Problem Set 2

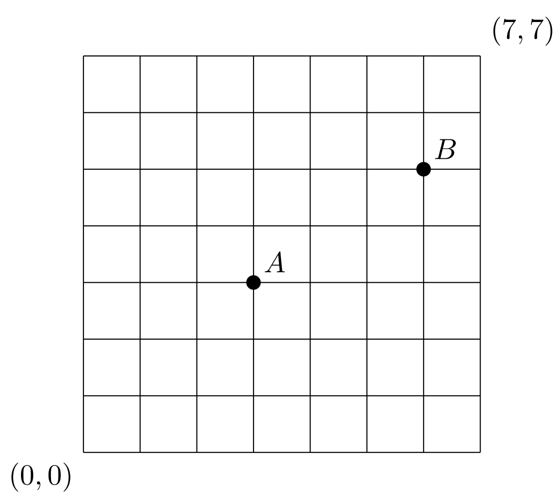
Due: Thursday, January 30th

Instructions: Answer each of the following questions and provide a justification for your answer. In addition to the points assigned below, you will receive 0-2 writing points for the entire problem set.

1. Suppose you are given a standard deck of cards.
   (a) (4 points) How many five card hands have at least one ace, but no more then three aces?
   (b) (4 points) How many five card hands have at least three jacks and no more then one ace?
   (c) (5 points) How many five cards hands have at least one heart or at least one diamond?

2. Suppose that you have 13 unmarked balls and 5 distinct boxes.
   (a) (2 points) How may ways are there to place the balls into the boxes such that every box has at least one ball?
   (b) (2 points) How may ways are there to place the balls into the boxes such that every box has at least two balls?

3. Suppose we have a square lattice of height 7 and length 7. Then we can consider the bottom left corner of the square lattice to be the origin and label it (0, 0).



Suppose you start at (0, 0) and want to end at (7, 7) and can only move up and right.

   (a) (2 points) How many paths are there that go through point A?
   (b) (2 points) How many paths are there that go through point B?
   (c) (2 points) How many paths are there that go through point A and B?
   (d) (2 points) How many paths are there that go through point A and **not** through B?
   (e) (2 points) How many paths are there that go through point B and **not** through A?

(f) (2 points) Compute the sum of your answers from parts (c),(d), and (e). Also, compute the sum of (a) and (b) minus your answer from part (c). Why are these two numbers the same?

4. Consider the combinatorial identity

$$\binom{n}{k} = \binom{n}{n-k}$$

where $n \geq 0$ and $0 \leq k \leq n$. Prove it in the following ways:

(a) (6 points) Prove it by using algebra. (But remember to still use words to explain what you are doing!)

(b) (8 points) Prove it using a counting argument.

5. (10 points) Prove the combinatorial identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

using a counting argument. (hint: Among the $n+1$ objects suppose that one of them is special. How does this help you divide the counting problem into two cases?)