Flux Conservative Klein Gordon

Lucas Timotheo Sanches <lucas.t.s.carneiro@gmail.com>

May 17, 2022

Abstract

This Thorn evolves the massive Klein Gordon field equation on top of a general (and possibly evolving) background spacetime. Once the spacetime initial data and evolution methods are set, the Klein Gordon field is computed on top of it. It's also possible take the field's contribution to the energy momentum tensor into account.

1 Introduction

This thorn aims to evolve the Klein-Gordon (KG) field on top of any background spacetime in a stable and well behaved manner. In order to do so, the KG equation is rewritten in first order flux-conservative (FC) form and the system characteristics are computed and analysed.

2 Physical System

The original KG equation for a scalar field (Φ) of mass m states that

$$\nabla_\mu \nabla^\mu \Phi - m^2 \Phi = 0. \tag{1}$$

It is well know that [1]

$$\nabla_\mu T^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^\mu) \tag{2}$$

which allows us to write

$$\nabla_\mu \nabla^\mu \Phi = \nabla_\mu (g^{\mu\nu} \nabla_\nu \Phi) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, g^{\mu\nu} \nabla_\nu \Phi) \tag{3}$$

and using the fact that $\nabla_\mu \Phi = \partial_\mu \Phi$, the KG equation becomes

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, g^{\mu\nu} \partial_\nu \Phi) - m^2 \Phi = 0. \tag{4}$$

To get to the flux conservative form of the equation, we first write out the outer spatial and time derivatives explicitly in Eq. (4) and obtain

$$\partial_t (\sqrt{-g}\, g^{t\nu} \partial_\nu \Phi) + \partial_i (\sqrt{-g}\, g^{i\nu} \partial_\nu \Phi) - \sqrt{-g}\, m^2 \Phi = 0 \tag{5}$$

where Latin indices label spatial dimensions. By defining

$$\Psi_i \equiv \partial_i \Phi \tag{6}$$

we can define a new function Π as

$$\Pi \equiv \sqrt{-g}\, g^{t\nu} \partial_\nu \Phi = \sqrt{-g}(g^{tt} \partial_t \Phi + g^{ti} \Psi_i), \tag{7}$$

which we can immediately invert to yield a time evolution equation for Φ:

$$\partial_t \Phi = \frac{1}{g^{tt}} \left(\frac{\Pi}{\sqrt{-g}} - g^{ti} \Psi_i \right). \tag{8}$$

By defining the flux vector F^i as

$$F^i \equiv \sqrt{-g}\, g^{i\nu} \partial_\nu \Phi = \sqrt{-g} \left\{ g^{ti} \left[\frac{1}{g^{tt}} \left(\frac{\Pi}{\sqrt{-g}} - g^{tj} \Psi_j \right) \right] + g^{ij} \Psi_j \right\}. \tag{9}$$

the original KG equation becomes

$$\partial_t \Pi + \partial_i F^i - \sqrt{-g}\, m^2 \Phi = 0, \tag{10}$$

which immediately yields a time evolution equation for Π. In order to complete the evolution system, we need to find time evolution equations for the Ψ_i variables which can be easily done using Eq. (6) and assuming that the Ψ_i functions are such that their second order partial derivatives commute:

$$\partial_t \Psi_i = \partial_t(\partial_i \Phi) = \partial_i(\partial_t \Phi) = \partial_i \left[\frac{1}{g^{tt}} \left(\frac{\Pi}{\sqrt{-g}} - g^{tj} \Psi_j \right) \right]. \tag{11}$$

We are now in possession of all necessary ingredients to evolve the system. Since we would like to use the `ADMBase` infrastructure it's advantageous to substituite metric components with ADM variables [2]. By doing that, we can write the evolution system as

$$\partial_t \Pi = \alpha \sqrt{\gamma}\, m^2 \Phi - \partial_i F^i \tag{12}$$

$$\partial_t \Psi_i = \partial_i \left(\beta^j \Psi_j - \frac{\alpha \Pi}{\sqrt{\gamma}} \right) \tag{13}$$

$$\partial_t \Phi = \left(\beta^i \Psi_i - \frac{\alpha \Pi}{\sqrt{\gamma}} \right) \tag{14}$$

where α is the spacetime lapse, β^i the contravariant shift vector and F^i the flux vector, given by

$$F^i = \alpha \sqrt{\gamma}\, \gamma^{ij} \Psi_j - \beta^i \Pi \tag{15}$$

3 Numerical Implementation

4 Using This Thorn

4.1 Obtaining This Thorn

4.2 Basic Usage

4.3 Special Behaviour

4.4 Interaction With Other Thorns

4.5 Examples

4.6 Support and Feedback

5 History

5.1 Thorn Source Code

5.2 Thorn Documentation

5.3 Acknowledgements

References

[1] Wald, Robert M, *General Relativity*, University of Chicago Press, 1984.

[2] Baumgarte, Thomas W and Shapiro, Stuart L, *Numerical relativity*, Cambridge University Press, 2010.