

## HW 21: Taylor polynomials

Due: Monday, December 2nd in SQRC by 9pm

**Learning Goals:**

- Estimate the error of Taylor polynomial approximations.

**Questions:**

1. (a) The 5th degree Taylor polynomial centered at  $x = 0$  is  $\tan(x) \approx x + x^3/3 + 2x^5/15$ .  
(b) Recall that  $\tan(x) = \sin(x)/\cos(x)$ . Multiply the 5th degree Taylor polynomial for  $\tan(x)$  from part a) by the 4th degree Taylor polynomial for  $\cos(x)$  and show that you get the 5th degree polynomial for  $\sin(x)$  (discarding higher degree terms).
2. Find the 5th degree Taylor polynomial centered at  $x = 0$  for the function  $f(x) = (1 + x)^{\frac{1}{2}}$ . Use this approximation to estimate  $\sqrt{1.1}$ . Use a calculator to check how accurate you are. (if your answer is not very accurate, maybe think about what value of  $x$  you are using).
3. Here is an integral we can't compute. Integrate the 6th degree Taylor series of  $\frac{1}{(1+x^2)^{\frac{1}{3}}}$  to approximate it instead. Feel free to use a computer to help but make sure to show your work.

$$\int_0^1 \frac{1}{(1+x^2)^{\frac{1}{3}}}$$

4. Use the Taylor polynomial approximation for  $\sin(x)$  to compute the limit

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x}.$$

5. (a) The first degree Taylor polynomial for  $e^x$  at  $x = 0$  is  $1 + x$ . Plot the remainder  $R_1(x) = e^x - (1 + x)$  over the interval  $-0.1 \leq x \leq 0.1$ . How does this graph demonstrate that  $R_1(x) = O(x^2)$  as  $x \rightarrow 0$ ?  
(b) There is a constant  $C_2$  for which  $R_1(x) \approx C_2 x^2$  when  $x \approx 0$ . Estimate the value of  $C_2$ .  
(c) Repeat these two steps for the second degree approximation  $1 + x + x^2/2$  by noticing that  $R_2(x) = O(x^3)$  and finding a constant  $C_3$ .
6. (a) Let  $f(x) = \ln(x)$ . Find the smallest bound  $M$  for which  $|f^{(4)}(x)| \leq M$  when  $|x - 1| \leq 0.5$ .  
(b) Let  $P_3(x)$  be the degree 3 Taylor polynomial for  $\ln(x)$  at  $x = 1$ , and let  $R_3(x)$  be the remainder  $R_3(x) = \ln(x) - P_3(x)$ . Find a number  $K$  for which  $|R_3(x)| \leq K|x - 1|^4$  for all  $x$  satisfying  $|x - 1| \leq .5$ .  
(c) If you use  $P_3(x)$  to approximate the value of  $\ln(x)$  in the interval  $0.5 \leq x \leq 1.5$ , how many digits of the approximation are correct?  
(d) Suppose we restrict the interval to  $|x - 1| \leq 0.1$ . Repeat parts (a) and (b), getting smaller values for  $M$  and  $K$ . Now how many digits of the polynomial approximation  $P_3(x)$  to  $\ln(x)$  are correct, if  $0.9 \leq x \leq 1.1$ ?