Lucas Waclawczyk

Decidability of Logical Theories

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Can we mechanize mathematics?

Inhalt

First-Order Logic

Th $(\mathbb{N}, +)$ – A Decidable Theory

Th $(\mathbb{N}, +, \times)$ – An Undecidable Theory

Gödel's Incompleteness Theorem

Conclusion

References

First-Order Logic

Syntax

$$\forall q \, \exists p \, \forall x \, \forall y \; \left[R_1(p,q) \wedge \left(\neg (R_1(x,1) \wedge R_1(y,1)) \vee R_2(x,y,p) \right) \right]$$

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- mostly "sentences" (no free variables)

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Simplified here:

- mostly "sentences" (no free variables)
- only prenex normal form (all quantifiers on the left)

Semantics

$$\forall q\, \exists p\, \forall x\, \forall y\ \left[R_1(p, q) \land \left(\neg (R_1(x, 1) \land R_1(y, 1)) \lor R_2(x, y, p) \right) \right]$$

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$\stackrel{!}{=}$ "There are infinitely many prime numbers."

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$$\text{vs.}\ \ [(x + x = y)\ \vee\ (x \geq y)]$$

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What Is Decidability?

- here: for logic (there is a more general definition)
- let $\mathcal{M}$ be a model, $\varphi \in \mathrm{L}(\mathcal{M})$

$$\mathrm{Th}(\mathcal{M}) \text{ decidable } :=$$

$$\text{there is an algorithm that decides whether } \varphi \text{ is true in } \mathcal{M}$$

Th $(\mathbb{N}, +)$ – A Decidable Theory

Theorem 1

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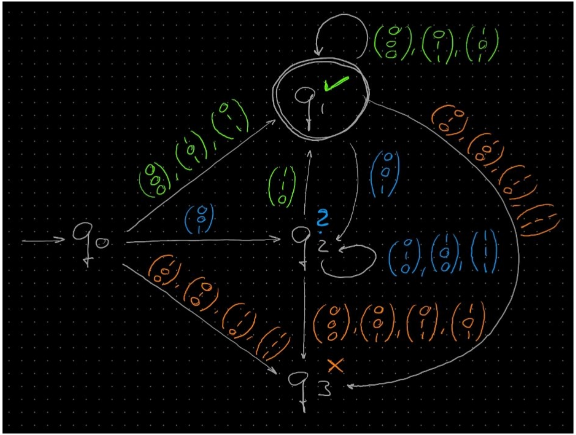
i.e., there is an algorithm that can decide,
whether a sentence $\varphi \in L(\mathbb{N}, +)$ is true or false.

Review: Automata



Example automaton that accepts all correct binary additions

Proof of Theorem 1

Idea: Construct an automaton that accepts an (almost) empty input iff the given sentence is true.

Let $i \in \mathbb{N} \setminus \{0\}$ and define $\Sigma_i := \{0,1\}^i$ and $\Sigma_0 := \{()\}$.
An example for a word in Σ_3:

$$\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \sim \quad \begin{pmatrix}1\\5\\6\end{pmatrix}$$

Proof of Theorem 1

Now, let

- $i \in \{0, \ldots, l\}$
- $\varphi = Q_1 x_1 \ldots Q_l x_l \, [\psi(x_1, \ldots, x_l)] \in \mathrm{L}(\mathbb{N}, +)$ where $Q_1, \ldots, Q_l \in \{\forall, \exists\}$

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- $\varphi_i := Q_{i+1} x_{i+1} \ldots Q_l x_l\, [\psi(x_1, \ldots, x_l)]$
  $\Rightarrow \varphi_0 = \varphi$
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$\Longrightarrow$ φ_l is only a Boolean expression.

Proof of Theorem 1

Construct an automaton A_l that behaves like $\varphi_l = \psi$, meaning A_l accepts exactly the tuples $(a_1, \dots, a_l) \in \mathbb{N}^l$ for which $\varphi_l(a_1, \dots, a_l)$ is true:

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- Take one addition automaton for each addition term in φ_l

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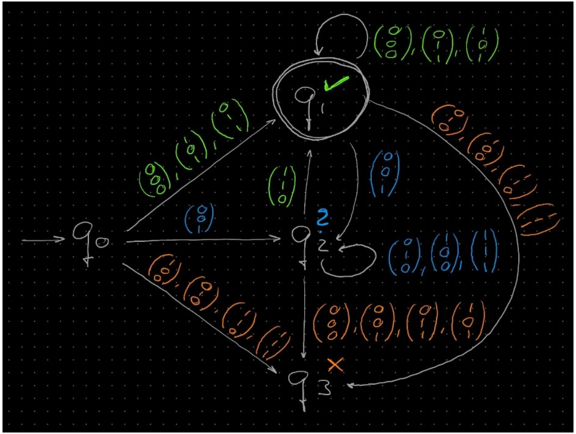

Addition automaton

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- Combine them:
  - automaton product for $\wedge$
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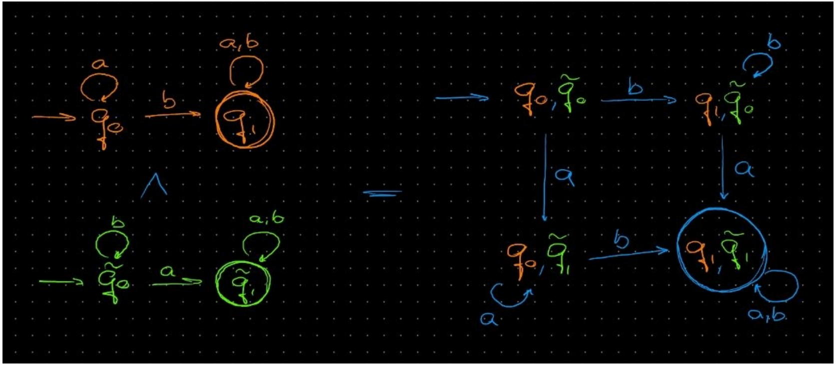

Conjunction of two simple automata

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Important: There is an algorithm that constructs A_l from φ_l

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If $Q_i = \exists$, construct automaton A_i from A_{i+1} by

- copying A_{i+1}
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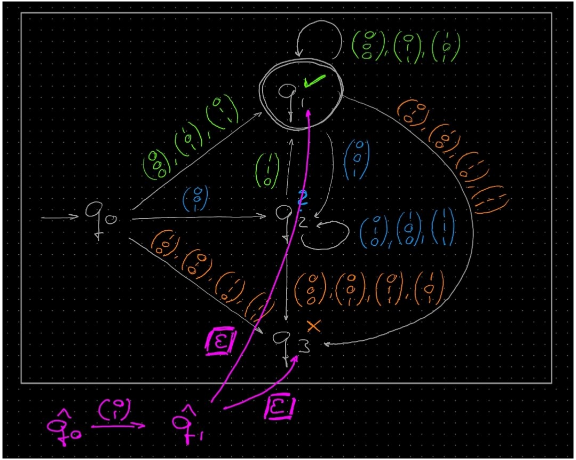

Example construction of non-deterministic guessing

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Let the algorithm return "$\varphi \in \mathrm{Th}(\mathbb{N}, +)$" $\quad \Leftrightarrow \quad A_0$ accepts input ()

Th $(\mathbb{N}, +, \times)$ – An Undecidable Theory

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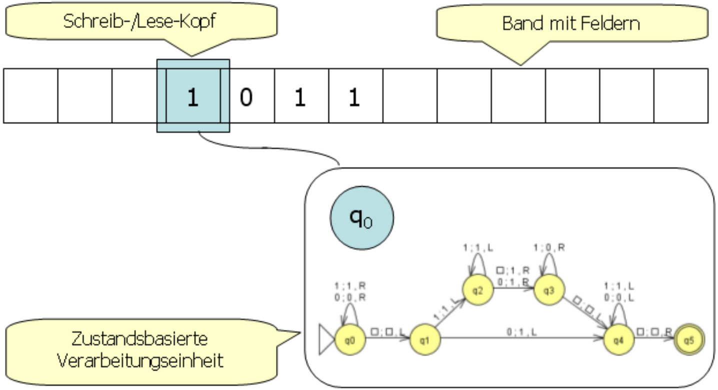
Turing Machines



Example turing machine
source: [2]

Proof Idea for Theorem 2

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Lemmas:

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Lemmas:

1. The provable statements of $\mathrm{Th}(\mathbb{N}, +, \times)$ are Turing recognizable.
   Proof idea: Just try all possible (suitably encoded) proofs.
2. There is a (true) statement in $\mathrm{Th}(\mathbb{N}, +, \times)$ that is not provable.
   Proof idea: Contradiction to Theorem 2 by using 1.

Gödel’s Incompleteness Theorem

A True, Unprovable Statement

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Proof: Let M be a Turing machine that operates as follows.

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$\Longrightarrow$ φ is the wanted statement.

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$\Longrightarrow$ Incompleteness ☺

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$\Longrightarrow$ Mathematics cannot be mechanized.

$\Longrightarrow$ No sound logical system can be complete.

References

References

[1] Martin Alessandro - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=75082873

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