

Homework Assignment 6*October 17, 2024*

Lucas Miguel Tassis

Exercise 1

We have to show that every finite subset $X \subseteq \mathbb{N}$ is first-order definable in the structure $(\mathbb{N}; <)$. Consider $X = \{x_1, \dots, x_k\}$. We want to define:

$$\varphi_X(x) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } x \in X \\ \text{false}, & \text{otherwise} \end{cases}$$

Thus,

$$\varphi_X(x) \stackrel{\text{def}}{=} \varphi_{\{0\}}(x) \vee \bigvee_{i=1; x_k \neq 0}^k \psi_{\{x_k\}}(x)$$

where,

$$\psi_{\{x_k\}}(x) \stackrel{\text{def}}{=} \exists y_1 \dots \exists y_{x_k} [\varphi_{\{0\}}(y_1) \wedge \varphi_{\text{succ}}(y_1, y_2) \wedge \dots \wedge \varphi_{\text{succ}}(y_{x_k}, x)]$$

and $\varphi_{\{0\}}(x)$, $\varphi_{\text{succ}}(x, y)$ are the wffs for the constant 0 and the successor function, respectively (proved in Exercises 1 and 2 of Lecture Slides 22 – Appendix).

Notice that $\psi_{\{x_k\}}$ can write a wff for any natural number $x_k \neq 0$. If $x_k = 0$ then $\varphi_{\{0\}}(x)$ will be true. Otherwise, if $x_k \neq 0$, then $\varphi_{\{0\}}(x_k)$ will be false, but $\psi_{\{x_k\}}$ will be true. Therefore, $\varphi_X(x)$ will be true for any $x \in X$, and false otherwise (because any $x \notin X$ will not be a part of $\varphi_X(x)$).

Exercise 2

We have to show that the predicate **prime** : $\mathbb{N} \rightarrow \{\text{true}, \text{false}\}$ is first order definable in the structure $(\mathbb{N}; |, +, 0)$. Thus, we need to define:

$$\text{prime}(n) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } n \text{ is a prime number} \\ \text{false}, & \text{otherwise} \end{cases}$$

For that, let us define equality first. We define **equal**(m, n), which is true if $m = n$, and false otherwise as:

$$\text{equal}(m, n) \stackrel{\text{def}}{=} (m|n) \wedge (n|m)$$

Additionally, we can define $1 \stackrel{\text{def}}{=} \text{succ}(0)$ (proved in Exercise 2 of Lecture Slides 22 – Appendix). Notice that we can prove $<$ using $(\mathbb{N}; +, 0)$, as shown in Exercise 4 of Lecture Slides 22 – Appendix (so we can use the successor result). Then, we can define **prime**(n) as:

$$\text{prime}(n) \stackrel{\text{def}}{=} \neg \text{equal}(n, 0) \wedge \neg \text{equal}(n, 1) \wedge \forall y (y|n \rightarrow \text{equal}(y, 1) \vee \text{equal}(y, n))$$

Exercise 3

The Lean template file with the solutions is available on [GitHub](#).

Exercise 4

The Lean template file with the solutions is available on [GitHub](#).

Problem 2

The Lean template file with the solutions is available on [GitHub](#).