Problem Set 3

Due: Friday, February 14th

Instructions: Do at least 7 of the following problems.

1. **Please Pack Your Knives and Vote** ⋆⋆⋆

   Let's talk about *Top Chef*. Judges eliminate the worst cheftestant (sorry) each week, but in some seasons the eliminated cheftestants have a chance to come back in the "Last Chance Kitchen": the cheftestants eliminated in the first two episodes face off head-to-head. The winner of that match takes on the cheftestant eliminated in the third episode, and the winner of *that* match goes off against the one eliminated in the next episode, and so on. Ultimately the winner of this sequential pairwise process is brought back into the main group when it's down to only a few competitors, but for simplicity's sake we'll pretend he or she is brought back at the *very* end of the competition to face off against the winner.

   Let's ignore evolution of skill during the competition (so each judge has a fixed ranking of the chefs from the very beginning) and formalize this as a hybrid method (*Top Chef* Voting, or TCV) blending together the ideas of Coombs and sequential pairwise: first we run Coombs's method to determine an agenda, sorting the candidates in the order they get kicked off and putting the winner(s) at the end (breaking ties by, let's say, alphabetical order), and *then* we run sequential pairwise voting using that agenda (and the same set of ballots from the judges). The winner(s) of TCV are the winners from the round of sequential voting. For example, consider this profile of five judges and five chefs.

Judge 1	Judge 2	Judge 3	Judge 4	Judge 5
Angelo	Carla	Elia	Elia	Dale
Bryan	Angelo	Angelo	Carla	Elia
Carla	Bryan	Bryan	Bryan	Angelo
Dale	Elia	Carla	Dale	Carla
Elia	Dale	Dale	Angelo	Bryan

   Dale is eliminated first with two last place votes, putting him first on the agenda. Next, Elia is eliminated and goes in the second agenda slot. Then Brian and Carla both have two last place votes and are eliminated simultaneously, entering the agenda in alphabetical order, and finally Angelo goes last in the agenda. So our agenda is (Dale, Elia, Brian, Carla, Angelo).

   Now following the agenda, Dale loses against Elia, then Elia beats Brian, Elia beats Carla, and Elia beats Angelo. So Elia is the winner under TCV.

   (a) Is TCV neutral?

   (b) Is TCV monotone?

   (c) Does TCV satisfy the "always a winner" criterion?

   (d) Does TCV satisfy the Condorcet criterion?

   (e) Does TCV satisfy the Pareto criterion?

   (f) Does TCV satisfy Independence of Irrelevant Alternatives?

2. **Weighted Copeland, Revisited** ⋆⋆

   Here is a repeat because I think it is an interesting and important problem:

   Consider the weighted Copeland Method where, for each arrow in the tournament graph, the winner gets points equal to the number of people who voted for her over the loser, and the loser gets points equal to the number of people who voted for them over the winner. (In the case of a tie, they both get points equal to the number of people who voted for them, i.e. half the total number of voters.)

   Surprise! This new version is nothing like Copeland's Method at all, but is in fact another familiar voting method in disguise. Which one, and why?

3. **Police Chiefs, Revisited** ⋆⋆

   Read this article about problem #8 from last week: `http://bit.ly/QRuCl5`

   The second round of voting was a modified Borda count, where each selectman gives two points to his first choice, one point to his second choice, and zero points to his third and fourth choices. Use this fact and the information in the article to prove that James Hicks was a Condorcet candidate.

4. **Condorcet Candidates and the Condorcet Paradox** ⋆⋆

   Prove that if a profile with an odd number of voters does not have a Condorcet winner, then there exists a set of three candidates in that profile who form the Condorcet (AKA rock-paper-scissors) paradox: A beats B, B beats C, and C beats A.

5. **Condorcet Losers** ⋆⋆

   A *Condorcet loser* is a candidate who would lose to every other candidate in a one-on-one simple majority race. Is it possible for a Condorcet loser to win an election...

   (a) under plurality voting?

   (b) under the instant runoff system?

   (c) under sequential pairwise voting?

   (d) under a dictatorship?

6. **Strong Connectivity and Agendas** ⋆⋆⋆

   A tournament graph is *strongly connected* if it's possible to follow a path from any vertex to any other vertex by following the arrows of the graph.

   On the previous assignment, we saw a profile where every candidate can win a sequential pairwise election if they get to choose the agenda. The goal of this problem is to show that this phenomenon happens exactly when the tournament graph of the profile is strongly connected.

   (a) Prove that if every candidate in a particular profile can win a sequential pairwise election (with the right agenda), then the tournament graph for that profile must be strongly connected.

   (b) Prove that if the tournament graph for a profile is strongly connected, then every candidate in that profile can win a sequential pairwise election with the right agenda.

7. **Possible tournaments** ⋆⋆

   In class we discussed a way to go from a voting profile to a tournament or a weighted tournament. Recall, a tournament is a complete, directed graph on n vertices. A weighted tournament is a complete, directed, weighted graph.

   (a) Given any tournament (i.e. a purely graph theory object), does there always exist a profile that produces that tournament? If not give an example, and if you can find necessary and sufficient conditions on the tournament to insure that it comes from a voting profile.

   (b) (optional but definitely doable ⋆⋆⋆) Given a *weighted* tournament does there always exist a profile that produces that weighted tournament? If not give an example, and if you can find necessary and sufficient conditions on the tournament to insure that it comes from a voting profile.

8. **Influential Coalitions** ⋆

   For a particular social choice function, let's say that a set of voters is an *influential coalition* if whenever everybody in that coalition lists A as their first choice, A is guaranteed to be the unique winner. So the unanimity condition means that the set of all voters is an influential coalition, but it might be the case that some smaller coalitions are also influential.

   (a) What are the influential coalitions under the plurality method?

   (b) What are the influential coalitions in a dictatorship?

   (c) What are the influential coalitions under the Borda count in a system with 4 candidates and 10 voters?

9. **Nearly a Dictatorship** ⋆

   Consider the following voting system, which we'll call a *near dictatorship*: one voter is preselected as the dictator, and two other voters are selected as her close advisors. If the advisors list the same candidate at the top of their lists, then that candidate is the social choice. Otherwise, the candidate at the top of the dictator's list is the social choice.

   (a) Is this method anonymous?

   (b) Is this method Pareto?

   (c) Is this method independent of irrelevant alternatives?

10. **Weakly Pareto** ⋆⋆

    Let's say that a social choice function is *weakly Pareto* if whenever every voter prefers A to B, B cannot be the unique winner (but it's okay if B ties for the win).

    (a) In math, we call something *weakly X* when we mean that everything which is X is also weakly X. Show that every Pareto social choice function is also weakly Pareto.

    On the other hand, some methods which aren't Pareto might still be weakly Pareto.

    (b) Is anti-plurality voting weakly Pareto?

    (c) Is sequential pairwise voting weakly Pareto?

11. **The Nomination Method** ⋆

The nomination method is a social choice function where *every* candidate who receives at least one first place vote is a winner. It results in a lot of ties, but it's great for candidates' self-esteem.

(a) Is the nomination method Pareto?

(b) Is the nomination method independent of irrelevant alternatives?

12. **Bloc Voting** ⋆⋆

A bloc voting method, like the electoral college, is a social choice function where the population is partitioned into blocs (think "states"), and each bloc is given a number of points (think "electoral votes"). Then the simple plurality method is used within each bloc, with the winner getting all the points from that bloc. If there is a tie in some state, the points are divided evenly between the winners. The social choice is the candidate who receives the most points total. Does bloc voting satisfy...

(a) Anonymity?

(b) Unanimity?

(c) Monotonicity?

(d) The Condorcet criterion?

(e) The majority criterion, which states that a candidate who receives a *majority* of first place votes is guaranteed to be the unique winner?