Exercise 1

(a) $(y \approx 0) \wedge (y \approx x) \vdash 0 \approx x$

1.	$(y \approx 0) \wedge (y \approx x)$	premise
2.	$y \approx 0$	$\wedge$e 1
3.	$y \approx x$	$\wedge$e 1
4.	$0 \approx x$	=e 2, 3

where, t_1 is y, t_2 is 0, and $\phi(y)$ is $y \approx x$

(b) $t_1 \approx t_2 \vdash (t + t_2) \approx (t + t_1)$

1.	$t_1 \approx t_2$	premise
2.	$t + t_1 \approx t + t_1$	=i
3.	$t + t_2 \approx t + t_1$	=e 1, 2

where, t_1 is t_1, t_2 is t_2, and $\phi(x)$ is $t + x \approx t + t_2$

Exercise 2

(a) $\exists x(S \to Q(x)) \vdash S \to \exists x Q(x)$

1.	$\exists x(S \to Q(x))$	premise
2.	S	assumption
3.	x_0	fresh x_0
4.	$S \to Q(x_0)$	assumption
5.	$Q(x_0)$	$\to$e 2, 3
6.	$\exists Q(x)$	$\exists$i 4
7.	$\exists Q(x)$	$\exists$e 1, 3–5
8.	$S \to \exists x Q(x)$	$\to$i 2–6

(b) $\forall x P(x) \to S \vdash \exists x(P(x) \to S)$

There are two solutions for this exercise, one using the identities and equivalences, and one using only natural deduction proof. I will show both, but the pure natural deduction proof I found on the web. The first one using some identities and equivalences is:

1. $\forall x P(x) \to S$ premise

2. $\neg \forall x P(x) \vee S$ 1, equivalence 3 (can be proved using truth table)

3. $\exists x \neg P(x) \vee S$ 2, provable identity 1

4. $\exists x (\neg P(x) \vee S)$ 3, prenex normal form (since x is not free on S we do not need to rename variables)

5. $\exists x (P(x) \to S)$ 4, equivalence 3 (can be proved using truth table)

The second one is using only natural deduction (found on the web):

1.	$\forall x P(x) \to S$	premise
2.	$\neg \exists x (P(x) \to S)$	assumption
3.	x_0	fresh x_0
4.	$\neg P(x_0)$	assumption
5.	$P(x_0)$	assumption
6.	$\bot$	$\neg$e 4, 5
7.	S	$\bot$e 6
8.	$P(x_0) \to S$	$\to$i 5–7
9.	$\exists x (P(x) \to S)$	$\exists x$i 8
10.	$\bot$	$\neg$e 2, 9
11.	$\neg \neg P(x_0)$	$\neg$i 4–10
12.	$P(x_0)$	$\neg \neg$e 11
13.	$\forall x P(x)$	$\forall x$i 3–12
14.	S	$\to$e 1, 13
15.	$P(t)$	assumption
16.	S	copy 14
17.	$P(t) \to S$	$\to$i 15–16
18.	$\exists x (P(x) \to S)$	$\exists x$i 17
19.	$\bot$	$\neg$e 2, 18
20.	$\neg \neg \exists x (P(x) \to S)$	$\neg$i 2–19
21.	$\exists x (P(x) \to S)$	$\neg \neg$e 20

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.