Energia conservada de uma métrica ADM

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad \text{onde} \quad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

$$\mathcal{L} = \frac{1}{2} \left\{ -N^2 \dot{t}^2 + \gamma_{ij} \left[\dot{x}^i \dot{x}^j + \beta^j \dot{x}^i \dot{t} + \beta^i \dot{t} \dot{x}^j + \beta^i \beta^j \dot{t}^2 \right] \right\}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \left\{ \begin{array}{l} \left[-N^2 + \gamma_{ij} \beta^i \beta^j \right] \dot{t}^2 + \left[\gamma_{ij} \beta^j \dot{x}^i + \gamma_{ij} \beta^i \dot{x}^j \right] \dot{t} \\ + \gamma_{ij} \dot{x}^i \dot{x}^j \end{array} \right\}$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} \left\{ \left[-N^2 + \gamma_{ij} \beta^i \beta^j \right] \dot{t}^2 + 2 \gamma_{ij} \beta^i \dot{x}^j \dot{t} + \gamma_{ij} \dot{x}^i \dot{x}^j \right\}$$

$$\Rightarrow \boxed{\begin{array}{l} \mathcal{L} = \frac{1}{2} \left[-N^2 + \gamma_{ij} \beta^i \beta^j \right] \dot{t}^2 \\ + \gamma_{ij} \beta^i \dot{x}^j \dot{t} + \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j \end{array}} \quad (1)$$

$$-\frac{\partial \mathcal{L}}{\partial \dot{t}} = -\left[-N^2 + \gamma_{ij} \beta^i \beta^j \right] \dot{t} - \gamma_{ij} \beta^i \dot{x}^j$$

$$\Rightarrow \boxed{E_G = \left[N^2 - \gamma_{ij} \beta^i \beta^j \right] \dot{t} - \gamma_{ij} \beta^i \dot{x}^j} \quad (2)$$

Temos também que $\dot{t} = E_L / N$, logo

$$E_G = \left[N^2 - \gamma_{ij} \beta^i \beta^j \right] \frac{E_L}{N} - \gamma_{ij} \beta^i \dot{x}^j$$

Adicionalmente temos

$$V^i = \frac{1}{N}\left(\frac{dx^i}{dt} + \beta^i\right) \Rightarrow \frac{dx^i}{dt} = N V^i - \beta^i$$

$$\frac{dx^i}{d\lambda} = \frac{dx^i}{dt}\frac{dt}{d\lambda} = (N V^i - \beta^i)\frac{E_L}{N}$$, logo

$$E_G = [N^2 - \gamma_{ij}\beta^i\beta^j]\frac{E_L}{N} - \gamma_{ij}\beta^i(N V^j - \beta^j)\frac{E_L}{N}$$

$$= \left[N^2 \cancel{- \gamma_{ij}\beta^i\beta^j} - N\gamma_{ij}\beta^i V^j \cancel{+ \gamma_{ij}\beta^i\beta^j}\right]\frac{E_L}{N}$$

$$\Rightarrow \boxed{E_G = (N - \gamma_{ij}\beta^i V^j)E_L} \quad (3)$$

A dedução a cima pressupõe que E_G é por unidade de massa. A eq. (3) compara duas coisas diferentes.
Vou refazer sem ser por unidade de massa

$$E_G = -p_\mu \xi^\mu = -g_{\mu\nu} p^\mu \xi^\nu$$

$$= -g_{\mu\nu} E_L (n^\mu + V^\mu)\delta^{t\nu}$$

$$= -g_{\mu t}(n^\mu + V^\mu)E_L$$

$$= -(g_{t\mu} n^\mu + g_{t\mu} V^\mu)E_L$$

$$= -(n_t + g_{ti} V^i)E_L$$

$$= -(-N + \beta_i V^i) E_L$$

$$= (N - \gamma_{ij}\beta^i V^j) E_L \quad \text{luego}$$

$$\boxed{E_G = (N - \gamma_{ij}\beta^i V^j) E_L} \quad (4)$$

$$p^\mu p_\mu = = g_{\mu\nu} p^\mu p^\nu = g_{\mu\nu} m^2 \dot{x}^\mu \dot{x}^\nu$$

$$= m^2 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \Rightarrow \boxed{p^\mu p_\mu = m^2 \delta} \quad (5)$$

$$p^\alpha p_\alpha = E^2 (n^\alpha + V^\alpha)(n_\alpha + V_\alpha)$$

$$= \Big[\underbrace{n^\alpha n_\alpha}_{=-1} + \underbrace{n^\alpha V_\alpha}_{=0} + \underbrace{V^\alpha n_\alpha}_{=0} + V^\alpha V_\alpha\Big] E^2$$

$$\Rightarrow p^\alpha p_\alpha = m^2 \delta = (V^\alpha V_\alpha - 1) E^2$$

$$\Rightarrow V^\alpha V_\alpha - 1 = \left(\frac{m}{E}\right)^2 \delta$$

$$\Rightarrow \boxed{V^\alpha V_\alpha = 1 + \delta \left(\frac{m}{E}\right)^2}$$