Exercise Sheet 4

Exercise 1

Let X be an inner product space (i.e. pre-Hilbert space) with the inner product $\langle \cdot, \cdot \rangle$. Prove that the formula $\|x\| = \sqrt{\langle x, x \rangle}$ defines a norm in X.

Exercise 2

Check that the following are Hilbert spaces:

(a) $\mathbb{R}^N$ with inner product $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$, where $x, y \in \mathbb{R}^N$.

(b) $L^2(\Omega)$ with the inner product

$$\langle f, g \rangle = \int_\Omega f(x)\overline{g(x)}dx \, .$$

for $f, g \in L^2(\Omega)$.

Exercise 3

Show that $\mathbb{R}^N$ with $\| \cdot \|_p$ norm is not a Hilbert space unless $p = 2$.

Exercise 4

Let $\Omega \subset \mathbb{R}^N$ be a compact set. Show that $C(\Omega)$ and $L^p(\Omega), 1 \leqslant p < \infty, p \neq 2$ with standard norms are not Hlibert spaces.

Exercise 5

Let X be a Hilbert space and $x, y \in X$. Prove that if $x \perp y$ then

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \, .$$

Extend the above formula to m mutually orthogonal vectors in X. Prove that if X is a real Hilbert space (i.e., with scalars from $\mathbb{R}$) then the opposite implication is also true. Give an example that this implication may be not true if X is a complex Hilbert space.