Energia conservada de uma métrica ADM

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad \text{onde} \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\mathcal{L} = \frac{1}{2}\left\{-N^2 \dot{t}^2 + \gamma_{ij}\left[\dot{x}^i \dot{x}^j + \beta^j \dot{x}^i \dot{t} + \beta^i \dot{t} \dot{x}^j + \beta^i \beta^j \dot{t}^2\right]\right\}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\left\{\left[-N^2 + \gamma_{ij}\beta^i\beta^j\right]\dot{t}^2 + \left[\gamma_{ij}\beta^j \dot{x}^i + \gamma_{ij}\beta^i \dot{x}^j\right]\dot{t} + \gamma_{ij}\dot{x}^i \dot{x}^j\right\}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}\left\{\left[-N^2 + \gamma_{ij}\beta^i\beta^j\right]\dot{t}^2 + 2\gamma_{ij}\beta^i \dot{x}^j \dot{t} + \gamma_{ij}\dot{x}^i \dot{x}^j\right\}$$

$$\Rightarrow \boxed{\mathcal{L} = \frac{1}{2}\left[-N^2 + \gamma_{ij}\beta^i\beta^j\right]\dot{t}^2 + \gamma_{ij}\beta^i \dot{x}^j \dot{t} + \frac{1}{2}\gamma_{ij}\dot{x}^i \dot{x}^j} \qquad (1)$$

$$-\frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left[-N^2 + \gamma_{ij}\beta^i\beta^j\right]\dot{t} - \gamma_{ij}\beta^i \dot{x}^j$$

$$\Rightarrow \boxed{E_G = \left[N^2 - \gamma_{ij}\beta^i\beta^j\right]\dot{t} - \gamma_{ij}\beta^i \dot{x}^j} \qquad (2)$$

Temos também que $\dot{t} = E_L/N$, logo

$$E_G = \left[N^2 - \gamma_{ij}\beta^i\beta^j\right]\frac{E_L}{N} - \gamma_{ij}\beta^i \dot{x}^j$$

Adicionalmente temos

$$V^i = \frac{1}{N}\left(\frac{dx^i}{dt} + \beta^i\right) \Rightarrow \frac{dx^i}{dt} = NV^i - \beta^i$$

$$\frac{dx^i}{d\lambda} = \frac{dx^i}{dt}\frac{dt}{d\lambda} = \left(NV^i - \beta^i\right)\frac{E_L}{N}$$, logo

$$E_G = \left[N^2 - \gamma_{ij}\beta^i\beta^j\right]\frac{E_L}{N} - \gamma_{ij}\beta^i\left(NV^j - \beta^j\right)\frac{E_L}{N}$$

$$= \left[N^2 \cancel{- \gamma_{ij}\beta^i\beta^j} - N\gamma_{ij}\beta^i V^j \cancel{+ \gamma_{ij}\beta^i\beta^j}\right]\frac{E_L}{N}$$

$$\Rightarrow \boxed{E_G = \left(N - \gamma_{ij}\beta^i V^j\right)E_L} \qquad (3)$$

Quantidade Conservada em geral

Seja K^μ tal que $\mathcal{L}_K g_{\mu\nu} = 0$ e uma geodésica de vetor tangente p^μ temos que a quantidade C dada por

$$\boxed{C = \alpha\, p_\mu K^\mu} \qquad (1)$$

é conservada ao longo da geodésica. Na formulação 3 + 1 temos

$$p^\mu = E_L\left(n^\mu + V^\mu\right)$$

Como temos $n^\alpha = \left(N^{-1}, -N^{-1}\beta^i\right)$, segue

$$p^\mu = E_L\left(\frac{1}{N} + \cancel{v^0}^{\to 0},\ \frac{-\beta^1}{N} + v^1,\ \frac{-\beta^2}{N} + v^2,\ \frac{-\beta^3}{N} + v^3 \right)$$

$$\Rightarrow \boxed{p^\mu = E_L\left(\frac{1}{N},\ v^i - \frac{\beta^i}{N} \right)} \qquad (2)$$

Luego

abreviar

$$C = \alpha g_{\mu\nu} p^\mu k^\nu = \left[g_{00} p^0 k^0 + g_{0i} p^0 k^i + g_{i0} p^i k^0 + g_{ij} p^i k^j \right]$$

$$\Rightarrow \boxed{C = \left[(\gamma_{ij} \beta^i \beta^j - \alpha^2) k^0 p^0 + \gamma_{ij} \beta^j k^i p^0 + \gamma_{ij} \beta^j p^i k^0 + g_{ij} p^i k^j \right.} \qquad (2)$$