ELECTRODYNAMICS HOMEWORK

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1 The Lorentz force is:

$$\vec{F}_L = e\left(\vec{E}(\vec{r},t) + \dot{\vec{q}}(t) \times \vec{B}(\vec{r},t)\right) \tag{1}$$

Using:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{2}$$

$$\vec{B} = \nabla \times \vec{A} \tag{3}$$

and $\vec{A} = \vec{A}(\vec{r},t)$, rewrite $\vec{F}_L$ in terms of ϕ and $\vec{A}$.

Solution:

Introducing the expressions of the fields in terms of the potential, the following expression is obtained:

$$\vec{F}_L = e\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \dot{\vec{q}}(t) \times \left(\nabla \times \vec{A}\right)\right) \tag{4}$$

Using the following vector identity:

$$\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}) \tag{5}$$

Follows:

$$\nabla(\dot{\vec{q}}\cdot\vec{A}) = (\dot{\vec{q}}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\dot{\vec{q}} + \dot{\vec{q}}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\dot{\vec{q}}) \tag{6}$$

The following terms are zero due to $\frac{\partial \dot{q}_i}{\partial q_k} = 0$, in other words due to the fact that $\dot{q}_i$ and q_k are independent coordinates.

$$(\vec{A}\cdot\nabla)\dot{\vec{q}} = 0 \tag{7}$$

$$\vec{A}\times(\nabla\times\dot{\vec{q}}) = 0 \tag{8}$$

Then

$$\dot{\vec{q}}\times(\nabla\times\vec{A}) = \nabla(\dot{\vec{q}}\cdot\vec{A}) - (\dot{\vec{q}}\cdot\nabla)\vec{A} \tag{9}$$

Finally the expression for the Lorentz force in terms of the vector potentials is:

$$\vec{F}_L = e\left(-\nabla\phi - \frac{\partial \vec{A}}{\partial t} + \nabla(\dot{\vec{q}}\cdot\vec{A}) - (\dot{\vec{q}}\cdot\nabla)\vec{A}\right) \tag{10}$$

2. Show that the Lagrangian:

$$\mathscr{L} = \frac{1}{2}m\dot{\vec{q}}^{\,2} - e\phi + e\,\dot{\vec{q}}\cdot\vec{A} \tag{11}$$

results into the right equation of motion with the Lorentz force.

Solution:

First the following derivatives are computed:

$$\frac{\partial \mathscr{L}}{\partial q_i} = -e\frac{\partial \phi}{\partial q_i} + e\,\dot{\vec{q}}\cdot\frac{\partial \vec{A}}{\partial q_i} \tag{12}$$

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{d}{dt}\left(m\dot{q}_i + eA_i\right) = m\ddot{q}_i + e\frac{dA_i}{dt} \tag{13}$$

The last expression can be simplified changing the total derivative for the expression in terms of partial derivatives:

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \dot{q}_x\frac{\partial A_i}{\partial x} + \dot{q}_y\frac{\partial A_i}{\partial y} + \dot{q}_z\frac{\partial A_i}{\partial z} = \frac{\partial A_i}{\partial t} + \dot{\vec{q}}\cdot\nabla A_i \tag{14}$$

Using the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = \frac{\partial \mathscr{L}}{\partial q_i} \tag{15}$$

Follows:

$$m\ddot{q}_i + e\left(\frac{\partial A_i}{\partial t} + \dot{\vec{q}}\cdot\nabla A_i\right) = -e\frac{\partial \phi}{\partial q_i} + e\,\dot{\vec{q}}\cdot\frac{\partial \vec{A}}{\partial q_i} \tag{16}$$

In vector form:

$$m\ddot{\vec{q}} + e\left(\frac{\partial \vec{A}}{\partial t} + \left(\dot{\vec{q}}\cdot\nabla\right)\vec{A}\right) = -e\nabla\phi + e\,\nabla\left(\dot{\vec{q}}\cdot\vec{A}\right) \tag{17}$$

Where it was used that $\left[\nabla\left(\dot{\vec{q}}\cdot\vec{A}\right)\right]_i = \dot{\vec{q}}\cdot\frac{\partial \vec{A}}{\partial q_i}$ because $\frac{\partial \dot{\vec{q}}}{\partial q_i} = 0$. Identifying $F = m\ddot{\vec{q}}$, follows:

$$F = -e\nabla\phi - e\frac{\partial \vec{A}}{\partial t} + e\,\nabla\left(\dot{\vec{q}}\cdot\vec{A}\right) - e\left(\dot{\vec{q}}\cdot\nabla\right)\vec{A} \tag{18}$$

Which is the Lorentz force found in (10).

3. Using:

$$\mathscr{H} = \sum_i p_i\dot{q}_i - \mathscr{L} \tag{19}$$

where p_i is the canonical momentum, derive $\mathscr{H}$ in terms of $\vec{p}$, $\vec{A}$ and ϕ.

Solution:

The canonical conjugated momentum is given by:

$$p_k = \frac{\partial \mathscr{L}}{\partial \dot{q}_k} = m\dot{q}_k + eA_k \tag{20}$$

Then the generalized velocities in terms of the canonical momentum are:

$$\dot{q}_k = \frac{p_k}{m} - \frac{e}{m}A_k \tag{21}$$

To write the Lagrangian in terms of the conjugated momentum the following quantities are computed:

$$\dot{\vec{q}}\cdot\dot{\vec{q}} = \left(\frac{\vec{p}}{m} - \frac{e}{m}\vec{A}\right)\cdot\left(\frac{\vec{p}}{m} - \frac{e}{m}\vec{A}\right) = \frac{1}{m^2}\left(\vec{p}^{\,2} + e^2\vec{A}^{\,2} - 2e\,\vec{p}\cdot\vec{A}\right) \tag{22}$$

$$\dot{\vec{q}}\cdot\vec{A} = \left(\frac{\vec{p}}{m} - \frac{e}{m}\vec{A}\right)\cdot\vec{A} = \frac{1}{m}\left(\vec{p}\cdot\vec{A} - e\vec{A}^{\,2}\right) \tag{23}$$

With this the Lagrangian becomes:

$$\mathscr{L} = \frac{1}{2m}\left(\vec{p}^{\,2} - e^2\vec{A}^{\,2}\right) - e\phi \tag{24}$$

To get the Hamiltonian in terms of the canonical momentum the following identity is needed:

$$\sum_i p_i\dot{q}_i = \vec{p}\cdot\dot{\vec{q}} = \vec{p}\cdot\left(\frac{\vec{p}}{m} - \frac{e}{m}\vec{A}\right) = \frac{1}{m}\left(\vec{p}^{\,2} - e\,\vec{p}\cdot\vec{A}\right) \tag{25}$$

Finally the Hamiltonian is given by:

$$\mathscr{H} = \frac{1}{2m}\left(\vec{p}^{\,2} + e^2\vec{A}^{\,2}\right) + e\phi - \frac{e}{m}\vec{p}\cdot\vec{A} = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi \tag{26}$$

$$\mathscr{H} = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + e\phi \tag{27}$$