Problem Set 4

Due: Friday, February 21st

Instructions: Do at least 7 of the following problems.

1. **More Impossibility Theorems** ⋆⋆

   Prove the following two theorems.

   (a) **Theorem:** Any social choice function that satisfies anonymity and neutrality must violate decisiveness.

   (b) **Theorem:** No social choice function involving at least three candidates satisfies both IIA, the Condorcet criterion, and always produces a unique winner.

2. **Weak Consistency** ⋆⋆

   In this question, we'll explore what happens when we merge two disjoint voting populations together to form a single population. This often happens when the number of congressional districts in a state changes, for example. If P_1 and P_2 are two different profiles for the same set of candidates (covering two different voting populations), then we'll denote by $P_1 + P_2$ the profile formed by merging the voting populations from P_1 and P_2. For example, if we have

   $$P_1 = \begin{array}{|c|c|c|c|} \hline 3 & 2 & 1 & 1 \\ \hline A & A & B & C \\ B & C & A & A \\ C & B & C & B \\ \hline \end{array} \quad , \quad P_2 = \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline A & B & B \\ B & A & C \\ C & C & A \\ \hline \end{array} \quad , \quad \text{then}$$

   $$P_1 + P_2 = \begin{array}{|c|c|c|c|c|} \hline 5 & 2 & 2 & 1 & 1 \\ \hline A & A & B & B & C \\ B & C & A & C & A \\ C & B & C & A & B \\ \hline \end{array}$$

   We say that a social choice procedure is *weakly consistent* if, whenever A is a winner for both profiles P_1 and P_2, then A is also a winner for the profile $P_1 + P_2$.

   (a) Is plurality voting weakly consistent?

   (b) Is the Borda count weakly consistent?

   (c) Is Hare's method weakly consistent?

3. **90% Pareto** ⋆⋆

   Let's say that a social choice function is *harshly Pareto* if whenever 90% or more of the voters prefer A to B, B must lose. Show that any harshly Pareto social choice function must fail the "always a winner" criterion.

4. **Edit Edit Edit** ⋆⋆

   One of the best ways to improve your proof writing skills is to edit. It helps you improve on exactly the things you need to work on most and allows for deeper understanding. For this problem choose two proof problems you wrote up on a previous homework that you think could use some polishing and rewrite them as best you can. As always, feel free to come talk with me if you want any feedback or help with the edits.

5. **Problem sets used to be cool, what happened?**

   Pick a problem that you have not attempted but wanted to do from a previous problem set. (you can do this for multiple problems if you want).

6. **Strategy-Proof Voting Methods** ⋆⋆

   Let's make a criterion that asks whether a system encourages voters to be honest.

   (a) Propose a mathematical definition for a social welfare function to be *strategy-proof* to encompass the idea that voters are happiest when they vote honestly.

   (b) Using your definition from part (a), is plurality voting strategy-proof?

   (c) Is the Borda count strategy-proof?

   (d) Is a dictatorship strategy-proof?

   (e) Any other methods you want to check?

7. **Dictating Coalitions** ⋆⋆⋆

   S is a dictating coalition if it can force x over y for *all* choices of alternatives x and y. What are all the possible dictating sets for the following voting systems:

   (a) Plurality

   (b) Instant runoff

   (c) Borda

   (d) Copeland

   (e) Any other methods you want to check?

8. **Splitting Hare's** ⋆⋆

   Side note: Instant runoff is often known as Hare's method. I need you to know this so I can make bad puns.

   The way instant runoff deals with with ties seems annoying, so let's try this new social choice function instead: in *decisive Hare*, whenever there is a tie for who has the fewest first-place votes, make a fork in the process by seeing what happens when you eliminate just one of those candidates. So for example, if there is a tie between A and B for fewest first-place votes, you continue the process by eliminating A and keeping B and, parallel to that, continue the process in a separate way by eliminating B and keeping A. Keep branching like this every time a tie occurs, keeping track of all possible tie resolutions. Anybody who wins along at least one of the forks is a winner of decisive Hare.

   (a) Find a single profile where three different candidates all tie for the win.

   (b) Is it possible for a Condorcet loser to win this method?

   (c) Is this new method favorite-friendly (as defined in Assignment #1, Problem #6)?

   (d) I was *this close* to calling it "Comb's Method" (because it's a fork for your Hare). Funny or nah?

9. **Intensity of Binary Alternatives** ⋆

   Intensity of binary alternatives (or IBA) is a criterion that might salvage the ideas of IIA without being quite as strict. A social welfare function satisfies IBA if the following always holds:

   Suppose A beats B in the social preference list generated by a certain profile. Create a new profile from the original profile by altering any number of ballots, but never changing the *relative intensity* of A versus B on any ballot: a voter may move the other candidates around at will, but he should never change whether he prefers A to B, or how far apart they are on the ballot. Then (and this is the condition required by IBA) A still must beat B in the social preference list generated by this new profile.

   (a) Prove (as was mentioned in class) that the Borda count satisfies IBA.

   (b) Prove that the modified Borda count on four candidates with point values 9, 4, 1, 0 for first, second, third, and fourth place (respectively) does *not* satisfy IBA.

10. **Disjoint Forcing** ⋆

    (In this problem, we are *not* assuming that the social welfare function in question satisfies Pareto or IIA. So you may not use any of the lemmas from our proof of Arrow's theorem, as they no longer apply.)

    Suppose that S forces x over y and T forces y over x. Prove that there must be some voter who is a member of both S and T.