Problem Set 6

Due: Friday, March 13th

Instructions: Do at least 10 of the following problems. **Many of the one star problems lead to lots of interesting questions and can easily be modified to become two or three star problems. I really encourage solutions that at least ask a follow up question if not answer them.**

1. **Bizarro U.N.** ⋆

   Consider a new version of the U.N. Security Council comprising eight non-permanent members and three permanent members, where the passage of a resolution requires at least seven "yes" votes and each permanent member has veto power. Describe this scenario as a weighted voting system. Explain where you got your numbers from, and prove that they indeed work.

2. **Rock the Veto** ⋆⋆

   Consider the following weighted voting system with six members:

   A has 10 votes, B has 9 votes, C has 7 votes, D and E both have 5 votes, and F has 4 votes. The quota is 25.

   Construct a new weighted voting system to give C veto power while changing nothing else. (In other words, the new winning coalitions are exactly the old winning coalitions that include C.) List the new weights and the new quota, and explain how you arrived at your answer.

3. **Votes of the Round Table** ⋆⋆

   (a) Suppose a council of five voters sits around a circular table, as pictured, to vote on a proposal. For a measure to pass, there must be at least three "yes" votes in a row around the table. For example {A,B,E} is a winning coalition, but {A,B,D} is not. Is this a weighted voting system?
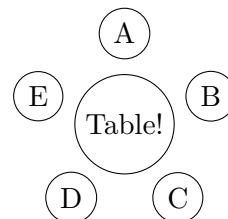
   (b) Now suppose one voter leaves and the table contracts, but the three-in-a-row condition is still maintained. Is this a weighted voting system?
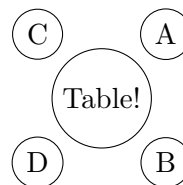
4. **An Independent Voice** ⋆

   In 2001, then-Republican Senator Jim Jeffords formally changed his party affiliation to Independent. After he did so, the composition of the Senate was 50 Democrats, 50 Republicans (including Vice President Dick Cheney), and 1 Independent.

   Consider this as a weighted voting system in which all members of a given party always vote the same way, and a bill needs 51 votes to pass. How powerful is Jeffords compared to the Republican and Democratic blocks? Explain.

5. **Who's the Dummy?** ⋆⋆

In a yes-no voting system, a *dummy voter*, like Luxembourg in the EEC, is a voter whose addition can never turn a losing coalition into a winning coalition—in other words, their vote never matters.

Consider a weighted voting system with four voters: Roger and Bert have 5 votes, Don has 2 votes, and Pete has 1 vote, for 13 votes total.

  (a) Choose a quota so that there are no dummies.
  (b) Choose a quota so that Pete is the only dummy.
  (c) Choose a quota so that Don and Pete are dummies, but Roger and Bert are not.

(In each case, explain why your choice of quota has the desired effect.)

6. **The Return of Monotonicity** ⋆

A yes-no voting system is *monotone* if the addition of a new "yes" vote can never turn a winning coalition into a losing coalition.

  (a) Suppose that in a monotone yes-no voting system, you are told that $\{a,b\}$, $\{b,d\}$, and $\{c,d,e\}$ are winning coalitions. Which of the following must also be winning coalitions? Explain.

        i. The coalition $\{a,b,d\}$.   iii. The coalition $\{a,c,d\}$.
        ii. The coalition $\{b,c\}$.   iv. The coalition $\{a,b,c,e\}$.

  (b) Prove that in any monotone yes-no voting system, each voter belongs to *at least* half of the winning coalitions.

7. **Growth of the European Economic Community** ⋆

In 1973, the European Economic Community expanded to a total of nine members, with weights as follows:

France	10	Belgium	5	England	10
Germany	10	Netherlands	5	Denmark	3
Italy	10	Luxembourg	2	Ireland	3

The new quota is 41 (out of a total 58 available points). Show that even though its vote is "diluted" points-wise, Luxembourg's power has increased! Specifically, construct a scenario in which Luxembourg's vote turns a losing coalition into a winning one.

8. **New England Power** ⋆⋆

Suppose New England secedes from the union and creates its own nation of six states with its own electoral college using the same numbers as they had before the secession:

Vermont	3	New Hampshire	4	Maine	4
Massachusetts	12	Connecticut	7	Rhode Island	4

Quota: 18

Compute the Banzhaf power index of each state.

9. **The Nassau County Board of Supervisors** ⋆⋆

John Banzhaf, a law professor and legal activist, first described his power index during a lawsuit against the Nassau County Board of Supervisors in New York. The board consisted of six members from five districts (two from Hempstead), using a weighted voting system with the following weights:

Hempstead	31	Hempstead	31	Oyster Bay	28
North Hempstead	21	Long Beach	2	Glen Cove	2

Quota: 58

The numbers were chosen because they were representative of the populations of the districts. Which residents do you think were unhappy with this arrangement? Explain your reasoning.

10. **The Shapley-Shubik Power Index** ⋆⋆

John Banzhaf is, it turns out, not the only player in this game. Another well-known power index is the Shapley-Shubik power index. The Shapley-Shubik index tries to measure the power of a voter x by asking the following question:

"Suppose the voters in a yes-no voting system are arranged in a random order from left to right, and the voters begin voting 'yes' from the left and heading towards the right. What are the odds that x's vote is the one that first causes the measure to pass?"

To compute the Shapley-Shubik index for a yes-no voting system with n voters, first write down all $n!$ permutations (orderings) of the voters. Then, for each ordering, add the voters to a coalition from left to right until the addition of some voter causes it to become a winning coalition. This voter is the "crucial voter" for that permutation. The Shapley-Shubick index for a voter is the number of permutations for which it is the crucial voter divided by $n!$.

Compute the Shapley-Shubik power index of each voter in the following weighted voting system:

Avon	5	Beverly	4	Chelsea	3	Dennis	1

Quota: 8

11. **You Can Always Get What You Want** ⋆

Consider the yes-no voting system with three voters a, b, and c where $\{a,b,c\}$, $\{a,b\}$, $\{a,c\}$, and $\{a\}$ are the winning coalitions and $\{b,c\}$, $\{b\}$, $\{c\}$, and $\{\}$ are the losing coalitions. Find the Banzhaf power index of each voter. What does this tell you?

12. **Different Weights, Same Power** ⋆

Give an example of a weighted voting system with four voters where each voter has a different weight, yet the system is anonymous.

13. **Congress: In Another Dimension!** ⋆⋆⋆

We have seen (er, okay, it depends when you read this sentence) that the U.S. Federal system with 537 voters—435 representatives, 100 senators, one vice president, and one president—cannot be expressed as a traditional weighted voting system. However, it can be expressed as what we might call a *two-dimensional* weighted voting system: each voter has an x-weight

and a y-weight (either or both of which might actually be zero), and the system has an x-quota and a y-quota. A coalition is winning if the sum of its voters' x-weights is at least the x-quota *and* the sum of its voters' y-weights is at least the y-quota.

Prove that the federal system is a two-dimensional weighted voting system by assigning x- and y-weights to each of the 537 voters and deciding on an x- and y-quota so that a bill passes exactly when it normally would:

(1) with at least half the House and exactly half the Senate and the president and vice President's support;

(2) with at least half the House and more than half the Senate and the president's support; or

(3) with at least two-thirds of the House and at least two-thirds of the Senate.

14. **Xenophilius Lovegood Endorses This Problem** ⋆⋆

On the right we have a picture of the Fano plane (also known as the "finite projective plane of order two" or "YEA WHY TRY HER WET WAR HAT"). The diagram consists of seven points (A, E, H, R, T, W, and Y) and seven "lines" (the three sides of the triangle, the three medians of the triangle, and a circle). Every point is on three lines, every line passes through three points, every two points are joined by one line, and every two lines meet at one point. It's also my favorite hypergraph.
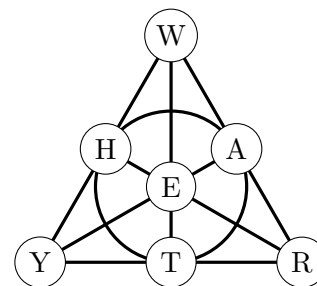
Consider a yes-no voting system with seven voters: Amy, Ernest, Hector, Rhoda, Titus, Winnie, and Yorick. A subset of the voters is a winning coalition if it includes three voters who are on a line. For example {Amy, Hector, Rhoda, Titus} is winning because it includes the three voters who lie on the circle, but {Ernest, Titus, Yorick} is losing because it only has three voters in it and they do not lie on a single line.

(a) Prove that this voting system is not anonymous by taking a winning coalition, and then swapping out one voter for another to create a losing coalition.

(b) Prove that this system *is* neutral in the following sense: if a set of voters S forms a losing coalition, then the set of voters *not* in S forms a winning coalition. In other words, prove that no matter how the votes are cast, there will *either* be a line of "yes" votes or a line of "no" votes. (By contrast, this is *not* the case in a round of Tic-Tac-Toe, because the game can be a tie.)

Hint: This is a little tricky, so you might try to break it up into two cases: (1) there are fewer than three "yes" votes; or (2) there are at least three "yes" votes. Prove that in both of these cases, there must be a line of voters who voted the same way.

15. **Big Necromancy** ⋆⋆ The Kingdom of Tudee is ruled by a three-member council. The members are Trina, Uma, and Vera, and their political beliefs can be modeled on two axes: Good versus Evil and Order versus Chaos.
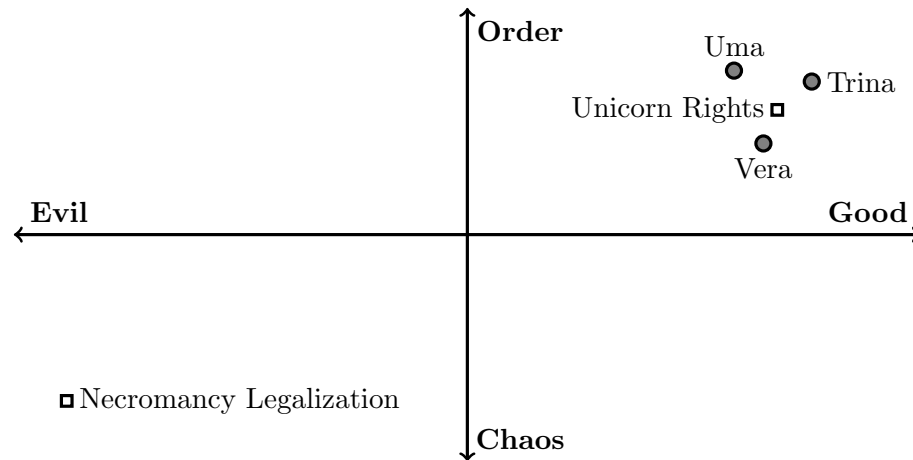
The system is as follows: any citizen can present an amendment to a bill, which moves the bill to any other point on the two-dimensional plane. If a majority of the councilors like the amendment (because it aligns more closely with their beliefs than the previous version), then it becomes the new version of the bill.

The three are currently considering a Unicorn Rights bill. You would like them to instead vote for Necromancy Legalization. Can you propose a series of amendments so that they end up on your preferred bill?