HW 21: Taylor polynomials

Due: Monday, December 2nd in SQRC by 9pm

Learning Goals:

- Estimate the error of Taylor polynomial approximations.

Questions:

1. (a) Find the 5th degree Taylor polynomial centered at $x = 0$ to the tangent function to get the 5th degree approximation. [Answer: $\tan(x) \approx x + x^3/3 + 2x^5/15$.]

   (b) Recall that $\tan(x) = \sin(x)/\cos(x)$. Multiply the 5th degree Taylor polynomial for $\tan(x)$ from part a) by the 4th degree Taylor polynomial for $\cos(x)$ and show that you get the 5th degree polynomial for $\sin(x)$ (discarding higher degree terms).

2. Find the 5th degree Taylor polynomial centered at $x = 0$ for the function $f(x) = (1 + x)^{\frac{1}{2}}$. Use this approximation to estimate $\sqrt{1.1}$. Use a calculator to check how accurate you are. (if your answer is not very accurate, maybe think about what value of x you are using).

3. Here is an integral we can't compute. Integrate the 6th degree Taylor series of $\frac{1}{(1+x^2)^{\frac{1}{3}}}$ to approximate it instead. Feel free to use a computer to help but make sure to show your work.

$$\int_0^1 \frac{1}{(1 + x^2)^{\frac{1}{3}}}$$

4. Use the Taylor polynomial approximation for $\sin(x)$ to compute the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

5. (a) The first degree Taylor polynomial for e^x at $x = 0$ is $1 + x$. Plot the remainder $R_1(x) = e^x - (1 + x)$ over the interval $-0.1 \leq x \leq 0.1$. How does this graph demonstrate that $R_1(x) = O(x^2)$ as $x \to 0$?

   (b) There is a constant C_2 for which $R_1(x) \approx C_2x^2$ when $x \approx 0$. Estimate the value of C_2.

   (c) Repeat these two steps for the second degree approximation $1 + x + x^2/2$ by noticing that $R_2(x) = O(x^3)$ and finding a constant C_3.

6. (a) Let $f(x) = \ln(x)$. Find the smallest bound M for which $|f^{(4)}(x)| \leq M$ when $|x-1| \leq 0.5$.

   (b) Let $P_3(x)$ be the degree 3 Taylor polynomial for $\ln(x)$ at $x = 1$, and let $R_3(x)$ be the remainder $R_3(x) = \ln(x) - P_3(x)$. Find a number K for which $|R(x)| \leq K|x - 1|^4$ for all x satisfying $|x - 1| \leq .5$.

   (c) If you use $P_3(x)$ to approximate the value of $\ln(x)$ in the interval $0.5 \leq x \leq 1.5$, how many digits of the approximation are correct?

   (d) Suppose we restrict the interval to $|x - 1| \leq 0.1$. Repeat parts (a) and (b), getting smaller values for M and K. Now how many digits of the polynomial approximation $P_3(x)$ to $\ln(x)$ are correct, if $0.9 \leq x \leq 1.1$?