Exercise Sheet 5

Exercise 1

Consider the real exponential integral, defined by

$$Ei(\lambda) := \int_{\lambda}^{\infty} \frac{e^{-t}}{t}\, dt$$

for large values of λ.

(a) Use integration by parts to show that

$$Ei(\lambda) = e^{-\lambda}\left(\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{2!}{\lambda^3} - \frac{3!}{\lambda^4} + ... + \frac{(-1)^{n-1}(n-1)!}{\lambda^n}\right) + r_n(\lambda)\,,$$

where

$$r_n(\lambda) = (-1)^n n! \int_{\lambda}^{\infty} \frac{e^{-t}}{t^{n+1}}\, dt\,.$$

(b) Show that for fixed n, $|r_n(\lambda)| \to 0$ as $\lambda \to \infty$.

(c) Show that for fixed n, $r_n(\lambda) = o(\frac{(n-1)!e^{-\lambda}}{\lambda^n})$ as $\lambda \to \infty$, and conclude that $Ei(\lambda)$ has the asymptotic expansion

$$Ei(\lambda) \approx e^{-\lambda}\left(\frac{1}{\lambda} - \frac{1}{\lambda^2} + \frac{2!}{\lambda^3} - \frac{3!}{\lambda^4} + ...\right)\,.$$

(d) Show that the asymptotic series in (c) diverges for fixed λ.

(e) Approximate the value $Ei(10)$. Determine how many terms in the asymptotic expansion of $Ei(\lambda)$ are needed to ensure that the error with respect to the value $Ei(10)$ provided by some scientific package is less than 10^{-8}. Compare the empirical number of these terms with the number estimated from the theoretical expression for $r_n(x)$.