2.4: The product rule

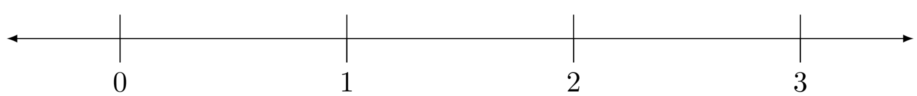
Goal: In this worksheet we will strengthen our understanding of what a derivative means and discover the product rule. We use pictures and concrete examples to develop our intuition. This worksheet also pays respects to the Boston molassacre of 1919 and raises awareness about the dangers of molasses flooding.

1. Your friend claims that the product rule says that $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$. How could you show your friend that they are mistaken using the function $f(x) = x^2$?

2. Molasses is starting to leak out of storage tank. The table below shows how many inches down the tank the molasses has dripped after t minutes have passed.

t (min)	0	2	4	6	8	10	12
x (in)	0	0.5	1	1.25	1.5	2	3

a) Using the number line below, draw tick-marks to show how far the molasses has flown every two minutes.



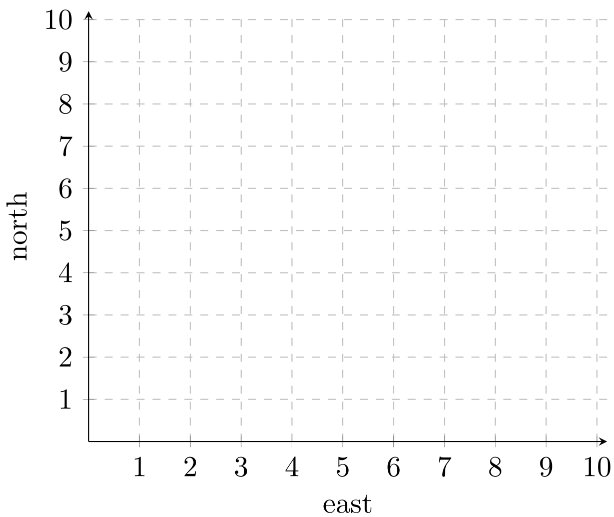
b) On the number, line shade the region between time $t = 2$ and $t = 4$. What does this length represent?

c) Estimate how fast the molasses was flowing at time $t = 3$.

d) Estimate how fast the molasses was flowing at time $t = 8$.

3. Suddenly the molasses tank bursts and molasses pours out! The molasses starts spreading away from the tank (located at $(0,0)$ in the plane) to the east and north in a rectangle that grows larger and larger. For the first ten seconds the number of feet the eastern edge of the rectangle has traveled after t seconds is given by the function $f(t) = 2t$ and the northern edge by the equation $g(t) = \frac{1}{2}t^2$. (the western and southern edges are always at $x = 0$ and $y = 0$ respectively). After ten seconds the molasses slows down and the eastern edge is given by $h(t)$ and the northern edge by $k(t)$.

   a) For times $t = 1, t = 2, t = 3$ and $t = 4$, draw the rectangle that the molasses has covered.



   b) Shade the region between the rectangles for times $t = 2$ and $t = 3$. What does this region represent? Cut this region into three rectangles. What is the area of each rectangle?

   c) Estimate the rate at which the molasses is covering the ground (units will be ft^2/sec) at time $t = 2.5$.

   d) Estimate the rate of molasses coverage at time $t = 3.5$.

   e) Write an expression for the how much more ground is covered by the molasses between times t_0 and t_1. Do this in two ways: (i) do it by taking the difference of the two larger rectangles, (ii) find the area of the three rectangles making up the difference as in part (b).

f) Do the same things as the previous question, but now assume the times are after $t = 10$ so we need to use the unknown functions $h(t)$ and $k(t)$. Your answer will be in terms of these functions.

g) Using your two answers from the part (e), let $t_1 = t_0 + h$ and compute the instantaneous rate of coverage by the molasses at time t_0 by dividing by h and taking a limit as h approaches 0. Feel free to recognize parts of your expression as derivatives and use that to help.

h) Now use your two answers from the part (f), let $t_1 = t_0 + h$ and compute the instantaneous rate of coverage by the molasses at time t_0 by dividing by h and taking a limit as h approaches 0. Feel free to recognize parts of your expression as derivatives and use that to help.

i) The two different ways of finding the instantaneous rate of change in the previous part is the product rule!

4. Here are some problems to practice using the product rule on:

   a) Compute the derivative of $f(x) = x(x + x^2)$ using the product rule.
   b) Compute the derivative of $g(x) = (\sqrt{x} + x)(1 + x^3)$ using the product rule.
   c) Suppose the price of widgets is \$20 and $20,000$ widgets are sold. If the price increases at a rate of \$1.25 per year and the quantity sold increases at a rate of 2000 per year, at what rate will revenue increase?
   d) Compute the derivative of $h(x) = \frac{x+2}{1-x^2}$ using the quotient rule.