Problem 4

Consider the one-dimensional version of the stady-state diffusion equation with spatially varying heat conductivity a, namely

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}(x)\right) = f(x) \tag{1}$$

for $x \in (0,1)$ and appropriate boundary condition. Here f is a given function corresponding to internal heat source. We are interested in solving the inverse problem that consists in determining the coefficient a in the above equation from measurements of u.

1. Derive explicit formula for the function a from equation (1), and discuss terms that may cause effect of instability in solving the considered problem.
2. Suppose that the boundary condition are

$$a(0)\frac{du}{dx}(0) = 0, \quad a(1)\frac{du}{dx}(1) = 1, \tag{2}$$

   the function $f(x) = -1$ and $u(x) = x$ for $x \in (0,1)$. Find a satisfying (1) and (2).
3. Assume that u is perturbed to $u_\delta(x) = x + \delta \sin\left(x/\delta^2\right)$, find a_δ that corresponds to these noisy measurements. Consider the limits of u_δ and a_δ as δ tends to 0 . Is the coefficient identification problem stable or unstable with respect to perturbations in the observation u?

Problem 5

Consider a body of mass m suspendended on a stiff spring with the Hooke constant k. If the body is displaced from its equilibrium position and released, it will execute an oscilatory motion which we will assume is damped by a resistive force which is proportional to the velocity. The constant of proporcionality c, is called the damping constant and the described problem is modeled by the following ordinary differential equation

$$m\frac{d^2u}{dx^2}(x) + c\frac{du}{dx}(x) + k\,u = 0 \tag{3}$$

with initial conditions $u(0) = u_0, du/dx(0) = u_1$. The above equation can be rewritten in the form

$$\frac{d^2u}{dx^2}(x) + a\frac{du}{dx}(x) + b\,u(x) = 0\,, \tag{4}$$

where $a = c/m$ and $b = k/m$. We are interested in the inverse problem of determining the mass m, the damping constant c and stiffness constant k from observations of the response u.

1. Analyze the uniqueness of the solution to the inverse problem of determining parameters a and b from data u.
2. Solve analytically the problem (3) with the initial conditions $u_0 = 1$, $u_1 = -1$, and with $m = 1, c = 2, k = 5$. Next, generate noisy measurements by adding uniformly distributed random numbers in $[-\delta, \delta]$ to the function u in N points corresponding to times $h, 2h, \ldots, Nh$, where h is the time step.
3. Show that integration of the equation (4) twice leads to

$$u(x) - u_0 - u_1 x + a\int_0^x (u(t) - u_0)\,dt + b\int_0^x u(t)(x-t)dt = 0\,.$$

   Using the midpoint rule, discretize the above equation on the grid $h, 2h, \ldots, Nh$. Next apply the method of least squares to estimate the parameters $a = c/m$ and $b = k/m$. Compare obtained estimates with the true parameters $a = 2$ and $b = 5$ for various choices of δ. Investigate the stability of the method with respect to δ.
4. Discretize the equation (4) using the central finite difference scheme and apply the method of least squares to estimate the parameters a and b. Compare obtained estimates with the true parameters $a = 2$ and $b = 5$ for various choices of δ. Investigate the stability of the method with respect to δ.
5. For the given noisy measurements u^δ as in Task 2, consider the minimization problem

$$\min_{a,b} \frac{1}{2}\|u - u^\delta\|^2 \qquad \text{subject to} \qquad \begin{cases} \dfrac{d^2u}{dx^2}(x) + a\dfrac{du}{dx}(x) + b\,u(x) = 0\,, \\ u(0) = u_0\,,\ \dfrac{du}{dx}(0) = u_1\,. \end{cases} \tag{5}$$

   Write a computer program to estimate values of a and b using the gradient descent method. To solve the corresponding differential equations apply the Runge-Kutta method.