Arrow's Theorem

Arrow's Theorem: The only social welfare function that is Pareto, Monotone, and IIA is dictatorship.

We will prove this theorem using a set of five Lemmas. Before we do this let's outline the strategy and give some definitions that will be helpful.

For an election with a fixed set of candidates and voting population (P), a subset $S \subseteq P$ can *force candidate* x *over candidate* y if whenever everyone in S votes for x over y, the social preference list will definitely have x over y regardless of what everyone else does. S is a dictating set if it can force x over y for *all* choices of alternatives x and y.

Crucial observation: to show that S forces x over y, all we need is a *particular* profile where everyone in S has x over y, everyone not in S has y over x, and the resulting social preference list has x over y. Because of independence this one profile covers all such profiles, and because of monotonicity it suffices to check the "worst case scenario" like this. (This is the only place where we use monotonicity, actually, and it turns out it can be salvaged without it.)

Proposition: If there are at least three alternatives, then any social welfare function satisfying IIA and Pareto won't produce ties.

We now present a five-lemma approach:

1. Lemma 1: Suppose S forces x over y, and z is some other candidate. Now split S into two subsets S_1 and S_2. Then either S_1 forces x over z, or S_2 forces z over y.

2. Lemma 2: Suppose S forces x over y and z is some other candidate. Then S forces x over z and S forces z over y.

3. Lemma 3: If S forces x over y, then S forces y over x.

4. Lemma 4: If S forces x over y, then S is a dictating set.

5. Lemma 5: If S is a dictating set and is partitioned into S_1 and S_2. Then either S_1 is a dictating set or S_2 is a dictating set.