Daily Prep Assignment for April 8th

Overview

To prepare for class

Preview activities:

- Preview activity:
  a) Recall from section 9.8 how to compute the arc length of a curve. Set up (but do not evaluate) an integral to compute the arc length of the curve defined by $\mathbf{r}(t) = (t^2, \sin(t), t)$ from $z = 0$ to $z = 3$.
  b) Now suppose the density function for this curve is $\delta(x, y, z) = 1$. What integral computes the mass of the curve described in part a)?
  c) What if the density function is given by $\delta(x, y, z) = 5$?
  d) What if the density function is given by $\delta(x, y, z) = z$?

Reading:

- Read the two sections on scalar line integrals and scalar surface integrals linked to on the course website.

Watching: Watch these additional resources if you need support reading the text.

1. Scalar line integrals and scalar surface integrals: `https://youtu.be/FXevK2LkLzk"`

During and after class

These problems are not in the book but I'm going to number them as if they were.

- Activity 11.10.2: Compute the mass of a wire that is the segment of $y = 3x + 2$ on $[1, 2]$ if the density function is $\delta(x, y) = 5x + 2y$.

- Activity 11.10.3: Compute the mass of a wire that is the curve defined by $r(t) = (\cos(t), \sin(t), t)$ for $0 \leq t \leq 4\pi$ if the density function is $\delta(x, y, z) = z$.

- Activity 11.10.4:
  a) Set up (but do not evaluate) an integral to compute the mass of a wire with density $\delta(x, y, z) = ye^x$ parameterized by $r(t) = (t^2, \sin(t), \frac{1}{t})$ with $0 \leq t \leq 2$.
  b) Discuss why you can't compute this integral by hand.
  c) Instead we are going to estimate the value of this integral using Riemann sums! It turns out this is what we often have to do in real world applications! It would be great if we understand how to do this conceptually so try breaking it up into the following steps.
     - First, break the curve up into four equal time intervals. Estimate the length of each interval by multiplying $|r'(t)|$ evaluated at some point on the interval (the speed) by the change in time for each interval.
     - Second, estimate the density of each interval by evaluating the density function at some point on that interval.

- Finally, estimate the mass of all four segments and add them together.

- Activity 11.10.5: Integrate the function $f(x, y, z) = z$ over the cone defined by $z = \sqrt{x^2 + y^2}$ and $0 \leq z \leq 2$.

- **Extra problem 1**: We have seen how to integrate in 2D, in polar, in 3D, in spherical, with arbitrary change of coordinates, over lines and over surface. Let's take a moment to reflect on what is similar about all these situations.

  a) Before going any further what similarities do you see between all of these integration problems? Is there a single method that binds them all together? What difference do you observe?

  b) More specifically, we saw that the Jacobian $\left|\dfrac{\partial(x, y)}{\partial(s, t)}\right|$ represents the amount we are stretching space by when we change from one coordinate to another. Compare and contrast that to how we use $|r'(t)|$ and $|r_s \times r_t|$ when computing integrals over curves or surfaces.

  c) Lastly, recall one of the first integrating techniques you ever learned: u-substitution. Think about the steps required when doing u-substitution and the steps required when doing a general change of coordinates as we learned in this class. Are they the same? are there differences?

- **Extra problem 2**: This problem illustrates how although we may not need to change coordinates to solve a problem it often makes them much easier.

  a) Consider the surface defined by $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$. This surface is called an ellipsoid. Make a sketch of this surface and describe it in your own words. Feel free to use appropriate technology to help visualize it.

  b) Find a change of coordinates that transforms the ellipsoid into a sphere.

  c) Compute the Jacobian for your change of coordinates.

  d) Compute the volume of the ellipsoid from part (a). If you wish to use geometric interpretations of an integral (like volumes of spheres) instead of evaluating iterated integrals feel free to do do so, making sure to write sentence to clarify your steps.

- **Optional question, you don't need to do this but you might find it interesting**: Let B be the unit ball in $\mathbb{R}^4$:

$$\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 \leq 1\}.$$

  $f(x, y, z, w) = \sqrt{x^2 + y^2 + z^2 + w^2}$, is the distance from (x, y, z, w) to the origin in $\mathbb{R}^4$. In this problem we will compute the average distance of a point in B to the origin. We will do this by evaluating

$$\frac{\iiiint_B f(x, y, z, w)\, dx\, dy\, dz\, dw}{\iiiint_B 1\, dx\, dy\, dz\, dw}.$$

  Before we begin, recall that the 2D ball is paramatrized by

$$(x, y) = (r\cos(\theta), r\sin(\theta)).$$

  Think about how we can use this to get a parameterization for the 3D ball in the following way:

$$(x, y, z) = ([r\cos(\theta)]\cos(\phi), [r\sin(\theta))]\cos(\phi), \sin(\phi)).$$

a) Devise an analog of polar or spherical coordinates for $\mathbb{R}^4$, such that the parameter domain for the 4D ball B is a 4-dimensional box. (just like the circle is a rectangle in polar coordinates and the sphere is a box in spherical coordinates). In other words, you should find a continuous function $T : \mathbb{R}^4 \to \mathbb{R}^4$, where $T(r, \theta, \phi, \psi) = (x, y, z, w)$. (*hint 1*: your answer will be very close to spherical coordinates. Think about how we got from spherical to polar. It also might help to think about the 4D ball as the 3D ball being revolved around the w-axis.) (*hint 2*: when you set ψ equal to 0 you should recover the parametrization of a sphere).

b) Compute the Jacobian for T.

c) Use the change of coordinates defined by T to compute the volume of B. Your answer should be $\pi^2/2$.)

d) Use T to compute the average value of f on B and thus reach our goal of computing the average distance of a point in B to the origin.