HW 21: Taylor polynomials

Due: Monday, December 2nd in SQRC by 9pm

Learning Goals:

- Estimate the error of Taylor polynomial approximations.

Questions:

1. The 5th degree Taylor polynomial centered at $x = 0$ is $\tan(x) \approx x + x^3/3 + 2x^5/15$. Recall that $\tan(x) = \sin(x)/\cos(x)$. Multiply the 5th degree Taylor polynomial for $\tan(x)$ from part a) by the 4th degree Taylor polynomial for $\cos(x)$ and show that you get the 5th degree polynomial for $\sin(x)$ (discarding higher degree terms).

$$\begin{aligned}\tan(x)\cdot\cos(x) &\approx (x + x^3/3 + 2x^5/15)\cdot(1 - x^2/2 + x^4/4!)\\ &= x + x^3/3 - x^3/2 + x^5/4! - x^5/6 + 2x^5/15\\ &= x - x^3/6 + x^5/120\\ &\approx \sin(x)\end{aligned}$$

2. Find the 5th degree Taylor polynomial centered at $x = 0$ for the function $f(x) = (1+x)^{\frac{1}{2}}$. Use this approximation to estimate $\sqrt{1.1}$. Use a calculator to check how accurate you are. (if your answer is not very accurate, maybe think about what value of x you are using).

The 5th degree Taylor polynomial is $1 + x/2 - x^2/8 + x^3/16 - 5x^4/128 + 7x^5/256$ which evaluates to 1.04881 when $x = 0.1$. The actual value is approx 1.04881. A common error might be to plug in $x = 1.1$ and get 1.46878.

3. Here is an integral we can't compute. Integrate the 6th degree Taylor series of $\frac{1}{(1+x^2)^{\frac{1}{3}}}$ to approximate it instead. Feel free to use a computer to help but make sure to show your work.

$$\int_0^1 \frac{1}{(1+x^2)^{\frac{1}{3}}}$$

The taylor polynomial is $1 - x^2/3 + (2x^4)/9 - (14x^6)/81$ which integrates to

$$\int_0^1 (1 - x^2/3 + (2x^4)/9 - (14x^6)/81)dx = 368/405 \approx 0.90864$$

4. Use the Taylor polynomial approximation for $\sin(x)$ to compute the limit

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} \approx \lim_{x\to 0} \frac{x - x^3/3! + O(x^5)}{x} = \lim_{x\to 0} 1 - x^2/3! + O(x^4) = 1$$

5. (a) The first degree Taylor polynomial for e^x at $x = 0$ is $1 + x$. Plot the remainder $R_1(x) = e^x - (1 + x)$ over the interval $-0.1 \le x \le 0.1$. How does this graph demonstrate that $R_1(x) = O(x^2)$ as $x \to 0$? (hint: Use Desmos to zoom in to the interval $-0.1 \le x \le 0.1$ and base your sketch off that graph.)

   The graph looks like a parabola.

   (b) There is a constant C_2 for which $R_1(x) \approx C_2 x^2$ when $x \approx 0$. Estimate the value of C_2.

   $C_2 \approx 0.5$ anything between 0.4 and 0.6 is fine.

   (c) Repeat these two steps for the second degree approximation $1 + x + x^2/2$ by noticing that $R_2(x) = O(x^3)$ and finding a constant C_3. (hint: this is a much better fit so you might want to zoom out to something like $-0.5 \le x \le 0.5$ to find to constant C_3 that works best.

   Something like $C_3 = 0.17$ but anything between 0.1 and 0.3 is fine.

6. (a) Let $f(x) = \ln(x)$. Find the smallest bound M for which $|f^{(4)}(x)| \le M$ when $|x-1| \le 0.5$.

   $f^{(4)}(x) = -6x^{-4}$ so the max it can be is when $x = 0.5$ and we get $-6/(0.5)^4 = -96$ so $M = 96$.

   (b) Let $P_3(x)$ be the degree 3 Taylor polynomial for $\ln(x)$ at $x = 1$, and let $R_3(x)$ be the remainder $R_3(x) = \ln(x) - P_3(x)$. Find a number K for which $|R(x)| \le K|x - 1|^4$ for all x satisfying $|x - 1| \le .5$.

   Applying Taylors theorem we get $K = M/4! = 4$

   (c) If you use $P_3(x)$ to approximate the value of $\ln(x)$ in the interval $0.5 \le x \le 1.5$, how many digits of the approximation are correct?

   The worst case is when $x = 0.5$ or $x = 1.5$ and in those cases the error $|R(x)| \le 4(0.5)^4 = 0.25$ This is not even one digit of accuracy.

   (d) Suppose we restrict the interval to $|x - 1| \le 0.1$. Repeat parts (a) and (b), getting smaller values for M and K. Now how many digits of the polynomial approximation $P_3(x)$ to $\ln(x)$ are correct, if $0.9 \le x \le 1.1$?

   Now M is $6/(0.9)^4 \approx 9.14$ so $K = M/4! \approx 0.381$ and the worst case is when $x = 0.9$ or $x = 1.1$ so $|R(x)| \le 0.0000381$ which is about 4 digits of accuracy.