Causality exercises 1: http://bayes.cs.ucla.edu/BOOK-2K/slides-hw1.pdf

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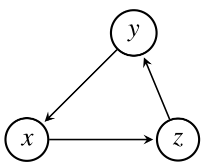
Exercise 1: Prove that the graphoid properties are satisfied if we interpret $(X \perp\!\!\!\perp Y|Z)$ to mean "all paths from a subset X of nodes are intercepted by a subset Z of nodes". The graphoid properties are:
Symmetry: $(X \perp\!\!\!\perp Y \mid Z) \Rightarrow (Y \perp\!\!\!\perp X \mid Z)$.
Decomposition: $(X \perp\!\!\!\perp YW \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid Z)$.
Weak union: $(X \perp\!\!\!\perp YW \mid Z) \Rightarrow (X \perp\!\!\!\perp Y \mid ZW)$.
Contraction: $(X \perp\!\!\!\perp Y \mid Z)$ & $(X \perp\!\!\!\perp W \mid ZY) \Rightarrow (X \perp\!\!\!\perp YW \mid Z)$.
Intersection: $(X \perp\!\!\!\perp W \mid ZY)$ & $(X \perp\!\!\!\perp Y \mid ZW) \Rightarrow (X \perp\!\!\!\perp YW \mid Z)$.

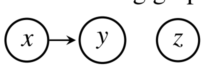
Interpretation:

- Here a "path" means an undirected path in a directed graph. If we interpret it to mean a directed path, then symmetry would fail on the following graph:

  Then for *directed* paths and $X = \{x\}$, $Y = \{y\}$ and $Z = \{z\}$, we have that every path from X to Y, it must be intercepted by an element of Z (i.e. $(X \perp\!\!\!\perp Y \mid Z)$, but there is a path from Y to X which does not intercept Z.

- For a path to "intercept" must mean something other than "shares a node with", since otherwise "intersection" would fail. Consider the following graph:

  Now let $X = \{x\}$, $Y = W = \{y\}$ and $Z = \{z\}$. Then with the "has a node in common" definition of "intercepted", the left hand side of the "Intersection" axiom is satisfied, but the right hand side is not. There is a unique path from X to Y that always includes the vertex $y \in Y = W$, and that path intercepts ZY (since in intercepts Y) and ZW, since it intercepts W.

  A reasonable definition that seems consistent with the rest of the text is that a set of nodes Z *intercepts* a path P, if there is a $z \in Z$ in the path, but it is not an endpoint.

Proof:
Symmetry: Assume that $(X \perp\!\!\!\perp Y \mid Z)$, and let P be a path from Y to X. Since the path is undirected, reversing P yields a path P' from X to Y. Since $(X \perp\!\!\!\perp Y \mid Z)$, P' must be intercepted by Z. Therefore P is intercepted by Z. Since P was arbitrary, we have $(Y \perp\!\!\!\perp X \mid Z)$, as desired.
Decomposition: Assume $(X \perp\!\!\!\perp YW \mid Z)$. Since every path from X to Y is also a path from X to $YW \supseteq Y$, it must be intercepted by Z, as desired.
Weak union: Assume $(X \perp\!\!\!\perp YW \mid Z)$. Let P be any path from X to Y. Then P is also a path from X to $YW \subseteq Y$, and therefore must be intercepted by Z. Since $Z \subseteq ZW$, P is also intercepted by ZW. So since P was arbitrary, we have $(X \perp\!\!\!\perp Y \mid ZW)$, as desired.
Contraction: Assume $(X \perp\!\!\!\perp Y \mid Z)$ and $(X \perp\!\!\!\perp W \mid ZY)$. Let P be a path from X to YW, and let $n \in YW$ be the terminal node. Then $n \in Y$ or $n \in W$. If the former, then Z must intercept P since we assumed $(X \perp\!\!\!\perp Y \mid Z)$. If $n \in W$, then since $X \perp\!\!\!\perp W \mid ZY)$, P is intercepted by Z or by Y. If Z, then we are done. If Y, then $n_i \in P$ is an element of Y for some i. This subpath of P is then a path from X to Y, which must be intercepted by Z by assumption. That means that P is also intercepted by Z. In either case, P is intercepted by Z, so we have: $(X \perp\!\!\!\perp YW \mid Z)$.
Intersection: Assume (1) $(X \perp\!\!\!\perp W \mid ZY)$ and (2) $(X \perp\!\!\!\perp Y \mid ZW)$, but that $(X \not\perp\!\!\!\perp YW \mid Z)$. Then there is a path P from X to YW which is not intercepted by Z. Let P_0 denote the *shortest* such path. If P terminates in W, then by (1) it is intercepted by ZY. P is not intercepted by Z, so it must be intercepted by Y at some point in the interior of the path, say at node $i < |P|$. But then $n_0, n_1, \ldots, n_i$ is also a path from X to YW that is strictly shorter than P, contradicting that P is the shortest path for which independence fails. The case where P terminates in Y is similar, and follows by symmetry.