CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury

Homework Assignment 9

November 7, 2024 — Lucas Miguel Tassis

Exercise 1

First, consider that we have the signature $\{R, \approx\}$, where $R(x_1, x_2)$ means that there exists an edge between vertices x_1 and x_2. Additionally, we also consider that the graph is undirected, *i.e.*, $\forall x_1 \forall x_2 (R(x, y) \to R(y, x))$.
Then, we can write an infinite set $\Gamma_{\text{bipartite}}$ of first order sentences such that, for every simple graph G, it holds that $G \models \Gamma_{\text{bipartite}}$ iff G is bipartite, as:

$$\Gamma_{\text{bipartite}} \stackrel{\text{def}}{=} \{\neg\varphi_{\{n\}} \mid n \geq 3; n \text{ is odd}\}$$

where,

$$\varphi_{\{n\}} \stackrel{\text{def}}{=} \exists x_1 \cdots \exists x_n \left[\left(\bigwedge_{1 \leq i,j \leq n} \neg(x_i \approx x_j)\right) \wedge \left(\bigwedge_{1 \leq i \leq n-1} R(x_i, x_{i+1})\right) \wedge R(x_n, x_1)\right]$$

Using the fact that G is bipartite iff every cycle in G has even length, we can create $\Gamma_{\text{bipartite}}$ as a set of first order sentences that say that there is **no** odd cycle in G. To create this set, we define $\varphi_{\{n\}}$, which is true iff there is a cycle of size n in the graph G:

i. $\bigwedge_{1 \leq i,j \leq n} \neg(x_i \approx x_j)$ guarantees that every element $x_1 \cdots x_n$ are distinct elements;

ii. $\left(\bigwedge_{1 \leq i \leq n-1} R(x_i, x_{i+1})\right) \wedge R(x_n, x_1)$ guarantees that there is a cycle of size n.

Therefore, by defining $\Gamma_{\text{bipartite}}$ as the set of $\neg\varphi_{\{n\}}$, for n odd, we created an infinite set $\Gamma_{\text{bipartite}}$ of first order sentences such that, for every simple graph G, it holds that $G \models \Gamma_{\text{bipartite}}$ iff G is bipartite.

Exercise 2

Considering the three sentences:

$$\varphi_1 \stackrel{\text{def}}{=} \forall x P(x, x)$$
$$\varphi_2 \stackrel{\text{def}}{=} \forall x \forall y (P(x, y) \to P(y, x))$$
$$\varphi_3 \stackrel{\text{def}}{=} \forall x \forall y \forall z (P(x, y) \wedge P(y, z) \to P(x, z))$$

We have to show that none of these sentences are semantically entailed by the other ones. To show that, we give an example of a P that satisfies each pair of sentences but not the one left.

i. $\{\varphi_1, \varphi_2\}$: we have reflexivity and symmetry, but not transitivity.

   Consider the set $S \stackrel{\text{def}}{=} \{a, b, c\}$ and the relation $P \stackrel{\text{def}}{=} \{(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a)\}$. In this example, we have reflexivity and symmetry, but we do not have transitivity. Take (b, a) and (a, c), we do not have (b, c).

ii. $\{\varphi_1, \varphi_3\}$: we have reflexivity and transitivity, but not symmetry.

   Consider the set $S \stackrel{\text{def}}{=} \{a, b, c\}$ and the relation $P \stackrel{\text{def}}{=} \{(a,a), (b,b), (c,c), (a,b), (b,c), (a,c)\}$. In this example, we have reflexivity and transitivity, but we do not have symmetry. Take (a, b), we do not have (b, a).

iii. $\{\varphi_2, \varphi_3\}$: we have symmetry and transitivity, but not reflexivity.

   Consider the set $S \stackrel{\text{def}}{=} \{a, b, c\}$ and the relation $P \stackrel{\text{def}}{=} \{(a,b), (b,a), (a,a), (b,b)\}$. In this example, we have symmetry and transitivity, but we do not have reflexivity. Take c, we do not have (c, c).

Thus, we showed that none of the sentences are semantically entailed by the other pair.

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.