TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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Decidability of Logical Theories

Proseminar Theoretical Computer Science // Dresden, May 16, 2020

Can we mechanize mathematics?

Inhalt

First-Order Logic

Syntax

$$\forall q\, \exists p\, \forall x\, \forall y\; \Big[R_1(p,q) \wedge \big(\neg (R_1(x,1) \wedge R_1(y,1)) \vee R_2(x,y,p) \big) \Big]$$

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- Special characters: $[,], (,)$

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$$\forall q\, \exists p\, \forall x\, \forall y \; \left[R_1(p,q) \land \left(\neg(R_1(x,1) \land R_1(y,1)) \lor R_2(x,y,p) \right) \right]$$

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Simplified here:

- mostly "sentences" (no free variables)

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- Special characters: [,], (,)

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- mostly "sentences" (no free variables)
- only prenex normal form (all quantifiers on the left)

Semantics

$$\forall q\, \exists p\, \forall x\, \forall y\ \left[R_1(p,q) \wedge \left(\neg(R_1(x,1) \wedge R_1(y,1)) \vee R_2(x,y,p)\right)\right]$$

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$$\overset{!}{=}\ \text{"There are infinitely many prime numbers."}$$

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$$= \quad \forall q\, \exists p\, \forall x,y\ [p > q \wedge (x,y > 1 \rightarrow xy \neq p)]$$

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- *model* $\mathcal{M}$
  - universe + assignment of relations
  - here $(\mathbb{N},\ >_2,\ (\times \neq)_3)$

Semantics

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$$\text{vs.}\ \ [(x + x = y)\ \vee\ (x \geq y)]$$

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Semantics

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$$\text{vs. } \forall y\, \exists x\ [\neg(xx \neq y)]$$

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What Is Decidability?

- here: for logic (there is a more general definition)
- let $\mathcal{M}$ be a model, $\varphi \in \mathrm{L}(\mathcal{M})$

$$\mathrm{Th}\,(\mathcal{M})\ \text{decidable} \ :=$$

$$\text{there is an algorithm that decides whether } \varphi \text{ is true in } \mathcal{M}$$

Th $(\mathbb{N}, +)$ – A Decidable Theory

Theorem 1

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Th $(\mathbb{N}, +)$ *is decidable.*

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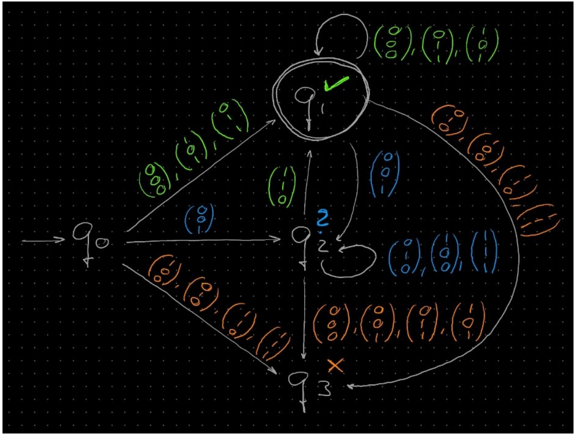
i.e., there is an algorithm that can decide,
whether a sentence $\varphi \in \mathrm{L}(\mathbb{N}, +)$ is true or false.

Review: Automata



Example automaton that accepts all correct binary additions

Proof of Theorem 1

Idea: Construct an automaton that accepts an (almost) empty input iff the given sentence is true.

Let $i \in \mathbb{N} \setminus \{0\}$ and define $\Sigma_i := \{0, 1\}^i$ and $\Sigma_0 := \{()\}$.
An example for a word in Σ_3:

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \begin{pmatrix}0\\0\\1\end{pmatrix} \begin{pmatrix}1\\1\\0\end{pmatrix} \quad \sim \quad \begin{pmatrix}1\\5\\6\end{pmatrix}$$

Proof of Theorem 1

Now, let

- $i \in \{0, \ldots, l\}$
- $\varphi = Q_1 x_1 \ldots Q_l x_l\, [\psi(x_1, \ldots, x_l)] \in \mathrm{L}(\mathbb{N}, +)$ where $Q_1, \ldots, Q_l \in \{\forall, \exists\}$

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- $\varphi_i := Q_{i+1} x_{i+1} \ldots Q_l x_l \, [\psi(x_1, \ldots, x_l)]$

  $\Rightarrow \varphi_0 = \varphi$

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$\Longrightarrow$ φ_l is only a Boolean expression.

Proof of Theorem 1

Construct an automaton A_l that behaves like $\varphi_l = \psi$, meaning A_l accepts exactly the tuples $(a_1, \ldots, a_l) \in \mathbb{N}^l$ for which $\varphi_l(a_1, \ldots, a_l)$ is true:

- Take one addition automaton for each addition term in φ_l
- Combine them:
  - automaton product for $\wedge$
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Important: There is an algorithm that constructs A_l from φ_l

Proof of Theorem 1

If $Q_i = \exists$, construct automaton A_i from A_{i+1} by

- copying A_{i+1}
- adding a new start state and one state for each character in Σ_i
- making A_i guess the right a_{i+1} non-deterministically

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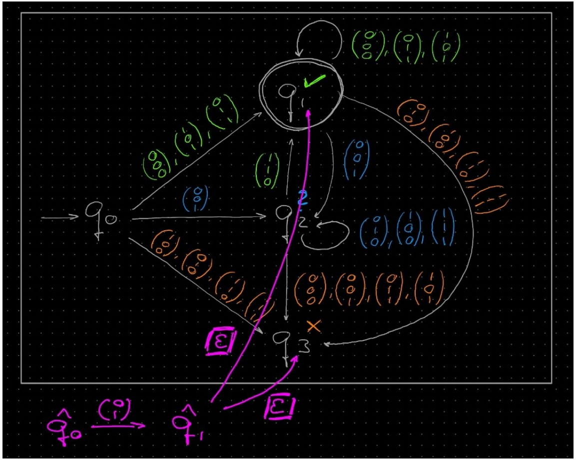

Example construction of non-deterministic guessing

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Let the algorithm return "$\varphi \in \mathrm{Th}(\mathbb{N}, +)$" $\quad \Leftrightarrow \quad A_0$ accepts input $()$

Th $(\mathbb{N}, +, \times)$ – An Undecidable Theory

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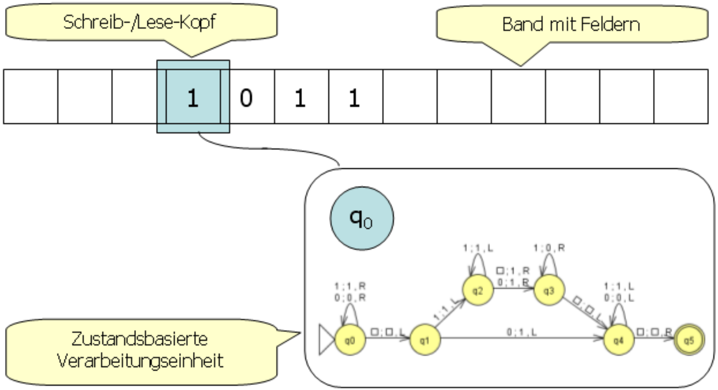
Turing Machines



Example turing machine
source: [2]

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Lemmas:

1. The provable statements of $\mathrm{Th}\,(\mathbb{N}, +, \times)$ are Turing recognizable.
   Proof idea: Just try all possible (suitably encoded) proofs.

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Lemmas:

1. The provable statements of Th $(\mathbb{N}, +, \times)$ are Turing recognizable.
   Proof idea: Just try all possible (suitably encoded) proofs.
2. There is a (true) statement in Th $(\mathbb{N}, +, \times)$ that is not provable.
   Proof idea: Contradiction to Theorem 2 by using 1.

Gödel’s Incompleteness Theorem

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Proof: Let M be a Turing machine that operates as follows.

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$\Longrightarrow$ φ is the wanted statement.

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$\Longrightarrow$ Mathematics cannot be mechanized.

$\Longrightarrow$ No sound logical system can be complete.

References

References

[1] Martin Alessandro - Own work, CC BY-SA 4.0, `https://commons.wikimedia.org/w/index.php?curid=75082873`

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