1) From the 4-D equation

$$\nabla_\mu \nabla^\mu \Phi - m^2 \Phi = 0 \qquad (1.1)$$

$$\nabla_\mu \left(g^{\mu\nu} \nabla_\nu \Phi \right) = \nabla_\mu \left(g^{\mu\nu} \partial_\nu \Phi \right)$$

$$= g^{\mu\nu} \nabla_\mu \left(\partial_\nu \Phi \right)$$

$$= g^{\mu\nu} \left[\partial_\mu \partial_\nu \Phi - \Gamma^\sigma_{\mu\nu} \partial_\sigma \Phi \right] \qquad (1.2)$$

Thus, we have from (1)

$$\boxed{g^{\mu\nu} \partial_\mu \partial_\nu \Phi - \Gamma^\sigma \partial_\sigma \Phi - m^2 \Phi = 0} \qquad (1.3)$$

where $\boxed{\Gamma^\mu \equiv g^{\alpha\beta} \Gamma^\mu_{\alpha\beta}}$ $\qquad (1.4)$

2) From the divergence formula

From Wald (3.4.10)

$$\nabla_\mu T^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, T^\mu \right) \qquad (2.1)$$

Thus

$$\nabla_\mu \nabla^\mu \Phi = \nabla_\mu \left(g^{\mu\nu} \nabla_\nu \Phi \right) = \nabla_\mu \left(g^{\mu\nu} \partial_\nu \Phi \right)$$

$$= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi \right)$$

And the K-G equation becomes

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi \right) - m^2 \Phi = 0 \qquad (2.2)$$

Let us expand (2.2) in time and space derivatives:

$$\partial_t \underbrace{\left(\sqrt{-g}\, g^{t\nu} \partial_\nu \Phi\right)}_{\equiv A} + \partial_i \underbrace{\left(\sqrt{-g}\, g^{i\nu} \partial_\nu \Phi\right)}_{\equiv B} - \sqrt{-g}\, m^2 \Phi = 0 \qquad (2.3)$$

Part A

$$\sqrt{-g}\, g^{t\nu} \partial_\nu \Phi = \sqrt{-g}\left(g^{tt} \partial_t \Phi + g^{ti} \partial_i \Phi\right) \equiv A \qquad (2.4)$$

Part B

$$\sqrt{-g}\, g^{i\nu} \partial_\nu \Phi = \sqrt{-g}\left(g^{ti} \partial_t \Phi + g^{ij} \partial_j \Phi\right) \equiv B^i \qquad (2.5)$$

To reduce the order of space derivatives, we define

$$\psi_i \equiv \partial_i \Phi \qquad (2.6)$$

$$\boxed{A = \Pi \equiv \sqrt{-g}\left(g^{tt} \partial_t \Phi + g^{ti} \psi_i\right)} \qquad (2.7)$$

Where (2.6) was used. From (2.7) we get

$$\boxed{\partial_t \Phi = \frac{\Pi}{\sqrt{-g}\, g^{tt}} - \frac{g^{ti}}{g^{tt}} \psi_i} \qquad (2.8)$$

And using (2.6) and (2.8) in (2.5)

$$B^i = \sqrt{-g}\left[g^{ti}\left(\frac{\Pi}{\sqrt{-g}\, g^{tt}} - \frac{g^{tj}}{g^{tt}} \psi_j\right) + g^{ij} \psi_j\right]$$

$$\Rightarrow \sqrt{-g}\left[\frac{g^{ti}}{g^{tt}}\left(\frac{\Pi}{\sqrt{-g}} - g^{tj} \psi_j\right) + g^{ij} \psi_j\right] = B^i \qquad (2.9)$$

The time derivatives of ψ_i can be computed using (2.6) and (2.8):

$$\partial_t \psi_i = \partial_t \partial_i \psi = \partial_i (\partial_t \psi)$$

$$\Rightarrow \boxed{\partial_t \psi_i = \partial_i \left(\frac{\Pi}{\sqrt{-g}\, g^{tt}} - \frac{g^{tj}}{g^{tt}} \psi_j \right)} \quad (2.10)$$

To compute the time derivative of Π, we use (2.7) in the K-G equation

$$\partial_t \Pi + \partial_i \beta^i - \sqrt{-g}\, m^2 \Phi = 0 \qquad (2.11)$$

$$\Rightarrow \boxed{\partial_t \Pi = -\partial_i \left\{ \sqrt{-g} \left[\frac{g^{ti}}{g^{tt}} \left(\frac{\Pi}{\sqrt{-g}} - g^{tj} \psi_j \right) + g^{ij} \psi_j \right] \right\} + \sqrt{-g}\, m^2 \Phi}$$

The system formed by (2.8), (2.10) and (2.11) can be solved to find Φ

Using that: $g^{tt} = -\alpha^{-2}$
$g^{ti} = \alpha^{-2} \beta^i$
$g^{ij} = \gamma^{ij} - \alpha^{-2} \beta^i \beta^j$
$\sqrt{-g} = \alpha \sqrt{\gamma}$

we have

$$\partial_t \Phi = \frac{-\alpha^{\cancel{2}} \Pi}{\cancel{\alpha} \sqrt{\gamma}} + \cancel{\alpha^2} \cancel{\alpha^{-2}} \beta^i \psi_i$$

$$\Rightarrow \boxed{\partial_t \Phi = \beta^i \psi_i - \frac{\alpha \Pi}{\sqrt{\gamma}}} \qquad (2.12)$$

$$\boxed{\partial_t \Psi_i = \partial_i \left(\beta^j \Psi_j - \frac{\alpha \Pi}{\sqrt{\gamma}} \right)} \quad (2.13)$$

$$B^i = \alpha \sqrt{\gamma} \left[\alpha^{-2} \beta^i \left(\beta^j \Psi_j - \frac{\alpha \Pi}{\sqrt{\gamma}} \right) + \left(\gamma^{ij} - \alpha^{-2} \beta^i \beta^j \right) \Psi_j \right]$$

$$= \alpha \sqrt{\gamma} \left[\alpha^{-2} \beta^i \beta^j \Psi_j - \frac{\beta^i \Pi}{\alpha \sqrt{\gamma}} + \gamma^{ij} \Psi_j - \alpha^{-2} \beta^i \beta^j \Psi_j \right]$$

$$\Rightarrow \boxed{B^i = \alpha \sqrt{\gamma}\, \gamma^{ij} \Psi_j - \beta^i \Pi} \quad (2.14)$$

And finally

$$\boxed{\partial_t \Pi = \alpha \sqrt{\gamma}\, m^2 \Phi - \partial_i B^i} \quad (2.15)$$

Ingoing Gaussian

An ingoing gaussian initial condition must satisfy

$$\partial_t \Phi - \partial_r \Phi = 0$$

If $r = \sqrt{x^2 + y^2 + z^2}$ and $\Phi(x, y, z)$

$$\partial_r = \frac{x}{r} \partial_x + \frac{y}{r} \partial_y + \frac{z}{r} \partial_z$$

$$\Rightarrow \partial_r \Phi = \frac{x^i}{r} \partial_i \Phi = \frac{x^i}{r} \Psi_i$$

$$\beta^i \Psi_i - \frac{\alpha \Pi}{\sqrt{\gamma}} - \frac{x^i}{r} \Psi_i = 0$$

$$\boxed{\Pi = \frac{\sqrt{\gamma}}{\alpha}\left(\beta^i - \frac{x^i}{\mu}\right)\psi_i} \qquad (2.16)$$