0.1 Model 2

A possibility is to extend the Sharpe-von Thadden model to add switching costs. The model abstracts away from product differentiation.

We adapt the proof ot

Let H_i^γ, $\gamma \in \{S, F\}$, denote the cumulative distribution function of the equilibrium mixed strategy of the inside bank given its information γ, and let H_o denote the c.d.f. of the equilibrium mixed strategy of the outside bank. As usual, H_i^γ and H_o are weakly monotone and continuous from the right, i.e., $H(\hat{r}) = \Pr(r \leq \hat{r})$ for each of the three mixed strategies. Define $H(r^-) = \lim_{t\uparrow r} H(t)$. Finally, let

Define

$$\ell_i^\gamma = \inf\{r \mid H_i^\gamma(r) > 0\}, \qquad \gamma \in \{S, F\}, \tag{A1}$$
$$u_i^\gamma = \sup\{r \mid H_i^\gamma(r) < 1\}, \qquad \gamma \in \{S, F\}, \tag{A2}$$
$$\ell_o = \inf\{r \mid H_o(r) > 0\}, \tag{A3}$$
$$u_o = \sup\{r \mid H_o(r) < 1\}. \tag{A4}$$

Without loss of generality, we restrict attention to interest rates in $[0, X_2/I_2 - 1]$.

The expected profits from quoting interest rate r are

$$P_i^\gamma(r) = \big(1 - H_o((r-\lambda)^-)\big)\big[p(\gamma)(1+r) - (1+\bar{r})\big], \qquad \gamma \in \{S, F\}, \tag{A5}$$
$$\begin{aligned} P_o(r) = {} & p\big(1 - H_i^S(r+\lambda)\big)\big[p(S)(1+r) - (1+\bar{r})\big] \\ & + (1-p)\big(1 - H_i^F(r+\lambda)\big)\big[p(F)(1+r) - (1+\bar{r})\big]. \end{aligned} \tag{A6}$$

For what follows it is useful to define $\hat{\ell}_o = \ell_o + \lambda$, $\hat{u}_o = u_o + \lambda$ and $\hat{H}_o$ the c.d.f. of the interest rates offered by the outside bank plus λ.

Step 1 $\ell_i^\gamma \geq r_\gamma$ for $\gamma \in \{S, F\}$.

Proof. Otherwise profits would be negative.

Step 2 $\ell_o \geq r_p$.

Proof. we know $r_f > r_p$, using step 1 $\ell_i^f \geq r_f > r_p$ hence any offer $r < r_p$ attracts at best both groups and at worse only the failures. Given that the cost is at best the pooling cost, the outside bank will not make offers lower than the pooling cost.

Step 3 $\ell_i^S \geq r_p + \lambda$.

Proof. Follows from Step 2. Any offer, by the inside bank, lower than that could be raised slightly without decreasing the probability of winning.

Step 4 $\hat{u}_o \geq u_i^S$.

Proof. Suppose $\hat{u}_o < u_i^S$, then the inside bank makes zero expected profits on all offers $r(s) \in (\hat{u}_0, u_i^s]$, however by step 3 the inside bank makes strictly positive profits on the S-firm.

Step 5 H_i^S is continuous on $[\ell_i^S, u_i^S)$.

Proof. Suppose that there is a $\hat{r} \in [\ell_i^S, u_i^S)$ at which H_i^S is discontinuous, i.e., with

$$H_i^S(\hat{r}^-) < H_i^S(\hat{r}).$$

Then, by Eq. (A.6), $P_o(\hat{r}^-) > P_o(\hat{r})$, because $p(S)(1+r) - (1+\bar{r}) > 0$ on $[\ell_i^S, u_i^S)$ by Step 3[1].

By the right-hand continuity of H_i^γ, $\gamma \in \{S, F\}$, there is an $\varepsilon > 0$ such that $H_o(\hat{r}^-) = H_o(r)$ is constant on $[\hat{r}, \hat{r}+\varepsilon]$. Therefore, P_i^S is continuous at $\hat{r}$ and strictly increasing on $[\hat{r}, \hat{r}+\varepsilon]$. Hence, H_i^S can have no mass on $[\hat{r}, \hat{r}+\varepsilon]$, which implies that $H_i^S(\hat{r}^-) = H_i^S(\hat{r})$. Contradiction.

Note that the proof of Step 5 does not apply to H_i^F, because we do not know whether the inside bank makes strictly positive profits on the F-firm.

Step 6. $u_i^S \geq \ell_i^F$.

Proof. Suppose that $u_i^S < \ell_i^F$. This implies that the inside bank never makes an offer $r \in (u_i^S, \ell_i^F)$.

(a) Suppose that $u_i^S < \hat{u}_o$. Then $\hat{H}_o$ can have no mass on $[u_i^S, \ell_i^F)$, because for every offer $r \in [u_i^S, \ell_i^F)$ the offer $\frac{1}{2}(r + \ell_i^F)$ would be strictly better for the outside banks. Then the (positive) mass of $\hat{H}_o$ on $[u_i^S, u_o]$ lies on $[\ell_i^F, \hat{u}_o]$. In particular, $\hat{H}_o$ is continuous at u_i^S[2].

Consider the following deviation from H_i^S: let $\delta > 0$ and $\varepsilon > 0$ be given and small. Let M_ε be the mass of H_i^S on $[u_i^S - \varepsilon, u_i^S]$. The deviation strategy is identical to H_i^S on $[\ell_i^S, u_i^S - \varepsilon)$, has zero mass on $[u_i^S - \varepsilon, \ell_i^F - \delta)$ and point mass M_ε on $\ell_i^F - \delta$. The expected net gain (given $\gamma = S$) from this deviation is not smaller than

$$M_\varepsilon \Big[(1 - \hat{H}_o(u_i^S))(\ell_i^F - u_i^S - \delta) - \big(\hat{H}_o(u_i^S) - \hat{H}_o(u_i^S - \varepsilon)\big) u_i^S \Big]. \tag{A.7}$$

The first of the two terms in Eq. (A.7) (which corresponds to the total gain from the deviation) is strictly positive for δ sufficiently small. The second term (corresponding to the total loss from the deviation) tends to 0 for $\varepsilon \to 0$ by the continuity of H_o at u_i^S. Hence, the deviation is strictly profitable for δ and ε small enough.

(b) Suppose that $u_i^S = \hat{u}_o$. Consider the following deviation from $\hat{H}_o$: let $\delta > 0$ and $\varepsilon > 0$ be given and small. Let N_ε be the mass of $\hat{H}_o$ on $[\hat{u}_o - \varepsilon, \hat{u}_o]$. Move all mass of $[\hat{u}_o - \varepsilon, \hat{u}_o)$ to $\ell_i^F - \delta$. Then the expected net gain from this deviation is not smaller than

$$N_\varepsilon \Big[\underbrace{(1-p)[(\ell_i^F - \delta) - (u_o + \lambda)]}_{Gain from \gamma = F} - p\big(H_i^S(\hat{u}_o) - H_i^S(\hat{u}_o - \varepsilon)\big)\big(p(S)(1+\hat{u}_o) - (1+\bar{r})\big) \Big]$$

where the second term now tends to 0 for $\varepsilon \to 0$ by Step 5.

Step 7. $u_i^F \leq \hat{u}_o$.

Proof. Suppose that $u_i^F > \hat{u}_o$. Then, for any offer $r \in (\hat{u}_o, u_i^F]$, the inside bank loses the F-firm with probability 1 (since $r > u_o + \lambda \geq r_o + \lambda$). Consequently, the inside bank makes zero profit on these offers. For this to be an equilibrium strategy, the inside bank must make zero expected profit

[1] Given that in step 3 we have that the expected profits of the incumbent when selling to S are strictly positive, the outside bank can switch some probability from $\hat{r}$ to $\hat{r} - \epsilon$ and increase their profits.

[2] The cdf will be flat on the $[u_i^S, \ell_i^F)$ interval.

on the F-firm everywhere in its support, which implies its lower bound must be the break-even rate, $\ell_i^F = r_F$[3].

However, this leads to a contradiction because the inside bank has a strategy available that guarantees strictly positive profits on the F-firm. Specifically, from Step 2 we know $\ell_o \geq r_p$. Thus, the outside bank never offers a rate below r_p. The inside bank can offer a fixed rate $r^* = r_F + \varepsilon$. Provided that $\varepsilon > 0$ is small enough such that $r_F + \varepsilon < r_p + \lambda$ (which is possible under the assumption that switching costs are non-trivial, i.e., $r_F < r_p + \lambda$), we have $r^* < \ell_o + \lambda$. By offering r^*, the inside bank retains the F-firm with probability 1 against any outside offer. Since the margin ε is positive, the expected profit is strictly positive. Thus, the condition $P_i^F = 0$ is impossible, and the assumption $u_i^F > \hat{u}_o$ must be false.

Step 8. $u_i^F = r_F + \lambda$.

Proof. Clearly, $u_i^F \geq r_F + \lambda$.[4] Suppose that $u_i^F - \lambda > r_F$. Since the outside bank can obtain strictly positive expected profits by choosing $r = \frac{1}{2}(r_F + (u_i^F - \lambda))$, it must make strictly positive profits also with H_o. By Step 7, $\hat{u}_o \geq u_i^F$ and given that $u_i^F \geq r_F + \lambda$ we have: $\hat{u}_o \geq u_i^F > r_F + \lambda \implies u_o > r_F$; hence, also H_i^F must make strictly positive expected profits.[5]

By Steps 4 and 7[6]:

$$P_o(u_o) = 0 \;\Rightarrow\; {}^{7}H_o(u_o^-) = 1 \;\Rightarrow\; {}^{8}P_i^S(\hat{u}_o) = P_i^F(\hat{u}_o) = 0 \Rightarrow {}^{9}H_i^S(\hat{u}_o^-) = H_i^F(\hat{u}_o^-) = 1 \;\Rightarrow\; {}^{10}P_o(u_o^-) = 0,$$

which is a contradiction to the finding that H_o makes strictly positive expected profits. Step 8 implies that $r_i(F) = r_F + \lambda$ with probability 1; in particular, the inside bank makes zero expected profits on the F-firm.

[3] If $\ell_i^F < r_F$ the firm makes losses and if $\ell_i^F > r_F$ given that the bank makes zero profits on the F-firms it means that it never lends, meaning that $\ell_i^F > \hat{u}_o$ but from step 2 $l_o \geq r_p$ hence the inside bank could offer r_p and make a profit on the F-firms.

[4] this proof is not correct, we can prove $u_i^F \geq r_F$ because the insider will not loose money on the f-firms, but not that also it covers the switching costs.Given that $\ell_o \geq r_F$, we have that $\hat{\ell}_o \geq r_F + \lambda$ hence for any interest rate lower than $\hat{\ell}_o$ there is a higher interest rate such that the inside bank always sells and makes a higher profit.

[5] Since the insider knows the type can always bid in $u_o - \varepsilon$ and win in some cases a positive profit.

[6] By Steps 4 ($\hat{u}_o \geq u_i^S$.) and 7 ($u_i^F \leq \hat{u}_o$), $\hat{u}_o$ is higher than any bid made by the insider, hence the outsider bidding u_o effectively matches the insider's highest possible effective price $\hat{u}_o$. Given that profits at u_o are 0 , they must be 0 on the whole support. Note that strategies have no atoms at the top due to undercutting.

[10] Given that profits are strictly positive for H_o and that at the top there are ostensibly profits, there is no probability of playing u_o (atom), by indifference among elements in the support.

[10] Given that the outsider always plays lower than u_o, when the insider plays $\hat{u}_o$ (effectively $u_o + \lambda$), they never retain the customer (since $r_i \geq r_o + \lambda$ always holds), hence profits are 0.

[10] Given that with $\hat{u}_o$ the insider makes zero profits and that they make positive profits with other bids, $\hat{u}_o$ cannot be in the active support of the insider.

[10] The outsider makes zero profits just below u_o because they win no customers (the insider has already finished bidding at $\hat{u}_o$).

Step 9. $u_o = r_F$ and $u_i^S = r_F + \lambda$.

Proof. First, we show $u_o \leq r_F$. Suppose $u_o > r_F$. Then the outsider makes strictly positive profits on F-firms at the top of the distribution. As shown in the proof of Step 8, this leads to a contradiction (specifically, it implies $P_o(u_o^-) = 0$ while equilibrium profits are positive). Thus, $u_o \leq r_F$.

Second, we show $u_o \geq r_F$. Suppose $u_o < r_F$. At the upper bound u_o, the outsider wins no S-firms (since $u_i^S \leq \hat{u}_o$, meaning the insider always retains S-firms against the bid u_o). Therefore, the only firms the outsider can possibly win at u_o are F-firms. However, since $u_o < r_F$, winning an F-firm yields a strictly negative profit ($u_o - r_F < 0$). Thus, the expected profit at u_o is non-positive. Since the outside bank can guarantee zero profit by not participating, and equilibrium profits must be constant in the support, we cannot have an upper bound $u_o < r_F$ that yields losses. Thus, $u_o \geq r_F$.

Combining these results, we have $u_o = r_F$. Finally, from Step 4 we have $u_i^S \leq u_o + \lambda$. In equilibrium, the supports must share the same effective upper bound (otherwise the player with the higher bound would lower it to increase winning probability without margin loss). Therefore, $u_i^S = u_o + \lambda = r_F + \lambda$.

Step 9.original $u_o = u_i^S = r_F$.

Proof. Suppose that $u_o > r_F$. Then choosing $r_i(F) = \frac{1}{2}(r_F + u_o)$ with probability 1 would yield the inside bank strictly positive expected profits on the F-firm[11].

The equality for u_i^S follows from Steps 4 ($u_o \geq u_i^s$) and 6 ($u_i^s \geq \ell_i^F$), which imply that $u_o \geq u_i^s \geq \ell_i^F$. We can use the part 1 of this same step to replace u_o with r_F to get $r_F \geq u_i^s \geq \ell_i^F$, moreover from step 8 we know that the F-strategy is only a point of mass: $\ell_i^F = u_i^F = r_F$ hence $u_i^s = r_F$.

Step 10.original The outside bank makes zero expected profits.

Proof. By Steps 8 and 9[12], Eq. (A.6) simplifies to

$$P_o(r) = p\big(1 - H_i^S(r)\big)\big[p(S)(1+r) - (1+\bar{r})\big] + (1-p)\big[p(F)(1+r) - (1+\bar{r})\big], \tag{A.8}$$

on $[\ell_o, u_o)$. By Step 5, P_o is continuous on $[\ell_o, u_o)$, and by Step 9 we have $P_o(u_o^-) = 0$.

Step 11.original $\ell_o = \ell_i^S = r_p$.

Proof. It is impossible that $\ell_o > \ell_i^S$, because then the inside bank would make strictly higher profits if it placed the mass of $[\ell_i^S, \ell_o]$ on ℓ_o. By a similar argument for the outside bank, $\ell_o < \ell_i^S$ is impossible. Finally, if $\ell_o > r_p$, the outside bank would make strictly positive expected profits, contradicting Step 10.

[11] In step 8 was shown that it makes zero profit on the F-firm

[12] When playing $u_o = r_F$ the outside bank receives the F-firms and makes zero profits, hence it makes zero profits in the whole support.