Exam 2

CS 2813 - Discrete Structures

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October 28, 2023

Question 1: *In one episode of the popular 90s American sitcom 'Friends', Joey and Rachel played the following coin tossing game: If heads, Rachel wins. If tails, Joey loses.*

(a) Who do you think won?

Rachel

(b) Write one compound proposition (in terms of p and q) that represents this game.

Let p be true if Rachel wins and q be true if Joey loses. p $\vee$ q

(c) Use one word from propositional logic to describe the outcome of the game/compound proposition.

Disjunctive

Question 2: *Prove or disprove that: Given two sets A and B,* $\overline{(A - B)} = \overline{A} \cup B$.

A - B = A $\cap$ $\overline{B}$
$\overline{A \cap \overline{B}} = \overline{A} \cup \overline{\overline{B}}$
$\overline{A}$ $\cup$ B, This proves the original statement.

Question 3:

(a) In bioinformatics, sequences are composed of nucleotides denoted with letters (A, T, G, and C). What is the number of distinct 7-nucleotide sequences that can be con- structed using the same nucleotides from the sequence GATTACA? Explain your answer.

G = 1 nucleotide, A = 3 nucleotides, T = 2 nucleotides, C = 1 nucleotides

There are 7 nucleotides with 1! * 3! * 2! * 1! = 12 duplicates which equals $\frac{7!}{12}$ = 420 distinct sequences.

(b) The term k-mer refers to all of a sequence's non-empty contiguous sequence of characters (subsequences) of length k, such that the sequence "APPLE" would have five '1-mers' i.e. monomers (A, P, P, L, and E), four 2-mers (AP, PP, PL, LE), three 3-mers (APP, PPL, and PLE), two 4-mers (APPL, PPLE), and one 5-mer (APPLE). Following this definition and example, and given a string or a sequence of length L how many k-mers does this string have for any specific k?

L - (k - 1)

(c) In the analysis of DNA sequences, CpG sites occur with high frequency in genomic regions called CpG islands (or CG islands), meaning the 2-mer 'CG' appears more often than other 2-mers in the sequence. If our sequence of length 10 contains exactly 3 such CG 2-mers, and assuming the nucleotides C and G are inseparable in the sequence and only appear together as

CG 2-mers (e.g. TATCGTCGCG). How many different sequences can we construct? Explain your answer.

Since there are 3 2-mers in the 10 nucleotides sequence, there are (10 - 3)! = 5040 sequences.

If there are either 3 A or T in the sequence, there are $\frac{5040}{3!*3!}$ = 140 different sequences.

If there are 2 A and T in the sequence, there are $\frac{5040}{3!*2!*2!}$ = 210 different sequences

Question 4: *Given a Boolean function, a Boolean sum of minterms can be formed that has the value 1 when this Boolean function has the value 1, and has the value 0 when the function has the value 0. The sum of minterms that represents the function is called the sum-of-products expansion or the disjunc- tive normal form of the Boolean function. It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums (maxterms, i.e. the term with the sum of N literals occurring exactly once). The resulting expansion is called the conjunctive normal form or product-of-sums expansion of the function.*

(a) Find the sum-of-products expansion of this Boolean function : F (x, y) = $\bar{x}$ + y

x	y	F(x, y)
F	F	T
F	T	T
T	F	F
T	T	T

$\overline{xy}$ + $\bar{x}$y + xy

(b) Find the product-of-sums expansion of F (x, y, z) = (x + z)y

x	y	z	F(x,y,z)
F	F	F	F
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	F
T	T	F	T
T	T	T	T

$(\bar{x} + \bar{y} + \bar{z})(\bar{x} + \bar{y}$ + z$)(\bar{x} + y + \bar{z})$(x + $\bar{y} + \bar{z}$)(x + $\bar{y}$ + z)

(c) Explain the strategy you used to solve each of (a) and (b) above.

For (a), I took the sum-of-products of all the terms that were true. For (b), I took the product-of-sums of all the terms that were false.

Question 5:

(a) How many functions are there from a set A of four elements to a set B with three elements? Explain your answer.

3^4 = 81 functions

For each element from set A, there's 3 elements from set B to map which equals to 81 functions.

(b) How many of these are one-to-one? Explain your answer.

None of the the functions are one-to-one because the cardinality of B is less than the cardinality of A.

(c) How many of these are onto? Explain your answer.

Using this equations from [1], the number of onto functions 3^4 - $\binom{3}{1}$(3-1)^4 + $\binom{3}{2}$(3-2)^4 = 81 - 48 + 3 = 36 functions.

Question 6: *Consider the following relations on the set of positive integers (Notes: greatest common divisor (gcd) of two or more numbers is the greatest common factor number that divides them, exactly. You can use examples in your justifications):*

$$R_1 = \{(x,y) \mid x + y > 10\}$$

$$R_2 = \{(x,y) \mid y \text{ divides } x\}$$

$$R_3 = \{(x,y) \mid \gcd(x,y) = 1\}$$

$$R_4 = \{(x,y) \mid x \text{ and } y \text{ have the same prime divisors}\}$$

(a) Which of these relations are reflexive? Justify your answers.

R_2 and R_4 are reflexive relations.

R_1: (1,1) does not satify x + y > 10

R_2: Any number divides into itself

R_3: (6,6) does not satisfy gcd(x,y) = 1

R_4: Any number will have the same divisors to itself

(b) Which of these relations are symmetric? Justify your answers.

R_1, R_3, and R_4 are symmetric relations.

R_1: Addition is commutative so any (x,y) that satifies x + y > 10 will be satified by (y,x)

R_2: (12,2) satifies the relation but (2,12) does not

R_3: gcd(x,y) = gcd(y,x)

R_4: (x,y) has the same divisors as (y,x)

(c) Which of these relations are antisymmetric? Justify your answers.

R_2 is an antisymmetric relation.

R_1, R_3, and R_4 are symmetric as explained from (b) which cannot make them antisymmetric.

(d) Which of these relations are transitive? Justify your answers.

R_1 and R_3 are not transitive relations because they do not satify the case where xRy and yRx lead to xRx.

R_2: For (12, 4) and (4, 2), (12,2) satifies the relation.

R_4: For (6, 12) and (12, 18), (6,18) share the same prime divisors.

REFERENCES

[1] Admin. (2023, August 29). Number of functions - formula and solved examples. BYJUS. https://byjus.com/jee/number-of-functions/#:~:text=If%20a%20set%20A%20has,of%20B%20should%20be%20used.