0.1 Model 1

Consider the simplest model of multi-product firms with switching costs, we assume that there are two periods and two products. For example one can think that initially a consumer opens a checking account (product 1) and later on she may take a loan (product 2). We denote the period/product by $t = 1, 2$.

There are J firms, indexed by j.

Consumer problem The per period utility of the consumer is given by:

$$u_{ijt} = \beta_j - \alpha p_{ijt} + \xi_{jt} + \mu_{ij} + \epsilon_{ijt} \tag{1}$$

where β_j represents vertical differentiaion, p_{ijt} is the price charged by firm j in period t, ξ_{jt} is a firm-specific demand shock, μ_{ij} is a persistent consumer-firm match value, and ϵ_{ijt} is an i.i.d. Type-I Extreme Value shock.

The term μ_{ij} is constant across periods for the same consumer-firm pair, creating persistent heterogeneity:

$$\mu_i = (\mu_{ij})_{j=1}^{J} \sim F_\mu, \quad \text{with } \mu_{ij} \text{ drawn once and fixed for both periods} \tag{2}$$

We assume that consumers are myopic, and that in case of switching they incur a switching cost s. Denote by j_t the firm chosen in period t. In period 1 the demand is given by:

$$D_{1j}(p_1) = \int_{\mu_i} \frac{\exp(\delta_{j1} + \mu_{ij})}{\sum_{j'} \exp(\delta_{j'1} + \mu_{ij'})} dF_{\mu_i} \tag{3}$$

where we use the fact that prices in the first period are not consumer-specific since consumers are ex-ante homogenous, firms do not observe μ_i when setting prices.

When setting prices in the second period firms know the consumer's choice in the first period. Hence they set price $p_{j2}(k)$ for consumers who chose firm k in period 1. The demand for consumers who chose firm k in period 1 is given by:

$$D_{2j}(k, p_2; p_1) = \int_{\mu_i} \frac{\exp(\delta_{j2k} + \mu_{ij} - \alpha s \cdot \mathbb{I}(j \neq k))}{\sum_{j'} \exp(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k))} dF_{\mu_i | j_1 = k} \tag{4}$$

where $\delta_{j2k} = \beta_j - \alpha p_{j2}(k) + \xi_{j2}$. Where the demand depends on p_1 since it determines the distribution of μ_i among consumers who chose firm k in period 1.

Firm problem In the second period, for each group of consumers who chose firm k in period 1, firms compete by setting prices $p_{j2}(k)$. Firm j chooses $p_{j2}(k)$ according to

$$\pi_{2j}(k; p_1) = \max_p D_{2j}(k, (p, p^*_{-j2}(k)))(p - c_2) \tag{5}$$

where $D_{2j}(k, \cdot)$ is the demand for firm j from consumers who previously bought from k.

In the first period, each firm j chooses p_{j1} to maximize total expected discounted profits. The firm anticipates how its period 1 price affects its period 1 market share and thus the size of its "locked-in" customer base in period 2. The maximization problem is:

$$\max_{p_{j1}} \Pi_j = D_{1j}(p_{j1}, p_{-j1}^*)(p_{j1} - c_1) + \sum_{k=1}^{J} D_{1k}(p_{j1}, p_{-j1}^*)\pi_{2j}(k; p_1) \tag{6}$$

Note that the second term sums over all possible first-period choices k, weighted by the mass of consumers D_{1k} who made that choice. This captures that by influencing D_{1j} (and rivals' D_{1k}), the firm changes the composition of the market in period 2.

Equilibrium FOC and Invest-Harvest Motive In the case where $\mu_{ij} = 0$ for all i, j (no persistent heterogeneity), there is no selection in period 1, hence $\pi_{2j}(k; p_1) = \pi_{2j}(k)$ independent of p_1. In this case we illustrate the invest-harvest motive more clearly.

The First Order Condition (FOC) with respect to p_{j1} reveals the dynamic incentives:

$$\frac{\partial \Pi_j}{\partial p_{j1}} = \underbrace{\frac{\partial D_{1j}}{\partial p_{j1}}(p_{j1} - c_1) + D_{1j}}_{\text{Period 1 Marginal Profit}} + \sum_{k=1}^{J} \frac{\partial D_{1k}}{\partial p_{j1}}\pi_{2j}(k) = 0 \tag{7}$$

We can decompose the summation. The FOC becomes:

$$\text{MR}_{1j} + \frac{\partial D_{1j}}{\partial p_{j1}}\pi_{2j}(j) + \sum_{k \neq j} \frac{\partial D_{1k}}{\partial p_{j1}}\pi_{2j}(k) = 0 \tag{8}$$

We can group the future effects without symmetry. Define the diversion weights

$$\omega_{jk} \equiv \frac{\frac{\partial D_{1k}}{\partial p_{j1}}}{-\frac{\partial D_{1j}}{\partial p_{j1}}}, \qquad k \neq j. \tag{9}$$

Under standard regularity conditions for differentiated-products demand, $\frac{\partial D_{1j}}{\partial p_{j1}} < 0$ and $\frac{\partial D_{1k}}{\partial p_{j1}} > 0$ for substitutes, implying $\omega_{jk} \geq 0$. Moreover, since $\sum_{k=1}^{J} D_{1k} = 1$, we have $\sum_{k=1}^{J} \frac{\partial D_{1k}}{\partial p_{j1}} = 0$, so $\sum_{k \neq j} \omega_{jk} = 1$.

Using these weights, the FOC can be rewritten as

$$\text{MR}_{1j} + \underbrace{\frac{\partial D_{1j}}{\partial p_{j1}}}_{(-)} \left[\pi_{2j}(j) - \sum_{k \neq j} \omega_{jk}\, \pi_{2j}(k) \right] = 0. \tag{10}$$

The bracketed term is the incremental value of acquiring a marginal period-1 customer: when p_{j1} falls, extra customers are drawn from rivals k in proportions ω_{jk}, and each such customer changes firm j's period-2 profit from the "poaching" level $\pi_{2j}(k)$ to the "incumbent" level $\pi_{2j}(j)$.

Interpretation:

1. **Harvest Motive (Period 2):** In the second period, consumers who chose firm j in period 1 face a switching cost s to leave. This grants firm j market power over its own base, allowing it to charge a higher price (a "rip-off" or "harvest" price) compared to the competitive poaching price. Thus, $\pi_{2j}(j) > \pi_{2j}(k)$ for $k \neq j$.

2. **Invest Motive (Period 1):** Equation (10) shows that the dynamic incentive depends on whether an incumbent customer is more valuable than a poached customer:

$$\pi_{2j}(j) \; > \; \sum_{k \neq j} \omega_{jk} \, \pi_{2j}(k). \tag{11}$$

This condition does <u>not</u> require symmetric firms or identical products; it only requires (i) demand substitution in period 1 so that $\omega_{jk} \geq 0$ and (ii) switching costs (or any state dependence) that make period-2 profits higher when the firm is the incumbent for that consumer. When the condition holds, the bracket in (10) is positive and, since $\frac{\partial D_{1j}}{\partial p_{j1}} < 0$, the entire dynamic term is negative.

To satisfy the FOC $= 0$, the static marginal profit MR_{1j} must be positive. This implies that the firm sets p_{j1} *lower* than the static monopoly price. The firm sacrifices period 1 margins (invests) to build a larger customer base (D_{1j}) from which it can extract higher rents in period 2 (harvest).

This model is essentially the same as Dube et al. (2009), but in a two-period setting.

1 Simulation: Equilibrium Computation (Duopoly)

We specialize to $J = 2$ firms. Because persistent heterogeneity (μ_{ij}) creates selection—consumers who chose firm k in Period 1 have systematically different μ_i draws—we use a simulation-based approach. We draw a panel of N consumers, each with fixed (μ_{i1}, μ_{i2}), and compute equilibrium prices by iterating best responses evaluated on this panel.

1.1 Parameters

- α: price sensitivity

- s: switching cost (monetary; enters utility as $-\alpha s$)

- c_1, c_2: marginal costs in periods 1 and 2

- β_j: firm-specific baseline valuation, $j = 1, 2$

- ξ_{jt}: firm-period demand shocks we assume them to be constant over time

- σ_μ: standard deviation of $\mu_{ij} \sim$ i.i.d. $N(0, \sigma_\mu^2)$

- N: number of simulated consumers

1.2 Step 0: Draw the Consumer Panel

Before solving for prices, draw and <u>fix</u> all random components:

1. For each firm $j = 1, 2$: draw $\xi_j \sim N(0, \sigma_\xi^2)$

2. For each consumer $i = 1, \ldots, N$ and firm $j = 1, 2$: draw $\mu_{ij} \sim N(0, \sigma_\mu^2)$

3. For each consumer i, firm j, and period t: draw $\epsilon_{ijt} \sim$ Type-I Extreme Value

These draws are held fixed throughout the price iteration, so that demand is a smooth (simulated) function of prices.

1.3 Step 1: Period 2 Equilibrium (given Period 1 choices)

For a given vector of Period 1 prices $p_1 = (p_{11}, p_{21})$, each consumer's Period 1 choice partitions the panel into two pools. Define:

$$\mathcal{S}_k(p_1) = \left\{ i : k = \arg\max_j (\delta_{j1} + \mu_{ij} + \epsilon_{ij1}) \right\}, \qquad k = 1, 2 \tag{12}$$

where $\delta_{j1} = \beta_j - \alpha p_{j1} + \xi_{j1}$.

For consumers in pool $\mathcal{S}_k$, Period 2 demand for firm j at prices $p_2(k) = (p_{12}(k), p_{22}(k))$ is:

$$\hat{D}_{2j}(k, p_2(k)) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} \mathbb{I}\left[j = \arg\max_{j'} (\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k) + \epsilon_{ij'2}) \right] \tag{13}$$

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where $\delta_{j2k} = \beta_j - \alpha p_{j2}(k) + \xi_{j2}$. Equivalently, we can integrate out ϵ analytically (using its known Type-I EV distribution) to obtain a smoother demand function:

$$\hat{D}_{2j}^{\text{smooth}}(k, p_2(k)) = \frac{1}{|\mathcal{S}_k|} \sum_{i \in \mathcal{S}_k} \frac{\exp(\delta_{j2k} + \mu_{ij} - \alpha s \cdot \mathbb{I}(j \neq k))}{\sum_{j'} \exp(\delta_{j'2k} + \mu_{ij'} - \alpha s \cdot \mathbb{I}(j' \neq k))} \quad (14)$$

This formulation is differentiable in prices and is preferable for the equilibrium computation. The ϵ_{ijt} draws are then only needed in Step 3 for computing realized outcomes.

Remark: Because $\mathcal{S}_k$ is a <u>selected</u> sample, consumers in $\mathcal{S}_k$ tend to have high μ_{ik}. This selection, combined with the switching cost s, generates stronger lock-in than either force alone.

Algorithm: For each pool $k = 1, 2$:

1. Initialize: $p_{j2}^{(0)}(k) = c_2 + 1$ for $j = 1, 2$

2. Iterate until convergence: for each firm j, search for the price p_{j2} that maximizes

$$\pi_j^{\text{pool } k}(p_{j2}) = \hat{D}_{2j}(k, (p_{j2}, p_{-j,2}^{(n)}))(p_{j2} - c_2) \quad (15)$$

   using a one-dimensional solver (e.g., golden section or `fminbnd`).

3. Store equilibrium prices $p_{j2}^*(k; p_1)$ and per-consumer profits $\hat{\pi}_{2j}(k; p_1) = \hat{D}_{2j}(k, p_2^*(k; p_1))(p_{j2}^*(k; p_1) - c_2)$.

Note: the standard logit markup formula $p = c + 1/[\alpha(1 - D)]$ does <u>not</u> apply here because demand is a mixed logit (integrated over the selected μ distribution), so we use numerical optimization instead.

1.4 Step 2: Period 1 Equilibrium

The key complication relative to the plain logit model is that the Period 2 equilibrium—and hence $\hat{\pi}_{2j}(k)$—depends on p_1 through the composition of the pools $\mathcal{S}_k(p_1)$. The firm's total profit is:

$$\hat{\Pi}_j(p_1) = \frac{|\mathcal{S}_j(p_1)|}{N}(p_{j1} - c_1) + \sum_{k=1}^{2} \frac{|\mathcal{S}_k(p_1)|}{N} \hat{\pi}_{2j}(k; p_1) \quad (16)$$

where $\hat{\pi}_{2j}(k; p_1)$ is the Period 2 equilibrium profit of firm j from pool k, which must be <u>re-solved</u> for each candidate p_1.

Algorithm (nested fixed point):

1. Initialize: $p_{j1}^{(0)} = c_1 + 1$ for $j = 1, 2$

2. Iterate until convergence: for each firm j, search for the p_{j1} that maximizes $\hat{\Pi}_j(p_{j1}, p_{-j,1}^{(n)})$. Each evaluation of $\hat{\Pi}_j$ requires:

   (a) Recompute pools $\mathcal{S}_k(p_{j1}, p_{-j,1}^{(n)})$ given the candidate p_{j1}

   (b) Solve the Period 2 equilibrium for each pool (Step 1)

   (c) Sum Period 1 and Period 2 profits

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Use a one-dimensional solver for firm j's best response.

3. Store equilibrium prices p_{j1}^*

Remark (computational cost): The nested structure (Period 2 equilibrium inside Period 1 search) is expensive because each evaluation of $\hat{\Pi}_j(p_1)$ requires a full Period 2 equilibrium solve. Two modifications can dramatically reduce the cost:

1. **Smooth pool assignments.** The hard partition $\mathcal{S}_k(p_1)$ via $\arg\max$ makes pool sizes jump discretely as p_1 changes. Instead, replace the indicator $\mathbb{I}[i \in \mathcal{S}_k]$ with the logit probability that consumer i chose firm k:

$$w_{ik}(p_1) = \frac{\exp(\delta_{k1} + \mu_{ik})}{\sum_{j'} \exp(\delta_{j'1} + \mu_{ij'})} \tag{17}$$

   This integrates out ϵ_{i1} analytically. The smoothed Period 2 demand from Step 1 becomes:

$$\hat{D}_{2j}^{\text{smooth}}(k, p_2(k); p_1) = \frac{\sum_{i=1}^{N} w_{ik}(p_1) \cdot \hat{D}_{2j}^i(k, p_2(k))}{\sum_{i=1}^{N} w_{ik}(p_1)} \tag{18}$$

   where $\hat{D}_{2j}^i$ is the same per-consumer logit probability from Step 1. Everything is now smooth in p_1.

2. **Solve all FOCs simultaneously.** Instead of nesting the Period 2 equilibrium inside the Period 1 search, stack all $2+2 \times J$ prices into a single vector $\mathbf{p} = (p_{11}, p_{21}, p_{12}(1), p_{22}(1), p_{12}(2), p_{22}(2)) \in \mathbb{R}^6$ and solve the system of 6 FOCs simultaneously using a nonlinear equation solver (`fsolve`). This eliminates the inner loop entirely: each Jacobian evaluation is a single pass over N consumers, and the solver typically converges in < 20 Newton steps.

These two modifications can be applied independently or together. Modification 1 alone makes the nested algorithm smoother and more robust. Modification 2 alone eliminates the nesting. Together, they yield a single smooth 6×6 system that is typically orders of magnitude faster than the nested approach.

1.5 Step 3: Compute Observables

Given equilibrium prices $p_1^*, p_2^*(k; p_1^*)$ and the consumer panel, compute:

- Market shares: $\hat{D}_{1j} = |\mathcal{S}_j|/N, \quad \hat{D}_{2j} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(j_2^i = j)$

- Switching rate: $\hat{\rho} = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}(j_1^i \neq j_2^i)$

- Transition matrix: $\hat{T}_{jk} = \Pr(j_2 = k \mid j_1 = j) = \frac{|\{i: j_1^i = j, j_2^i = k\}|}{|\mathcal{S}_j|}$

- Firm profits: $\hat{\Pi}_j = \hat{D}_{1j}(p_{j1}^* - c_1) + \sum_{k=1}^{2} \hat{D}_{1k} \hat{\pi}_{2j}(k)$

- Period 2 price premium: $\Delta p_2 = p_{j2}^*(j; p_1^*) - p_{j2}^*(k; p_1^*)$ for $k \neq j$ (the harvest markup)

1.6 Verification

The numerical solution should satisfy:

- No firm can profitably deviate in either period (check via grid search around equilibrium prices)

- When $s = 0$, $\sigma_\mu = 0$, and $\beta_1 = \beta_2$, $\xi_{jt} = 0$: switching rate $\approx 1/2$ and $p_1^* = p_2^*$ (static symmetric logit duopoly)

- When $\sigma_\mu = 0$: the model reduces to plain logit and prices should match the closed-form logit markup results

- When $s \to \infty$: switching rate $\to 0$

- Increasing σ_μ (holding s fixed) should <u>reduce</u> the switching rate, since persistent preferences reinforce switching costs

- Period 2 incumbent price > entrant price: $p_{j2}^*(j) > p_{j2}^*(k \neq j)$