Exam 3 Prep

1. Define $t : \mathbb{R}^2 \to \mathbb{R}$ by $t(x, y) = x + y$.

   i. What is the range of t?
   ii. Is t onto? Why/Why not?
   iii. What is the pre-image of zero?
   iv. What is the pre-image of 1?
   v. Is t one-to-one? Why/Why not?
   vi. If t is not one-to-one, please change the domain of t to a subset of $\mathbb{R}^2$ so that t restricted to this new subset is one-to-one AND has the same range as t.

   Solution:

   i. The range of t is all of $\mathbb{R}$. For $s \in \mathbb{R}$ then $(s, 0)$ maps to s.
   ii. Function t is onto as shown in part (i) above.
   iii. The pre-image of zero is $\{(x, y) \in \mathbb{R}^2 \ : \ x + y = 0\}$. This is the line $y = -x$ in $\mathbb{R}^2$.
   iv. The pre-image of one is $\{(x, y) \in \mathbb{R}^2 \ : \ x + y = 1\}$. This is the line $y = -x + 1$ in $\mathbb{R}^2$.
   v. Function t is not injective as $(0, 1)$ and $(-1, 2)$ both map to 1, for example. (in fact an entire line of points maps to 1 as noted before)
   vi. If we restrict t to the line $y = x$ in $\mathbb{R}^2$ we get the desired properties. For suppose $t(x_1, x_1) = t(x_2, x_2)$ for some pair of points (x_1, x_1) and (x_2, x_2) on $y = x$. Then we get $2x_1 = 2x_2$ so $x_1 = x_2$. This shows t is injective on this line. And an arbitrary point $s \in \mathbb{R}$ is mapped to by $(s/2, s/2)$, so t is still surjective.

2. Use set builder notation to write the set of all natural numbers that leave a remainder of 3 when divided by 5. **Solution:** $\{n \in \mathbb{Z}^+ \ : \ n = 3 \pmod 5\}$

3. Is the following statement true? Please prove or give a counterexample.

$$\text{“}A \subseteq C \text{ if and only if } A \cup (B \cap C) = (A \cup B) \cap C\text{”}$$

   Solution: The statement is true.

   Proof:

   ($\Rightarrow$) Suppose $A \subseteq C$. We now element chase to show $A \cup (B \cap C) = (A \cup B) \cap C$.

   ($\subseteq$) Let $x \in A \cup (B \cap C)$. So $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in C$. So $x \in A \cup B$ and $x \in C$. So $x \in (A \cup B) \cap C$ as needed. Now if $x \in B \cap C$ then $x \in B$. This means $x \in A \cup B$ and $x \in C$. So $x \in (A \cup B) \cap C$ once more.

   ($\supseteq$) Let $x \in (A \cup B) \cap C$. Then $x \in A \cup B$ and $x \in C$. Suppose $x \in A$. Then we are done as $x \in A \cup (B \cap C)$. Now suppose $x \in B$. Since $x \in C$ as well we get $x \in B \cap C$. Thus $x \in A \cup (B \cap C)$ as needed.

   ($\Leftarrow$) We argue directly. Suppose $A \cup (B \cap C) = (A \cup B) \cap C$. Now let $x \in A$. We need to element chase x into C to conclude $A \subseteq C$. Now since $x \in A$ we know $x \in A \cup (B \cap C)$. Also, since $A \cup (B \cap C) = (A \cup B) \cap C$ we can conclude $x \in (A \cup B) \cap C$. However for this to be true x must be in both $A \cup B$ and C. Since we have shown $x \in C$ the proof is complete.

   QED.

4. Let $f : A \to B$ be a function, and let $C \subseteq D \subseteq B$. Define the pre-image, or inverse image, $f^{-1}(E)$ of some subset $E \subseteq B$ as

$$f^{-1}(E) = \{x \in A | f(x) \in E\}$$

Use element chasing to show that $f^{-1}(C) \subseteq f^{-1}(D)$.

Solution: How fun!

Proof: We element chase. Let $x \in f^{-1}(C)$. Then $f(x) \in C$. Now $C \subseteq D$ so $f(x) \in D$. This means $x \in f^{-1}(D)$ and we are done. QED.

5. Let A be a nonempty set. Suppose $g : A \to \mathbb{R}$ is an injective function. Define a relation $\sim$ on A in the following way. For $a, b \in A$,

$$a \sim b \text{ if and only if } g(a) \leq g(b)$$

   a. Please prove, or disprove by giving a counterexample, that $\sim$ is reflexive.
   b. Please prove, or disprove by giving a counterexample, that $\sim$ is symmetric.
   c. Please prove, or disprove by giving a counterexample, that $\sim$ is transitive.

Solution:

a. We show $\sim$ is reflexive.

Proof: Let $a \in A$. Then $g(a) = g(a)$ so $g(a) \leq g(a)$ is true. Thus $a \sim a$. QED

b. We show $\sim$ is not always symmetric.

Counterexample: If A contains more than one distinct element and those elements are related this fails. Take $a, b \in A$ with $a \neq b$ and assume $a \sim b$. Because g is injective $g(a) \neq g(b)$ so we have $g(a) < g(b)$. Then it is not the case that $g(b) \leq g(a)$. So $b \nsim a$.

c. We show $\sim$ is transitive.

Proof: Suppose $a, b, c \in A$, $a \sim b$ and $b \sim c$. Then $g(a) \leq g(b)$ and $g(b) \leq g(c)$. By transitivity of inequality, $g(a) \leq g(c)$. So $a \sim c$ as needed.

6. Let $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be a function given by $F(n) = \{m \in \mathbb{N} : m | n\}$.

   a. Please compute $F(10)$.
   b. Please determine whether or not F is one-to-one. Justify your conclusion with a proof or counterexample.
   c. Please determine whether or not F is onto. Justify your conclusion with a proof or counterexample.

Solution:

a. $F(10) = \{1, 2, 5, 10\}$.

b. Function F is injective.

Proof: Suppose $n, m \in \mathbb{N}$ and $F(n) = F(m)$. We note that the largest element of $F(n)$ is n and the largest element of $F(m)$ is m. Since $F(n) = F(m)$ these largest elements must be the same number so $n = m$. QED.

c. This function is not surjective. Observe that every natural number is divisible by 1 and itself. So the empty set, an element of $\mathcal{P}(\mathbb{N})$, is not in the image of F.