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Analytic solutions for optimal statistical arbitrage trading

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ABSTRACT

In this paper we derive analytic formulae for statistical arbitrage trading where the security price follows an Ornstein–Uhlenbeck process. By framing the problem in terms of the first-passage time of the process, we derive expressions for the mean and variance of the trade length and the return. We examine the problem of choosing an optimal strategy under two different objective functions: the expected return, and the Sharpe ratio. An exact analytic solution is obtained for the case of maximising the expected return.

1. Introduction

Statistical arbitrage trading has previously been examined by various authors [1–6]. The goal of this type of trading is to develop highly automated trading strategies that take a probabilistic approach to trading. These strategies engage in high frequency trading using algorithms based on stochastic methods to identify price inefficiencies in the market. The use of such an approach has increased substantially in recent years as a greater understanding of the stochastic behavior of financial markets has developed through empirical investigation and phenomenological modeling. This has largely been driven by an increase in interdisciplinary research by physicists and has allowed for the development of increasingly sophisticated models for price behavior.

A common approach when performing this type of trading is to construct a stationary, mean reverting synthetic asset as a linear combination of securities. One example is the method of pairs trading which has been the focus of several recent studies [2,3,7]. This approach allows for the construction of a trading strategy where trades are entered when the process reaches an extreme value and exited when the process reverts to some equilibrium value.

Most commonly used methods for investing do not address the significance of the role of time, mainly due to the fact that modern portfolio theory [8,9] is based on single-period models. However, when considering statistical based strategies that engage in high frequency trading, the time between trades, i.e. trading frequency, becomes an important quantity to consider. The importance of the role that time plays in financial markets has been explored by many studies, for instance: data seasonality [10,11], market activity time [12–14], and waiting times between orders and trades [15–19]. In the context of trading strategies it is crucial to consider not only the return per trade but also the time over which the returns take place. In such a setting it is imperative to consider transaction costs, because whether inefficiencies can be successfully traded depends on the cost of trading. Continuous time trading strategies were presented in Ref. [5] to provide a mathematical framework for the construction and analysis of statistical arbitrage methods. It was shown that in this framework that optimal strategies balance return per trade and transaction cost with the stochastic trading frequency.

When implementing these types of strategies, speed of computation is vital, as calculations are often required to be performed in real time. A high frequency trading desk may perform thousands of transactions each day on hundreds of different securities. In this situation, numerical methods, such as simulation or quadrature, may not be fast enough to update

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calculations within the required time constraints. In such an environment there is a need for analytic and approximate solutions.

In this paper we present analytic formulae and solutions for calculating optimal statistical arbitrage strategies with transaction costs. We assume that the traded security is described by an Ornstein–Uhlenbeck process [20]. The resulting analysis provides a mathematical model which can be used to explore the relationships between variables and offer insight into the dynamics of trading strategies. We construct a continuous trading strategy for the Ornstein–Uhlenbeck process and express the trade length and return of the strategy in terms of the first-passage time of the process. Using known solutions for the first-passage time, we derive analytic solutions for: the expected return; the variance of the return; and the expected trade length of the strategy. Optimal trading strategies can be found by constructing objective functions that are expressed in terms of the expected value and variance of the return. We derive an analytic solution for the strategy that maximises expected return. The solution is shown to satisfy a real valued integral equation. We present an approximate solution via a Taylor series expansion that is in close agreement with the exact solution. We prove that for optimal trading, the trading bands are symmetric about the mean of the traded security. We formulate expressions for the Sharpe ratio [9] of the strategy and show how it can be maximised. Results are illustrated using an example from an earlier work, in which, numerical methods were used to evaluate first-passage time distributions. The model allows for the examination of the impact of transaction cost on trade length and return, in order to determine whether a strategy can be successful.

The rest of the paper is as follows. In Section 2 we define the continuous time trading strategy for the Ornstein–Uhlenbeck process. The trade length, expected return, and variance of the return are formulated in terms of the first-passage time of the Ornstein–Uhlenbeck process. We use known expressions for the moments of the first-passage time to derive analytic formulae for the mean and variance of the trade length and return. In Section 3 we construct optimal trading strategies for the Ornstein–Uhlenbeck process by maximising the expected return and maximising the Sharpe ratio. An analytic solution to the problem is obtained in the case of maximising the expected return. In Section 4 we present the results for the optimal strategies applied to real world data. Section 5 concludes and summarises the main results of the paper.

2. Continuous time trading

A continuous trading strategy comprises a sequence of individual trades performed on a continuous time stochastic process. Consequently, many of the important quantities related to the trading strategy are functions of the frequency at which these trades take place. The trading frequency is specified by how many times the strategy trades per unit of time. This value is dependent on how long it takes in total to move from one trade entry point to the next, passing though the exit point along the way. We model the price of the traded security p_t as,

$$p_t = \mathrm{e}^{X_t}; \qquad X_{t_0} = x_0, \tag{1}$$

where X_t satisfies the following stochastic differential equation,

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \eta \mathrm{d}W_t, \tag{2}$$

where $\alpha > 0$, $\eta > 0$, and W_t is the Wiener process. A continuous time trading strategy is defined by entering a trade when $X_t = a$, exiting the trade at $X_t = m$, and waiting until the process returns to $X_t = a$, to complete the trading cycle. Such a strategy can be thought of as periodic, since the actions are repeated between trade entry points. However, since X_t is a stochastic process, the time taken to complete the trade cycle will be a random variable $\mathcal{T}$. We refer to $\mathcal{T}$ as the total trade length. Thus, the behavior of $\mathcal{T}$ will largely determine the properties of the strategy.

We assume that $a < m$ and decompose the total trade length into sub-intervals,

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2,$$

where $\mathcal{T}_1$ is the time taken for the process to travel from a to m and $\mathcal{T}_2$ is the time taken from m back to a. Here the variables $\mathcal{T}_1, \mathcal{T}_2$ can be identified as first-passage times for the process X_t. Since X_t is a Markov process, the passage times $\mathcal{T}_1, \mathcal{T}_2$ are independent. Furthermore, it is well known that the first-passage time for the Ornstein–Uhlenbeck process has finite mean and variance [21]. Thus the mean and variance of $\mathcal{T}$ may be written as,

$$\mathbb{E}[\mathcal{T}] = \mathbb{E}[\mathcal{T}_1] + \mathbb{E}[\mathcal{T}_2], \tag{3}$$

$$\mathbb{V}[\mathcal{T}] = \mathbb{V}[\mathcal{T}_1] + \mathbb{V}[\mathcal{T}_2]. \tag{4}$$

Expressions for the return per unit time and variance of the return per unit time of the strategy, can be formulated in terms of the trading frequency. Let $r(a, m, c)$ be the return per trade as a function of entry and exit levels and transaction cost. Then, the expected value and the variance of the return per unit time for the strategy are given by,

$$\mu(a, m, c, t) = r(a, m, c)\mathbb{E}\left[N_t\right]/t, \tag{5}$$

$$\sigma^2(a, m, c, t) = r(a, m, c)^2\mathbb{V}\left[N_t\right]/t, \tag{6}$$

where the variable N_t represents the number of trades over a time interval of length t. In this representation, N_t is the counting process for a renewal process where the interarrival times are given by the trade cycle length $\mathcal{T}$. Since the value of $\mathcal{T}$ is independent and identically distributed for each trade cycle, the renewal density theorem and the central limit theorem [22] give,

$$\lim_{t\to\infty} \mathbb{E}[N_t] = t/\mathbb{E}[\mathcal{T}], \tag{7}$$

$$\lim_{t\to\infty} \mathbb{V}[N_t] = \mathbb{V}[\mathcal{T}]t/\mathbb{E}[\mathcal{T}]^3. \tag{8}$$

The return for a single trade can be expressed as a function of the exit and entry values, minus the total transaction costs associated with the trade. Since the variable X_t represents the log-price, the function $r(a, m, c) = (m - a - c)$ gives the continuously compound rate of return for a single trade accounting for transaction cost. It is clear that, for a strategy to be profitable, the return gained by moving from a to m must first exceed the transaction costs. By combining the return per trade with Eqs. (5)–(8), we obtain,

$$\mu(a, m, c) = r(a, m, c)/\mathbb{E}\left[\mathcal{T}\right], \tag{9}$$

$$\sigma^2(a, m, c) = r(a, m, c)^2\mathbb{V}\left[\mathcal{T}\right]/\mathbb{E}\left[\mathcal{T}\right]^3. \tag{10}$$

Note that, since the process X_t is stationary, the return per trade is deterministic. However the time frame over which the return is realised is stochastic. Depending on the properties of X_t and $\mathcal{T}$, a trade may take a long time before reaching the exit level and experience a significant deviation away from the exit level during that time. The expressions for the strategy return and variance may be calculated using well-known identities for the first-passage time of the Ornstein–Uhlenbeck process.

The first-passage time of the Ornstein Uhlenbeck process has been studied extensively in the literature. It is used in a wide variety of fields such as: biology [23]; disease modeling for HIV infection [24]; and finance in the context modeling default risk [25,26]. It is well known that no closed form solution for the first-passage time density currently exist, except for the special case when the system is symmetric [27]. However, the moments of the first-passage time have been investigated and calculated for various special cases [21,28–31]. Due to our formulation of the problem, we are able to use these known identities to evaluate both the expected trade length and the variance of the trade length. By applying the transformation $Y_t = X_t\sqrt{2\alpha}/\eta$ via Itô's lemma and performing the time dilation $\tau = \alpha t$ we can transform the problem to the dimensionless system,

$$\mathrm{d}Y_\tau = -Y_\tau \mathrm{d}\tau + \sqrt{2}\mathrm{d}W_\tau,$$

with trade entry level $\bar{a} = a\sqrt{2\alpha}/\eta$, exit level $\bar{m} = m\sqrt{2\alpha}/\eta$, and transaction cost $\bar{c} = c\sqrt{2\alpha}/\eta$. This transformation expresses the amplitude of the process in terms of the steady state standard deviation, $\theta = \eta/\sqrt{2\alpha}$. Note that the trade length rescales to $T = \alpha\mathcal{T}$. To simplify analysis we will work with the dimensionless system. For a process started at $y = y_0$ with barrier at $y = b$ we define the first-passage time $T_{b,y_0} = \inf\{t \geq 0 : Y_t > b|Y_0 = y_0\}$. The first-passage time has been shown by Refs. [29,30] to have mean,

$$\mathbb{E}[T_{b,y_0}] = \begin{cases} \phi_1(b) - \phi_1(y_0); & y_0 < b, \\ \phi_1(-b) - \phi_1(-y_0); & y_0 > b, \end{cases} \tag{11}$$

where

$$\phi_1(z) = \frac{1}{2}\sum_{k=1}^{\infty} \Gamma(k/2)\left(\sqrt{2}z\right)^k/k!.$$

Similarly the variance of the first-passage time is due to Ref. [31] and is given by,

$$\mathbb{V}[T_{0,y_0}] = \begin{cases} \phi_1(b)^2 - \phi_2(b) + \phi_2(y_0) - \phi_1(y_0)^2; & y_0 < b, \\ \phi_1(-b)^2 - \phi_2(-b) + \phi_2(-y_0) - \phi_1(-y_0)^2; & y_0 > b, \end{cases} \tag{12}$$

where

$$\phi_2(z) = \frac{1}{2}\sum_{k=1}^{\infty} \Gamma\left(k/2\right)\Psi(k/2)\left(\sqrt{2}z\right)^k/k!,$$

and $\Psi(x) = \psi(x) - \psi(1)$ and $\psi(x)$ is the digamma function,

$$\psi(z) = \frac{1}{\Gamma(z)}\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(z).$$

Since we are using an Ornstein–Uhlenbeck process that is symmetric about $Y_t = 0$ we also have that $T_{b,0} = T_{-b,0}$ and $T_{0,y_0} = T_{0,-y_0}$. Using these results with $\bar{a} < \bar{m}$ the required first-passage times are,

$$T_1 = T_{\bar{m},\bar{a}},$$

$$T_2 = T_{\bar{a},\bar{m}} = T_{-\bar{a},-\bar{m}}.$$

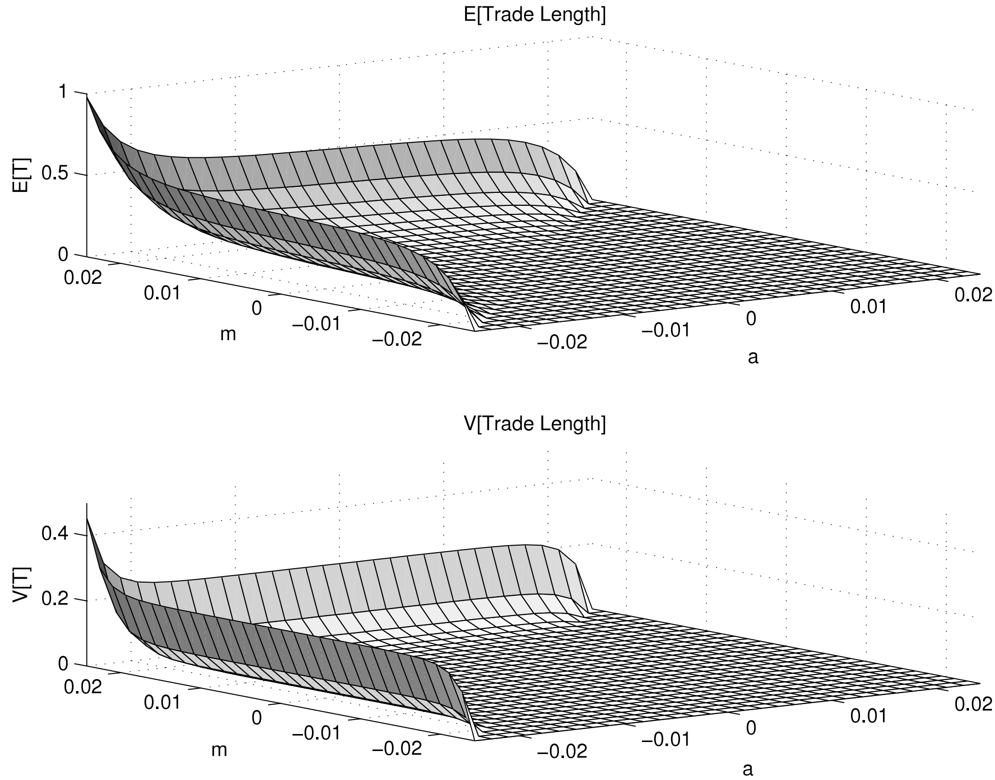

Fig. 1. The expected value and the variance of the trade length as a function of trade entry and exit level. This example uses parameters $\alpha = 180.9670$, $\eta = 0.1538$, $c = 0.001$.

From Eqs. (3) and (11) we can write the expected trade length as,

$$\mathbb{E}[T] = \pi\left(\mathrm{Erfi}\left(\bar{m}/\sqrt{2}\right) - \mathrm{Erfi}\left(\bar{a}/\sqrt{2}\right)\right),$$

where $\mathrm{Erfi}(x) = \mathrm{i}\mathrm{Erf}(\mathrm{i}x)$ is the imaginary error function. Likewise from Eqs. (4) and (12) the variance is,

$$\mathbb{V}[T] = w_1(\bar{m}) - w_1(\bar{a}) - w_2(m) + w_2(\bar{a}),$$

where

$$w_1(z) = \left(\frac{1}{2}\sum_{k=1}^{\infty}\Gamma\left(k/2\right)\left(\sqrt{2}z\right)^k/k!\right)^2 - \left(\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\Gamma\left(k/2\right)\left(\sqrt{2}z\right)^k/k!\right)^2$$

$$w_2(z) = \sum_{k=1}^{\infty}\Gamma\left((2k-1)/2\right)\Psi\left((2k-1)/2\right)\left(\sqrt{2}z\right)^{(2k-1)}/(2k-1)!.$$

Hence, for the continuous trading strategy under the parametric system of Eqs. (1) and (2) we can obtain the closed form solution for the expected trade length as,

$$\mathbb{E}\left[\mathcal{T}\right] = \frac{\pi}{\alpha}\left(\mathrm{Erfi}\left(m\sqrt{\alpha}/\eta\right) - \mathrm{Erfi}\left(a\sqrt{\alpha}/\eta\right)\right). \tag{13}$$

Similarly the closed form solution of the trade length variance is,

$$\mathbb{V}[\mathcal{T}] = \frac{\left(w_1\left(\frac{m\sqrt{2\alpha}}{\eta}\right) - w_1\left(\frac{a\sqrt{2\alpha}}{\eta}\right) - w_2\left(\frac{m\sqrt{2\alpha}}{\eta}\right) + w_2\left(\frac{a\sqrt{2\alpha}}{\eta}\right)\right)}{\alpha^2}. \tag{14}$$

Fig. 1 displays surface plots for the expected trade length and the variance of the trade length for different entry and exit levels. This figure indicates the nonlinear behavior of the trade length as the trading levels become further apart. Using Eqs. (9) and (13) we can obtain the analytic form of the expected return for the strategy,

$$\mu(a, m, c) = \frac{\alpha(m-a-c)}{\pi\left(\mathrm{Erfi}\left(\frac{m\sqrt{\alpha}}{\eta}\right) - \mathrm{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)\right)}. \tag{15}$$

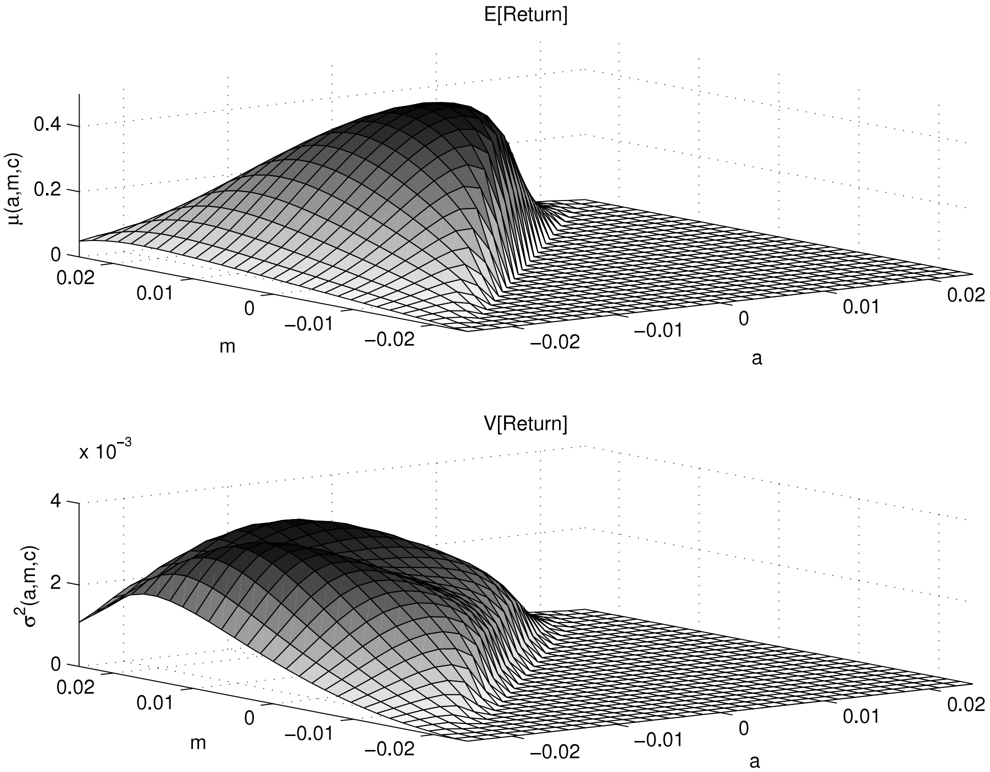

Fig. 2. The expected return and the variance of the return as a function of trade entry and exit level in the positive half place. This example uses parameters $\alpha = 180.9670$, $\eta = 0.1538$, $c = 0.001$.

Likewise from Eqs. (10) and (14), the variance of the strategy return is,

$$\sigma^2(a, m, c) = \alpha(m - a - c)^2 \frac{\left(w_1\left(\frac{m\sqrt{2\alpha}}{\eta}\right) - w_1\left(\frac{a\sqrt{2\alpha}}{\eta}\right) - w_2\left(\frac{m\sqrt{2\alpha}}{\eta}\right) + w_2\left(\frac{a\sqrt{2\alpha}}{\eta}\right)\right)}{\pi^3\left(\mathrm{Erfi}\left(\frac{m\sqrt{\alpha}}{\eta}\right) - \mathrm{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)\right)^3}. \tag{16}$$

These analytic formulae, for the mean and variance, allow us to determine the properties of the strategy in terms of the trade entry and exit levels and transaction costs. Fig. 2 displays a surface plot of the expected return and variance of the return as a function of entry and exit level with a fixed transaction cost. Although the return per trade $r(a, m, c)$ is linear, it is apparent that the nonlinear behavior of the trading frequency has a strong influence on the profitability of the trading strategy. It is also interesting that the variance of the return displays a bimodal shape which is symmetric around $m = -a$. The equations for the mean and variance of the return allow us to determine the trading bands that optimise the trading strategy.

3. Optimal strategies

To calculate an optimal trading strategy, we seek to identify trade entry and exit levels that maximise some objective function for a given transaction cost. Suitable objective functions can be constructed as functions of the strategies expected return and variance. In this section we present two examples of optimal strategy choice: maximising the expected return; and maximising the Sharpe ratio.

3.1. Maximum expected return

We wish to find the values of a and m that maximise the expected return $\mu(a, m, c)$ given in Eq. (15). Differentiating Eq. (15) with respect to a and m and setting the derivatives equal to zero we obtain the following equations,

$$\sqrt{\frac{4\pi}{\alpha\eta^2}}\mathrm{e}^{\frac{\alpha a^2}{\eta^2}}(m - a - c) - \frac{\pi}{\alpha}\left(\mathrm{Erfi}\left(\frac{m\sqrt{\alpha}}{\eta}\right) - \mathrm{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)\right) = 0,$$

$$\sqrt{\frac{4\pi}{\alpha\eta^2}}\mathrm{e}^{\frac{\alpha m^2}{\eta^2}}(m - a - c) - \frac{\pi}{\alpha}\left(\mathrm{Erfi}\left(\frac{m\sqrt{\alpha}}{\eta}\right) - \mathrm{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)\right) = 0.$$

It is clear that $\frac{\partial\mu}{\partial a} = \frac{\partial\mu}{\partial m} = 0$ requires that $a^2 = m^2$. Since we have assumed that $a < m$, this implies that $a < 0$ and $m = -a$. Thus, the optimal entry and exit bands must be symmetric about zero. This is a somewhat significant result since the paradigm approach for trading a mean reverting process is to use asymmetric bands, entering a trade when the process exhibits a two standard deviation event and exiting when it returns to zero [7]. Accordingly the maximum expected return is given by,

$$\mu^*(a, c) = \frac{\alpha(2a + c)}{2\pi \operatorname{Erfi}\left(a\sqrt{\alpha}/\eta\right)},$$

where the optimal value of a satisfies the following equation,

$$e^{\frac{\alpha a^2}{\eta^2}}(2a + c) - \eta\sqrt{\frac{\pi}{\alpha}} \operatorname{Erfi}\left(a\sqrt{\alpha}/\eta\right) = 0. \tag{17}$$

It is relatively straightforward to find the root of this equation numerically. As an alternative to the numerical approach, we can construct an approximate solution of the above equation via Taylor series. Taking a third order Taylor series of Eq. (17) around $a = 0$ and simplifying, we obtain,

$$c + \frac{\alpha c}{\eta^2}a^2 + \frac{4\alpha}{3\eta^2}a^3 = 0.$$

This cubic equation can be solved and the appropriate negative real root chosen to yield the following approximation that is valid for small values of $a\sqrt{\alpha}/\eta$,

$$a = -\frac{c}{4} - \frac{c^2\alpha}{4\left(c^3\alpha^3 + 24c\alpha^2\eta^2 - 4\sqrt{3c^4\alpha^5\eta^2 + 36c^2\alpha^4\eta^4}\right)^{1/3}} - \frac{\left(c^3\alpha^3 + 24c\alpha^2\eta^2 - 4\sqrt{3c^4\alpha^5\eta^2 + 36c^2\alpha^4\eta^4}\right)^{1/3}}{4\alpha}. \tag{18}$$

3.2. Maximum Sharpe ratio

In the previous example, maximising the expected return puts no constraints on the risk associated with the return. To account for risk, we consider maximising the Sharpe ratio. The Sharpe ratio was proposed by Ref. [9] in the context of single-period portfolio theory. By considering the variance as a measure of risk, the ratio acts as a measure for the relative return per unit of risk. The Sharpe ratio is defined as the expected return in excess of the risk-free rate normalised by the standard deviation of the return. In the context of continuous time trading we define an analogous Sharpe ratio as,

$$S = (\mu - r^*)/\sigma, \tag{19}$$

where

$$r^* = r_f/\mathbb{E}[\mathcal{T}],$$

and r_f is the risk-free rate of return. Thus, r^* corresponds to the risk-free rate over the same time period as the return produced by the trading strategy. This provides a measure of the strategy's efficiency in generating returns in excess of a risk-free investment on an equivalent time scale. Using Eqs. (19), (15), and (16) we have,

$$S(a, m, c, r_f) = (m - a - c - r_f)\sqrt{\frac{\mathbb{E}[\mathcal{T}]}{(m - a - c)^2\mathbb{V}[\mathcal{T}]}}.$$

Inserting the corresponding values for the expected trade length and variance of trade length we obtain,

$$S(a, m, c, r_f) = \frac{m - a - c - r_f}{\sqrt{(m - a - c)^2}}\sqrt{\frac{\alpha\pi\left(\operatorname{Erfi}\left(\frac{m\sqrt{\alpha}}{\eta}\right) - \operatorname{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)\right)}{\left(w_1\left(\frac{m\sqrt{2\alpha}}{\eta}\right) - w_1\left(\frac{a\sqrt{2\alpha}}{\eta}\right) - w_2\left(\frac{m\sqrt{2\alpha}}{\eta}\right) + w_2\left(\frac{a\sqrt{2\alpha}}{\eta}\right)\right)}}.$$

Fig. 3 displays a plot of Sharpe ratio as a function of trade entry and exit bands. By considering the expected value and the variance of $\mathcal{T}$ as functions of a and m, we can use the derivatives, $\frac{\partial S}{\partial a} = \frac{\partial S}{\partial m} = 0$, to obtain the following equations,

$$-\frac{1}{m - a - c - r_f} + \frac{\frac{\partial}{\partial a}\mathbb{E}[\mathcal{T}]}{2\mathbb{E}[\mathcal{T}]} + \frac{1}{m - a - c} - \frac{\frac{\partial}{\partial a}\mathbb{V}[\mathcal{T}]}{2\mathbb{V}[\mathcal{T}]} = 0,$$

$$-\frac{1}{m - a - c - r_f} - \frac{\frac{\partial}{\partial m}\mathbb{E}[\mathcal{T}]}{2\mathbb{E}[\mathcal{T}]} + \frac{1}{m - a - c} + \frac{\frac{\partial}{\partial m}\mathbb{V}[\mathcal{T}]}{2\mathbb{V}[\mathcal{T}]} = 0.$$

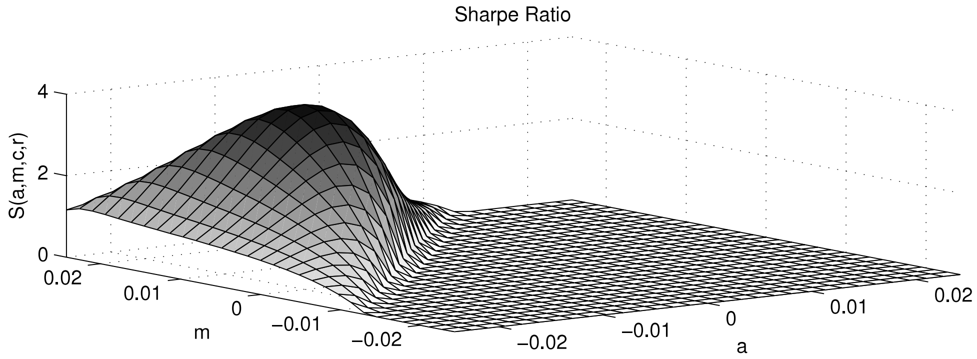

Fig. 3. The Sharpe ratio as a function of trade entry and exit level in the positive half place. This example uses parameters $\alpha = 180.9670$, $\eta = 0.1538$, $c = 0.001$, $r_f = 0.01$.

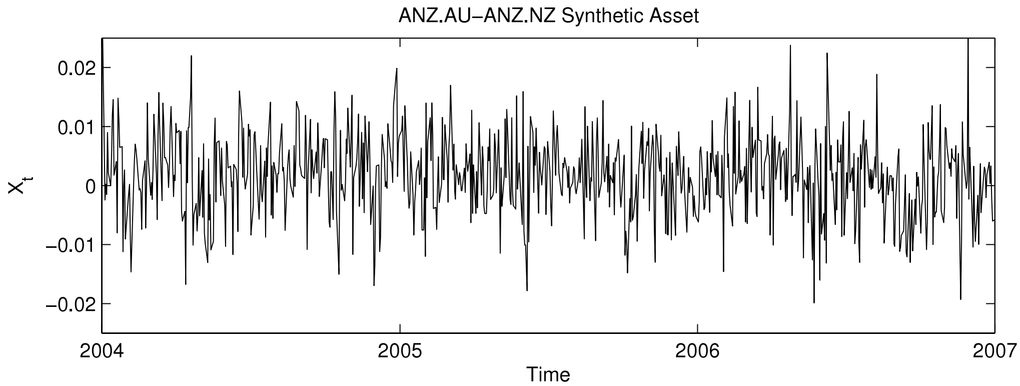

Fig. 4. The synthetic asset constructed from dual-listed ANZ securities on the Australian and New Zealand stock exchanges.

From the equations for the expected time Eq. (13) and variance Eq. (14) we have,

$$\frac{\partial}{\partial a}\mathbb{E}[\mathcal{T}] = -\frac{\partial}{\partial m}\mathbb{E}[\mathcal{T}],$$
$$\frac{\partial}{\partial a}\mathbb{V}[\mathcal{T}] = -\frac{\partial}{\partial m}\mathbb{V}[\mathcal{T}].$$

Thus, it is clear that we must again have $a < 0$ and $m = -a$ as a condition to optimise the strategy. By noting that $w_1(-z) = -w_1(z)$ and $w_2(-z) = -w_2(z)$, these results allow us to express the maximum Sharpe ratio in terms of one less variable,

$$S^*(a, c, r_f) = \frac{-(2a + c + r_f)\sqrt{\alpha\pi\,\mathrm{Erfi}\left(\frac{a\sqrt{\alpha}}{\eta}\right)}}{\sqrt{(2a + c)^2\left(w_1\left(\frac{a\sqrt{2\alpha}}{\eta}\right) + w_2\left(\frac{a\sqrt{2\alpha}}{\eta}\right)\right)}}, \tag{20}$$

which can be maximised to find a. Although these equations are less tractable than those obtained when maximising the expected return, there is still a great advantage in having an analytic form for the Sharpe ratio. To find the maximum one need only apply a straightforward optimisation routine to Eq. (20).

4. Results

Here we present the results for the optimisation discussed in the previous section. To illustrate the results we use the example presented in Ref. [5] which constructs a stationary synthetic asset as a linear combination of the dual-listed securities for ANZ Banking Group Ltd (ANZ.AX, ANZ.NZ). These securities are traded on the Australian and New Zealand stock exchanges simultaneously. Fig. 4 displays a plot of the synthetic asset. A full description of this example is given by Ref. [5], which finds the parameters for the Ornstein–Uhlenbeck process to be $\alpha = 180.9670$ and $\eta = 0.1538$.

The expected return of the strategy is maximised via Eq. (17) in order to obtain the optimal value of a for a given transaction cost c. Fig. 5 displays the solution to Eq. (17) along with the approximation formula for a. This figure shows

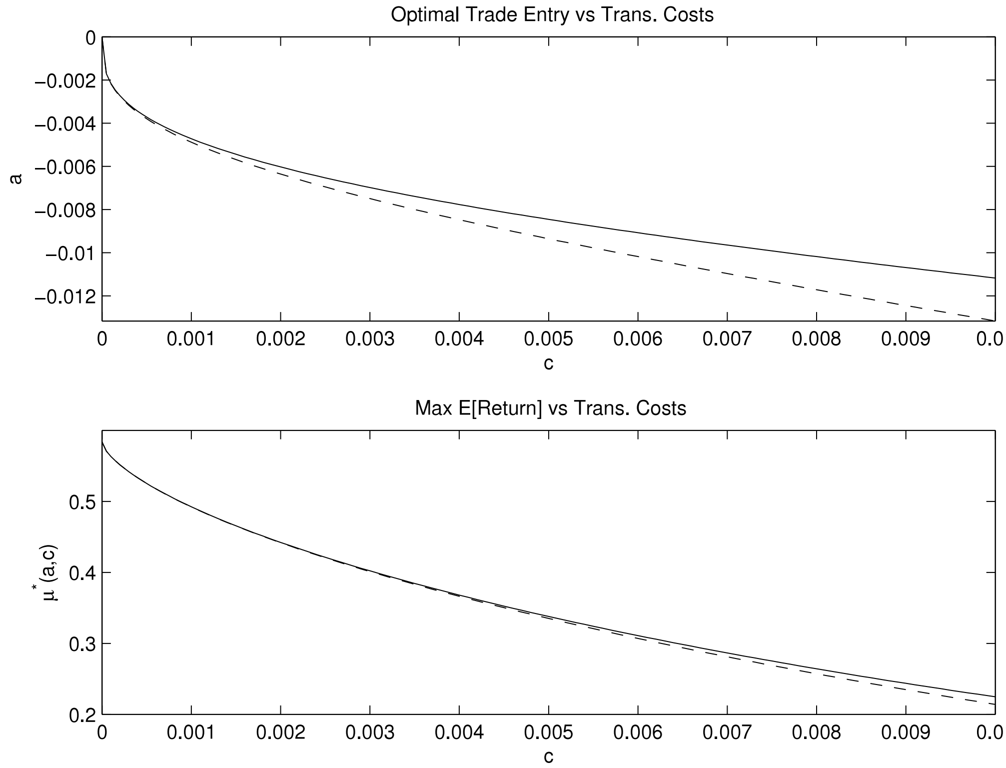

Fig. 5. The optimal trading bands and corresponding maximum return of the strategy as a function of transaction cost. The solid line represents the numerical solution while the dashed line represents the analytic approximation.

Table 1
Results for maximising the expected return with $\alpha = 180.9670$, $\eta = 0.1538$.

c	a	a^{approx}	μ^*	μ^{approx}
0.0010	−0.0047152	−0.0048750	0.49236	0.492250
0.0020	−0.0060232	−0.0063549	0.44219	0.441760
0.0030	−0.0069778	−0.0074910	0.40214	0.401160
0.0040	−0.0077651	−0.0084682	0.36797	0.366200
0.0050	−0.0084523	−0.0093531	0.33790	0.335130
0.0060	−0.0090721	−0.0101780	0.31095	0.306960
0.0150	−0.0133720	−0.0165920	0.14860	0.127780
0.0175	−0.0143940	−0.0182670	0.11961	0.094441
0.0200	−0.0153880	−0.0199340	0.09536	0.067177

that despite being a low order approximation, Eq. (18) is in good agreement with the exact solution for small values of a. When maximising the expected return these small values of a also correspond to small values of c, thus the approximation is valid for the case of small transaction costs. Further, the optimal return, μ^* obtained with the approximate solution provides a close match to that obtained with Eq. (17). Results for the optimal strategy with different transaction costs are shown in Table 1.

In the case of maximising the Sharpe ratio we hold the transaction cost at a constant level, $c = 0.001$, and examine the optimal choice of entry and exit point as a function of the risk-free rate r_f. The optimal trading level a is obtained by maximised Eq. (20). Fig. 6 displays the optimal entry value a, and the corresponding maximum Sharpe ratio as a function of r_f. These figures illustrate how the efficiency of the strategy is affected by the existence of a risk-free investment over an equivalent time scale. As the risk-free return increases, the strategy becomes less attractive on a return per unit risk basis. Table 2 presents the results of the optimisation together with the expected return and the variance in each case.

5. Summary

In this paper we have presented analytic formulae for statistical arbitrage trading. We derived closed form solutions for the mean and variance of the trade length and the strategy return. These results allow for the description and analysis of trading strategies, including the effects of transaction costs. Analytic solutions for constructing optimal trading strategies were obtained by maximising the expected return. It was shown analytically that in the optimal case the trade entry and

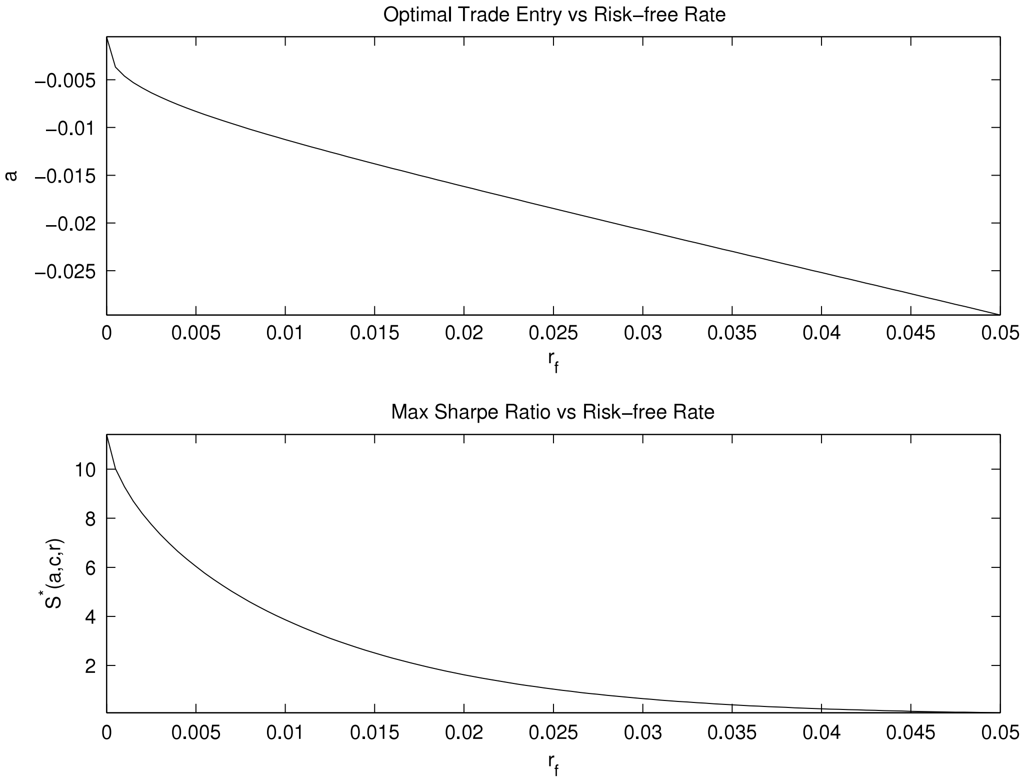

Fig. 6. The optimal trading bands and corresponding maximum return of the strategy as a function of the risk-free rate r_f. This example uses $c = 0.001$.

Table 2
Results for maximising the Sharpe ratio with $\alpha = 180.9670$, $\eta = 0.1538$, $c = 0.001$.

r_f	a	S^*	μ	σ^2
0.0010	−0.0046075	9.2824	0.49231	0.0021698
0.0020	−0.0058600	8.1947	0.48757	0.0023423
0.0030	−0.0068033	7.3497	0.47782	0.0024544
0.0040	−0.0076014	6.6449	0.46638	0.0025421
0.0050	−0.0083134	6.0371	0.45405	0.0026159
0.0060	−0.0089681	5.5027	0.44115	0.0026800
0.0150	−0.0138010	2.5102	0.31271	0.0029518
0.0175	−0.0150020	2.0220	0.27554	0.0029215
0.0200	−0.0161780	1.6249	0.23890	0.0028357

exit levels are symmetric about zero. A formula for maximising the Sharpe ratio was derived. This formula provides a way to construct an optimal strategy incorporating a measure of risk, namely the variance of the return. It was shown that the trade entry and exit levels are also symmetric when maximising the Sharpe ratio. We examined how the optimal choice varies with the risk-free rate of return.

Analytic solutions such as those provided in this paper are of use when implementing statistical arbitrage strategies that engage in high frequency trading, due to the necessity to perform calculations in real time. We note that the results obtained in this paper rely on the assumption that the security is described by an Itô diffusion process, namely the Ornstein–Uhlenbeck process, which is Gaussian. It is well known that financial data displays non-Gaussian behavior and therefore the model will not accurately represent real world behavior. However, this approach offers a way to investigate and understand how the important system variables relate to each other. In particular, it indicates the importance that time plays in determining an optimal trading strategy. The method can also be applied to non-Gaussian processes, such as the generalised Ornstein–Uhlenbeck process which is driven by a Levy noise. However, analytic results may not be so forthcoming.

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