CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury

Homework Assignment 12

December 5, 2024 — Lucas Miguel Tassis

Exercise 1

The exercise asks us to define $X \sim Y$ by asserting the existence of a unary function F from X to Y which is both injective and surjective. One possible way is:

$$X \sim Y \stackrel{\text{def}}{=} \exists F.\Big(\big(\forall x \in X.\forall y \in X.\forall z \in Y.(F(x) \approx z \wedge F(y) \approx z \to x \approx y)\big)\wedge \\ \big(\forall y \in Y.\exists x \in X.(F(x) \approx y)\big)\Big)$$

The first line says that F is injective from X to Y, and the second says that F is surjective from X to Y.

Exercise 2

1. The first item asks us to define a 2nd-order sentence $\Psi_{\text{countable-infty}}(Y)$ s.t. $\mathcal{A} \models \Psi_{\text{countable-infty}}$ iff $\mathcal{A}$ is countably infinite. Similarly to the sentence in the slide, we can define as:

$$\Psi_{\text{countable-infty}}(Y) \stackrel{\text{def}}{=} \Psi_{\text{infty}}(Y) \wedge \big(\forall X \subseteq Y.\ \Psi_{\text{infty}}(X) \to (X \sim Y)\big),$$

where

$$\Psi_{\text{infty}}(X) \stackrel{\text{def}}{=} \exists F.\Big(\big(\forall x \in X.\forall y \in X.\forall z \in X.(F(x) \approx z \wedge F(y) \approx z \to x \approx y)\big)\wedge \\ \big(\exists y \in X.\forall x \in X.\neg(F(x) \approx y)\big)\Big),$$

and $X \sim Y$ is as defined in Exercise 1. In this case, the model should be infinite, and also there is a bijection from every subset X of Y from X to Y.

2. The second item asks us to define a 2nd-order sentence $\Psi_{\text{uncountable}}(Y)$ s.t. $\mathcal{A} \models \Psi_{\text{uncountable}}$ iff $\mathcal{A}$ is uncountably infinite. We can define the sentence as:

$$\Psi_{\text{uncountable}}(Y) \stackrel{\text{def}}{=} \Psi_{\text{infty}}(Y) \wedge \neg\big(\forall X \subseteq Y.\ \Psi_{\text{infty}}(X) \to (X \sim Y)\big),$$

where $\Psi_{\text{infty}}(Y)$ and $X \sim Y$ are the same as defined in Item 1. We can also rewrite the sentence by pushing the negation:

$$\Psi_{\text{uncountable}}(Y) \stackrel{\text{def}}{=} \Psi_{\text{infty}}(Y) \wedge \big(\exists X \subseteq Y.\ \Psi_{\text{infty}}(X) \wedge \neg(X \sim Y)\big),$$

In this case, the model should still be infinite, however, there is not a bijection from every subset X of Y from X to Y.

Exercise 3

The exercise asks us to define a second-order wff $\theta(x, y)$, such that $\theta(x, y)$ iff "no binary predicate Y can discern x and y". One possible way is:

$$\theta(x, y) \stackrel{\text{def}}{=} \forall Y. \Big(\forall c. \big(Y(x, c) \leftrightarrow Y(y, c) \land Y(c, x) \leftrightarrow Y(c, y)\big)\Big)$$

The idea behind the wff is that x and y are identical to θ iff x and y satisfy the same binary predicates, and each iff in the wff guarantees this.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Exercise 5

The Lean template file with the solutions is available on GitHub.

Exercise 6

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.