TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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# **Decidability** of Logical Theories

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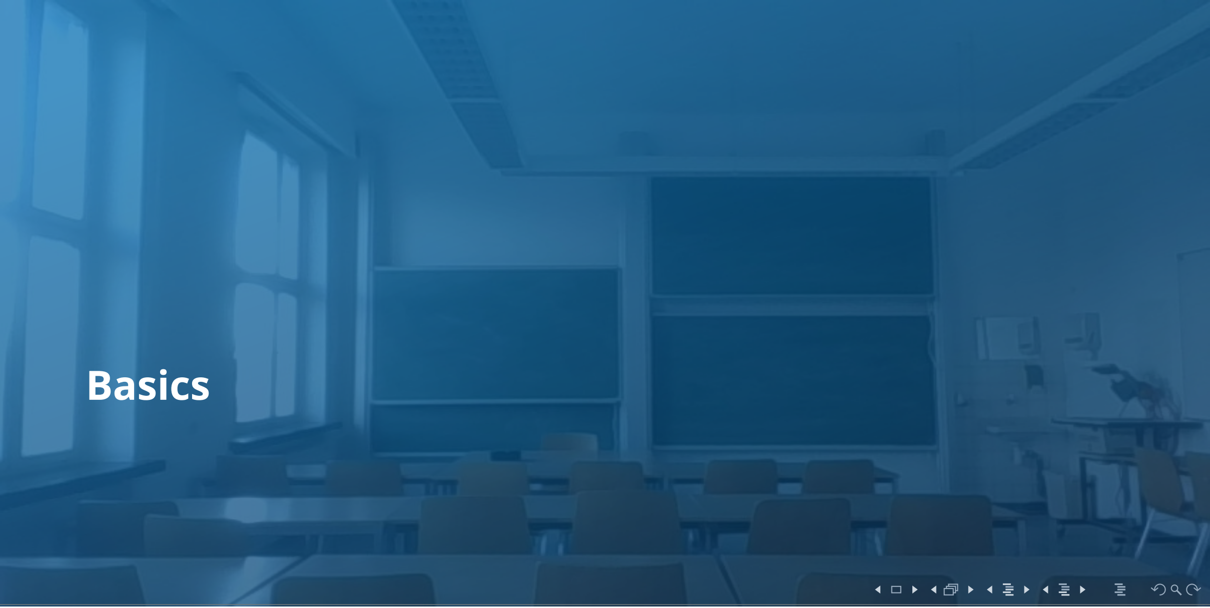
# Inhalt



# Basics

# First-Order Logic

$$\forall q\, \exists p\, \forall x\, \forall y\ \left[ R_1(p,q) \land \left( \neg (R_1(x,1) \land R_1(y,1)) \lor R_2(x,y,p) \right) \right]$$

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$$= \quad \forall q\, \exists p\, \forall x,y\ [p > q \wedge (x,y > 1 \to xy \neq p)]$$

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# What Is Decidability?

For logic (there is a more general definition):

- $\mathcal{M}$ model, $\varphi \in \mathrm{L}(\mathcal{M})$
- $\mathrm{Th}(\mathcal{M})$ decidable $:=$ there is an algorithm that decides whether $\varphi$ is true in $\mathcal{M}$

# Th $(\mathbb{N}, +)$ – A Decidable Theory

# Theorem 1

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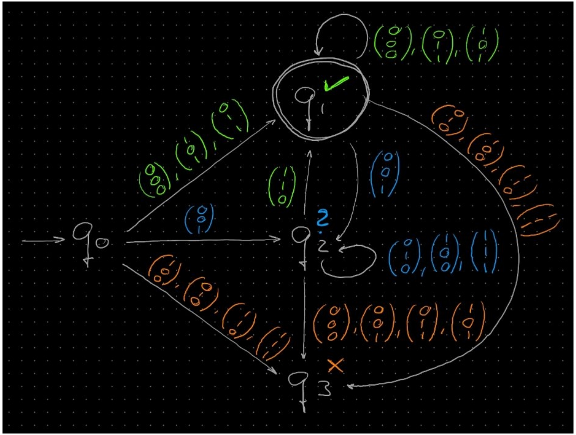
i.e., there is an algorithm that can decide, whether a sentence $\varphi \in \mathrm{L}(\mathbb{N},+)$ is true or false.

# Review: Automata




Example automaton that accepts all correct binary additions

# Proof of Theorem 1

Let $i \in \mathbb{N} \setminus \{0\}$ and define $\Sigma_i := \{0,1\}^i$ and $\Sigma_0 := \{()\}$.
An example for a word in $\Sigma_2$:

$$\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \sim \quad \begin{pmatrix}1\\5\\6\end{pmatrix}$$

# Proof of Theorem 1

Now, let

- $i \in \{1, \ldots, l\}$
- $\varphi = Q_1 x_1 \ldots Q_l x_l \, [\psi(x_1, \ldots, x_l)] \in \mathrm{L}(\mathbb{N}, +)$ where $Q_1, \ldots, Q_l \in \{\forall, \exists\}$
- $\varphi_i := Q_{i+1} x_{i+1} \ldots Q_l x_l \, [\psi(x_1, \ldots, x_l)]$ and by convention $\varphi_l := \psi$
- $\varphi_i(a_1, \ldots, a_i) := Q_{i+1} x_{i+1} \ldots Q_l x_l \, [\psi(a_1, \ldots, a_i, x_{i+1}, \ldots, x_l)]$

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$\implies$ $\varphi_l$ is only a Boolean expression.

# Proof of Theorem 1

- Calculate one addition automaton for each addition term in $\varphi_l$
- Combine them:
  - automaton product for $\wedge$
  - automaton union for $\vee$
  - automaton complement for $\neg$

  in a way that they behave like $\varphi_l$, i. e. that the resulting automaton accepts tuples $(a_1, \dots, a_l) \in \mathbb{N}^l$ for which $\varphi_l(a_1, \dots, a_l)$ is true.

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**Important:** There is an algorithm that constructs $A_l$ from $\varphi_l = \psi$.

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If $Q_i = \exists$, construct automaton $A_i$ from $A_{i+1}$ by

- copying $A_{i+1}$
- adding a new start state and one state for each character in $\Sigma_i$
- making $A_i$ guess the right $a_{i+1}$ non-deterministically

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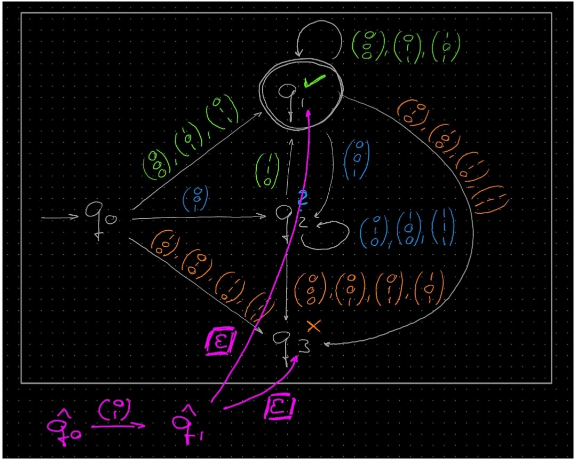

Example construction of non-deterministic guessing

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Let the algorithm return "$\varphi \in \mathrm{Th}\,(\mathbb{N}, +)$" $\quad \Leftrightarrow \quad A_0$ accepts input ()

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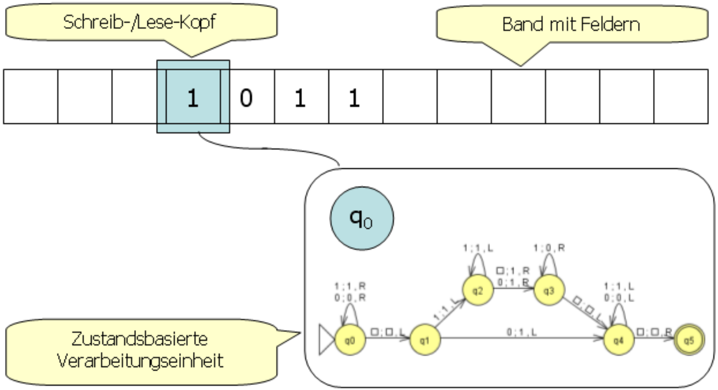
i.e., there is no algorithm that can decide, whether a sentence $\varphi \in$ L$(\mathbb{N}, +)$ is true or false.

# Turing Machines




Example turing machine
source: [2]

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   **Proof idea:** Just try all possible (suitably encoded) proofs.
2. There is a true statement in $\mathrm{Th}(\mathbb{N}, +, \times)$ that is not provable.
   **Proof idea:** Contradiction to Theorem 2 by using 1.

# Gödel’s Incompleteness Theorem

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$\Longrightarrow \quad \neg\exists x[\varphi_{M,0}]$ is the wanted statement.

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$\Longrightarrow$ Mathematics cannot be mechanized.

$\Longrightarrow$ No sound logical system can be complete.

# References

# References


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