Exercise Sheet 3

Exercise 1

Check that following are normed spaces:

(a) $\mathbb{R}^N$ with Euclidean norm $\|x\|_2 = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$ for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$,

(b) $C(\Omega)$, where Ω is a compact subset of $\mathbb{R}^N$, with $\|f\| = \max_{x\in\Omega} |f(x)|$

(c) $L^p(\Omega)$ for $1 \leqslant p \leqslant \infty$, where Ω is an open subset of $\mathbb{R}^N$, with

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p dx\right)^{1/p} & \text{if } 1 \leqslant p < \infty\,, \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| & \text{if } p = \infty\,. \end{cases}$$

Exercise 2

Check that alternative norms in $\mathbb{R}^N$ are

(a) $\|x\|_\infty = \max_{1\le i\le N} |x_i|$ for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$,

(b) $\|x\|_p = \left(\sum_{i=1}^N |x_i|^p\right)^{1/p}$ for $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $1 \le p < \infty$.

For $N = 2$ draw unit balls for $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$.

Exercise 3

We say that norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in X are equivalent if there exist positive constants C_1 and C_2 such that

$$C_1\|x\|_2 \leqslant \|x\|_1 \leqslant C_2\|x\|_2 \quad \text{for all } x \in X$$

Prove that if the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in X are equivalent then convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\|\cdot\|_2$ implies convergence of $\{x_n\}_{n\in\mathbb{N}}$ in $\|\cdot\|_1$ and, conversely, convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\|\cdot\|_1$ implies convergence of $\{x_n\}_{n\in\mathbb{N}}$ in $\|\cdot\|_2$. What can we say about $\{x_n\}_{n\in\mathbb{N}}$ if for some $C_2 > 0$ the inequality

$$\|x\|_1 \leqslant C_2\|x\|_2 \quad \text{for all } x \in X$$

holds?

Exercise 4

Let $\|\cdot\|_p$ denote the p-norm in $\mathbb{R}^2$. Show that $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are equivalent norms.

Exercise 5

Consider the norms $\|f\|_{L^1} = \int_0^1 |f(x)|dx$ and $\|f\|_{C^0} = \max_{x\in[0,1]} |f(x)|$ on $X = C[0,1]$. Show that these two norms are not equivalent. *Hint*: Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by $f_n(x) = x^n$.

Exercise 6

Consider the family of functions $\{f_n\}_{n\in\mathbb{N}}$ that are piecewise defined as

$$f_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} - \frac{1}{n}, \\ \frac{1}{2} + \frac{n}{2}\left(x - \frac{1}{2}\right) & \frac{1}{2} - \frac{1}{n} < x \le \frac{1}{2} + \frac{1}{n}, \\ 1 & \frac{1}{2} + \frac{1}{n} < x \le 1. \end{cases}$$

Plot $f_n(x)$ for general n, and show that this family $\{f_n\}_{n\in\mathbb{N}}$ converges in $L^2[0,1]$ but not in $L^\infty[0,1]$.

Exercise 7

Show that $X = C[0,1]$ equipped with the L^1-norm $\|f\|_{L^1} = \int_0^1 |f(x)|dx$ is not complete. *Hint:* Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by

$$f_n(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} - \frac{1}{n} \\ nx + 1 - \frac{n}{2} & \frac{1}{2} - \frac{1}{n} \le x \le \frac{1}{2}, \\ 1 & \frac{1}{2} \le x \le 1 \end{cases} \qquad \text{for } n \ge 2\,.$$

Exercise 8

Define $C^1[a,b] := \{f : [a,b] \to \mathbb{R} : f, f' \in C[a,b]\}$. Show that the set $C^1[a,b]$ equipped with the norm $\|\cdot\|_{C^0}$ is not complete. Prove that the set $C^1[a,b]$ with the norm

$$\|f\|_{C^1} := \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |f'(x)|$$

is a Banach space.