(a)

(b)

(c)

Figure 14.1.1: Finding area under a curve in space.

lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\,dx$ as $\int_I f(x)\,dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

14.1   Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 14.1.1(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \le x \le 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below the curve drawn with the solid line? In other words, what is the area of the region above C and under the the surface f? This region is shown in Figure 14.1.1(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 14.1.1(c), four rectangles have been drawn over the curve C. The bottom corners of each rectangle lie on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height $\times$ width", we assert that the

$$\text{area above } C \approx \sum (\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights $\times$

Notes:

widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 11.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a height. Thus

$$\text{area under } f \text{ and above } C \approx \sum (\text{heights} \times \text{widths});$$

$$\text{area under } f \text{ and above } C = \lim_{||\Delta s|| \to 0} \sum f(c_i) \Delta s_i$$

$$= \int_C f(s) \, ds. \tag{14.1}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\iint_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s) \, ds$" as meaning "sum up, along a curve C, function values $f(s) \times$ small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 14.1 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 14.1.1   Line Integral Over A Scalar Field

Let C be a smooth curve parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s) \, ds = \lim_{||\Delta s|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the ith subinterval, Δs_i is the width of the ith subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

Note: Definition 14.1.1 uses the term **scalar field** which has not yet been defined. Its meaning is discussed in the paragraph preceding Definition 14.3.1 when it is compared to a **vector field**.

Notes:

When C is a **closed** curve, i.e., a curve that ends at the same point at which it starts, we use

$$\oint_C f(s)\ ds \quad \text{instead of} \quad \int_C f(s)\ ds.$$

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

This definition of the line integral doesn't really say anything new. If C is a curve and s is the arc–length parameter of C on $a \leq s \leq b$, then

$$\int_C f(s)\ ds = \int_a^b f(s)\ ds.$$

The real difference with this integral from the standard "$\int_a^b f(x)\ dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. *If* we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before. Unfortunately, parametrizing a curve in terms of the arc–length parameter is usually very difficult, so we must develop a method of evaluating line integrals using a different parametrization.

Given a curve C, find any parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t) \rangle$.

In Section 11.5, we defined the arc–length parameter in Equation 11.1 as

$$s(t) = \int_0^t \| \vec{r}\,'(u) \|\ du.$$

By the Fundamental Theorem of Calculus, $ds = \| \vec{r}\,'(t) \|\ dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x,y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\ ds = \int_a^b f\big(g(t), h(t)\big)\| \vec{r}\,'(t) \|\ dt.$$

We restate this as a theorem, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Notes:

> **Theorem 14.1.1   Evaluating a Line Integral Over A Scalar Field**
>
> - Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t) \rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$, where f is continuous over C. Then
>
> $$\int_C f(s)\, ds = \int_a^b f\big(g(t), h(t)\big) \| \vec{r}'(t) \|\, dt.$$
>
> - Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$, where f is continuous over C. Then
>
> $$\int_C f(s)\, ds = \int_a^b f\big(g(t), h(t), k(t)\big) \| \vec{r}'(t) \|\, dt.$$

To be clear, the first point of Theorem 14.1.1 can be used to find the area under a surface $z = f(x, y)$ and above a curve C. We will later give an understanding of the line integral when C is a curve in space.

Let's do an example where we actually compute an area.

Example 14.1.1   Evaluating a line integral: area under a surface over a curve.
Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 14.1.2.

Solution   Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x + 1$), we can let $x = t$, $y = 2t + 1$, and write $\vec{r}(t) = \langle t, 2t + 1 \rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t + 1) = \cos(t) + \sin(2t + 1) + 2$. We also need $\| \vec{r}'(t) \|$; with $\vec{r}'(t) = \langle 1, 2 \rangle$, we have $\| \vec{r}'(t) \| = \sqrt{5}$. Thus $ds = \sqrt{5}\, dt$.

The area we seek is

$$\int_C f(s)\, ds = \int_{-1}^{1} \big(\cos(t) + \sin(2t + 1) + 2 \big) \sqrt{5}\, dt$$

$$= \sqrt{5} \Big(\sin(t) - \frac{1}{2} \cos(2t + 1) + 2t \Big) \Big|_{-1}^{1}$$

$$\approx 14.418 \text{ units}^2.$$
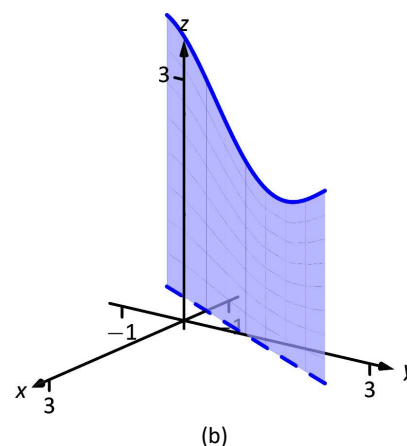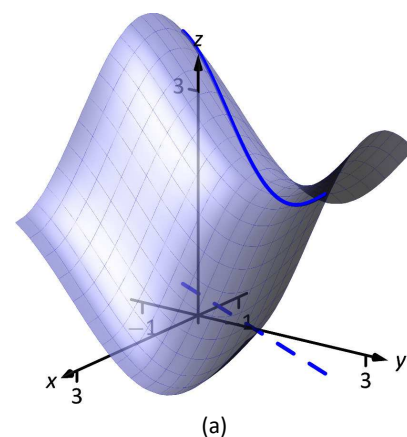
(a)

(b)

Figure 14.1.2: Finding area under a curve in Example 14.1.1.
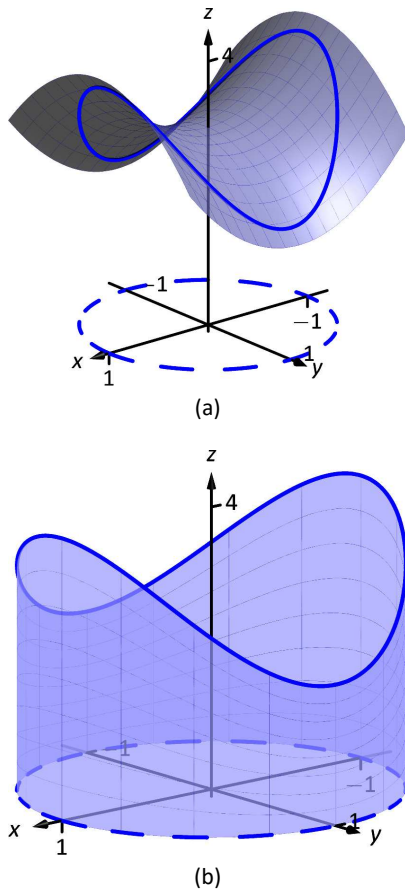
Notes:

(a)

(b)

Figure 14.1.3: Finding area under a curve in Example 14.1.2.

We will practice setting up and evaluating a line integral in another example, then find the area described at the beginning of this section.

Example 14.1.2   **Evaluating a line integral: area under a surface over a curve.**
Find the area over the unit circle in the *x-y* plane and under the surface $f(x, y) = x^2 - y^2 + 3$, shown in Figure 14.1.3.

SOLUTION   The curve *C* is the unit circle, which we will describe with the parametrization $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$. We find $||\vec{r}'(t)|| = 1$, so $ds = 1\,dt$.

We find the values of *f* over *C* as $f(x, y) = f(\cos t, \sin t) = \cos^2 t - \sin^2 t + 3$. Thus the area we seek is (note the use of the $\oint f(s)\,ds$ notation):

$$\oint_C f(s)\,ds = \int_0^{2\pi} \left(\cos^2 t - \sin^2 t + 3 \right)\,dt$$

$$= 6\pi.$$

(Note: we may have approximated this answer from the start. The unit circle has a circumference of 2π, and we may have guessed that due to the apparent symmetry of our surface, the average height of the surface is 3.)

We now consider the example that introduced this section.

Example 14.1.3   **Evaluating a line integral: area under a surface over a curve.**
Find the area under $f(x, y) = 1 - \cos(x)\sin(y)$ and over the parabola $y = x^2$, from $-1 \le x \le 1$.

SOLUTION   We parametrize our curve *C* as $\vec{r}(t) = \langle t, t^2 \rangle$ for $-1 \le t \le 1$; we find $||\vec{r}'(t)|| = \sqrt{1 + 4t^2}$, so $ds = \sqrt{1 + 4t^2}\,dt$.

Replacing *x* and *y* with their respective functions of *t*, we have $f(x, y) = f(t, t^2) = 1 - \cos(t)\sin(t^2)$. Thus the area under *f* and over *C* is found to be

$$\int_C f(s)\,ds = \int_{-1}^{1} \left(1 - \cos(t)\sin\left(t^2\right) \right) \sqrt{1 + t^2}\,dt.$$

This integral is impossible to evaluate using the techniques developed in this text. We resort to a numerical approximation; accurate to two places after the decimal, we find the area is

$$= 2.17.$$

Notes:

We give one more example of finding area.

Example 14.1.4   Evaluating a line integral: area under a curve in space.
Find the area above the *x-y* plane and below the helix parametrized by $\vec{r}(t) = \langle \cos t, 2\sin t, t/\pi \rangle$, for $0 \le t \le 2\pi$, as shown in Figure 14.1.4.

SOLUTION   Note how this is problem is different than the previous examples: here, the height is not given by a surface, but by the curve itself.

We use the given vector-valued function $\vec{r}(t)$ to determine the curve C in the *x-y* plane by simply using the first two components of $\vec{r}(t)$: $\vec{c}(t) = \langle \cos t, 2\sin t \rangle$. Thus $ds = \|\vec{c}\,'(t)\| \, dt = \sqrt{\sin^2 t + 4\cos^2 t}\, dt$.

The height is not found by evaluating a surface over C, but rather it is given directly by the third component of $\vec{r}(t)$: t/π. Thus

$$\oint_C f(s)\, ds = \int_0^{2\pi} \frac{t}{\pi} \sqrt{\sin^2 t + 4\cos^2 t}\, dt \approx 9.69,$$

where the approximation was obtained using numerical methods.

Figure 14.1.4: Finding area under a curve in Example 14.1.4.

Note how in each of the previous examples we are effectively finding "area under a curve", just as we did when first learning of integration. We have used the phrase "area *over* a curve C and under a surface," but that is because of the important role C plays in the integral. The figures show how the curve C defines another curve on the surface $z = f(x, y)$, and we are finding the area under that curve.

Properties of Line Integrals

Many properties of line integrals can be inferred from general integration properties. For instance, if k is a scalar, then $\int_C k f(s)ds = k \int_C f(s)ds$.

One property in particular of line integrals is worth noting. If C is a curve composed of subcurves C_1 and C_2, where they share only one point in common (see Figure 14.1.5(a)), then the line integral over C is the sum of the line integrals over C_1 and C_2:

$$\int_C f(s)\, ds = \int_{C_1} f(s)\, ds + \int_{C_2} f(s)\, ds.$$

This property allows us to evaluate line integrals over some curves C that are not smooth. Note how in Figure 14.1.5(b) the curve is not smooth at D, so by our definition of the line integral we cannot evaluate $\int_C f(s)ds$. However, one can evaluate line integrals over C_1 and C_2 and their sum will be the desired quantity.
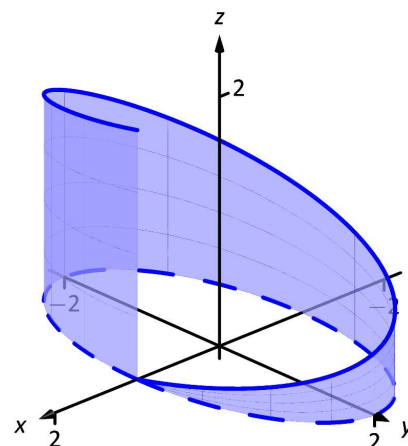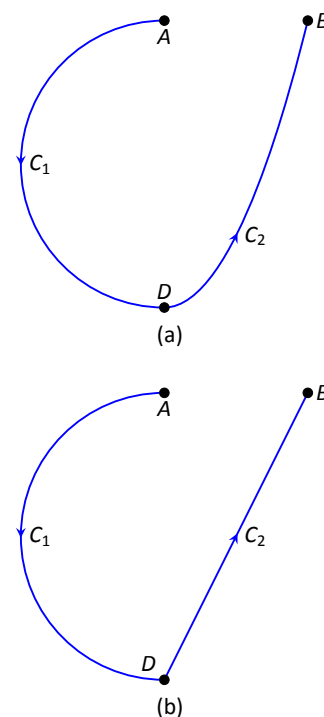

(a)

(b)

Figure 14.1.5: Illustrating properties of line integrals.

Notes:

A curve C that is composed of two or more smooth curves is said to be **piecewise smooth**. In this chapter, any statement that is made about smooth curves also holds for piecewise smooth curves.

We state these properties as a theorem.

Theorem 14.1.2   Properties of Line Integrals Over Scalar Fields

1. Let C be a smooth curve parametrized by the arc–length parameter s, let f and g be continuous functions of s, and let k_1 and k_2 be scalars. Then

$$\int_C \left(k_1 f(s) + k_2 g(s) \right) \, ds = k_1 \int_C f(s) \, ds + k_2 \int_C g(s) \, ds.$$

2. Let C be piecewise smooth, composed of smooth components C_1 and C_2. Then

$$\int_C f(s) \, ds = \int_{C_1} f(s) \, ds + \int_{C_2} f(s) \, ds.$$

Mass and Center of Mass

We first learned integration as a method to find area under a curve, then later used integration to compute a variety of other quantities, such as arc length, volume, force, etc. In this section, we also introduced line integrals as a method to find area under a curve, and now we explore one more application.

Let a curve C (either in the plane or in space) represent a thin wire with variable density $\delta(s)$. We can approximate the mass of the wire by dividing the wire (i.e., the curve) into small segments of length Δs_i and assume the density is constant across these small segments. The mass of each segment is density of the segment $\times$ its length; by summing up the approximate mass of each segment we can approximate the total mass:

$$\text{Total Mass of Wire } = \sum \delta(s_i) \Delta s_i.$$

By taking the limit as the length of the segments approaches 0, we have the definition of the line integral as seen in Definition 14.1.1. When learning of the line integral, we let $f(s)$ represent a height; now we let $f(s) = \delta(s)$ represent a density.

We can extend this understanding of computing mass to also compute the center of mass of a thin wire. (As a reminder, the center of mass can be a useful

---

Notes:

piece of information as objects rotate about that center.) We give the relevant formulas in the next definition, followed by an example. Note the similarities between this definition and Definition 13.6.4, which gives similar properties of solids in space.

Definition 14.1.2 Mass, Center of Mass of Thin Wire

Let a thin wire lie along a smooth curve C with continuous density function $\delta(s)$, where s is the arc length parameter.

1. The **mass** of the thin wire is $M = \int_C \delta(s)\, ds.$

2. The **moment about the y-z plane** is $M_{yz} = \int_C x\delta(s)\, ds.$

3. The **moment about the x-z plane** is $M_{xz} = \int_C y\delta(s)\, ds.$

4. The **moment about the x-y plane** is $M_{xy} = \int_C z\delta(s)\, ds.$

5. The **center of mass** of the wire is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M} \right).$$

Example 14.1.5 Evaluating a line integral: calculating mass.

A thin wire follows the path $\vec{r}(t) = \langle 1+\cos t, 1+\sin t, 1+\sin(2t)\rangle$, $0 \leq t \leq 2\pi$. The density of the wire is determined by its position in space: $\delta(x, y, z) = y + z$ gm/cm. The wire is shown in Figure 14.1.6, where a light color indicates low density and a dark color represents high density. Find the mass and center of mass of the wire.

SOLUTION   We compute the density of the wire as

$$\delta(x, y, z) = \delta\big(1+\cos t, 1+\sin t, 1+\sin(2t)\big) = 2 + \sin t + \sin(2t).$$

We compute ds as

$$ds = \|\,\vec{r}'(t)\,\|\, dt = \sqrt{\sin^2 t + \cos^2 t + 4\cos^2(2t)}\, dt = \sqrt{1 + 4\cos^2(2t)}\, dt.$$

Thus the mass is

$$M = \oint_C \delta(s)\, ds = \int_0^{2\pi} \big(2 + \sin t + \sin(2t)\big)\sqrt{1 + 4\cos^2(2t)}\, dt \approx 21.08\text{gm}.$$
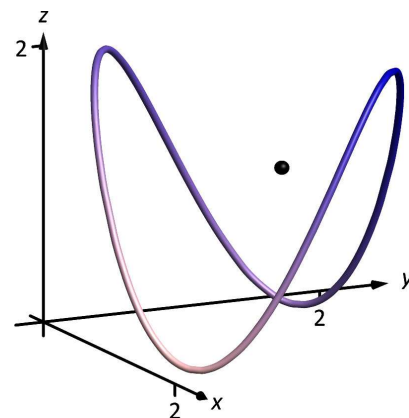

Figure 14.1.6: Finding the mass of a thin wire in Example 14.1.5.

Notes:

We compute the moments about the coordinate planes:

$$M_{yz} = \oint_C x\delta(s)\, ds = \int_0^{2\pi} (1 + \cos t)(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)}\, dt \approx 21.08.$$

$$M_{xz} = \oint_C y\delta(s)\, ds = \int_0^{2\pi} (1 + \sin t)(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)}\, dt \approx 26.35$$

$$M_{xy} = \oint_C z\delta(s)\, ds = \int_0^{2\pi} (1 + \sin(2t))(2 + \sin t + \sin(2t))\sqrt{1 + 4\cos^2(2t)}\, dt \approx 25.40$$

Thus the center of mass of the wire is located at

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right) \approx (1, 1.25, 1.20),$$

as indicated by the dot in Figure 14.1.6. Note how in this example, the curve C is "centered" about the point $(1, 1, 1)$, though the variable density of the wire pulls the center of mass out along the y and z axes.

We end this section with a callback to the Integration Review that preceded this section. A line integral looks like: $\int_C f(s)\, ds$. As stated before the definition of the line integral, this means "sum up, along a curve C, function values $f(s) \times$ small arc lengths." When $f(s)$ represents a height, we have "height × length = area." When $f(s)$ is a density (and we use $\delta(s)$ by convention), we have "density (mass per unit length) × length = mass."

In the next section, we investigate a new mathematical object, the *vector field*. The remaining sections of this chapter are devoted to understanding integration in the context of vector fields.

Notes: