CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury

Homework Assignment 6

October 17, 2024 — Lucas Miguel Tassis

Exercise 1

We have to show that every finite subset $X \subseteq \mathbb{N}$ is first-order definable in the structure $(\mathbb{N}; <)$. Consider $X = \{x_1, \cdots, x_k\}$. We want to define:

$$\varphi_X(x) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } x \in X \\ \text{false}, & \text{otherwise} \end{cases}$$

Thus,

$$\varphi_X(x) \stackrel{\text{def}}{=} \bigvee_{i=1}^{k} \psi_{\{x_k\}}(x)$$

where,

$$\psi_{\{x_k\}}(x) \stackrel{\text{def}}{=} \exists y_0 \cdots \exists y_{x_k - 1} \left[\varphi_{\{0\}}(y_0) \wedge \varphi_{\text{SUCC}}(y_0, y_1) \wedge \cdots \wedge \varphi_{\text{SUCC}}(y_{x_k-1}, x) \right]$$

and $\varphi_{\{0\}}(x)$, $\varphi_{\text{SUCC}}(x, y)$ are the wffs for the constant 0 and the successor function, respectively (proved in Exercises 1 and 2 of Lecture Slides 22 – Appendix).

Notice that $\psi_{\{x_k\}}$ can write a wff for any natural number x_k. Therefore, $\varphi_X(x)$ will be true for any $x \in X$, and false otherwise (because any $x \notin X$ will not be a part of $\varphi_X(x)$).

PS: When $x_k = 0$ we would like to have $\psi_{\{0\}}(x) = \varphi_{\{0\}}(x)$, but I did not know if I could break the expression in cases when $x_k = 0$ and when $x_k \neq 0$. If we can, I think a more formal definition (or at least more "organized") of $\psi_{\{x_k\}}(x)$ would be:

$$\psi_{\{x_k\}}(x) \stackrel{\text{def}}{=} \begin{cases} \varphi_{\{0\}}(x), & \text{if } x_k = 0 \\ \exists y_1 \cdots \exists y_{x_k} \left[\varphi_{\{0\}}(y_1) \wedge \varphi_{\text{SUCC}}(y_1, y_2) \wedge \cdots \wedge \varphi_{\text{SUCC}}(y_{x_k}, x) \right], & \text{otherwise} \end{cases}$$

Exercise 2

We have to show that the predicate `prime` : $\mathbb{N} \to \{\text{true}, \text{false}\}$ is first order definable in the structure $(\mathbb{N}; |, +, 0)$. Thus, we need to define:

$$\texttt{prime}(n) \stackrel{\text{def}}{=} \begin{cases} \text{true}, & \text{if } n \text{ is a prime number} \\ \text{false}, & \text{otherwise} \end{cases}$$

For that, let us define equality first. We define $\texttt{equal}(m, n)$, which is true if $m = n$, and false otherwise as:

$$\texttt{equal}(m, n) \stackrel{\text{def}}{=} (m|n) \wedge (n|m)$$

Additionally, we can define $1 \stackrel{\text{def}}{=} \texttt{succ}(0)$ (proved in Exercise 2 of Lecture Slides 22 – Appendix). Notice that we can prove $<$ using $(\mathbb{N}; +, 0)$, as shown in Exercise 4 of Lecture Slides 22 – Appendix (so we can use the successor result). Then, we can define `prime`(n) as:

$$\texttt{prime}(n) \stackrel{\text{def}}{=} \neg\texttt{equal}(n,0) \land \neg\texttt{equal}(n,1) \land \forall y(y|n \to \texttt{equal}(y,1) \lor \texttt{equal}(y,n))$$

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.