Problem Set 5

Due: Friday, February 28th

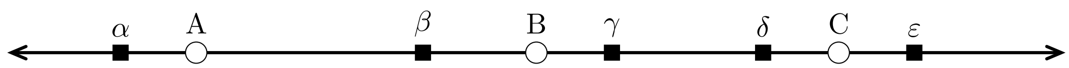
Instructions: Do at least 7 of the following problems.

1. **Positional versus Pairwise** ⋆⋆

   Devise an example of a profile involving 5 candidates that results in a tie among all candidates no matter what positional voting method is used and yet where candidate A is the unique winner of the Copeland method

2. **A Geometric Approach to Political Views** ⋆⋆⋆

   Let's consider a new way of viewing voter preferences: the views of each candidate and voter are plotted as points on some map (a "metric space" if you want to be fancy), like this one-dimensional example below. Candidates are white circles labeled by capital Roman letters, and voters are black squares labeled by lowercase Greek letters.



   In the one-dimensional model, perhaps views to the left are more liberal and those to the right are more conservative. A voter's ballot is determined by his spacial relationship to the candidates: his first choice is the candidate ideologically closest to him, his second choice is the second-closest candidate, and so on. In the above example, we get the following profile:

Voter α	Voter β	Voter γ	Voter δ	Voter ε
A	B	B	C	C
B	A	C	B	B
C	C	A	A	A

   For simplicity's sake, let's assume that our map is finely-tuned enough that no voter is exactly equidistant from two candidates. That is, while it's perfectly possible that whatever voting system we use may produce ties, each voter's *ballot* gives a strict ranking of the candidates, because no two candidates are exactly the same distance away from him. It turns out that with these assumptions, it is easy to find Condorcet candidates on a one-dimensional map:

   (a) Suppose there are an odd number of voters, no two of whom are in the same exact spot on a one-dimensional map. The *median voter* is the (unique) voter who has the same number of voters to her left as she does to her right. (So in our earlier example, γ is the median voter.) Prove that the candidate closest to the median voter is always a Condorcet winner.

   Unfortunately, actual political views are more complicated than this. We've got plenty of dimensions to consider when evaluating candidates: social policy, foreign policy, economic policy, environmental policy, saxophone proficiency, *et cetera*. Even in two dimensions, it turns out we are no longer guaranteed a Condorcet winner:

   (b) Draw a two-dimensional political map with three candidates and three voters such that the resulting profile yields the Condorcet (rock-paper-scissors) paradox.

3. **From Cardinal Voting to Approval Voting** $\star\star$

   We can turn a cardinal ballot into an approval ballot in the following way: say that a voter approves of a candidate whenever he gives them a score of 5 or above, and disapproves of the candidate whenever he gives them a score less than 5. Give an example of a single cardinal voting profile with three candidates where the cardinal preference order (obtained by comparing the total number of points each candidate receives) is (A, B, C), but the corresponding approval preference order (obtained by comparing the number of "yes" votes each candidate receives) is (C, B, A).

4. **One-Two Ballots** $\star$

   A *one-two ballot* is a ballot in which each voter marks her first choice and second choice, but that's it. With a one-two method we can compute the plurality winner (by just looking at the first choices), but we cannot in general use two-round voting (at least in the usual way), because if a voter does not rank either finalist as her top two then we do not know which one she prefers.

   (a) Describe what variations of the Borda count we can use with a one-two ballot.

   (b) Describe how you might implement an analogue of instant runoff with a one-two ballot.

   (c) How might we implement something like Copeland's method?

   (d) Suppose an election has exactly three candidates. How does the information from a one-two ballot compare with the information from a regular preference ballot?

5. **An Irrational Voter** $\star\star$

   Consider the following Condorcet profile with ten voters and five candidates. Prove that some voter is behaving irrationally. By irrationally I mean there personal preference order has a cycle in it.

A	B
4	6

A	C
9	1

A	D
6	4

A	E
7	3

B	C
4	6

B	D
5	5

B	E
2	8

C	D
4	6

C	E
8	2

D	E
1	9

6. **Cumulative Voting and Minority Populations** $\star$

   Cumulative voting is often favored in elections with multiple winners (e.g. a city council race with several open seats) as a way of promoting representation of minority populations. Why is cumulative voting a good choice for this?

7. **Partial Ballots** $\star\star$

   In this problem, we'll consider what happens when we let voters rank as many candidates as they choose. When a voter does not rank a candidate, we will assume that he likes that candidate less than every listed candidate.

   A *partial ballot* is an ordering of some of the candidates. For example, with three candidates, there are fifteen possible partial ballots:

A	B	C	A B	A C
B A	B C	C A	C B	A B C
A C B	B A C	B C A	C A B	C B A

Likewise, a *tabulated partial profile* is a listing of how many voters choose each partial ballot, and a *partial social welfare function* is a function that maps a partial profile to a ranking of the candidates (perhaps with ties). In other words, it's like a regular social welfare function, but the amount of information we get from each voter is somewhat reduced.

(a) Create a definition for a partial social welfare function to be Pareto, analogous to the usual Pareto criterion.

(b) Create a definition for a partial social welfare function to be IIA, analogous to the usual IIA criterion.

(c) Do you think Arrow's theorem still applies in this new context? If so, explain how our original proof might be adapted (you don't actually have to prove the whole thing again, but rather just explain what general changes we would make). If not, give an example of a partial social welfare function that is not a dictatorship but which satisfies your new criteria for Pareto and IIA.

8. **Split Approval Voting** $\star$

Split approval voting (SAV) is an approval voting system in which a voter who approves of k candidates gives $1/k$ of a point to each of them, and a candidate who approves of nobody gives no points.

(a) Prove that SAV is monotone.

(b) Prove that SAV is anonymous.

(c) Prove that SAV is neutral.

(d) Prove, by way of example, that SAV does <u>not</u> satisfy IIA.

9. **Instant Runoff and Pairwise Ballots** $\star\star$

We've seen that some social choice procedures (like the Borda count) can still be computed using the data from pairwise ballots. In this problem, you will prove that instant runoff is *not* such a procedure. To do this, find two tabulated profiles P_1 and P_2 which lead to the same pairwise profiles, but where P_1 and P_2 have different winners under instant runoff.

10. **Condorcet's Jury** $\star\star$

The Marquis de Condorcet believed that, while individual voters may disagree, there must exist somewhere a true ideal for what is the *correct* decision between two choices, and that this correctness is transitive (i.e. free of the Condorcet paradox). This might seem like a preposterous belief when it comes to individual preferences like pizza toppings, but when the voters are deciding on what is best for the public good, it is at least plausible that every such situation has a "correct" answer, even if it might be difficult to ascertain.

One consequence of this philosophy is that, if you believe that voters on average have a better-than-50% chance of making correct decisions, then the question becomes not *what* the majority of voters believe, but rather *how large* that majority is. The wider the majority, the more we can be confident that the majority is correct. What does this imply about the relative merits of voting methods we've seen so far? How does it affect your view of what criteria are important?

11. **Edit Edit Edit** ⋆⋆

    Choose two proof problems you wrote up on a previous homework that you think could use some polishing and rewrite them as best you can. As always, feel free to come talk with me if you want any feedback or help with the edits.

12. **Old Problems**

    Pick a problem that you have not attempted but wanted to do from a previous problem set. (you can do this for multiple problems if you want).