

Homework Assignment 5*October 10, 2024*

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Exercise 1**(a)** $(y \approx 0) \wedge (y \approx x) \vdash 0 \approx x$

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|----|--------------------------------------|--------------|
| 1. | $(y \approx 0) \wedge (y \approx x)$ | premise |
| 2. | $y \approx 0$ | $\wedge e$ 1 |
| 3. | $y \approx x$ | $\wedge e$ 1 |
| 4. | $0 \approx x$ | $=e$ 2, 3 |

where, t_1 is y , t_2 is 0, and $\phi(y)$ is $y \approx x$ **(b)** $t_1 \approx t_2 \vdash (t + t_2) \approx (t + t_1)$

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|----|---------------------------|-----------|
| 1. | $t_1 \approx t_2$ | premise |
| 2. | $t + t_1 \approx t + t_1$ | $=i$ |
| 3. | $t + t_2 \approx t + t_1$ | $=e$ 1, 2 |

where, t_1 is t_1 , t_2 is t_2 , and $\phi(x)$ is $t + x \approx t + t_1$ **Exercise 2****(a)** $\exists x(S \rightarrow Q(x)) \vdash S \rightarrow \exists xQ(x)$

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|----|---------------------------------|----------------------|
| 1. | $\exists x(S \rightarrow Q(x))$ | premise |
| 2. | S | assumption |
| 3. | x_0 | fresh x_0 |
| 4. | $S \rightarrow Q(x_0)$ | assumption |
| 5. | $Q(x_0)$ | $\rightarrow e$ 2, 4 |
| 6. | $\exists xQ(x)$ | $\exists i$ 5 |
| 7. | $\exists xQ(x)$ | $\exists e$ 1, 3–6 |
| 8. | $S \rightarrow \exists xQ(x)$ | $\rightarrow i$ 2–7 |

(b) $\forall xP(x) \rightarrow S \vdash \exists x(P(x) \rightarrow S)$

There are two solutions for this exercise, one using the identities and equivalences, and one using only natural deduction proof. I will show both, but the pure natural deduction proof was based on web search. The first one using some identities and equivalences is:

1. $\forall x P(x) \rightarrow S$ premise
2. $\neg \forall x P(x) \vee S$ 1, [equivalence 1](#) (can be proved using truth table)
3. $\exists x \neg P(x) \vee S$ 2, [provable identity 1](#)
4. $\exists x (\neg P(x) \vee S)$ 3, [prenex normal form](#) (since x is not free on S we do not need to rename variables)
5. $\exists x (P(x) \rightarrow S)$ 4, [equivalence 1](#) (can be proved using truth table)

The second one is using only natural deduction (based on web search):

1.	$\forall x P(x) \rightarrow S$	premise
2.	$\neg \exists x (P(x) \rightarrow S)$	assumption
3.	x_0	fresh x_0
4.	$\neg P(x_0)$	assumption
5.	$P(x_0)$	assumption
6.	\perp	\neg e 4, 5
7.	S	\perp e 6
8.	$P(x_0) \rightarrow S$	\rightarrow i 5–7
9.	$\exists x (P(x) \rightarrow S)$	\exists xi 8
10.	\perp	\neg e 2, 9
11.	$\neg \neg P(x_0)$	\neg i 4–10
12.	$P(x_0)$	$\neg \neg$ e 11
13.	$\forall x P(x)$	\forall xi 3–12
14.	S	\rightarrow e 1, 13
15.	$P(t)$	assumption
16.	S	copy 14
17.	$P(t) \rightarrow S$	\rightarrow i 15–16
18.	$\exists x (P(x) \rightarrow S)$	\exists xi 17
19.	\perp	\neg e 2, 18
20.	$\neg \neg \exists x (P(x) \rightarrow S)$	\neg i 2–19
21.	$\exists x (P(x) \rightarrow S)$	$\neg \neg$ e 20

Exercise 3

The Lean template file with the solutions is available on [GitHub](#).

Exercise 4

The Lean template file with the solutions is available on [GitHub](#).