Counting Arguments

Here are some examples of counting arguments:

1. We want to show $n! = \binom{n}{k}k!(n-k)!$. We do this by claiming that both sides count the same thing. In this case both sides count the number of ways to order n things.

   The left hand side counts by first choosing one object to put first, which has n choices. Then choosing another to be second has $n-1$ choices since we already picked one. Going on in this way we get $n \cdot (n-1) \cdot (n-2) \cdots 1 = n!$ ways of ordering the n objects.

   On the other hand we could order them as follows: First choose k of them to put in the front and $n-k$ to put in the back. Next, we need to decide on an ordering of those first k things and there are $k!$ ways to do that. Finally, we order the last $n-k$ things and there are $(n-k)!$ ways of doing that. Using the multiplication principle this gives us $\binom{n}{k}k!(n-k)!$ total ways of ordering.

2. We want to show $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$. We do this by showing both sides count the number of ways to choose $k+1$ things from $n+1$.

   The left hand side is by definition.

   For the right hand side, pick one of the $n+1$ objects and mark it. When we choose $k+1$ of all the $n+1$ things we are in one of two cases: either we pick that marked object or we don't. In the case that we picked that marked object, we still need to pick k other objects from the remaining n objects and there are $\binom{n}{k}$ ways of doing that. If we did not chose the marked object then we still need to pick $k+1$ objects from the remaining n and there are $\binom{n}{k+1}$ ways of doing that. Because these two cases are exclusive we add them to get the total number of ways to choose $k+1$ things from $n+1$. This gives us $\binom{n}{k} + \binom{n}{k+1}$ total ways.

3. We want to show $\binom{5}{5}\binom{7}{4} + \binom{5}{4}\binom{7}{5} + \binom{5}{3}\binom{7}{6} + \binom{5}{2}\binom{7}{7} = \binom{12}{9}$. We will show that both sides count the number of ways to choose 9 things from 12.

   The right hand side is by definition the number of ways to choose 9 things from 12.

   On the left hand side we count by first breaking up the twelve things into a group of 5 and a group of 7. When we choose the 9 things there are four cases to consider, (i) we choose 5 from the first group and 4 from the second, (ii) 4 from the first group and 5 from the second, (iii) 3 from the first group and 6 from the second, or (iv) 2 from the fist group and 7 from the second. They are all exclusive cases so we add them together to get the left hand side.

4. We show that $\binom{n+1}{k+1} = \sum_{i=0}^{n-k} \binom{k+i}{k}$ by demonstrating that both sides of the equation count the number of ways to choose $k+1$ objects from $n+1$.

   The left hand side equals this by definition.

   To help illustrate the counting method on the right hand side, first line up all $n+1$ objects in a row. For any way of choosing $k+1$ objects from this line, there will be one chosen object that is farthest to the right and we call this object the special object.

   The special object must be at least at spot $k+1$ because otherwise we could not fit the remaining k objects that we still need chose to the left of it. In general we denote the position of the special object as $k+1+i$. So when $i=0$ the special object is as far left as it can be, in spot $k+1$. When $i = n-k$ the special object is in spot $k+1+(n-k) = n+1$ which is as far right as it can be.

We break up our counting into cases based on where the special object is. Equivalently these cases correspond to value of i from 0 to $n-k$.

There are $k+i$ spots to the left of the special object for a given value of i. So if we know that the special object is chosen with value i and we are choosing a total of $k+1$ objects we must choose the remaining k from the $k+i$ spots to the left of the special object. In other words there are $\binom{k+i}{k}$ ways to choose the $k+1$ objects from the $n+1$ total objects so that the special object is in position $k+1+i$.

If we sum all of these cases together (indexed from $i=0$ to $i=n-k$ as noted earlier) we get a total of

$$\sum_{i=0}^{n-k} \binom{k+i}{k}$$

ways. This finishes the counting argument.