Problem Set 6

Due: Thursday, February 13th

Instructions: Answer each of the following questions and provide a justification for your answer. In addition to the points assigned below, you will receive 0-2 writing points for the entire problem set.

1. In class we showed that $\displaystyle\sum_{i=0}^{n}\binom{n}{i} = 2^n$. Below is an outline of how to prove this with a counting argument.

   (a) What is a quantity that $\displaystyle\sum_{i=0}^{n}\binom{n}{i}$ counts? What I am *not* looking for here is something like "it counts the sum of...", but rather what I want something like "it counts the number of ways to choose...". Try to express it in a single sentence. To be clear, our goal is to find something that the right hand side will also count so making it as simple as you can and not depend on a sum is probably a good idea.

   (b) Justify your answer to part (a) by explaining how $\displaystyle\sum_{i=0}^{n}\binom{n}{i}$ counts it.

   (c) Now show that the right hand side also counts the same thing. (hint: it probably helps to think about the right hand side as a large product $2 \cdot 2 \cdots 2$. Products naturally arise from a counting algorithm that first makes one choice, and then another, and so on. So whatever the right hand side is doing, it is probably repeatedly choosing between two alternatives.)

2. Prove that for all positive integers n that

$$\sum_{i=1}^{n} i^2 = 1 + 4 + 9 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. Prove that the sum of the fist n odd numbers is n^2. That is, prove

$$\sum_{i=1}^{n} (2i-1) = 1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

4. Use induction to prove that $2^n > n^2$ for $n \geq 5$.

5. Consider the function $f(x) = xe^x$. Let $f^{(n)}(x)$ denote the n-th derivative of $f(x)$.For example, let $f^{(2)}(x) = (xe^x)'' = (e^x + xe^x)' = 2e^x + xe^x$.Use the induction to prove: $f^{(n)}(x) = (x+n)e^x$ for all $n \geq 1$.