Problem Set 15

Due: Thursday, April 9th

Instructions: Answer each of the following questions and provide a justification for your answer. In addition to the points assigned below, you will receive 0-2 writing points for the entire problem set.

1. Prove that a function $f : A \to B$ is a bijection if and only if there exists a function $g : B \to A$ such that for all $a \in A$ we have $g(f(a)) = a$ and for all $b \in B$ we have $f(g(b)) = b$.

2. In this problem you will complete a proof that a set and its power set must have different cardinalities. Here is the claim:

   Theorem: Let A be a set. Then $|A| \neq |\mathcal{P}(A)|$.

   To prove this claim argue two cases. In Case 1 suppose A is empty and show that the claim is true. In Case 2 suppose that A is not empty. Now argue by contradiction. Carefully write down what the contradiction assumption $|A| = |\mathcal{P}(A)|$ implies. I won't tell you the whole proof, but I'll give you a few major pieces. You will need to consider the set

   $$D = \{a \in A \ : \ a \notin f(a)\}$$

   for a particular function $f : A \to \mathcal{P}(A)$. (Do you see where this function comes from?) Notice that D is an element of $\mathcal{P}$ so it is in the codomain of f. Use surjectivity of f to get a special element $d \in A$. Now ask yourself the question, is d an element of D? This should give a fun contradiction.

3. Prove that $\mathcal{P}(\mathbb{N})$ is uncountably infinite. (Use the previous problem, this should be a short write-up.)

4. Are $|\mathbb{N}|$ and $|\mathbb{R}|$ the only possibilities for the cardinality of an infinite set? If yes, why is this so? If no, what other possibilities are there? A well-reasoned paragraph is sufficient for this problem.