Problem Set 8

Due: Friday, April 3rd

Instructions: Do at least 7 of the following problems.

1. **Apportionment you missed ?**

   Do a problem from problem set 7 that you did not get a chance to do.

2. **The Balinski-Young Algorithm** $\star\star\star$

   In 1982, mathematicians Michael Balinski and Peyton Young developed a new apportionment method, although for various reasons they do not recommend actually using it in Congress. The method works as follows:

   - Compute the Jefferson critical divisors for each state and sort them all together from highest to lowest. Each state starts with zero seats.
   - Do this h times:
     (i) Look at the highest critical divisor in your list.
     (ii) Check what would happen if you gave a seat to that state by comparing it to the upper quota:

     $$\lceil s_i \rceil = \left\lceil \frac{\text{state population}}{\text{country population}} \times \text{total seats allotted so far} \right\rceil$$

     (iii) If the new seat would cause that state to have *more than* its upper quota, skip them for now and try the next state down the list.
     (iv) The first state on the list for which the new seat will *not* exceed upper quota gets the seat. Cross that divisor off your list.

   (a) Does this method satisfy the quota property? (Obviously it never violates upper quota, so you just need to check lower quota.)

   (b) Does this method exhibit the Alabama paradox? Explain.

   (c) Does this method exhibit the population paradox? Explain.

3. **Hill's Optimality** $\star\star\star$

   Prove that Hill's method is optimal in the following sense:

   Suppose under Hill's method that state i with population p_i receives a_i seats, and state j with population p_j receives a_j seats. Suppose further that state i is better represented than state j, or in other words that $a_i/p_i > a_j/p_j$. This means that the per capita representation of state i is greater than that of state j by a certain percentage. Then (this is what you're trying to prove) if a single seat were transferred from state i to state j, this would cause state j to have a greater per capita representation than state i by an even *larger* percentage.

4. **Evaluating the Current State of Apportionment** $\star\star$

   Knowing everything we know about the apportionment problem, what changes (if any) to our current system (Hill's method, $h = 435$) would you recommend? Explain in detail.

5. **Mowing the Lawn Now** $\star$

   Cake-cutting problems are sometimes referred to as the *division of goods*. The following problem is then a "division of bads":

   A house with n people has a large backyard, and the residents would like to split up the task of mowing the lawn so that nobody feels that his own portion is larger than $1/n$. Construct an analogue of the moving knife algorithm to solve this problem (perhaps a "moving clothesline"?), and prove that it works as intended.

6. **Proving Envy** $\star$

   Prove that the moving knife algorithm is not envy-free explicitly by stating specific flavor preferences of three players, drawing a cake, explaining what happens when the algorithm is applied, and showing that some player envies another's share.

7. **A Disproportionate Method** $\star$

   Construct a non-random cake-cutting procedure for three players where *no player* is guaranteed their fair share. (All the cake must be distributed to the players; you can't just throw it out the window and declare the algorithm a success.)

   Give examples of to show that your algorithm actually does this. In other words, for your fixed algorithm, apply it to three different cakes and show that each player is unhappy in at least one situation.

8. **Strict Proportionality** $\star$

   A cake distribution for n players is *strictly proportional* if every player believes his own share is *more* than $1/n$. Is it always possible to find such a distribution? Explain.

9. **How Much Do I Get?** $\star\star$

   Consider the following algorithm:

   - April cuts a cake into what she considers to be equal thirds.
   - Ben cuts each of those pieces into what he considers to be equal halves.
   - The three players select pieces in the following order: Chris, Ben, April, April, Ben, Chris. So each player gets two of the six pieces.

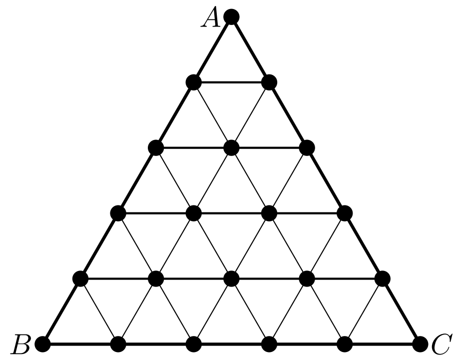
   (a) Is April guaranteed one third of the cake?

   (b) Is Ben guaranteed one third of the cake?

   (c) Is Chris guaranteed one third of the cake?

10. **A Totally Unrelated Problem** $\star\star\star$

   Suppose you have a big triangle like the one below, cut up into small triangles. Paint the corners of the big triangle amethyst, beige, and chartreuse, as in the picture below:



Then paint all the other dots in this picture, obeying the following rules:

- All dots on the left side of the big triangle must be amethyst or beige.
- All dots on the right side of the big triangle must be amethyst or chartreuse.
- All dots on the bottom side of the big triangle must be beige or chartreuse.
- Dots in the middle may be any color.

Is there necessarily a small triangle whose corners are all different colors?