Exercise Sheet 6

Exercise 1

In order to overcome the ill-posednes of the problem of differentiation of noisy data $f_\delta : [0,1] \to \mathbb{R}$, one can apply the Tikhonov regularization. This technique consists in minimizing of the functional

$$J(u) := \frac{1}{2} \int_0^1 (u(x) - f_\delta(x))^2 \, dx + \frac{\alpha}{2} \int_0^1 \left| \frac{du}{dx}(x) \right|^2 dx \tag{1}$$

over all functions u in the Sobolev space $H_0^1(0,1) := \left\{ u \in L^2(0,1) : \frac{du}{dx} \in L^2(0,1),\ u(0) = u(1) = 0 \right\}$. Here $\alpha > 0$ is the regularization parameter.
Prove that the functional J has a unique minimum, which is achieved at the function $u_\alpha \in H_0^1(0,1)$, which is a weak solution to the problem

$$\begin{cases} u_\alpha(x) - \alpha \dfrac{d^2u}{dx^2}(x) = f_\delta(x) \quad x \in (0,1), \\ u_\alpha(0) = u_\alpha(1) = 0\,. \end{cases} \tag{2}$$

Given the noisy data f_δ, write a computer program to solve the above problem on a uniform grid with N inner nodes $h, 2h, \ldots, 1-h$, where $h = 1/(N+1)$. Draw plots of functions f, f_δ and u in one figure, and next, their derivatives in the new one. Fix N and vary the regularization parameter α. Investigate the behavior of the errors as a function of α and δ.
Next consider the problem of an optimal value choice for the regularization parameter α in the Tikhonov model (1). To that end, assume that $\|n_\delta\|_{L^2(0,1)} = \delta$ and derive a bound in terms of δ and α of the L^2-norm of the difference

$$\frac{du_\alpha}{dx} - \frac{df}{dx}\,,$$

where u_α is a unique solution to the problem (2) and f is the true data. Next, choose α so that the error in the solution would be minimal. *Hint:* At some point you may need to use the inequality $2ab \leq a^2 + b^2$.

Exercise 2

Consider the one-dimensional version of the stady-state diffusion equation with spatially varying heat conductivity a, namely

$$-\frac{d}{dx}\left(a(x) \frac{du}{dx}(x) \right) = f(x) \tag{3}$$

for $x \in (0,1)$ and appropriate boundary condition. Here f is a given function corresponding to internal heat source. We are interested in solving the inverse problem that consists in determining the coefficient a in the above equation from measurements of u.
Derive explicit formula for the function a from equation (3), and discuss terms that may cause effect of instability in solving the considered problem. Suppose that the boundary condition are

$$a(0)\frac{du}{dx}(0) = 0, \quad a(1)\frac{du}{dx}(1) = 0\,, \tag{4}$$

$f(x) = -1$ and $u(x) = x$ for $x \in (0,1)$. Find a satisfying (3) and (4). Assume that u is perturbed to $u_\delta(x) = \delta \sin\left(x/\delta^2\right) + x$, find a_δ that corresponds to these noisy measurements. Consider the limits of u_δ and a_δ as δ tends to 0. Is the coefficient identification problem stable or unstable with respect to perturbations in the observation u?
Consider the stability estimates in the stady-state diffusion model (3). Assume that functions u_1 and u_2 are solutions to the problem (3) with coefficients a_1 and a_2, respectively. Find the L^2 -norm estimate of the difference $a_1 - a_2$. In which function space should u lie so that the problem of coefficient estimation in the model (3) would be stable?