Exercise Sheet 2

Exercise 1

Consider the equation

$$\pi - x + \frac{1}{2}\sin\left(\frac{x}{2}\right) = 0$$

for $x \in [0, 2\pi]$. Show that this equation has a unique solution $x^\star \in [0, 2\pi]$. Rewrite this equation as the fixed point problem and apply the method of successive approximation to find an approximate value of $x^\star$. Estimate the number of iterations required to achieve 10^{-2} accuracy.

Exercise 2

Apply Newton's method to solve the equation $x^2 - p = 0$ for some $p > 0$. Show that this method is a particular case of the fixed point iteration method. Determine the interval $[a, b]$ such that for any starting point $x_0 \in (a, b)$, the sequence $\{x_n\}$ generated by Newton's method will converge to $\sqrt{p}$.

Exercise 3

Consider the integral equation

$$f(x) = x + \frac{1}{4}\int_0^{\pi/2} f(y)\cos x \, dy\,.$$

Show that this equation has a unique solution in $C[0, \pi/2]$. Find an approximate solution to the above integral equation by using the method of successive approximation. Write the result in possibly compact form.

Exercise 4

Consider the initial value problem

$$\begin{cases} x'(t) &= 2(x(t)+1) \quad \text{for } t > 0\,, \\ x(0) &= 0\,. \end{cases}$$

Apply the method of successive approximation to find the approximate sequence $\{x_n(t)\}$. Analyse the convergence of this sequence as $n \to \infty$. Compare your results with the analytical solution.