Definition 8.1.9: A graph G is said to be *connected* if for any two vertices $x, y \in V(G)$ there exists a path $x, v_0, v_1, \ldots, y$.

So nearly all the graphs we have looked at to this point are connected but the one in Figure 8.1.9 is not. It happens to be in three "pieces." But we need a better way to talk about the pieces.

Definition 8.1.10: A maximal connected subgraph of a graph G is called a *component* of G. These are sometimes called *connected components* for emphasis.

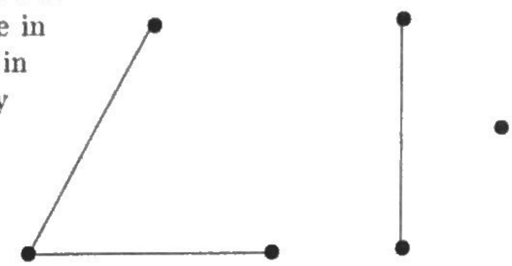


Figure 8.1.9

Recall what we mean by maximal. It does not mean the biggest. That is maximum. Maximal means that we cannot add anything further to it and still satisfy the given conditions. Thus the example has three components and the fourth subgraph in Figure 8.1.8 has two components. A connected graph then must have exactly one component. Also, a vertex can be a subgraph or a component all to itself. Such a vertex is called *isolated* and has degree zero.

——Exercises 8.1——

1) Let's consider the graph of Figure 8.1.1 with the vertices labelled so we can tell them apart as in Figure 8.1.10. a) Give an example of a walk of length 5 which is not a trail; b) a trail of length 5 which is not a path; c) a path of length 5. d) How long is a longest possible path? e) Is such a path unique? f) Give an example of a circuit which is not a cycle. g) What is the shortest possible cycle? h) the longest possible cycle?

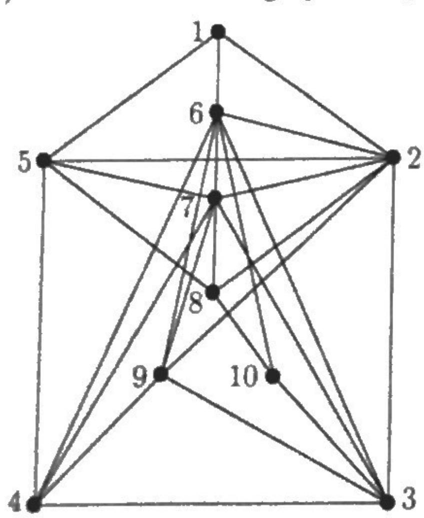


Figure 8.1.10

2) Let G be a graph which models, in a very simple way, the United States. Let the set of vertices be the 50 states plus the District of Columbia. Let two vertices be adjacent if the two states they represent have a common border. For example, Alabama and Mississippi are adjacent, while Minnesota and Ohio are not. a) Is G connected? b) How many components does G have? c) How many vertices of degree 1? (These are often called *pendant* ver-

edges? b) Then how would you describe the degree of a vertex? c) Is this graph necessarily connected? d) Can circuits exist? e) If so, how? If not, why not?

10) Determine whether each of the following is a degree sequence for a graph (not a multigraph). If so, draw such a graph (which may not be connected); if not, prove it.
   a) 1,1,1,1,1,1,6
   b) 1,3,5,7,7,7,7,7
   c) 1,2,3,4,4,5
   d) 2,3,4,5,6
   e) 3,3,4,5,5,5,6,6,7,7
   f) 0,0,0,1,1,1,1,2,2,2

11) The two graphs shown in Figure 8.1.12 are two versions of the same graph. Label the vertices in each so that the edges correspond. For example, if $\{1, 2\}$ is an edge in one graph it must also be an edge in the other graph.

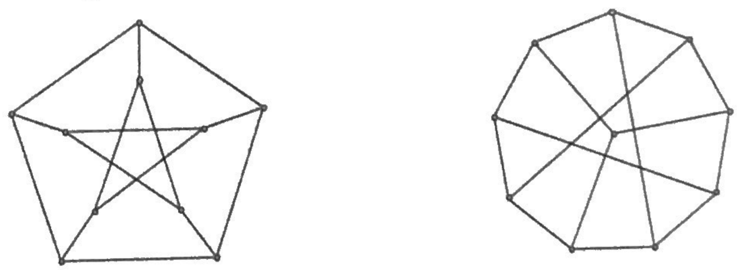


Figure 8.1.12

12) Let's consider a bit of graph theory known as graph labelling. Normally, a labelling is defined as a function ϕ from the vertices of a graph to some subset of the integers. These are considered the vertex labels. Then each edge is labelled with the absolute value of the difference between the labels on its end-vertices. See the graph in Figure 8.1.13 for an example. The point is to label the vertices so that some pattern emerges in the labels on the edges. One such labelling is called *graceful*. In this case, suppose that the graph G has n edges. Then the labelling is called graceful if $\phi : V(G) \to \{0, 1, 2, 3, \ldots, n\}$ is one-to-one and if the resulting labels on the edges are exactly $\{1, 2, 3, \ldots, n\}$. A triangle with its vertices labelled 0, 1, 3 is an example. Your job is to find a graceful labelling for each of the four graphs in Figure 8.1.14.

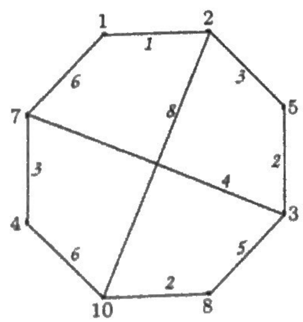


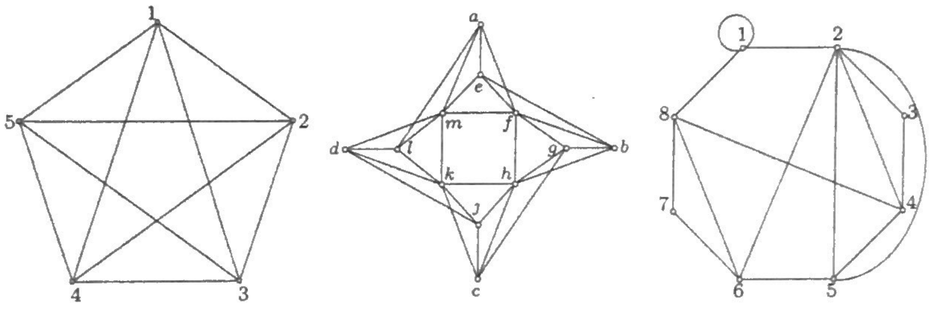
Figure 8.1.13

Theorem 8.2.4: A connected graph G is semi-Eulerian if and only if there are exactly two vertices of odd degree in $V(G)$.

Now for a geographical/historical note. If you're looking for Königsberg on a current map, you won't find it. What with the border changes following World War II, Germany lost all of East Prussia to Poland and the Soviet Union, the latter gaining control of Königsberg and giving it a new name–Kaliningrad. But since the Soviet Union no longer exists, either, you will find Kaliningrad in the Kaliningrad Oblast, a part of Russia, but detached from the rest of the country, lying between Poland and Lithuania.

——Exercises 8.2——

1) Find an Eulerian circuit in each of the graphs of Figure 8.2.4.



a b c

Figure 8.2.4

2) Can an Eulerian graph have a cut-vertex? If not, explain. If so, give an example including the Eulerian circuit.

3) Can an Eulerian graph have a bridge? If not, explain. If so, give an example including the Eulerian circuit.

4) Write a formal proof of Theorem 8.2.4.

5) Consider the graph G in Figure 8.2.5. A quick check of the degrees of the vertices tells us that G is neither Eulerian nor semi-Eulerian. Thus if we ask whether the figure can be drawn without lifting your pencil off the paper, we know the answer. However, we could still have an interesting puzzle by asking, "If you can't draw it in one continuous line, then what is the minimum number of lines necessary to reproduce the figure?" Answer this question for G and give a reason for your answer.

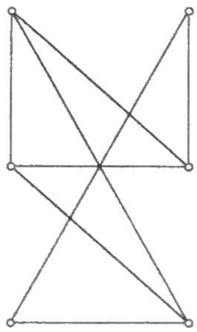


Figure 8.2.5

6) State and prove a theorem answering the question posed in the previous

exercise for a connected graph G with exactly $2k$ vertices of odd degree. What about a graph with exactly $2k+1$ vertices of odd degree?

7) Repeat the previous exercise for a disconnected graph.