Math 490: Mathematics of Social Choice

Here is a little refresher about proving things in case you are out of the habit. When you think of a proof, perhaps you're picturing something like this:

> **Theorem:** $\forall x, y \in \mathbb{R}^+, \exists n \in \mathbb{N}$ s.t. $nx > y$.
>
> *Proof:* $n := \lceil y/x \rceil + 1$. $n > y/x \Rightarrow nx > y$. $\square$

But most of the time, a proof will look more like this:

> **Theorem:** A tree graph on n vertices has exactly $n - 1$ edges.
>
> *Proof:* Suppose we draw such a tree from scratch by starting with a collection of n vertices and drawing the edges one by one. Each time we draw an edge, its endpoints must be disconnected: otherwise, if there already were a path between them, then this path together with the new edge would form a cycle, and trees can't contain cycles. So each edge drawn merges two connected components together, reducing the total number of components by one. Since we start with n connected components (one for each vertex) and finish with one connected component (the tree itself), the number of edges drawn must be exactly $n - 1$.

Just like in other disciplines, we write proofs because we want to communicate ideas to people who don't yet know or understand them. Fancy symbols like $\exists, \mathbb{Q}, \hookrightarrow, \nsim$, and $\lightning$ are only useful when you and your audience are already fluent in them, and even then it's common for notation to obscure rather than illuminate. Writing in whole sentences allows me to understand exactly what you're saying, and it makes it easier for you to read through your work to look for holes.

Right, holes. The big difference between a mathematical proof and an argument you might make in another discipline is that—assuming you've written it carefully—nobody should be able to look at your proof and say, "But what about...." To avoid this, you need to make sure you haven't assumed anything false just because it seems obvious. Sometimes this is subtle.

Okay, but how do you prove things in the first place? Here are some more ideas to keep in mind.

- To prove that X is possible, you just need to give an example where it happens.
- To prove that X is impossible, you need to make a real argument:
  - One way is to use the other information in the situation to deduce that X does not occur, but this can be tough.
  - Another subtly different way is to assume X *does* occur, and then follow its implications to find a paradox. This is called **proof by contradiction**.
- To prove "If X, then Y," you want to assume X and try to deduce Y. It is *not* sufficient to

give an example of a situation where X and Y are both true; you want to show that *every* time X occurs, so does Y.

- Another way of proving "If X, then Y" is to assume that Y is *not* true, and then use this to deduce that X is also not true. This is called **proof by contrapositive**, but it's closely related to proof by contradiction as well.
- There is only one way to disprove "If X, then Y": you need to give a situation where X is true and Y is false.
- The statement "X if and only if Y", sometimes written "X iff Y", means "If X then Y <u>and</u> if Y then X." In other words, in every situation, X and Y are both true or both false. Proving this usually requires two proofs: one for "if X then Y" and another for "if Y then X."