

Homework Assignment 10

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Lucas Miguel Tassis

Exercise 1

Yes, $\text{Th}(\mathcal{M}) = \{\varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{M} \models \varphi\}$ is deductively closed. Since we are considering first-order logic, then we have that $\mathcal{M} \models \varphi \rightarrow \mathcal{M} \vdash \varphi$ by *completeness*. Therefore, we can say that $\text{Th}(\mathcal{M})$ is deductively closed, because for every $\mathcal{M} \models \varphi$, we also have $\mathcal{M} \vdash \varphi$ by using completeness.

Exercise 2

1. The first-order sentence will be defined as:

$$\varphi_1 \stackrel{\text{def}}{=} \forall x (B(x) \vee G(x) \vee P(x) \vee Y(x))$$

2. The first-order sentence will be defined as:

$$\begin{aligned} \varphi_2 \stackrel{\text{def}}{=} \forall x \Big(& (B(x) \wedge \neg G(x) \wedge \neg P(x) \wedge \neg Y(x)) \vee \\ & (\neg B(x) \wedge G(x) \wedge \neg P(x) \wedge \neg Y(x)) \vee \\ & (\neg B(x) \wedge \neg G(x) \wedge P(x) \wedge \neg Y(x)) \vee \\ & (\neg B(x) \wedge \neg G(x) \wedge \neg P(x) \wedge Y(x)) \Big) \end{aligned}$$

3. The first-order sentence will be defined as:

$$\begin{aligned} \varphi_3 \stackrel{\text{def}}{=} \forall x \forall y \Big(& (\neg(x \approx y) \wedge R(x, y)) \rightarrow \big[(B(x) \wedge (G(y) \vee P(y) \vee Y(y))) \vee \\ & (G(x) \wedge (B(y) \vee P(y) \vee Y(y))) \vee \\ & (P(x) \wedge (B(y) \vee G(y) \vee Y(y))) \vee \\ & (Y(x) \wedge (B(y) \vee G(y) \vee P(y))) \big] \Big) \end{aligned}$$

4. Considering that \mathcal{M} is an infinite planar graph, we have from Hint 2 that every finite subgraph of \mathcal{M} is also planar. Additionally, we have from Hint 1, that every finite planar graph is four-colorable. Thus, since \mathcal{M} is finitely satisfiable (every finite subgraph has a four-coloring from Hints 1 and 2), then \mathcal{M} must also be four-colorable by *compactness*.

Exercise 3

The Lean template file with the solutions is available on [GitHub](#).

Exercise 4

The Lean template file with the solutions is available on [GitHub](#).

Problem 2

The Lean template file with the solutions is available on [GitHub](#).