# Problem Set 1 Solutions

1. **Trump v. Clinton v. Johnson v. Stien** ⋆

In this problem we will look at the 2016 presidential election.

| Candidates | Popular vote | Simple (if you want) |
|---|---|---|
| Clinton | 65,853,514 | 65.9 |
| Trump | 62,984,828 | 63 |
| Johnson | 4,489,341 | 4.5 |
| Stein | 1,457,218 | 1.5 |

(a) Make an educated guess about the full voter preference profile. You will have to make some choices here, for example would you put set every Johnson voter's second choice as Trump or would you split them between Trump and Clinton in some way? You don't have to be too careful or complicated, just make some guesses and see what happens.

**Solution:** To keep it simple let's just have four types of voters. There is no right or wrong way to do this but here is one way of doing it.

| 65.9 | 63 | 4.5 | 1.5 |
|---|---|---|---|
| C | T | J | S |
| J | J | T | C |
| S | C | C | J |
| T | S | S | T |

(b) Compute the social welfare result with the following voting methods: plurality, Borda count, instant runoff, Copeland, Coombs. Is there a Condorcet winner?

**Solution:**

i. Plurality: C>T>J>S.
ii. Borda: J (407.7) > C (403.1) > T (333.9) > S (205.3)
iii. Instant runoff: T > C > J > S
iv. Copeland: J (3 wins) > T (2 wins) > C (1 win) > S (0 wins)
v. Coombs: J > T > C > S

(c) Any thoughts or surprises?

**Solution:** Lots of thoughts. The first is that the results depend a lot on the voting method. It seem like Gary Johnson (considered an outsider candidate in the election) has a very legitimate argument for being the best candidate.

On a more personal note, these results shake a lot of the common narratives that people use when discussing elections. It is often brought up that Clinton won the popular vote, but notice that she did not win a majority of the first place votes, just more than anyone else. If you believe that plurality is not the best voting system the argument that she "won" the popular vote becomes a lot less enticing. I'm not saying Johnson or Trump was a better candidate, I'm just saying the usual narrative seems to be misleading.

As a last note, perhaps if we did use a different voting system the candidates would look a little different. Perhaps instead of just these 4 candidates others would still be in the race like Bernie Sanders or Jeb Bush. That might change the above results a lot. It is interesting to think how our choice of election procedure influences who even is a candidate.

Those are just the thoughts off the top of my head. What else does this example make you wonder about?

2. **2008 Flashback** ⋆

   In the spring of 2008, after John McCain effectively clinched the GOP nomination but while Hillary Clinton and Barack Obama were still vying for the Democratic spot, there were three plausible candidates for the next president of the United States. List the six possible voter profiles (rankings) of these candidates, and give reasonable estimates for what portion of the voting population would choose each profile. Then use your estimates to determine who would win an election between those three if the electoral college were abandoned in favor of each of the following methods:

   (a) Plurality voting.
   (b) The Hare method.
   (c) The Borda count.
   (d) The Coombs method.

   **Solution:** This problem is similar to problem 1 and the solution depends a lot on what you chose so I will skip over this one.

3. **Anonymity on the National Stage** ⋆

   Are nationwide presidential elections in the United States (using our current voting system) anonymous? Are they neutral? Argue why or why not.

   **Solution:** Before we state our proof we clarify the voting system we are using. To simplify things we ignore some of the stranger peculiarities of the electoral college (such as delegates not voting for their intended candidate, check out the Wikipedia page on faithless electors). Each state uses a plurality vote to determine which candidate will receive all of the electoral votes for that state. Each state has a number of electoral votes equal to the number of representatives in the house (which is roughly proportional to population) plus two. For example California has 55 electoral votes while Oregon has 7.

   We claim that this voting system is not anonymous. We do this by constructing an example of a set of voter preferences in which swapping the ballots of two voters changes the result of the election. Consider a set of voter preferences in which the plurality vote for California has the Republican candidate winning by a single vote and the plurality vote for Oregon has the Democrat candidate winning by a single vote. If we swap a republican voter from California with a Democrat voter from Oregon, then California will now be won by the Democratic candidate while Oregon will be won by the Republican. But this change in electoral votes favors the Democrat and if the rest of the state voted in such a way that the race was close it would change the winner of the national election.

   We claim that this voting system is neutral. Suppose we have a set of voter preferences. Plurality is a neutral voting system so if we switch the candidates on every voters preference list, every state that was voting for a given candidate will switch to the other candidate. This will result in the elector votes also swapping between those two candidates. Because the electoral votes determine the outcome of the election the results will also switch, which is what neutrality requires.

4. **Anonymity in the Senate** $\star\star$

   Forgetting for the moment about filibusters and supermajorities and whatnot, consider the Senate as a straightforward voting system between two alternatives. (Namely, whether a bill should pass or not pass.) It has 101 voters: 100 senators who vote yes or no under a simple majority system, and the vice president who casts a tie-breaking vote whenever the senators are split 50-50. Is this system anonymous? Explain your answer carefully.

   **Solution:** We claim that this voting system is anonymous. In fact we claim something stronger: that this voting system is equivalent to a plurality. Indeed, consider an election in which yes gets $x$ votes and $x \geq 51$. We will show that yes wins. If at least 51 of the yes votes are from senators then yes wins because it has a majority of the senators. The only other case to consider is if exactly 50 of the yes votes are from senators and 1 is the vice president. In this case since there is a tie among senators the tie breaker is the vice president so yes still wins. This shows that if a majority of voters (regardless of if they include the vice president or not) vote for yes or no then that option will win. Plurality is anonymous so this voting system is anonymous.

5. **Votes and Recreation** $\star$

   April, Ben, Chris, and Donna are running for a seat on the Pawnee city council. A record high turnout of seven voters show up to vote for the local election. Their profiles are as follows:

| Voter 1 | Voter 2 | Voter 3 | Voter 4 | Voter 5 | Voter 6 | Voter 7 |
|---|---|---|---|---|---|---|
| A | A | B | B | C | A | D |
| D | C | D | A | B | B | B |
| B | D | C | D | D | D | C |
| C | B | A | C | A | C | A |

   Determine the winner of the election for each of the following voting methods:

   (a) Plurality method.
   **Solution:** A > B > C = D

   (b) The Borda count.
   **Solution:** B > A = D > C

   (c) Instant-runoff method.
   **Solution:** B > A > C = D

   (d) Coombs method.
   **Solution:** B > D > A = C

6. **Neutrality, Anonymity, and Orderings** $\star$

   Suppose a neutral and anonymous social welfare function produces the preference list (A,B,C) for the following profile:

| Voter 1 | Voter 2 | Voter 3 |
|---|---|---|
| C | A | B |
| B | C | C |
| A | B | A |

(a) What preference list would the same social welfare function produce on this profile?

| Voter 1 | Voter 2 | Voter 3 |
|---|---|---|
| C | A | B |
| A | B | A |
| B | C | C |

**Solution:** Looking at the votes we see that if we exchange candidates A and C and swap voters 1 and 2 the original voting profile turns into the one above. Thus, the initial result of A>B>C will turn into C>B>A.

(b) Give a profile for which the same social welfare function would produce the preference list (C,A,B).

**Solution:** To find the desired ballot we start with our original ballots and then replace candidates appropriatly. If we replace A with C, B with A, and C with B the ballot becomes

| Voter 1 | Voter 2 | Voter 3 |
|---|---|---|
| B | C | A |
| A | B | B |
| C | A | C |

and by neutrality it has the desired outcome.

7. **Variations on Borda** $\star\star$

For this problem, consider the following voter profiles:

| Voter 1 | Voter 2 | Voter 3 | Voter 4 |
|---|---|---|---|
| B | C | C | A |
| A | A | D | D |
| D | B | B | B |
| C | D | A | C |

(a) Find the winner using the Borda count.

**Solution:** A > B = C > D

(b) Here are four other ways to score the candidates. Do as many of these calculations as you find helpful.

- Suppose we made a new version of the Borda count, where each first place vote gets 8 points, each second place vote gets 4 points, each third place vote gets $-4$ points, and each fourth place vote gets $-8$ points. Redo the election using this new method.
  **Solution:** A > C > B = D
- Redo it again, this time using the points $-1$, $-5$, $-9$, $-13$ for (respectively) first, second, third, and fourth place.
  **Solution:** A > B = C > D
- And again, this time using the points 9, 4, 1, and 0.
  **Solution:** C > A > B > D
- Once more now, with values 5, 4, 3, and 2.
  **Solution:** A > B = C > D

(c) Propose a condition on the new set of point values that will guarantee the results are the same as the original method. Prove your answer.

**Solution:** Suppose a sequence $a_0, a_1, a_2, \ldots$ is defined by two numbers $a_0$ and $d > 0$ so that $a_k = a_0 + kd$. We claim that any point system that gives the $k$'th from last place $a_k$ points is equivalent to the Borda count. To be clear, this means last place gets $a_1$ points, second to last gets $a_1 + d$ points, third from last gets $a_1 + 2d$ points and so on.

Proof: Suppose we have such a system and a preference list for $n$ voters. If we subtract $na_0$ from each candidates final score the order of the scores will not change and we will have the same results as a point system that assigns $a_n = kd$ points to the $k$'th from last place. Now if we scale all the candidates scores by $1/d$ the order of the scores will not change and we will have the same results as a point system that assigns $a_n = k$ points to the $k$'th from last place. But that is exactly the Borda count. So we have shown that the Borda count will have the outcome as our original point system given by $a_n = a_0 + kd$.

(d) Is the Borda count the same as Eric's point system?

**Solution:** Yes. In Eric's point system the lowest score wins. By negating all of the scores given this criterion becomes the highest score wins. After negation Eric's system assigns $-k$ points to the $k$'th from first place, or $n-k$ from the $k$'th from last. This is a sequence with $a_0 = -n-1$ and $d = 1$ so by the previous part it is equilvalent to Borda count.

8. **Iterated Plurality Voting** ⋆⋆

It's very easy to convert a social welfare function into a social choice function: just call the candidate(s) in first place the winner(s), and ignore the rest. Converting a social choice function into a social welfare function is harder, but here's how we can do it (at least for functions meeting the "always a winner" criterion):

Run the social choice function on your candidates, and put the winner(s) in first place. Now delete them from your profile (that is, delete the candidate from every ballot, moving everyone else up accordingly). Run your social choice procedure *again*, and put the winner(s) in second place. Keep doing this until you've placed all the candidates.

Use this process to turn plurality-the-social-choice-function into a social welfare function on the following tabulated profile. Are your results the same as if you had used the usual version of plurality as a social welfare function?

| 10 | 5 | 3 | 3 | 2 |
|---|---|---|---|---|
| A | B | C | C | D |
| B | C | B | D | C |
| D | D | D | B | B |
| C | A | A | A | A |

**Solution:** The normal results would be A > C > B > D. With iterated plurality we get A > B > D > C, which differs in every spot except the first!

9. **Counting Voting Systems** ⋆⋆⋆

Remember that a function from a set of $a$ inputs to a set of $b$ outputs is a procedure that assigns each input to exactly one output. Some inputs may lead to the same output, but no single input can lead to more than one output. There are $b$ possible choices of output for each input, so the number of possible functions is $b \times b \times b \times \cdots \times b$ ($a$ times) $= b^a$. The goal of this problem is to use this fact to count the number of possible social welfare functions in an election with $c$ candidates and $v$ voters.

(a) In an election with $c$ candidates, how many possible ballots are there? For purposes of this problem, let's disallow ties.

**Solution:** There are $c!$ ways to arrange the $c$ candidates.

(b) Use your previous answer to determine how many possible profiles (sequences of $v$ ballots) there are.

**Solution:** Each voter has $c!$ options so there are $(c!)^v$ many voting profiles.

(c) How many possible outputs are there? Again, do not allow ties: an output of a social welfare function is simply an ordering of the $c$ candidates.

**Solution:** Once again there are $c!$ ways to order the candidates.

(d) Use your previous answers to determine how many possible social welfare functions there are. You counted the inputs in part (b) and the outputs in part (c).

**Solution:** For each profile there are $c!$ options for an output so there are $(c!)^{(c!)^v}$ many functions.

(e) (Optional and difficult) What if you allow ties?

**Solution:** No one attempted this so I'll leave it for now.

10. **Weighted Copeland** $\star\star$

In class we discussed the weighted Copeland method. For each pairwise comparison, give the winner points equal to the number of votes they won by. The candidate with the most points wins.

This method is actually the same as another voting method we are already studying. Find its doppelganger and prove that they are equal social welfare functions.

**Solution:** We will discuss this in class or on another homework.