I think this model is better than the model of Sharpe (model 2) because it is more general, for example the consumer type is allowed to have a generic distribution whereas in the case of Sharpe's model there are only two type of consumers. Moreover the equilibrium proof is easier and there is more litereature on this types of model(Engelbrecht-Wiggans et al., 1983; Hendricks and Porter, 1988), hence it would be easier to present and readers probably would prefer it over the Sharpe's model.

1 Model

Adapting Engelbrecht-Wiggans et al. (1983)

This version incorporates switching costs. Let bank 1 be the incumbent, which has a relationship with the borrower. The borrower incurs a switching cost $\lambda > 0$ if they choose bank 2. This borrower chooses bank 2 if $r_1 > r_2 + \lambda$, otherwise it chooses bank 1.

Define h to be default probability, there is an informed bank (bank 1) and an uninformed bank (bank 2), which is smoothly distributed in the population according to the cdf $F(h)$ and pdf $f(h)$. The strategies are $r_1(h) = \sigma(h) : h \to r$ and $G(x) = \Pr(r_2 \leq x)$, which are interest rates.

Assume that in equilibrium σ is an increasing function, and denote by $\tau : r \to h$ its inverse, $\tau(\sigma(h)) = h$.

Then expected profits of bank 1 are:

$$\begin{aligned} \pi_1(r_1(h)) &= \Pr(\sigma(h) \leq r_2 + \lambda) \cdot [(1-h)\sigma(h) - 1] \\ &= \Pr(r_2 \geq \sigma(h) - \lambda) \cdot [(1-h)\sigma(h) - 1] \\ &= [1 - G(\sigma(h) - \lambda)] \cdot [(1-h)\sigma(h) - 1] \end{aligned} \tag{1}$$

and the expected profits of bank 2 are:

Bank 2 wins if $r_2 < \sigma(h) - \lambda$, which is $\sigma(h) > r_2 + \lambda$.

$$\begin{aligned} \pi_2(r_2) &= \Pr(\sigma(h) > r_2 + \lambda) \cdot E[(1-h) \cdot r_2 - 1 \mid \sigma(h) > r_2 + \lambda] \\ &= \Pr(h > \tau(r_2 + \lambda)) \cdot E[(1-h) \cdot r_2 - 1 \mid h > \tau(r_2 + \lambda)] \\ &= [1 - F(\tau(r_2 + \lambda))] \cdot [E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1] \end{aligned} \tag{2}$$

Assume that bank 2 makes zero profits[1], then we have:

$$[1 - F(\tau(r_2 + \lambda))] \cdot [E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1] = 0 \tag{3}$$

since the winning probability is not zero, then the expected profits have to be zero.

$$E[(1-h) \mid h > \tau(r_2 + \lambda)] \cdot r_2 - 1 = 0 \implies r_2 = \frac{1}{E[(1-h) \mid h > \tau(r_2 + \lambda)]} \tag{4}$$

Let $k = \tau(r_2 + \lambda)$, which means $\sigma(k) = r_2 + \lambda$. Therefore $r_2 = \sigma(k) - \lambda$. Substituting:

$$\sigma(k) - \lambda = \frac{1}{E[1 - h | h > k]}$$

[1] Would have to be proved, but I am confident that it is true since the uninformed firm in auction models always makes zero profits.

Hence,

$$\sigma(h) = \lambda + \mu(h)^{-1} \tag{5}$$

where $\mu(h) = E[1 - H|H > h]$.

Then we can use profit maximization by the first firm, the FOC of equation 1 are:

$$\begin{aligned}
-g(\sigma(h) - \lambda)[(1-h)\sigma(h) - 1] + [1 - G(\sigma(h) - \lambda)][1 - h] = 0 \\
\frac{1-h}{[(1-h)\sigma(h) - 1]} = \frac{g(\sigma(h) - \lambda)}{[1 - G(\sigma(h) - \lambda)]} = -\frac{d}{d\sigma}[\log(1 - G(\sigma(h) - \lambda))] \\
\frac{1 - \tau(r)}{[(1 - \tau(r))r - 1]} = -\frac{d}{d\sigma}[\log(1 - g(r - \lambda))]
\end{aligned} \tag{6}$$

Integrating both sides from $\underline{r} = \sigma(\underline{h})$ to a given r, where $G(\underline{r} - \lambda) = 0$ we have:

$$\begin{aligned}
-[\log(1 - G(r - \lambda)) - \log(1 - \underbrace{G(\underline{r} - \lambda))}_{=0}] &= \int_{\underline{r}}^{r} \frac{1 - \tau(u)}{[(1 - \tau(u))u - 1]} du \\
-\log(1 - G(r - \lambda)) &= \int_{\underline{r}}^{r} \frac{1 - \tau(u)}{[(1 - \tau(u))u - 1]} du \\
G(r - \lambda) &= 1 - \exp\left[-\int_{\underline{r}}^{r} \frac{1 - \tau(u)}{[(1 - \tau(u))u - 1]} du\right]
\end{aligned} \tag{7}$$

1.1 Case with $\lambda = 0$

From equation 5 we have that:

$$\sigma(h) = \frac{1}{E[(1 - H) \mid H \leq h]} \equiv \frac{1}{\mu(h)} \tag{8}$$

Then we can obtain the mixed strategy of bank 2 by a change of variables, consider $u = \sigma(t) \implies \tau(u) = t, du = \sigma'(t)dt$, then the limits of integration change from $[\underline{r}, r]$ to $\underline{h}, \tau(r)]$. Substituting into equation 7 we have:

$$G(\sigma(h)) = 1 - \exp\left[-\int_{\underline{h}}^{h} \frac{1 - t}{[(1 - t)\sigma(t) - 1]} \sigma'(t)dt\right] \tag{9}$$

given that $\sigma(t) = 1/\mu(t)$, we have $\sigma'(t) = -\mu'(t)/\mu(t)^2$, replacing in the equation above:

$$G(\sigma(h)) = 1 - \exp\left[\int_{\underline{h}}^{h} \frac{1 - t}{\frac{1-t}{\mu(t)} - 1} \frac{\mu'(t)}{\mu(t)^2} dt\right] = 1 - \exp\left[\int_{\underline{h}}^{h} \frac{(1 - t)\mu'(t)}{(1 - t - \mu(t))\mu(t)} dt\right] \tag{10}$$

1.2 Example

To derive a closed-form solution, we assume that the default risk h is uniformly distributed on $[\underline{h}, \bar{h}]$, where $0 < \underline{h} < \bar{h} < 1$. This ensures that even the safest borrower has positive default risk and that no borrower defaults with certainty. The conditional expectation is then:

$$E[1-h \mid h>k] = \frac{\int_k^{\bar{h}}(1-x)dx}{\int_k^{\bar{h}} dx} = \frac{\left[(x-\frac{x^2}{2})\right]_k^{\bar{h}}}{\bar{h}-k} = \frac{(\bar{h}-\frac{\bar{h}^2}{2})-(k-\frac{k^2}{2})}{\bar{h}-k}$$

Simplifying the numerator:

$$\bar{h}-\frac{\bar{h}^2}{2}-k+\frac{k^2}{2} = (\bar{h}-k)-\frac{\bar{h}^2-k^2}{2} = (\bar{h}-k)-\frac{(\bar{h}-k)(\bar{h}+k)}{2} = (\bar{h}-k)\left(1-\frac{\bar{h}+k}{2}\right)$$

Therefore:

$$E[1-h \mid h>k] = 1-\frac{\bar{h}+k}{2} = \frac{2-\bar{h}-k}{2}$$

Substituting this into equation 4 with $k=\tau(r_2+\lambda)$:

$$r_2 = \frac{1}{(2-\bar{h}-\tau(r_2+\lambda))/2} = \frac{2}{2-\bar{h}-\tau(r_2+\lambda)}$$

We can solve this for τ. Let $r=r_2+\lambda$, so $r_2=r-\lambda$. The equation becomes:

$$r-\lambda = \frac{2}{2-\bar{h}-\tau(r)} \implies 2-\bar{h}-\tau(r) = \frac{2}{r-\lambda} \implies \tau(r) = 2-\bar{h}-\frac{2}{r-\lambda}$$

This is the inverse of bank 1's strategy. To find the strategy $\sigma(h)$ itself, we set $h=\tau(r)$ and solve for r:

$$h = 2-\bar{h}-\frac{2}{r-\lambda} \implies r-\lambda = \frac{2}{2-\bar{h}-h} \implies r=\sigma(h)=\lambda+\frac{2}{2-\bar{h}-h}$$

This is the equilibrium pricing function for the informed bank. It prices at a markup over the switching cost λ, where the markup depends on the borrower's risk and the upper bound $\bar{h}$.

Verification: We can verify this makes economic sense:

- For the safest borrower ($h=\underline{h}$): $\sigma(\underline{h}) = \lambda+\frac{2}{2-\bar{h}-\underline{h}}$
- For the riskiest borrower ($h=\bar{h}$): $\sigma(\bar{h}) = \lambda+\frac{2}{2-2\bar{h}} = \lambda+\frac{1}{1-\bar{h}}$
- Since $\bar{h}<1$, the rate is finite and positive for all borrowers in the support.
- The profit margin for type h is $(1-h)\sigma(h)-1 = (1-h)\left(\lambda+\frac{2}{2-\bar{h}-h}\right)-1$.

Finally, we can use our expression for $\tau(u)$ to find the explicit distribution G for bank 2 from equation 7. We have:

$$1-\tau(u) = 1-\left(2-\bar{h}-\frac{2}{u-\lambda}\right) = \bar{h}-1+\frac{2}{u-\lambda}$$

The integrand becomes:

$$\frac{1-\tau(u)}{[(1-\tau(u))u-1]} = \frac{\bar{h}-1+\frac{2}{u-\lambda}}{\left(\bar{h}-1+\frac{2}{u-\lambda}\right)u-1}$$

Let $A = \bar{h}-1$ (note $A < 0$ since $\bar{h} < 1$). Then:

$$\begin{aligned}
\frac{A+\frac{2}{u-\lambda}}{\left(A+\frac{2}{u-\lambda}\right)u-1} &= \frac{A(u-\lambda)+2}{(u-\lambda)} \cdot \frac{1}{\frac{[A(u-\lambda)+2]u-(u-\lambda)}{u-\lambda}} \\
&= \frac{A(u-\lambda)+2}{Au^2 - A\lambda u + 2u - u + \lambda} \\
&= \frac{A(u-\lambda)+2}{Au^2 + (2 - A\lambda - 1)u + \lambda} \\
&= \frac{Au - A\lambda + 2}{Au^2 + (1 - A\lambda)u + \lambda}
\end{aligned}$$

Substituting back $A = \bar{h}-1$:

$$\begin{aligned}
&= \frac{(\bar{h}-1)u - (\bar{h}-1)\lambda + 2}{(\bar{h}-1)u^2 + (1-(\bar{h}-1)\lambda)u + \lambda} \\
&= \frac{(\bar{h}-1)(u-\lambda) + 2}{(\bar{h}-1)u^2 + (1-(\bar{h}-1)\lambda)u + \lambda}
\end{aligned}$$

This integral is complex in the general case. For a cleaner closed-form solution, consider the case where $\bar{h} = 1$ (so default probability ranges from $\underline{h} > 0$ to 1). With $\bar{h} = 1$:

$$\sigma(h) = \lambda + \frac{2}{2-1-h} = \lambda + \frac{2}{1-h}$$

and

$$\tau(r) = 2 - 1 - \frac{2}{r-\lambda} = 1 - \frac{2}{r-\lambda}$$

The lower bound of integration is $\underline{r} = \sigma(\underline{h}) = \lambda + \frac{2}{1-\underline{h}}$.

Now we compute:

$$1 - \tau(u) = 1 - \left(1 - \frac{2}{u-\lambda}\right) = \frac{2}{u-\lambda}$$

The integrand becomes:

$$\frac{1-\tau(u)}{(1-\tau(u))u-1} = \frac{\frac{2}{u-\lambda}}{\frac{2u}{u-\lambda}-1} = \frac{\frac{2}{u-\lambda}}{\frac{2u-(u-\lambda)}{u-\lambda}} = \frac{2}{u+\lambda}$$

The integral becomes:

$$\int_{\underline{r}}^{r} \frac{2}{u+\lambda} du = 2\ln(u+\lambda)\Big|_{\underline{r}}^{r} = 2\ln\left(\frac{r+\lambda}{\underline{r}+\lambda}\right) = \ln\left(\left(\frac{r+\lambda}{\underline{r}+\lambda}\right)^2\right)$$

Substituting into equation 7:

$$-\log(1 - G(r - \lambda)) = \ln\left(\left(\frac{r + \lambda}{\underline{r} + \lambda}\right)^2\right)$$
$$\implies G(r - \lambda) = 1 - \left(\frac{\underline{r} + \lambda}{r + \lambda}\right)^2$$

Let $x = r - \lambda$, so $r = x + \lambda$. Bank 2's strategy $G(x) = \Pr(r_2 \leq x)$ is:

$$G(x) = 1 - \left(\frac{\underline{r} + \lambda}{x + 2\lambda}\right)^2$$

where $\underline{r} + \lambda = 2\lambda + \frac{2}{1-\underline{h}}$.

The support for bank 2's offers starts where $G(x) = 0$, which occurs at $x = \underline{r} - \lambda = \frac{2}{1-\underline{h}}$.

1.3 Switchers

What is the probability a bank switches?

1.4 Equilibrium computation

Previosuly we derived the equilibrium conditions (see equations 5 and 7) for the informed incumbent bank and the uninformed entrant bank.

For a generic distribution $F(h)$ with support $[h_{min}, h_{max}]$, we cannot obtain closed-form solutions. Instead, we use numerical methods to compute the equilibrium strategies.

1.4.1 Step 1: Define the conditional expectation function

For each type h, we need to compute the conditional expectation:

$$\mu(h) = E[1 - H \mid H > h] = \frac{\int_h^{h_{max}} (1-t) f(t) dt}{1 - F(h)} \tag{11}$$

Properties:

- $\mu(h)$ is decreasing in h (higher types have worse expected repayment)
- $\mu(h_{min}) = E[1 - H]$ (unconditional expectation)
- As $h \to h_{max}$: $\mu(h) \to 1 - h_{max}$

Empirical computation: Given N draws from F: $\{h_1, ..., h_N\}$, we compute μ on a grid of M points $\{h_1^{grid}, ..., h_M^{grid}\}$:

$$\mu(h_k^{grid}) = \frac{1}{|\{i : h_i > h_k^{grid}\}|} \sum_{i: h_i > h_k^{grid}} (1 - h_i) \tag{12}$$

1.4.2 Step 2: Support of strategies

The incumbent's offers range from:

$$\underline{r} = \sigma(h_{min}) = \lambda + \frac{1}{E[1-H]} \tag{13}$$

$$\bar{r} = \sigma(h_{max}) = \lambda + \frac{1}{1 - h_{max}} \tag{14}$$

The entrant's offers have support $[x_{min}, \infty)$ where $x_{min} = \underline{r} - \lambda = \frac{1}{E[1-H]}$.

1.4.3 Step 3: Compute the inverse function $\tau(r)$

From $\sigma(\tau(r)) = r$, we need to find $\tau(r)$ for each $r \in [\underline{r}, \bar{r}]$. Given $\sigma(h) = \lambda + 1/\mu(h)$:

$$r = \lambda + \frac{1}{\mu(\tau(r))} \implies \mu(\tau(r)) = \frac{1}{r - \lambda} \tag{15}$$

Numerical solution: For each r in a grid, solve for $\tau(r)$ by finding h such that $\mu(h) = 1/(r - \lambda)$ using interpolation or root-finding.

1.4.4 Step 4: Compute the entrant's mixed strategy

From equation 7, we compute:

$$G(r-\lambda) = 1 - \exp\left[-\int_{\underline{r}}^{r} \frac{1-\tau(u)}{(1-\tau(u))u-1} du\right] \tag{16}$$

Numerical algorithm:

1. Create a grid of M points: $r_1 = \underline{r}, r_2, ..., r_M$ in $[\underline{r}, \bar{r}]$
2. For each r_j, compute $\tau(r_j)$ as described in Step 3
3. Compute the integrand: $I(r_j) = \frac{1-\tau(r_j)}{(1-\tau(r_j))r_j-1}$
4. Use numerical integration (e.g., trapezoidal rule):

$$\int_{\underline{r}}^{r_j} I(u)du \approx \sum_{k=1}^{j-1} \frac{I(r_k) + I(r_{k+1})}{2}(r_{k+1} - r_k) \tag{17}$$

5. Compute: $G(r_j - \lambda) = 1 - \exp\left[-\int_{\underline{r}}^{r_j} I(u)du\right]$

1.4.5 Step 5: Sample from the entrant's strategy

To simulate the model with N borrowers:

1. Draw N borrowers: $h_i \sim F(h)$ for $i = 1, ..., N$
2. For each borrower i, compute incumbent's offer: $r_1^i = \sigma(h_i) = \lambda + 1/\mu(h_i)$
3. Draw entrant's offers from G using inverse transform sampling:
   - Draw $U_i \sim \text{Uniform}[0, 1]$
   - Find r_2^i such that $G(r_2^i) = U_i$ by interpolation
4. Borrower i switches if $r_2^i + \lambda < r_1^i$

1.4.6 Verification

The numerical solution should satisfy:

- Bank 2's expected profit is zero for all r_2 in its support
- Bank 1 is indifferent over its support for each type h
- G is a valid CDF: $G(x_{min}) = 0$ and $\lim_{x\to\infty} G(x) = 1$