1 Model Setup

We take model 4 and assume that the repayment probability is distributed according to a parametric family of functions. Specifically, we assume the Kumaraswamy distribution.

1.1 Kumaraswamy Distribution

The probability density function is:

$$f(h) = \alpha\beta h^{\alpha-1}(1-h^{\alpha})^{\beta-1} \tag{1}$$

The cumulative distribution function is:

$$F(h) = 1-(1-h^{\alpha})^{\beta} \tag{2}$$

Note that for $\alpha = \beta = 1$, this reduces to the uniform distribution. The expected value is given by:

$$\mathbb{E}(\alpha, \beta) = \beta B\left(1+\frac{1}{\alpha}, \beta\right) = \frac{\beta\Gamma(1+\frac{1}{\alpha})\Gamma(\beta))}{\Gamma(1+\frac{1}{\alpha}+\beta)} \tag{3}$$

where $B(\cdot)$ is the beta function and Γ is the gamma function.

1.2 Parameters to Identify

The elements we need to identify from our model are:

- λ: Switching cost
- α, β: Distribution of types

1.3 Observable Data

For each borrower $i = 1, ..., N$, we observe:

- r_i^{win}: The interest rate charged by the winning bank (i.e., the bids conditional on winning)
- $W_i \in \{1, 2\}$: The bank from which the customer took the loan, where $W_i = 1$ indicates the incumbent won and $W_i = 2$ indicates the entrant won
- $D_i \in \{0, 1\}$: Whether the borrower defaulted ($D_i = 1$) or repaid ($D_i = 0$)

2 Estimation Strategy with Limited Observables

In practice, the researcher observes only a subset of the model's variables.

2.1 Unobservable Variables

The following are *not* observed by the researcher:

1. **Latent type** h_i: The borrower's true default probability
2. **Loser's bid**: The interest rate offered by the bank that lost (either r_1 if entrant won, or r_2 if incumbent won)
3. **Bank profits**: The realized profits are not directly observed

The parameter vector to be estimated is $\theta = (\alpha, \beta, \lambda, h_{min}, h_{max})$.

2.2 Likelihood Function

Given the model structure, we can construct the likelihood of observing the data. For borrower i, we observe (r_i^{win}, W_i, D_i) but not the latent type h_i.

Incumbent wins ($W_i = 1$): When the incumbent wins, they play the pure strategy $r_1 = \sigma(h)$. Since $\sigma(\cdot)$ is strictly increasing, observing r_i^{win} when the incumbent wins reveals the borrower's type uniquely:

$$h_i = \tau(r_i^{win}) \equiv \sigma^{-1}(r_i^{win}) \tag{4}$$

To construct the likelihood, we trace back through the data generating process. The likelihood for observation i when the incumbent wins is:

$$\mathcal{L}_i(r_i^{win}, D_i, W_i = 1; \theta) = \Pr(r_i^{win}, D_i, W_i = 1; \theta) \tag{5}$$

Step 1: Density of the latent type. The borrower's type is drawn from:

$$h_i \sim f(h; \alpha, \beta) \tag{6}$$

Step 2: Incumbent's offer. Given type h_i, the incumbent offers:

$$r_1 = \sigma(h_i; \theta) = \lambda + \frac{1}{\mu(h_i)} \tag{7}$$

This is a deterministic function, not random.

Step 3: Winning probability. The entrant draws r_2 from $G(\cdot)$. The incumbent wins if $\sigma(h_i) \leq r_2 + \lambda$, which occurs when $r_2 \geq \sigma(h_i) - \lambda$. Thus:

$$\Pr(W_i = 1 \mid h_i; \theta) = \Pr(r_2 \geq \sigma(h_i) - \lambda) = 1 - G(\sigma(h_i) - \lambda; \theta) \tag{8}$$

Step 4: Default outcome. Conditional on type h_i, default is Bernoulli:

$$\Pr(D_i \mid h_i) = h_i^{D_i}(1 - h_i)^{1-D_i} \tag{9}$$

Step 5: Change of variables. We observe r_i^{win}, not h_i. To convert from the density of h_i to the density of r_i^{win}, we use the transformation:

$$r_i^{win} = \sigma(h_i) \implies h_i = \tau(r_i^{win}) \tag{10}$$

The density of r_i^{win} (induced by the density of h_i) is:

$$f_r(r_i^{win}) = f(\tau(r_i^{win}); \alpha, \beta) \cdot \left| \frac{d\tau}{dr} \right|_{r=r_i^{win}} \tag{11}$$

This is the standard change-of-variables formula from probability theory: the Jacobian term $|\tau'(r)|$ adjusts for the rate at which density "stretches" or "compresses" under the transformation.

Complete likelihood: Combining all components, the joint density of observing $(r_i^{win}, D_i, W_i = 1)$ is:

$$\begin{aligned}\mathcal{L}_i(r_i^{win}, D_i, W_i = 1; \theta) &= f_r(r_i^{win}) \cdot \Pr(W_i = 1 \mid h_i; \theta) \cdot \Pr(D_i \mid h_i) \\ &= f(\tau(r_i^{win}); \alpha, \beta) \cdot \left|\tau'(r_i^{win})\right| \cdot [1 - G(r_i^{win} - \lambda; \theta)] \\ &\quad \cdot h_i^{D_i}(1 - h_i)^{1-D_i}\end{aligned} \tag{12}$$

Expressing as a function of observables only: Since h_i is not observed, we substitute $h_i = \tau(r_i^{win})$ everywhere. The likelihood is now purely a function of the observables and the parameters:

$$\begin{aligned}\mathcal{L}_i(r_i^{win}, D_i, W_i = 1; \theta) &= f(\tau(r_i^{win}); \alpha, \beta) \cdot \left|\tau'(r_i^{win})\right| \cdot [1 - G(r_i^{win} - \lambda; \theta)] \\ &\quad \cdot [\tau(r_i^{win})]^{D_i}[1 - \tau(r_i^{win})]^{1-D_i}\end{aligned} \tag{13}$$

Interpretation: This expression has four components:

1. $f(\tau(r_i^{win}); \alpha, \beta)$: How likely is the inferred type $\tau(r_i^{win})$ under the parametric distribution?
2. $|\tau'(r_i^{win})|$: The Jacobian adjusting for the nonlinear transformation from types to rates
3. $1 - G(r_i^{win} - \lambda; \theta)$: The probability the incumbent wins against the entrant's mixed strategy
4. $[\tau(r_i^{win})]^{D_i}[1 - \tau(r_i^{win})]^{1-D_i}$: The probability of observing the realized default outcome given the inferred type

Key insight: When the incumbent wins, observing the winning rate r_i^{win} allows us to perfectly infer the latent type through the invertible relationship $h_i = \tau(r_i^{win})$. This makes the likelihood computation straightforward—we simply evaluate all model components at the inferred type.

Entrant wins ($W_i = 2$): When the entrant wins, the situation is fundamentally different from the incumbent wins case. The entrant uses a mixed strategy $G(x)$, so the winning rate r_i^{win} is a random draw from this distribution rather than a deterministic function of type h. Moreover, we don't observe the incumbent's bid, so we cannot infer h_i directly. Instead, we must integrate over all possible types that could have led to this outcome.

Step-by-step derivation:

Step 1: Write the joint probability. The likelihood for observation i when the entrant wins is:

$$\mathcal{L}_i(r_i^{win}, D_i, W_i = 2; \theta) = \Pr(r_i^{win}, D_i, W_i = 2; \theta) \tag{14}$$

Step 2: Decompose using the law of total probability. Since we don't observe h_i when the entrant wins, we integrate it out:

$$\mathcal{L}_i(r_i^{win}, D_i, W_i = 2; \theta) = \int_{h_{min}}^{h_{max}} \Pr(r_i^{win}, D_i, W_i = 2 \mid h; \theta) \cdot f(h; \alpha, \beta)\, dh \tag{15}$$

Step 3: Factor the conditional probability. We factor the joint probability of observing rate r_i^{win}, the entrant winning, and default outcome D_i:

$$\Pr(r_i^{win}, D_i, W_i = 2 \mid h; \theta) = \Pr(r_i^{win}, W_i = 2 \mid h; \theta) \cdot \Pr(D_i \mid h) \tag{16}$$

where:

- $\Pr(r_i^{win}, W_i = 2 \mid h; \theta) = g(r_i^{win}; \theta) \cdot \mathbf{1}\{r_i^{win} < \sigma(h; \theta) - \lambda\}$ is the joint probability that the entrant offers rate r_i^{win} *and* wins. The entrant draws from G with density g, and wins if and only if their draw is below $\sigma(h) - \lambda$. Therefore, the joint event "offer r_i^{win} and win" has probability $g(r_i^{win}) \cdot \mathbf{1}\{r_i^{win} < \sigma(h) - \lambda\}$.
- $\Pr(D_i \mid h) = h^{D_i}(1 - h)^{1-D_i}$ is the probability of the observed default outcome given type h

Step 4: Substitute into the integral. Combining Steps 2 and 3:

$$\begin{aligned}\mathcal{L}_i(r_i^{win}, D_i, W_i = 2; \theta) = \int_{h_{min}}^{h_{max}} & g(r_i^{win}; \theta) \cdot \mathbf{1}\{r_i^{win} < \sigma(h; \theta) - \lambda\} \\ & \cdot h^{D_i}(1 - h)^{1-D_i} \cdot f(h; \alpha, \beta)\, dh \end{aligned} \tag{17}$$

Remark 1 (Equivalent conditional formulation). The expression above works directly with the joint density $\Pr(r_i^{win}, W_i = 2 \mid h) = g(r_i^{win})\mathbf{1}\{r_i^{win} < \sigma(h) - \lambda\}$. An equivalent way to write the same likelihood is to condition on the event that the entrant wins. Since

$$\Pr(W_i = 2 \mid h; \theta) = \Pr(r_2 < \sigma(h; \theta) - \lambda) = G(\sigma(h; \theta) - \lambda; \theta), \tag{18}$$

we have

$$\Pr(r_i^{win} \mid W_i = 2, h; \theta) = \frac{g(r_i^{win}; \theta)\, \mathbf{1}\{r_i^{win} < \sigma(h; \theta) - \lambda\}}{G(\sigma(h; \theta) - \lambda; \theta)}. \tag{19}$$

Plugging this into $\Pr(r_i^{win}, W_i = 2 \mid h) = \Pr(r_i^{win} \mid W_i = 2, h) \Pr(W_i = 2 \mid h)$ recovers the joint formulation used above (so one should not multiply by $G(\sigma(h) - \lambda)$ in addition to the joint term).

Step 5: Since $\sigma(\cdot)$ is strictly increasing, the condition $r_i^{win} < \sigma(h) - \lambda$ is equivalent to $h > \tau(r_i^{win} + \lambda)$, where $\tau = \sigma^{-1}$. The integral simplifies to:

$$\mathcal{L}_i(r_i^{win}, D_i, W_i = 2; \theta) = g(r_i^{win}; \theta) \cdot \int_{\tau(r_i^{win}+\lambda)}^{h_{max}} h^{D_i}(1 - h)^{1-D_i} \cdot f(h; \alpha, \beta)\, dh \tag{20}$$

This integrates only over types h for which the entrant's observed offer r_i^{win} actually beats the incumbent.

Interpretation: This expression has two components:

1. $g(r_i^{win};\theta)$: The density that the entrant offers the observed rate r_i^{win}
2. $\int_{\tau(r_i^{win}+\lambda)}^{h_{max}} h^{D_i}(1-h)^{1-D_i} \cdot f(h;\alpha,\beta)\, dh$: The integral over types h for which the entrant's offer r_i^{win} beats the incumbent (i.e., $\sigma(h) > r_i^{win} + \lambda$), weighted by the probability of the observed default outcome and the prior density of h

The key insight is that observing the entrant's winning rate r_i^{win} restricts the set of possible borrower types: only types with $h > \tau(r_i^{win} + \lambda)$ could have generated this outcome, since lower types would have incumbents offering rates that beat r_i^{win}.

Key difference from $W_i = 1$: When the entrant wins, we cannot uniquely identify h_i from observables. The same winning rate r_i^{win} could arise from any type h, as long as the entrant happens to draw r_i^{win} from their mixed strategy. Therefore, the likelihood involves integrating over the uncertainty about the borrower's true type, weighted by how likely each type is to produce the observed outcome $(r_i^{win}, W_i = 2, D_i)$.

Sample likelihood: The likelihood for the full sample is:

$$\mathcal{L}(\{r_i^{win}, D_i, W_i\}_{i=1}^N;\theta) = \prod_{i=1}^N \mathcal{L}_i(r_i^{win}, D_i, W_i;\theta) \tag{21}$$

where $\mathcal{L}_i(r_i^{win}, D_i, W_i;\theta)$ equals the expression for $W_i = 1$ or $W_i = 2$ depending on which bank wins.

The log-likelihood is:

$$\ell(\{r_i^{win}, D_i, W_i\}_{i=1}^N;\theta) = \sum_{i=1}^N \log \mathcal{L}_i(r_i^{win}, D_i, W_i;\theta) \tag{22}$$

2.3 Estimation Procedure

Maximum Likelihood Estimation:

1. For a given parameter vector θ:
   - Compute the equilibrium strategies $\sigma(h;\theta)$ and $G(x;\theta)$ numerically (following the algorithm in Section "Equilibrium computation" in model4_adapting Engelbrecht et al.tex)
   - For each observation (r_i^{win}, D_i, W_i), evaluate the likelihood $\mathcal{L}_i(r_i^{win}, D_i, W_i;\theta)$. When $W_i = 2$, this requires numerical integration over h
2. Maximize the log-likelihood:

$$\hat{\theta} = \arg\max_{\theta} \ell(\{r_i^{win}, D_i, W_i\}_{i=1}^N;\theta) \tag{23}$$

   using numerical optimization (e.g., Nelder-Mead, BFGS)
3. Compute standard errors from the inverse Hessian or via bootstrap

2.4 Identification

Key identification arguments:

- **Switching cost λ:** Identified from the gap between incumbent and entrant rates. In equilibrium, the incumbent charges $r_1(h) = \lambda + 1/\mu(h)$ while the entrant's support starts at $x_{min} = 1/E[1-H]$. The minimum observed incumbent rate minus the minimum observed entrant rate reveals λ.

Proposition 2 (Identification of λ)**.** *The switching cost λ is identified from the difference between the minimum equilibrium rate of the incumbent and the minimum equilibrium rate of the entrant:*

$$\lambda = r_1^{min} - r_2^{min} \tag{24}$$

Proof. **Step 1: Define h_{min}.** Let h_{min} denote the lower bound of the support of the distribution of default probabilities $F(h)$, i.e., $h_{min} = \inf\{h : F(h) > 0\}$. Since $\sigma(h)$ is strictly increasing in h, the safest borrower h_{min} receives the lowest incumbent rate.

Step 2: Compute r_1^{min}. Define $r_1^{min} \equiv \sigma(h_{min})$ as the minimum rate offered by the incumbent. From the equilibrium strategy:

$$r_1^{min} = \sigma(h_{min}) = \lambda + \frac{1}{\mu(h_{min})} \tag{25}$$

where $\mu(h_{min}) = E[1 - H \mid H > h_{min}] = E[1 - H]$, since conditioning on $H > h_{min}$ is vacuous (all types satisfy this). Therefore:

$$r_1^{min} = \lambda + \frac{1}{E[1-H]} \tag{26}$$

Step 3: Compute r_2^{min}. Define r_2^{min} as the lower bound of the support of the entrant's mixed strategy $G(x)$, i.e., $r_2^{min} = \inf\{x : G(x) > 0\}$.

From the entrant's mixed strategy (equation **??**):

$$G(r - \lambda) = 1 - \exp\left[-\int_{\underline{r}}^{r} \frac{1-\tau(u)}{(1-\tau(u))u - 1} du\right] \tag{27}$$

At $r = \underline{r} = \sigma(h_{min})$, the integral evaluates to zero, so $G(\underline{r} - \lambda) = 1 - \exp(0) = 0$. Thus the support of G begins at $x = \underline{r} - \lambda$, i.e.:

$$r_2^{min} = \underline{r} - \lambda = \sigma(h_{min}) - \lambda = \frac{1}{\mu(h_{min})} = \frac{1}{E[1-H]} \tag{28}$$

Step 4: Compute the difference. Subtracting (28) from (26):

$$r_1^{min} - r_2^{min} = \left(\lambda + \frac{1}{E[1-H]}\right) - \frac{1}{E[1-H]} = \lambda \tag{29}$$

□

Practical implication: In the data, one can estimate λ by computing the difference between the minimum observed winning rate when the incumbent wins and the minimum observed winning rate when the entrant wins:

$$\hat{\lambda} = \min_{i:W_i=1} r_i^{win} - \min_{i:W_i=2} r_i^{win} \tag{30}$$

- **Distribution parameters** (α, β)**:** Identified from:
  1. The distribution of winning rates conditional on winner identity
  2. The correlation between winning rates and default outcomes
  3. The switching probability (fraction of borrowers choosing the entrant)
- **Support bounds** (h_{min}, h_{max})**:** The observed range of rates (conditional on winner) restricts the support of h via the equilibrium pricing function $\sigma(h)$.

3 Random thoughts

- Switching costs does not generate heterogeneity in the rates paid by old customers, whereas information asymmetries do. Maybe the variance of rates paid by old customers can tell us about the degree of learning by the bank.

- If we can determine the switching probability implied by the model, then we can compare it to the observed in the data and this can reveal something about the relative importance of switching costs vs information asymmetries. For example if there is no information asymmetries, then the equilibrium is in pure strategies and there is no switching. If there are information asymmetries there will be switching.

- Given that we obseve the winning bids conditional on winning, we can try to obtain some moments from the model and compare them to the data.

- Maybe if one calculates the probability of switching conditional on ex-post repayment there are some moments that we can try to match. Note that the lemons are the ones that switch, hence switchers should have a lower repayment rate than non-switchers.

- Note that the home bank plays a pure strategy. For any h they play $\sigma(h)$ and whenever the home bank wins we observe $\sigma(h)$, and the home bank has a higher likelihood of winning when h is low. if we adjust for this winning probability we can obtain the distribution of $\sigma(h)$ and then given a switching cost obtain the distribution of h.

- To identify the switching costs we can use the difference between the lowest bid by the entrant and the lowest rate by the incumbent. The difference should be equal to the switching cost.