In this pdf we adapt model 4 to make it structural. Incorporate preference heterogeneity.

1 Model

There are two banks, consumer utility, conditional on the consumer having previously used bank j_0, is:

$$u_{ij}(j_0) = -\alpha r_{ij} - \lambda \cdot 1(j \neq j_0) + \varepsilon_{ij}$$

Define h to be default probability, there is an informed bank (bank 1) and an uninformed bank (bank 2), which is smoothly distributed in the population according to the cdf $F(h)$ and pdf $f(h)$. The strategies are $r_1(h) = \sigma(h) : h \to r$ and $G(x) = \Pr(r_2 \leq x)$, which are interest rates.

Given the interest rates the probability of choosing the incumbent is:

$$s(\sigma(h), r_2) = \frac{\exp(-\alpha\sigma(h))}{\exp(-\alpha\sigma(h)) + \exp(-\alpha r_2 - \lambda)}$$

Then expected profits of bank 1 are:

$$\begin{aligned}
\pi_1(r_1(h)) &= \int [(1-h)\sigma(h) - 1] s(\sigma(h), r_2) dG(r_2) \\
&= [(1-h)\sigma(h) - 1] \int s(\sigma(h), r_2) dG(r_2)
\end{aligned}$$

The expected profits of bank 2 are:

$$\pi_2(r_2) = \int_h [1 - s(\sigma(h), r_2)][(1-h)r_2 - 1] dF(h) \tag{1}$$

2 Equilibrium Definition and Existence

Definition 1 (Equilibrium)**.** An equilibrium consists of a pure strategy $\sigma^* : [h_{min}, h_{max}] \to \mathbb{R}_+$ for the incumbent and a mixed strategy (CDF) $G^* : \mathbb{R}_+ \to [0,1]$ for the entrant such that:

1. For each $h \in [h_{min}, h_{max}]$, $\sigma^*(h)$ maximizes the incumbent's expected profits:

$$\sigma^*(h) \in \arg\max_{r_1 \geq 0} [(1-h)r_1 - 1] \int s(r_1, r_2)\, dG^*(r_2) \tag{2}$$

2. G^* maximizes the entrant's expected profits: for all r_2 in the support of G^*,

$$\pi_2(r_2; \sigma^*, G^*) = \max_{r_2' \geq 0} \int_{h_{min}}^{h_{max}} [1 - s(\sigma^*(h), r_2')][(1-h)r_2' - 1]\, dF(h) \tag{3}$$

We now state the assumptions required for the proof and establish existence.

Assumption 2. The distribution F of default probabilities has support $[h_{min}, h_{max}] \subset (0,1)$, with F absolutely continuous and density f bounded away from zero on $[h_{min}, h_{max}]$.

Assumption 3. Interest rates are restricted to a compact set $[0, \bar{r}]$ where $\bar{r} > \frac{1}{1-h_{max}}$ is sufficiently large that no bank would ever find it profitable to offer a rate above $\bar{r}$.

Theorem 4 (Equilibrium Existence)**.** *Under Assumptions 2–3, the model admits a Bayesian Nash equilibrium* (σ^*, G^*)*.*

Proof. We construct an operator whose fixed point is an equilibrium and apply Schauder's fixed point theorem.

Step 1: Define the strategy spaces.

Let $\mathcal{S} = \{\sigma : [h_{min}, h_{max}] \to [0, \bar{r}] \mid \sigma \text{ is measurable}\}$ denote the set of incumbent strategies, and let $\mathcal{G} = \{G : [0, \bar{r}] \to [0,1] \mid G \text{ is a CDF}\}$ denote the set of entrant mixed strategies. The product space $\mathcal{S} \times \mathcal{G}$ is the joint strategy space.

Endow $\mathcal{S}$ with the L^1 topology (under the measure induced by F) and $\mathcal{G}$ with the topology of weak convergence. Both spaces are convex. The set $\mathcal{G}$ is compact under weak convergence by Prokhorov's theorem, since all CDFs are supported on the compact set $[0, \bar{r}]$. The set $\mathcal{S}$ can be restricted to uniformly bounded measurable functions on $[h_{min}, h_{max}]$ taking values in $[0, \bar{r}]$, which is convex and compact in the weak-$*$ topology of $L^\infty([h_{min}, h_{max}])$.

Step 2: Define the best-response operator.

Define the best-response mapping $T : \mathcal{S} \times \mathcal{G} \to \mathcal{S} \times \mathcal{G}$ by $T(\sigma, G) = (T_1(\sigma, G), T_2(\sigma, G))$ where:

Incumbent's best response. For each h, define:

$$T_1(\sigma, G)(h) = \arg\max_{r_1 \in [0,\bar{r}]} [(1-h)r_1 - 1] \int_0^{\bar{r}} s(r_1, r_2)\, dG(r_2) \tag{4}$$

where $s(r_1, r_2) = \frac{\exp(-\alpha r_1)}{\exp(-\alpha r_1) + \exp(-\alpha r_2 - \lambda)}$.

Note that for each fixed h and G, the objective function is:

$$\phi(r_1; h, G) = [(1-h)r_1 - 1] \underbrace{\int_0^{\bar{r}} \frac{\exp(-\alpha r_1)}{\exp(-\alpha r_1) + \exp(-\alpha r_2 - \lambda)} dG(r_2)}_{\equiv S(r_1, G)} \tag{5}$$

The term $(1-h)r_1 - 1$ is linear (hence concave) in r_1. The market share $S(r_1, G)$ is a log-concave function of r_1 (since the logit share is log-concave in own price). Therefore $\phi(r_1; h, G)$ is the product of a non-negative concave function and a log-concave function on the region where $(1-h)r_1 - 1 \geq 0$, which ensures that the maximizer is unique for each h.[1]

Entrant's best response. The entrant's expected profit given σ is:

$$\Pi_2(r_2; \sigma) = \int_{h_{min}}^{h_{max}} [1 - s(\sigma(h), r_2)][(1-h)r_2 - 1]\, dF(h) \tag{6}$$

Define $T_2(\sigma, G)$ as any CDF G' supported on the set of maximizers of $\Pi_2(r_2; \sigma)$. Since Π_2 is continuous in r_2 on $[0, \bar{r}]$, the set of maximizers is nonempty and closed. The entrant randomizes

[1] If $(1-h)r_1 - 1 < 0$ the bank makes negative expected profit and sets a rate to avoid this region. At $r_1 = 1/(1-h)$ the margin is zero; the optimum lies weakly above this.

over this set (if it contains more than one point, any mixture over maximizers is a valid best response).

Step 3: Verify the conditions of Schauder's fixed point theorem.

We verify the three conditions: (i) the domain is a nonempty, compact, convex subset of a locally convex topological vector space; (ii) the mapping T maps this set into itself; and (iii) T is continuous.

(i) Compactness and convexity. The set $\mathcal{S} \times \mathcal{G}$ (with the product topology) is convex by construction. Compactness follows from: $\mathcal{G}$ is compact under weak convergence (Prokhorov), and $\mathcal{S}$ (uniformly bounded measurable functions valued in $[0, \bar{r}]$) is compact in the weak-$*$ topology.

(ii) T maps $\mathcal{S} \times \mathcal{G}$ into itself. For any $(\sigma, G) \in \mathcal{S} \times \mathcal{G}$: $T_1(\sigma, G)(h) \in [0, \bar{r}]$ for all h (the maximizer lies in the compact action set), so $T_1(\sigma, G) \in \mathcal{S}$. Similarly, $T_2(\sigma, G)$ is a CDF on $[0, \bar{r}]$, so $T_2(\sigma, G) \in \mathcal{G}$.

(iii) Continuity of T. We argue each component is continuous.

For T_1: The objective function $\phi(r_1; h, G)$ is jointly continuous in (r_1, G) for each h (by the continuous mapping theorem applied to the integral $\int s(r_1, r_2) dG(r_2)$ under weak convergence of $G_n \to G$) and continuous in σ (since T_1 depends on G but not on σ directly—the incumbent optimizes pointwise for each h). By Berge's Maximum Theorem, the maximizer $T_1(\sigma, G)(h)$ is upper hemicontinuous in G. Since the maximizer is unique (by the concavity argument above), upper hemicontinuity implies continuity.

For T_2: The entrant's profit $\Pi_2(r_2; \sigma)$ is continuous in σ (under L^1 convergence) by the dominated convergence theorem, since $s(\sigma(h), r_2)$ is continuous in $\sigma(h)$ and bounded. Again by Berge's Maximum Theorem, the best-response correspondence is upper hemicontinuous.

Step 4: Apply Schauder's Fixed Point Theorem.

Since $\mathcal{S} \times \mathcal{G}$ is a nonempty, compact, convex subset of a locally convex space and $T : \mathcal{S} \times \mathcal{G} \to \mathcal{S} \times \mathcal{G}$ is continuous, Schauder's Fixed Point Theorem guarantees the existence of a fixed point $(\sigma^*, G^*) = T(\sigma^*, G^*)$. By construction, this fixed point constitutes a Bayesian Nash equilibrium. $\square$

Remark 5 (Role of product differentiation). The logit demand structure is essential for the proof. In model 4 (without product differentiation), the borrower deterministically chooses the lower effective price, creating discontinuities in demand. The logit smoothing ensures that:

1. Market shares $s(r_1, r_2)$ are continuous (indeed C^∞) in both rates
2. Profit functions are continuous, enabling application of Berge's Maximum Theorem
3. The best-response mapping is single-valued (generically), avoiding the need for fixed point theorems for correspondences (e.g., Kakutani)

As $\alpha \to \infty$, the model converges to model 4 where the borrower chooses deterministically, and the equilibrium converges to the mixed-strategy equilibrium derived previously.