Exam 1 Prep

The first exam will cover all material we have discussed so far in the course: counting, number theory, and proof techniques like contradiction and induction. The in-class portion of the exam will be 50 minutes. It will ask for basic recall like "what is the definition of $a|b$" or short questions like "how many poker hands have at least one ace in them, how many have at least an ace and a jack, how many...". Anything I think you can do in less than 50 minutes is fair game.

I will send out the other portion of the exam on Monday afternoon and it will be due on Thursday by class time. For that portion you can use your notes and the book but no other sources. You may not discuss that portion of the exam with anyone but me. The out of class portion will involve longer tasks. For example I imagine proofs by induction, proofs that require some cleverness that you might need some time to think through, counting proofs, or longer counting arguments would be on that portion.

The problem complexity will be roughly the same as the homework, but that means they will range from fairly straight forward to less straight forward. I think the best preparation is to review the homework, review feedback from the graders, and read the book. Here are some example problems if you want more than is in the book.

1 Counting

1. How many (positive integer) divisors does 2940 have? What about 3150?
2. In how many ways can the following poker hands occur?
   (a) Straight flush - 5 consecutive ranks, all in the same suit, ace ranks high, i.e. ace, 2, 3, 4, 5 isn't valid.
   (b) Four of a kind - all 4 cards of a single rank plus one other card.
   (c) Full house - 3 cards of one rank and 2 cards of another rank.
   (d) Flush - 5 cards of the same suit but don't count the straight flushes.
   (e) Straight - 5 consecutive ranks, suits don't matter, but don't count the straight flushes.
   (f) Three of a kind - 3 cards of a single rank, and two other cards of distinct ranks.
   (g) Two pair - 2 cards of one rank, 2 cards of another rank, and 1 card of a distinct rank.
3. (a) In how many ways can you give away 12 apples and 10 pears to 4 people so that each person gets at least one apple and one pear?
   (b) In how many ways can you give away 8 apples, 9 pears and 10 peaches to 3 people so that each person gets at least one piece of each fruit?
4. Lucas is in a gardening store looking at pepper plants. There are 20 different varieties. She will get back home with 8 pepper plants.
   (a) In how many ways can she choose 8 plants of mutually distinct varieties?
   (b) In how many ways can she make her acquisition if she is *not* committed to getting 8 plants of mutually distinct varieties?

2 Proof-writing

1. Go over the proofs that $\sqrt{2}$ is irrational, and that there are infinitely many primes.
2. Go through all the proof-writing feedback you got on your homework.
3. Prove that $\sqrt{p}$, for p a prime integer, is irrational.
4. Let $n > 4$ be an integer. Show that n divides $(n-1)!$ if and only if n is not prime.
5. Integers a, b and c are the sides of a right-angled triangle. Prove that at least one of a, b and c is divisible by 3.
6. The equation $a^3 + 2a = 2$ has exactly one real solution. Show that a is irrational.
7. Show that the equation $5a^2 - 2b^2 = 1$ has no integer solutions.
8. Let $x \neq 1$ be a real number and let $n \geq 0$ be an integer. Use induction to prove that
$$(1+x)(1+x^2)(1+x^4)(1+x^8)...(1+x^{2^n}) = \frac{1-x^{2^{n+1}}}{1-x}.$$
9. Use induction to show that for integers $k_1, k_2, ..., k_n$ the product
$$(4k_1+1)(4k_2+1)....(4k_n+1)$$
leaves remainder 1 after division by 4.
10. The (basic) Triangle Inequality states that
$$|a+b| \leq |a| + |b|$$
for all numbers (and sometimes even vectors) a, b. Use mathematical induction to show the following, more general version of the Triangle Inequality:
$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n| \ \text{ for all } \ a_1, a_2, ..., a_n.$$
11. Let x be some real number with $0 < x < 1$. Show that
$$(1+x)^n \geq nx^n \ \text{ for all integers } \ n \geq 1.$$
12. Let $a \big| (b-1)$. Use mathematical induction to show that for all $n \geq 1$ we have
$$a \big| (b^n - 1).$$