TECHNISCHE UNIVERSITÄT DRESDEN

DRESDEN concept

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Decidability of Logical Theories

Proseminar Theoretical Computer Science // Dresden, June 17, 2020

Can we mechanize mathematics?

Inhalt

First-Order Logic

Syntax

$$\forall q\, \exists p\, \forall x\, \forall y\ \left[R_1(p,q) \wedge \Big(\neg \big(R_1(x,1) \wedge R_1(y,1) \big) \vee R_2(x,y,p) \Big) \right]$$

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- Special characters: [,], (,)

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$$\forall q\, \exists p\, \forall x\, \forall y \; \left[R_1(p,q) \land \left(\neg (R_1(x,1) \land R_1(y,1)) \lor R_2(x,y,p) \right) \right]$$

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Simplified here:

- mostly "sentences" (no free variables)

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Simplified here:

- mostly "sentences" (no free variables)
- only prenex normal form (all quantifiers on the left)

Semantics

$$\forall q\, \exists p\, \forall x\, \forall y \;\left[R_1(p,q) \wedge \left(\neg (R_1(x,1) \wedge R_1(y,1)) \vee R_2(x,y,p) \right) \right]$$

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$\overset{!}{=}$ "There are infinitely many prime numbers."

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$$= \quad \forall q\, \exists p\, \forall x,y \; [p > q \wedge (x,y > 1 \rightarrow xy \neq p)]$$

- *universe $\mathcal{U}$*
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- *model $\mathcal{M}$*
  - universe + assignment of relations
  - here $(\mathbb{N},\ >_2,\ (\times \neq)_3)$

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$$\forall q\, \exists p\, \forall x,y\ \ [p > q \wedge (x,y > 1 \rightarrow xy \neq p)]$$

$$\text{vs. } [(x + x = y) \vee (x \geq y)]$$

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  - sentences that make sense in $\mathcal{M}$
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$$\text{vs.}\ \ \forall y\, \exists x\ \ [\neg(xx \neq y)]$$

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  - here $\mathrm{L}(\mathbb{N},\ >_2,\ (\times \neq)_3)$
- *theory* $\mathrm{Th}\,(\mathcal{M})$
  - true sentences formed with $\mathcal{M}$
  - here $\mathrm{Th}\,(\mathbb{N},\ >_2,\ (\times \neq)_3)$

What Is Decidability?

- here: for logic (there is a more general definition)
- let $\mathcal{M}$ be a model, $\varphi \in \mathrm{L}(\mathcal{M})$

$$\mathrm{Th}\,(\mathcal{M})\ \text{decidable} \ :=$$

$$\text{there is an algorithm that decides whether } \varphi \text{ is true in } \mathcal{M}$$

Th $(\mathbb{N}, +)$ – A Decidable Theory

Theorem 1

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Th $(\mathbb{N}, +)$ *is decidable.*

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i.e., there is an algorithm that can decide,
whether a sentence $\varphi \in \mathrm{L}(\mathbb{N}, +)$ is true or false.

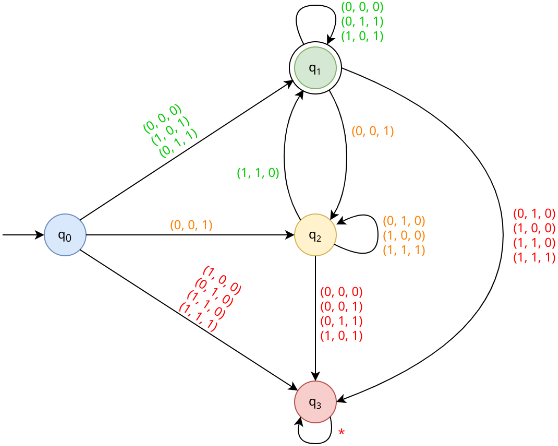
Proof of Theorem 1

Idea: Construct an automaton that accepts an (almost) empty input iff the given sentence is true.

Let $i \in \mathbb{N} \setminus \{0\}$ and define $\Sigma_i := \{0,1\}^i$ and $\Sigma_0 := \{()\}$.
An example for a word in Σ_3:

$$\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} \quad \sim \quad \begin{pmatrix}3\\9\\12\end{pmatrix}$$

Review: Automata



Example automaton that accepts $+_3$ encoded in Σ_3

Proof of Theorem 1

Now, let

- $k \in \{0, \ldots, l\}$
- $\varphi = Q_1 x_1 \ldots Q_l x_l \, [\psi(x_1, \ldots, x_l)] \in \mathrm{L}(\mathbb{N}, +)$ where $Q_1, \ldots, Q_l \in \{\forall, \exists\}$

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- $\varphi_k := Q_{k+1} x_{k+1} \dots Q_l x_l\, [\psi(x_1, \dots, x_l)]$
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- $\varphi_k(a_1, \ldots, a_k) := Q_{k+1} x_{k+1} \ldots Q_l x_l \, [\psi(a_1, \ldots, a_k, x_{k+1}, \ldots, x_l)]$
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$\Longrightarrow$ φ_l is only a Boolean expression.

Proof of Theorem 1

Construct an automaton A_l that behaves like $\varphi_l = \psi$, meaning A_l accepts exactly the tuples $(a_1, \ldots, a_l) \in \mathbb{N}^l$ for which $\varphi_l(a_1, \ldots, a_l)$ is true:

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- Take one addition automaton for each addition term in φ_l

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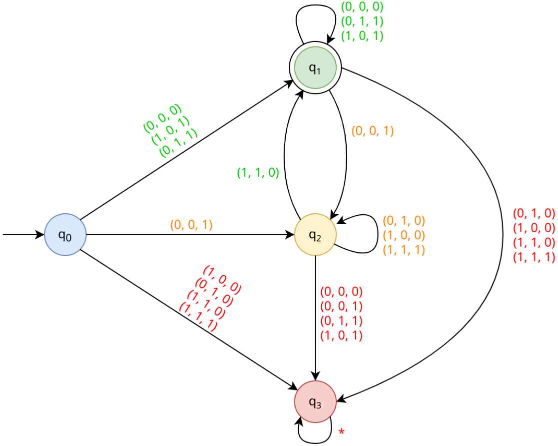

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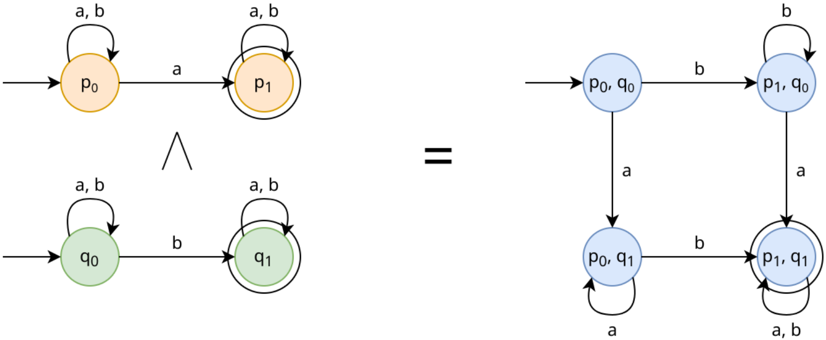

Conjunction of two simple automata

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  in a way that they behave like φ_l.

Important: There is an algorithm that constructs A_l from φ_l.

Proof of Theorem 1

If $Q_l = \exists$, construct automaton A_{l-1} from A_l by

- copying states of A_l
- adding a new starting state
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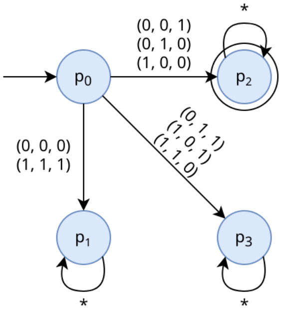

Example construction of non-deterministic guessing

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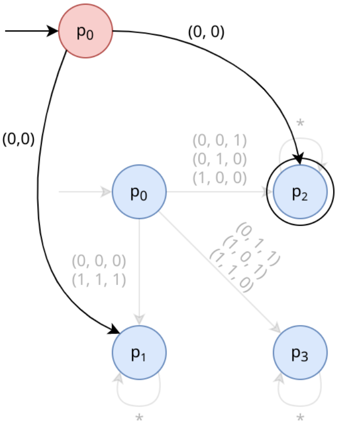

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Let the algorithm return "$\varphi \in \mathrm{Th}(\mathbb{N}, +)$" $\Leftrightarrow$ A_0 accepts input $()$

$\mathrm{Th}(\mathbb{N}, +, \times)$ – An Undecidable Theory

Theorem 2

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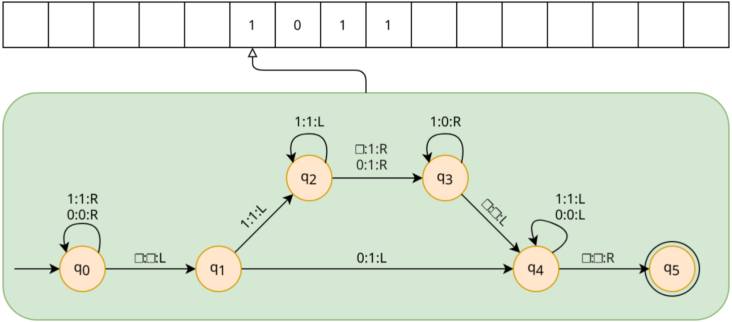
Turing Machines



Example turing machine
adapted from: [2]

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$\Longrightarrow$ $\quad$ The formulas $\exists x\,[\varphi_{M,w}(x)] \in \mathrm{Th}\,(\mathbb{N}, +, \times)$ are decidable.

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Lemmas:

1. The provable statements of $\mathrm{Th}\,(\mathbb{N}, +, \times)$ are Turing recognizable.
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Lemmas:

1. The provable statements of Th $(\mathbb{N}, +, \times)$ are Turing recognizable.
   Proof idea: Just try all possible (suitably encoded) proofs.
2. There is a (true) statement in Th $(\mathbb{N}, +, \times)$ that is not provable.
   Proof idea: Contradiction to Theorem 2 by using 1.

Gödel's Incompleteness Theorem

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- Delete the input
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$\Longrightarrow \quad \varphi$ is the wanted statement.

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$\Longrightarrow$ No sound arithmetical system can be complete.

$\Longrightarrow$ Mathematics cannot be mechanized.

References

References

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