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| CS 511 Formal Methods, Fall 2024 | Instructor: Assaf Kfoury |

# Homework Assignment 2

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## Exercise 1

(Some ideas were from the example of the proof of length of two lists on Wikipedia)
The exercise asks us to use structural induction on $t \in A^*$ to prove the property $P(t)$ defined by:

$$P(t) \stackrel{\text{def}}{=} \text{for all } s \in A^* \text{ it holds that } \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$$

To prove this property by structural induction, we also have to use the definition of the reverse function. Using the definition on page 7 in Lecture Slides 6, we have that $\text{reverse}(s \cdot x) \stackrel{\text{def}}{=} x \cdot \text{reverse}(s)$ and $\text{reverse}(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$.

**Base step:** In the base case, we have that t = $\varepsilon$ (empty string), thus:

$$\begin{aligned}
\text{reverse}(s \cdot t) &= \text{reverse}(s \cdot \varepsilon) \\
&= \text{reverse}(s) \\
&= \varepsilon \cdot \text{reverse}(s) \\
&= \text{reverse}(t) \cdot \text{reverse}(s)
\end{aligned}$$

Thus, we have that the property holds up in the base case.

**Inductive step:** Consider that we denote the string $t = w \cdot x$, where $x$ is the last character of the string $t$. To prove by structural induction, we will assume that $P(w)$ is true, and prove that the property still holds for $P(t = w \cdot x)$. If we apply $\text{reverse}(s \cdot t)$:

$$\begin{aligned}
\text{reverse}(s \cdot t) &= \text{reverse}(s \cdot (w \cdot x)) \\
&= \text{reverse}((s \cdot w) \cdot x) \\
&= x \cdot \text{reverse}(s \cdot w) && \text{reverse definition} \\
&= x \cdot (\text{reverse}(w) \cdot \text{reverse}(s)) && \text{induction hypothesis} \\
&= (x \cdot \text{reverse}(w)) \cdot \text{reverse}(s) \\
&= \text{reverse}(w \cdot x) \cdot \text{reverse}(s) \\
&= \text{reverse}(t) \cdot \text{reverse}(s)
\end{aligned}$$

Since the property still holds for $P(t)$, we proved by structural induction that:

$$P(t) \stackrel{\text{def}}{=} \text{for all } s \in A^* \text{ it holds that } \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$$

# Exercise 2

(From LCS, page 87: Exercise 1.4.15)
We have to use mathematical induction on $n$ to show that:

$$((\phi_1 \land (\phi_2 \land (\cdots \land \phi_n) \cdots) \to \psi) \to ((\phi_1 \to (\phi_2 \to (\cdots (\phi_n \to \psi) \cdots))))$$

**Base step:** First, we have to show that the statement holds for the base case, which is $n = 1$. In this case, we have $(\phi_1 \to \psi) \to (\phi_1 \to \psi)$, which is true. So the statement holds for the base case.

**Inductive step:** Now, we have to show that if the statement is holds for $n = k$ (induction hypothesis), then $k + 1$ also holds. Using natural deduction on $k + 1$:

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| 1. | $((\phi_1 \land (\phi_2 \land (\cdots \land (\phi_k \land \phi_{k+1}) \cdots)) \to \psi$ | premise |
| 2. | $(\phi_1 \land (\phi_2 \land (\cdots \land \phi_k) \cdots)$ | assumption |
| 3. | $\phi_{k+1}$ | assumption |
| 4. | $(\phi_1 \land (\phi_2 \land (\cdots \land (\phi_k \land \phi_{k+1}) \cdots)$ | $\land$i 2,3 |
| 5. | $\psi$ | $\to$e 1,4 |
| 6. | $\phi_{k+1} \to \psi$ | $\to$i 3–5 |
| 7. | $(\phi_1 \land (\phi_2 \land (\cdots \land \phi_k) \cdots) \to (\phi_{k+1} \to \psi)$ | $\to$i 2–6 |
| 8. | $(\phi_1 \to (\phi_2 \to (\cdots (\phi_k \to (\phi_{k+1} \to \psi) \cdots)))$ | induction hypothesis |

Thus, we showed by mathematical induction that the statement holds.

# Problem 1

## (b) (LEM) is derivable from (PBC)

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| 1. | $\neg(p \lor \neg p)$ | assumption |
| 2. | $p$ | assumption |
| 3. | $p \lor \neg p$ | $\lor$i$_1$ 2 |
| 4. | $\bot$ | $\neg$e 1, 3 |
| 5. | $\neg p$ | $\neg$i 2–4 |
| 6. | $p \lor \neg p$ | $\lor$i$_2$ 5 |
| 7. | $\bot$ | $\neg$e 1, 6 |
| 8. | $p \lor \neg p$ | PBC 1–7 |

### (c) ($\neg\neg$E) is derivable from (LEM)

The only way I found of deriving ($\neg\neg$E) from (LEM) was by also using *disjunctive syllogism* (Wikipedia). Disjunctive syllogism is: $p \vee q, \neg p \vdash q$. I tried proving it (in order to use it in the exercise), but couldn't do it without using ($\neg\neg$E). I don't know if there is another way of proving disjunctive syllogism without either ($\neg\neg$E) or (PBC). Most of the places I searched also used one of the two to prove it (for example). I decided to put the proof here because it was the only one I could find, but I think it might involve the use of other rules beside LEM (which probably goes against the idea of the exercise).

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|---|---|---|
| 1. | $\neg p \vee p$ | LEM |
| 2. | $\neg\neg p$ | assumption |
| 3. | $p$ | disjunctive syllogism |
| 4. | $\neg\neg p \to p$ | $\to$i 2–3 |

## Exercise 3

The Lean template file with the solutions is available on GitHub.

## Exercise 4

The Lean template file with the solutions is available on GitHub.

## Problem 2

The Lean template file with the solutions is available on GitHub.