Beliefs identification

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Adverse selection can be mitigated by some factors, for example Handel (2013) shows that there is inertia in the choice of health insurance plans, which weakens the link between private information and choices. Crawford et al. (2018), in the market for credit, shows that market power weakens adverse selection by increasing the value of the marginal borrower with respect to the average borrower.

1 Model with distorted survival beliefs and guaranteed annuity

Consider an individual i has initial wealth (savings) $W > 0$ at $t = 1$.

The true survival probability between any two consecutive periods is $x_i \in (0, 1)$. Individual beliefs about survival are distorted: the perceived one-period survival probability is

$$\hat{x}_i \equiv \theta x_i,$$

with $\theta > 0$ and $\hat{x}_i \in (0, 1)$.

Individuals also have a heterogenous bequest motive β_i, with CDF F_β on $[0, \infty)$.

The individual has three options.

The first option is no annuity (N), self-insurance with savings. The second option is an immediate annuity (A) that pays $F > 0$ in each period the individual is alive, with no bequest. The third option is a guaranteed annuity (G) that pays $F_g > 0$ in each period the individual is alive; in addition, if the individual dies before T the annuity continues paying to the beneficiaries.

The corresponding values of the options are: $V^N(x, \theta, \beta_i, W)$ for no annuity, $V^A(x, \theta; F)$ for immediate annuity, and $V^G(x, \theta, \beta_i; F_g)$ for guaranteed annuity. For a microfoundation of these value functions see appendix 5.1.

Given $(x, \theta, \beta_i, W, F, F_g)$, the individual chooses the option with highest expected utility:

$$\text{Option chosen} = \arg\max \left\{V^N(x, \theta, \beta_i, W),\ V^A(x, \theta; F),\ V^G(x, \theta, \beta_i; F_g)\right\}.$$

Non-identification The observed choice is

$$D_i \in \{A, G, N\} \quad \text{with} \quad D_i = \arg\max\{V^A, V^G, V^N\}.$$

We observe the joint distribution of (x_i, D_i); primitives of interest are θ and F_β.

In this case we can prove a non-identification result, see section 5.2.

Identification with price shifters

Previously we showed that data on (x_i, D_i) is not sufficient to identify (θ, F_β)

Adding exogenous price variation. Now allow the annuity payments to depend on an observable "price state" r (e.g. an interest rate or term-structure shifter):

$$F = F(r), \qquad F_g = F_g(r),$$

with r observed by the econometrician and exogenous:

$$r \perp (\beta, x, \theta),$$

and with sufficient support on some compact interval $\mathcal{R}$.

Now the value functions will also be indexed by r, for each (x, r, θ) we obtain two cutoffs

$$0 \leq \beta_1(x, r; \theta) \leq \beta_2(x, r; \theta) \leq \infty,$$

with

$$D = \begin{cases} A, & \beta \leq \beta_1(x, r; \theta), \\ G, & \beta_1(x, r; \theta) < \beta \leq \beta_2(x, r; \theta), \\ N, & \beta > \beta_2(x, r; \theta), \end{cases}$$

and choice probabilities

$$p_A(x, r) = F_\beta\big(\beta_1(x, r; \theta)\big), \tag{1}$$

$$p_A(x, r) + p_G(x, r) = F_\beta\big(\beta_2(x, r; \theta)\big). \tag{2}$$

Why price variation helps: breaking the simple relabeling. Without r, the previous non-identification argument rests on the fact that

$$p_A(x) = F_\beta(\beta_1(x; \theta))$$

only pins down F_β on the *image* of $x \mapsto \beta_1(x; \theta)$, and for any alternative θ_1 we can reparametrize F_β along that image. With r, we now observe $p_A(x, r)$ and $p_A(x, r) + p_G(x, r)$ for a *two-dimensional* regressor $z = (x, r)$, and the relevant composition is

$$p_A(z) = F_\beta(\beta_1(z; \theta)), \qquad p_A(z) + p_G(z) = F_\beta(\beta_2(z; \theta)).$$

A sufficient condition to break the relabeling is:

(C1) Rich price support and invertibility. For each $\theta > 0$, the mappings $z \mapsto \beta_j(z; \theta)$, $j = 1, 2$, are continuous and strictly monotone in r for all x, with images that overlap across different values of $(x, r) \in \mathcal{X} \times \mathcal{R}$. Formally, for any two points z_1, z_2 there exist z_3, z_4 such that

$$\beta_1(z_1; \theta) = \beta_2(z_3; \theta), \qquad \beta_2(z_2; \theta) = \beta_1(z_4; \theta),$$

and $z \mapsto \beta_j(z; \theta)$ are invertible on their images.

Intuitively, $(C1)$ says that as interest rates change, the thresholds $\beta_1(x, r; \theta)$ and $\beta_2(x, r; \theta)$ sweep over the support of β in a way that generates *overlapping evaluations* of F_β at common β values coming from different (x, r) pairs.

Proof of Identification under (C1)

Let (θ_0, F_β^0) be the true parameters generating the observed choice probabilities $p_A(x, r)$ and $p_G(x, r)$. Suppose there exists an alternative pair (θ_1, F_β^1) that is observationally equivalent. That is, for all $(x, r) \in \mathcal{X} \times \mathcal{R}$, the following equalities hold:

$$F_\beta^1(\beta_1(x, r; \theta_1)) = p_A(x, r) = F_\beta^0(\beta_1(x, r; \theta_0)) \tag{3}$$

$$F_\beta^1(\beta_2(x, r; \theta_1)) = p_A(x, r) + p_G(x, r) = F_\beta^0(\beta_2(x, r; \theta_0)) \tag{4}$$

We define the index $j \in \{1, 2\}$ to refer to the relevant margin, where β_1 governs the choice between A and G, and β_2 governs the choice between G and N. We can summarize (3) and (4) as:

$$F_\beta^1(\beta_j(x, r; \theta_1)) = F_\beta^0(\beta_j(x, r; \theta_0)) \quad \text{for } j \in \{1, 2\}. \tag{5}$$

Step 1: Constructing the Mapping ϕ_j

Fix an arbitrary survival probability $x \in \mathcal{X}$ and let b be a bequest level in the image of $\beta_j(x, \cdot; \theta_1)$. By Assumption (C1), the function $r \mapsto \beta_j(x, r; \theta_1)$ is continuous and strictly monotone. Therefore, for a fixed x, there exists a unique inverse price function, denoted as $r_j^*(b, x; \theta_1)$, such that:

$$\beta_j(x, r_j^*(b, x; \theta_1); \theta_1) = b \tag{6}$$

Substitute the price $r = r_j^*(b, x; \theta_1)$ into the observational equivalence condition (5):

$$F_\beta^1(b) = F_\beta^0(\beta_j(x, r_j^*(b, x; \theta_1); \theta_0)) \tag{7}$$

We define the composite mapping $\phi_j(b, x; \theta_0, \theta_1)$ as the cutoff value implied by the true parameter θ_0 at the state (x, r_j^*):

$$\phi_j(b, x; \theta_0, \theta_1) \equiv \beta_j(x, r_j^*(b, x; \theta_1); \theta_0) \tag{8}$$

Using this definition, equation (7) becomes:

$$F_\beta^1(b) = F_\beta^0(\phi_j(b, x; \theta_0, \theta_1)) \tag{9}$$

Step 2: Contradiction via Variation in x

The left-hand side of (9) depends only on b. Consequently, the right-hand side must be invariant to x. This implies that for any two survival probabilities $x, x' \in \mathcal{X}$:

$$F_\beta^0(\phi_j(b, x; \theta_0, \theta_1)) = F_\beta^0(\phi_j(b, x'; \theta_0, \theta_1)) \tag{10}$$

Since F_β^0 is a strictly increasing CDF (on the relevant support), (10) requires:

$$\phi_j(b, x; \theta_0, \theta_1) = \phi_j(b, x'; \theta_0, \theta_1) \tag{11}$$

Substituting the definition from (8), condition (11) requires that the true cutoff $\beta_j(\cdot; \theta_0)$ and the alternative cutoff $\beta_j(\cdot; \theta_1)$ move in perfect synchronization across x and r. Given the non-linear interaction between beliefs θx and prices r in the value functions (specifically the ratio of annuity flows to bequest utility), this condition generically fails unless $\theta_1 = \theta_0$. Thus, θ is point identified.

To prove this, note that equation 11 requires :

$$\frac{\partial \phi_j(b, x; \theta_0, \theta_1)}{\partial x} = 0 \quad \text{for all } x. \tag{12}$$

Differentiating the definition of ϕ_j in (8) with respect to x:

$$\frac{\partial \phi_j}{\partial x} = \frac{\partial \beta_j(\theta_0)}{\partial x} + \frac{\partial \beta_j(\theta_0)}{\partial r} \cdot \frac{\partial r_j^*}{\partial x} \tag{13}$$

From the definition of the inverse price r_j^* in (6), we apply the Implicit Function Theorem to find $\partial r_j^* / \partial x$:

$$\frac{\partial \beta_j(\theta_1)}{\partial x} + \frac{\partial \beta_j(\theta_1)}{\partial r} \frac{\partial r_j^*}{\partial x} = 0 \implies \frac{\partial r_j^*}{\partial x} = -\frac{\partial \beta_j(\theta_1)/\partial x}{\partial \beta_j(\theta_1)/\partial r} \tag{14}$$

Substituting (14) into (13), the condition $\partial \phi_j / \partial x = 0$ is equivalent to:

$$\frac{\partial \beta_j(\theta_0)/\partial x}{\partial \beta_j(\theta_0)/\partial r} = \frac{\partial \beta_j(\theta_1)/\partial x}{\partial \beta_j(\theta_1)/\partial r} \tag{15}$$

To see that (15) does not generally hold when $\theta_1 \neq \theta_0$, it is convenient to consider the case with three periods and $j = 1$ (see section 5.1), in which case:

$$\beta_1(x, r; \theta) = f(\theta x)\, g(r), \qquad f(u) \equiv \frac{1 + u + u^2}{1 - u}, \quad g(r) \equiv \frac{F(r)^\alpha - F_g(r)^\alpha}{F_g(r)}.$$

In which case replacing in equation 15 we have:

$$\frac{f'(\theta_0 x) x g(r)}{f(\theta_0 x) g'(r)} = \frac{f'(\theta_1 x) x g(r)}{f(\theta_1 x) g'(r)} \implies \frac{f'(\theta_0 x)}{f(\theta_0 x)} = \frac{f'(\theta_1 x)}{f(\theta_1 x)} \tag{16}$$

using $\frac{f'(u)}{f(u)} = \frac{-u^2+2u+2}{1-u^3}$ we have:

$$\frac{-(\theta_0 x)^2 + 2(\theta_0 x) + 2}{1 - (\theta_0 x)^3} = \frac{-(\theta_1 x)^2 + 2(\theta_1 x) + 2}{1 - (\theta_1 x)^3} \tag{17}$$

which is not generally true. Therefore, equation (15) holds if and only if $\theta_1 = \theta_0$. Thus, θ_0, F_β^0 are identified. $\square$

The core identification problem is distinguishing whether individuals are buying annuities because they have low bequest motives (low β) or because they are optimistic about their survival (high belief $\hat{x} = \theta x$). In a static setting with fixed prices, these two forces are indistinguishable: a high-bequest pessimist might make the same choice as a low-bequest optimist. We break this observational equivalence by exploiting exogenous variation in annuity prices (driven by interest rates, r). The key insight is that beliefs (θx) and bequest motives (β) interact differently with price changes.Specifically, the "exchange rate" between annuity income and bequest value depends non-linearly on beliefs. When interest rates rise, annuity payouts increase. A rational agent ($\theta = 1$) and a biased agent ($\theta \neq 1$) will re-evaluate the trade-off between the immediate annuity and the guaranteed annuity differently because their perceived "effective price" of the guarantee depends on their survival scaling θ. By observing how the demand for each product shifts across different price levels for the same underlying survival type x, we can trace out a "marginal rate of substitution" curve. The shape of this curve is unique to the specific belief parameter θ.

2 literature review

- O'Dea and Sturrock (2023) studies the same but is not really the interaction between private info and beliefs, rather that people are pessimistic, but do not study the interaction of beliefs with private information.

3 Estimation

For estimation we can use maximum likelihood.

The likelihood contribution of individual i is:

$$\mathcal{L}_i(\theta, F_\beta) = p_A(x_i; \theta, F_\beta)^{\mathbb{1}(D_i=A)} \cdot p_G(x_i; \theta, F_\beta)^{\mathbb{1}(D_i=G)} \cdot p_N(x_i; \theta, F_\beta)^{\mathbb{1}(D_i=N)}.$$

Then, the likelihood function is:

$$\mathcal{L}(\theta, F_\beta) = \prod_{i=1}^{N} \mathcal{L}_i(\theta, F_\beta). \tag{18}$$

4 Other thoughts

- We observe choices, which are a function of beliefs and preferences. Jingi asked me why would we care about identifying beliefs separately from preferences. Is there any policy relevance to this? I think there is none but is just very interesting to separate it. But I should think about welfare implications, for example that there are a lot of people who are actually buying insurance because they do not take into account their private information.

References

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O'Dea, C., & Sturrock, D. (2023). Survival Pessimism and the Demand for Annuities. *The Review of Economics and Statistics*, *105*(2), 442–457. https://doi.org/10.1162/rest_a_01048

5 Appendix

5.1 Microfoundations for value functions

Time is $t = 1, 2, 3$. Each individual i has initial wealth (savings) $W > 0$ at $t = 1$.

The true survival probability between any two consecutive periods is $x_i \in (0, 1)$. Individual beliefs about survival are distorted: the perceived one-period survival probability is

$$\hat{x}_i \equiv \theta x_i,$$

with $\theta > 0$ and $\hat{x}_i \in (0, 1)$. Period utility from consumption is

$$u(c) = c^{\alpha}, \qquad \alpha \in (0, 1).$$

The bequest weight β_i is heterogeneous with CDF F_β on $[0, \infty)$. Bequest utility is linear:

$$v(B) = \beta_i B, \qquad \beta_i \geq 0.$$

We assume no time discounting.

The individual has three options at $t = 1$. The first option is no annuity (N), self-insurance with savings. The second option is an immediate annuity (A) that pays $F > 0$ in each period the individual is alive, with no bequest. The third option is a guaranteed annuity (G) that pays $F_g > 0$ in each period the individual is alive; in addition, if the individual dies between periods 1 and 2, the contract pays a guaranteed amount F_g at $t = 2$ as a pure bequest (no consumption).

1. No annuity: consumption–savings problem and value. If the individual does not buy an annuity, she chooses a consumption–savings plan:

$$c_1, c_2, c_3 \geq 0$$

subject to $c_1 + c_2 + c_3 = W$ and $c_1, c_2, c_3 \geq 0$.

Given beliefs $\hat{x} = \theta x$, the expected utility from a given plan (c_1, c_2, c_3) is

$$U^N(c_1, c_2, c_3; x, \theta, \beta_i) = u(c_1) + (1 - \hat{x})\,\beta_i(W - c_1) + \hat{x}\Big[u(c_2) + (1 - \hat{x})\,\beta_i(W - c_1 - c_2) + \hat{x}\,u(c_3)\Big].$$

The value of self-insurance is

$$V^N(x, \theta, \beta_i, W) \equiv \max_{\substack{c_1, c_2, c_3 \geq 0 \\ c_1 + c_2 + c_3 = W}} U^N(c_1, c_2, c_3; x, \theta, \beta_i).$$

Sometimes we also use s_t for savings at the beginning of period t, for instance $s_1 = W - c_1$.

2. Immediate annuity (A). If the individual buys the immediate annuity, she uses all wealth W to purchase a contract that pays F in each period she is alive. There is no bequest from the annuity.

Expected utility under the immediate annuity is

$$V^A(x, \theta) = \big(1 + \hat{x} + \hat{x}^2\big)\,F^{\alpha}.$$

3. Guaranteed annuity (G). If the individual buys the guaranteed annuity, she uses all wealth W to purchase a contract that:

- pays F_g each period she is alive (as with an immediate annuity),
- if she dies between $t = 1$ and $t = 2$, pays a guaranteed amount F_g at $t = 2$ as a bequest (no consumption).

Hence, the expected utility under the guaranteed annuity is

$$\begin{aligned} V^G(x, \theta, \beta_i) &= u(F_g) + \hat{x}\,u(F_g) + \hat{x}^2\,u(F_g) + (1 - \hat{x})\,v(F_g) \\ &= \big(1 + \hat{x} + \hat{x}^2\big)\,F_g^{\alpha} + (1 - \hat{x})\,\beta_i F_g. \end{aligned}$$

4. Choice among N, A, and G. Given (x, θ, β_i, W), the individual chooses the option with highest expected utility:

$$\text{Option chosen} = \arg\max\big\{V^N(x, \theta, \beta_i, W),\, V^A(x, \theta),\, V^G(x, \theta, \beta_i)\big\}.$$

5.2 Non-identification result

Assumption 1 (Monotone partition in β_i)**.** *Assume that there exist cutoffs* $0 \leq \beta_1(x; \theta) \leq \beta_2(x; \theta)$ *such that,*

$$D_i = \begin{cases} A, & \text{if } \beta_i \leq \beta_1(x_i; \theta), \\ G, & \text{if } \beta_1(x_i; \theta) < \beta_i \leq \beta_2(x_i; \theta), \\ N, & \text{if } \beta_i > \beta_2(x_i; \theta). \end{cases}$$

Assumption 1 is intuitive, it says that individuals with a higher bequest motive will buy less insurance since they do not mind leavin savings to their beneficiaries. A sufficient condition is that $s_1(\beta_i = 0; x) > F_g$. [1]

Define the conditional choice probabilities

$$p_k(x) \equiv \Pr(D_i = k \mid x_i = x), \qquad k \in \{A, G, N\}.$$

By Assumption 1, the model implies:

$$p_A(x) = \Pr(\beta_i \le \beta_1(x;\theta)) = F_\beta\big(\beta_1(x;\theta)\big), \tag{19}$$

$$p_A(x) + p_G(x) = \Pr(\beta_i \le \beta_2(x;\theta)) = F_\beta\big(\beta_2(x;\theta)\big), \tag{20}$$

$$p_N(x) = 1 - \big[p_A(x) + p_G(x)\big] = 1 - F_\beta\big(\beta_2(x;\theta)\big). \tag{21}$$

Non-identification of (θ, F_β) (nonparametric F_β). Assume:

(A1) For each $\theta > 0$, the functions $x \mapsto \beta_1(x;\theta)$ and $x \mapsto \beta_2(x;\theta)$ are strictly monotone and continuous on the support of x, hence invertible with inverses $b \mapsto x_1^\theta(b)$ and $b \mapsto x_2^\theta(b)$.

(A2) The images of $\beta_1(\cdot;\theta)$ and $\beta_2(\cdot;\theta)$ are disjoint (or can be made disjoint by restricting attention to suitable subsets of x).

In section 5.3 we provide intuitive conditions under which this assumptions are satisfied.

Suppose the data are generated by some true pair (θ_0, F_β^0), yielding observed choice probabilities

$$p_A(x) = F_\beta^0\big(\beta_1(x;\theta_0)\big), \qquad p_A(x) + p_G(x) = F_\beta^0\big(\beta_2(x;\theta_0)\big),$$

for all x.

Proposition 2 (Non-identification)**.** *Under (A1)–(A2), for any alternative $\theta_1 > 0$ there exists a CDF F_β^1 such that the pair (θ_1, F_β^1) generates the same conditional choice probabilities $\{p_A(x), p_G(x), p_N(x)\}$ for all x. Hence (θ, F_β) is not point-identified.*

Proof. Fix $\theta_1 > 0$ and consider the cutoff functions $\beta_1(x;\theta_1)$ and $\beta_2(x;\theta_1)$. By (A1), define inverses $x_1^{\theta_1}(b)$ and $x_2^{\theta_1}(b)$ on the images of $\beta_1(\cdot;\theta_1)$ and $\beta_2(\cdot;\theta_1)$, respectively, such that

$$\beta_1\big(x_1^{\theta_1}(b);\theta_1\big) = b, \qquad \beta_2\big(x_2^{\theta_1}(b);\theta_1\big) = b.$$

Define F_β^1 on these images by

$$\begin{aligned} F_\beta^1(b) &\equiv p_A\big(x_1^{\theta_1}(b)\big), && \text{for } b \in \mathrm{Im}\big(\beta_1(\cdot;\theta_1)\big), \\ F_\beta^1(b) &\equiv p_A\big(x_2^{\theta_1}(b)\big) + p_G\big(x_2^{\theta_1}(b)\big), && \text{for } b \in \mathrm{Im}\big(\beta_2(\cdot;\theta_1)\big). \end{aligned}$$

[1] Given that $\frac{\partial V^G}{\partial \beta_i} = (1-x)\beta_i F_g$ and that $\frac{\partial V^N}{\partial \beta_i} > \beta_i(1-x)[s_1(\beta_i) + x s_2(\beta_i)]$ if $s_1(\beta_i = 0; x) > F_g$ then, given that savings are increasing on β_i, we have that $s_1(\beta_i; x) > F_g$. Then $\frac{\partial V^G}{\partial \beta_i}(1-x) = \beta_i F_g < \beta_i(1-x)[s_1(\beta_i) + x s_2(\beta_i)] < \frac{\partial V^N}{\partial \beta_i}$

Disjointness of the images (A2) guarantees that F_β^1 is well-defined. Monotonicity of p_A and $p_A + p_G$ in x and of $x_1^{\theta_1}$, $x_2^{\theta_1}$ in b implies F_β^1 is nondecreasing in b on the union of these images, and it can be extended arbitrarily (but monotonically) to a full CDF on $[0, \infty)$.

Now, for any x,

$$F_\beta^1\big(\beta_1(x;\theta_1)\big) = p_A(x), \qquad F_\beta^1\big(\beta_2(x;\theta_1)\big) = p_A(x) + p_G(x).$$

Hence the model-implied probabilities under (θ_1, F_β^1) satisfy

$$p_A(x;\theta_1, F_\beta^1) = F_\beta^1\big(\beta_1(x;\theta_1)\big) = p_A(x),$$

and

$$p_A(x;\theta_1, F_\beta^1) + p_G(x;\theta_1, F_\beta^1) = F_\beta^1\big(\beta_2(x;\theta_1)\big) = p_A(x) + p_G(x),$$

so $p_G(x;\theta_1, F_\beta^1) = p_G(x)$ and $p_N(x;\theta_1, F_\beta^1) = 1 - p_A(x) - p_G(x) = p_N(x)$ for all x.

Therefore the observed conditional choice probabilities do not uniquely determine (θ, F_β): for each $\theta_1 > 0$ we can construct a CDF F_β^1 that exactly reproduces the same $\{p_A(x), p_G(x), p_N(x)\}_x$.

□

5.3 Microfoundations for (A1) and (A2)

In this subsection we show how the cutoff properties used in Step 3 follow from the underlying value functions, under mild assumptions on primitives, instead of being imposed directly on the reduced-form objects $\beta_1(\cdot;\theta)$, $\beta_2(\cdot;\theta)$.

Explicit expression and monotonicity of $\beta_1(x;\theta)$. Recall that the A–G cutoff $\beta_1(x;\theta)$ is defined implicitly by

$$V^A(x,\theta) \;=\; V^G(x,\theta,\beta_1(x;\theta)).$$

Replacing for the value functions, and rearranging we have that the cutoff solves:

$$\beta_1(x;\theta) = \frac{1+\hat{x}+\hat{x}^2}{1-\hat{x}} \cdot \frac{F^\alpha - F_g^\alpha}{F_g}, \tag{22}$$

Under our maintained primitives $\hat{x} \in (0,1)$, $F > 0$, $F_g > 0$, and $u(c) = c^\alpha$ with $\alpha \in (0,1)$, (22) is well-defined and continuous in x for any $\theta > 0$. In addition, if we assume that the guaranteed annuity pays a (weakly) lower flow than the plain annuity,

$$F > F_g > 0, \tag{23}$$

then $F^\alpha - F_g^\alpha > 0$ and the factor $(F^\alpha - F_g^\alpha)/F_g$ is strictly positive.

To study monotonicity, define

$$f(\hat{x}) \;\equiv\; \frac{1+\hat{x}+\hat{x}^2}{1-\hat{x}}, \qquad \hat{x} \in (0,1).$$

Then

$$f'(\hat{x}) \;=\; \frac{(1+2\hat{x})(1-\hat{x}) + (1+\hat{x}+\hat{x}^2)}{(1-\hat{x})^2} \;=\; \frac{2+2\hat{x}-\hat{x}^2}{(1-\hat{x})^2}.$$

For $\hat{x} \in (0,1)$, the numerator satisfies $2 + 2\hat{x} - \hat{x}^2 > 2 > 0$ and the denominator is positive, so $f'(\hat{x}) > 0$ on $(0,1)$. Since $\hat{x} = \theta x$ with $\theta > 0$, the composite $x \mapsto f(\theta x)$ is strictly increasing and continuous on the support of x. Combining this with (22) and (23), we obtain:

Lemma 3 (Microfoundations for the A–G cutoff)**.** *Under $\hat{x} = \theta x \in (0,1)$, $F > F_g > 0$, and $u(c) = c^\alpha$ with $\alpha \in (0,1)$, the A–G cutoff $\beta_1(x;\theta)$ is given by* (22) *and is continuous and strictly increasing in x for any fixed $\theta > 0$.*

This shows that the monotonicity and continuity of $\beta_1(\cdot;\theta)$ in Assumption (A1) are direct implications of the primitives.

A sufficient condition for the monotonicity of $\beta_2(x;\theta)$ is that $s_1(\beta_i = 0; x) > F_g$. Given that $\frac{\partial V^G}{\partial \beta_i} = (1-x)\beta_i F_g$ and that $\frac{\partial V^N}{\partial \beta_i} > \beta_i(1-x)[s_1(\beta_i) + x s_2(\beta_i)]$ if $s_1(\beta_i = 0; x) > F_g$ then, given that savings are increasing on β_i, we have that $s_1(\beta_i; x) > F_g$. Then $\frac{\partial V^G}{\partial \beta_i}(1-x) = \beta_i F_g < \beta_i(1-x)[s_1(\beta_i) + x s_2(\beta_i)] < \frac{\partial V^N}{\partial \beta_i}$

A simple sufficient condition for (A2). Recall that (A2) requires the images of the cutoff functions $\beta_1(\cdot;\theta)$ and $\beta_2(\cdot;\theta)$ to be disjoint (or made disjoint by restricting the range of x). A convenient and transparent sufficient condition is a *uniform separation* of the two cutoffs in β.

Let $\mathcal{X}$ denote the support of x and assume $\mathcal{X}$ is compact.

Assumption 4 (Uniform separation of cutoffs)**.** *For a given $\theta > 0$, suppose there exists a constant $\delta > 0$ such that*

$$\beta_2(x;\theta) - \beta_1(x;\theta) \;\geq\; \delta \qquad \textit{for all } x \in \mathcal{X}.$$

Lemma 5 (Assumption 4 implies (A2))**.** *Fix $\theta > 0$ and suppose Assumption 4 holds. Then the images $\mathrm{Im}(\beta_1(\cdot;\theta))$ and $\mathrm{Im}(\beta_2(\cdot;\theta))$ are disjoint. In particular, condition* (A2) *holds.*

Proof. Because $\mathcal{X}$ is compact and $\beta_j(\cdot;\theta)$ is continuous for $j = 1,2$, each image $\mathrm{Im}(\beta_j(\cdot;\theta))$ is a compact interval in $\mathbb{R}$. Define

$$\overline{\beta}_1 \equiv \sup_{x \in \mathcal{X}} \beta_1(x;\theta), \qquad \underline{\beta}_2 \equiv \inf_{x \in \mathcal{X}} \beta_2(x;\theta).$$

By Assumption 4,

$$\beta_2(x;\theta) \;\geq\; \beta_1(x;\theta) + \delta \quad \text{for all } x.$$

Taking the infimum over x on the left-hand side and the supremum over x on the right-hand side yields

$$\underline{\beta}_2 \;\geq\; \overline{\beta}_1 + \delta.$$

In particular, $\overline{\beta}_1 < \underline{\beta}_2$. Hence

$$\mathrm{Im}(\beta_1(\cdot;\theta)) \;\subseteq\; (-\infty, \overline{\beta}_1], \qquad \mathrm{Im}(\beta_2(\cdot;\theta)) \;\subseteq\; [\underline{\beta}_2, \infty),$$

and the two images are disjoint because the right endpoint of the first is strictly smaller than the left endpoint of the second. This is exactly the content of (A2). □

5.4 Microfoundations for C1

[WORK IN PROGRESS]

Support of the price state and plausibility of (C1)

We now formalize what we mean by "sufficient support" of the price state r, and we provide simple primitive conditions under which condition (C1) is satisfied. Throughout, recall that r affects payoffs only through the annuity payments $(F(r), F_g(r))$, is observed by the econometrician, and is exogenous: $r \perp (x, \beta, \theta)$.:contentReference[oaicite:0]index=0

Formalizing "sufficient support on a compact interval $\mathcal{R}$". Let $\mathcal{R} \subset \mathbb{R}$ denote the support of r.

Assumption 6 (Rich support of the price state)**.** *There exist real numbers $\underline{r} < \overline{r}$ such that:*

1. *$\mathcal{R} = [\underline{r}, \overline{r}]$ is a non-degenerate compact interval.*
2. *The distribution of r has* full support *on $\mathcal{R}$; that is, for any subinterval $[a, b] \subseteq [\underline{r}, \overline{r}]$ with $a < b$,*
$$\Pr\big(r \in [a, b] \mid x, \beta\big) > 0.$$
Equivalently, the conditional density $f_{r|x,\beta}(r \mid x, \beta)$ is strictly positive on $(\underline{r}, \overline{r})$.

Assumption 6 formalizes "sufficient support" as saying that the price state r can take any value in a whole interval $[\underline{r}, \overline{r}]$ with positive probability.

Primitive conditions for (C1). We want conditions stated in terms of primitives $(F(r), F_g(r))$ and the value functions (V^A, V^G, V^N) under which (C1) is satisfied.

Assumption 7 (Price monotonicity of annuity flows)**.** *The annuity flows respond smoothly and non-degenerately to r:*

1. *$F(r)$ and $F_g(r)$ are continuously differentiable on $\mathcal{R}$.*
2. *$F(r)$ and $F_g(r)$ are strictly monotone in r (either both increasing or both decreasing), and their ratio varies with r:*
$$\frac{d}{dr}\left(\frac{F(r)^\alpha - F_g(r)^\alpha}{F_g(r)}\right) \neq 0 \quad \textit{for all } r \in (\underline{r}, \overline{r}).$$

Assumption 7 says that interest-rate movements generate genuine shifts in the relative attractiveness of the plain and guaranteed annuities: as r changes, the "implicit price" of the guarantee (in units of $F_g(r)$) moves in a strictly monotone way.

Under our microfoundations, the A–G cutoff is explicitly

$$\beta_1(x,r;\theta) = \frac{1+\hat{x}+\hat{x}^2}{1-\hat{x}} \cdot \frac{F(r)^\alpha - F_g(r)^\alpha}{F_g(r)}, \qquad \hat{x} = \theta x,$$

so Assumption 7 immediately implies that for each fixed (x,θ), $r \mapsto \beta_1(x,r;\theta)$ is continuous and strictly monotone on $\mathcal{R}$, and, by Assumption 6, its image is an interval of β-values with non-empty interior.

For the G–N cutoff, recall that $\beta_2(x,r;\theta)$ is defined by

$$V^N(x,\theta,\beta_2(x,r;\theta),W) = V^G(x,r,\theta,\beta_2(x,r;\theta)),$$

where

$$V^G(x,r,\theta,\beta) = (1+\hat{x}+\hat{x}^2)F_g(r)^\alpha + (1-\hat{x})\beta\, F_g(r), \quad \hat{x} = \theta x.$$

[Why does β_2 satisfy the assumptions]