Problem Set 1

Due: Friday, January 31st

Instructions: Do at least 7 of the following problems. Problems are rated on a three-star system. ⋆ problems are more or less straightforward, usually asking you to show that you've got a basic understanding of the material but also give a fun way to apply it. They are a great way to get started! ⋆⋆ problems ask you to analyze the material from a new angle, and usually that means proving something about a new concept. ⋆⋆⋆ problems do this and more. Attempt all of them, regardless of the rating; I may think a problem is difficult, but I am also often wrong. In the end I expect you to solve a mix of problems. You can also do more than I suggested. You won't get more points but I'll count the best ones for grading purposes.

1. **Trump v. Clinton v. Johnson v. Stien** ⋆

   In this problem we will look at the 2016 presidential election.

Candidates	Popular vote	Simple (if you want)
Clinton	65,853,514	65.9
Trump	62,984,828	63
Johnson	4,489,341	4.5
Stein	1,457,218	1.5

   (a) Make an educated guess about the full voter preference profile. You will have to make some choices here, for example would you put set every Johnson voter's second choice as Trump or would you split them between Trump and Clinton in some way? You don't have to be too careful or complicated, just make some guesses and see what happens.

   (b) Compute the social welfare result with the following voting methods: plurality, Borda count, instant runoff, Copeland, Coombs. Is there a Condorcet winner?

   (c) Any thoughts or surprises?

2. **2008 Flashback** ⋆

   In the spring of 2008, after John McCain effectively clinched the GOP nomination but while Hillary Clinton and Barack Obama were still vying for the Democratic spot, there were three plausible candidates for the next president of the United States. List the six possible voter profiles (rankings) of these candidates, and give reasonable estimates for what portion of the voting population would choose each profile. Then use your estimates to determine who would win an election between those three if the electoral college were abandoned in favor of each of the following methods:

   (a) Plurality voting.

   (b) The Hare method.

   (c) The Borda count.

   (d) The Coombs method.

   (Note: feel free to replace these three candidates with another set from a more recent scenario, if you think it would be more interesting. Perhaps Mitt Romney, Newt Gingrich, and Rick Santorum fighting for the 2012 Republican nomination?)

3. **Anonymity on the National Stage** ⋆

   Are nationwide presidential elections in the United States (using our current voting system) anonymous? Are they neutral? Argue why or why not.

4. **Anonymity in the Senate** ⋆⋆

   Forgetting for the moment about filibusters and supermajorities and whatnot, consider the Senate as a straightforward voting system between two alternatives. (Namely, whether a bill should pass or not pass.) It has 101 voters: 100 senators who vote yes or no under a simple majority system, and the vice president who casts a tie-breaking vote whenever the senators are split 50-50. Is this system anonymous? Explain your answer carefully.

5. **Votes and Recreation** ⋆

   April, Ben, Chris, and Donna are running for a seat on the Pawnee city council. A record high turnout of seven voters show up to vote for the local election. Their profiles are as follows:

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5	Voter 6	Voter 7
A	A	B	B	C	A	D
D	C	D	A	B	B	B
B	D	C	D	D	D	C
C	B	A	C	A	C	A

   Determine the winner of the election for each of the following voting methods:

   (a) Plurality method.

   (b) The Borda count.

   (c) Instant-runoff method.

   (d) Coombs method.

6. **Neutrality, Anonymity, and Orderings** ⋆

   Suppose a neutral and anonymous social welfare function produces the preference list (A,B,C) for the following profile:

Voter 1	Voter 2	Voter 3
C	A	B
B	C	C
A	B	A

   (a) What preference list would the same social welfare function produce on this profile?

Voter 1	Voter 2	Voter 3
C	A	B
A	B	A
B	C	C

   (b) Give a profile for which the same social welfare function would produce the preference list (C,A,B).

7. **Variations on Borda** ⋆⋆

   For this problem, consider the following voter profiles:

Voter 1	Voter 2	Voter 3	Voter 4
B	C	C	A
A	A	D	D
D	B	B	B
C	D	A	C

(a) Find the winner using the Borda count.

(b) Here are four other ways to score the candidates. Do as many of these calculations as you find helpful.

- Suppose we made a new version of the Borda count, where each first place vote gets 8 points, each second place vote gets 4 points, each third place vote gets -4 points, and each fourth place vote gets -8 points. Redo the election using this new method.
- Redo it again, this time using the points -1, -5, -9, -13 for (respectively) first, second, third, and fourth place.
- And again, this time using the points 9, 4, 1, and 0.
- Once more now, with values 5, 4, 3, and 2.

(c) Propose a condition on the new set of point values that will guarantee the results are the same as the original method. Prove your answer.

(d) Is the Borda count the same as Eric's point system?

8. **Iterated Plurality Voting** ⋆⋆

It's very easy to convert a social welfare function into a social choice function: just call the candidate(s) in first place the winner(s), and ignore the rest. Converting a social choice function into a social welfare function is harder, but here's how we can do it (at least for functions meeting the "always a winner" criterion):

Run the social choice function on your candidates, and put the winner(s) in first place. Now delete them from your profile (that is, delete the candidate from every ballot, moving everyone else up accordingly). Run your social choice procedure *again*, and put the winner(s) in second place. Keep doing this until you've placed all the candidates.

Use this process to turn plurality-the-social-choice-function into a social welfare function on the following tabulated profile. Are your results the same as if you had used the usual version of plurality as a social welfare function?

10	5	3	3	2
A	B	C	C	D
B	C	B	D	C
D	D	D	B	B
C	A	A	A	A

9. **Counting Voting Systems** ⋆⋆⋆

Remember that a function from a set of a inputs to a set of b outputs is a procedure that assigns each input to exactly one output. Some inputs may lead to the same output, but no single input can lead to more than one output. There are b possible choices of output for each input, so the number of possible functions is $b \times b \times b \times \cdots \times b$ (a times) $= b^a$. The goal of this problem is to use this fact to count the number of possible social welfare functions in an election with c candidates and v voters.

(a) In an election with c candidates, how many possible ballots are there? For purposes of this problem, let's disallow ties.

(b) Use your previous answer to determine how many possible profiles (sequences of v ballots) there are.

(c) How many possible outputs are there? Again, do not allow ties: an output of a social welfare function is simply an ordering of the c candidates.

(d) Use your previous answers to determine how many possible social welfare functions there are. You counted the inputs in part (b) and the outputs in part (c).

(e) (Optional and difficult) What if you allow ties?

10. **Weighted Copeland** $\star\star$

In class we discussed the weighted Copeland method. For each pairwise comparison, give the winner points equal to the number of votes they won by. The candidate with the most points wins.

This method is actually the same as another voting method we are already studying. Find its doppelganger and prove that they are equal social welfare functions.