Problem Set 2

Due: Friday, Febuary 7th

Instructions: Do at least 7 of the following problems.

1. **Non-Imposition** $\star$

   Say that a social choice function (defined on a specific set of voters and candidates) satisfies the *non-imposition* criterion if every one of those candidates can be a <u>unique</u> winner for the right profile.

   (a) Explain how non-imposition is a desirable characteristic of voting systems.

   (b) Prove that if a social choice function satisfies the Condorcet criterion, then it satisfies non-imposition.

   (c) Give an example of a neutral social choice function which does *not* satisfy non-imposition.

2. **Monotone for Two Candidates** $\star\star$

   Here are two possible definitions for *monotone* in a two candidate election.

   Definition 1: A social welfare function is monotone if for any voter preference list, if a new voter is added that prefers A over B (resp. B over A) then candidate A (resp. B) will do no worse in this election then before that voter was added.

   Definition 2: A social welfare function is monotone if for any voter preference list, if a voter that prefers B over A (resp. A over B) changes their vote to A over B (resp. B over A) then candidate A (resp. B) will do no worse in this election then before that voter switched.

   (a) Are these two definitions equivalent? If so prove it and if not give an example of a social welfare function that satisfies one but not the other.

   (b) (optional) Are the analogues definitions for elections with more than two candidates? If so state them. Are they still equivalent?

3. **Prove May's Theorem** $\star\star$

   In class we gave a "poof" of May's Theorem: in an election with 2 alternatives, majority rule is only system satisfying all 4 properties. Turn this poof into a proof.

4. **Nobody's Favorite** $\star\star$

   Let's say a voting method is *favorite-friendly* if the winner must have gotten at least one first-place vote. Determine, with proof, which of the following methods are favorite-friendly.

   (a) Plurality voting.

   (b) The instant runoff method.

   (c) The Borda count.

   (d) The Coombs method.

   (e) A dictatorship.

5. **Anti-majority Candidates** ⋆⋆

   Let's say a candidate is the *anti-majority candidate* if the majority (not just a plurality) of voters rank that candidate in last place. Is it possible for the anti-majority candidate to win an election...

   (a) under plurality voting?

   (b) under the Coombs method?

   (c) under the Borda count?

   (d) under sequential pairwise voting?

6. **More Voting Criterion and Voting Methods** ⋆(⋆(⋆))

   Are there any other voter criterion you think we should consider? What methods of ours meet this criterion? Are there more social welfare functions you want to consider? What criterion does that method meet?

7. **Constructing a Profile from Pairwise Results** ⋆⋆⋆

   In an election with three candidates, suppose you know how every possible *pair* of candidates would perform in a one-on-one race. (In other words, you know what the final results would be if A ran against B, or if B ran against C, and if A ran against C.) Is this always enough information to determine a unique profile for the voters (that is, the number of voters who would choose each of the six possible ballots in a three-way race)? If so, explain how you would use this information to find the profile. If not, give an example of two different profiles which would lead to the same result in any one-on-one matchup. And if you're up for it, generalize your results to an election with four or more candidates. Is your answer the same?

8. **Selecting a Police Chief** ⋆⋆

   Based on a real-life scenario! The Board of Selectmen in a small suburban Massachusetts town is picking a new chief of police. The five members of the board will be choosing between four possible candidates. Devise a good voting method for them to use, keeping in mind the following criteria:

   • The method should be deterministic (so no random chance) and should always result in a single winner.

   • The method should be reasonably easy to explain to the public.

   • The method should *not* require all selectmen to list their full ballots; because town politics, being nonpartisan, are (ostensibly) friendly and the forum is open, it would be painful for the candidates to have to hear that they are the last-place choices of the selectmen.

   • Preferably your method is also anonymous (in the mathematical sense). But if not, you might want to use the fact that one selectman is designated as the chairman.

   Explain your system, and argue why it is a good one to use in this situation.

9. **Setting the Agenda on Snacks** ⋆⋆

   Suppose three of your friends are deciding what snack to buy (Airheads, Bugles, Cheez-Its, or Doritos). Their preferences are as follows:

Xerxes	Yvonne	Zander
A	C	B
B	A	D
D	B	C
C	D	A

The three of them decide to use sequential pairwise voting to determine a winner, and as an unbiased third party, you get to pick the agenda. At least, they think you're unbiased; in actuality, your choice of the agenda gives you complete control over the winner, and you intend to use it.

(a) Pick an agenda such that the winner is Airheads.

(b) Pick an agenda such that the winner is Bugles.

(c) Pick an agenda such that the winner is Cheez-Its.

(d) Pick an agenda such that the winner is Doritos.

10. **Condorcet and Changing Agendas** ⋆⋆

(a) Prove that if a sequential pairwise voting election has a Condorcet candidate, then she will always win regardless of the agenda.

(b) Prove that if a sequential pairwise voting election does *not* have a Condorcet candidate, then there are two agendas which result in different winners.