Arrow's Theorem

Arrow's Theorem: The only social welfare function that is Pareto, Monotone, and IIA is dictatorship.

We will prove this theorem using a set of five Lemmas. Before we do this let's outline the strategy and give some definitions that will be helpful.

For an election with a fixed set of candidates and voting population (P), a subset $S \subseteq P$ can *force candidate* x *over candidate* y if whenever everyone in S votes for x over y, the social preference list will definitely have x over y regardless of what everyone else does. S is a dictating set if it can force x over y for *all* choices of alternatives x and y.

Crucial observation: to show that S forces x over y, all we need is a *particular* profile where everyone in S has x over y, everyone not in S has y over x, and the resulting social preference list has x over y. Because of independence this one profile covers all such profiles, and because of monotonicity it suffices to check the "worst case scenario" like this. (This is the only place where we use monotonicity, actually, and it turns out it can be salvaged without it.)

Proposition: If there are at least three alternatives, then any social welfare function satisfying IIA and Pareto won't produce ties.

We now present a five-lemma approach:

1. Lemma 1: Suppose S forces x over y, and z is some other candidate. Now split S into two subsets S_1 and S_2. Then either S_1 forces x over z, or S_2 forces z over y.

2. Lemma 2: Suppose S forces x over y and z is some other candidate. Then S forces x over z or S forces z over y.

3. Lemma 3: If S forces x over y, then S forces y over x.

4. Lemma 4: If S forces x over y, then S is a dictating set.

5. Lemma 5: If S is a dictating set and is partitioned into S_1 and S_2. Then either S_1 is a dictating set or S_2 is a dictating set.

The proof.

1. Also, ties. **Proposition**: If there are at least three alternatives, then any social welfare function satisfying IIA and Pareto won't produce ties. *Proof:* Suppose in fact we do have ties, namely x and y tied in some scenario. Let S be the set of voters voting for x over y, so we have part of the ballot like this:

$$\begin{array}{ccc|ccc} & S & & & \text{not } S & \\ \hline x & & x & y & & y \\ & \cdots & & & \cdots & \\ y & & y & x & & x \end{array}$$

   And it leads to x and y tied. Now let's insert another candidate z into the ballots as such:

$$\begin{array}{ccc|ccc} & S & & & \text{not } S & \\ \hline x & & x & z & & z \\ z & \cdots & z & y & \cdots & y \\ y & & y & x & & x \end{array}$$

   The result of this must have z before y (by Pareto), and therefore also above x (since it's tied with y). In particular, IIA now tells us that if everyone in S has x over z and everyone not in S has x under z, then x is under z.

   On the other hand:

$$\begin{array}{ccc|ccc} & S & & & \text{not } S & \\ \hline x & & x & y & & y \\ y & \cdots & y & z & \cdots & z \\ z & & z & x & & x \end{array}$$

   This time by Pareto we have z under y (and also x), but this violates IIA from the above. So no ties. Whew.

2. **Lemma 1**: Suppose S forces x over y, and z is some other candidate. Now split S into two subsets S_1 and S_2. Then either S_1 forces x over z, or S_2 forces z over y.

   Proof: What happens here?

$$\begin{array}{c|c|l} S_1 & S_2 & \text{everyone else} \\ \hline x & z & y \\ y & x & z \\ z & y & x \end{array}$$

What happens? Well, since everyone in $S = S_1 \cup S_2$ has x over y, x must be over y. So now either z is above y (in which case S_2 forces z over y) or z is under y (in which case it's also under x, so S_1 forces x over z).

3. **Lemma 2**: Suppose S forces x over y and z is some other candidate. Then S forces x over z or S forces z over y.

   Proof: First split S into two sets: $S_1 = S$ and $S_2 = \emptyset$. Then by Lemma 1, either S forces x over z or $\emptyset$ forces z over y. But the latter is impossible: $\emptyset$ can't force anything or else you'd violate Pareto. So S forces x over z.

   Now split S into $S_1 = \emptyset$ and $S_2 = S$. By Lemma 1, either $\emptyset$ forces x over z (impossible) or S forces z over y, so S forces z over y.

4. **Lemma 3**: If S forces x over y, then S forces y over x.

   Proof: Suppose S forces x over y. Let z be some other candidate. So S forces x over z by lemma 2. Now y is some alternative besides x and z, so again by lemma 2, S forces y over z. Finally, x is some candidate besides y and z, so by lemma 2 again, S forces y over x.

5. **Lemma 4**: If S forces x over y, then S is a dictating set.

   Proof: We need to show that for any arbitrary candidates u and v (possibly the same as x or y), S can force u over v. Two cases!

   Case 1: $v = x$. Well, we know S can force $v = x$ over y, so by lemma 3 S can force v under y, so by lemma 2 S can force v under anything, including u.

   Case 2: $v \neq x$. Well, S forces x over y, so S can force x over v, so S can force u over v.

6. **Lemma 5**: If S is a dictating set and is partitioned into S_1 and S_2. Then either S_1 is a dictating set or S_2 is a dictating set.

   Proof: Lemma 1 (a dictating set must force x over y, after all) says that either S_1 forces x over z or S_2 forces z over y. But lemma 4 says that in the first case, S_1 is a dictating set, and in the second, S_2 is a dictating set. So one of them is a dictating set.

7. *Proof of Arrow's Theorem*: We know by Pareto that P (the whole population) is a dictating set. Break it up into two (strictly) smaller sets: one of them is still dictating by lemma 5. Break that into smaller sets, and one of them is still dictating. Keep doing this until you find a dictating set with one person. They're the dictator.