top$ is the parameter vector and $y = \begin{bmatrix}y_1 & y_2 & \dots & y_M\end{bmatrix}^\top$ is the vector of measurement points.

b) Solve the optimization problem from a) analytically.

c) Numerically compute the parameters $\beta_{1,2,3,4}$ in MATLAB for the data set available on ILIAS.

Problem 3: Zermelo's Problem

Consider a boat on the ocean. The boat shall travel from an initial point to an end point in shortest time. We assume the magnitude of the boat's velocity vector to be constant, so the boat is steered only by influencing its heading angle. Additionally, there are strong currents that affect the boat's velocity.



a) Propose suitable equations of motions for the boat and denote the input with u.

b) Formulate the problem as an optimal control problem.

Problem 4: Shortest path on a sphere

Consider a unit sphere

$$S^2 := \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

in the three dimensional space and let $P_0 = \begin{bmatrix}x_0 & y_0 & z_0\end{bmatrix}^\top$ and $P_f = \begin{bmatrix}x_f & y_f & z_f\end{bmatrix}$ be two given points on the sphere. Formulate the problem of finding the shortest path on the sphere from P_0 to P_f as an optimal control problem.