CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury

Homework Assignment 10

November 14, 2024 — Lucas Miguel Tassis

Exercise 1

Yes, $\mathrm{Th}(\mathcal{M}) = \{\varphi \mid \varphi$ is a first-order sentence s.t $\mathcal{M} \models \varphi\}$ is deductively closed. Since we are considering first-order logic, then we have that $\mathcal{M} \models \varphi \to\ \mathcal{M} \vdash \varphi$ by *completeness*. Therefore, we can say that $\mathrm{Th}(\mathcal{M})$ is deductively closed, because for every $\mathcal{M} \models \varphi$, we also have $\mathcal{M} \vdash \varphi$ by using completeness.

Exercise 2

1. The first-order sentence will be defined as:

$$\varphi_1 \stackrel{\text{def}}{=} \forall x(B(x) \lor G(x) \lor P(x) \lor Y(x))$$

2. The first-order sentence will be defined as:

$$\begin{aligned}\varphi_2 \stackrel{\text{def}}{=} \forall x\Big(&\big(B(x) \land \neg G(x) \land \neg P(x) \land \neg Y(x)\big)\lor\\ &\big(\neg B(x) \land G(x) \land \neg P(x) \land \neg Y(x)\big)\lor\\ &\big(\neg B(x) \land \neg G(x) \land P(x) \land \neg Y(x)\big)\lor\\ &\big(\neg B(x) \land \neg G(x) \land \neg P(x) \land Y(x)\big)\Big)\end{aligned}$$

3. The first-order sentence will be defined as:

$$\begin{aligned}\varphi_3 \stackrel{\text{def}}{=} \forall x\forall y\Big(\big(\neg(x \approx y) \land R(x,y)\big) \to \big[&\big(B(x) \land \neg B(y)\big)\lor\\ &\big(G(x) \land \neg G(y)\big)\lor\\ &\big(P(x) \land \neg P(y)\big)\lor\\ &\big(Y(x) \land \neg Y(y)\big)\big]\Big)\end{aligned}$$

4. Considering that $\mathcal{M}$ is an infinite planar graph, we have from Hint 2 that every finite subgraph of $\mathcal{M}$ is also planar. Additionally, we have from Hint 1, that every finite planar graph is four-colorable. Thus, since $\mathcal{M}$ is finitely satisfiable (every finite subgraph has a four-coloring from Hints 1 and 2), then $\mathcal{M}$ must also be four-colorable by *compactness*.

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.

Problem 2

The Lean template file with the solutions is available on GitHub.