Quaternions

Warning: this material is entirely optional and only for your enjoyment. Proceed only if you want to learn something really cool.

The *complex numbers* are built from the real numbers by adding a symbol i with the property that $i \cdot i = -1$. The canonical way to write a complex number is in the form $a + bi$. The complex numbers are just like other numbers, you can add, subtract, multiply, and divide them. For example

$$\begin{aligned}(a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Notice that after I multiply I regroup the terms so they are in the canonical form of $a + bi$.

I want to ask about an even stranger set of numbers called the *quaternions*. They are of the form $a+bi+cj+dk$ with the defining properties that $i \cdot i = -1, j \cdot j = -1, k \cdot k = -1$ and $ijk = -1$.

(a) How can we write ij in the "canonical form" $a + bi + cj + dk$. (hint: use the defining equation $ijk = -1$ and $i \cdot i = -1, j \cdot j = -1, k \cdot k = -1$).

(b) What about ji? (warning: the quaternions at not commutative, $ij \neq ji$. Once again, use the defining equations).

(c) What about ikj?

(d) A quaternion is *purely imaginary* if it is of the form $bi + cj + dk$. Compute the multiplication of two purely imaginary quaternions and write the product in canonical form:

$$(u_1 i + u_2 j + u_3 k)(v_1 i + v_2 j + v_3 k).$$

Sometimes we compare an imaginary quaternion $u_1 i + u_2 j + u_3 k$ to the vector $\langle u_1, u_2, u_3 \rangle$ in $\mathbb{R}^3$. How does the product you just computed relate to vector computations? (*hint: what is the real part of the vector you just found? What is the imaginary part?*)

Remark: People really use quaternions! They are the historical reason we use the letters **i, j, k** for the standard basis of $\mathbb{R}^3$!!! A common use for quaternions today is in computer graphics where quaternions give the most efficient way of computing rotations in space.

If you want to learn more and watch some videos that will blow your mind check these out:

`https://eater.net/quaternions`