14.6 Surface Integrals

Consider a smooth surface $\mathcal{S}$ that represents a thin sheet of metal. How could we find the mass of this metallic object?

If the density of this object is constant, then we can find mass via "mass= density $\times$ surface area," and we could compute the surface area using the techniques of the previous section.

What if the density were not constant, but variable, described by a function $\delta(x, y, z)$? We can describe the mass using our general integration techniques as

$$\text{mass} = \iint_{\mathcal{S}} dm,$$

where dm represents "a little bit of mass." That is, to find the total mass of the object, sum up lots of little masses over the surface.

How do we find the "little bit of mass" dm? On a small portion of the surface with surface area ΔS, the density is approximately constant, hence $dm \approx \delta(x, y, z)\Delta S$. As we use limits to shrink the size of ΔS to 0, we get $dm = \delta(x, y, z)dS$; that is, a little bit of mass is equal to a density times a small amount of surface area. Thus the total mass of the thin sheet is

$$\text{mass} = \iint_{\mathcal{S}} \delta(x, y, z)\ dS. \tag{14.3}$$

To evaluate the above integral, we would seek $\vec{r}(u, v)$, a smooth parametrization of $\mathcal{S}$ over a region R of the u-v plane. The density would become a function of u and v, and we would integrate $\iint_R \delta(u, v)\, ||\ \vec{r}_u \times \vec{r}_v\ ||\ dA$.

The integral in Equation (14.3) is a specific example of a more general construction defined below.

Definition 14.6.1 Surface Integral

Let $G(x, y, z)$ be a continuous function defined on a surface $\mathcal{S}$. The **surface integral of G on $\mathcal{S}$** is

$$\iint_{\mathcal{S}} G(x, y, z)\ dS.$$

Surface integrals can be used to measure a variety of quantities beyond mass. If $G(x, y, z)$ measures the static charge density at a point, then the surface integral will compute the total static charge of the sheet. If G measures the amount of fluid passing through a screen (represented by $\mathcal{S}$) at a point, then the surface integral gives the total amount of fluid going through the screen.

Notes:

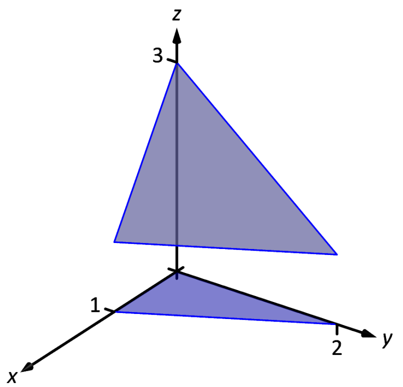

Figure 14.6.1: The surface whose mass is computed in Example 14.6.1.

Example 14.6.1 Finding the mass of a thin sheet

Find the mass of a thin sheet modeled by the plane $2x + y + z = 3$ over the triangular region of the x-y plane bounded by the coordinate axes and the line $y = 2-2x$, as shown in Figure 14.6.1, with density function $\delta(x, y, z) = x^2+5y+z$, where all distances are measured in cm and the density is given as gm/cm^2.

SOLUTION We begin by parameterizing the planar surface $\mathcal{S}$. Using the techniques of the previous section, we can let $x = u$ and $y = v(2 - 2u)$, where $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Solving for z in the equation of the plane, we have $z = 3 - 2x - y$, hence $z = 3 - 2u - v(2 - 2u)$, giving the parametrization $\vec{r}(u, v) = \langle u, v(2 - 2u), 3 - 2u - v(2 - 2u)\rangle$.

We need $dS = \|\,\vec{r}_u \times \vec{r}_v\,\|\, dA$, so we need to compute $\vec{r}_u$, $\vec{r}_v$ and the norm of their cross product. We leave it to the reader to confirm the following:

$$\vec{r}_u = \langle 1, -2v, 2v - 2\rangle, \quad \vec{r}_v = \langle 0, 2 - 2u, 2u - 2\rangle,$$

$$\vec{r}_u \times \vec{r}_v = \langle 4 - 4u, 2 - 2u, 2 - 2u\rangle \quad \text{and} \quad \|\,\vec{r}_u \times \vec{r}_v\,\| = 2\sqrt{6}\sqrt{(u-1)^2}.$$

We need to be careful to not "simplify" $\|\,\vec{r}_u \times \vec{r}_v\,\| = 2\sqrt{6}\sqrt{(u-1)^2}$ as $2\sqrt{6}(u-1)$; rather, it is $2\sqrt{6}|u - 1|$. In this example, u is bounded by $0 \leq u \leq 1$, and on this interval $|u - 1| = 1 - u$. Thus $dS = 2\sqrt{6}(1 - u)dA$.

The density is given as a function of x, y and z, for which we'll substitute the corresponding components of $\vec{r}$ (with the slight abuse of notation that we used in previous sections):

$$\begin{aligned}\delta(x, y, z) &= \delta\big(\vec{r}(u, v)\big)\\ &= u^2 + 5v(2 - 2u) + 3 - 2u - v(2 - 2u)\\ &= u^2 - 8uv - 2u + 8v + 3.\end{aligned}$$

Thus the mass of the sheet is:

$$\begin{aligned}M &= \iint_{\mathcal{S}} dm\\ &= \iint_R \delta\big(\vec{r}(u, v)\big)\,\|\,\vec{r}_u \times \vec{r}_v\,\|\; dA\\ &= \int_0^1\int_0^1 \big(u^2 - 8uv - 2u + 8v + 3\big)\big(2\sqrt{6}(1 - u)\big)\, du\, dv\\ &= \frac{31}{\sqrt{6}} \approx 12.66 \text{ gm.}\end{aligned}$$

Notes: