Exercise Sheet 1

Exercise 1

Prove the following inequalities and in each case give the condition under which the equality holds.

(a) Cauchy's inequality.

$$ab \le \frac{a^2}{2} + \frac{b^2}{2} \quad \text{for } a, b \in \mathbb{R}\,.$$

Hint: $0 \le (a-b)^2$.

(b) Cauchy's inequality with ε.

$$ab \le \varepsilon a^2 + \frac{b^2}{4\varepsilon} \quad \text{for } a, b \in \mathbb{R},\ \varepsilon > 0\,.$$

Hint: $ab = ((2\varepsilon)^{1/2} a)\left(\frac{b}{(2\varepsilon)^{1/2}}\right)$.

(c) Young's inequality.
Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad \text{for } a, b \in \mathbb{R}$$

and equality holds if and only if $a^p = b^q$.
Hint 1: $x \mapsto e^x$ is convex, $ab = e^{\ln a + \ln b} = e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q}$.
Hint 2: For given $a, b > 0$ calculate area of regions

$$R_1 = \left\{0 \le x \le a,\ 0 \le y \le x^{p-1}\right\}\,, \quad R_2 = \left\{0 \le y \le b,\ 0 \le x \le y^{q-1}\right\}\,,$$

and compare with the area of the rectangle $R = [0, a] \times [0, b]$.

(d) Hölder's inequality.
Let $\Omega \subset \mathbb{R}^N$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. If

$$\int_\Omega |f(x)|^p dx\,,\ \int_\Omega |g(x)|^q dx < \infty\,,$$

then

$$\int_\Omega |f(x)g(x)| dx \le \left(\int_\Omega |f(x)|^p dx\right)^{1/p} \left(\int_\Omega |g(x)|^q dx\right)^{1/q}.$$

Hint: Consider two cases:

(i) $\int_\Omega |f(x)|^p dx > 0\,,\ \int_\Omega |g(x)|^q dx > 0$.
Define $\bar{f}(x) = f(x)/\left(\int_\Omega |f(x)|^p\, dx\right)^{1/p}$ and $\bar{g}(x) = g(x)/\left(\int_\Omega |g(x)|^q\, dx\right)^{1/q}$.
Apply Young's inequality to $|\bar{f}(x)||\bar{g}(x)|$, integrate over Ω.

(ii) one of the integrals, $\int_\Omega |f(x)|^p\, dx$ or $\int_\Omega |g(x)|^q\, dx$, vanishes.

(e) Minkowski's inequality.
Let $\Omega \subset \mathbb{R}^N$, $1 < p < \infty$. If

$$\int_\Omega |f(x)|^p\, dx\,,\ \int_\Omega |g(x)|^p\, dx < \infty\,,$$

then

$$\left(\int_\Omega |f(x) + g(x)|^p\, dx\right)^{1/p} \le \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p} + \left(\int_\Omega |g(x)|^p\, dx\right)^{1/p}.$$

Hint:

(i) $|f(x) + g(x)|^p = |f(x) + g(x)|^{p-1}(|f(x) + g(x)|) \le |f(x) + g(x)|^{p-1}(|f(x)| + |g(x)|)$.

(ii) $\frac{p-1}{p} + \frac{1}{p} = 1$, apply Hölder's inequality with appropriate exponents $1 < p', q' < \infty$ to

$$\int_\Omega |f(x) + g(x)|^{p-1}|f(x)|\, dx \quad \text{and} \quad \int_\Omega |f(x) + g(x)|^{p-1}|g(x)|\, dx\,.$$