Counting Arguments

Here are some examples of counting arguments:

1. We want to show $n! = \binom{n}{k}k!(n-k)!$. We do this by claiming that both sides count the same thing. In this case both sides count the number of ways to order n things. The left hand side counts by first choosing one object to put first, which has n choices. Then choosing another to be second has $n-1$ choices since we already picked one. Going on in this way we get $n \cdot (n-1) \cdots 1 = n!$ ways of ordering.

   On the other hand we could order them as follows: First choose k of them to put in the front of the order and $n-k$ to put in the back. Next, we need to decide on an ordering of those first k things and there are $k!$ ways to do that. Last, we order the last $n-k$ things and there are $(n-k)!$ ways of doing that. Using the multiplication principle this gives us $\binom{n}{k}k!(n-k)!$ total ways of ordering.

2. We want to show $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. We do this by showing both sides count the number of ways to choose $k+1$ things from $n+1$. The left hand side is immediate.

   For the right hand side, pick one of the $n+1$ objects and mark it. Now when we choose $k+1$ of all the things we are in one of two cases: either we pick that marked object or we don't. In the case that we picked that marked object, we still need to pick k from the remaining n objects and there are $\binom{n}{k}$ ways of doing that. If we did not chose the marked object then we picked some $k+1$ objects from the remaining n and there are $\binom{n}{k+1}$ ways of doing that. Because these two cases are exclusive we add them to get the total number of ways to choose $k+1$ things from $n+1$.

3. We want to show $\binom{5}{5}\binom{7}{4} + \binom{5}{4}\binom{7}{5} + \binom{5}{3}\binom{7}{6} + \binom{5}{2}\binom{7}{7} = \binom{12}{9}$. We will show that both sides count the number of ways to choose 9 things from 12.

   The right hand side is exactly the number of ways to choose 9 things from 12. On the left hand side we first break up the twelve things into a group of 5 and a group of 7. When we choose the 9 things there are four cases to consider, (i) we choose 5 from the first group and 4 from the second, (ii) 4 from the first group and 5 from the second, (iii) 3 from the first group and 6 from the second, or (iv) 2 from the fist group and 7 from the second. They are all exclusive so we add them together.