1 Model

We extend Sharpe (1990) by introducing switching costs, following the equilibrium characterization of Von Thadden (2004).

1.1 Setup

A continuum of firms seek financing for two sequential projects with fixed investment I^t, $t = 1, 2$. Each project returns $R \cdot I^t$ with probability p_q and 0 otherwise, where $q \in \{L, H\}$ denotes firm quality with $p_L < p_H$. The proportion of high-quality firms is $\theta \in (0, 1)$. Firms do not know their own quality.

Risk-neutral banks compete à la Bertrand with unlimited funds at rate $\bar{r}$. Banks initially do not know firm quality, but a bank financing the first project (the inside bank) perfectly observes the outcome $\gamma \in \{S, F\}$ (success or failure). Outside banks observe nothing ($\phi = 0$).

Switching cost. If a firm switches to an outside bank in period 2, it incurs cost $\lambda \geq 0$. Thus, facing inside offer r_i and outside offer r_o, the firm switches iff $r_o + \lambda < r_i$.

1.2 Timing

Period 1: Banks offer rates r_j^1. The firm borrows, invests, and repays $(1 + r^1)I^1$ iff $\gamma = S$.
Period 2: The inside bank offers $r_i(\gamma)$; outside banks offer r_o. The firm chooses, paying r_i if staying or $r_o + \lambda$ if switching. The project realizes and is repaid if successful.
Assumptions: No long-term contracts; limited liability; if indifferent, the firm stays.

1.3 Benchmark Rates

Define success probabilities:

$$p = \theta p_H + (1 - \theta) p_L,$$
$$p(S) = \frac{\theta p_H^2 + (1 - \theta) p_L^2}{p}, \qquad p(F) = \frac{\theta(1 - p_H)p_H + (1 - \theta)(1 - p_L)p_L}{1 - p},$$

satisfying $p(S) > p > p(F)$. The zero-profit rates are:

$$1 + r_S = \frac{1 + \bar{r}}{p(S)}, \qquad 1 + r_p = \frac{1 + \bar{r}}{p}, \qquad 1 + r_F = \frac{1 + \bar{r}}{p(F)},$$

with $r_S < r_p < r_F$. We assume $(1 + r_F)I^2 \leq R \cdot I^2$.

1.4 Results

Let H_i^γ, $\gamma \in \{S, F\}$, denote the cumulative distribution function of the equilibrium mixed strategy of the inside bank given its information γ, and let H_o denote the c.d.f. of the equilibrium mixed strategy of the outside bank.

Proposition 1. *The bidding game in stage 4 has a unique mixed-strategy equilibrium. The equilibrium strategies are characterized as follows:*

1. ***The Inside Bank:***
   - *For F-firms, the inside bank plays a pure strategy, offering $r_i^F = r_F + \lambda$ with probability 1.*
   - *For S-firms, the inside bank randomizes according to an atomless cumulative distribution function H_i^S on the interval $[r_p + \lambda, r_F + \lambda]$, given by:*

$$H_i^S(r) = \frac{(r - \lambda) - r_p}{p(S)(1 + r - \lambda) - (1 + \bar{r})}$$

2. ***The Outside Bank:***
   - *The outside bank randomizes over the interval $[r_p, r_F]$.*
   - *The distribution H_o is continuous on $[r_p, r_F)$ but has a mass point (atom) at the upper bound r_F.*
   - *For $r \in [r_p, r_F)$, the cumulative distribution function is:*

$$H_o(r) = \frac{p(S)(r - r_p)}{p(S)(1 + r + \lambda) - (1 + \bar{r})}$$

   - *The size of the atom at r_F is $1 - \lim_{r \nearrow r_F} H_o(r) = 1 - \frac{p(S)(r_F - r_p)}{p(S)(1+r_F+\lambda)-(1+\bar{r})} = 1 - \frac{p(S)(1+\lambda+r_p)-(1+\bar{r})}{p(S)(1+r_F+\lambda)-(1+\bar{r})}$.*

Proof. The proof is provided in Appendix 2.1. □

Proposition 2 (Equilibrium Profits)**.** *In the unique mixed-strategy equilibrium, the expected profits are:*

1. ***Outside Bank:*** *The outside bank earns zero expected profit:*

$$\Pi_o = 0.$$

2. ***Inside Bank:*** *The inside bank's expected profits differ by firm type:*
   - ***On S-firms:*** *The inside bank earns a profit equal to the informational rent plus the value of the switching cost:*

$$\Pi_i^S = p(S)(r_p - r_S + \lambda).$$

   - ***On F-firms:*** *The inside bank earns a profit solely from extracting the switching cost when the outsider bids at the cap r_F:*

$$\Pi_i^F = p(F)\lambda \left(\frac{\lambda + r_p - r_S}{\lambda + r_F - r_S} \right).$$

Proof. The proof is provided in Appendix 2.2 □

1.5 Equilibrium Switching Probabilities

In this section, we calculate the probability that a firm switches from the inside bank to the outside bank in equilibrium. A switch occurs if the outside bank offers a rate r_o such that $r_o + \lambda < r_i$, where r_i is the rate offered by the inside bank.

1. Switching Probability for F-firms The inside bank plays a pure strategy for F-firms, setting $r_i^F = r_F + \lambda$. Thus, an F-firm switches if and only if the outside bank's offer satisfies:

$$r_o + \lambda < r_F + \lambda \implies r_o < r_F.$$

The outside bank plays a mixed strategy on $[r_p, r_F]$ with an atom at r_F. The probability of offering strictly less than r_F is $H_o(r_F^-)$. Using the expression from Eq. (A.12) and the limit derived in Proposition 2:

$$\text{Prob(Switch} \mid F) = H_o(r_F^-) = \frac{\lambda + r_p - r_S}{\lambda + r_F - r_S}.$$

This represents the probability that the outside bank does *not* play the atom at the top.

2. Switching Probability for S-firms For S-firms, the inside bank randomizes over $[r_p + \lambda, r_F + \lambda]$. A switch occurs if $r_o < r_i - \lambda$. Let $\tilde{r}_i = r_i - \lambda$ be the effective inside rate, which is distributed according to $F_i(r) = H_i^S(r + \lambda)$ on $[r_p, r_F]$. The probability of switching is:

$$\text{Prob(Switch} \mid S) = \int_{r_p}^{r_F} H_o(r)\, dF_i(r).$$

Substituting the equilibrium distributions $H_o(r)$ and $F_i(r)$:

$$\text{Prob(Switch} \mid S) = \int_{r_p}^{r_F} \left(\frac{p(S)(r - r_p)}{p(S)(1 + r + \lambda) - (1 + \bar{r})} \right) d \left(\frac{r - r_p}{p(S)(1 + r) - (1 + \bar{r})} \right).$$

This integral captures the interaction between the two mixed strategies. Since the outside bank bids more aggressively (lower rates) than the inside bank's effective rate (due to the atom r_F), S-firms are retained with higher probability than F-firms.

3. Aggregate Switching Probability The total probability of observing a switch in equilibrium is the weighted average of the type-specific probabilities:

$$\text{Prob(Switch)} = p \cdot \text{Prob(Switch} \mid S) + (1 - p) \cdot \text{Prob(Switch} \mid F).$$

where p is the prior probability of the firm being type S.

2 Appendix

2.1 Equilibrium proof

Before starting the proof, we define some notation.

Let H_i^γ, $\gamma \in \{S, F\}$, denote the cumulative distribution function of the equilibrium mixed strategy of the inside bank given its information γ, and let H_o denote the c.d.f. of the equilibrium mixed strategy of the outside bank. As usual, H_i^γ and H_o are weakly monotone and continuous from the right, i.e., $H(\hat{r}) = \Pr(r \leq \hat{r})$ for each of the three mixed strategies. Define $H(r^-) = \lim_{t \uparrow r} H(t)$. Finally, let

Define

$$\ell_i^\gamma = \inf\{r \mid H_i^\gamma(r) > 0\}, \qquad \gamma \in \{S, F\}, \tag{A1}$$
$$u_i^\gamma = \sup\{r \mid H_i^\gamma(r) < 1\}, \qquad \gamma \in \{S, F\}, \tag{A2}$$
$$\ell_o = \inf\{r \mid H_o(r) > 0\}, \tag{A3}$$
$$u_o = \sup\{r \mid H_o(r) < 1\}. \tag{A4}$$

Without loss of generality, we restrict attention to interest rates in $[0, X_2/I_2 - 1]$.

The expected profits from quoting interest rate r are

$$P_i^\gamma(r) = \big(1 - H_o((r-\lambda)^-)\big)\big[p(\gamma)(1+r) - (1+\bar{r})\big], \qquad \gamma \in \{S, F\}, \tag{A5}$$
$$\begin{aligned} P_o(r) = {} & p\big(1 - H_i^S(r+\lambda)\big)\big[p(S)(1+r) - (1+\bar{r})\big] \\ & + (1-p)\big(1 - H_i^F(r+\lambda)\big)\big[p(F)(1+r) - (1+\bar{r})\big]. \end{aligned} \tag{A6}$$

For what follows it is useful to define $\hat{\ell}_o = \ell_o + \lambda$, $\hat{u}_o = u_o + \lambda$ and $\hat{H}_o$ the c.d.f. of the interest rates offered by the outside bank plus λ.

Step 1

$\ell_i^\gamma \geq r_\gamma$ for $\gamma \in \{S, F\}$.

Proof. Otherwise profits would be negative.

Step 2

$\ell_o \geq r_p$.

Proof. we know $r_f > r_p$, using step 1 $\ell_i^f \geq r_f > r_p$ hence any offer $r < r_p$ attracts at best both groups and at worse only the failures. Given that the cost is at best the pooling cost, the outside bank will not make offers lower than the pooling cost.

Step 3

$\ell_i^S \geq r_p + \lambda$.

Proof. Follows from Step 2. Any offer, by the inside bank, lower than that could be raised slightly without decreasing the probability of winning.

Step 4

$\hat{u}_o \geq u_i^S$.

Proof. Suppose $\hat{u}_o < u_i^S$, then the inside bank makes zero expected profits on all offers $r(s) \in (\hat{u}_0, u_i^s]$, however by step 3 the inside bank makes strictly positive profits on the S-firm.

Step 5

H_i^S is continuous on $[\ell_i^S, u_i^S)$.

Proof. Suppose that there is a $\hat{r} \in [\ell_i^S, u_i^S)$ at which H_i^S is discontinuous, i.e., with

$$H_i^S(\hat{r}^-) < H_i^S(\hat{r}).$$

Then, by Eq. (A.6), $P_o(\hat{r}^-) > P_o(\hat{r})$, because $p(S)(1+r) - (1+\bar{r}) > 0$ on $[\ell_i^S, u_i^S)$ by Step 3[1].

By the right-hand continuity of H_i^γ, $\gamma \in \{S, F\}$, there is an $\varepsilon > 0$ such that $H_o(\hat{r}^-) = H_o(r)$ is constant on $[\hat{r}, \hat{r}+\varepsilon]$. Therefore, P_i^S is continuous at $\hat{r}$ and strictly increasing on $[\hat{r}, \hat{r}+\varepsilon]$. Hence, H_i^S can have no mass on $[\hat{r}, \hat{r}+\varepsilon]$, which implies that $H_i^S(\hat{r}^-) = H_i^S(\hat{r})$. Contradiction.

Note that the proof of Step 5 does not apply to H_i^F, because we do not know whether the inside bank makes strictly positive profits on the F-firm.

Step 6

$u_i^S \geq \ell_i^F$.

Proof. Suppose that $u_i^S < \ell_i^F$. This implies that the inside bank never makes an offer $r \in (u_i^S, \ell_i^F)$.

(a) Suppose that $u_i^S < \hat{u}_o$. Then $\hat{H}_o$ can have no mass on $[u_i^S, \ell_i^F)$, because for every offer $r \in [u_i^S, \ell_i^F)$ the offer $\frac{1}{2}(r + \ell_i^F)$ would be strictly better for the outside banks. Then the (positive) mass of $\hat{H}_o$ on $[u_i^S, u_o]$ lies on $[\ell_i^F, \hat{u}_o]$. In particular, $\hat{H}_o$ is continuous at u_i^S[2].

Consider the following deviation from H_i^S: let $\delta > 0$ and $\varepsilon > 0$ be given and small. Let M_ε be the mass of H_i^S on $[u_i^S - \varepsilon, u_i^S]$. The deviation strategy is identical to H_i^S on $[\ell_i^S, u_i^S - \varepsilon)$, has zero mass on $[u_i^S - \varepsilon, \ell_i^F - \delta)$ and point mass M_ε on $\ell_i^F - \delta$. The expected net gain (given $\gamma = S$) from this deviation is not smaller than

$$M_\varepsilon \Big[(1 - \hat{H}_o(u_i^S))(\ell_i^F - u_i^S - \delta) - \big(\hat{H}_o(u_i^S) - \hat{H}_o(u_i^S - \varepsilon)\big) u_i^S \Big]. \tag{A.7}$$

The first of the two terms in Eq. (A.7) (which corresponds to the total gain from the deviation) is strictly positive for δ sufficiently small. The second term (corresponding to the total loss from the deviation) tends to 0 for $\varepsilon \to 0$ by the continuity of H_o at u_i^S. Hence, the deviation is strictly profitable for δ and ε small enough.

(b) Suppose that $u_i^S = \hat{u}_o$. Consider the following deviation from $\hat{H}_o$: let $\delta > 0$ and $\varepsilon > 0$ be given and small. Let N_ε be the mass of $\hat{H}_o$ on $[\hat{u}_o - \varepsilon, \hat{u}_o]$. Move all mass of $[\hat{u}_o - \varepsilon, \hat{u}_o)$ to $\ell_i^F - \delta$. Then the expected net gain from this deviation is not smaller than

$$N_\varepsilon \Big[\underbrace{(1-p)[(\ell_i^F - \delta) - (u_o + \lambda)]}_{Gain from \gamma = F} - p\big(H_i^S(\hat{u}_o) - H_i^S(\hat{u}_o - \varepsilon)\big)\big(p(S)(1+\hat{u}_o) - (1+\bar{r})\big) \Big]$$

where the second term now tends to 0 for $\varepsilon \to 0$ by Step 5.

[1] Given that in step 3 we have that the expected profits of the incumbent when selling to S are strictly positive, the outside bank can switch some probability from $\hat{r}$ to $\hat{r} - \epsilon$ and increase their profits.

[2] The cdf will be flat on the $[u_i^S, \ell_i^F)$ interval.

Step 7.

$u_i^F \leq \hat{u}_o$.

Proof. Suppose that $u_i^F > \hat{u}_o$. Then, for any offer $r \in (\hat{u}_o, u_i^F]$, the inside bank loses the F-firm with probability 1 (since $r > u_o + \lambda \geq r_o + \lambda$). Consequently, the inside bank makes zero profit on these offers.

Given that, from step 4 $u_i^s \leq \hat{u}_o$, the outsider when setting u_o gets only F-firms. Since the inside firm makes zero profits then $\hat{u}_o \leq r_F$, if this was not the case then there is a $\varepsilon > 0$ such that $r_F + \varepsilon < u_o + \lambda$ and the insider can sell at a profit. But if $\hat{u}_o \leq r_F$ then the outside firms makes a loss when selling at the upper range, hence actually $\hat{u}_o > r_F$ and then the insider has to be making a profit with the F-firms.

However, this leads to a contradiction because the inside bank has a strategy available that guarantees strictly positive profits on the F-firm. Specifically, from Step 2 we know $\ell_o \geq r_p$. Thus, the outside bank never offers a rate below r_p. The inside bank can offer a fixed rate $r^* = r_F + \varepsilon$. Provided that $\varepsilon > 0$ is small enough such that $r_F + \varepsilon < r_p + \lambda$ (which is possible under the assumption that switching costs are non-trivial, i.e., $r_F < r_p + \lambda$), we have $r^* < \ell_o + \lambda$. By offering r^*, the inside bank retains the F-firm with probability 1 against any outside offer. Since the margin ε is positive, the expected profit is strictly positive. Thus, the condition $P_i^F = 0$ is impossible, and the assumption $u_i^F > \hat{u}_o$ must be false.

Step 8 (equivalent to step 9 in the original proof).

$u_o = r_F$ and $u_i^S = r_F + \lambda$.

Proof. First, we prove $u_o \leq r_F$ by contradiction. Suppose $u_o > r_F$. Since the upper bound is above the break-even rate, the outside bank must make strictly positive expected profits in equilibrium ($P_o > 0$). From Steps 4 and 7, we know that the inside bank's supports satisfy $u_i^S \leq \hat{u}_o$ and $u_i^F \leq \hat{u}_o$. Thus, at the outsider's maximum bid u_o, the switching condition $r_i > u_o + \lambda$ is never met (except possibly via ties). Standard undercutting arguments imply that the inside bank cannot have atoms at $\hat{u}_o$. Therefore, an outside bid of u_o wins with probability zero. Consequently, the profit at the top is zero: $P_o(u_o) = 0$. In a mixed strategy equilibrium, all strategies in the support must yield the same expected profit. Thus, $P_o(u_o) = 0$ implies $P_o(r) = 0$ for all r, contradicting the condition that $P_o > 0$. Therefore, the assumption $u_o > r_F$ must be false. We conclude $u_o \leq r_F$.

Second, we prove $u_o \geq r_F$. Suppose $u_o < r_F$. Consider the profit at the upper bound u_o. Since $u_i^S \leq \hat{u}_o$, the inside bank retains S-firms against the bid u_o with probability 1. Thus, the outsider wins no S-firms at this price. The only firms the outsider can possibly win at u_o are F-firms. However, since $u_o < r_F$, winning an F-firm yields a strictly negative profit ($u_o - r_F < 0$). Thus, the expected profit at u_o is non-positive. Since the outside bank can guarantee zero profit by not participating, it cannot be optimal to set an upper bound $u_o < r_F$ that yields losses. Therefore, $u_o \geq r_F$.

Combining these results, we have $u_o = r_F$.

Finally, we establish $u_i^S = r_F + \lambda$. From Step 4, we have $u_i^S \leq \hat{u}_o = r_F + \lambda$. Suppose for contradiction that $u_i^S < r_F + \lambda$. Then there exists a range of outside bids $r_o \in (u_i^S - \lambda, r_F]$ such that $r_o + \lambda > u_i^S$. For any such bid r_o:

- The outsider wins zero S-firms, because the inside bank always offers $r_i(S) \leq u_i^S < r_o + \lambda$.

- The outsider might win F-firms (if the insider bids high enough), but since the price is $r_o \leq r_F$, winning *only* F-firms yields non-positive profit ($r_o - r_F \leq 0$).

Thus, bids in this range yield non-positive expected profit. Since the equilibrium profit is non-negative, including these dominated strategies in the support is suboptimal. A rational outsider would lower u_o to eliminate these bids. Therefore, there can be no gap, and we must have $u_i^S = r_F + \lambda$.

Step 9 (equivalent to step 8 in the original proof).

$u_i^F = r_F + \lambda$.

Proof. Clearly, $u_i^F \geq r_F$.[3] Suppose that $u_i^F > r_F + \lambda$. Since the outside bank can obtain strictly positive expected profits by choosing a fixed rate $r = \frac{1}{2}(r_F + u_i^F - \lambda)$ (which satisfies $r > r_F$ and $r < u_i^F - \lambda$), it must make strictly positive expected profits in equilibrium ($P_o > 0$).

From Step 7, we know $\hat{u}_o \geq u_i^F$. Combining this with our assumption ($u_i^F > r_F + \lambda$), we have:

$$\hat{u}_o \geq u_i^F > r_F + \lambda \implies u_o + \lambda > r_F + \lambda \implies u_o > r_F.$$

Since $u_o > r_F$, the outside bank bids above the break-even rate for F-firms. Consequently, H_i^F must also yield strictly positive expected profits (as the insider can undercut $\hat{u}_o$ slightly and win with a positive margin).

However, from Steps 4 and 7, we know $\hat{u}_o$ is the upper bound for both firm types ($u_i^S \leq \hat{u}_o$ and $u_i^F \leq \hat{u}_o$). Standard undercutting arguments imply that H_i^S and H_i^F cannot have atoms at $\hat{u}_o$. Therefore, we have the following implication chain at the top of the support[4]:

$$P_o(u_o) = 0 \;\Rightarrow\; {}^{5}H_o(u_o^-) = 1 \;\Rightarrow\; {}^{6}P_i^S(\hat{u}_o) = P_i^F(\hat{u}_o) = 0 \Rightarrow {}^{7}H_i^S(\hat{u}_o^-) = H_i^F(\hat{u}_o^-) = 1 \;\Rightarrow\; {}^{8}P_o(u_o^-) = 0,$$

This result ($P_o(u_o^-) = 0$) contradicts the earlier finding that H_o makes strictly positive expected profits. Thus, the assumption $u_i^F > r_F + \lambda$ must be false, implying $u_i^F \leq r_F + \lambda$.

Finally, to prove that $u_i^F < r_F + \lambda$ is not true, we use proof by contradiction. Suppose that $u_i^F < r_F + \lambda$. Using the fact shown in Step 8, we have that $u_o = r_F$ and $u_i^S = r_F + \lambda$, this assumption implies $u_i^F < u_o + \lambda$. Consider the outside bank's expected profit for bids in the gap interval $r_o \in (u_i^F - \lambda, u_o]$.

- **F-firms:** The outsider wins zero F-firms. Since the inside bank stops bidding for F-firms at u_i^F, the switching condition $r_i > r_o + \lambda$ requires $r_i > u_i^F$, which occurs with probability 0.
- **S-firms:** The outsider wins S-firms with positive probability. Since $u_i^S = u_o + \lambda$, the insider places mass above $r_o + \lambda$. Thus, the outsider captures the "good" firms while avoiding the "bad" ones.

[3] The inside bank will not charge below the break-even cost r_F as this guarantees a loss.
[4] $P_o(u_o) = 0$ because the firm when pricing at the top of the support never wins.
[8] Since P_o is strictly positive in equilibrium (as shown above), but zero at u_o, u_o cannot be in the active support (by the indifference principle).
[8] When the insider plays $\hat{u}_o$ (effective price $u_o + \lambda$), they lose to the outsider (who plays $\leq u_o$) with probability 1, yielding 0 profit.
[8] Since the insider makes positive profits elsewhere, they cannot put probability mass on $\hat{u}_o$, which yields 0 profit.
[8] Just below u_o, the outsider wins no customers because the insider has already finished bidding at $\hat{u}_o$ (probability of winning drops to 0).

Since S-firms are strictly profitable at the rate r_F (which is the break-even rate for the riskier F-firms), the outside bank earns strictly positive profits in this range. However, at the precise upper bound u_o, the outsider wins zero S-firms (as $u_i^S = u_o + \lambda$, undercutting applies) and zero F-firms. Thus $P_o(u_o) = 0$. This creates a contradiction: profits cannot be strictly positive just below u_o and zero at u_o in an equilibrium support. Therefore, the gap cannot exist, and we must have $u_i^F = r_F + \lambda$.

Step 10.

The outside bank makes zero expected profits.
Proof. By Steps 9 and 8 we have $u_o = r_F$ and $u_i^S = u_i^F = r_F + \lambda$. Since the inside bank's strategies H_i^S and H_i^F have no atoms at the top of the support (due to standard undercutting arguments), we have:

$$\lim_{r \nearrow r_F} H_i^S(r+\lambda) = H_i^S(r_F+\lambda) = 1 \quad \text{and} \quad \lim_{r \nearrow r_F} H_i^F(r+\lambda) = H_i^F(r_F+\lambda) = 1.$$

Substituting these limits into the outside bank's profit function (Eq. A.6) as r approaches $u_o = r_F$:

$$\begin{aligned}
\lim_{r \nearrow r_F} P_o(r) &= \lim_{r \nearrow r_F} \Big(p\big(1 - H_i^S(r+\lambda)\big)\big[p(S)(1+r) - (1+\bar{r})\big] \\
&\quad + (1-p)\big(1 - H_i^F(r+\lambda)\big)\big[p(F)(1+r) - (1+\bar{r})\big] \Big) \\
&= p(0)\big[p(S)(1+r_F) - (1+\bar{r})\big] + (1-p)(0)\big[p(F)(1+r_F) - (1+\bar{r})\big] \\
&= 0.
\end{aligned}$$

Since H_o is an equilibrium mixed strategy, the expected profit $P_o(r)$ must be constant for all r in the active support $[\ell_o, u_o)$. Therefore, $P_o(r) = 0$ for all r in the support.

Step 11.

$\ell_o = r_p$ and $\ell_i^S = r_p + \lambda$.
Proof. First, we show that the lower bounds satisfy $\ell_i^S = \ell_o + \lambda$. Suppose $\ell_i^S > \ell_o + \lambda$. Then the outside bank could raise its lowest offers from ℓ_o to $\ell_o + \varepsilon$ (where $\ell_o + \varepsilon + \lambda < \ell_i^S$) and still undercut the inside bank's entire distribution with probability 1. This would strictly increase the profit margin without reducing market share, contradicting the optimality of ℓ_o. Suppose $\ell_i^S < \ell_o + \lambda$. Then the inside bank could raise its lowest offers from ℓ_i^S to $\ell_i^S + \varepsilon$ (where $\ell_i^S + \varepsilon < \ell_o + \lambda$) and still retain the customer with probability 1 against any outside offer. This would strictly increase profits, contradicting the optimality of ℓ_i^S. Thus, we must have $\ell_i^S = \ell_o + \lambda$.

Second, we prove $\ell_o = r_p$. From Step 2, we know $\ell_o \geq r_p$. Suppose strictly that $\ell_o > r_p$. Consider the outside bank's profit at the lower bound ℓ_o. Since $\ell_o + \lambda = \ell_i^S$, the bid ℓ_o is effectively strictly lower than any bid r_i in the inside bank's support (except exactly at the boundary, which has zero mass). Consequently, at ℓ_o, the outside bank satisfies the switching condition $r_i > \ell_o + \lambda$ with probability 1 for both S and F firms. The outside bank thus captures the entire pool of borrowers. The expected profit at this lower bound is the pooling profit:

$$P_o(\ell_o) = p[p(S)(1+\ell_o) - (1+\bar{r})] + (1-p)[p(F)(1+\ell_o) - (1+\bar{r})].$$

Since $\ell_o > r_p$ (where r_p is defined as the zero-profit pooling rate), this profit is strictly positive. However, Step 10 established that the equilibrium profit P_o is zero. This contradiction implies $\ell_o = r_p$. It immediately follows that $\ell_i^S = r_p + \lambda$.

Step 12.

H_o is continuous on $[r_p, r_F)$.[9]
Proof. Suppose that H_o has an atom at some $\hat{r} \in [r_p, r_F)$, i.e., $H_o(\hat{r}^-) < H_o(\hat{r})$. Then, for the inside bank, the expected profit function $P_i^S(r)$ would jump upward at $r = \hat{r} + \lambda$. Specifically, because of the atom at $\hat{r}$, $P_i^S(\hat{r} + \lambda) > P_i^S(r)$ for r in a left-neighborhood of $\hat{r} + \lambda$. Consequently, the inside bank would not play strategies in a small interval just below $\hat{r} + \lambda$. This implies H_i^S is constant on some interval $(\hat{r} + \lambda - \varepsilon, \hat{r} + \lambda)$. Substituting this into the outside bank's profit function (Eq. A.6), since $H_i^S(r + \lambda)$ is constant for $r \in (\hat{r} - \varepsilon, \hat{r})$, $P_o(r)$ is strictly increasing in this interval (as the markup $(1 + r)$ increases while the probability of winning remains fixed). Since P_o is strictly increasing on $(\hat{r} - \varepsilon, \hat{r})$, the outside bank would not put mass on the lower part of this interval, pushing the mass up to $\hat{r}$. However, if P_o is strictly increasing up to $\hat{r}$, then $P_o(\hat{r}^-) < P_o(\hat{r})$ (assuming continuity of distributions elsewhere). But if H_o has an atom at $\hat{r}$, standard indifference arguments usually require P_o to be constant or flat. More formally, if H_i^S is constant near $\hat{r} + \lambda$, the outside bank has an incentive to bid slightly higher (up to $\hat{r}$) to capture the higher margin without losing market share. This contradicts the equilibrium requirement that players are indifferent over the support. Thus, H_o can have no mass on $\hat{r}$.

Step 13.

H_i^S is strictly increasing on $[r_p + \lambda, r_F + \lambda]$ and H_o is strictly increasing on $[r_p, r_F)$. [10]
Proof. Suppose that H_i^S is constant on some interval $[\alpha, \beta] \subset [r_p + \lambda, r_F + \lambda]$. Let $[a, b] \supseteq [\alpha, \beta]$ be the maximal such interval. By Step 5 and the definition of $\ell_i^S = r_p + \lambda$, we must have $a > r_p + \lambda$. Since $H_i^S(r)$ is constant for $r \in [a, b]$, the term $H_i^S(r_o + \lambda)$ in the outside bank's profit function (Eq. A.6) is constant for outside offers $r_o \in [a - \lambda, b - \lambda]$. Consequently, $P_o(r_o)$ is strictly increasing on $[a - \lambda, b - \lambda]$ (as the markup increases while the winning probability is constant). Therefore, the outside bank will place no mass on $[a - \lambda, b - \lambda)$ (it would prefer the upper endpoint). Thus, H_o is constant on $[a - \lambda, b - \lambda)$. Now consider the inside bank's profit $P_i^S(r_i)$ for $r_i \in [a, b)$. Since H_o is constant on $[a - \lambda, b - \lambda)$, the term $H_o((r_i - \lambda)^-)$ is constant. This implies P_i^S is strictly increasing on $[a, b]$, so the inside bank would not put mass on $[a, b]$. This contradicts the definition of $[a, b]$ as a "flat" spot in the cumulative distribution (which implies no mass), unless the interval extends all the way to a boundary where the logic changes. However, gaps in support are ruled out by standard "no gap" arguments (if there is a gap, the lower bound of the upper segment is suboptimal).

Step 14 (not in the original proof)

$u_i^F = \ell_i^F = r_F + \lambda$.
Proof.

[9] this proof was not checked thoroughly
[10] this proof was not checked thoroughly

From equation (A.5) we have that:

$$P_i^S(r) = (1 - H_o(r-\lambda)^-)[p(S)(1+r) - (1+\bar{r})] \tag{1}$$
$$P_i^F(r) = (1 - H_o(r-\lambda)^-)[p(F)(1+r) - (1+\bar{r})]. \tag{2}$$

note that from Step 13 we know that H_i^S is strictly increasing on $[r_p+\lambda, r_F+\lambda]$, which means that $P_i^S(r)$ is constant on this interval. Hence for any r in this interval, we have:

$$\begin{aligned} P_i^F(r) &= (1 - H_o(r-\lambda)^-)\frac{[p(S)(1+r) - (1+\bar{r})]}{[p(S)(1+r) - (1+\bar{r})]}[p(F)(1+r) - (1+\bar{r})] \\ &= \frac{P_i^S(r)}{[p(S)(1+r) - (1+\bar{r})]}[p(F)(1+r) - (1+\bar{r})] = P_i^S(r)\frac{[p(F)(1+r) - (1+\bar{r})]}{[p(S)(1+r) - (1+\bar{r})]} \end{aligned} \tag{3}$$

the first term of the right hand side is constant in r as established before. The second term is increasing in r given $p(S) > p(F)$, which just means that the success of the project is greater if the first period project was good than if it was bad. [11] Given that $P_i^F(r)$ is increasing in r over the support $[r_p+\lambda, r_F+\lambda]$, the inside bank maximizes its profit on F-firms by placing all probability mass at the upper bound of the support. Hence, we have that $u_i^F = \ell_i^F = r_F + \lambda$.[12]

The last step has completed the characterization of the mixed strategies, because it implies that P_i^S and P_o are constant on their respective supports. Specifically, $P_o(r)$ is constant on $[r_p, r_F)$ and $P_i^S(r)$ is constant on $[r_p+\lambda, r_F+\lambda]$. By the continuity of H_o on $[r_p, r_F)$ and H_i^S on $[r_p+\lambda, r_F+\lambda)$, we obtain therefore from Eqs. (A.5) and (A.6) for the outside rate $r \in [r_p, r_F)$:

$$(1 - H_o(r))[p(S)(1+r+\lambda) - (1+\bar{r})] = c, \tag{A.9}$$
$$p(1 - H_i^S(r+\lambda))[p(S)(1+r) - (1+\bar{r})] + (1-p)[p(F)(1+r) - (1+\bar{r})] = 0. \tag{A.10}$$

Note that in Eq. (A.10), we use the fact that $H_i^F(r+\lambda) = 0$ for $r < r_F$ (as the insider places the F-firm mass at $r_F + \lambda$).

[11]

$$P_i^F(r) = P_i^S(r)\frac{[p(F)(1+r) - (1+\bar{r})]}{[p(S)(1+r) - (1+\bar{r})]}$$

hence, using the fact that $P_i^S(r)$ is constant in r we have that:

$$\frac{dP_i^F(r)}{dr} = P_i^S(r)\frac{d}{dr}\frac{[p(F)(1+r) - (1+\bar{r})]}{[p(S)(1+r) - (1+\bar{r})]} \implies \frac{dP_i^F(r)}{dr} = P_i^S(r)\frac{d}{dr}\frac{(1+\bar{r})\cdot[p(S) - p(F)]}{[p(S)(1+r) - (1+\bar{r})]^2}$$

[12] Later on we will see that the outside firm actually makes positive profits from the F-firms because the outside firm plays r_F with positive probability mass.

The constant c in Eq. (A.9) can be determined by substituting the lower bound $r = r_p$ (where $H_o(r_p) = 0$) into Eq. (A.9):

$$c = p(S)(1 + r_p + \lambda) - (1 + \bar{r}) = p(S)(r_p + \lambda - r_S).$$

Straightforward manipulations of Eqs. (A.9) and (A.10) then yield the equilibrium distributions:

$$H_i^S(r + \lambda) = \frac{r - r_p}{p(S)(1 + r) - (1 + \bar{r})}, \tag{A.11}$$

$$H_o(r) = \frac{p(S)(r - r_p)}{p(S)(1 + r + \lambda) - (1 + \bar{r})}. \tag{A.12}$$

for $r \in [r_p, r_F)$. One easily checks that $\lim_{r \to r_F} H_i^S(r + \lambda) = 1$ (using the standard Sharpe identity $\frac{r_F - r_p}{p(S)(r_F - r_S)} = 1$), hence, H_i^S is continuous on its domain. On the other hand, Eq. (A.12) shows that $H_o(r_F^-) < 1$. Specifically, the denominator contains the term $+\lambda$, keeping the value strictly below 1. Thus, H_o has a mass point (atom) at the upper bound r_F.

This identifies a unique mixed strategy profile. Because both players randomize over the intervals $[r_p, r_F]$ (outsider) and $[r_p + \lambda, r_F + \lambda]$ (insider), there are no profitable deviations from these strategies for either player. Proposition 2 is therefore proved.

2.2 Calculation of expected profits

1. Outside Bank: From Step 10, we established that $P_o(r) = 0$, in section 2.1, for all r in the support. Thus, the total expected profit is zero.

2. Inside Bank (S-firms): The inside bank is indifferent among all strategies in its support $[r_p + \lambda, r_F + \lambda]$. We calculate the profit at the lower bound $r = r_p + \lambda$. At this price, the inside bank wins with probability 1 (since the outsider never bids below r_p). The profit is:

$$\Pi_i^S = P_i^S(r_p + \lambda) = 1 \cdot [p(S)(1 + r_p + \lambda) - (1 + \bar{r})].$$

Using the definition $1 + r_S = \frac{1+\bar{r}}{p(S)}$, we substitute $(1 + \bar{r}) = p(S)(1 + r_S)$:

$$\Pi_i^S = p(S)(1 + r_p + \lambda) - p(S)(1 + r_S) = p(S)(r_p - r_S + \lambda).$$

3. Inside Bank (F-firms): The inside bank plays the pure strategy $r = r_F + \lambda$. It wins only when the outside bank plays its atom at r_F. The probability of winning is the size of the outsider's atom: $\alpha = 1 - H_o(r_F^-)$. From Eq. (A.12), we have:

$$H_o(r_F^-) = \frac{p(S)(r_F - r_p)}{p(S)(1 + r_F + \lambda) - (1 + \bar{r})}.$$

Substituting $(1 + \bar{r}) = p(S)(1 + r_S)$, the denominator becomes $p(S)(r_F - r_S + \lambda)$. Thus:

$$H_o(r_F^-) = \frac{r_F - r_p}{r_F - r_S + \lambda}.$$

The probability of winning is:

$$\alpha = 1 - \frac{r_F - r_p}{r_F - r_S + \lambda} = \frac{(r_F - r_S + \lambda) - (r_F - r_p)}{r_F - r_S + \lambda} = \frac{\lambda + r_p - r_S}{\lambda + r_F - r_S}.$$

The profit margin at $r_F + \lambda$ is:

$$\text{Margin} = p(F)(1 + r_F + \lambda) - (1 + \bar{r}).$$

Using $1 + r_F = \frac{1+\bar{r}}{p(F)}$, this simplifies to:

$$\text{Margin} = p(F)\left(\frac{1 + \bar{r}}{p(F)} + \lambda\right) - (1 + \bar{r}) = p(F)\lambda.$$

Therefore, $\Pi_i^F = \alpha \cdot \text{Margin} = p(F)\lambda\left(\frac{\lambda + r_p - r_S}{\lambda + r_F - r_S}\right)$.