Problem 2

Consider a hanging cable with the weight density u. By appropriate idealization of the problem, one can find that the vertical displacement f is given by

$$f(x) = \int_0^1 k(x,y)u(y)dy$$

for $x \in (0,1)$, where the kernel k reads

$$k(x,y) = \begin{cases} \frac{y}{T}(1-x) & \text{if } y \in [0,x]\,, \\ \frac{x}{T}(1-y) & \text{if } y \in (x,1]\,. \end{cases}$$

Here the constant $T > 0$ denotes the tension in the cable.

1. Using the Leibnitz integral rule, show that if f satisfies the above integral equation, then f is a solution of the boundary value problem
$$\begin{cases} Tf''(x) + u(x) = 0 \quad x \in (0,1)\,, \\ f(0) = f(1) = 0\,. \end{cases}$$
2. Next, show that the inverse problem of determining the density u from the measurements of f is ill-posed because it is not stable. That is, show that a small perturbation $f_\delta(x) = (x-1)\sin(x) + \delta(x-1)\sin(x/\delta)$, $0 < \delta \ll 1$, in the deflection $f(x) = (x-1)\sin(x)$ is accounted for by a large perturbation in u.

Problem 3

Consider the Fredholm integral equation of the first kind in one space dimension, given by

$$f(x) = \int_{-1}^1 k(x,y)u(y)\,dy \tag{1}$$

for $x \in (-1,1)$ with the kernel

$$k(x,y) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{|x-y|^2}{2\sigma^2}\right)\,,$$

where $\sigma > 0$ is a parameter.

1. Apply the trapezoidal rule to approximate the integral equation (1) by the $M \times N$ linear system
$$K\vec{u} = \vec{f}, \tag{2}$$
where the vector $\vec{u} \in \mathbb{R}^N$ is meant to approximate $[u(y_1), \ldots, u(y_N)]^T$, the vector $\vec{f} = [f(x_1), \ldots, f(x_M)]^T$ and $K \in \mathbb{R}^{M\times N}$ is the kernel discretization matrix.
2. Can you find the vector $\vec{u}$ by resolving the system (2)? Explain why.
3. Consider the function $u : [-1,1] \to \mathbb{R}$ given by the formula $u(x) = \chi_{[-1/2,1/2]}(x)$. Determine the vector $\vec{f}$ by (2) for the given $\vec{u}$. Find an approximate solution to (2) using the method of least squares
$$\min_{\vec{u}\in\mathbb{R}^N} \frac{1}{2}\|K\vec{u} - \vec{f}\|^2\,.$$
Consider the noisy data $\vec{f^\delta} = \vec{f} + \vec{n}^\delta$.
4. Apply the truncated singular value decomposition to reconstruct $\vec{u}$ from the exact and noisy data, $\vec{f}$ and $\vec{f^\delta}$.
5. Apply the Tiknonov regularization
$$\min_{\vec{u}\in\mathbb{R}^N} \frac{1}{2}\|K\vec{u} - \vec{f}\|^2 + \frac{\alpha}{2}\|\vec{u}\|^2\,.$$
to reconstruct $\vec{u}$ from the exact and noisy data, $\vec{f}$ and $\vec{f^\delta}$.
6. Solve the equation (1) using the convolution theorem and the discrete Fourier transform for the data as in the above tasks.