CS 511 Formal Methods, Fall 2024	Instructor: Assaf Kfoury

Homework Assignment 2

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Exercise 1

(Some ideas were from the example of the proof of length of two lists on Wikipedia)
The exercise asks us to use structural induction on $t \in A^*$ to prove the property $P(t)$ defined by:

$$P(t) \stackrel{\text{def}}{=} \text{for all } s \in A^* \text{ it holds that } \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$$

To prove this property by structural induction, we also have to use the definition of the reverse function. Using the definition on page 7 in Lecture Slides 6, we have that $\text{reverse}(s{\cdot}x) \stackrel{\text{def}}{=} x{\cdot}\text{reverse}(s)$ and $\text{reverse}(\varepsilon) \stackrel{\text{def}}{=} \varepsilon$.

Base step: In the base case, we have that t = ε (empty string), thus:

$$\begin{aligned}
\text{reverse}(s \cdot t) &= \text{reverse}(s \cdot \varepsilon) \\
&= \text{reverse}(s) \\
&= \varepsilon \cdot \text{reverse}(s) \\
&= \text{reverse}(t) \cdot \text{reverse}(s)
\end{aligned}$$

Thus, we have that the property holds up in the base case.

Inductive step: Consider that we denote the string $t = w \cdot x$, where x is the last character of the string t. To prove by structural induction, we will assume that $P(w)$ is true, and prove that the property still holds for $P(t = w \cdot x)$. If we apply $\text{reverse}(s \cdot t)$:

$$\begin{aligned}
\text{reverse}(s \cdot t) &= \text{reverse}(s \cdot (w \cdot x)) \\
&= \text{reverse}((s \cdot w) \cdot x) \\
&= x \cdot \text{reverse}(s \cdot w) && \text{reverse definition} \\
&= x \cdot (\text{reverse}(w) \cdot \text{reverse}(s)) && \text{induction hypothesis} \\
&= (x \cdot \text{reverse}(w)) \cdot \text{reverse}(s) \\
&= \text{reverse}(w \cdot x) \cdot \text{reverse}(s) \\
&= \text{reverse}(t) \cdot \text{reverse}(s)
\end{aligned}$$

Since the property still holds for $P(t)$, we proved by structural induction that:

$$P(t) \stackrel{\text{def}}{=} \text{for all } s \in A^* \text{ it holds that } \text{reverse}(s \cdot t) = \text{reverse}(t) \cdot \text{reverse}(s)$$

Exercise 2

(From LCS, page 87: Exercise 1.4.15)
We have to use mathematical induction on n to show that:

$$((\phi_1 \wedge (\phi_2 \wedge (\cdots \wedge \phi_n)\cdots) \to \psi) \to ((\phi_1 \to (\phi_2 \to (\cdots (\phi_n \to \psi)\cdots))))$$

Base step: First, we have to show that the statement holds for the base case, which is $n = 1$. In this case, we have $(\phi_1 \to \psi) \to (\phi_1 \to \psi)$, which is true. So the statement holds for the base case.

Inductive step: Now, we have to show that if the statement is holds for $n = k$ (induction hypothesis), then $k + 1$ also holds. Using natural deduction on $k + 1$:

1.	$((\phi_1 \wedge (\phi_2 \wedge (\cdots \wedge (\phi_k \wedge \phi_{k+1})\cdots)) \to \psi$	premise
2.	$(\phi_1 \wedge (\phi_2 \wedge (\cdots \wedge \phi_k)\cdots)$	assumption
3.	ϕ_{k+1}	assumption
4.	$(\phi_1 \wedge (\phi_2 \wedge (\cdots \wedge (\phi_k \wedge \phi_{k+1})\cdots)$	$\wedge$i 2,3
5.	ψ	$\to$e 1,4
6.	$\phi_{k+1} \to \psi$	$\to$i 3–5
7.	$(\phi_1 \wedge (\phi_2 \wedge (\cdots \wedge \phi_k)\cdots) \to (\phi_{k+1} \to \psi)$	$\to$i 2–6
8.	$(\phi_1 \to (\phi_2 \to (\cdots (\phi_k \to (\phi_{k+1} \to \psi)\cdots)))$	induction hypothesis

Thus, we showed by mathematical induction that the statement holds.

Problem 1

(b) (LEM) is derivable from (PBC)

(c) (¬¬E) is derivable from (LEM)

Exercise 3

The Lean template file with the solutions is available on GitHub.

Exercise 4

The Lean template file with the solutions is available on GitHub.