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Oligopolistic pricing with heterogeneous consumer search

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Abstract

A finite number of identical stores sell a homogeneous good to consumers with heterogeneous search costs who search sequentially with perfect recall and without replacement. Consumers' search rules are optimal with respect to the stores' pricing strategies, and consumers must visit a store to obtain information about its actual price. A symmetric Nash equilibrium (SNE) always exists. The model exhibits a variety of equilibrium behaviors between monopoly pricing and marginal-cost pricing. The competitiveness of a market depends crucially on the shape of the search cost distribution, moreso than on the number of competing firms.

Keywords: Pricing; Search; Oligopoly; Heterogeneity

JEL classification: D43; D83; C72

1. Introduction

Three decades ago Stigler (1961) launched a research program to investigate the effects of costly information acquisition by consumers, arguing that price dispersion would arise. Rothschild (1973) criticized Stigler's specific model because it did not take account of the firms' side of the market. He implored economists to develop equilibrium models in which

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both consumers and firms use all their information and act optimally; doing so with Stigler's model results in a single-price equilibrium. Diamond (1971) demonstrated that if all consumers have strictly positive search costs, then the unique Nash equilibrium for identical stores selling a homogeneous good is the monopoly price. The literature then focused on developing models that exhibit equilibrium price dispersion (e.g. Braverman, 1980; Salop and Stiglitz, 1977; Reinganum, 1979; and Wilde and Schwartz, 1979). The present paper provides two main insights: (i) the Diamond result disappears when some consumers love to shop, and (ii) the competitiveness of a market depends crucially on the shape of the search cost distribution, moreso than on the number of competing firms.

Despite numerous past studies, our understanding of the nature of oligopolistic pricing in the face of costly information acquisition has been inadequate. What conditions are necessary and sufficient for monopoly pricing, competitive pricing, single-price equilibria, or dispersed-price equilibria? How does the distribution of search costs in the consumer population affect the outcomes? How do alternative consumer search protocols and information structures affect the equilibria? What is the effect of the number of stores? While we have insights from specific models, the diverse features of those models make it difficult to draw any general conclusions.

Table 1 displays the key features of a sample of models, selected to represent the spectrum of features and results. There is insufficient space in this paper to discuss each of these model features, so we focus on the two most crucial features. A comparison of the different model results will be presented in Section 7.

The first feature concerns the information available to consumers. Models in the literature differ sharply in what they assume consumers know before search. An exogenous portion of the consumers may be fully informed while the remaining consumers are completely uninformed (Varian, 1980) or parially informed (Wilde and Schwartz, 1979). However, the predominant distinction in Table 1 is between the so-called Stackelberg ('Stkb' in Table 1) paradigm and the Nash paradigm. Under the 'Stackelberg paradigm', consumers know the 'market distribution' of actual prices being charged but do not know which store is charging which price. For example, if there are N stores whose symmetric mixed-strategy is to draw a random price from a probability distribution $F(p)$, these N independent random draws give N actual prices, and the frequency distribution of these actual prices, say $M(p)$, is called the *market distribution*. Note that given a finite number of stores, $M(p)$ can be quite different from $F(p)$. Consumers are assumed to know $M(p)$ and to choose optimal search rules with respect to $M(p)$. Thus, the Stackelberg paradigm gives consumers a significant amount of information before searching, which provides a greater incentive for stores to

Table 1

Paper	Cons. info.	Search protocol	Search costs	Cons. demand	No. of stores	Prod. costs
Salop–Stiglitz '77	Stkb	AoN	TT	UC	N*	U
Braverman '80	Stkb	AoN	Al+	D	N*	U
Rob '85	Stkb	SwR	Al	UI	∞	K
Stiglitz '87, Appendix A	Stkb	Seq	Al	D	N	K
Stiglitz '87, Appendix B	Stkb	SwR	Al	D	N	K
Burdett–Judd '83	Nash	Par	Hc	UC	∞	K
Axell '77	Nash	SwR	Al	D	∞	Co
Stahl '89a	Nash	Seq	TT	D	N	K
Stahl '95	Nash	Seq	Al+	D	N	K

Key to consumer information:
Stkb – Stackelberg paradigm
Nash – Nash paradigm
Key to search protocol:
AoN – All or nothing
Seq – Sequential (without replacement)
SwR – Sequential with replacement
Par – Parallel search
Key to search costs:
Al – Atomless distribution over consumers
Al+ – Atomless distribution except possibly for an atom of shoppers
Hc – Homogeneous $c > 0$
TT – Two types with costs c_1 and c_2
Key to consumer demand:
UC – Unit demand up to a choke price
UI – Unit demand even at infinite prices
D – regular downward-sloping demand
Key to number of stores:
N* – long-run free-entry zero-profit case only
Key to production costs:
U – identical U-shaped average costs
K – identical constant marginal cost
Co – Convex total cost function

set lower prices. *In contrast*, under the 'Nash paradigm', the consumer's search rule is optimal with respect to the stores' NE pricing strategies, $F(p)$, and consumers have no information (before search) about the market distribution $M(p)$.

The character of equilibrium pricing depends crucially on the choice of paradigm: Nash or Stackelberg. In the language of game theory, the issue is how to specify the 'information set' at the consumer's decision node. If the purpose of our model is to study the issues of information acquisition by consumers, then the natural answer is that, prior to searching, consumers should have no information regarding actual prices. Any alternative informational assumptions should be modeled either as the product of consumer search or as strategic actions taken by the stores (e.g. Butters, 1977) or other agents in the model. In other words, the Nash paradigm is the preferred modeling choice. Moreover, the Nash paradigm avoids a host of technical problems that haunt the Stackelberg paradigm, e.g. troublesome integer conditions (typically overlooked) in constructing dispersed equilibria, non-stationary optimal search rules when search is without replacement, and non-existence of any equilibrium in many cases.[1] Diamond's (1971) and Axell's (1977) dynamic models with lagged consumer learning provide a further defense of the Nash paradigm.

The second most important feature of models concerns the search costs of consumers. We would like to know how the equilibria are affected by the distribution of search costs in the consumer population. An important issue is the influence of 'shoppers' (herein defined as consumers with non-positive search costs). In our casual experience, there are many consumers who seem to enjoy shopping and indeed search all the stores in their market area. Satisfaction from shopping itself is equivalent to lowering the search cost—perhaps below zero for some consumers. Since consumers with negative search costs would always obtain full information from all stores before purchasing, the simplest way to capture this phenomenon would be to permit an atom of fully informed shoppers. Another interpretation of an atom of shoppers is as a coalition of consumers who share their search information. Indeed, we maintain that this is a better way to model non-search information transmission than using the Stackelberg paradigm. Therefore, we believe it is important to have a model that can accommodate an atom of shoppers (with strictly positive mass less than one). Unfortunately, an atom of shoppers creates existence problems for all the models in Table 1 except the last two.

In response to these observations, this paper studies a game-theoretic model of oligopolistic pricing with heterogeneous consumer search costs under the Nash paradigm. There is a finite number of identical stores selling a homogeneous good with constant marginal costs. Consumers search sequentially without replacement and with perfect recall. The consumers are identical except for search costs, so the consumer environment is specified by a representative demand function and a distribution of search costs (including possibly an atom of shoppers). This framework is presented more fully in Section 2, and several necessary properties of symmetric Nash equilibria (SNE) are derived.

[1] A major contributing factor to non-existence is the inability of stores to effect mixed-strategies vis-à-vis uninformed consumers, since the Stackelberg paradigm always gives consumers a discrete-point distribution of actual prices. Rob (1985) avoids this discreteness by assuming a continuum of stores each independently choosing a single price and, by the law of large numbers, collectively generating a smooth market distribution.

We prove in Section 3 that there always exists an SNE. In Section 4 we derive results for the case when there is no atom of shoppers. We identify when there is a continuum of pure-strategy SNE which includes the monopoly price, and when the monopoly price is the unique SNE. We also identify a class of search cost distributions for which all the SNE are pure-strategy SNE.

When there is an atom of shoppers (with mass less than one), an SNE distribution of prices must be atomless. These SNE are further characterized in Section 5, revealing the effect of shoppers on the SNE. Asymptotic properties (as the number of stores increases) are developed in Section 6. The results are compared with those of Table 1, and our cumulative understanding is discussed in Section 7. All proofs are gathered in Appendix B.

2. Formal description of the model

N stores sell a homogeneous good and have identical linear homogeneous production technologies. Quoted prices, p, are net of the common marginal cost. We will be analyzing SNE exclusively, and $F(p)$ will denote an SNE distribution of prices.

Consumers must visit a store to learn about that store's actual price. They search with perfect recall and without replacement. Each visit (beyond the first) incurs a cost c. Consumers are identical except for search costs, which are distributed in the population according to a cumulative distribution function $\mu(c)$. We know that if search costs are bounded away from zero, then the unique SNE is the monopoly price. Also, if all consumers have zero search costs (and are fully informed), then the unique SNE is the Bertrand price (0). To focus exclusively on other cases, we assume that the minimum search cost is 0, and that the maximum search cost, $\bar{c}$, is strictly positive. We further assume that μ is absolutely continuous on $(0, \bar{c})$. The case of no atom of shoppers will be analyzed in Section 4, while the case of a positive atom will be analyzed in Section 5.

Consumers have a demand function $D(p)$ for the good. We assume that $D'(p) \leq 0$, $D(p) \geq 0$, $0 < D(0) < \infty$, and that $\int_0^\infty D(p)\mathrm{d}p < \infty$.[2] $R(p) \equiv pD(p)$ is the *revenue function*. We assume that $R(p)$ is continuous, has a unique maximum at $\hat{p} < \infty$, and is strictly increasing and continuously differentiable for all $p < \hat{p}$. Interpreting $R(p)$ as revenue per consumer, the model is invariant to the number of consumers.

[2] The requirement that $D(0) < \infty$ is reasonable since prices are net of marginal cost. The assumption of bounded consumer surplus is weaker than assuming there exists a positive price p' such that $D(p) = 0$ for all $p \geq p'$.

2.1. Optimal consumer search rule

Using consumer surplus as the measure of consumer benefits, when the lowest observed price is r, the expected gain from an additional search from the distribution $F(p)$ is

$$H(r,F) \equiv \int_0^r \int_x^r D(p)\mathrm{d}p\mathrm{d}F(x) = \int_0^r D(x)F(x)\mathrm{d}x \ , \tag{2.1}$$

where the second equality follows from integration by parts. Then, the *consumer's reservation price*, $r(c,F)$, is defined implicitly by

$$H[r(c,F),F] = c \ , \tag{2.2}$$

or, if Eq. (2.2) has no root, then we define $r(c,F) = \infty$. For example, if $D(x) = 1$ and F is concentrated at p^*, then $r(c,F) = p^* + c$.

Assuming perfect and costless recall (see Kohn and Shavell, 1974), it is optimal to (i) search if the lowest observed price exceeds $r(c,F)$; (ii) stop and purchase $D(p)$ if the lowest observed price is less than $r(c,F)$; and (iii) search or stop if the lowest observed price equals $r(c,F)$.

The results are invariant to how the indifferent case (iii) is resolved for $c > 0$ types, and for the 'shoppers' ($c = 0$ types) when $\mu(0) = 0$. However, when there is an atom of shoppers [i.e. $\mu(0) > 0$], the specification of case (iii) is substantive. If the shoppers were to stop at the first price at or below $r(c,F)$, then the monopoly price would always be a pure-strategy SNE; but if shoppers were to visit at least two stores before purchasing, then there would be no pure-strategy SNE.

The purpose of allowing an atom of shoppers is to capture in a simple manner the notion that some consumers may enjoy shopping, in which case they would visit all the stores. Therefore, we *assume* that all shoppers visit all stores before making any purchases.[3] Given this assumption, it is expositionally convenient to specify that all types continue to search when the lowest observed price equals $r(c,F)$.

Thus, the optimal rule for all consumer types is: STOP, if either the lowest observed price is *less than* $r(c,F)$, or all N stores have been visited; SEARCH, otherwise.

2.2. Expected profits and SNE

Let $\Psi(p,F)$ denote the expected share of all consumers purchasing from a store charging price p. An expression for $\Psi(p,F)$ is derived in Appendix A.

[3] We could specify that shoppers visit at least two (or three ...) stores before purchasing; the resulting SNE would be the same for all such specifications (as in Burdett and Judd, 1983, p. 966).

The *expected profits* of any store charging price p when the SNE distribution is F are

$$E\pi(p,F) \equiv R(p)\Psi(p,F) . \tag{2.3}$$

Definition 2.1. $F(p)$ is an SNE if $E\pi(p,F) \leq \max_p pE\pi(p,F)$ for all p with equality for F-a.e. p.[4]

2.3. Necessary properties of SNE

Let 'supp$F$' denote the support of F, let $\underline{p}$ denote the minimum price in supp$F$, and let $\bar{p}$ denote the maximum price in supp$F$.

Lemma 2.1. If F is an SNE, then "(a)" F is atomless for all $p > \underline{p}$, and "(b)" furthermore, when $\mu(0) > 0$, F is atomless at $\underline{p}$ as well.

This is a common result for search models. Intuitively, when there is a positive mass of consumers searching for prices lower than a price at which there is an atom, say $\tilde{p}$, then stores can gain a discrete increase in profits by undercutting $\tilde{p}$ slightly.

An immediate consequence of Lemma 2.1 is

Proposition 2.1. If $\mu(0) > 0$, then there are no pure-strategy SNE.

Using Lemma 2.1, it is shown in Appendix B that, given an SNE F, the share of all consumers purchasing from a store charging price p, $\Psi(p,F)$, and expected profits, $E\pi(p,F)$, are continuous in p.[5]

The next result puts bounds on the range of possible SNE prices.

Lemma 2.2. If F is an SNE, then $\bar{p} \leq \min\{\hat{p}, r(\bar{c}, F)\}$, and $\underline{p} > 0$.

Recalling that $r(\bar{c}, F)$ is the reservation price for the consumer with the highest search cost, stores never price above the highest reservation price of consumers nor the monopoly price, and store prices are bounded above the Bertrand price (0). The latter result implies that SNE expected profits are strictly positive.

[4] 'F-a.e.' stands for 'F almost everywhere' and means that the set of p for which the equality does not hold has F-probability zero.

[5] By virtue of the continuity of $E\pi(\cdot, F)$, we can drop the 'F-a.e.' language in the description of an SNE F: that is, if F is an SNE distribution, then $E\pi(p, F) \leq E\pi(\bar{p}, F)$ with equality for all $p \in \text{supp}F$.

3. Existence of SNE

It is convenient first to derive a characterization of the upper bound of the support, $\bar{p}$, for mixed-strategy SNE. These are necessary conditions that stem from the requirement that $E\pi(\bar{p}, F) \geq E\pi(p, F)$. Let $\bar{\mu} \equiv \mu[H(\bar{p}, F)]$.

Proposition 3.1. If F is an SNE, then "(a)" $\mu[H(p, F)] \geq 1 - (1 - \bar{\mu})R(\bar{p})/R(p)$ *for all* $p > \bar{p}$; *"(b)"*

$$\frac{\partial \mu[H(\bar{p}, F)]/\partial c^{+}}{1 - \bar{\mu}} \geq \frac{\bar{p}R'(\bar{p})}{R^{2}(\bar{p})};$$

and "(c)" when $\underline{p} < \bar{p}$,

$$\frac{\bar{p}R'(\bar{p})}{R^{2}(\bar{p})} = \frac{\partial \mu[H(\bar{p}, F)]/\partial c^{-}}{1 - \bar{\mu}}.[6]$$

There are two interesting byproducts of Proposition 3.1. First, part (a) puts a lower bound on the search cost distribution for $c \geq H(\bar{p}, F)$. Thus, if F is an SNE for a given μ, then F is also an SNE for any $\hat{\mu}$ which is identical with μ for all $c < H(\bar{p}, F)$ and satisfies (a) for $c \geq H(\bar{p}, F)$. For example, all the mass above $H(\bar{p}, F)$, namely $1 - \bar{\mu}$, could be concentrated at $c = H(\bar{p}, F)$. On the other hand, a mixed-strategy SNE with $\bar{p} < \hat{p}$ can be upset by violating (a); e.g. by putting a slight downward kink in μ at $c = H(\bar{p}, F)$.[7]

Secondly, from the proof (in Appendix B), we can infer the following restriction on the class of possible SNE price distributions. This restriction will be important in Section 6.

Corollary 3.1. If $F(p)$ *is an SNE, then as* $p \to \bar{p}$ *from below, the limit of* $[1 - F(p)]^{N-2}F'(p)$ *is zero.*

One implication of this corollary is that if $N = 2$, then $F'(\bar{p}) = 0$, i.e. for a duopoly, an SNE price distribution must have a vanishing density as prices approach the upper bound $\bar{p}$. An example of a distribution that cannot be an SNE for any finite N and any consumer environment is $F(p) = 1 - [1 - \ln(2 - p)]^{-1}$ for $p \in [1, 2]$.

When there are no shoppers [i.e. $\mu(0) = 0$], the payoff function $E\pi(p, F)$

[6] For any function $y(x)$, $\partial y/\partial x^{+}$ denotes the right-hand derivative, and $\partial y/\partial x^{-}$ denotes the left-hand derivative.

[7] Intuitively, prices higher than $\bar{p}$ must be unprofitable, which depends on the marginal customers that would be lost to further search. A downward kink in μ would cause a discrete reduction in the losses without affecting the marginal gains, so prices higher than $\bar{p}$ would be profitable.

is continuous for all (p, F), and thus an SNE exists (Glicksberg, 1952). It is also easy to verify that the monopoly price $\hat{p}$ is always an SNE for this case (see Corollary 4.1 below). When there are shoppers [i.e. $\mu(0) > 0$], the payoff function is continuous *except* at $p = \underline{p}$ when $F(\underline{p}) > 0$. Therefore, we cannot directly apply Glicksberg's theorem. Moreover, there is no pure-strategy SNE. A new existence proof is constructed in Appendix B for this important case.

Theorem 3.1. Given our assumptions on the optimal search rule of shoppers, $\mu(c)$, $D(p)$ and $R(p)$, there exists an SNE for all N.

4. Characterization of pure-strategy SNE for atomless μ

By virtue of Proposition 2.1, pure-strategy SNE exist only if there is no atom of shoppers. We henceforth assume in this section that $\mu(0) = 0$. Also, note that if $\tilde{p}$ is a pure-strategy SNE, then, from Eq. (2.1): $H(p, F) = \int_{\tilde{p}}^{p} D(x)\mathrm{d}x$ for $p \geq \tilde{p}$.

Proposition 4.1. Given $\mu(0) = 0$, a necessary and sufficient condition for $\tilde{p}$ to be an SNE is that

$$\mu\left[\int_{\tilde{p}}^{p} D(x)\mathrm{d}x\right] > 1 - R(\tilde{p})/R(p)$$

for all $p > \tilde{p}$.

Note that both sides of this inequality begin at zero for $p = \tilde{p}$ and are increasing in p (at least up to $\hat{p}$). Hence, the right-hand side puts a lower bound on the cumulative search cost distribution. Moreover, if $\tilde{p} = \hat{p}$, then the right-hand side is negative for all $p > \hat{p}$, so the inequality is satisfied. Hence,

Corollary 4.1. Given $\mu(0) = 0$, the monopoly price $\hat{p}$ is an SNE.

Note that this Diamond monopoly-pricing SNE may be one of a continuum of SNE—an issue that will be explored below. When there is a continuum of SNE, we may assume that a convention arises, serving as a self-fulfilling common belief about which pure-strategy SNE will be played.

While Proposition 4.1 provides a means of computationally verifying a pure-strategy SNE, it is nevertheless useful to have some further characterizations of the set of pure-strategy SNE for interesting classes of search cost distributions and revenue functions. To this end, we introduce the following definitions.

Revenue Condition (RC). $pR'(p)/R^2(p)$ is strictly decreasing for all $p < \hat{p}$.

Non-Decreasing Hazard Rate (NDHR). $\mu'(c)/[1-\mu(c)]$ is non-decreasing.

RC is satisfied for all concave (and linear) demand functions, as well as many convex demand functions. For example, consider the family of demand functions $D(p) = (1-p)^{\beta}$, with $\beta > 0$; they satisfy RC and all our assumptions on $D(\cdot)$ and $R(\cdot)$, and the demand functions are convex for all $\beta > 1$.[8]

Define

$$p^* \equiv \min\{p \in [0, \hat{p}] \,|\, pR'(p)/R^2(p) \leq \mu'(0)\} . \tag{4.1}$$

Given $R'(\hat{p}) = 0$, and noting that $\lim_{p\to 0} pR'(p)/R^2(p) = +\infty$, it follows that p^* is uniquely defined. Furthermore, $0 < p^* < \hat{p}$, whenever $\infty > \mu'(0) > 0$.[9]

Proposition 4.2. "(a)" If $\tilde{p}$ is a pure-strategy SNE, then $\tilde{p} \geq p^*$. "(b)" If RC and NDHR hold and $\mu(0) = 0$, then every $\tilde{p} \in [p^*, \hat{p}]$ is a pure-strategy SNE.

Corollary 4.2. As $\mu'(0) \to 0$, $p^* \to \hat{p}$, so the set of pure-strategy SNE shrinks to the monopoly price, and the monopoly price is the unique pure-strategy SNE when $\mu'(0) = 0$.

We see that not only is the monopoly price always an SNE, but whenever there is a positive density of shoppers and RC and NDHR hold, there is a multiplicity of pure-strategy SNE.[10] Moreover, the Diamond monopoly-pricing result arises as a limiting case as the density of shoppers vanishes.[11]

[8] I am indebted to Raymond Deneckere for bringing this family of demand functions to my attention.

[9] To accommodate unit demand up to some price $\hat{p}$, we define $p^* = \hat{p}$ for all $\mu'(0) \leq 1/\hat{p}$.

[10] Observing that the results are *independent of the number of stores N*, it is apparent that Propositions 4.1 and 4.2 are similar to those of Rob (1985, lemma 3 and theorem 1), even though Rob has a continuum of stores, and consumers purchase only one unit at any price. However, Rob rules out all the pure-strategy SNE except the lowest, p*, by virtue of the Stackelberg paradigm.

[11] The case of an infinite density of shoppers merits comment. Eq. (4.1) implies that $p^* = 0$ whenever $\mu'(0) = \infty$. But then the hazard rate cannot be non-decreasing, so Proposition 4.2(b) does not apply. In other words, an infinite density of shoppers does *not* imply that all $\tilde{p} \in [0, \hat{p}]$ are pure-strategy SNE. Indeed, by Lemma 2.2, $\underline{p} > 0$ always. On the other hand, it is possible to have a sequence of search cost distributions such that Proposition 4.2(b) holds, and $p^* \to 0$. For example, let $\mu(c) = 1 - \exp(-nc)$, and let $n \to \infty$, so $p^*(n) \to 0$. Note, however, that this sequence of search cost distributions converges weakly (see Billingsley, 1979) to the degenerate distribution with unit mass at zero, which we have excluded from our study by assuming $\bar{c} > 0$.

The continuum of pure-strategy SNE identified by Proposition 4.2 used both the RC and NDHR assumptions. Since the latter is the less pleasing restriction, it is interesting to see if a continuum of pure-strategy SNE exists under our assumptions for more general search cost distributions.

Proposition 4.3. Given RC and $\mu'(0) > 0 = \mu(0)$, there exists a $p' \in (0, \hat{p})$ such that every $p \in [p', \hat{p}]$ is a pure-strategy SNE.[12]

Having characterized pure-strategy SNE for the case of atomless search cost distributions, the next two propositions address whether we have the total picture or if there are mixed-strategy SNE as well.

Proposition 4.4. "(a)" If $\mu'(0) = 0 = \mu(0)$, then the monopoly price $\hat{p}$ is the unique SNE. "(b)" If $\mu(0) = 0$ and F is an SNE, then $F(\underline{p}) > 0$.

While we have already seen that when there is a zero density of shoppers the monopoly price is the unique pure-strategy SNE (see Corollary 4.1), we now see via Proposition 4.4(a) that there are no mixed-strategy SNE in this case. Intuitively, when there is a zero density of consumers with zero search costs, there would be a first-order gain in revenue per sale with no accompanying loss of sales volume from shading prices up from the lower bound of a mixed-strategy price distribution—making such a mixed-strategy SNE impossible.

Proposition 4.4(b) is the converse of Lemma 2.1(b), which stated that $F(\underline{p}) = 0$ whenever $\mu(0) > 0$. A consequence of this dichotomy is that if a sequence of NE converges as $\mu(0)$ approaches zero from above, it does not converge pointwise to an SNE of the shopperless model. At best the convergence would be weak.

Proposition 4.5. Given RC, NDHR, and $\mu(0) = 0$, the only SNE are those pure-strategy SNE of Proposition 4.2 with $\tilde{p} \in [p^*, \hat{p}]$.

In other words, when the search cost distribution has a non-decreasing hazard rate and no atoms, there are no mixed-strategy SNE, and every $\tilde{p} \in [p^*, \hat{p}]$ is a pure-strategy SNE independent of N. To understand this result, first note that raising (lowering) prices induces two effects: (i) a loss (gain) in sales volume, and (ii) a gain (loss) in revenue per sale. These two effects must be balanced at every price in the support of a mixed-strategy SNE. Second observe that the NDHR condition implies that the sales

[12] In contrast, Braverman (1980) shows that under the Stackelberg paradigm and the AoN search protocol, these search cost conditions are incompatible with the existence of single-price equilibria.

volume effect is increasing in price, while RC implies that the revenue effect is decreasing in price. If the two effects are balanced at high prices, then at lower prices the revenue gain outweighs the volume loss, so higher prices are more profitable; hence, a mixed-strategy SNE is impossible. From the formal proof, these two conditions are more than sufficient for this result.

5. Further characterization of SNE

A distinguishing characteristic of a pure-strategy SNE is that consumers do not search; they simply purchase from the first store they visit. For other SNE (when they exist), the next result is that there always is a positive mass of consumers who search several stores with positive probability (and thereby become partially informed), and a positive mass of consumers who do not search [namely, $1-\bar{\mu}$, where $\bar{\mu} \equiv \mu[H(\bar{p}, F)]$, i.e. those with search costs too large to justify searching].

Proposition 5.1. If F is an SNE and $\underline{p} < \bar{p}$, *then* $\bar{\mu} \in (0, 1)$.

We saw in Section 4 that the Diamond monopoly-pricing result is a limiting case when there is no atom and no positive density of shoppers. We will now show that the Bertrand result is also a limiting case. Let δ_0 denote the probability measure with unit mass at zero, and let $\{\mu_n\}$ be a sequence of search cost distributions. We also expand our notation and let $F(p; \mu_n)$ denote an SNE when the search cost distribution is μ_n.

Proposition 5.2. If $\mu_n \to \delta_0$ *pointwise, then* $F(p; \mu_n)$ *converges weakly to* δ_0.

Weak convergence of μ_n is not sufficient as the following example shows. Suppose $\mu_n(0) = 0 = \mu_n'(0)$, while μ_n converges weakly to δ_0. By Proposition 4.4(a), for all n the monopoly price is the unique SNE, so clearly $F(p; \mu_n)$ does not converge in any sense to the Bertrand outcome.

On the other hand, pointwise convergence of μ_n is not necessary for the limit to be the Bertrand outcome. Indeed, a stronger form of convergence of the SNE can be obtained, namely convergence of the support to zero.

Proposition 5.3. If "(a)" $\bar{c}_n \to 0$, *or if "(b)" the hazard rate of* μ_n *is non-decreasing,* $\mu_n(0) > 0$ *and* $\mu_n'(0) \to \infty$, *then* $\bar{p}_n \to 0$.

An example for part (b) is $\mu_n(c) = 1 - (1-\lambda)\exp(-nc)$, for $\lambda \in (0, 1)$. This example demonstrates the dramatic influence of an arbitrarily small atom of shoppers, for when $\lambda = 0$ [and hence $\mu(0) = 0$], the monopoly price is always a SNE.

We turn now to further analysis of the effect of shoppers on the SNE. As we saw in Sections 3 and 4, atomless search cost distributions have a continuum of SNE, whereas an atom of shoppers will likely have a unique SNE. Thus, there seems to be a significant qualitative difference between having no atom of shoppers and having an arbitrarily small atom of shoppers. To investigate this phenomenon further, we consider a fixed atomless distribution of search costs μ, perturb this search cost distribution by adding an atom of shoppers, compute the SNE for this perturbed game, and take the limit of these (typically unique) SNE as the atom of shoppers vanishes.

Proposition 5.4. Assume $\mu(0)=0$. Let $\mu_n(c) \equiv (1/n) + [(n-1)/n]\mu(c)$, and let $F^*(p)$ be a limit of the sequence of SNE $F(p;\mu_n)$. "(a)" $F^*(p)$ is an SNE of the unperturbed game. "(b)" Moreover, if RC and NDHR hold, then $F^*(p)$ has unit mass at p^*, Eq. (4.1).

In other words, given RC, NDHR, and an atomless search cost distribution, the limit of the SNE of the game perturbed with an atom of shoppers is the lowest pure-strategy p^* of the unperturbed game (with no atom of shoppers). We, henceforth, call p^* the 'shopper-robust' SNE for the atomless μ. From Eq. (4.1), we can see that this shopper-robust price is closer to the monopoly price as the density at zero, $\mu'(0)$, becomes smaller (since there are not enough low search cost consumers to make price cutting worthwhile), and is closer to marginal cost (0) as $\mu'(0)$ becomes larger (since then unilateral price-cutting yields larger gains in market share), but it is independent of the number of stores N.

6. Asymptotic behavior (as $N \to \infty$)

We are interested in the behavior of equilibrium pricing as $N \to \infty$, for several reasons. First, the literature has given considerable attention to the limiting case as a representation of 'perfect competition'. Second, whether pricing converges to a Walrasian equilibrium as N becomes large has potential policy implications. Third, the sensitivity of the equilibrium pricing to the consumer search cost distribution is most clear for the limiting case.

Note that an alternative interpretation of an equilibrium with N stores and positive profits is like a 'long-run' equilibrium in which the fixed costs are exactly equal to those positive (variable) profits. If we had a unique equilibrium that was decreasing in N, then the effects of increasing N would be isomorphic to the long-run effects of decreasing the fixed costs. However, we hasten to caution the reader that our model does not generally have a unique equilibrium, so the dynamics of the entry game could be extremely complex (as in any repeated game).

Consider first the case when there is no atom of shoppers. The characterization results of Section 4 did not depend on N. Hence, the monopoly price $\hat{p}$ is always an SNE (Corollary 4.1), and if RC holds, there is a $p' < \hat{p}$ such that every $p \in [p', \hat{p}]$ is a pure-strategy SNE (Proposition 4.3). It is shown in Stahl (1989b) that each of these pure-strategy SNE are also SNE of the 'limit model' with $N = \infty$. In other words, with no atom of shoppers, there is no tendancy for $F(p; N)$ towards marginal cost pricing as $N \to \infty$. However, unlike the finite model, the limit model can have a mixed-strategy SNE price distribution with atoms above $\underline{p}$. In other words, the limit model admits behavior that is qualitatively different from the finite model.

In the presence of an atom of shoppers, there does not exist an SNE of the limit model.[13] On the other hand, by Theorem 3.1, the finite model has an SNE for all N. Thus, the model is discontinuous at $N = \infty$. Nonetheless, it is interesting to inquire about the limit of $F(p; N)$ as $N \to \infty$, because it reveals the behavior of the SNE for large N.

Proposition 6.1. Given RC, and a weakly convergent sequence of strictly mixed-strategy SNE $F(p; N)$ as $N \to \infty$, the limit $F^*(p)$ has unit mass at p'' defined by $p''R'(p'')/R^2(p'') = \mu'(0)/[1 - \mu(0)]$.

The premise of Propostion 6.1 entails that there exists a strictly mixed-strategy SNE for all N, a condition that we have established only when there is an atom of shoppers. With no atom of shoppers, Proposition 6.1 may be vacuous, leaving us with the pure-strategy SNE of Proposition 4.3. In other words, Proposition 6.1 asserts that given RC, as the number of stores increases without bound, mixed-strategy SNE either cease to exist or converge weakly to a pure-strategy at p''.

Note that if $\mu'(0) = 0$, then $p'' = \hat{p}$ (the monopoly price). Hence, even though an atom of shoppers rules out pure-strategy SNE for finite N, Proposition 6.1 establishes that when there is a zero density of shoppers, the mixed-strategy SNE converge to monopoly pricing as the number of stores increases without bound. This is a generalization of the anti-competitive effect of entry found in Stahl (1989a) where there was an atom of shoppers and $\mu'(c) = 0$ for all $c \in (0, \bar{c})$. The intuition is similar: with a large number of stores, price cuts succeed in winning market share only if the price is low enough to attract searching consumers, but this happens with probability of

[13] Non-existence is due to the search rule for shoppers. As pointed out in Subsection 2.1, if shoppers were to stop at the first store with $p = \underline{p}$, then the monopoly price would always be an SNE. We ruled this out by assuming that shoppers would always become fully informed because they enjoy shopping.

the order of $(1-F)^N$, which vanishes exponentially and the number of consumers so attracted by the lowest prices vanishes when $\mu'(0)=0$, while at higher prices the market share of high search cost consumers decreases only geometrically as $1/N$.

On the other hand, if $0<\mu'(0)<\infty$, then $0<p''\leq p^*<\hat{p}$. Thus, as the number of stores increases without bound, the mixed-strategy SNE (if any exist for all N) converge to a pure-strategy price between marginal-cost pricing and monopoly pricing. Moreover, convergence to marginal-cost pricing is possible only if $\mu'(0)=\infty$.[14] Again, we see the dramatic effect of shoppers, without which monopoly pricing could prevail for all N.

So far the only *search-cost-invariant* restriction on a mixed-strategy SNE F is that $0<\underline{p}<\bar{p}$, and Corollary 3.1. The next result establishes that when there are many stores there are no other search-cost-invariant restrictions. To begin, we take the SNE requirement that $\mathrm{E}\pi(p,F;N,\mu)=\mathrm{E}\pi(\bar{p},F;N,\mu)$ for all $p\in\mathrm{supp}F$, set $\partial\mathrm{E}\pi(p,F;N,\mu)/\partial p=0$, and solve for the search cost density:

$$\mu'(c)=$$
$$\left[\frac{(1-\bar{\mu})R(\bar{p})\dfrac{r(c,F)R'[r(c,F)]}{R^3[r(c,F)]}-\dfrac{N(N-1)\mu(c)}{D[r(c,F)]}\{1-F[r(c,F)]\}^{N-2}F'[r(c,F)]}{1-[1-F(p)]^N-N[1-F(p)]^{N-1}F(p)}\right]. \tag{6.1}$$

Eq. (6.1) is an ordinary differential equation with initial condition $\mu(0)=0$, and terminal condition $\mu[H(\bar{p},F)]=\bar{\mu}$. Motivated by Corollary 3.1, we define the following restriction.

Endpoint Condition (EP). A distribution F satisfies EP if, for sufficiently large N, the limit as $p\to\bar{p}$ from below $[1-F(p)]^{N-2}F'(p)$ is zero.

Proposition 6.2. Suppose F is a probability distribution with $0<\underline{p}<\bar{p}\leq\hat{p}$. "(a)" There is a sequence of μ_N such that as $N\to\infty$, the SNE $F(\cdot\,;N,\mu_N)$ converges to F. "(b)" Moreover, if EP is satisfied, then the sequence of μ_N can be constructed so $F(\cdot\,;N,\mu_N)=F$ for all N sufficiently large.

Thus, any $F(p)$ with $0<\underline{p}<\bar{p}\leq\hat{p}$, can be approximated by SNE of finite models, given a suitable search cost distribution. Moreover, when there is a

[14] If we admit an atom of shoppers to ensure that Proposition 6.1 is not vacuous, is it reasonable to go one step further and assume $\mu'(0)=\infty$? In this case, entry would drive prices towards marginal costs. Recalling that the atom of shoppers is really a modeling artifact to capture a distribution of search costs that extends to negative c (to accommodate shopping enjoyment), an infinite density at $c=0$ does not seem natural. Also note that since μ is a cumulative distribution function, the right-hand derivative is well-defined at $c=0$, even when there is an atom at 0.

large number of stores, almost any price distribution (subject to EP) can be an SNE.

7. Conclusions

We have investigated a model of oligopolistic pricing among a finite number of identical stores selling a homogeneous good when consumers have heterogeneous search costs and use sequential search rules that are optimal with respect to the stores' equilibrium pricing strategies (the Nash paradigm). We also permitted an atom of shoppers to capture the phenomenon of consumers who derive satisfaction from the process of shopping itself.

This model allows us to answer a number of new questions concerning the effect of certain characteristics of the search cost distribution on the equilibrium price distribution, especially the effect of shoppers, the empirical restrictions on the possible equilibrium price distributions, and the effect of the number of stores.

When there is no atom of shoppers, there generally is a continuum of pure-strategy SNE which includes the monopoly price and possibly a continuum of mixed-strategy SNE as well. If the search cost density vanishes at zero search costs, then the monopoly price is the unique SNE. However, when there is an atom of shoppers, there is no pure-strategy SNE. Thus, even a small number of consumers who enjoy shopping is sufficient to destroy the monopolistic SNE. As the atom of shoppers increases toward unit mass, the SNE converges to marginal-cost pricing. Thus, our model exhibits a variety of equilibrium behaviors between monopoly pricing and marginal-cost pricing, depending on the search cost distribution.

When the search cost distribution has a non-decreasing hazard rate and no atom of shoppers, the set of SNE consists entirely of pure-strategy SNE. If an atom of shoppers is introduced, there ceases to be a pure-strategy SNE—only mixed-strategy SNE exist. As the atom of shoppers vanishes, the mixed-strategy SNE converge to the lowest pure-strategy SNE of the atomless case, which we dub the shopper-robust SNE of this case.

Given a finite density of shoppers, there is no tendency for prices to approach marginal cost as the number of stores increases. In particular, with no atom of shoppers, there is always a continuum of pure-strategy SNE which includes the monopoly price, and this continuum is independent of the number of stores. As the number of stores increases without bound, mixed-strategy SNE either cease to exist or converge to a pure strategy. If there is a positive atom but a zero density of shoppers, entry pushes the SNE towards monopoly pricing.

For sufficiently large finite models, we established that virtually any price distribution could be an SNE for some consumer search cost distribution. Thus, when there is a large number of stores, the observed market price distribution alone cannot be used to test the model—information about the search cost distribution is crucial.

Let us now compare these results with those of the models in Table 1, and attempt to integrate this information into a coherent understanding of the effects of costly information acquisition by consumers in a 'competitive' market for a homogeneous good. The pertinent questions are of the form: Under what conditions will there be monopolistic pricing (marginal cost pricing, other single-price equilibria, and price dispersion)?

Monopoly pricing is always an equilibrium under the Nash paradigm when there is no atom of shoppers, and it is the unique equilibrium when there is also no positive search cost density at zero (Axell, 1977, and this paper). Burdett and Judd (1983) using the Nash paradigm and optimal parallel search show that monopoly pricing is always an equilibrium.[15] We can also get monopoly pricing under the Stackelberg paradigm when search costs are sufficiently high (Salop and Stiglitz, 1977); however, Braverman (1980) demonstrated that monopoly pricing in their model was an artifact of their consumer demand assumption.

First-best pricing (i.e. marginal cost pricing at minimum average cost) occurs under the Stackelberg paradigm and then only in some situations with a sufficiently large atom of shoppers, sufficiently flat average costs, and an (unlikely) integer solution to the optimal number of stores (Salop and Stiglitz, 1977; Braverman, 1980).[16] Combining this observation with our finding that marginal-cost pricing does not arise under the Nash paradigm with a finite density of shoppers, we conclude that costly information almost surely implies a departure from first-best pricing.

When can other single-price equilibria arise? *A necessary condition in all models is the absense of an atom of shoppers*. The role of the density of shoppers varies with the models. Braverman (1980) using the Stackelberg paradigm and the AoN search protocol concludes that a zero density of shoppers is also necessary, but the AoN search protocol is driving Braverman's result. Stiglitz (1987, Appendix B) using sequential search with replacement concludes that a zero density of shoppers is not necessary. He also shows that single-price equilibria arise when search costs are strictly

[15] This is consistent with our model, because Burdett and Judd assume shoppers stop at the first store offering their reservation price instead of sampling at least two stores. In their model of noisy search, with a positive probability of sampling at least two stores, they get a dispersed pricing result very similar to Stahl (1989a).

[16] The Stackelberg paradigm rather than the AoN search protocol seems to be the driving force behind this result. Braverman and Dixit (1981) show that if there is an infinite density of shoppers in Braverman's model, then the unique equilibrium is first-best pricing (assuming the integer condition is satisfied).

increasing in the number of searches. Rob (1985) (requiring a positive density of shoppers to keep prices bounded given unit demand at *any* price) proves the existence of a unique single-price equilibrium for specific search cost distributions. Using the Nash paradigm, we have shown (given no atom of shoppers) that a positive density of shoppers is necessary and sufficient for single-price equilibria other than monopoly pricing.

The existence of price dispersion is a central question in this literature. There are many ways to generate dispersed pricing.[17] Of course, we must avoid those environments that generate only single-price equilibria (e.g. under the Nash paradigm, search cost distributions with a zero density of shoppers or a non-decreasing hazard rate). *A necessary condition for all models is that some consumers have positive search costs*, i.e. $\bar{c} > 0$. (i) Under the Nash paradigm, an atom of shoppers or a positive density of shoppers with a decreasing hazard rate are sufficient conditions for dispersed-price equilibria. This paper has shown how to construct a search cost distribution that will support any price distribution that satisfies mild regularity conditions. (ii) Under the Stackelberg paradigm, Rob (1985) provides formulae for computing dispersed-price equilibria in a model with a continuum of stores. (Stiglitz, 1987, Appendix B, obtains a similar result). Finite Stackelberg models with U-shaped average costs can generate a special kind (and only this kind) of price dispersion called two-price equilibria (TPE) under some conditions, namely the (unlikely) integer condition, plus search cost conditions. For a finite Stackelberg model with constant marginal costs and search with replacement, Stiglitz (1987, Appendix A) provides an example of a TPE, but he provides no general conditions for the existence of dispersed-price equilibria for any of the models.

Finally, we come to the question of the effect of the number of stores on the equilibrium. In the case of homogeneous stores, does entry promote more competitive pricing? Putting aside the continuum models in which number of stores must be interpreted as mass of stores, this question has been largely ignored. Stahl (1989a) demonstrated that the SNE price distribution converges to the monopoly price as $N \rightarrow \infty$ (see also Rosenthal, 1980). A similar result holds in this paper when there is a zero density of shoppers. Stiglitz (1987) found convergence to marginal-cost pricing under the Stackelberg paradigm when search is with replacement. If search costs are increasing with the number of searches, he found that the range of

[17] Models with heterogeneous production costs can generate price dispersion. Essentially, each store charges its monopoly price—setting marginal revenue equal to marginal cost. Reinganum (1979) showed this within the Nash paradigm, even with homogeneous strictly positive search costs, using downward-sloping demand functions with elasticities less than -1. MacMinn (1980) showed that the result could be extended to permit unit demand up to a choke price, given that search costs are uniform on $[0, \bar{c}]$. Presumably, this result holds for more general search cost distributions provided there is a positive density of shoppers.

single-price equilibria increases with N and in the limit includes the entire interval between marginal costs and the monopoly price. In other cases, he found ambiguous answers. In contrast, we found that when there is no atom of shoppers (as in Stiglitz's model), there is always a range of single-price equilibrium that includes the monopoly price independent of N even though search costs do not increase with the number of searches. Although both papers show the possibility of more monopolistic outcomes as N increases, the reasons are entirely different: Stiglitz's result is driven by the increasing search costs, while our result is driven by the Nash paradigm.

In closing, our model provides a single theory of the phenomena of monopoly pricing, other single-price equilibria, and dispersed-price equilibria as products of the search cost distribution, while avoiding the precarious integer conditions and informational assumptions of the Stackelberg paradigm models.

Appendix A: Derivation of each store's share of consumers

As a matter of notation, let $F(r)$ denote the probability that $p \leqslant r$; let $F^0(r)$ denote the probability that $p < r$; and let $F(\{r\}) \equiv F(r) - F^0(r)$ denote the probability that $p = r$.

Fixing c, each store's share of these c-type consumers can be computed as follows. Let p_j be the price charged by store j. First recall that a consumer will stop and purchase from the first store with price $p_j < r(c, F)$. Such a store could capture this consumer on her first sampling, second sampling,..., or Nth sampling. We take a typical store j, and consider two cases.

(1) If $p_j < r(c, F)$, then the total share of consumers is

$$\psi(r(c, F), F) \equiv \frac{1}{N} \sum_{K=0}^{N-1} [1 - F^0(r(c, F))]^K . \tag{A.1}$$

(2) If $p_j \geqslant r(c, F)$, let $p_{\min} \equiv \min\{p_k | k \neq j\}$. No sale will occur on a consumer's first visit to store j. Of course, if $p_j > p_{\min}$, then store j sells nothing to these c-type consumers. If $p_j < p_{\min}$, then the consumer will return to store j and purchase. If $p_j = p_{\min}$, we assume that the consumer has an equal chance of purchasing from any of the stores charging this minimum price.[18] Then, the probability that store j will capture a typical c-type consumer is

[18] Alternatively, we could assume that if the consumer's Nth visit is to a store offering $p_{\min}$, then she stops; otherwise, she goes to one of the $p_{\min}$ stores with equal probability. This rule leads to more complicated share formulae but no difference in the equilibrium results because no such ties occur in equilibrium.

$$\nu(p_j, F) \equiv \sum_{m=1}^{N} \frac{1}{m} \cdot \frac{(N-1)!}{(m-1)!(N-m)!} [1 - F(p_j)]^{N-m} [F(\{p_j\})]^{m-1} . \tag{A.2}$$

$\Psi(p, F)$ denotes the share of all consumers purchasing from a store charging price p. Noting that $p > (<) r(c, F)$ as $H(p, F) > (<) c$, we can compute $\Psi(p, F)$ by integrating $\nu(p_j, F)$ with respect to μ over $c \in [0, H(p, F)]$, integrating $\psi(r(c, F), F)$ over $c > H(p, F)$, and summing the two integrals:

$$\Psi(p, F) \equiv \nu(p, F) \cdot \mu[H(p, F)] + \int_{H(p,F)^+}^{\bar{c}} \psi(r(c, F), F)\mu'(c)\mathrm{d}c . \tag{A.3}$$

It is straightforward to verify that $\Psi(\cdot, F)$ is non-increasing.

Appendix B: Proofs

Proof of Lemma 2.1. Straightforward; see Stahl (1989b).

Proof of Lemma 2.2. (a) Suppose to the contrary that $\bar{p} > r(\bar{c}, F) \equiv \bar{r}$. But then for $p = \bar{p}$, consumers do not stop and (by Lemma 2.1) with probability one do not return, implying that $\mathrm{E}\pi(\bar{p}, F) = 0$. Note that $\bar{r} > r(0, F) = \underline{p}$. Hence, for some $p < \bar{r}$, a mass of consumers will stop so $\mathrm{E}\pi(p, F) > 0$: a contradiction. (b) By assumption, $R(p) < R(\hat{p})$ for $p > \hat{p}$. Then, since $\Psi(p, F)$ is non-increasing, $\mathrm{E}\pi(p, F) < \mathrm{E}\pi(\hat{p}, F)$ for all $p > \hat{p}$; therefore $\bar{p} \leq \hat{p}$. (c) Suppose to the contrary that $\underline{p} = 0$. Then from Eq. (2.3), $\mathrm{E}\pi(\underline{p}, F) = 0$. But $\Psi(p, F) > 0$ for $p \in (0, \varepsilon)$ for some $\varepsilon > 0$; hence, $\mathrm{E}\pi(\bar{p}, F) > 0$ for all $p \in (0, \varepsilon)$, a contradiction of the supposition that $\underline{p} = 0$. Q.E.D.

Using Lemmas 2.1 and 2.2, Eq. (A.3) can be simplified to

$$\Psi(p, F) = [1 - F(p)]^{N-1} \cdot \mu[H(p, F)] + y(p, F) , \tag{B.1}$$

where

$$y(p, F) \equiv \int_{H(p,F)^+}^{\bar{c}} \psi(r(c, F), F)\mu'(c)\mathrm{d}c.$$

Since $F(\cdot)$ is atomless above $\underline{p}$, it has a well-defined density, $F'(\cdot)$, almost everywhere, so we can compute the slope of $\Psi(p, F; N, \mu)$ for $\underline{p} < p < \bar{p}$:

$$\partial\Psi(p,F)/\partial p^{+} = -(N-1)\mu[H(p,F)]\cdot[1-F(p)]^{N-2}F'(p) - \mu'[H(p,F)]\cdot D(p)\cdot Z[F(p),N]/N\,, \tag{B.2}$$

where

$$Z[F(p),N] \equiv 1-[1-F(p)]^{N} - N[1-F(p)]^{N-1}F(p).$$

Note that $Z(\cdot,N)$ is strictly increasing with $Z(0,N)=0$ and $Z(1,N)=1$.

Proof of Proposition 3.1. Since we must have $\mathrm{E}\pi(p,F)<\mathrm{E}\pi(\bar{p},F)$ for all $p>\bar{p}$, part (a) is immediate and also implies that

$$\partial\mathrm{E}\pi(\bar{p},F)/\partial p^{+} \leqslant 0\,. \tag{B.3}$$

Differentiating Eq. (2.3):

$$\frac{\partial\mathrm{E}\pi(p,F)}{\partial p} = R(p)D(p)\Psi(p,F)\left(\frac{pR'(p)}{R^2(p)} + \frac{\partial\Psi(p,F)/\partial p}{D(p)\Psi(p,F)}\right). \tag{B.4}$$

From Eq. (B.2), the right-hand derivative is $\partial\mathrm{E}\pi(\bar{p},F)/p^{+} = [\bar{p}R'(\bar{p})/R^2(\bar{p}) - \mu'/(1-\bar{\mu})]$; hence, by (B.3), we obtain part (b) of the Proposition. When $\underline{p}<p$, by Lemma 2.1 there is some $\varepsilon>0$ such that $\mathrm{E}\pi(p,F)=\mathrm{E}\pi(\bar{p},F)$ for all $p\in[\bar{p}-\varepsilon,\bar{p}]$, which implies that

$$\partial\mathrm{E}\pi(\bar{p},F)/p^{-}=0\,. \tag{B.5}$$

To derive the left-hand derivative, we need to consider two situations. First, suppose the limit, as $p\to\bar{p}$ from below, of $[1-F(p)]^{N-2}F'(p)$ is zero. Then part (c) of the Proposition follows by substitution of Eq. (B.2) into Eq. (B.4) and using (B.5). Second, suppose the limit, as $p\to\bar{p}$ from below, of $[1-F(p)]^{N-2}\cdot F'(p)$ is positive. Then, from (B.2) and (B.4), $\partial\mathrm{E}\pi(\bar{p},F)/\partial p^{-} < \partial\mathrm{E}\pi(\bar{p},F)/\partial p^{+} \leqslant 0$, which implies that $\bar{p}$ is dominated by a lower p, a contradiction. Q.E.D.

Proof of Theorem 3.1. Let A_n denote the set of n endpoints of a partition of $[0,\hat{p}]$ into $n-1$ equal intervals. Let Δ_n be the simplex in $\Re^n$, and interpret $f\in\Delta_n$ as a probability measure on A_n. Each such f defines a cumulative probability distribution: $F(f)$. Let $u(p,f)\equiv\mathrm{E}\pi(p,F(f))$ and let $\mathrm{E}u(g,f)\equiv\Sigma_{p\in A_n}u(p,f)g(p)$. Since $\mathrm{E}u(\cdot,\cdot)$ is continuous, by standard arguments there exists an $f_n\in\Delta_n$ such that $\mathrm{E}u_n\equiv\mathrm{E}u(f_n,f_n)\geqslant\mathrm{E}u(g,f_n)$ for all $g\in\Delta_n$. That is, f_n is an SNE of this finite game. Let F_n denote the cumulative probability distribution generated by f_n. Since the family of probability measures on $[0,\hat{p}]$ is compact in the topology of weak convergence, for every sequence there is a convergent subsequence (see Billingsley, 1968). Let F^* be such a limit distribution. We will show that $\int\mathrm{E}\pi(p,F^*)\mathrm{d}F^*\geqslant$

$\int \mathrm{E}\pi(p, F^*)\mathrm{d}G$ for all probability measures G on $[0, \hat{p}]$; that is, F^* is an SNE of the original game.

Since the monopoly price is always an SNE when $\mu(0) = 0$, it suffices to consider the case when $\mu(0) > 0$. There are three steps to this proof.

(1) Let $\underline{p}_n$ ($\bar{p}_n$) denote the lower (upper) bound of the support of F_n. We claim that if $\underline{p}_n \to \underline{p}$, then $\underline{p} > 0$. Suppose to the contrary that $\underline{p}_n \to 0$. Then $\mathrm{E}u_n \to 0$. From (A.1) and (B.1), $\Psi(p, F_n) \geq \{1 - \mu[H(p, F_n)]\}/N$. Since $r(\bar{c}, F_n)$ is strictly bounded above 0, there is a $p > 0$ and a $\delta > 0$ such that $R(p)\Psi(p, F_n) > \delta$, while $\mathrm{E}u_n \to 0$, a contradiction. Hence, $\underline{p} > 0$.

(2) We claim that F^* is atomless. Suppose to the contrary that $F^*(\{\tilde{p}\}) = \alpha > 0$. Let $p_n \equiv \max\{p \in A_n | F_n(p) < F_n(\tilde{p})\}$ and $q_n \equiv \min\{p \in A_n | F_n(p) > F_n(\tilde{p})\}$. Clearly, $p_n < q_n$, and both converge to $\tilde{p}$; furthermore, for n sufficiently large, $F_n(q_n) - F_n^0(p_n) \geq \alpha/2$. Define $\Delta u_n \equiv u(p_n, f_n) - u(q_n, f_n)$. It is straightforward to show that $H(p, F)$ is jointly continuous, and that the integrand of Eq. (B.1), which defines $y(p, F)$, is uniformly integrable. Therefore, $R(p_n) \to R(\tilde{p})$, $\mu[H(p_n, F_n)] \to \mu[H(\tilde{p}, F^*)]$, and $y(p_n, F_n) \to y(\tilde{p}, F^*)$. Hence, for any $\varepsilon > 0$, there is an n' such that for $n > n'$, $\Delta u_n < R(\tilde{p}) \cdot \mu[H(\tilde{p}, F^*)] \cdot \{[1 - F_n^0(p_n)]^{N-1} - [1 - F_n(q_n)]^{N-1}\}/N + \varepsilon$. Since $F_n(q_n) - F_n^0(p_n) \geq \alpha/2$, the term in braces is bounded below zero; since $\tilde{p} \geq \underline{p} > 0$, $R(\tilde{p}) > 0$; and since $\mu(0) > 0$, $\mu[H(\tilde{p}, F^*)] > 0$. Hence, for n sufficiently large, $\Delta u_n < 0$, which contradicts the optimality of f_n. Therefore, F^* is atomless.

(3) It follows from the second step that $u(p, f_n) \to \mathrm{E}\pi(p, F^*)$ and that $\mathrm{E}u_n \to Eu^* \equiv \int \mathrm{E}\pi(p, F^*)\mathrm{d}F^*$. Now suppose there is some p such that $\mathrm{E}\pi(p, F^*) > \mathrm{E}u^*$. But by continuity, for sufficiently large n, $u(p, f_n) > \mathrm{E}u_n$, a contradiction of the optimality of f_n. Therefore, $\mathrm{E}u^* \geq \int \mathrm{E}\pi(p, F^*)\mathrm{d}G$ for all probability measures G on $[0, \hat{p}]$. Q.E.D.

Proof of Proposition 4.1. Immediate from the requirement that $R(p)\Psi(p, F) = \mathrm{E}\pi(p, F) < \mathrm{E}\pi(\tilde{p}, F) = R(\tilde{p})/N$ for all $p \neq \tilde{p}$. Q.E.D.

Proof of Proposition 4.2. Suppose F is concentrated at $\tilde{p}$. From Proposition 3.1, a necessary condition for $\tilde{p}$ to be a pure-strategy SNE is that $\tilde{p}R'(\tilde{p})/R^2(\tilde{p}) \leq \mu'(0)$. Thus, by the definition of p^*, we must have $\tilde{p} \geq p^*$. RC implies that all $\tilde{p} \in [p^*, \hat{p}]$ satisfy this necessary condition. From Eq. (B.4), $\partial \mathrm{E}\pi(p, F)/\partial p$ is proportional to

$$\left(\frac{pR'(p)}{R^2(p)} - \frac{\mu'[H(p, F)]}{1 - \mu[H(p, F)]}\right)$$

for $p > \tilde{p}$. Then, given NDHR and F concentrated at $\tilde{p}$, $\partial \mathrm{E}\pi(p, F)/\partial p < 0$ for all $p > \tilde{p}$, so $\mathrm{E}\pi(p, F) < \mathrm{E}\pi(\tilde{p}, F)$ for all $p > \tilde{p}$. Q.E.D.

Proof of Proposition 4.3. Lemma 2.1 implies $p' > 0$. From Eqs. (B.2) and (B.3), given F concentrated at $\hat{p}$, $\partial \mathrm{E}\pi(\hat{p}, F)/\partial p^+ = -\hat{p}D^2(\hat{p})\mu'(0) < 0$. By continuity, there exists a $p' \in [\underline{p}, \hat{p})$ such that if F is concentrated at p', then for all $p > p'$, $\partial \mathrm{E}\pi(p, F)/\partial p^+ < 0$ and, hence, $\mathrm{E}\pi(p, F) < E\pi(p', F)$. Q.E.D.

Proof of Proposition 4.4.

(a) From Eq. (B.2), $\partial\Psi(\underline{p}, F)/\partial p^+ = -\mu'(0) \cdot D(\underline{p}) \cdot Z[F(\underline{p}), N]/N$. Hence, if $\mu'(0) = 0$, then $\partial\Psi(\underline{p}, F)/\partial p^+ = 0$, so $\partial E\pi(\underline{p}, F)/\partial p^+ > 0$ for all $\underline{p} < \hat{p}$, which leaves only the pure-strategy SNE at $\hat{p}$.

(b) Suppose, to the contrary, that $F(\underline{p}) = 0$. Then, again $\partial \mathrm{E}\pi(\underline{p}, F)/\partial p^+ > 0$, which implies that $\underline{p}$ is dominated by slightly higher prices. Q.E.D.

Proof of Proposition 4.5. Suppose, to the contrary, that there is a mixed-strategy SNE with $\underline{p} < \bar{p}$. Clearly, we must have $\partial \mathrm{E}\pi(\underline{p}, F)/\partial p^+ = 0$. Using Eqs. (B.2) and (B.3), this implies that $\underline{p}R'(\underline{p})/R^2(\underline{p}) = \mu'(0)Z[F(\underline{p}), N]/N\Psi(\underline{p}, F)$. Since $Z \leq 1$, and from Eq. (B.1), $N\Psi(\underline{p}, F) \geq 1 - \mu[H(\underline{p}, F)]$, we have $\underline{p}R'(\underline{p})/R^2(\underline{p}) \leq \mu'(0)/[1 - \mu(0)] \leq \mu'[H(\bar{p}, F)]/(1 - \bar{\mu}) = \bar{p}R'(\bar{p})/R^2(\bar{p})$, where the last inequality follows from a non-decreasing hazard rate and the last equality follows from Proposition 3.1. Given $\underline{p} < \bar{p}$ and RC, it is impossible to satisfy these necessary conditions. Q.E.D.

Proof of Proposition 5.1. Given $\underline{p} < \bar{p}$, $H(\bar{p}, F) > 0$, so $\bar{\mu} > 0$. Now, suppose to the contrary that $\bar{\mu} = 1$, i.e. $H(\bar{p}, F) = \bar{c}$. But then from (B.1), $\Psi(\bar{p}, F) = 0$, which contradicts the fact that $\mathrm{E}\pi(\bar{p}, F) > 0$ is an SNE. Q.E.D.

Proof of Proposition 5.2. By pointwise convergence of μ_n, $\mu_n(0) \to 1$; hence, $F_n \equiv F(p; \mu_n)$ is strictly mixed. Moreover, $\mu_n(0) \to 1$ implies that $\bar{\mu}_n \to 1$, and therefore $\mathrm{E}\pi(\bar{p}_n, F_n) = (1 - \bar{\mu}_n)R(\bar{p}_n)/N \to 0$. Consequently, $\underline{p}_n \to 0$. Now if $(\bar{p}_n - \underline{p}_n) \to 0$, we are done, so suppose there is an $\alpha > 0$ such that $(\bar{p}_n - \underline{p}_n) \geq \alpha$ for all n. Let $p > 0$ be any price such that for n sufficiently large, $p \in (\underline{p}_n, \bar{p}_n)$. Then $\mathrm{E}\pi(p, F_n) = R(p)\{[1 - F_n(p)]^{N-1} \cdot \mu[H(p, F_n)] + y(p, F_n)\} = \mathrm{E}\pi(\bar{p}_n, F_n)$, which vanishes as $n \to \infty$. Since $y(p, F_n) \geq 0$, we must have $y(p, F_n) \to 0$, and since $\mu[H(p, F_n)] \to 1$, we must have $F_n(p) \to 1$ as $n \to \infty$. Q.E.D.

Proof of Proposition 5.3.

(a) Since $H(\bar{p}_n, F_n) \leq \bar{c}_n$, from Eq. (2.1) it follows that $\bar{p}_n \to 0$.

(b) Given NDHR and an atom of shoppers, F_n is non-degenerate, so by Proposition 3.1, $\bar{p}_n$ must satisfy: $\bar{p}_nR'(\bar{p}n)/R^2(\bar{p}_n) = \mu_n'[H(\bar{p}_n, F_n)]/(1 - \bar{\mu}_n) \geq \mu_n'(0) \to \infty$; therefore, $\bar{p}_n \to 0$. Q.E.D.

Proof of Proposition 5.4.

(a) From Proposition 2.1, $F_n(p) \equiv F(p; \mu_n)$ is atomless. That F^* is atomless above $\underline{p}$ can be proven using an argument similar to the proof of Theorem 3.1. This implies that $\mathrm{E}\pi(p, F^*; \mu)$ is continuous in p. It is then easy to show that F^* is an SNE. (For details, see Stahl, 1989b.)

(b) From (a), F^* is an SNE of the unperturbed game, and from Proposition 4.2, F^* must have unit mass concentrated at some $p \in [p^*, \hat{p}]$. Since each μ_n has a non-decreasing hazard rate, from Proposition 3.1, $\bar{p}_n R'(\bar{p}_n)/R^2(\bar{p}_n) = \mu_n'[H(\bar{p}_n, F_n)]/(1-\bar{\mu}_n) \geq \mu_n'(0)/[1-\mu_n(0)] = \mu'(0)$. The first inequality follows from NDHR. Then, given RC, $\bar{p}_n < p^*$ for all n; hence, $\bar{p}_n$ must converge to p^*. Hence, it must be that F^* has unit mass at p^*. Q.E.D.

The proof of the next result uses an expression for $F(p)$ that we now present. Recall that we must have $\mathrm{E}\pi(p, F) = \mathrm{E}\pi(\bar{p}, F)$ for all $p \in \mathrm{supp}F$. Then, using Eq. (B.1) and Eq. (2.3):

$$F(p) = 1 - \left[\frac{(1-\bar{\mu})R(\bar{p})/R(p) - Ny(p, F)}{N\mu[H(p, F)]}\right]^{1/(N-1)} \tag{B.6}$$

for all $p \in \mathrm{supp}F$. Recall that $Ny(\bar{p}, F) = 1 - \bar{\mu}$.

Proof of Proposition 6.1. Let $F_N \equiv F(\cdot\,; N)$, $\bar{\mu}_N \equiv \mu[H(\bar{p}_N, F_N)]$, where $\bar{p}_N$ is the upper bound of $\mathrm{supp}F_N$. From Eq. (B.6), for every $p \in \mathrm{supp}F_N$, the SNE distribution has the form: $F_N(p) = 1 - \{w(p, N)/N\}^{1/(N-1)}$. Note that since $\mathrm{d}F_N/\mathrm{d}p \geq 0$, $\mathrm{d}w/p \leq 0$. Let $\{\bar{p}_k\}$ be any convergent subsequence of $\{\bar{p}_N\}$, let $\bar{p} \equiv \lim_{k\to\infty}\{\bar{p}_k\}$, and define $w^*(p) \equiv \lim_{k\to\infty} w(p, k)$. It is straightforward to verify that (i) $w^*(\bar{p}) = 0$, and (ii) $\mathrm{d}w^*(\bar{p})/\mathrm{d}p < 0$. Hence, for all $p < \bar{p}$, $w^*(p) > 0$, which implies that for sufficiently large k and $p < \bar{p} - 1/k$, $w(p, k) > 0$; hence, $F_N(p) \to 0$ for all $p < \bar{p}$. It follows that $H(\bar{p}_k, F_k) \to 0$. From Proposition 3.1(c), $\bar{p}_k R'(\bar{p}_k)/R^2(\bar{p}_k) = \mu'[H(\bar{p}_k, F_k)]/(1-\bar{\mu}_k)$ for all N, so in the limit: $\bar{p}R'(\bar{p})/R^2(\bar{p}) = \mu'(0)/[1-\mu(0)]$. By RC, there is a unique price (which we denote as p'') that satisfies this last equation. Hence, $\bar{p}_k \to p''$. Since this holds for any convergent subsequence, $F^*(p)$ has unit mass at p''. Q.E.D.

Proof of Proposition 6.2.

(1) We will prove the second assertion first. Given EP, the second term of Eq. (6.1) converges to zero as $N \to \infty$ for all $c < H(\bar{p}, F)$ (i.e. $p < \bar{p}$). Moreover, the denominator converges to 1. Therefore, Eq. (6.1) converges to

$$\mu'(c) = (1-\bar{\mu})R(\bar{p})\left(\frac{r(c, F)R'[r(c, F)]}{R^3[r(c, F)]}\right). \tag{B.7}$$

Given $\bar{\mu}$ as a parameter, Eq. (B.7) can be integrated to produce a solution, denoted $\mu(c; \bar{\mu})$ that satisfies $\mu(0; \bar{\mu}) = \mu(0)$. The other boundary condition is $\mu[H(\bar{p}, F); \bar{\mu}] = \bar{\mu}$. Let $\gamma(\bar{\mu})$ denote the integral of the right-hand side of Eq. (B.7) over the interval $c \in [0, H(\bar{p}, F)]$. Observe that $\gamma(0) > 0$, $\gamma' < 0$, and $\gamma(1) = 0$; therefore, there exists a unique fixed point $\bar{\mu}(F) = \gamma[\bar{\mu}(F)]$. Let $\mu(c, F) \equiv \mu(c; \bar{\mu}(F))$, so $\mu(\cdot, F)$ solves Eq. (B.7) and the boundary conditions. We can complete the specification of $\mu(\cdot, F)$ for $c \geqslant H(\bar{p}, F)$ by placing an atom of mass $(1 - \bar{\mu}(F))$ at $c = H(\bar{p}, F)$.

By continuity, for sufficiently large N, Eq. (6.1) defines a positive search cost density function, and can be solved for the seach cost distribution, denoted $\mu_N(c, F)$, which also satisfies the boundary conditions. To complete the specification of $\mu_N(c, F)$ put mass $(1 - \bar{\mu}_N)$ at $c = H(\bar{p}, F)$. Thus, the SNE of the finite case $F(p; N, \mu_N) = F(p)$ for all N.

(2) In the absense of the endpoint condition assumed in part (1), for any small $\varepsilon > 0$, define $S(p, \varepsilon) \equiv \{y : F(p - \varepsilon) - \varepsilon \leqslant y \leqslant F(p + \varepsilon) + \varepsilon\}$. The graph of $S(\cdot, \varepsilon)$ covers the graph of $F(\cdot)$ with a disc with a radius of at least ε. Therefore, we can find a strictly increasing continuously differentiable function $G(p, \varepsilon)$ with (i) $|\partial G/\partial p| < 1/\varepsilon$, (ii) $G(\underline{p}, \varepsilon) \geqslant \varepsilon/2$, and (iii) $G(\bar{p}, \varepsilon) = 1$ the graph of which is completely contained within the graph of $S(p, \varepsilon)$. Note that as $\varepsilon \to 0$, $G(p, \varepsilon) \to F(p)$ for F-a.e. p. Next, we can choose an N_ε sufficiently large, such that the magnitude of second term of Eq. (6.1) is less than $\varepsilon/N_\varepsilon$ times the magnitude of the first term of Eq. (6.1), and, therefore Eq. (6.1) defines a legitimate search cost distribution, μ_{N_ε}, such that the SNE $F(p; N_\varepsilon, \mu_{N_\varepsilon}) = G(p, \varepsilon)$. Q.E.D.

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