Problem Set 7

Due: Friday, March 13th

Instructions: Do at least 7 of the following problems. I highly encourage you to do one of the first four or at least consider them.

1. **Jefferson and Adams with Proportional Representation** $\star\star$

   In a proportional representation system (for example the Netherlands' Second Chamber and Israel's Knesset), voters vote for a single *party*, and then seats are apportioned to each party according to how many votes it receives.

   A particular concern in such a system is how the apportionment method treats large and small parties. Over time, a party may choose to merge with another party, or to split into two smaller parties. We must consider what incentives our apportionment method gives to these unions and schisms.

   If the choice is between the divisor methods of Adams and Jefferson, which do you think is more appropriate for a proportional representation system? Explain, using what you know about the biases of those systems.

2. **A Shocking Paradox** $\star$

   Consider the following imaginary census:

State	Population
Arkantana	1,450,000
Best Virginia	3,400,000
Connectidots	5,150,000
Total	10,000,000

   (a) Compute the Hamilton apportionment with house size $h = 10$.

   (b) Compute the Hamilton apportionment with house size $h = 11$.

   (c) Why is this surprising?

3. **Another Shocking Paradox** $\star$

   Again consider the census from the previous question.

   (a) If you haven't already done so, compute the Hamilton apportionment with house size $h = 10$.

   Suppose a new state, Dakota, joins the union with population 2,600,000. We might expect Dakota to have either 2 or 3 representatives.

   (b) Perhaps it should have two seats: let $h = 12$ and compute the new Hamilton apportionment for these four states.

   (c) Hm. Perhaps it should have three seats: let $h = 13$ and compute the new Hamilton apportionment for these four states.

   (d) Why are your results to part (c) surprising?

4. **Yet Another Shocking Paradox** $\star$

   Once again consider the census from question 4.

   (a) If you haven't already done so, compute the Hamilton apportionment with house size $h = 10$.

   Suppose a new census is taken and the following numbers are recorded:

State	Population
Arkantana	1,470,000
Best Virginia	3,380,000
Connectidots	4,650,000
Total	9,500,000

   (b) Compute the Hamilton apportionment again with these new numbers, using $h = 10$.

   (c) Why is this surprising?

5. **A Chicken Wing in Every Pot** $\star$

   Four friends split a \$17 order of chicken wings, paying according to how much cash they happen to have on hand at the moment: Barbara pays \$9, Frank pays \$4, Garland pays \$3, and Terry pays \$1. There are 25 wings in the order, and they would like to split the wings according to how much each person paid. (You cannot cut a wing into pieces; sorry.)

   (a) Compute the Hamilton apportionment of the wings.

   (b) Compute the Jefferson apportionment of the wings.

6. **The Minnesota Republican Caucus** $\star$

   Like many states with a caucus system, Minnesota's allocation of delegates in the Republican primary is rather confusing. Here are the results from 2012:

Candidate	Votes	Delegates
Santorum	21,988	2
Paul	13,282	32
Romney	8,240	1
Gingrich	5,263	0
Total	48,916	35

   Sure, whatever, Minnesota.

   Let's suppose these same 35 delegates were instead allocated to the candidates using a method that is *not* completely insane.

   (a) Compute the allocation of delegates using Hamilton's method.

   (b) Compute the allocation of delegates using Jefferson's method.

   (c) How closely do your results align with the original allocation?

7. **Rounding Way Up** $\star\star$

   Give an example of a Hamilton apportionment in which a state whose standard quota is 2.1 receives 3 seats. (Use whatever values of n, h, and p you so desire.)

8. **The Method of Lowndes** $\star\star$

   In the 1821, Congressman William Lowndes of South Carolina introduced a new apportionment scheme meant to address the biases of Jefferson's method. Like Hamilton, Lowndes looks at the standard quotas before deciding whether to round each one up or down. This time, the bonus seats are distributed in the way that makes the biggest congressional district as small as possible. For a particular h, the method can be described as follows:

   – Compute the standard quota (s_i) of each state.
   – Round each down to the lower quota.
   – Count the deficit of seats, namely h minus the sum of the lower quotas.
   – For each state, divide the decimal part of s_i by the integer part of s_i. This measures how many people would be funneled into each district if the state did *not* get one of the bonus seats.
   – Give the bonus seats to the states for which this ratio is largest.

   For example, here's the procedure used on yet another small census with $h = 10$:

State	Population	Quota	Integer	Decimal	Ratio
Awaii	144,000	1.44	1	.44	.44
Blorida	351,000	3.51	3	.51	.17
Corgia	505,000	5.05	5	.05	.01

   The "Quota" column is determined as usual, by dividing the population of the state by the total population and multiplying by h. This number is then separated into an integer and decimal part, and the "Ratio" column gives the value of the decimal part divided by the integer part.

   We first give each state its lower quota, or in other words the integer part: 1 for Awaii, 3 for Blorida, 5 for Corgia. That's 9 total, leaving one bonus seat to give it out, and it goes to Awaii because they have the highest corresponding ratio (even though Blorida had a higher decimal part). So Awaii gets 2 seats, Blorida 3, and Corgia 5.

   (a) Use Lowndes's method on the census data from the 1790s (see back page).
   (b) What do you notice about the bonus seats?

9. **A Quota for Large States** $\star\star$

   Consider the quota method (like that of Hamilton or Lowndes) which rounds up the larger values of s_i and rounds down the smaller ones. Use this method on the 1800s census data (see back page).

10. **An Agreement Between Hamilton and Webster** $\star\star$

    Notice that when using a house size of 120 for the 1790 census, Hamilton's method happens to round every state's quota to the *nearest* integer. Prove that whenever this occurs, Hamilton's method and Webster's method must give the same apportionment.

    Be careful! Remember that Hamilton's is a quota method while Webster's is a divisor method.

11. **The Dreaded Ties** ⋆⋆

So far, we've often assumed that it is possible to compare various quantities—populations, standard quotas, and so on—and conclude that one is larger than the other. This makes sense on a practical scale: while it's possible for an apportionment method to result in a "tie" which keeps it from determining a result, this is so unlikely that, in fact, it has never actually happened, so it's not worth worrying about.

Oh, wait, except we're in a math class! Let's worry about it anyway!

(a) Use—or rather, attempt to use—Hamilton's method on the following census with $h = 20$. What goes wrong?

State	Population
State of Abstraction	182,417
State of Bliss	332,417
State of Confusion	485,166
Total	1,000,000

(b) Attempt to use Jefferson's method on the following census with $h = 12$. What goes wrong?

State	Population
State of Anguish	150,000
State of Being	375,000
State of Calm	475,000
Total	1,000,000

12. **A Possibly-New Method** ⋆⋆

Consider the following apportionment method:

Begin with an intended house size h. Compute the standard quota ($h \cdot p_i/p$) for each state, and round each of those standard quotas down. Then, instead of allocating the bonus seats, simply use each of the rounded-down quotas as your apportionment. (Note: the house size you end up with here is *not* the original h you used.)

Does this method satisfy the quota property? Carefully explain your answer.

13. **Unknown Unknowns** ⋆⋆

Consider the following incomplete census for a country of three states:

State	Population
Ashingtah	140,000
Boregon	?
Coloraska	?
Total	1,000,000

Using $h = 10$, Ashingtah has a standard quota of 1.4, so under Hamilton's method we would expect it to receive either one or two seats.

(a) Fill in populations for the other two states so that Ashingtah receives one seat.

(b) Fill in populations for the other two states so that Ashingtah receives two seats.

14. **The Return of Condorcet** $\star\star\star$

The Marquis de Condorcet, ever full of surprises, had his own idea for an apportionment method. Like Jefferson, Adams, and others, he advocates a divisor method. The Condorcet method is to divide each state's population by the divisor d, and then round each quantity down if the the decimal part is less than .4, and up if the decimal part is greater than or equal to .4. (Why .4? Your guess is as good as mine.)

Use this method on the 1810s census data (see back page) to obtain a house size of $h = 181$.

(You will probably want the aid of a computer. A spreadsheet program may be useful. If you do use technology, explain your method and include your work.)

15. **Strategic Choices of the Divisor** $\star$

Choose three states from the 1820s census data (see back page): one small, one medium, one large. We would like to make an apportionment of these seats using Jefferson's method so that the house size is at most fifty, but every state has at least one seat.

(a) If you were from the small state, what value of d would you propose? Explain.

(b) If you were from the large state, what value of d would you propose? Explain.

16. **Practicing Critical Divisors** $\star\star$

Choose a state from the 1830s census data (see back page).

(a) Compute the first five Jefferson critical divisors of your state.

(b) Compute the first five Webster critical divisors of your state.

(c) Compute the first five Dean critical divisors of your state.

Census Data for the 1790s through the 1830s

Listed below are the results of the first five censuses. Censi? Censorum? Anyway, some questions ask you to refer to this table, so here it is. A "–" indicates that a state had not yet joined the Union.

State	1790s	1800s	1810s	1820s	1830s
Alabama	–	–	–	11,147	262,508
Connecticut	236,841	250,622	261,818	275,208	297,665
Delaware	55,540	61,812	71,004	70,943	75,432
Georgia	70,835	138,807	210,346	281,126	429,811
Illinois	–	–	–	54,843	157,147
Indiana	–	–	–	147,102	343,031
Kentucky	68,705	204,822	374,287	513,623	621,832
Louisiana	–	–	–	125,779	171,904
Maine	–	–	–	298,335	399,454
Maryland	278,514	306,610	335,946	364,389	405,843
Massachusetts	475,327	574,564	700,745	523,287	610,408
Mississippi	–	–	–	62,320	110,358
Missouri	–	–	–	62,496	130,419
New Hampshire	141,822	183,855	214,460	244,161	269,326
New Jersey	179,570	206,181	241,222	274,551	319,922
New York	331,589	577,805	953,043	1,368,775	1,918,578
North Carolina	353,523	424,785	487,971	556,821	639,747
Ohio	–	–	230,760	581,434	937,901
Pennsylvania	432,879	601,863	809,773	1,049,313	1,348,072
Rhode Island	68,446	68,970	76,888	83,038	97,194
South Carolina	206,236	287,131	336,569	399,351	455,025
Tennessee	–	100,169	243,913	390,769	625,263
Vermont	85,533	154,465	217,895	235,764	280,657
Virginia	630,560	747,362	817,615	895,303	1,023,503
Total	3,615,920	4,889,823	6,584,255	8,969,878	11,931,000