0.1 Model 1

Consider the simplest model of multi-product firms with switching costs, we assume that there are two periods and two products. For example one can think that initially a consumer opens a checking account (product 1) and later on she may take a loan (product 2). We denote the period/product by $t = 1, 2$.

There are J firms, indexed by j.

Consumer problem The per period utility of the consumer is given by:

$$u_{ijt} = \beta_j - \alpha p_{ijt} + \xi_{jt} + \mu_{ij} + \epsilon_{ijt} \tag{1}$$

where β_j represents vertical differentiaion, p_{ijt} is the price charged by firm j in period t, ξ_{jt} is a firm-specific demand shock, μ_{ij} is a persistent consumer-firm match value, and ϵ_{ijt} is an i.i.d. Type-I Extreme Value shock.

The term μ_{ij} is constant across periods for the same consumer-firm pair, creating persistent heterogeneity:

$$\mu_i = (\mu_{ij})_{j=1}^{J} \sim F_\mu, \quad \text{with } \mu_{ij} \text{ drawn once and fixed for both periods} \tag{2}$$

We assume that consumers are myopic, and that in case of switching they incur a switching cost s. Denote by j_t the firm chosen in period t. In period 1 the demand is given by:

$$D_{1j}(p_1) = \int_{\mu_i} \frac{\exp(\delta_{j1} + \mu_{ij})}{\sum_{j'} \exp(\delta_{j'1} + \mu_{ij'})} dF_{\mu_i} \tag{3}$$

where we use the fact that prices in the first period are not consumer-specific since consumers are ex-ante homogenous, firms do not observe μ_i when setting prices.

When setting prices in the second period firms know the consumer's choice in the first period. Hence they set price $p_{j2}(k)$ for consumers who chose firm k in period 1. The demand for consumers who chose firm k in period 1 is given by:

$$D_{2j}(k, p_2; p_1) = \int_{\mu_i} \frac{\exp(\delta_{j2k} + \mu_{ij} - \alpha s \cdot \mathbb{I}(j \neq k))}{\sum_{j'} \exp(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k))} dF_{\mu_i|j_1=k} \tag{4}$$

where $\delta_{j2k} = \beta_j - \alpha p_{j2}(k) + \xi_{j2}$. Where the demand depends on p_1 since it determines the distribution of μ_i among consumers who chose firm k in period 1.

Firm problem In the second period, for each group of consumers who chose firm k in period 1, firms compete by setting prices $p_{j2}(k)$. Firm j chooses $p_{j2}(k)$ according to

$$\pi_{2j}(k; p_1) = \max_p D_{2j}(k, (p, p^*_{-j2}(k)))(p - c_2) \tag{5}$$

where $D_{2j}(k, \cdot)$ is the demand for firm j from consumers who previously bought from k.

In the first period, each firm j chooses p_{j1} to maximize total expected discounted profits. The firm anticipates how its period 1 price affects its period 1 market share and thus the size of its "locked-in" customer base in period 2. The maximization problem is:

$$\max_{p_{j1}} \Pi_j = D_{1j}(p_{j1}, p^*_{-j1})(p_{j1} - c_1) + \sum_{k=1}^{J} D_{1k}(p_{j1}, p^*_{-j1})\pi_{2j}(k; p_1) \tag{6}$$

Note that the second term sums over all possible first-period choices k, weighted by the mass of consumers D_{1k} who made that choice. This captures that by influencing D_{1j} (and rivals' D_{1k}), the firm changes the composition of the market in period 2.

Equilibrium FOC and Invest-Harvest Motive In the case where $\mu_{ij} = 0$ for all i, j (no persistent heterogeneity), there is no selection in period 1, hence $\pi_{2j}(k; p_1) = \pi_{2j}(k)$ independent of p_1. In this case we illustrate the invest-harvest motive more clearly.

The First Order Condition (FOC) with respect to p_{j1} reveals the dynamic incentives:

$$\frac{\partial \Pi_j}{\partial p_{j1}} = \underbrace{\frac{\partial D_{1j}}{\partial p_{j1}}(p_{j1} - c_1) + D_{1j}}_{\text{Period 1 Marginal Profit}} + \sum_{k=1}^{J} \frac{\partial D_{1k}}{\partial p_{j1}} \pi_{2j}(k) = 0 \tag{7}$$

We can decompose the summation. The FOC becomes:

$$\text{MR}_{1j} + \frac{\partial D_{1j}}{\partial p_{j1}} \pi_{2j}(j) + \sum_{k \neq j} \frac{\partial D_{1k}}{\partial p_{j1}} \pi_{2j}(k) = 0 \tag{8}$$

We can group the future effects without symmetry. Define the diversion weights

$$\omega_{jk} \equiv \frac{\frac{\partial D_{1k}}{\partial p_{j1}}}{-\frac{\partial D_{1j}}{\partial p_{j1}}}, \qquad k \neq j. \tag{9}$$

Under standard regularity conditions for differentiated-products demand, $\frac{\partial D_{1j}}{\partial p_{j1}} < 0$ and $\frac{\partial D_{1k}}{\partial p_{j1}} > 0$ for substitutes, implying $\omega_{jk} \geq 0$. Moreover, since $\sum_{k=1}^{J} D_{1k} = 1$, we have $\sum_{k=1}^{J} \frac{\partial D_{1k}}{\partial p_{j1}} = 0$, so $\sum_{k \neq j} \omega_{jk} = 1$.

Using these weights, the FOC can be rewritten as

$$\text{MR}_{1j} + \underbrace{\frac{\partial D_{1j}}{\partial p_{j1}}}_{(-)} \left[\pi_{2j}(j) - \sum_{k \neq j} \omega_{jk}\, \pi_{2j}(k) \right] = 0. \tag{10}$$

The bracketed term is the incremental value of acquiring a marginal period-1 customer: when p_{j1} falls, extra customers are drawn from rivals k in proportions ω_{jk}, and each such customer changes firm j's period-2 profit from the "poaching" level $\pi_{2j}(k)$ to the "incumbent" level $\pi_{2j}(j)$.

Interpretation:

1. **Harvest Motive (Period 2):** In the second period, consumers who chose firm j in period 1 face a switching cost s to leave. This grants firm j market power over its own base, allowing it to charge a higher price (a "rip-off" or "harvest" price) compared to the competitive poaching price. Thus, $\pi_{2j}(j) > \pi_{2j}(k)$ for $k \neq j$.

2. **Invest Motive (Period 1):** Equation (10) shows that the dynamic incentive depends on whether an incumbent customer is more valuable than a poached customer:

$$\pi_{2j}(j) \; > \; \sum_{k \neq j} \omega_{jk} \, \pi_{2j}(k). \tag{11}$$

This condition does <u>not</u> require symmetric firms or identical products; it only requires (i) demand substitution in period 1 so that $\omega_{jk} \geq 0$ and (ii) switching costs (or any state dependence) that make period-2 profits higher when the firm is the incumbent for that consumer. When the condition holds, the bracket in (10) is positive and, since $\frac{\partial D_{1j}}{\partial p_{j1}} < 0$, the entire dynamic term is negative.

To satisfy the FOC $= 0$, the static marginal profit MR_{1j} must be positive. This implies that the firm sets p_{j1} *lower* than the static monopoly price. The firm sacrifices period 1 margins (invests) to build a larger customer base (D_{1j}) from which it can extract higher rents in period 2 (harvest).

This model is essentially the same as Dube et al. (2009), but in a two-period setting.

1 Simulation: Equilibrium Computation (Duopoly)

We specialize to $J = 2$ firms. Because persistent heterogeneity (μ_{ij}) creates selection—consumers who chose firm k in Period 1 have systematically different μ_i draws—we use a simulation-based approach. We draw a panel of N consumers, each with fixed (μ_{i1}, μ_{i2}), and compute equilibrium prices by iterating best responses evaluated on this panel.

1.1 Parameters

- α: price sensitivity
- s: switching cost (monetary; enters utility as $-\alpha s$)
- c_1, c_2: marginal costs in periods 1 and 2
- β_j: firm-specific baseline valuation, $j = 1, 2$
- ξ_{jt}: firm-period demand shocks we assume them to be constant over time
- σ_μ: standard deviation of $\mu_{ij} \sim$ i.i.d. $N(0, \sigma_\mu^2)$
- N: number of simulated consumers

1.2 Step 0: Draw the Consumer Panel

Before solving for prices, draw and <u>fix</u> all random components:

1. For each firm $j = 1, 2$: draw $\xi_j \sim N(0, \sigma_\xi^2)$
2. For each consumer $i = 1, \ldots, N$ and firm $j = 1, 2$: draw $\mu_{ij} \sim N(0, \sigma_\mu^2)$
3. For each consumer i, firm j, and period t: draw $\epsilon_{ijt} \sim$ Type-I Extreme Value

These draws are held fixed throughout the price iteration, so that demand is a smooth (simulated) function of prices.

1.3 Step 1: Period 2 Equilibrium (given Period 1 choices)

For a given vector of Period 1 prices $p_1 = (p_{11}, p_{21})$, each consumer's Period 1 choice partitions the panel into two pools. Define:

$$\mathcal{S}_k(p_1) = \left\{ i : k = \arg\max_j \left(\delta_{j1} + \mu_{ij} + \epsilon_{ij1} \right) \right\}, \qquad k = 1, 2 \tag{12}$$

where $\delta_{j1} = \beta_j - \alpha p_{j1} + \xi_{j1}$.

For consumers in pool $\mathcal{S}_k$, Period 2 demand for firm j at prices $p_2(k) = (p_{12}(k), p_{22}(k))$ is:

$$\hat{D}_{2j}(k, p_2(k)) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} \mathbb{I} \left[j = \arg\max_{j'} \left(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k) + \epsilon_{ij'2} \right) \right] \tag{13}$$

where $\delta_{j2k} = \beta_j - \alpha p_{j2}(k) + \xi_{j2}$. Equivalently, we can integrate out ϵ analytically (using its known Type-I EV distribution) to obtain a smoother demand function:

$$\hat{D}_{2j}^{\text{smooth}}(k, p_2(k)) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} \frac{\exp(\delta_{j2k} + \mu_{ij} - \alpha s \cdot \mathbb{I}(j \neq k))}{\sum_{j'} \exp(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k))} \tag{14}$$

This formulation is differentiable in Period 2 prices and is preferable for the equilibrium computation. Denote the per-consumer Period 2 choice probability (the summand above) by:

$$\hat{D}_{2j}^{i}(k, p_2(k)) \equiv \frac{\exp(\delta_{j2k} + \mu_{ij} - \alpha s \cdot \mathbb{I}(j \neq k))}{\sum_{j'} \exp(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k))} \tag{15}$$

so that the smooth demand with hard partition is simply $\hat{D}_{2j}^{\text{smooth}}(k, p_2(k)) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} \hat{D}_{2j}^{i}(k, p_2(k))$.

Smooth pool assignments: integrating out ϵ_{i1}. The formulation above still relies on the hard partition $\mathcal{S}_k(p_1)$ via arg max, which makes pool memberships—and hence demand—jump discretely as p_1 changes. We can also integrate out ϵ_{i1} analytically, yielding a demand function that is smooth in p_1 as well.

Start from the theoretical Period 2 demand (equation 4), which conditions on $\mu_i \mid j_1 = k$. By Bayes' rule the conditional density of μ_i among consumers who chose firm k is:

$$dF_{\mu_i|j_1=k} = \frac{\Pr(j_1 = k \mid \mu_i)}{D_{1k}(p_1)}\, dF_{\mu_i} \tag{16}$$

Since ϵ_{i1} is Type-I Extreme Value, the conditional probability of choosing firm k given μ_i has the logit form:

$$w_{ik}(p_1) \equiv \Pr(j_1 = k \mid \mu_i) = \frac{\exp(\delta_{k1} + \mu_{ik})}{\sum_{j'} \exp(\delta_{j'1} + \mu_{ij'})} \tag{17}$$

and $D_{1k}(p_1) = \int w_{ik}(p_1)\, dF_{\mu_i}$. Substituting the Bayes' rule expression into equation (4):

$$\begin{aligned} D_{2j}(k, p_2; p_1) &= \int \hat{D}_{2j}^{i}(k, p_2(k)) \cdot \frac{w_{ik}(p_1)}{D_{1k}(p_1)}\, dF_{\mu_i} \\ &= \frac{\int \hat{D}_{2j}^{i}(k, p_2(k)) \cdot w_{ik}(p_1)\, dF_{\mu_i}}{\int w_{ik}(p_1)\, dF_{\mu_i}} \end{aligned} \tag{18}$$

where the second line uses $D_{1k}(p_1) = \int w_{ik}\, dF_{\mu_i}$. Approximating both integrals via sample averages over the N draws of μ_i:

$$\hat{D}_{2j}^{\text{smooth}}(k, p_2(k); p_1) = \frac{\sum_{i=1}^{N} w_{ik}(p_1) \cdot \hat{D}_{2j}^{i}(k, p_2(k))}{\sum_{i=1}^{N} w_{ik}(p_1)} \tag{19}$$

This is equivalent to the hard-partition formula, but with the indicator $\mathbb{I}[i \in \mathcal{S}_k]$ replaced by the smooth weight $w_{ik}(p_1)$, and the sum running over <u>all</u> N consumers rather than only those in $\mathcal{S}_k$. Since w_{ik} is differentiable in p_1, demand is now smooth in <u>both</u> Period 1 and Period 2 prices, which makes the equilibrium computation more robust. The ϵ_{ijt} draws are then only needed in Step 3 for computing realized outcomes.

Remark (selection): Because $w_{ik}(p_1)$ is larger for consumers with high μ_{ik}, the weighted average naturally captures the selection effect: consumers who are more likely to have chosen firm k in Period 1 receive more weight in the Period 2 demand calculation. This selection, combined with the switching cost s, generates stronger lock-in than either force alone.

Algorithm: For each pool $k = 1, 2$:

1. Initialize: $p_{j2}^{(0)}(k) = c_2 + 1$ for $j = 1, 2$

2. Iterate until convergence: for each firm j, search for the price p_{j2} that maximizes

$$\pi_j^{\text{pool}\,k}(p_{j2}) = \hat{D}_{2j}(k, (p_{j2}, p_{-j,2}^{(n)}))(p_{j2} - c_2) \tag{20}$$

   using a one-dimensional solver (e.g., golden section or `fminbnd`).

3. Store equilibrium prices $p_{j2}^*(k; p_1)$ and per-consumer profits $\hat{\pi}_{2j}(k; p_1) = \hat{D}_{2j}(k, p_2^*(k; p_1))(p_{j2}^*(k; p_1) - c_2)$.

Period 2 FOC and markup formula. Because $\hat{D}_{2j}^i$ has the logit form, its derivative with respect to p_{j2} takes the familiar shape:

$$\frac{\partial \hat{D}_{2j}^i}{\partial p_{j2}} = -\alpha\, \hat{D}_{2j}^i \big(1 - \hat{D}_{2j}^i\big) \tag{21}$$

Differentiating the smooth demand (19) with respect to p_{j2} (the weights w_{ik} do not depend on Period 2 prices):

$$\frac{\partial \hat{D}_{2j}^{\text{smooth}}}{\partial p_{j2}} = \frac{\sum_{i=1}^N w_{ik}\,(-\alpha)\, \hat{D}_{2j}^i (1 - \hat{D}_{2j}^i)}{\sum_{i=1}^N w_{ik}} \tag{22}$$

The FOC for the Period 2 profit $\hat{D}_{2j}^{\text{smooth}}(p_{j2} - c_2)$ is:

$$\frac{\partial \hat{D}_{2j}^{\text{smooth}}}{\partial p_{j2}}(p_{j2} - c_2) + \hat{D}_{2j}^{\text{smooth}} = 0 \tag{23}$$

Substituting and noting that the denominators $\sum_i w_{ik}$ cancel:

$$-\alpha(p_{j2} - c_2)\sum_i w_{ik}\, \hat{D}_{2j}^i (1 - \hat{D}_{2j}^i) + \sum_i w_{ik}\, \hat{D}_{2j}^i = 0 \tag{24}$$

Solving for the markup:

$$p_{j2}(k) - c_2 = \frac{1}{\alpha} \cdot \frac{\sum_i w_{ik}\, \hat{D}_{2j}^i}{\sum_i w_{ik}\, \hat{D}_{2j}^i (1 - \hat{D}_{2j}^i)} = \frac{1}{\alpha\, \bar{\sigma}_{jk}} \tag{25}$$

where

$$\bar{\sigma}_{jk} \equiv \frac{\sum_i w_{ik}\, \hat{D}^i_{2j}\,(1-\hat{D}^i_{2j})}{\sum_i w_{ik}\, \hat{D}^i_{2j}} \tag{26}$$

is a weighted average of $(1-\hat{D}^i_{2j})$, with weights proportional to $w_{ik} \cdot \hat{D}^i_{2j}$—i.e., consumers who are both likely to be in pool k and likely to buy from firm j receive the most weight.

This generalizes the standard logit markup $p = c + 1/[\alpha(1-D)]$. In the plain logit case ($\mu_{ij} = 0$ for all i, j), all consumers are identical so $\hat{D}^i_{2j} = D_{2j}$ for every i, and $\bar{\sigma}_{jk} = 1 - D_{2j}$, recovering the standard formula. With heterogeneity ($\sigma_\mu > 0$), the markup reflects the composition of the pool: because selected consumers have high μ_{ik} (and hence high $\hat{D}^i_{2k}$, low $\hat{D}^i_{2j}$ for $j \neq k$), the incumbent's $\bar{\sigma}_{jk}$ when $j = k$ tends to be smaller than $1 - D_{2j}$, leading to a higher markup than the plain logit formula would predict.

Note that (25) is an implicit equation since $\hat{D}^i_{2j}$ itself depends on p_{j2} through δ_{j2k}. It can be solved via fixed-point iteration (updating p_{j2} from the right-hand side given current $\hat{D}^i_{2j}$) or used as a one-dimensional root-finding problem, which is faster than unconstrained optimization.

1.4 Step 2: Period 1 Equilibrium

The key complication relative to the plain logit model is that the Period 2 equilibrium—and hence $\hat{\pi}_{2j}(k)$—depends on p_1 through the weights $w_{ik}(p_1)$, which determine the composition of the consumer pools. Using the smooth formulation from Step 1, the firm's total profit is:

$$\hat{\Pi}_j(p_1) = \hat{D}_{1j}(p_1)(p_{j1} - c_1) + \sum_{k=1}^{2} \hat{D}_{1k}(p_1)\, \hat{\pi}_{2j}(k; p_1) \tag{27}$$

where $\hat{D}_{1k}(p_1) = \frac{1}{N}\sum_{i=1}^{N} w_{ik}(p_1)$ and $\hat{\pi}_{2j}(k; p_1)$ is the Period 2 equilibrium profit of firm j from pool k, computed using the smooth demand (19). Both terms are smooth functions of p_1, but $\hat{\pi}_{2j}(k; p_1)$ must be re-solved for each candidate p_1.

Algorithm (nested fixed point):

1. Initialize: $p^{(0)}_{j1} = c_1 + 1$ for $j = 1, 2$

2. Iterate until convergence: for each firm j, search for the p_{j1} that maximizes $\hat{\Pi}_j(p_{j1}, p^{(n)}_{-j,1})$. Each evaluation of $\hat{\Pi}_j$ requires:

   (a) Recompute pools $\mathcal{S}_k(p_{j1}, p^{(n)}_{-j,1})$ given the candidate p_{j1}

   (b) Solve the Period 2 equilibrium for each pool (Step 1)

   (c) Sum Period 1 and Period 2 profits

   Use a one-dimensional solver for firm j's best response.

3. Store equilibrium prices p^*_{j1}

Remark (computational cost): The nested structure (Period 2 equilibrium inside Period 1 search) is expensive because each evaluation of $\hat{\Pi}_j(p_1)$ requires a full Period 2 equilibrium solve. The smooth pool assignments from Step 1 already ensure that $\hat{\Pi}_j$ is differentiable in p_1, which makes the nested best-response iteration more robust. A further speedup is to **solve all FOCs simultaneously**: stack all $2 + 2 \times J$ prices into a single vector $\mathbf{p} = (p_{11}, p_{21}, p_{12}(1), p_{22}(1), p_{12}(2), p_{22}(2)) \in \mathbb{R}^6$ and solve the system of 6 FOCs simultaneously using a nonlinear equation solver (`fsolve`). This eliminates the inner loop entirely: each Jacobian evaluation is a single pass over N consumers, and the solver typically converges in < 20 Newton steps. Combined with the smooth formulation (19), this yields a single smooth 6×6 system that is typically orders of magnitude faster than the nested approach.

1.5 Step 3: Compute Observables

Given equilibrium prices $p_1^*, p_2^*(k; p_1^*)$ and the consumer panel, compute:

- Market shares: $\hat{D}_{1j} = |\mathcal{S}_j|/N, \quad \hat{D}_{2j} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}(j_2^i = j)$
- Switching rate: $\hat{\rho} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{I}(j_1^i \neq j_2^i)$
- Transition matrix: $\hat{T}_{jk} = \Pr(j_2 = k \mid j_1 = j) = \frac{|\{i : j_1^i = j, j_2^i = k\}|}{|\mathcal{S}_j|}$
- Firm profits: $\hat{\Pi}_j = \hat{D}_{1j}(p_{j1}^* - c_1) + \sum_{k=1}^{2} \hat{D}_{1k}\, \hat{\pi}_{2j}(k)$
- Period 2 price premium: $\Delta p_2 = p_{j2}^*(j; p_1^*) - p_{j2}^*(k; p_1^*)$ for $k \neq j$ (the harvest markup)

1.6 Verification

The numerical solution should satisfy:

- No firm can profitably deviate in either period (check via grid search around equilibrium prices)
- When $s = 0$, $\sigma_\mu = 0$, and $\beta_1 = \beta_2$, $\xi_{jt} = 0$: switching rate $\approx 1/2$ and $p_1^* = p_2^*$ (static symmetric logit duopoly)
- When $\sigma_\mu = 0$: the model reduces to plain logit and prices should match the closed-form logit markup results
- When $s \to \infty$: switching rate $\to 0$
- Increasing σ_μ (holding s fixed) should <u>reduce</u> the switching rate, since persistent preferences reinforce switching costs
- Period 2 incumbent price > entrant price: $p_{j2}^*(j) > p_{j2}^*(k \neq j)$