Adapting Engelbrecht-Wiggans et al. (1983)

Define h to be default probability, there is an informed bank (bank 1) and an uninformed bank (bank 2). The strategies are $r_1(h) = \sigma(h) : h \to r$ and $G(x) = \Pr(r_2 \leq x)$, which are interest rates.

Assume that in equilibrium σ is an increasing function, and denote by $\tau : r \to h$ its inverse, $\tau(\sigma(h)) = h$.

Then expected profits of bank 1 are:

$$\begin{aligned}
\pi_1(r_1(h)) &= \Pr(\sigma(h) \leq r_2) \cdot [(1-h)\sigma(h) - 1] \\
&= [1 - \Pr(\sigma(h) > r_2)] \cdot [(1-h)\sigma(h) - 1] \\
&= [1 - G(\sigma(h))] \cdot [(1-h)\sigma(h) - 1] \qquad (1)
\end{aligned}$$

and the expected profits of bank 2 are:

$$\begin{aligned}
\pi_2(r_2) &= \Pr(r_2 \leq \sigma(H)) \cdot E[(1-H) \cdot r_2 - 1 \mid r_2 \leq \sigma(H)] \\
&= \Pr(H \geq \tau(r_2)) \cdot E[(1-H) \cdot r_2 - 1 \mid H \geq \tau(r_2)] \\
&= [1 - F(\tau(r_2))] \cdot E[(1-H) \cdot r_2 - 1 \mid H \geq \tau(r_2)] \\
&= [1 - F(\tau(r_2))] \cdot (E[(1-H) \mid H \geq \tau(r_2)] \cdot r_2 - 1) \qquad (2)
\end{aligned}$$

Proposition 1 (Zero profits of the uninformed bank). Assume H has an atomless distribution F with support $[\underline{h}, \bar{h}] \subseteq [0,1]$. Assume the informed bank uses a strictly increasing strategy σ with range $[\sigma(\underline{h}), \sigma(\bar{h})]$ and define $\bar{r} \equiv \sigma(\bar{h}) < \infty$. If, in addition, Bank 2 randomizes over a connected support with $\sup \operatorname{supp}(G) = \bar{r}$, then Bank 2 earns zero expected profit. In particular, every r_2 in the support of G yields zero expected profit.

Remark 1 (On the boundary condition $\sup \operatorname{supp}(G) = \bar{r}$). The equality $\sup \operatorname{supp}(G) = \bar{r}$ is a boundary condition tying together two equilibrium objects. It does <u>not</u> follow from the primitives stated so far: if Bank 1's highest type optimally chooses to lose for sure (e.g., because any competitive rate yields non-positive profits), then one can have $\sup \operatorname{supp}(G) < \sigma(\bar{h})$ and Bank 2's winning probability need not vanish at the top of its support.

In applications, this boundary condition is typically implied by an additional structural feature such as a borrower reservation rate (an exogenous cap on feasible/accepted interest rates) or a parameter restriction ensuring even the highest type of Bank 1 strictly prefers quoting some competitive rate to losing for sure.

Proof. **Step 1 (Outside option implies nonnegative equilibrium profit).** Bank 2 can always choose an interest rate strictly above $\bar{r}$. Since Bank 1 never offers more than $\bar{r}$ under σ, this deviation loses for sure and yields profit 0. Therefore Bank 2's equilibrium expected profit satisfies $\bar{\pi}_2 \geq 0$.

Step 2 (Indifference on the support). In a mixed-strategy equilibrium, Bank 2 must be indifferent across all $r_2 \in \operatorname{supp}(G)$; otherwise it would assign zero probability to strictly suboptimal rates. Hence there exists a constant $\bar{\pi}_2$ such that $\pi_2(r_2) = \bar{\pi}_2$ for all $r_2 \in \operatorname{supp}(G)$. **Step 3 (Profits approach 0 near the top of the support).** Because $\sup \operatorname{supp}(G) = \bar{r}$, for each n there exists $r_n \in \operatorname{supp}(G)$ such that $\bar{r} - \frac{1}{n} < r_n \leq \bar{r}$. Hence we can pick a sequence $r_n \uparrow \bar{r}$ in $\operatorname{supp}(G)$. Since

σ is strictly increasing, $\tau(r_n) \uparrow \bar{h}$. Because F is atomless, it is continuous at $\bar{h}$, so the winning probability satisfies

$$\Pr(H \geq \tau(r_n)) = 1 - F(\tau(r_n)) \longrightarrow 1 - F(\bar{h}) = 0.$$

Conditional on winning we have $H \in [\tau(r_n), \bar{h}] \subseteq [0,1]$, so $(1-H) \in [0,1]$ and, since $r_n \leq \bar{r}$,

$$-(1) \leq (1-H) \cdot r_n - 1 \leq \bar{r} - 1.$$

Thus the conditional margin $E[(1-H) \cdot r_n - 1 \mid H \geq \tau(r_n)]$ is uniformly bounded. Multiplying this bounded term by the winning probability (which converges to 0) yields $\pi_2(r_n) \to 0$.

Step 4 (Combine). Since $\pi_2(r_n) = \bar{\pi}_2$ for all n and $\pi_2(r_n) \to 0$, we have $\bar{\pi}_2 = 0$. Together with Step 2 this implies $\pi_2(r_2) = 0$ for all $r_2 \in \operatorname{supp}(G)$. $\square$

Since Bank 2 earns zero profits in equilibrium, we have $\pi_2(r_2) = 0$ for all $r_2 \in \operatorname{supp}(G)$. For any r_2 in the (relative) interior of the support, the winning probability $1 - F(\tau(r_2))$ is strictly positive, so the conditional expected margin must be zero:

$$E[(1-H) \mid H \geq \tau(r_2)] \cdot r_2 - 1 = 0 \implies r_2 = \frac{1}{E[(1-H) \mid H \geq \tau(r_2)]} \tag{3}$$

Then we can use profit maximization by the first firm; the FOC of equation 1 are:

$$\begin{aligned} -g(\sigma(h))[(1-h)\sigma(h) - 1] + [1 - G(\sigma(h))][1-h] &= 0 \\ \frac{1-h}{[(1-h)\sigma(h) - 1]} = \frac{g(\sigma(h))}{[1 - G(\sigma(h))]} &= -\frac{d}{dr}[\log(1 - G(r))]\bigg|_{r=\sigma(h)} \\ \frac{1-\tau(r)}{[(1-\tau(r))r - 1]} &= -\frac{d}{dr}[\log(1 - G(r))] \end{aligned}$$

[1] Integrating both sides from $\underline{r} = \sigma(\underline{h})$ to a given r, where $G(\underline{r}) = 0$ we have:

$$\begin{aligned} -[\log(1 - G(r)) - \log(1 - \underbrace{G(\underline{r})}_{=0})] &= \int_{\underline{r}}^{r} \frac{1-\tau(u)}{[(1-\tau(u))u - 1]} du \\ -\log(1 - G(r)) &= \int_{\underline{r}}^{r} \frac{1-\tau(u)}{[(1-\tau(u))u - 1]} du \\ G(r) &= 1 - \exp\left[-\int_{\underline{r}}^{r} \frac{1-\tau(u)}{[(1-\tau(u))u - 1]} du\right] \end{aligned} \tag{4}$$

Let $k = \tau(r_2)$ the the default probability such that the home bank offers r_2, then we have $r_2 = \sigma(k)$ and we replace into equation 3

$$\sigma(h) = \frac{1}{E[(1-H) \mid H \geq h]} \equiv \frac{1}{\mu(h)} \tag{5}$$

[1] The second line uses the fact that the hazard rate $g(r)/[1 - G(r)] = -d\log(1 - G(r))/dr$. The third line follows by substituting $h = \tau(r)$ and $\sigma(h) = r$.

Then we can obtain the mixed strategy of bank 2 by a change of variables, consider $u = \sigma(t) \implies \tau(u) = t, du = \sigma'(t)dt$, then the limits of integration change from $[\underline{r}, r]$ to $[\underline{h}, \tau(r)]$. Substituting into equation 4 we have:

$$G(\sigma(h)) = 1 - \exp\left[-\int_{\underline{h}}^{h} \frac{1-t}{[(1-t)\sigma(t) - 1]}\sigma'(t)dt\right] \tag{6}$$

given that $\sigma(t) = 1/\mu(t)$, we have $\sigma'(t) = -\mu'(t)/\mu(t)^2$, replacing in the equation above:

$$G(\sigma(h)) = 1 - \exp\left[\int_{\underline{h}}^{h} \frac{1-t}{\frac{1-t}{\mu(t)} - 1}\frac{\mu'(t)}{\mu(t)^2}dt\right] = 1 - \exp\left[\int_{\underline{h}}^{h} \frac{(1-t)\mu'(t)}{(1-t-\mu(t))\mu(t)}dt\right] \tag{7}$$