Análise de Sistemas de Potência

Aula 02: Conceitos Fundamentais de Sistemas Elétricos

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Sumário

Phasors

A sinusoidal voltage or current at constant frequency is characterized by two parameters:

- **A maximum value;**
- **A phase angle.**

A voltage $v(t)$, given by:

$$v(t) = \mathrm{V}_{\max} \cos(\omega t + \delta)$$

Has a maximum value $\mathrm{V}_{\max}$ and a phase angle δ when referenced to $\cos(\omega t)$.

Phasors

The rootmean-square (rms) value, also called *effective value*, of the sinusoidal voltage is:

$$V = \frac{V_{\mathbf{max}}}{\sqrt{2}}$$

Euler's identity, $e^{j\phi} = \cos\phi + j\sin\phi$, can be used to express a sinusoid in terms of a phasor. For the above voltage, we have:

$$\begin{aligned} v(t) &= \mathrm{Re}\left[\mathrm{V}_{\max} e^{j(\omega t+\delta)}\right] \\ &= \mathrm{Re}\left[\sqrt{2}\left(\mathrm{V}e^{j\delta}\right)e^{j\omega t}\right] \end{aligned}$$

Where $j = \sqrt{-1}$ and *Re* denotes "real part of".

Phasors

The rms phasor representation of the voltage is given in three forms:

- Exponential;
- Polar;
- Rectangular.

In math notation:

$$V = \underbrace{\mathrm{V} e^{j\partial}}_{\textbf{exponential}} = \underbrace{\mathrm{V}\underline{/\partial}}_{\textbf{polar}} = \underbrace{\mathrm{V}\cos\partial + j\,\mathrm{V}\sin\partial}_{\textbf{rectangular}}$$

Phasors

A phasor can be easily converted from one form to another.

Conversion from polar to rectangular is shown in the phasor diagram of Figure bellow.

Euler's identity can be used to convert from exponential to rectangular form.

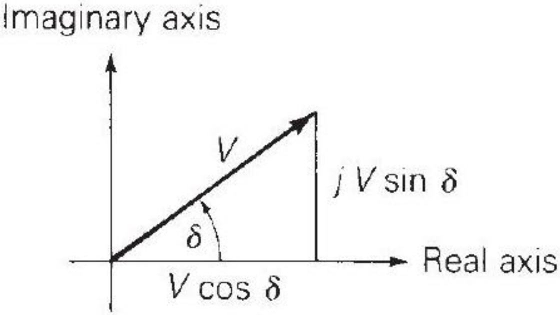
As an example, the voltage:

$$v(t) = 169.7 \cos\left(\omega t + 60^\circ\right) \;\; \textbf{volts}$$

Has a maximum value $V_{max} = 169.7$ **volts, a phase angle** $\delta = 60^\circ$ **when referenced to** $\cos(\omega t)$**, and an rms phasor representation in polar form of:**

$$V = 120\underline{/60^\circ} \quad \textbf{volts}$$

Phasors



Phasors

Also, the current:

$$i(t) = 100\cos\left(\omega t + 45^\circ\right)\ \mathrm{A}$$

Has a maximum value $\mathrm{I}_{\mathrm{max}} = 100\ \mathrm{A}$**, an rms value** $\mathrm{I} = 100/\sqrt{2} = 70.7\ \mathrm{A}$**, a phase angle of** 45°**, and the following phasor representation:**

$$I = 70.7\underline{/45}^\circ = 70.7e^{j45} = 50 + j50 \quad \mathrm{A}$$

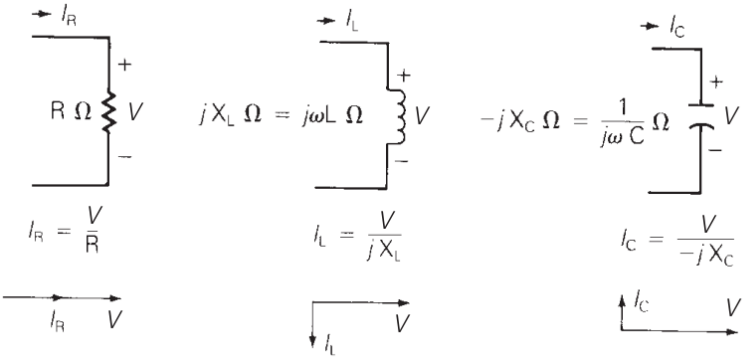
Phasors

The relationships between the voltage and current phasors for the three passive elements:

- Resistor (R);
- Inductor (L);
- Capacitor (C).

Are summarized in Figure bellow, where sinusoidal-steady-state excitation and constant values of R, L, and C are assumed.

Phasors



I_R
$R\ \Omega$
V
$I_R = \frac{V}{R}$
I_L
$jX_L\ \Omega = j\omega L\ \Omega$
$I_L = \frac{V}{jX_L}$
I_C
$-jX_C\ \Omega = \frac{1}{j\omega C}\ \Omega$
$I_C = \frac{V}{-jX_C}$

Instantaneous Power in Single Phase AC Circuits

Some Considerations

Power is the rate of change of energy with respect to time. T

he unit of power is a watt, which is a joule per second.

The instantaneous power in watts absorbed by an electrical load is the product of the instantaneous voltage across the load in volts and the instantaneous current into the load in amperes.

Assume that the load voltage is:

$$v(t) = \mathrm{V}_{\mathbf{max}} \cos(\omega t + \partial) \text{ } \mathbf{volts}$$

Purely Resistive Load

For a purely resistive load, the current into the load is in phase with the load voltage, $I = V/R$, and the current into the resistive load is

$$i_R(t) = \mathrm{I}_{\mathrm{R\,max}} \cos(\omega t + \delta) \quad \mathrm{A}$$

Where $\mathrm{I}_{\mathrm{R\,max}} = \mathrm{V}_{\mathrm{max}}/\mathrm{R}$**. The instantaneous power absorbed by the resistor is**

$$\begin{aligned} p_{\mathrm{R}}(t) &= v(t) i_{\mathrm{R}}(t) = \mathrm{V}_{\mathrm{max}} \mathrm{I}_{\mathrm{R\,max}} \cos^2(\omega t + \delta) \\ &= \frac{1}{2} V_{\mathrm{max}} I_{\mathrm{R\,max}} \{1 + \cos[2(\omega t + \delta)]\} \\ &= \mathrm{V}\mathrm{I}_{\mathrm{R}} \{1 + \cos[2(\omega t + \delta)]\} \quad \mathrm{W} \end{aligned}$$

Purely Resistive Load

The instantaneous power absorbed by the resistor has the average value:

$$P_R = VI_R = \frac{V^2}{R} = I_R^2 R \quad W$$

Plus a double-frequency term $VI_R \cos[2(\omega t + \delta)]$**, that, as a purely sinusoid wave form, has a zero average value.**

Purely Inductive Load

For a purely inductive load, the current lags the voltage by 90°, $I_L = V/(jX_L)$**, and:**

$$i_L(t) = \mathrm{I}_{\mathrm{L\,max}} \cos\left(\omega t + \delta - 90^\circ\right) \mathrm{A}$$

Where $\mathrm{I}_{\mathrm{Lmax}} = \mathrm{V}_{\mathrm{max}}/\mathrm{X}_{\mathrm{L}}$**, and** $\mathrm{X}_{\mathrm{L}} = \omega \mathrm{L}$ **is the inductive reactance.**

The instantaneous power absorbed by the inductor is:

$$\begin{aligned} p_L(t) &= v(t) i_L(t) = \mathrm{V}_{\mathrm{max}} \mathrm{I}_{\mathrm{L\,max}} \cos(\omega t + \delta) \cos\left(\omega t + \delta - 90^\circ\right) \\ &= \frac{1}{2} \mathrm{V}_{\mathrm{max}} \mathrm{I}_{\mathrm{L\,max}} \cos\left[2(\omega t + \delta) - 90^\circ\right] \\ &= \mathrm{V}\mathrm{I}_{\mathrm{L}} \sin[2(\omega t + \delta)] \mathrm{W} \end{aligned}$$

The instantaneous power absorbed by the inductor is a double-frequency sinusoid with zero average value.

Purely Capacitive Load

For a purely capacitive load, the current leads the voltage by 90°, $I_c = V/(-jX_C)$, **and:**

$$i_C(t) = I_{C\max}\cos(\omega t + \delta + 90^\circ) \quad \mathrm{A}$$

Where $I_{C\max} = V_{\max}/X_C$, **and** $X_C = 1/(\omega C)$ **is the capacitive reactance. The instantaneous power absorbed by the capacitor is:**

$$\begin{aligned} p_C(t) &= v(t)i_C(t) = V_{\max}I_{C\max}\cos(\omega t + \delta)\cos(\omega t + \delta + 90^\circ) \\ &= \frac{1}{2}V_{\max}I_{C\max}\cos\left[2(\omega t + \delta) + 90^\circ)\right] \\ &= -VI_C\sin[2(\omega t + \delta)] \quad \mathrm{W} \end{aligned}$$

The instantaneous power absorbed by a capacitor is also a double-frequency sinusoid with zero average value.

General RLC Load

For a general load composed of RLC elements under sinusoidal-steady-state excitation, the load current is of the form:

$$i(t) = \mathrm{I}_{\max} \cos(\omega t + \beta) \quad \mathrm{A}$$

Using the identity: $\cos \mathrm{A} \cos \mathrm{B} = \frac{1}{2}[\cos(\mathrm{A} - \mathrm{B}) + \cos(\mathrm{A} + \mathrm{B})]$, **the instantaneous power absorbed by the load is calculated as:**

$$\begin{aligned} p(t) &= v(t) i(t) = \mathrm{V}_{\max} \mathrm{I}_{\max} \cos(\omega t + \delta) \cos(\omega t + \beta) \\ &= \frac{1}{2} \mathrm{V}_{\max} \mathrm{I}_{\max} \{\cos(\delta - \beta) + \cos[2(\omega t + \delta) - (\delta - \beta)]\} \\ &= \mathrm{VI} \cos(\delta - \beta) + \mathrm{VI} \cos(\delta - \beta) \cos[2(\omega t + \delta)] \\ &\quad + \mathrm{VI} \sin(\delta - \beta) \sin[2(\omega t - \delta)] \\ p(t) &= \mathrm{VI} \cos(\delta - \beta)\{1 + \cos[2(\omega t + \delta)]\} + \mathrm{VI} \sin(\delta - \beta) \sin[2(\omega t - \delta)] \end{aligned}$$

General RLC Load

$$p(t) = VI\cos(\delta-\beta)\{1+\cos[2(\omega t+\delta)]\} + VI\sin(\delta-\beta)\sin[2(\omega t-\delta)]$$

Letting $I\cos(\delta-\beta) = I_R$ **and** $I\sin(\delta-\beta) = I_X$ **gives**

$$p(t) = \underbrace{VI_R\{1+\cos[2(\omega t+\delta)]\}}_{p_R(t)} + \underbrace{VI_X\sin[2(\omega t+\delta)]}_{p_x(t)}$$

The instantaneous power absorbed by the load has two components:

One can be associated with the power $p_R(t)$ **absorbed by the resistive component of the load, and the other can be associated with the power** $p_X(t)$ **absorbed by the reactive (inductive or capacitive) component of the load.**

General RLC Load

The first component $p_R(t)$ is identical to power absorbed by a purely resistive load, where $I_R = I\cos(\delta - \beta)$ is the component of the load current in phase with the load voltage.

The phase angle $(\delta - \beta)$ represents the angle between the voltage and current.

The second component $p_X(t)$ is identical to power absorbed by a capacitive or a inductive load, where $I_x = I\sin(\delta - \beta)$ is the component of load current 90° out of phase with the voltage.

Real Power

The above Power Equation shows that the instantaneous power $p_R(t)$ absorbed by the resistive component of the load is a double-frequency sinusoid with average value P given by

$$\mathrm{P} = \mathrm{VI_R} = \mathrm{VI}\ \cos(\delta - \beta)\ \mathrm{W}$$

The average power P is also called *real power or active power*.

Power Factor

The term $\cos(\delta - \beta)$ is called the power factor. The phase angle $(\delta - \beta)$, which is the angle between the voltage and current, is called the power factor angle.

For dc circuits, the power absorbed by a load is the product of the dc load voltage and the dc load current;

For ac circuits, the average power absorbed by a load is the product of the rms load voltage V, rms load current I, and the power factor $\cos(\delta - \beta)$.

Power Factor

For inductive loads, the current lags the voltage, which means β is less than δ, and the power factor is said to be lagging.

For capacitive loads, the current leads the voltage, which means β is greater than δ, and the power factor is said to be leading.

By convention, the power factor $\cos(\delta - \beta)$ is positive.

If $|\delta - \beta|$ is greater than 90°, then the reference direction for current may be reversed, resulting in a positive value of $\cos(\delta - \beta)$.

Reactive Power

The instantaneous power absorbed by the reactive part of the load, given by the component $p_X(t)$, is a double-frequency sinusoid with zero average value and with amplitude Q given by:

$$Q = VI_X = VI\sin(\delta - \beta) \text{ var}$$

The term Q is given the name *reactive power*.

Although it has the same units as real power, the usual practice is to define units of reactive power as volt-amperes reactive, or var.

Complex Power

For circuits operating in sinusoidal-steady-state, real and reactive power are conveniently calculated from complex power:

Let the voltage across a circuit element be $V = \mathrm{V}\underline{/\delta}$, and the current into the element be $I = \mathrm{I}\underline{/\beta}$.

Then the complex power S is the product of the voltage and the conjugate of the current:

$$S = VI^* = [\mathrm{V}\underline{/\delta}]\ [\mathrm{I}\underline{/\beta}]^* = \mathrm{VI}\underline{/\delta - \beta}$$
$$= \mathrm{VI}\cos(\delta - \beta) + j\mathrm{VI}\sin(\delta - \beta)$$

Where $(\delta - \beta)$ is the angle between the voltage and current.

Complex Power

Comparing the equations in last slides, $\mathcal{S}$ is recognized as:

$$\mathcal{S} = \mathrm{P} + j\mathrm{Q}$$

The magnitude $\mathrm{S} = \mathrm{VI}$ of the complex power $\mathcal{S}$ is called the apparent power.

Although it has the same units as P and Q, it is common practice to define the units of apparent power S as volt-amperes or VA.

The real power P is obtained by multiplying the apparent power $\mathrm{S} = \mathrm{VI}$ by the power factor $p.f. = \cos(\delta - \beta)$.

Complex Power

$$S = \mathrm{P} + j\mathrm{Q}$$

If P is positive, then the circuit element *absorbs positive real power*.

If P is negative, the circuit element *delivers positive real power*.

If Q is positive, the circuit element *absorbs positive reactive power*.

If Q is negative, the circuit element *delivers positive reactive power*.

Complex Power

The complex power absorbed by any of the three load elements (RLC) can be calculated as follows (assume a load voltage $V = V\angle\delta$):

$$\textbf{Resistor: } S_R = VI_R^* = [V\angle\delta]\left[\frac{V}{R}\angle-\delta\right] = \frac{V^2}{R}$$

$$\textbf{Inductor: } S_L = VI_L^* = [V\angle\delta]\left[\frac{V}{-jX_L}\angle-\delta\right] = +j\frac{V^2}{X_L}$$

$$\textbf{Capacitor: } S_C = VI_C^* = [V\angle\delta]\left[\frac{V}{jX_C}\angle-\delta\right] = -j\frac{V^2}{X_C}$$

Complex Power

The real power $\mathrm{P} = \mathrm{Re}(S)$ **absorbed by a passive load is always positive.**

The reactive power $\mathrm{Q} = \mathrm{Im}(S)$ **absorbed by a load may be either positive or negative.**

When the load is inductive, the current lags the voltage, which means β **is less than** δ**, and the reactive power absorbed is positive.**

When the load is capacitive, the current leads the voltage, which means β **is greater than** δ**, and the reactive power absorbed is negative; or, alternatively, the capacitive load delivers positive reactive power.**

Complex Power

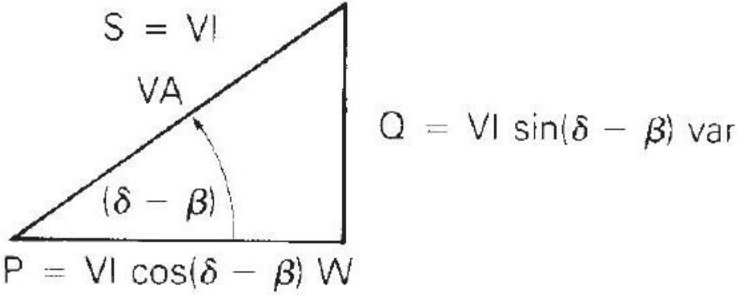
Complex power can be summarized graphically by use of the power triangle.

The apparent power S, real power P, and reactive power Q form the three sides of the power triangle.

The power factor angle $(\delta - \beta)$ is also shown, and the following expressions can be obtained:

$$
\begin{aligned}
S &= \sqrt{P^2 + Q^2} \\
(\delta - \beta) &= \tan^{-1}(Q/P) \\
Q &= P\tan(\delta - \beta) \\
\textbf{p.f.} &= \cos(\delta - \beta) = \frac{P}{S} = \frac{P}{\sqrt{P^2 + Q^2}}
\end{aligned}
$$

Complex Power



Network Equations

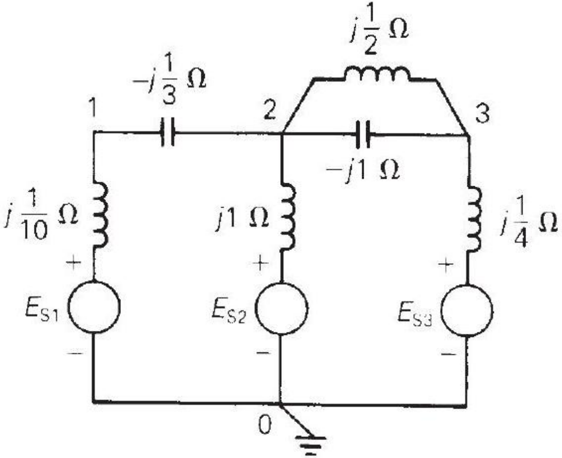
For circuits operating in sinusoidal-steady-state, Kirchhoff's current law (KCL) and voltage law (KVL) apply to phasor currents and voltages.

Thus the sum of all phasor currents entering any node is zero and the sum of the phasor voltage drops around any closed path is zero.

Network analysis techniques based on Kirchhoff's laws are useful for analyzing such circuits. Some of more know of these thechnics are:

- Nodal analysis,
- Mesh or loop analysis,
- Superposition,
- Source transformations,
- Thevenin's theorem or Norton's theorem.

Network Equations



Network Equations

Various computer solutions of power system problems are formulated from nodal equations, which can be systematically applied to circuits.

Nodal equations are written in the following three steps:

STEP 1 For a circuit with $(N+1)$ nodes (also called buses), *select one bus as the reference bus* and define the voltages at the remaining buses with respect to the reference bus.

Network Equations

The example circuit has four buses - that is, $N + 1 = 4$ or $N = 3$.

Bus 0 is selected as the reference bus, and bus voltages V_{10}, V_{20}, and V_{30} are then defined with respect to bus 0.

STEP 2 ***Transform each voltage source*** **in series with an impedance to an equivalent current source in parallel with that impedance.**

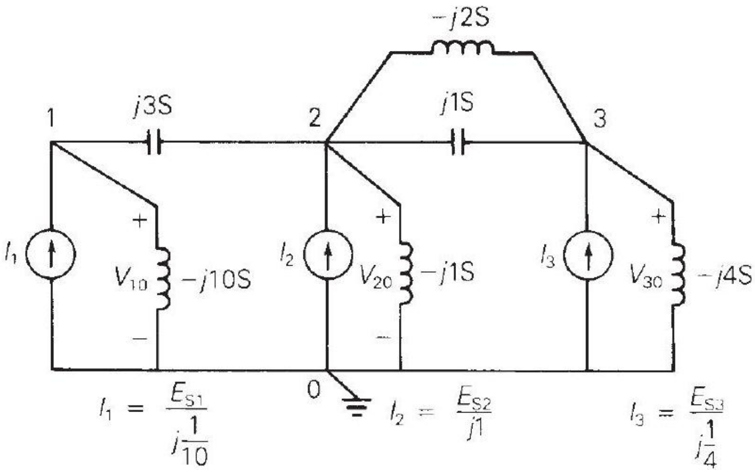
Also, ***show admittance values instead of impedance values*** **on the circuit diagram. Each current source is equal to the voltage source divided by the source impedance.**

Network Equations

STEP 3 *Write nodal equations* in matrix format as follows:

$$\begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & Y_{23} & \cdots & Y_{2N} \\ Y_{31} & Y_{32} & Y_{33} & \cdots & Y_{3N} \\ \vdots & \vdots & \vdots & & \vdots \\ Y_{N1} & Y_{N2} & Y_{N3} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_{10} \\ V_{20} \\ V_{30} \\ \vdots \\ V_{N0} \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix}$$

Network Equations



Network Equations

Using matrix notation:

$$\mathbf{Y} \cdot \mathbf{V} = \mathbf{I}$$

Where: $\boldsymbol{Y}$ **is the** $N \times N$ **bus admittance matrix,**

$\boldsymbol{V}$ **is the column vector of** N **bus voltages, and**

$\boldsymbol{I}$ **is the column vector of** N **current sources.**

Network Equations

The elements Y_{kn}, of the bus admittance matrix Y are formed as follows:

diagonal elements: $Y_{kk} =$ **sum of admittances connected to bus k** $(k = 1, 2, \ldots, N)$

off-diagonal elements: $Y_{kn} =$ **sum of admittances connected between buses** k **and** n **with** $k \neq n)$

Network Equations

The diagonal element Y_{kk} is called the self-admittance or the driving-point admittance of bus k,

The off-diagonal element Y_{kn} for $k \neq n$ is called the mutual admittance or the transfer admittance between buses k and n.

Since $Y_{kn} = Y_{nk}$, the matrix Y is symmetric.

The advantage of this method of writing nodal equations is that a digital computer can be used both to generate the admittance matrix Y and to solve for the unknown bus voltage vector V.

Network Equations

Once a circuit is specified with the reference bus and other buses identified, the circuit admittances and their bus connections become computer input data for calculating the elements Y_{kn}.

After Y is calculated and the current source vector I is given as input, standard computer programs for solving simultaneous linear equations can then be used to determine the bus voltage vector V.

That is, the system has a closed form of a classical Linear System.

Notations Used in the text

When double subscripts are used to denote a voltage in this text, the voltage shall be that at the node identified by the first subscript with respect to the node identified by the second subscript.

Also, a current I_{ab} shall indicate the current from node a to node b.

Voltage polarity marks $(+/-)$ and current reference arrows $(\rightarrow)$ are not required when double subscript notation is employed.

The polarity marks for V_{10}, V_{20}, and V_{30}, although not required, are shown for clarity.

The reference arrows for sources I_1, I_2, and I_3 are required, however, since single subscripts are used for these currents.

Matrices and vectors shall be indicated in this text by boldface type (for example, $\boldsymbol{Y}$ or $\boldsymbol{V}$).

Balanced Three Phase Systems

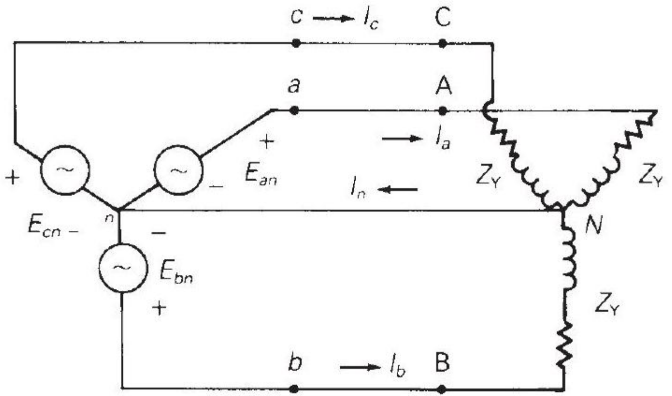


FIGURE 2.10

Circuit diagram of a three-phase Y-connected source feeding a balanced-Y load

A three-phase, Y-connected (or "wye-connected") voltage source feeding a balanced-Y-connected load.

Balanced Three-Phase Circuits

For a Y connection, the neutrals of each phase are connected.

The source neutral connection is labeled bus n and the load neutral connection is labeled bus N.

The three-phase source is assumed to be ideal since source impedances are neglected.

Also neglected are the line impedances between the source and load terminals, and the neutral impedance between buses n and N.

The three-phase load is balanced, which means the load impedances in all three phases are identical.

Balanced Line-To-Neutral Voltages

The terminal buses of the three-phase source are labeled a, b, and c, and the source line-to-neutral voltages are labeled E_{an}, E_{bn}, and E_{cn}.

The source is balanced when these voltages have equal magnitudes and an equal 120°-phase difference between any two phases.

An example of balanced three-phase line-to-neutral voltages is

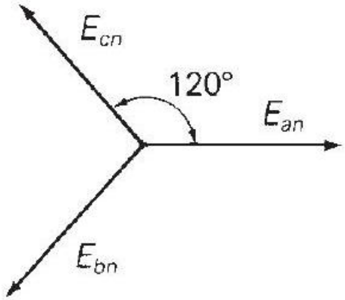
$$E_{an} = 10\underline{/0^\circ}$$
$$E_{bn} = 10\underline{/-120^\circ} = 10\underline{/+240^\circ}$$
$$E_{cn} = 10\underline{/+120^\circ} = 10\underline{/-240^\circ} \text{ volts}$$

Where the line-to-neutral voltage magnitude is 10 volts and E_{an} is the reference phasor.

The phase sequence is called positive sequence or abc sequence when E_{an} leads E_{bn} by 120° and E_{bn} leads E_{cn} by 120°.

The phase sequence is called negative sequence or *acb* sequence when E_{an} leads E_{cn} by 120° and E_{cn} leads E_{bn} by 120°.

The voltages in the examble above are positive-sequence voltages, since E_{an} leads E_{bn} by 120°. The corresponding phasor diagram is shown in Figure.



Balanced Line-to-Line Voltages

The voltages E_{ab}, E_{bc}, and E_{ca} between phases are called *line-to-line voltages*. Writing a KVL equation for a closed path around buses a, b, and n:

$$E_{ab} = E_{an} - E_{bn}$$

For the line-to-neutral voltages, we have:

$$E_{ab} = 10\underline{/0^\circ} - 10\underline{/-120^\circ} = 10 - 10\left[\frac{-1 - j\sqrt{3}}{2}\right] = 10 \cdot \frac{3}{2} + j10 \cdot \frac{\sqrt{3}}{2}$$

$$E_{ab} = \sqrt{3}(10)\left(\frac{\sqrt{3} + j1}{2}\right) = \sqrt{3}\left(10\underline{/30^\circ}\right) \quad \textbf{volts}$$

Balanced Three Phase Systems

Similarly, the line-to-line voltages E_{bc} and E_{ca} are

$$\begin{aligned} E_{bc} &= E_{bn} - E_{cn} = 10\underline{/-120^\circ} - 10\underline{/+120^\circ} \\ &= \sqrt{3}\left(10\underline{/-90^\circ}\right) \qquad \textbf{volts} \\ E_{ca} &= E_{cn} - E_{an} = 10\underline{/+120^\circ} - 10\underline{/0^\circ} \\ &= \sqrt{3}\left(10\underline{/150^\circ}\right) \;\; \textbf{volts} \end{aligned}$$

These line-to-line voltages E_{ab}, E_{bc} and E_{ca} are also balanced, since they have equal magnitudes of $\sqrt{3}(10)$ volts and 120° displacement between any two phases.

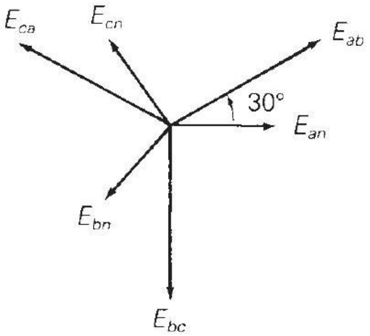
Balanced Three Phase Systems

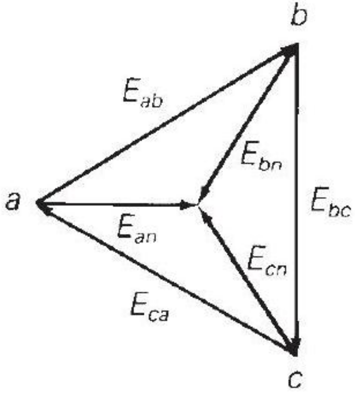
Comparison of these line-to-line voltages with the line-to-neutral voltages leads to the following conclusion:

In a balanced three-phase Y-connected system with positive-sequence sources, the line-to-line voltages are $\sqrt{3}$ times the line-to-neutral voltages and lead by 30°. That is,

$$
\begin{aligned}
E_{ab} &= \sqrt{3}\, E_{an} \underline{/+30^\circ} \\
E_{bc} &= \sqrt{3}\, E_{bn} \underline{/+30^\circ} \\
E_{ca} &= \sqrt{3}\, E_{cn} \underline{/+30^\circ}
\end{aligned}
$$

Balanced Three Phase Systems





Positive-sequence line-to-neutral and line-to-line voltages in a balanced threephase, Y-connected system.

The clockwise sequence of the vertices *abc* indicates positive-sequence voltages.

Balanced Three Phase Systems

Since the balanced line-to-line voltages form a closed triangle in Figure 2.12, their sum is zero.

In fact, the sum of line-to-line voltages $(E_{ab} + E_{bc} + E_{ca})$ is always zero, even if the system is unbalanced, since these voltages form a closed path around buses a, b, and c.

Also, in a balanced system, the sum of the line-to-neutral voltages $(E_{an} + E_{bn} + E_{cn})$ equals zero.

Balanced Line Currents

When the impedance between the source and load neutrals is neglected, buses n and N are at the same potential, $E_{nN} = 0$.

Accordingly, a separate KVL equation can be written for each phase, and the line currents can be written by inspection:

$$
\begin{aligned}
I_a &= E_{an}/Z_Y \\
I_b &= E_{bn}/Z_Y \\
I_c &= E_{cn}/Z_Y
\end{aligned}
$$

Balanced Three Phase Systems

For example, if each phase of the Y-connected load has an impedance $Z_Y = 2\angle 30^\circ \Omega$, then:

$$I_a = \frac{10\angle 0^\circ}{2\angle 30^\circ} = 5\angle -30^\circ \mathrm{A}$$

$$I_b = \frac{10\angle -120^\circ}{2\angle 30^\circ} = 5\angle -150^\circ \mathrm{A}$$

$$I_c = \frac{10\angle +120^\circ}{2\angle 30^\circ} = 5\angle 90^\circ \mathrm{A}$$

Balanced Three Phase Systems

The line currents are also balanced, since they have equal magnitudes of 5 A and 120° displacement between any two phases.

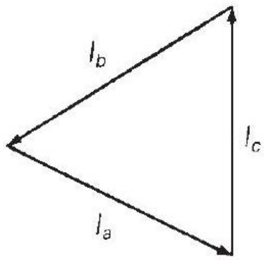
The neutral current I_n is determined by writing a KCL equation at bus N:

$$I_n = I_a + I_b + I_c$$

Using the line currents calculated above, give:

$$I_n = 5\underline{/-30^\circ} + 5\underline{/-150^\circ} + 5\underline{/90^\circ}$$
$$I_n = 5\left(\frac{\sqrt{3} - j1}{2}\right) + 5\left(\frac{-\sqrt{3} - j1}{2}\right) + j5 = 0$$

Balanced Three Phase Systems



Phasor diagram of line currents in a balanced three-phase system

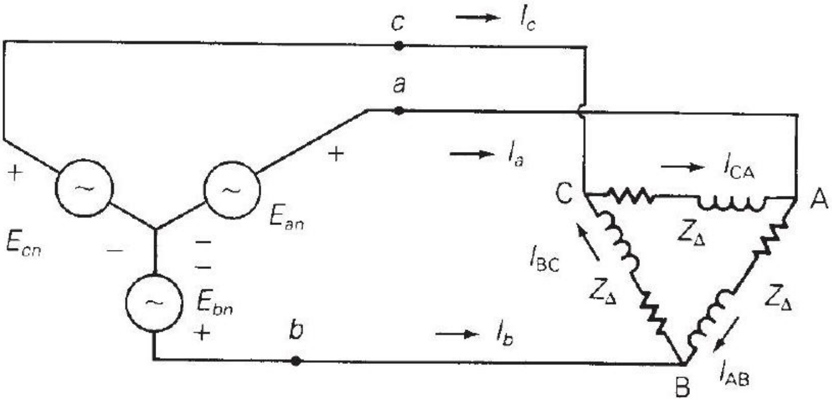
Balanced Three Phase Systems

In general, the sum of any balanced three-phase set of phasors is zero, since balanced phasors form a closed triangle.

Thus, although the impedance between neutrals n and N is assumed to be zero, the neutral current will be zero for any neutral impedance ranging from short circuit (0Ω) to open circuit ($\infty\Omega$), as long as the system is balanced.

If the system is not balanced-which could occur if the source voltages, load impedances, or line impedances were unbalanced - then the line currents will not be balanced, and a neutral current I_n may flow between buses n and N.

Balanced - Δ Loads



Circuit diagram of a Y-connected source feeding a balanced- Δ load

Balanced Three Phase Systems

Figure shows a three-phase Y-connected source feeding a balanced-Δ-connected load.

For a balanced- Δ connection, equal load impedances Z_Δ are connected in a triangle whose vertices form the buses, labeled A, B, and C.

Important! The Δ connection does not have a neutral bus.

Balanced

Since the line impedances are neglected in Figure, the source line-to-line voltages are equal to the load line-to-line voltages, and the Δ-load currents I_{AB}, I_{BC} **and** I_{CA} **are:**

$$I_{\mathrm{AB}} = E_{ab}/Z_{\Delta}$$

$$I_{\mathrm{BC}} = E_{bc}/Z_{\Delta}$$

$$I_{\mathrm{CA}} = E_{ca}/Z_{\Delta}$$

Balanced Three Phase Systems

For example, if the line-to-line voltages are given by:

$$E_{ab} = \sqrt{3}\,(10\underline{/30^\circ}) \qquad \textbf{volts}$$
$$E_{bc} = \sqrt{3}\,(10\underline{/-90^\circ}) \qquad \textbf{volts}$$
$$E_{ca} = \sqrt{3}\,(10\underline{/150^\circ}) \qquad \textbf{volts}$$

And if $Z_\Delta = 5\angle 30^\circ \Omega$**, then the** Δ**-load currents are:**

$$I_{\text{AB}} = \sqrt{3}\left(\frac{10\angle 30^\circ}{5\angle 30^\circ}\right) = 3.464\angle 0^\circ \text{A}$$
$$I_{\text{BC}} = \sqrt{3}\left(\frac{10\angle -90^\circ}{5\angle 30^\circ}\right) = 3.464\angle -120^\circ \text{A}$$
$$I_{\text{CA}} = \sqrt{3}\left(\frac{10\angle 150^\circ}{5\angle 30^\circ}\right) = 3.464\angle +120^\circ \text{A}$$

Balanced Three Phase Systems

Also, the line currents can be determined by writing a KCL equation at each bus of the Δ load, as follows:

$$I_a = I_{AB} - I_{CA} = 3.464\angle 0^\circ - 3.464\angle 120^\circ = \sqrt{3}\cdot 3.464\underline{/-30^\circ}$$
$$I_b = I_{BC} - I_{AB} = 3.464\angle -120^\circ - 3.464\angle 0^\circ = \sqrt{3}\cdot 3.464\underline{/-150^\circ}$$
$$I_c = I_{CA} - I_{BC} = 3.464\angle 120^\circ - 3.464\angle -120^\circ = \sqrt{3}\cdot 3.464\underline{/+90^\circ}$$

Balanced Three Phase Systems

Important Conclusions are:

Both the Δ-load currents given and the line currents are balanced.

Thus the sum of balanced Δ-load currents $(I_{\mathrm{AB}} + I_{\mathrm{BC}} + I_{\mathrm{CA}})$ equals zero.

The sum of line currents $(I_a + I_b + I_c)$ is always zero for a Δ-connected load, even if the system is unbalanced, since there is no neutral wire.

Balanced Three Phase Systems

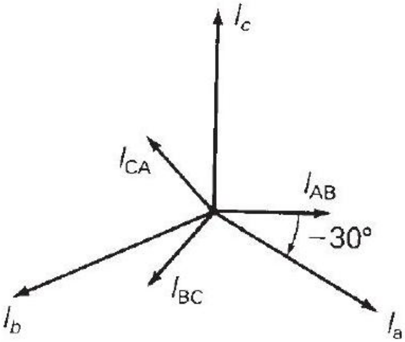
Comparison of these equations leads to the following conclusion:

For a balanced- Δ load supplied by a balanced positive-sequence source, the line currents into the load are $\sqrt{3}$ times the Δ-load currents and lag by 30°. That is:

$$I_a = \sqrt{3} I_{\mathrm{AB}} \angle -30^\circ$$
$$I_b = \sqrt{3} I_{\mathrm{BC}} \angle -30^\circ$$
$$I_c = \sqrt{3} I_{\mathrm{CA}} \angle -30^\circ$$

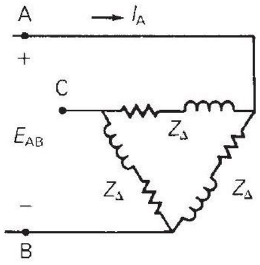
Balanced Three Phase Systems

This result is summarized in Figure above:

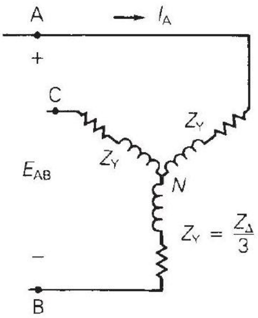


Phasor diagram of line currents and load currents for a balanced- Δ load.

$\Delta - Y$ Conversion For Balanced Loads



(a) Balanced-Δ load.



(b) Equivalent balanced-Y load.

Conversion.

Figure above shows the conversion of a balanced-Δ load to a balanced-Y load.

If balanced voltages are applied, then these loads will be equivalent as viewed from their terminal buses A, B, and C when the line currents into the Δ load are the same as the line currents into the Y load.

For the Δ load:

$$I_A = \sqrt{3} I_{AB} \angle{-30^\circ} = \frac{\sqrt{3} E_{AB} \angle{-30^\circ}}{Z_\Delta}$$

And for the Y load:

$$I_A = \frac{E_{AN}}{Z_Y} = \frac{E_{AB} \angle{-30^\circ}}{\sqrt{3} Z_Y}$$

Conversion.

Comparison of these equations indicates that I_A will be the same for both the Δ and Y loads when:

$$\frac{\sqrt{3}E_{AB}\angle{-30^\circ}}{Z_\Delta} = \frac{E_{AB}\angle{-30^\circ}}{\sqrt{3}Z_Y}$$

$$\frac{\sqrt{3}}{Z_\Delta} = \frac{1}{\sqrt{3}Z_Y}$$

$$Z_Y = \frac{Z_\Delta}{3}$$

Also, the other line currents I_B and I_C into the Y load will equal those into the Δ load when $Z_Y = Z_\Delta/3$, since these loads are balanced.

Conversion

Thus a balanced- Δ load can be converted to an equivalent balanced-Y load by dividing the Δ-load impedance by 3.

The angles of these Δ - and equivalent Y-load impedances are the same.

Similarly, a balanced-Y load can be converted to an equivalent balanced- Δ load using $Z_\Delta = 3 \cdot Z_Y$.

Equivalent Line-To-Neutral Diagrams

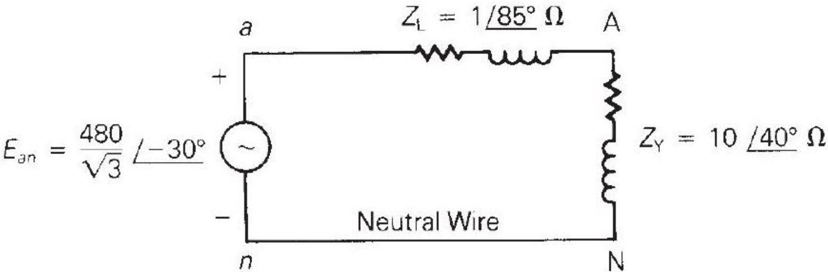
When working with balanced three-phase circuits, only one phase need be analyzed.

Δ loads can be converted to Y loads, and all source and load neutrals can be connected with a zero-ohm neutral wire without changing the solution.

Then one phase of the circuit can be solved. The voltages and currents in the other two phases are equal in magnitude to and $\pm 120^\circ$ out of phase with those of the solved phase.

Equivalent.

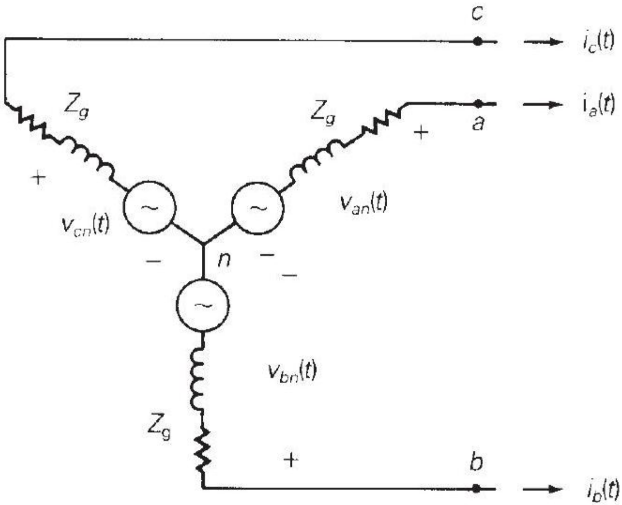
Figure above shows an equivalent line-to-neutral diagram for one phase of a particular 3φ circuit.



Equivalent line-to-neutral diagram for a circuit.

Obs.: When discussing three-phase systems in this text, voltages shall be rms line-to-line voltages unless otherwise indicated. This is standard industry practice.

Instantaneous Power: Balanced Three-Phase Generators



Instantaneous Power: Balanced Three-Phase Generators

Figure above shows a Y-connected generator represented by three voltage sources with their neutrals connected at bus n and by three identical generator impedances Z_g.

Assume that the generator is operating under balanced steady-state conditions with the instantaneous generator terminal voltage given by:

$$v_{an}(t) = \sqrt{2}\, \mathrm{V}_{\mathrm{LN}} \cos(\omega t + \partial) \quad \textbf{volts}$$

Instantaneous Power: Balanced Three-Phase Generators

And with the instantaneous current leaving the positive terminal of phase a given by:

$$i_a(t) = \sqrt{2}\mathrm{I_L}\cos(\omega t + \beta)\mathrm{A}$$

Where $\mathrm{V_{LN}}$ is the rms line-to-neutral voltage and $\mathrm{I_L}$ is the rms line current.

The instantaneous power $p_a(t)$ delivered by phase a of the generator is:

$$\begin{aligned} p_a(t) &= v_{an}(t)i_a(t) \\ &= 2\,\mathrm{V_{LN}I_L}\cos(\omega t + \delta)\cos(\omega t + \beta) \\ &= \mathrm{V_{LN}I_L}\cos(\delta - \beta) + \mathrm{V_{LN}I_L}\cos(2\omega t + \delta + \beta) \quad \mathrm{W} \end{aligned}$$

Instantaneous Three Phase Power

Assuming balanced operating conditions, the voltages and currents of phases b and c have the same magnitudes as those of phase a and are $+120^\circ$ out of phase with phase a.

Therefore the instantaneous power delivered by phase b is:

$$\begin{aligned} p_b(t) &= 2\,\mathrm{V_{LN}I_L}\cos\left(\omega t + \delta - 120^\circ\right)\cos\left(\omega t + \beta - 120^\circ\right) \\ &= \mathrm{V_{LN}I_L}\cos(\delta - \beta) + \mathrm{V_{LN}I_L}\cos\left(2\omega t + \delta + \beta - 240^\circ\right) \quad \mathrm{W} \end{aligned}$$

Instantaneous Three Phase Power

And by phase *c*:

$$\begin{aligned} p_c(t) &= 2\,\mathrm{V_{LN}I_L}\cos\left(\omega t + \delta + 120^\circ\right)\cos\left(\omega t + \beta + 120^\circ\right) \\ &= \mathrm{V_{LN}I_L}\cos(\delta - \beta) + \mathrm{V_{LN}I_L}\cos\left(2\omega t + \delta + \beta + 240^\circ\right) \quad \mathrm{W} \end{aligned}$$

The total instantaneous power $p_{3\phi}(t)$ delivered by the three-phase generator is the sum of the instantaneous powers delivered by each phase.

Instantaneous Three Phase Power

The total instantaneous power $p_{3\phi}(t)$ is:

$$\begin{aligned} p_{3\phi}(t) &= p_a(t) + p_b(t) + p_c(t) \\ &= 3\,\mathrm{V_{LN}I_L}\cos(\delta - \beta) + \mathrm{V_{LN}I_L}[\cos(2\omega t + \delta + \beta) \\ &\quad + \cos\left(2\omega t + \delta + \beta - 240^\circ\right) \\ &\quad + \cos\left(2\omega t + \delta + \beta + 240^\circ\right)]\ \mathrm{W} \end{aligned}$$

The three cosine terms within the brackets can be represented by a balanced set of three phasors.

Therefore, the sum of these three terms is zero for any value of δ, for any value of β, and for all values of t. Equation of $p_{3\phi}(t)$ then reduces to:

$$p_{3\phi}(t) = P_{3\phi} = 3\, V_{LN} I_L \cos(\delta - \beta) \quad W$$

Equation above can be written in terms of the line-to-line voltage V_{LL} instead of the line-to-neutral voltage V_{LN}.

Under balanced operating conditions:

$$V_{LN} = V_{LL}/\sqrt{3} \text{ and } P_{3\phi} = \sqrt{3}\, V_{LL} I_L \cos(\delta - \beta) \quad W$$

Inspection of above equation leads to the following conclusion:

The total instantaneous power delivered by a three-phase generator under balanced operating conditions is not a function of time, but a constant:

$$p_{3\phi}(t) = \mathrm{P}_{3\phi}$$

Complex Power: Balanced Three-Phase Generators

The phasor representations of the voltage and current presented before are:

$$V_{an} = \mathrm{V}_{\mathrm{LN}}\underline{/\delta} \quad \mathrm{V}$$
$$I_a = \mathrm{I}_{\mathrm{L}}\underline{/\beta} \quad \mathrm{A}$$

Where I_a leaves positive terminal "a "of the generator.

The complex power S_a delivered by phase a of the generator is:

$$\begin{aligned} S_a &= V_{an} I_a^* = \mathrm{V}_{\mathrm{LN}}\mathrm{I}_{\mathrm{L}}\underline{/(\delta-\beta)} \\ &= \mathrm{V}_{\mathrm{LN}}\mathrm{I}_{\mathrm{L}}\cos(\delta-\beta) + j\,\mathrm{V}_{\mathrm{LN}}\mathrm{I}_{\mathrm{L}}\sin(\delta-\beta) \end{aligned}$$

Complex Three Phase Power

Under balanced operating conditions, the complex powers delivered by phases b and c are identical to S_a, and the total complex power $S_{3\phi}$ delivered by the generator is:

$$
\begin{aligned}
S_{3\phi} &= S_a + S_b + S_c = 3S_a \\
&= 3\,\mathrm{V_{LN}I_L}\underline{/(\delta - \beta)} \\
&= 3\,\mathrm{V_{LN}I_L}\cos(\delta - \beta) + j3\,\mathrm{V_{LN}I_L}\sin(\delta - \beta)
\end{aligned}
$$

Complex Three Phase Power

In terms of the total real and reactive powers:

$$S_{3\phi} = \mathrm{P}_{3\phi} + j\mathrm{Q}_{3\phi}$$

Where

$$\begin{aligned} \mathrm{P}_{3\phi} &= \mathrm{Re}\left(S_{3\phi}\right) = 3\, \mathrm{V}_{\mathrm{LN}}\mathrm{I}_{\mathrm{L}} \cos(\delta - \beta) \\ &= \sqrt{3}\, \mathrm{V}_{\mathrm{LL}}\mathrm{I}_{\mathrm{L}} \cos(\delta - \beta) \quad \mathrm{W} \end{aligned}$$

And

$$\begin{aligned} \mathrm{Q}_{3\phi} &= \mathrm{Im}\left(S_{3\phi}\right) = 3\, \mathrm{V}_{\mathrm{LN}}\mathrm{I}_{\mathrm{L}} \sin(\delta - \beta) \\ &= \sqrt{3}\, \mathrm{V}_{\mathrm{LL}}\mathrm{I}_{\mathrm{L}} \sin(\delta - \beta) \quad \mathrm{var} \end{aligned}$$

Complex Three Phase Power

Also, the total apparent power is:

$$S_{3\phi} = \left|S_{3\phi}\right| = 3\,V_{LN}I_{L} = \sqrt{3}\,V_{LL}I_{L} \quad \text{VA}$$

Complex Power: Balanced-Y And Balanced- Δ Impedance Loads

Power Equations presented before are also valid for balanced-Y and Δ impedance loads.

For a balanced-Y load, the equations for line-to-neutral voltage across the phase a load impedance and the current entering the positive terminal of that load impedance are the same. And that is all.

Complex Three Phase Power

For a balanced- Δ load, the line-to-line voltage across the phase $a - b$ load impedance and the current into the positive terminal of that load impedance can be represented by:

$$V_{ab} = \mathrm{V_{LL}} \angle \delta \quad \mathrm{V}$$
$$I_{ab} = \mathrm{I_{\Delta}} \angle \beta \quad \mathrm{A}$$

Where $\mathrm{V_{LL}}$ is the rms line-to-line voltage and $\mathrm{I_{\Delta}}$ is the rms Δ-load current.

The complex power S_{ab} absorbed by the phase $a - b$ load impedance is then

$$S_{ab} = V_{ab} I_{ab}^{*} = \mathrm{V_{LL}} \mathrm{I_{\Delta}} / (\delta - \beta)$$

Complex Three Phase Power

The total complex power absorbed by the Δ load is:

$$
\begin{aligned}
S_{3\phi} &= S_{ab} + S_{bc} + S_{ca} = 3S_{ab} \\
&= 3\,\mathrm{V}_{\mathrm{LL}}\mathrm{I}_{\Delta}/(\delta - \beta) \\
&= 3\,\mathrm{V}_{\mathrm{LL}}\mathrm{I}_{\Delta}\cos(\delta - \beta) + j3\,\mathrm{V}_{\mathrm{LL}}\mathrm{I}_{\Delta}\sin(\delta - \beta)
\end{aligned}
$$

Complex Three Phase Power

Rewriting equations in terms of the total real and reactive power:

$$
\begin{aligned}
S_{3\phi} &= \mathrm{P}_{3\phi} + j\mathrm{Q}_{3\phi} \\
\mathrm{P}_{3\phi} &= \mathrm{Re}\left(S_{3\phi}\right) = 3\,\mathrm{V_{LL}I_{\Delta}}\cos(\delta-\beta) \\
&= \sqrt{3}\,\mathrm{V_{LL}I_{L}}\cos(\delta-\beta)\mathrm{W} \\
\mathrm{Q}_{3\phi} &= \mathrm{Im}\left(S_{3\phi}\right) = 3\,\mathrm{V_{LL}I_{\Delta}}\sin(\delta-\beta) \\
&= \sqrt{3}\,\mathrm{V_{LL}I_{L}}\sin(\delta-\beta) \qquad \textbf{var}
\end{aligned}
$$

Where the Δ-load current I_{Δ} is expressed in terms of the line current $\mathrm{I_L} = \sqrt{3}\mathrm{I}_{\Delta}$.

Complex Three Phase Power

Also, the total apparent power is:

$$S_{3\phi} = \left|S_{3\phi}\right| = 3\, V_{LL} I_{\Delta} = \sqrt{3}\, V_{LL} I_{L} VA$$

These Equations developed for the balanced- Δ load are identical to equations developed for balanced-Y loads.