Quaternions

The *complex numbers* are built from the real numbers by adding a symbol i with the property that $i^2 = -1$. The canonical way to write a complex number is in the form $a + bi$. The complex numbers are just like other numbers, you can add, subtract, multiply, and divide them. For example

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i.$$

Notice that after I multiply I regroup the terms so they are in the canonical form of $a + bi$. I want to ask about an even stranger set of numbers called the *quaternions*. They are of the form $a + bi + cj + dk$ where $i^2 = j^2 = k^2 = -1$ and $ijk = -1$.

(a) For warm ups, write ij in the "canonical form" $a + bi + cj + dk$. What about ji? What about ikj?

(b) A quaternion is *purely imaginary* if it is of the form $bi + cj + dk$. Compute the multiplication of two purely imaginary quaternions and write the product in canonical form:

$$(u_1 i + u_2 j + u_3 k)(v_1 i + v_2 j + v_3 k).$$

Sometimes we compare an imaginary quaternion $u_1 i + u_2 j + u_3 k$ to the vector $\langle u_1, u_2, u_3 \rangle$ in $\mathbb{R}^3$. How does the product you just computed relate to vector computations? (*hint: what is the real part of the vector you just found? What is the imaginary part?*)

Remark: People really use quaternions! They are the reason we use the letters **i, j, k** for the standard basis of $\mathbb{R}^3$!!! A common use for quaternions today is in computer graphics where quaternions give the most efficient way of computing rotations in space.