Exercise Solutions for 9.1 to 9.6

Activity 9.1.2

Activity 9.1.2. Identify the domain of each of the following functions. Draw a picture of each domain in the xy-plane.

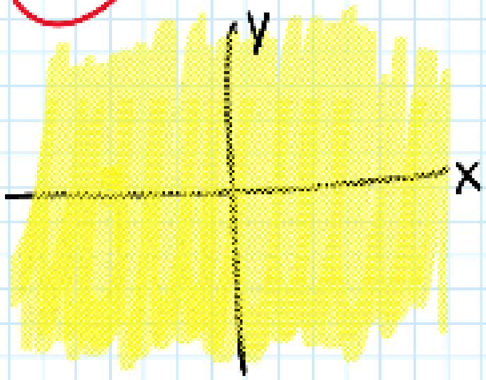
a. $f(x, y) = x^2 + y^2$

b. $f(x, y) = \sqrt{x^2 + y^2}$

c. $Q(x, y) = \dfrac{x + y}{x^2 - y^2}$

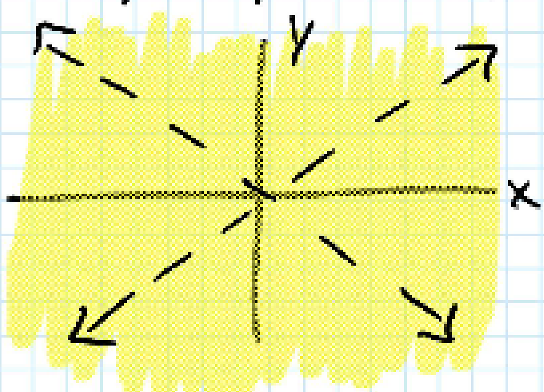
d. $s(x, y) = \dfrac{1}{\sqrt{1 - xy^2}}$

a. $(-\infty, \infty)$

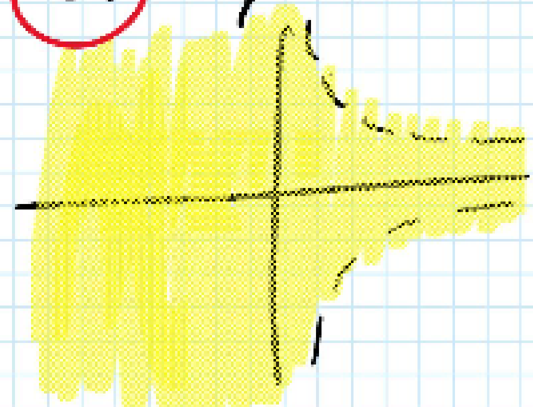


b. $(-\infty, \infty)$

c. $(-\infty, \infty)$, but $|x| \neq |y|$



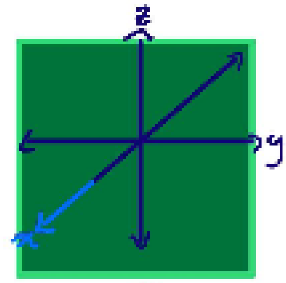
d. $xy^2 < 1$



9.1.4

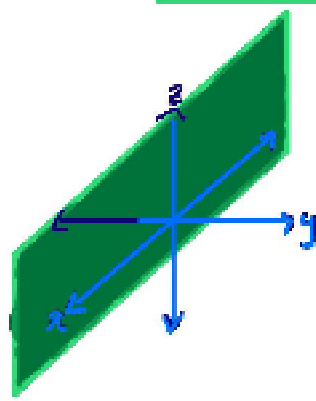
a) (x,y,z) $x=2$

Vertical Plane at $x=2$



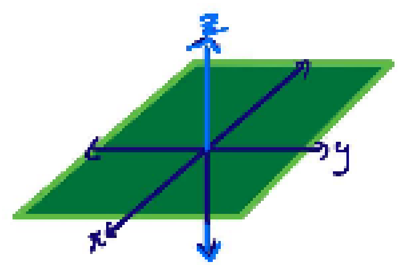
b) (x,y,z) $y=-1$

Slanted Plane at $y=-1$



c) (x,y,z) $z=0$

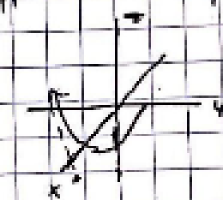
Horizontal plane
at $z=0$



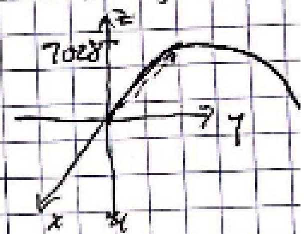
9.1.6) a) The function along this trace appears to be parabolic following

$z = \frac{\sin(1-2)}{9.8} x^2$

a →



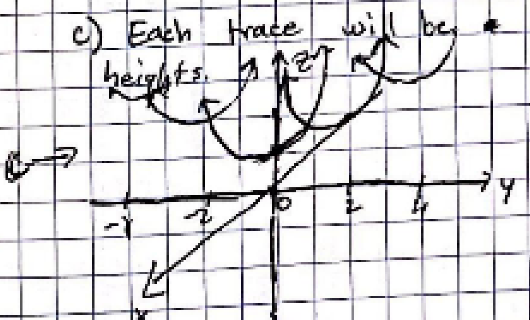
b →



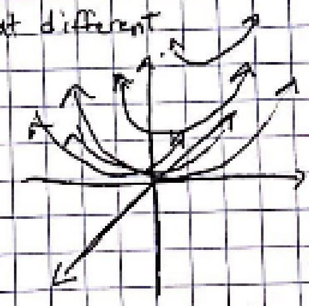
b) The function is the concave down portion of a sine curve.

c) Each trace will be parabolas at different heights.

c →



d →



d) It will be the same as part c but rotated 90°

e) A bowl-like shape sitting 1 unit above the origin (paraboloid) ıl map of the Porcupine Mountains in

points you can find;
ermine a path of steepest ascent that

ermine the least steep path that leads to

Steepest Path

9.3.3

a) Length of $u = \langle 1, 2, -3 \rangle$, $|u| = \sqrt{u \cdot u} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14} \approx 3.74$

b) $\cos\theta = \dfrac{u \cdot v}{|u| \cdot |v|} \Rightarrow \theta = \cos^{-1}\left(\dfrac{u \cdot v}{|u||v|}\right)$, $u = \langle 1, 2 \rangle$, $v = \langle 4, -1 \rangle$

$u \cdot v = (1 \cdot 4) + (2 \cdot -1) = 4 - 2 = 2$

$|u| = \sqrt{u \cdot u} = \sqrt{1^2 + 2^2} = \sqrt{5}$

$|v| = \sqrt{v \cdot v} = \sqrt{4^2 + (-1)^2} = \sqrt{17}$

$\theta = \cos^{-1}\left(\dfrac{2}{\sqrt{5} \cdot \sqrt{17}}\right) = 77.5°$

c) $\cos\theta = \dfrac{y \cdot z}{|y||z|} \Rightarrow \theta = \cos^{-1}\left(\dfrac{y \cdot z}{|y||z|}\right)$, $y = \langle 1, 2, -3 \rangle$, $z = \langle -2, 1, 1 \rangle$

$y \cdot z = (1 \cdot -2) + (2 \cdot 1) + (-3 \cdot 1) = -2 + 2 - 3 = -3$

$|y| = \sqrt{y \cdot y} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

$|z| = \sqrt{z \cdot z} = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{4 + 1 + 1} = \sqrt{6}$

$\theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{14} \cdot \sqrt{6}}\right) = 109.1°$

d) right angle $= 90° = \pi/2$ rad

$|u||v|\cos\theta = u \cdot v$

$|u||v|\cos(90) = u \cdot v$

$0 = u \cdot v$

If 2 vectors are normal, the dot product will always be 0.

e) acute angle is $\theta < 90°$ or $\theta < \frac{\pi}{2}$, let θ_1 be $< 90°$

$|u||v|\cos\theta_1 = u \cdot v$

$(+x)|u||v| = u \cdot v$

If two vectors are acute, their dot product will be positive

f) obtuse angle is $\theta > 90°$ or $\theta > \pi/2$, let $\theta_2 > 90°$

$|u||v|\cos\theta_2 = u \cdot v$

$(-x)|u||v| = u \cdot v$

If two vectors are obtuse, their dot product will be negative

• 9.4.2

$\vec{u} = \langle 0,1,3\rangle \quad \vec{v} = \langle 2,-1,0\rangle$

a. $\vec{u}\times\vec{v} = (0+3)\vec{i} - (0-6)\vec{j} + (0-2)\vec{k} = \langle 3, 6, -2\rangle$

b. $\vec{u}\cdot(\vec{u}\times\vec{v}) = (0\cdot 3) + (1\cdot 6) + (3\cdot -2) = 0$

$\vec{v}\cdot(\vec{u}\times\vec{v}) = (2\cdot 3) + (-1\cdot 6) + (0\cdot -2) = 0$.

- $\vec{u}\times\vec{v}$ is orthogonal to the plane on which $\vec{u}$ and $\vec{v}$ lie

c. $\vec{v}\times\vec{i} = (0-0)\vec{i} - (0-0)\vec{j} + (0+1)\vec{k} = \langle 0,0,1\rangle$

d. $(\vec{u}\times\vec{v})\times\vec{i} = (0-0)\vec{i} - (0+2)\vec{j} + (0-6)\vec{k} = \langle 0,-2,-6\rangle$

$\vec{u}\times(\vec{v}\times\vec{i}) = (1-0)\vec{i} - (0-0)\vec{j} + (0-0)\vec{k} = \langle 1,0,0\rangle$

↳ Since $(\vec{u}\times\vec{v})\times\vec{i} \neq \vec{u}\times(\vec{v}\times\vec{i})$, meaning cross products are NOT associative.

e. $\vec{u}\times\vec{u} = (3-3)\vec{i} - (0-0)\vec{j} + (0-0)\vec{k} = \langle 0,0,0\rangle$

$\boxed{9.4.3}$

ⓐ $|u \times v| = |u||v| \sin\theta$

$u = \langle 1,3,-2 \rangle \quad v = \langle 3,0,1 \rangle$

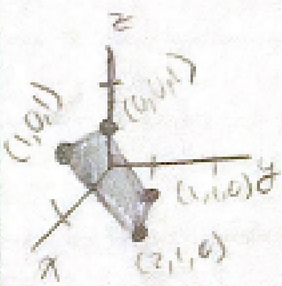
$$\begin{pmatrix} i & j & k & i & j \\ 1 & 3 & -2 & 1 & 3 \\ 3 & 0 & 1 & 3 & 0 \end{pmatrix} \qquad 3i - 6j - 9k - j$$

$u \times v = \langle 3, -7, -9 \rangle$

$|u \times v| = \sqrt{3^2 + 7^2 + 9^2} = \sqrt{139}$

★ area of parallelogram formed by 2 vectors = length of their cross prod

ⓑ area of parallelogram whose vertices are $(1,0,1)$, $(0,0,1)$, $(2,1,0)$, and $(1,1,0)$



$\underline{\text{base}}$
$a = \langle 2,1,0 \rangle - \langle 1,1,0 \rangle = \langle 1,0,0 \rangle$

$\underline{\text{height}}$
$b = \langle 2,1,0 \rangle - \langle 1,0,1 \rangle = \langle 1,1,-1 \rangle$

$$\begin{vmatrix} i & j & k & i & j \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 1 \end{vmatrix} \qquad 0i + k + j$$

$\langle 0, 1, 1 \rangle$

$\text{area} = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

9.4.4

a) $u \times v$: $\begin{vmatrix} i & j & k \\ 3 & 5 & -1 \\ 2 & -2 & 1 \end{vmatrix} = i(5-2) - j(3+2) + k(-6-10) = 3i - 5j - 16k \rightarrow \langle 3, -5, -16 \rangle$

$\|\langle 3, -5, -16 \rangle\| = \sqrt{9+25+256} = \sqrt{290}$ $\boxed{\left\langle \frac{3}{\sqrt{290}}, \frac{-5}{\sqrt{290}}, \frac{-16}{\sqrt{290}} \right\rangle}$

$v \times w$: $\begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 3 & 5 & -1 \end{vmatrix} = i(2-5) - j(-2-3) + k(10+6) = -3i + 5j + 16k$

$\boxed{\left\langle \frac{-3}{\sqrt{290}}, \frac{5}{\sqrt{290}}, \frac{16}{\sqrt{290}} \right\rangle}$

b) $|(u \times v) \cdot w|$

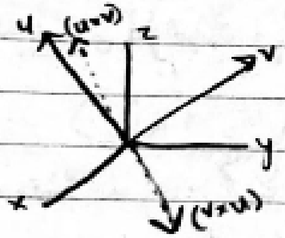
$u \times v = \langle 3, -5, -16 \rangle$ $(u \times v) \cdot w = \begin{bmatrix} 3 \\ -5 \\ -16 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = 9 - 15 - 16 = |-22| \rightarrow \boxed{22}$

c) $(4,1,0) - (0,1,2) = (4,0,-2)$

$(-2,2,2) - (0,1,2) = (-2,1,0)$

$\begin{vmatrix} i & j & k \\ 4 & 0 & -2 \\ -2 & 1 & 0 \end{vmatrix} = i(0+2) - j(0-4) + k(4) \longrightarrow \boxed{(2,4,4)}$

d)



e) $(a \times b) \cdot c = 0$ ← volume = 0, so on the same plane

$\downarrow$

$(1, 3, -2) \times (2, 1, -4)$

$\begin{vmatrix} i & j & k \\ 1 & 3 & -2 \\ 2 & 1 & -4 \end{vmatrix} = i(-12+2) - j(-4+4) + k(1-6) = -10i + 0j - 5k \rightarrow (-10, 0, -5)$

$\begin{bmatrix} -10 \\ 0 \\ -5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 + 0 + 0 = 0$ ✓

Yes, the vectors lie in the same plane. The magnitude of the cross product of a and b represents a plane (parallelogram). Because the dot product of this and c is equal to zero indicates that c has the same depth as a and b and is therefore on the same plane.

9.5.2

a) Find v for $\mathcal{L}$, $v = \overrightarrow{P_1P_2} = P_2 - P_1 = \langle -2, 1, -2 \rangle - \langle 1, 2, -1 \rangle = \langle -3, -1, -1 \rangle$

b) $r(t) = P_0 + tv = P_1 + tv = \langle 1, 2, -1 \rangle + \langle -3, -1, -1 \rangle t$

c) Direction of s(t) is $\langle 6, 2, 2 \rangle$; since it is a scalar multiple of v for r(t), $\langle 6, 2, 2 \rangle$ is parallel to $\langle -3, -1, -1 \rangle$

d) Yes - r(t) passes through the point $\langle -5, 0, -3 \rangle$

See: $r(2) = \langle 1, 2, -1 \rangle + 2\langle -3, -1, -1 \rangle = \langle -5, 0, -3 \rangle$

AND the direction $\langle 6, 2, 2 \rangle$ is parallel to $\langle -3, -1, -1 \rangle$

See: $-2\langle -3, -1, -1 \rangle = \langle 6, 2, 2 \rangle$

9.5.3

a. $x(t) = 1 + 3t$, $y(t) = 2 + t$, $z(t) = -1 + t$

b. the point does not, because there is no value t that solves the parametric equation for (1, 2, 1)

c. the direction is (4, -3, 2)

d. if L and K intersect, then
$x(t) = x(s)$, $y(t) = y(s)$, and $z(t) = z(s)$
$1 + 3t = 11 + 4s$, $2 + t = 1 - 3s$, $-1 + t = 3 + 2s$

$s = -1$, $t = 2$
they intersect at the point (7, 4, 1)

9.5.4

a) $2(x-0)-(y-2)+(z-4)=0 \Rightarrow P_1$

b) $2(2-0)-(0-2)+(2-4)$

$4+2-2$

$4 \neq 0$, therefore $(2,0,2)$ not on P_1

c) parallel planes have the same normal vectors

$P_2: 2(x-3)-(y-0)+(z-4)=0$

d) $P_3: x+2y-2z$

A line perpendicular to P_3 will have a direction vector the same as P_3's normal vector

$x(t)=2+t \quad y(t)=2t \quad z(t)=2-2t$

e) intersection is where $x(t)=x$, $y(t)=y$, $z(t)=z$

$(2+t)+2(2t)-2(2-2t)=7$

$2+t+4t-4+4t=7$

$9t=7-2+4$

$9t=9$

$t=1$

$x(1)=2+1=3 \quad y(1)=2(1)=2 \quad z(1)=2-2(1)=0$

$(3,2,0)$

9.5.5

a) $\overrightarrow{P_0P_1} = P_1 - P_0 = \langle 0, -2, 0 \rangle$

$\overrightarrow{P_0P_2} = P_2 - P_0 = \langle -1, -1, 4 \rangle$

b) $n \cdot \overrightarrow{PP_0} = 0$

normal vector → line orthogonal to every point on a plane

$\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} i & j & k \\ 0 & -2 & 0 \\ -1 & -1 & 4 \end{vmatrix}$

$i \begin{vmatrix} -2 & 0 \\ -1 & 4 \end{vmatrix} - j \begin{vmatrix} 0 & 0 \\ -1 & 4 \end{vmatrix} + k \begin{vmatrix} 0 & -2 \\ -1 & -1 \end{vmatrix}$

$\boxed{\langle -8, 0, -2 \rangle}$

c) $-8(x-1) + 0(y-2) + -2(z+1) = 0$

d) $m = \langle -3, 4, 2 \rangle$

$(1, -3, 5)$

$(3, -2, 6)$

e) $\cos\theta = \dfrac{n \cdot m}{|n||m|}$

$\langle -8, 0, -2 \rangle \cdot \langle -3, 4, 2 \rangle$

$-24 + 0 - 4 = \dfrac{20}{\sqrt{64+4}\ \sqrt{9+16+4}} = \left(\dfrac{20}{\sqrt{29}\sqrt{68}}\right) \cos^{-1}$

$\boxed{\theta = 63^\circ}$

9.6.2

a. This curve is the unit circle. It starts at the top of the circle and completes one full clockwise rotation as t goes from 0 to 2π.

b. This curve is exactly like the curve from part a, but it completes 2 full rotations instead of just one.

c. This curve is also the unit circle, but it begins at the leftmost point and completes one full counterclockwise rotation.

d. This curve is a counterclockwise unit circle like c, but it begins at the rightmost point and rotates more than 6 times, continuously speeding up as t increases.

9.6.3

a) $\vec{r}(t) = \langle t\cos(t), t\sin(t) \rangle$

A spiral originating from the center, similar look to fibonacci sequence.

b) $\vec{r}(t) = \langle \sin(t)\cos(t), t\sin(t) \rangle$

mis-shapen infinity loop where they cross at the origin and the smaller loop is on top

c) $\vec{r}(t) = \langle \sin(5t), \sin(4t) \rangle$

moves in an oval that is slighty curved causing it to resemble a quilt pillow

d) $\vec{r}(t) = \langle t^2\sin(t)\cos(t), 0.9\,t\cos(t^2), \sin(t) \rangle$

It is a loop that resembles random motion in 3D

e) $\vec{r}(t) = \langle t\cos^2(t), t\sin(t^2)\cos(t) \rangle$

looks like smoke coming from food in Cartoons

Activity 9.6.4. Consider the paraboloid defined by $f(x, y) = x^2 + y^2$.

a. Find a parameterization for the $x = 2$ trace of f. What type of curve does this trace describe?

b. Find a parameterization for the $y = -1$ trace of f. What type of curve does this trace describe?

c. Find a parameterization for the level curve $f(x, y) = 25$. What type of curve does this trace describe?

d. How do your responses change to all three of the preceding questions if you instead consider the function g defined by $g(x, y) = x^2 - y^2$? (Hint for generating one of the parameterizations: $\sec^2(t) - \tan^2(t) = 1$.)

a) $\vec{r}(t) = \langle 2, t, f(2,t) \rangle = \langle 2, t, t^2+4 \rangle$

This describes a parabola shifted up 4

b) $\vec{r}(t) = \langle t, -1, f(t, -1) \rangle = \langle t, -1, t^2+1 \rangle$

This describes a parabola shifted up 1

c) $\vec{r}(t) = \langle 5\cos t, 5\sin t \rangle$

Describes a circle of radius 5

d) a) $\vec{r}(t) = \langle 2, t, 4-t^2 \rangle$

b) $\vec{r}(t) = \langle t, -1, t^2-1 \rangle$

c) $\vec{r}(t) = \langle 5\sec t, 5\tan t \rangle$