ELECTRODYNAMICS
HOMEWORK # 1

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1. Prove:

$$\nabla \times (\nabla \times \vec{G}) = \nabla(\nabla \cdot \vec{G}) - \nabla^2 \vec{G} \tag{1}$$

Proof:

$$[\nabla \times (\nabla \times \vec{G})]_i = \epsilon_{ijk}\partial_j(\nabla \times \vec{G})_k \tag{2}$$

To prove this it is useful to look at it by components (2).

$$(\nabla \times \vec{G})_k = \epsilon_{km\ell}\partial_m G_\ell \tag{3}$$

$$\epsilon_{ijk}\partial_j(\nabla \times \vec{G})_k = \epsilon_{ijk}\partial_j\epsilon_{km\ell}\partial_m G_\ell = \epsilon_{ijk}\epsilon_{km\ell}\partial_j\partial_m G_\ell \tag{4}$$

Using the identity (3) and introducing it in (2) follows (4) using the fact that scalar commute.

$$\epsilon_{ijk}\epsilon_{km\ell} = \delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm} \tag{5}$$

$$\epsilon_{ikj}\epsilon_{km\ell}\partial_j\partial_m G_\ell = \delta_{im}\delta_{j\ell}\partial_j\partial_m G_\ell - \delta_{i\ell}\delta_{jm}\partial_j\partial_m G_\ell = \partial_\ell\partial_i G_\ell - \partial_m\partial_m G_i \tag{6}$$

With the aide of the identity (5) and introducing it in equation (4), it follows expression (6).

$$\partial_\ell\partial_i G_\ell - \partial_m\partial_m G_i = \partial_i\partial_\ell G_\ell - \partial_m\partial_m G_i \tag{7}$$

Clairaut's theorem allows to exchange the order of derivatives in line (7).

$$\partial_i\partial_\ell G_\ell - \partial_m\partial_m G_i = \partial_i\nabla \cdot \vec{G} - \nabla^2 G_i \tag{8}$$

$$\nabla \times (\nabla \times \vec{G}) = \nabla(\nabla \cdot \vec{G}) - \nabla^2 \vec{G} \tag{9}$$

Using the fact that $\partial_\ell G_\ell = \nabla \cdot \vec{G}$ and $\partial_m\partial_m = \nabla^2$ follows equation (8). It is seen from (8) that (9) is the vector with components (2). With proofs the identity (1).

2. Prove:

$$\nabla(\vec{F} \cdot \vec{G}) = (\vec{F} \cdot \nabla)\vec{G} + (\vec{G} \cdot \nabla)\vec{F} + \vec{F} \times (\nabla \times \vec{G}) + \vec{G} \times (\nabla \times \vec{F}) \tag{10}$$

Proof:

$$[\vec{F} \times (\nabla \times \vec{G})]_i = \epsilon_{ijk}F_j(\nabla \times \vec{G})_k = \epsilon_{ijk}\epsilon_{km\ell}F_j\partial_m G_\ell \tag{11}$$

$$= (\delta_{im}\delta_{j\ell} - \delta_{i\ell}\delta_{jm})F_j\partial_m G_\ell = (\partial_i G_\ell)F_\ell - F_m\partial_m G_i = (\partial_i G_\ell)F_\ell - (\vec{F} \cdot \nabla)G_i \tag{12}$$

$$(\partial_i G_\ell)F_\ell = [\vec{F} \times (\nabla \times \vec{G})]_i + (\vec{F} \cdot \nabla)G_i \tag{13}$$

First it is useful to find the relation given by equation (13) which follows using by examining the components of the vector $\vec{F} \times (\nabla \times \vec{G})$.

$$(\partial_i F_\ell)G_\ell = [\vec{G} \times (\nabla \times \vec{F})]_i + (\vec{G} \cdot \nabla)F_i \tag{14}$$

As the procedure done to obtain relation (13) was done using arbitrary vector functions, the labels can be exchanged ($G \longrightarrow F$) and the identity (14) follows automatically.

$$[\nabla(\vec{F}\cdot\vec{G})]_i = \partial_i(F_jG_j) = (\partial_i G_\ell)F_\ell + (\partial_i F_\ell)G_\ell \tag{15}$$

Equation (15) follows from looking at the components of the vector $\nabla(\vec{F}\cdot\vec{G})$ and computing the derivative by means of the product rule.

$$[\nabla(\vec{F}\cdot\vec{G})]_i = \partial_i(F_jG_j) = [\vec{F}\times(\nabla\times\vec{G})]_i + (\vec{F}\cdot\nabla)G_i + [\vec{G}\times(\nabla\times\vec{F})]_i + (\vec{G}\cdot\nabla)F_i \tag{16}$$

Introducing the relations (13) and (14) in (15) i get expression (16). From expression (16) it can be seen that follows equation (10).

3. Prove:

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \oint_{\partial V} \phi\nabla\psi\cdot\hat{n}dA \tag{17}$$

Proof:

$$\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi \tag{18}$$

To prove relation (17) it suffices to show that the identity (18) holds. Then using the divergence theorem relation (17) follows.

$$\nabla\cdot(\phi\nabla\psi) = \partial_i(\phi\partial_i\psi) = \partial_i\phi(\partial_i\psi) + \phi\partial_i\partial_i\psi = \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi \tag{19}$$

$$\int_V (\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)d^3x = \int_V \nabla\cdot(\phi\nabla\psi)d^3x = \oint_{\partial V} \phi\nabla\psi\cdot\hat{n}dA \tag{20}$$

4. Prove:

$$\int_V f(\nabla\cdot\vec{F})d^3x = \oint_{\partial V} f\vec{F}\cdot\hat{n}\; dA - \int_V \vec{F}\cdot\nabla f\; d^3x \tag{21}$$

Proof:

$$\nabla\cdot(f\vec{F}) = f\nabla\cdot\vec{F} + \vec{F}\cdot\nabla f \tag{22}$$

To prove (21) first I prove (22).

$$\nabla\cdot(f\vec{F}) = \partial_i(fF_i) = F_i\partial_i f + f\partial_i F_i = \vec{F}\cdot\nabla f + f\nabla\cdot\vec{F} \tag{23}$$

$$f\nabla\cdot\vec{F} = \nabla\cdot(f\vec{F}) - \vec{F}\cdot\nabla f \tag{24}$$

Equation (23) is computed using the product rule for derivatives. Relation (24) rearranges the terms of (23) leaving a relation for the term of interest, $f\nabla\cdot\vec{F}$.

$$\int_V f\nabla\cdot\vec{F}d^3x = \int_V \nabla\cdot(f\vec{F})d^3x - \int_V \vec{F}\cdot\nabla f d^3x \tag{25}$$

$$\int_V \nabla\cdot(f\vec{F})d^3x = \oint_{\partial V} f\vec{F}\cdot\hat{n}\; dA \tag{26}$$

$$\int_V f\nabla\cdot\vec{F}d^3x = \oint_{\partial V} f\vec{F}\cdot\hat{n}\; dA - \int_V \vec{F}\cdot\nabla f d^3x \tag{27}$$

Using the identity (24) and the divergence theorem follows relation (27). This was the desired relation.