CS 511 Formal Methods, Fall 2024 — Instructor: Assaf Kfoury

Homework Assignment 11

November 21, 2024 — Lucas Miguel Tassis

Exercise 1

(a) $(\forall x \forall y(S(x,y) \to S(y,x))) \to (\forall x \neg S(x,x))$

The formula states that if S is symmetric, then S is not reflexive. A trivial model that does not satisfy this formula is: $\mathcal{M} = (A, S^{\mathcal{M}})$, where $A \stackrel{\text{def}}{=} \{a\}$ be the domain of the model, and $S^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a,a)\}$.

(b) $\exists y((\forall x P(x)) \to P(y))$

The formula is valid, using the tableaux as proof:

$$\begin{array}{c} \neg\exists y((\forall x P(x)) \to P(y)) \\ | \\ \neg(\forall x P(x) \to P(a)) \\ | \\ \forall x P(x) \\ | \\ \neg P(a) \\ | \\ P(a) \\ | \\ \mathbf{X} \end{array}$$

Since all the paths are closed, the negation of the formula is a contradiction, thus the formula is valid.

Exercise 2

(c) $(\forall x(P(x) \to \exists y Q(y))) \to (\forall x \exists y(P(x) \to Q(y)))$

The formula is valid, using the tableaux as proof:

$$
\begin{array}{c}
\neg(\forall x(P(x) \to \exists y Q(y))) \to (\forall x \exists y(P(x) \to Q(y))) \\
\forall x(P(x) \to \exists y Q(y)) \\
\neg \forall x \exists y(P(x) \to Q(y)) \\
\neg \exists y(P(a) \to Q(y)) \\
P(a) \to \exists y Q(y) \\
\neg(P(a) \to Q(a)) \\
P(a) \\
\neg Q(a) \\
\begin{array}{cc}
\neg P(a) & \exists y Q(y) \\
\mathbf{X} & Q(b) \\
 & \neg(P(a) \to Q(b)) \\
 & P(a) \\
 & \neg Q(b) \\
 & \mathbf{X}
\end{array}
\end{array}
$$

Since all the paths are closed, the negation of the formula is a contradiction, thus the formula is valid.

(d) $(\forall x \exists y(P(x) \to Q(y))) \to (\forall x(P(x) \to \exists y Q(y)))$

The formula is valid, using the tableaux as proof:

$$
\begin{array}{c}
\neg((\forall x \exists y(P(x) \to Q(y))) \to (\forall x(P(x) \to \exists y Q(y)))) \\
\forall x \exists y(P(x) \to Q(y)) \\
\neg \forall x(P(x) \to \exists y Q(y))) \\
\neg(P(a) \to \exists y Q(y)) \\
P(a) \\
\neg \exists y Q(y) \\
\exists y(P(a) \to Q(y)) \\
P(a) \to Q(b) \\
\forall y \neg Q(y) \\
\neg Q(b) \\
\begin{array}{cc}
\neg P(a) & Q(b) \\
\mathbf{X} & \mathbf{X}
\end{array}
\end{array}
$$

Since all the paths are closed, the negation of the formula is a contradiction, thus the formula is valid.

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.