Problem 1

Consider the problem of calculating the derivative of a function which contains additive noise. That is, instead of the true data f we only have the function f^δ, such that

$$f^\delta(x) = f(x) + n^\delta(x)$$

for $x \in (0,1)$ and $f^\delta(0) = f(0) = 0, f^\delta(1) = f(1) = 0$, where

$$n^\delta(x) = \sqrt{2}\delta \sin(2\pi k x)$$

represents the data noise. Here $k \in \mathbb{N}$ can be arbitrary large and $\delta \in \mathbb{R}$ is some small positive number.

1. Calculate the L^2-norm and the L^∞-norm of the differences $f - f^\delta$ and $\frac{df}{dx} - \frac{df^\delta}{dx}$, to conclude that the problem of differentiation of a noisy data is ill-posed.
2. For $f(x) = \sin(2\pi x)$ and several different values of δ and k, draw the graphs of the functions f and f^δ in one figure, and their derivatives $\frac{df}{dx}$ and $\frac{df^\delta}{dx}$ in another figure.
3. Assuming that $f \in C^\mu(0,1)$ with $\mu = 2$ or 3, estimate the L^∞-norm of the difference

$$\frac{df}{dx} - \frac{f^\delta(\cdot + h) - f^\delta(\cdot - h)}{2h}\,. \tag{1}$$

   That is, estimate the error that you introduce when you compute numerically the derivative of f^δ using the Euler central difference scheme on a uniform grid with N inner nodes $h, 2h, \ldots, 1-h$, where $h = 1/(N+1)$.
4. Draw the graph of L^∞-norm of the difference (1) as a function of the discretization step h. Comment your results.

In order to overcome the ill-posednes of the problem of differentiation of noisy data $f^\delta : [0,1] \to \mathbb{R}$, one can apply the Tikhonov regularization. This technique consists in minimizing of the functional

$$J(f) := \frac{1}{2}\int_0^1 \left(f(x) - f^\delta(x)\right)^2 dx + \frac{\alpha}{2}\int_0^1 \left|\frac{df}{dx}(x)\right|^2 dx \tag{2}$$

over all functions f in the space $H_0^1(0,1) := \left\{f \in L^2(0,1) : \frac{df}{dx} \in L^2(0,1),\ f(0) = f(1) = 0\right\}$. Here $\alpha > 0$ is the regularization parameter. Using standard results of variational calculus, it can be shown that the functional J has a unique minimum, which is achieved at the function $f_\alpha \in H_0^1(0,1)$, which is a weak solution to the problem

$$\begin{cases} f_\alpha(x) - \alpha\,\dfrac{d^2 f_\alpha}{dx^2}(x) = f^\delta(x) \quad x \in (0,1), \\ f_\alpha(0) = f_\alpha(1) = 0\,. \end{cases} \tag{3}$$

5. Given the noisy data f^δ, write a computer program to solve the problem (3) on a uniform grid with N inner nodes $h, 2h, \ldots, 1-h$, where $h = 1/(N+1)$ using, e.g., the finite difference method. Draw plots of functions f, f^δ and f_α in one figure, and their derivatives in another figure.
6. Fix δ, k, N and vary the regularization parameter α. Investigate the behavior of the L^2-norm of the differences $f_\alpha - f$ and $\frac{df_\alpha}{dx} - \frac{df}{dx}$ as a function of α.
7. Consider the problem of an optimal value choice for the regularization parameter α in the Tikhonov model (2). To that end, apply the estimate of the L^2-norm of the difference $\frac{df_\alpha}{dx} - \frac{df}{dx}$ derived on the lecture.