Exercise Sheet 6

Exercise 1

Use the WKB method to find an approximate solution to the initial value problem

$$\begin{cases} y'' - \lambda(1+x^2)^2 y = 0\,, \\ y(0) = 0\,,\ y'(0) = 1\,, \end{cases}$$

for $\lambda \gg 1$.

Exercise 2

Show that the large eigenvalues of the problem

$$\begin{cases} y'' + \lambda(\pi + x)^4 y = 0\,, \\ y(0) = 0\,,\ y(\pi) = 0\,, \end{cases}$$

are given approximately by

$$\lambda = \frac{9n^2}{49\pi^4}$$

for large $n \in \mathbb{Z}$. Find the corresponding eigenfunctions.

Exercise 3

Show that the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

can be transformed into the equation

$$u'' + r(x)u = 0$$

with $r(x) = q(x) - \frac{1}{2}p'(x) - \frac{1}{4}p^2(x)$ by the Liouville transformation

$$u(x) = y(x)\exp\left(\frac{1}{2}\int_a^x p(s)ds\right)\,.$$

Exercise 4

The time-independent Schrödinger equation for a one-dimensional system is given by

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x), \qquad \text{for } -\infty < x < +\infty$$

where $\hbar$ is the reduced Planck constant, m is the mass of the particle, $V(x)$ is the potential energy, E is the total energy, and $\psi(x)$ is the wavefunction. We rewrite this equation as

$$\psi''(x) + k^2(x)\psi(x) = 0,$$

where $k(x) = \sqrt{\frac{2m}{\hbar^2}\left(E - V(x)\right)}$. Because $\hbar$ is extremely small, we see that $\lambda = 2m/\hbar^2 \gg 1$. Use the WKB method to find an approximation of ψ for $V(x) < E$ and $V(x) > E$.

Consider a particle of the mass m in a one-dimensional potential energy box (or an infinite potential well),

$$V(x) = \begin{cases} 0 & \text{if } 0 \le x \le a, \\ +\infty & \text{otherwise.} \end{cases}$$

Using the WKB approximation, derive the wavefunction ψ inside the potential well. Apply the WKB quantization condition given by

$$\int_{x_1}^{x_2} k(x)dx = \left(n + \frac{1}{2}\right)\pi\hbar,$$

where $x_1 = 0$ and $x_2 = a$ are the classical turning points (boundaries of the well), to find the allowed energy levels E_n of the particle. Compare your results with the exact energy levels derived from the exact solution to the Schrödinger equation.