Problem (operations puzzle): Given a sequence of positive integers $S = (a_0, a_1, \dots, a_{n-1})$ and a desired value C, construct a sequence of signs $s_0, \dots, s_{n-1}$ such that:

$$\sum_{i=0}^{n-1} s_i \cdot a_i = C$$

Theorem: This problem is NP-complete.

Proof:

1. First note that any possible solution can be checked in linear time, so the problem is in NP.
2. Now we must find an NP-complete problem such that we can write an algorithm using an oracle for the operations puzzle, where:
   A. The size of operations puzzles are polynomial in the size of the original problem instance, and
   B. The number of calls to the operations puzzle oracle is polynomial in the size of the original problem instance.

For this puzzle, we will use the "Subset Sum" problem, which is NP-complete ([https://en.wikipedia.org/wiki/Subset_sum_problem (https://en.wikipedia.org/wiki/Subset_sum_problem)](https://en.wikipedia.org/wiki/Subset_sum_problem)), in particular, the variant where all the numbers are positive (which is also NP-complete). The problem statement is similar:

> In its most general formulation, there is a multiset S of integers and a target sum T, and the question is to decide whether any subset of the integers sum to precisely T.

Given an instance $S = (a_0, \dots, a_{n-1}), T$ of the positive subset sum problem, let:

$$\begin{aligned}
L &= \left\lceil \lg\left(\sum_{s \in S} s\right) \right\rceil + 3 \\
q_i &:= 2^{L+2i} \qquad \text{(a power of two larger than 4 times the sum of S)} \\
C &:= 2T + \sum_i q_i \\
b_i &:= a_i + q_i
\end{aligned}$$

Now define a new sequence of numbers $S' = (a_0, \dots, a_{n-1}, b_0, \dots b_{n-1})$.

Claim: The operations puzzle problem (S', C) has a solution if and only if the subset-sum problem (S, T) has a solution.

Proof of claim: (sketch) First note that for any solution to the operations puzzle, the signs $s_n, \ldots s_{2n-1}$ must all be 1, since the terms q_i occur on both sides of the equation, and cannot be formed as a sum of any of the other terms. (They're too large to be a sum of the s_i, and are too far apart from each other for any q_i to be a sum of other q_j's.)

(⇒) If s_i $(0 \leq i < n)$ is -1 in the operations solution, then the two a_i terms cancel. This corresponds to a_i not being a member of the subset that sums to T. Similarly, if $s_i = 1$, then this corresponds to including the number in the sum, since the a_i term is included twice. Thus there we can construct a subsequence $S_0 := (a_i \in S | s_i = 1) \subseteq S$ of the original sequence such that:

$$\left(\sum_{a \notin S_0} (-a + a) \right) + \sum_{a \in S_0} 2a + \sum_i q_i = 2T + \sum_i q_i$$

Canceling like terms and dividing by 2, we obtain:

$$\sum_{a \in S_0} a = T$$

(⇒) Given $S_0 \subseteq S$ such that $\sum_{a \in S_0} a = T$, then define signs $s_i = -1$ if $a_i \notin S_0$, $s_i = 1$ if $a_i \in S_0$ $(0 \leq i < n)$, and $s_i = +1$ otherwise. By a similar argument to the ⇒ case, this is a solution to the transformed operations puzzle instance.