CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury
Homework Assignment 6	
October 17, 2024	Lucas Miguel Tassis

Exercise 1

We have to show that every finite subset $X \subseteq \mathbb{N}$ is first-order definable in the structure $(\mathbb{N}; <)$. Consider $X = \{x_1, \cdots, x_k\}$. We want to define:

$$\varphi_X(x) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } x \in X \\ \text{false}, & \text{otherwise} \end{cases}$$

Thus,

$$\varphi_X(x) \stackrel{\text{def}}{=} \bigvee_{i=1}^{k} (\varphi_{\{0\}}(x_k) \vee \psi(x_k))$$

where,

$$\psi(x_k) \stackrel{\text{def}}{=} \exists y_1 \cdots \exists y_{x_k} \left[\varphi_{\{0\}}(y_1) \wedge \varphi_{\text{SUCC}}(y_1, y_2) \wedge \cdots \wedge \varphi_{\text{SUCC}}(y_{x_k}, x_k)\right]$$

and $\varphi_{\{0\}}(x)$, $\varphi_{\text{SUCC}}(x, y)$ are the wffs for the constant 0 and the successor function, respectively (proved in Exercises 1 and 2 of Lecture Slides 22 – Appendix).

Notice that ψ can write a wff for any natural number $x_k \neq 0$. If $x_k = 0$ then $\varphi_{\{0\}}(x_k)$ will be true, and the value of $\psi(x_k)$ does not matter (because the $\vee$ will be true). Otherwise, if $x_k \neq 0$, then $\varphi_{\{0\}}(x_k)$ will be false, but $\psi(x_k)$ will "build" the natural number x_k, and $(\varphi_{\{0\}}(x_k) \vee \psi(x_k))$ will be true. Therefore, $\bigvee_{i=1}^{k}(\varphi_{\{0\}}(x_k) \vee \psi(x_k))$ will cover all the $x_k \in X$, and $\varphi_X(x)$ will be true for any $x \in X$, and false otherwise (because any $x \notin X$ will not be a part of $\varphi_X(x)$).

Exercise 2

We have to show that the predicate $\texttt{prime} : \mathbb{N} \to \{\text{true}, \text{false}\}$ is first order definable in the structure $(\mathbb{N}; |, +, 0)$. Thus, we need to define:

$$\texttt{prime}(n) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } n \text{ is a prime number} \\ \text{false}, & \text{otherwise} \end{cases}$$

For that, let us define equality first. We define $\texttt{equal}(m, n)$, which is true if $m = n$, and false otherwise as:

$$\texttt{equal}(m, n) \stackrel{\text{def}}{=} (m|n) \wedge (n|m)$$

Additionally, we can define $1 \stackrel{\text{def}}{=} \texttt{succ}(0)$ (proved in Exercise 2 of Lecture Slides 22 – Appendix). Notice that we can prove $<$ using $(\mathbb{N}; +, 0)$, as shown in Exercise 4 of Lecture Slides 22 – Appendix (so we can use the successor result). Then, we can define $\texttt{prime}(\text{n})$ as:

$$\texttt{prime}(n) \stackrel{\text{def}}{=} \neg\texttt{equal}(n, 0) \wedge \neg\texttt{equal}(n, 1) \wedge \forall y (y|n \to \texttt{equal}(y, 1) \vee \texttt{equal}(y, n))$$

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.