Problem Set 9

Due: Friday, April 10th

Instructions: Do at least 4 of the following problems.

1. **Would You Rather?** $\star$

   (a) In the two-player cut-and-choose algorithm, would you rather be the cutter or the chooser? (Does it depend on the situation? If so, explain.)

   (b) In the three-player trimming algorithm, would you rather be April (the first cutter), Ben (the first trimmer), or Chris (the second trimmer)?

2. **Shore Enough** $\star\star$

   Jim, Dan, and Landry bought a piece of beachfront property and would like to split it amongst themselves. The property is bounded to the north by the ocean, to the south by a road, and to the west and east by property lines perpendicular to the road. Given a map of the property, each one draws two north-south lines to cut the land into what he thinks are three equally valuable pieces. A possible set of cuts is shown in the diagram below, with Jim's cuts represented by dotted lines, Dan's by dashed lines, and Landry's by wavy lines:
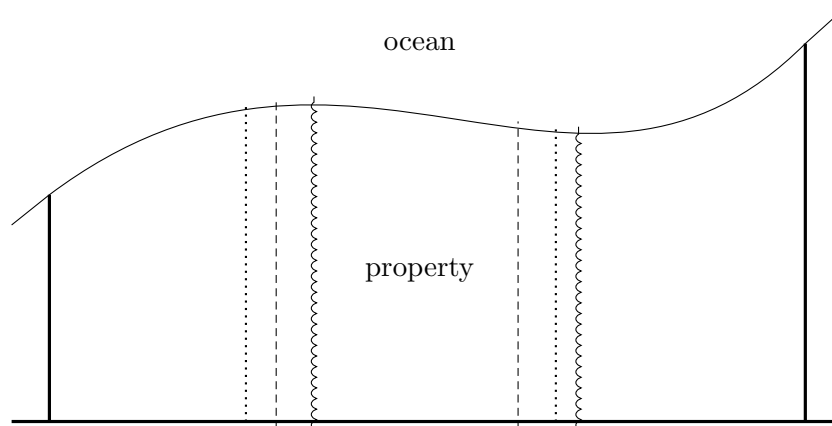

   The above example has an easy solution: Jim gets his leftmost piece, Dan gets his middle piece, and Landry gets his rightmost piece. Nobody's shares overlap, and there's even some space left over!

   (a) Is it always possible to give each person one of the thirds he drew with no overlap, no matter where the vertical cuts are drawn? If so, explain how; if not, give an example where it is impossible.

   (b) What if the property is to be split between more people?

3. **Matching with Friends** $\star$

   One way of finding a proportional cake distribution for a given set of pieces of cake is to find a *perfect matching* on a *bipartite graph*: in the diagrams below, each of n gray diamonds

represents a person and each of n white circles represents a slice of cake. A line connects a diamond to a circle if that person thinks the value of that slice is at least $1/n$. For which of these diagrams does a *perfect matching* exist? In other words, is there a way to pair up the people with the slices so everyone thinks their own slice is at least $1/n$ of the cake? Either find a matching, or prove that one does not exist.
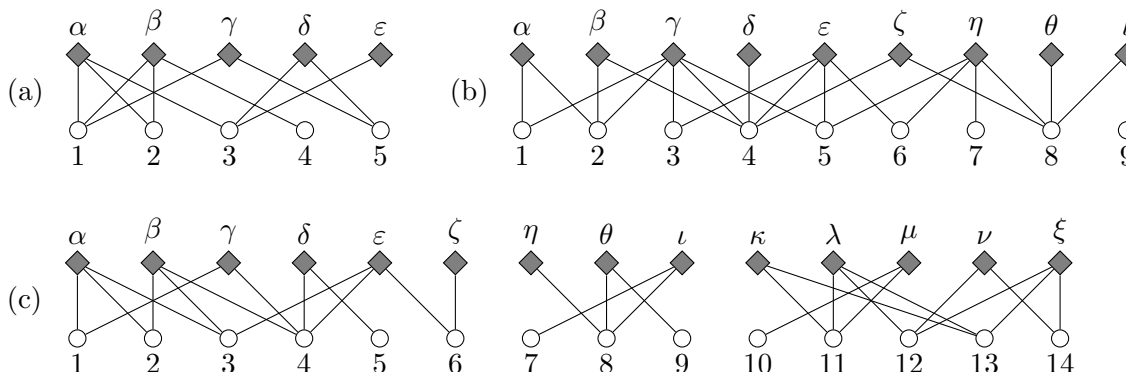
4. **Adjusted Development** ⋆⋆

George and Lucille are going through a divorce, and are deciding how to split a set of possessions between them. They decide to use the adjusted winner procedure, and submit the following valuations to their lawyer Bob.

George	Item	Lucille
18	Model home	29
14	Family cabin	25
15	Stair car	4
8	Yacht	12
28	Banana stand	6
9	Cornballer	2
1	Roomba	8
4	Air horn	12
3	Dead dove	2

How should the items be divided? (One of them may need to be split; explain what ratio is appropriate, and how that split might work.)

5. **Unequal Cuts** ⋆⋆

Suppose Arya, Bran, and Catelyn would like to divide a cake in an unequal way: 25% for Arya, 35% for Bran, and 40% for Catelyn.

(a) Adapt the moving knife procedure so that each person gets (in their opinion) at least the percentage assigned to them.

(b) Does your procedure extend to other divisions, or does it only work for certain values?

6. **Muffin Division** ⋆⋆

Anna, Bill, and Claudette have received a batch of eighteen muffins as a gift—six each of fig, bacon-corn, and avocado—and are deciding how to split them amongst themselves. Their preferences are as follows:

- Anna is vegetarian, so the bacon-corn muffins are worthless to her. She thinks the others are equally good.
- Bill is allergic to nuts and the fig muffins contain almonds, but he's equally happy with the others.
- Claudette is willing to eat all kinds of muffins, but she thinks the avocado muffins are twice as delicious as the others.

Use the Selfridge-Conway envy-free procedure to divide the muffins. (Here "cutting" just means arranging muffins into piles, but you are allowed to break a muffin into smaller pieces if necessary.) Is your result equitable? Is it efficient?

7. **A Real-Life Application** ⋆⋆

Find an opportunity to use any of our fair division procedures (for divisible or indivisible, homogeneous or heterogeneous goods) in your daily life. Describe how it played out.