Problem Set 8

Due: Thursday, February 27th

Instructions: Answer each of the following questions and provide a justification for your answer. In addition to the points assigned below, you will receive 0-2 writing points for the entire problem set.

1. Let F_n denote the number of ways to tile a 2 by n chessboard with dominoes. Prove that for $n \geq 3$ that $F_n = F_{n-1} + F_{n-2}$.

   Example: there are 3 ways to tile a 2 by 3 chess board, so $F_3 = 3$.

   

2. Here we practice the Division Algorithm and Euclidean Algorithm in the context of polynomials with *real* coefficients.

   (a) Find the greatest common divisor of $x^2 + x + 1$ and $x^2 - x + 1$.

   (b) Find two polynomials $a(x)$ and $b(x)$ so that

   $$GCD(x^2 + x + 1, x^2 - x + 1) = (x^2 + x + 1)a(x) + (x^2 - x + 1)b(x)$$

   (c) Find two polynomials $c(x)$ and $d(x)$ so that

   $$(x^2 + x + 1)c(x) + (x^2 - x + 1)d(x) = 2 - 2x$$

   (hint: use part b)

   (d) Find the greatest common divisor of $x^5 - 1$ and $x^2 - 1$.

   (e) Find two polynomials $a(x)$ and $b(x)$ so that

   $$GCD(x^5 - 1, x^2 - 1) = (x^5 - 1)a(x) + (x^2 - 1)b(x)$$

3. Prove that for all positive integers a, b and k that $\text{GCD}(k, ab) = 1$ if and only if $\text{GCD}(k, a) = 1$ and $\text{GCD}(k, b) = 1$.

4. Suppose you have a 5 liter jug and a 7 liter jug. We can perform any of the following moves:

   - Fill a jug completely with water.
   - Transfer water from one jug to another, stopping if the other jug is filled.
   - Empty a jug of water if it is completely filled. (we can't empty partially filled jugs)

   The goal is to end up with one jug having exactly 1 liter of water. How do we do this?