Adapting Engelbrecht-Wiggans et al. (1983)

This version incorporates switching costs. Let bank 1 be the incumbent, which has a relationship with the borrower. The borrower incurs a switching cost $\lambda > 0$ if they choose bank 2. This borrower chooses bank 2 if $r_1 > r_2 + \lambda$, otherwise it chooses bank 1.

Define h to be default probability, there is an informed bank (bank 1) and an uninformed bank (bank 2), which is smoothly distributed in the population according to the cdf $F(h)$ and pdf $f(h)$. The strategies are $r_1(h) = \sigma(h) : h \to r$ and $G(x) = \Pr(r_2 \leq x)$, which are interest rates.

Assume that in equilibrium σ is an increasing function, and denote by $\tau : r \to h$ its inverse, $\tau(\sigma(h)) = h$.

Then expected profits of bank 1 are:

$$\begin{aligned}
\pi_1(r_1(h)) &= \Pr(\sigma(h) \leq r_2 + \lambda) \cdot [(1-h)\sigma(h) - 1] \\
&= \Pr(r_2 \geq \sigma(h) - \lambda) \cdot [(1-h)\sigma(h) - 1] \\
&= [1 - G(\sigma(h) - \lambda)] \cdot [(1-h)\sigma(h) - 1]
\end{aligned} \tag{1}$$

and the expected profits of bank 2 are:

Bank 2 wins if $r_2 < \sigma(h) - \lambda$, which is $\sigma(h) > r_2 + \lambda$.

$$\begin{aligned}
\pi_2(r_2) &= \Pr(\sigma(h) > r_2 + \lambda) \cdot E[(1-h) \cdot r_2 - 1 \mid \sigma(h) > r_2 + \lambda] \\
&= \Pr(h > \tau(r_2 + \lambda)) \cdot E[(1-h) \cdot r_2 - 1 \mid h > \tau(r_2 + \lambda)] \\
&= [1 - F(\tau(r_2 + \lambda))] \cdot [E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1]
\end{aligned} \tag{2}$$

Assume that bank 2 makes zero profits[1], then we have:

$$[1 - F(\tau(r_2 + \lambda))] \cdot [E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1] = 0 \tag{3}$$

since the winning probability is not zero, then the expected profits have to be zero.

$$E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1 = 0 \implies r_2 = \frac{1}{E[(1-h) \mid h > \tau(r_2 + \lambda)]} \tag{4}$$

Let $k = \tau(r_2 + \lambda)$, which means $\sigma(k) = r_2 + \lambda$. Therefore $r_2 = \sigma(k) - \lambda$. Substituting:

$$\sigma(k) - \lambda = \frac{1}{E[1 - h | h > k]}$$

Hence,

$$\sigma(h) = \lambda + \mu(h)^{-1} \tag{5}$$

where $\mu(h) = E[1 - H | H > h]$.

[1] Would have to be proved, but I am confident that it is true since the uninformed firm in auction models always makes zero profits.

Then we can use profit maximization by the first firm, the FOC of equation 1 are:

$$\begin{aligned}
-g(\sigma(h)-\lambda)[(1-h)\sigma(h)-1]+[1-G(\sigma(h)-\lambda)][1-h]=0 \\
\frac{1-h}{[(1-h)\sigma(h)-1]} = \frac{g(\sigma(h)-\lambda)}{[1-G(\sigma(h)-\lambda)]} = -\frac{d}{d\sigma}[\log(1-G(\sigma(h)-\lambda))] \\
\frac{1-\tau(r)}{[(1-\tau(r))r-1]} = -\frac{d}{d\sigma}[\log(1-g(r-\lambda))] \qquad (6)
\end{aligned}$$

Integrating both sides from $\underline{r}=\sigma(\underline{h})$ to a given r, where $G(\underline{r}-\lambda)=0$ we have:

$$\begin{aligned}
-[\log(1-G(r-\lambda))-\log(1-\underbrace{G(\underline{r}-\lambda))}_{=0}] &= \int_{\underline{r}}^{r}\frac{1-\tau(u)}{[(1-\tau(u))u-1]}du \\
-\log(1-G(r-\lambda)) &= \int_{\underline{r}}^{r}\frac{1-\tau(u)}{[(1-\tau(u))u-1]}du \\
G(r-\lambda) &= 1-\exp\left[-\int_{\underline{r}}^{r}\frac{1-\tau(u)}{[(1-\tau(u))u-1]}du\right] \qquad (7)
\end{aligned}$$

0.1 Case with $\lambda = 0$

From equation 5 we have that:

$$\sigma(h)=\frac{1}{E[(1-H)\mid H\leq h]}\equiv\frac{1}{\mu(h)} \qquad (8)$$

Then we can obtain the mixed strategy of bank 2 by a change of variables, consider $u=\sigma(t) \implies \tau(u)=t, du=\sigma'(t)dt$, then the limits of integration change from $[\underline{r}, r]$ to $\underline{h}, \tau(r)]$. Substituting into equation 7 we have:

$$G(\sigma(h))=1-\exp\left[-\int_{\underline{h}}^{h}\frac{1-t}{[(1-t)\sigma(t)-1]}\sigma'(t)dt\right] \qquad (9)$$

given that $\sigma(t)=1/\mu(t)$, we have $\sigma'(t)=-\mu'(t)/\mu(t)^2$, replacing in the equation above:

$$G(\sigma(h))=1-\exp\left[\int_{\underline{h}}^{h}\frac{1-t}{\frac{1-t}{\mu(t)}-1}\frac{\mu'(t)}{\mu(t)^2}dt\right]=1-\exp\left[\int_{\underline{h}}^{h}\frac{(1-t)\mu'(t)}{(1-t-\mu(t))\mu(t)}dt\right] \qquad (10)$$

0.2 Example

To derive a closed-form solution, we assume that the default risk h is uniformly distributed on $[\underline{h},\bar{h}]$, where $0<\underline{h}<\bar{h}<1$. This ensures that even the safest borrower has positive default risk and that no borrower defaults with certainty. The conditional expectation is then:

$$E[1-h\mid h>k]=\frac{\int_k^{\bar{h}}(1-x)dx}{\int_k^{\bar{h}}dx}=\frac{\left[(x-\frac{x^2}{2})\right]_k^{\bar{h}}}{\bar{h}-k}=\frac{(\bar{h}-\frac{\bar{h}^2}{2})-(k-\frac{k^2}{2})}{\bar{h}-k}$$

Simplifying the numerator:

$$\bar{h} - \frac{\bar{h}^2}{2} - k + \frac{k^2}{2} = (\bar{h} - k) - \frac{\bar{h}^2 - k^2}{2} = (\bar{h} - k) - \frac{(\bar{h} - k)(\bar{h} + k)}{2} = (\bar{h} - k)\left(1 - \frac{\bar{h} + k}{2}\right)$$

Therefore:

$$E[1 - h \mid h > k] = 1 - \frac{\bar{h} + k}{2} = \frac{2 - \bar{h} - k}{2}$$

Substituting this into equation 4 with $k = \tau(r_2 + \lambda)$:

$$r_2 = \frac{1}{(2 - \bar{h} - \tau(r_2 + \lambda))/2} = \frac{2}{2 - \bar{h} - \tau(r_2 + \lambda)}$$

We can solve this for τ. Let $r = r_2 + \lambda$, so $r_2 = r - \lambda$. The equation becomes:

$$r - \lambda = \frac{2}{2 - \bar{h} - \tau(r)} \implies 2 - \bar{h} - \tau(r) = \frac{2}{r - \lambda} \implies \tau(r) = 2 - \bar{h} - \frac{2}{r - \lambda}$$

This is the inverse of bank 1's strategy. To find the strategy $\sigma(h)$ itself, we set $h = \tau(r)$ and solve for r:

$$h = 2 - \bar{h} - \frac{2}{r - \lambda} \implies r - \lambda = \frac{2}{2 - \bar{h} - h} \implies r = \sigma(h) = \lambda + \frac{2}{2 - \bar{h} - h}$$

This is the equilibrium pricing function for the informed bank. It prices at a markup over the switching cost λ, where the markup depends on the borrower's risk and the upper bound $\bar{h}$.

Verification: We can verify this makes economic sense:

- For the safest borrower ($h = \underline{h}$): $\sigma(\underline{h}) = \lambda + \frac{2}{2 - \bar{h} - \underline{h}}$
- For the riskiest borrower ($h = \bar{h}$): $\sigma(\bar{h}) = \lambda + \frac{2}{2 - 2\bar{h}} = \lambda + \frac{1}{1 - \bar{h}}$
- Since $\bar{h} < 1$, the rate is finite and positive for all borrowers in the support.
- The profit margin for type h is $(1 - h)\sigma(h) - 1 = (1 - h)\left(\lambda + \frac{2}{2 - \bar{h} - h}\right) - 1$.

Finally, we can use our expression for $\tau(u)$ to find the explicit distribution G for bank 2 from equation 7. We have:

$$1 - \tau(u) = 1 - \left(2 - \bar{h} - \frac{2}{u - \lambda}\right) = \bar{h} - 1 + \frac{2}{u - \lambda}$$

The integrand becomes:

$$\frac{1 - \tau(u)}{[(1 - \tau(u))u - 1]} = \frac{\bar{h} - 1 + \frac{2}{u - \lambda}}{\left(\bar{h} - 1 + \frac{2}{u - \lambda}\right)u - 1}$$

Let $A = \bar{h} - 1$ (note $A < 0$ since $\bar{h} < 1$). Then:

$$\begin{aligned}
\frac{A + \frac{2}{u-\lambda}}{\left(A + \frac{2}{u-\lambda}\right) u - 1} &= \frac{A(u-\lambda)+2}{(u-\lambda)} \cdot \frac{1}{\frac{[A(u-\lambda)+2]u-(u-\lambda)}{u-\lambda}} \\
&= \frac{A(u-\lambda)+2}{Au^2 - A\lambda u + 2u - u + \lambda} \\
&= \frac{A(u-\lambda)+2}{Au^2 + (2 - A\lambda - 1)u + \lambda} \\
&= \frac{Au - A\lambda + 2}{Au^2 + (1 - A\lambda)u + \lambda}
\end{aligned}$$

Substituting back $A = \bar{h} - 1$:

$$\begin{aligned}
&= \frac{(\bar{h}-1)u - (\bar{h}-1)\lambda + 2}{(\bar{h}-1)u^2 + (1 - (\bar{h}-1)\lambda)u + \lambda} \\
&= \frac{(\bar{h}-1)(u-\lambda) + 2}{(\bar{h}-1)u^2 + (1 - (\bar{h}-1)\lambda)u + \lambda}
\end{aligned}$$

This integral is complex in the general case. For a cleaner closed-form solution, consider the case where $\bar{h} = 1$ (so default probability ranges from $\underline{h} > 0$ to 1). With $\bar{h} = 1$:

$$\sigma(h) = \lambda + \frac{2}{2 - 1 - h} = \lambda + \frac{2}{1-h}$$

and

$$\tau(r) = 2 - 1 - \frac{2}{r-\lambda} = 1 - \frac{2}{r-\lambda}$$

The lower bound of integration is $\underline{r} = \sigma(\underline{h}) = \lambda + \frac{2}{1-\underline{h}}$.

Now we compute:

$$1 - \tau(u) = 1 - \left(1 - \frac{2}{u-\lambda}\right) = \frac{2}{u-\lambda}$$

The integrand becomes:

$$\frac{1-\tau(u)}{(1-\tau(u))u - 1} = \frac{\frac{2}{u-\lambda}}{\frac{2u}{u-\lambda} - 1} = \frac{\frac{2}{u-\lambda}}{\frac{2u-(u-\lambda)}{u-\lambda}} = \frac{2}{u+\lambda}$$

The integral becomes:

$$\int_{\underline{r}}^{r} \frac{2}{u+\lambda} du = 2\ln(u+\lambda)\Big|_{\underline{r}}^{r} = 2\ln\left(\frac{r+\lambda}{\underline{r}+\lambda}\right) = \ln\left(\left(\frac{r+\lambda}{\underline{r}+\lambda}\right)^2\right)$$

Substituting into equation 7:

$$-\log(1 - G(r - \lambda)) = \ln\left(\left(\frac{r + \lambda}{\underline{r} + \lambda}\right)^2\right)$$
$$\implies G(r - \lambda) = 1 - \left(\frac{\underline{r} + \lambda}{r + \lambda}\right)^2$$

Let $x = r - \lambda$, so $r = x + \lambda$. Bank 2's strategy $G(x) = \Pr(r_2 \leq x)$ is:

$$G(x) = 1 - \left(\frac{\underline{r} + \lambda}{x + 2\lambda}\right)^2$$

where $\underline{r} + \lambda = 2\lambda + \frac{2}{1-\underline{h}}$.

The support for bank 2's offers starts where $G(x) = 0$, which occurs at $x = \underline{r} - \lambda = \frac{2}{1-\underline{h}}$.

0.3 Switchers

What is the probability a bank switches?

0.4 Equilibrium computation

Previosuly we derived the equilibrium conditions (see equations 5 and 7) for the informed incumbent bank and the uninformed entrant bank.

For a generic distribution $F(h)$ with support $[h_{min}, h_{max}]$, we cannot obtain closed-form solutions. Instead, we use numerical methods to compute the equilibrium strategies.

0.4.1 Step 1: Define the conditional expectation function

For each type h, we need to compute the conditional expectation:

$$\mu(h) = E[1 - H \mid H > h] = \frac{\int_h^{h_{max}} (1-t) f(t) dt}{1 - F(h)} \tag{11}$$

Properties:

- $\mu(h)$ is decreasing in h (higher types have worse expected repayment)
- $\mu(h_{min}) = E[1 - H]$ (unconditional expectation)
- As $h \to h_{max}$: $\mu(h) \to 1 - h_{max}$

Empirical computation: Given N draws from F: $\{h_1, ..., h_N\}$, we compute μ on a grid of M points $\{h_1^{grid}, ..., h_M^{grid}\}$:

$$\mu(h_k^{grid}) = \frac{1}{|\{i : h_i > h_k^{grid}\}|} \sum_{i: h_i > h_k^{grid}} (1 - h_i) \tag{12}$$

0.4.2 Step 2: Support of strategies

The incumbent's offers range from:

$$\underline{r} = \sigma(h_{min}) = \lambda + \frac{1}{E[1 - H]} \tag{13}$$

$$\bar{r} = \sigma(h_{max}) = \lambda + \frac{1}{1 - h_{max}} \tag{14}$$

The entrant's offers have support $[x_{min}, \infty)$ where $x_{min} = \underline{r} - \lambda = \frac{1}{E[1-H]}$.

0.4.3 Step 3: Compute the inverse function $\tau(r)$

From $\sigma(\tau(r)) = r$, we need to find $\tau(r)$ for each $r \in [\underline{r}, \bar{r}]$. Given $\sigma(h) = \lambda + 1/\mu(h)$:

$$r = \lambda + \frac{1}{\mu(\tau(r))} \implies \mu(\tau(r)) = \frac{1}{r - \lambda} \tag{15}$$

Numerical solution: For each r in a grid, solve for $\tau(r)$ by finding h such that $\mu(h) = 1/(r - \lambda)$ using interpolation or root-finding.

0.4.4 Step 4: Compute the entrant's mixed strategy

From equation 7, we compute:

$$G(r-\lambda) = 1 - \exp\left[-\int_{\underline{r}}^{r} \frac{1-\tau(u)}{(1-\tau(u))u-1} du\right] \tag{16}$$

Numerical algorithm:

1. Create a grid of M points: $r_1 = \underline{r}, r_2, ..., r_M$ in $[\underline{r}, \bar{r}]$
2. For each r_j, compute $\tau(r_j)$ as described in Step 3
3. Compute the integrand: $I(r_j) = \frac{1-\tau(r_j)}{(1-\tau(r_j))r_j-1}$
4. Use numerical integration (e.g., trapezoidal rule):

$$\int_{\underline{r}}^{r_j} I(u)du \approx \sum_{k=1}^{j-1} \frac{I(r_k)+I(r_{k+1})}{2}(r_{k+1}-r_k) \tag{17}$$

5. Compute: $G(r_j-\lambda) = 1 - \exp\left[-\int_{\underline{r}}^{r_j} I(u)du\right]$

0.4.5 Step 5: Sample from the entrant's strategy

To simulate the model with N borrowers:

1. Draw N borrowers: $h_i \sim F(h)$ for $i = 1, ..., N$
2. For each borrower i, compute incumbent's offer: $r_1^i = \sigma(h_i) = \lambda + 1/\mu(h_i)$
3. Draw entrant's offers from G using inverse transform sampling:
   - Draw $U_i \sim \text{Uniform}[0,1]$
   - Find r_2^i such that $G(r_2^i) = U_i$ by interpolation
4. Borrower i switches if $r_2^i + \lambda < r_1^i$

0.4.6 Verification

The numerical solution should satisfy:

- Bank 2's expected profit is zero for all r_2 in its support
- Bank 1 is indifferent over its support for each type h
- G is a valid CDF: $G(x_{min}) = 0$ and $\lim_{x\to\infty} G(x) = 1$