Physica A 388 (2009) 2865–2873

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Optimal trading strategies for Itô diffusion processes

William K. Bertram

Tribeca Investment Partners, Sydney NSW 2000, Australia

ARTICLE INFO

Article history:
Received 2 February 2009
Received in revised form 25 March 2009
Available online 10 April 2009

PACS:
89.65.Gh
02.50.Ey

Keywords:
Econophysics
Stochastic processes
First passage time

ABSTRACT

In this paper we present a method for determining optimal trading strategies for Itô diffusion processes. By framing the problem in terms of the first passage time for the process we derive distribution and density functions for the trade length and use these functions to calculate the expected trading frequency for the strategy. The expected value and the variance of the rate of profit are obtained as functions of the return per trade and trading frequency. We present two measures for trade drawdown which may be used as constraints when determining an optimal strategy. The optimal strategy is calculated for the Ornstein–Uhlenbeck process by maximising the expected rate of profit.

1. Introduction

Recent times have seen an increasing interest in the use of systematic, quantitative based trading strategies. This interest has been driven mainly by hedge funds and the proprietary trading desks of investment banks. These strategies are based on exploiting statistical anomalies in security prices and are usually referred to as *statistical arbitrage*. Such an approach attempts to profit from discrepancies in the relative prices of securities caused by market inefficiencies. Strategies of this type rely heavily on the analysis of market data. As markets are traded more heavily these opportunities become harder to exploit and to employ such strategies successfully it becomes necessary to engage in high frequency algorithmic trading. A common approach when performing this type of trading is to construct a stationary, mean-reverting synthetic asset as a linear combination of securities. One example is the method of *pairs trading* which has been the focus of several recent studies [1–3]. Another example is where a futures contract is traded against one or more securities. This type of trading is referred to as *index arbitrage*, where the goal of the trader is to capitalise on any price discrepancy between index futures and the underlying basket of index constituents. For various reasons, such as liquidity and transaction cost, it is possible to profit from such a trade. In the examples listed above, the goal is to construct a tradable stationary process so that trades are entered when the process reaches an extreme value, and exited when the process reverts to some mean value. The cost of trading is crucial when considering such strategies. Since market inefficiencies are generally small in magnitude, any potential profit could be wiped out by transaction costs. Indeed, transaction costs are one reason why inefficiencies remain.

In order to execute a trading strategy in an optimal sense it is necessary to understand the characteristics of the stochastic process that drives the target security. In practice it is customary for traders to take a heuristic approach and enter a trade when the security prices reach a 'sufficiently large' deviation. To date, there has been no widely accepted study addressing the question of what constitutes the optimal deviation size. While there have been several recent works examining the development and implementation of such trading strategies [4–7], there has been little research done on the topic of how to execute a strategy optimally in the presence of transaction costs. The problem of determining when to enter and exit a trade can be expressed as a function of two random variables: the return per trade; and frequency at which trades take place.

E-mail addresses: williamb@maths.usyd.edu.au, william.bertram@yahoo.com.au.

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doi:10.1016/j.physa.2009.04.004

This approach naturally leads to the examination of the effect of transaction costs on trading. If one trades too frequently, or for too little return, then the cumulative effect of transaction costs can outweigh any potential profit. Conversely, if one trades for a larger return per trade, with a lower trading frequency, then the strategy may not accumulate profits at an optimal rate. Further, there are constraints that must also be considered when choosing a trading strategy, one of the most important being the *drawdown* on a position. The drawdown is defined as being the maximum negative mark-to-market return experienced over the life of a trade. If a trade is entered at the wrong time, then it may be subject to events such as margin calls if the drawdown becomes too large. Effects such as these should be considered when determining an optimal trading strategy.

The analysis of optimal trading strategies can often be expressed in terms of first passage time problems. The first passage time of a process is the time it takes the process to reach a given level for the first time starting from some given initial value. The work of Ref. [1] suggested using the first passage time of the Ornstein–Uhlenbeck process to work out the time with greatest probability and used this as an optimal exit time in an investigation of a pairs trading strategy.

In this paper we examine the problem of constructing an optimal trading strategy for an asset whose price is described by an Itô diffusion process. Although we use the Ornstein–Uhlenbeck process as an example for illustrative purposes, the method can easily be applied to other types of Itô diffusions such the CIR/Square Bessel process [8], CEV process [9], even non-stationary processes such as arithmetic Brownian motion. Previous research has only examined such strategies in special cases, for instance [2] considers the Gaussian white noise case while [1] use an Ornstein–Uhlenbeck process with unit standard deviation and reversion rate. The problem is formulated in terms of the first passage time distribution of the process. Trading frequency is shown to be a function of two first passage times which represent the time taken to exit from an existing trade, and the time taken to enter a new trade. By using the theory of first passage times we express the total trading time as the convolution of the solutions to two systems of Fokker–Planck equations. We present a general framework to choose optimal trade entry and exit levels. These levels are optimal in the sense that they maximise the expected rate of profit subject to transaction costs and constraints on drawdown. We formulate expressions for the drawdown of a trade in terms of both maximum expected drawdown and maximum 95% quantile drawdown. By using the theory of first passage times we are able to express the problem in terms of partial differential equations. Thus, the problem can be solved numerically without resorting to simulation based methods. We note that the purpose of this paper is not to propose specific models for combinations of assets or to characterise the behaviour of such real-world data, rather the authors aim is to focus on the effects that trading constraints and transaction costs have on the returns of a strategy. Our approach may be extended to incorporate more realistic non-Gaussian processes such as the Levy process [10] or the Continuous Time Random Walk (CTRW) [11] by using simulation based calculations to derive the necessary distribution functions.

The rest of the paper is as follows, in the next section we calculate expected trade length and expected trade frequency where the asset being traded follows an Itô diffusion process. By solving two first passage time problems and taking a convolution of the solutions we calculate the distribution and density functions for the total trading time. Using the expected trading frequency we calculate the expected rate of profit and the variance of the rate of profit as functions of trade entry and exit levels. In Section 3 we derive two measures of trade drawdown: the *Maximum Expected Drawdown*; and the *Maximum Q_{95} Drawdown*. These measures are calculated as functions of trade entry and exit levels. In Section 4 we find the optimal trading levels that maximise the expected rate of profit for the Ornstein–Uhlenbeck process and present the results. Section 5 concludes by summaries the main results of the study.

2. Expected trading frequency

Since a trading strategy comprises a sequence of individual trades, many of the important quantities related to the trading strategy can be expressed as functions of the frequency at which these trades take place, for instance profit and loss. The trading frequency is specified by how many times the strategy trades per unit of time. This value is dependent on how long it takes in total to move from one trade entry point to the next, passing though the exit point along the way. In order to analyse the total time between trade entry points we will consider two subintervals: the time from trade entry until exit $\mathcal{T}_{exit}$; and the time from exit until a new trade is entered $\mathcal{T}_{enter}$. We may write the total time as,

$$\mathcal{T}_{total} = \mathcal{T}_{exit} + \mathcal{T}_{enter}. \tag{1}$$

The variable $1/\mathcal{T}_{total}$ represents the frequency at which trades take place. If we model the price of the traded security, p_t as,

$$p_t = \mathrm{e}^{X_t}; \qquad X_{t_0} = x_0, \tag{2}$$

where X_t is an Itô diffusion process, then the random variables $\mathcal{T}_{exit}$ and $\mathcal{T}_{enter}$ can be identified as first passage times, for which we can evaluate distribution and density functions.

Recall that the first passage time $\mathcal{T}$ of a stochastic process X_t is defined to be the time at which the process first reaches the boundaries $X_{\mathcal{T}} = x_L$ or $X_{\mathcal{T}} = x_U$ having started from a given initial point $X_{t_0} = x_0$, where $x_L < x_0 < x_U$. As in [12] the first passage time is formally defined as the random variable,

$$\mathcal{T}_{[x_L,x_U]}(x_0) = \sup_{t\geq t_0}\{t \mid x_L < X_t < x_U; X_0 = x_0\}, \tag{3}$$

with $x_L < x_0 < x_U$. Let X_t be a general Itô diffusion process satisfying the stochastic differential equation,

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}W_t; \quad X_{t_0} = x_0, \tag{4}$$

where W_t is the Wiener process. The transition densities $p(x, t|x_0, t_0)$, for the process satisfy the Fokker–Planck equation [13],

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\mu(x,t)p(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\sigma^2(x,t)p(x,t)\right], \tag{5}$$

with initial condition $p(x, t_0|x_0, t_0) = \delta(x-x_0)$. If Eq. (5) is solved with absorbing boundary conditions at $x = x_L$ and $x = x_U$,

$$p(x_L, t) = 0, \tag{6}$$
$$p(x_U, t) = 0, \tag{7}$$

then the integral $\int_{x_L}^{x_U} p(x, t|x_0, t_0)\mathrm{d}x$ represents the probability that the process has not yet reached either of the boundaries x_L or x_U at time t. Therefore the cumulative distribution function for the first passage time is given by,

$$G_{[x_L,x_U]}(t|x_0, t_0) = 1 - \int_{x_L}^{x_U} p(x, t|x_0, t_0)\mathrm{d}x, \tag{8}$$

with corresponding density function,

$$g_{[x_L,x_U]}(t|x_0, t_0) = \frac{\partial}{\partial t} G_{[x_L,x_U]}(t|x_0, t_0) = -\int_{x_L}^{x_U} \frac{\partial}{\partial t} p(x, t|x_0, t_0)\mathrm{d}x. \tag{9}$$

Further, it can also be shown [15] that the distribution function $G_{[x_L,x_U]}(t|x_0, t_0)$ satisfies the partial differential equation,

$$-\frac{\partial G}{\partial t_0} = -\mu(x_0, t_0)\frac{\partial G}{\partial x_0} + \frac{1}{2}\sigma^2(x_0, t_0)\frac{\partial^2 G}{\partial x_0^2}, \tag{10}$$

in the region $x_L \le x_0 \le x_U$ subject to the initial and boundary conditions,

$$G_{[x_L,x_U]}(t|x_0, t) = 0, \tag{11}$$
$$G_{[x_L,x_U]}(t|x_L, t_0) = 1, \tag{12}$$
$$G_{[x_L,x_U]}(t|x_U, t_0) = 1. \tag{13}$$

By choosing appropriate values for x_0, x_L and x_U we can obtain the distribution and density functions for the trade exit and entry times.

The general trading strategy is to enter a trade by purchasing a security at some level a and to exit by selling the security when it reaches some level m, where $a < m$. In the case where we are trading a stationary mean-reverting process we will be able to repeat these actions as the process continues to pass between these levels. The issue in question is how to determine the values of a and m such that we implement the strategy in an optimal way, taking into account trading costs and limits on funding requirements. To address this problem we must first formulate expressions for the trading frequency of the strategy in terms of the trade entry and exit times. The exit time can be characterised as the first passage time of the process X_t starting at $x_0 = a$ with a single barrier at $x = m$. We write the exit time as $\mathcal{T}_{exit} = \mathcal{T}_{[-\infty,m]}(a)$ with density function $g_{[-\infty,m]}(t|a, t_0)$. Likewise, the entry time consists of the first passage time of the process starting from $x_0 = m$ with barrier at the trade entry point $x = a$. The entry time is written as $\mathcal{T}_{entry} = \mathcal{T}_{[a,\infty]}(m)$ and its density is given by $g_{[a,\infty]}(t|m, t_0)$.

By limiting our analysis to Itô diffusion processes, the Markov property ensures that the two random variables, $\mathcal{T}_{exit}$ and $\mathcal{T}_{entry}$ are independent. Thus the density of the sum of random time variables in Eq. (1) is given by convolution,

$$f(t; m, a) = g_{[-\infty,m]}(t|a, t_0) \otimes g_{[a,\infty]}(t|m, t_0). \tag{14}$$

By using the density function, $f(t; m, a)$, we can evaluate the expected trade length,

$$\mathbb{E}\left[\mathcal{T}_{total}\right] = \int_0^\infty t f(t; m, a)\mathrm{d}t, \tag{15}$$

and the expected trade frequency and variance

$$\mathbb{E}\left[\frac{1}{\mathcal{T}_{total}}\right] = \int_0^\infty \frac{1}{t} f(t; m, a)\mathrm{d}t, \tag{16}$$

$$\mathbb{V}\left[\frac{1}{\mathcal{T}_{total}}\right] = \int_0^\infty \frac{1}{t^2} f(t; m, a)\mathrm{d}t - \mathbb{E}\left[\frac{1}{\mathcal{T}_{total}}\right]^2. \tag{17}$$

We may also use an alternative approach to obtain the expected trading frequency. For an Itô diffusion process the first and second moments of the first passage time $T = \mathbb{E}[\mathcal{T}]$ and $V = \mathbb{E}[\mathcal{T}^2]$, were shown by Ref. [12] to satisfy the system of second order differential equations,

$$\frac{\sigma^2(x,t)}{2}T''(x) + \mu(x,t)T'(x) = -1, \tag{18}$$

$$\frac{\sigma^2(x,t)}{2}V''(x) + \mu(x,t)V'(x) = -2T, \tag{19}$$

where $T(x_L) = V(x_L) = 0$ and $T(x_U) = V(x_U) = 0$. The theory of renewal processes then allows us to obtain the expected trading frequency as,

$$\mathbb{E}\left[\frac{1}{\mathcal{T}}\right] \sim \frac{1}{\mathbb{E}[\mathcal{T}]}. \tag{20}$$

By solving for the cases $\mathcal{T}_{[-\infty,m]}(a)$ and $\mathcal{T}_{[a,\infty]}(m)$ we can calculate the expected total trade time as,

$$\mathbb{E}[\mathcal{T}_{total}] = \mathbb{E}[\mathcal{T}_{exit}] + \mathbb{E}[\mathcal{T}_{entry}]. \tag{21}$$

This approach is simpler to implement numerically as it avoids the calculation of the convolution in Eq. (14). Note that although we can obtain the second moment of the first passage time, there is no result equivalent to Eq. (20) that would allow us to calculate the variance of the trading frequency in this manner.

The return on a single trade is given by the difference between the exit and entry prices minus the total transaction costs associated with the trade. Since the variable X_t represents the log-price, the function $r(a, m, c) = (m - a - c)$ gives the continuously compound rate of return for a single trade, where c is the transaction cost per trade measured in percent. By multiplying this value by the expected frequency we obtain the expressions for the expected rate of profit for the strategy and its variance,

$$\mu_P = r(a,m,c)\mathbb{E}\left[\frac{1}{\mathcal{T}_{total}}\right] = (m-a-c)\int_0^\infty \frac{1}{t}f(t;m,a)\mathrm{d}t, \tag{22}$$

$$\sigma_P = r(a,m,c)^2\mathbb{V}\left[\frac{1}{\mathcal{T}_{total}}\right] = (m-a-c)^2\int_0^\infty \frac{1}{t^2}f(t;m,a)\mathrm{d}t - \mu_P^2. \tag{23}$$

3. Measures for trade drawdown

In financial terms, the drawdown refers to the magnitude of the mark-to-market value of a trade where the current value of the position is less than the entry value. Mathematically this value can be expressed as,

$$D_t = (a - X_t)^+ = \begin{cases} 0; & X_t > a, \\ a - X_t; & X_t \le a, \end{cases} \tag{24}$$

where X_t is the current value of the security and a is the value of the security at which the trade was entered. Thus, the drawdown indicates the size of an unrealised loss for the trade. In the case of a stationary process such as the Ornstein–Uhlenbeck process the drawdown should not matter since, in theory, the process is guaranteed to revert to the exit level if we wait long enough. However, the drawdown is an important consideration when trading due to operational constraints such as margin requirements, capital limits, profit reporting and risk management. Indeed, there have been several famous instances where hedge funds have run into difficulties with trade drawdown even when convergence to the exit level has been guaranteed, for example in fixed income markets [16]. This means that drawdown should be considered as a constraint in the choice of optimal entry/exit points.

We propose two possible measures for trade drawdown that may be used as constraints for choosing a strategy. The first is the *Maximum Expected Drawdown* while the second is the *Maximum Q_{95} Drawdown*. Consider the function $p(x, t|a, t_0)$ which satisfies Eq. (5) with boundary conditions $p(m, t) = 0$ and $p(-\infty, t) = 0$ where $-\infty < a \le m$. Note that while $p(x, t|a, t)$ is not a true probability density function since $\int_{-\infty}^{m} p(x, t|a, t_0)\mathrm{d}x \neq 1$, the integral $\int_{-\infty}^{a} p(x, t|a, t_0)\mathrm{d}x$ represents the probability that the process is less than its initial value, a, conditional on the process not having come into contact with the barrier at m before time t [14]. Further, the integral

$$\int_{-\infty}^{a} xp(x,t|a,t_0)\mathrm{d}x, \tag{25}$$

yields the average value of the process in the region $[-\infty, a]$ at time t, conditional on the process not having come into contact with the barrier at m.

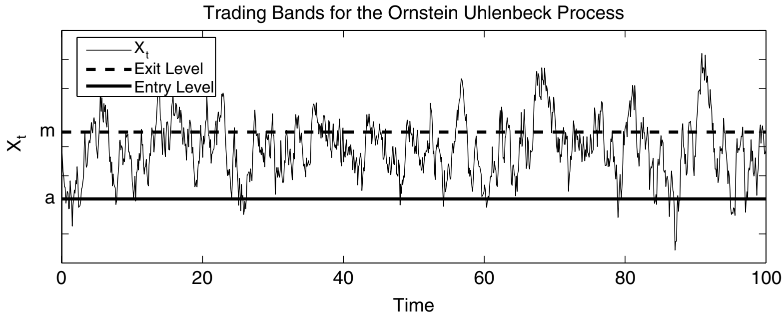


Fig. 1. Ornstein–Uhlenbeck process with trade entry level at $X_t = a$ and trade exit level at $X_t = m$.

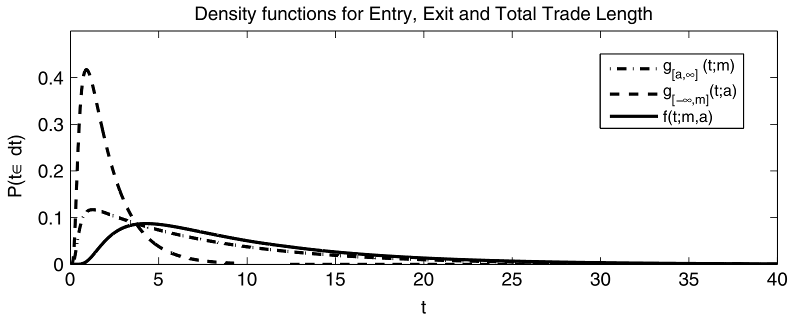


Fig. 2. Density functions $g_{[-\infty,m]}(t|a, t_0)$, $g_{[a,\infty]}(t|m, t_0)$ and $f(t; m, a)$ for the Ornstein–Uhlenbeck process.

By combining Eqs. (24) and (25) we may formulate the maximum expected drawdown as,

$$\max_t \mathbb{E}[D_t] = \max_t \int_{-\infty}^{a} (a - x) p(x, t|a, t_0) \mathrm{d}x, \tag{26}$$

where the maximum is taken over all times $t_0 < t < \infty$.

As an alternative to the maximum expected drawdown we can obtain an expression for the drawdown in terms of the maximum 95% quantile bands. In this approach, we find the value Q_t such that the absorbing process X_t is above this level with 95% probability, $\mathbf{P}[X_t > Q_t] \geq 0.95$. The maximum Q_{95} drawdown is then defined as,

$$Q_{95} = \max_t (a - Q_t)^+, \tag{27}$$

where Q_t satisfies,

$$\int_{-\infty}^{Q_t} p(x, t|a, t_0) \mathrm{d}x = 0.05. \tag{28}$$

Thus Q_{95} represents the maximum drawdown that could occur before exit time with a 95% level of confidence.

4. Results for the Ornstein–Uhlenbeck process

In this example we model the traded security with the Ornstein–Uhlenbeck process,

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}W_t; \quad X_0 = x_0, \tag{29}$$

where σ is the instantaneous standard deviation and α is the rate of reversion. The steady-state standard deviation of the above process is given by,

$$\theta = \frac{\sigma}{\sqrt{2\alpha}}. \tag{30}$$

The Ornstein–Uhlenbeck process is used in practice to model tradable synthetic assets known as *spreads*. These synthetic assets are constructed as a linear combination of two or more securities and are traded in the areas of *index arbitrage* and *pairs trading* [1,2]. We are required to find the entry level a and exit level m, where $a < m$, that maximises a chosen objective function. For illustrative purposes we will maximise the expected rate of profit, although the analysis could also be carried out by maximising other functions such as the Sharpe Ratio [17]. Notice that this example that increasing the distance

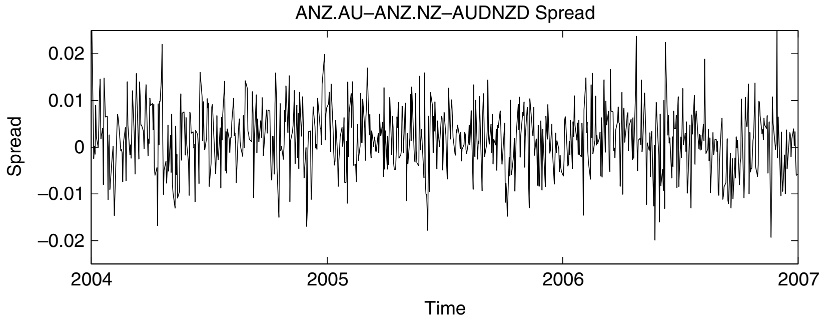


Fig. 3. The spread for dual-listed ANZ securities on the Australian and New Zealand stock exchanges.

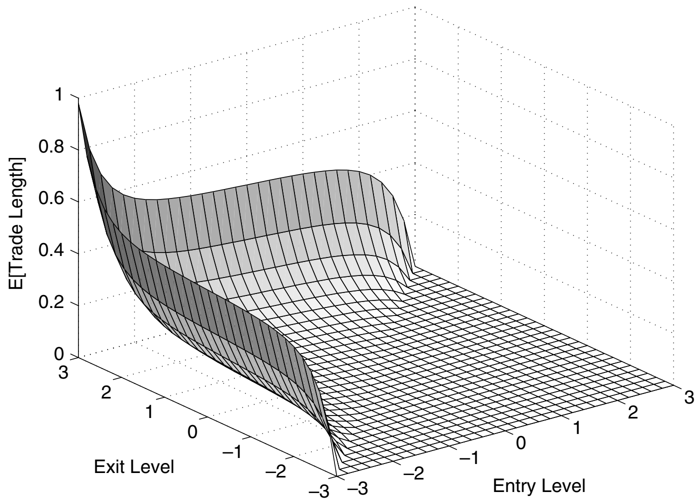


Fig. 4. The expected trade length as a function of trade entry and exit level.

between the entry and exit levels increases the return for a single trade and simultaneously decreases the number of trades that take place per unit time. Fig. 1 displays a sample path for the Ornstein–Uhlenbeck process including sample entry and exit bands at $a = -1.8$, $m = 0.5$. For this process, Eq. (10) is

$$-\frac{\partial G}{\partial t_0} = \alpha x_0 \frac{\partial G}{\partial x_0} + \frac{1}{2}\sigma^2 \frac{\partial^2 G}{\partial x_0^2}. \tag{31}$$

The above equation can be solved numerically and the derivative taken with respect to t, as in Eq. (9), to obtain $g_{[-\infty,m]}(t|a, t_0)$ and $g_{[a,\infty]}(t|m, t_0)$. The density function for the total trade length, $f(t; m, a)$, is shown in Fig. 2 for values of $a = -1.8$, $m = 0.5$. Similarly the differential equation Eq. (18) for the alternative calculation of the expected trade length is,

$$\frac{\sigma^2}{2} T''(x) - \alpha x T'(x) = -1 \tag{32}$$

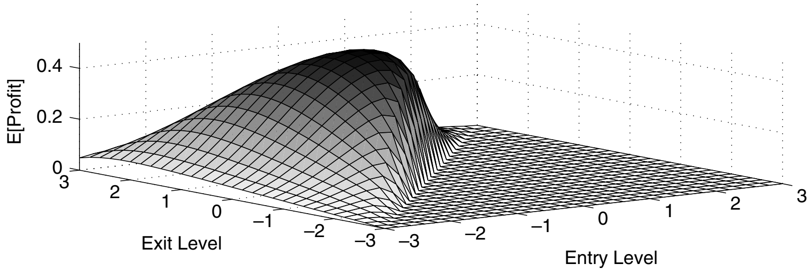
which we solve for the cases: $T(-\infty) = T(m) = 0$; and $T(a) = T(\infty) = 0$.

We illustrate this example by fitting Eq. (29) to a spread constructed from the *dual-listed* security, ANZ Bank (ANZ). This security is traded simultaneously on both the Australian and New Zealand stock exchanges. Let p_{AU}, p_{NZ} denote the prices of the Australian and New Zealand stocks respectively and p_{FX} denote the Australian-to-New Zealand dollar exchange rate. We can construct a synthetic spread by buying the Australian listed stock, selling the New Zealand listed stock and buying New Zealand dollars in a 1:1:1 capital ratio. If we write the spread as,

$$X_t = \log p_{AU} - \log p_{NZ} + \log p_{FX}, \tag{33}$$

then changes in X_t will yield the continuously compound rate of return on the trade. Fig. 3 displays the dual-listed ANZ spread for the period 2004–2007 using daily observations. Using regression we estimate the corresponding model parameters as $\alpha = 180.9670$ and $\sigma = 0.1538$ with $\mathrm{d}t \approx 1/250$.

In Fig. 4 we show a surface plot for expected trade length as a function of the entry and exit levels where we have used the values $\alpha = 180.9670$, $\sigma = 0.1538$, $0 < t < 1$ and steady-state standard deviation $\theta = 0.0081$. Here the axes have



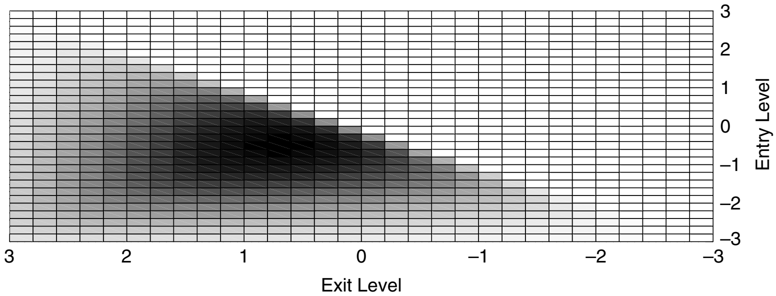


Fig. 5. The expected rate of profit as a function of trade entry/exit levels. The transaction cost is $c = 0.001$.

been normalised to display the entry and exit in units of the process' steady-state standard deviation. This figure shows the nonlinear increase in trade length as the entry and exit bands become further and further apart.

Fig. 5 displays the expected profit rate as a function of entry and exit levels. Here we have included a transaction cost (TC) value $c = 0.001$. We can see from this figure that there is clear maximum. In order to maximise profit for the Ornstein–Uhlenbeck process with the stated parameters, this result suggests having entry/exit bands at approximately 0.8 standard deviations either side of the mean. This figure clearly shows the extent to which the positions of the trade entry/exit bands affect the profitability of the strategy. This result is of particular interest as it provides a way to quantify the practical task of choosing between having a larger return per trade or trading at a higher frequency. When such processes are traded by practitioners, the standard approach is to place entry bands at 2 standard deviations from the mean and have the exit band at the mean itself [3].

Fig. 6 displays the maximum value of the expected profit rate as a function of transaction costs. As expected the highest return is obtained when transaction costs are zero. The function displayed in this figure provides us with a way to analyse whether a trading strategy will be profitable given a particular cost for trading. In Fig. 7 we show the maximum expected drawdown for different entry and exit levels. This figure shows how the drawdown is determined, not only by the position of the entry level, but also by the trading frequency. If the bands are set close together then the drawdown can be reduced, although transaction costs would need to be quite small for the strategy to be profitable. Table 1 presents the results for the optimisation of the expected return rate for different values of transaction cost. As the transaction costs increase we see that the rate of profit, μ_P decreases as the optimal entry and exit bands are made further and further apart to offset the cost of trading. This is also reflected as an increase in the expected trade length. We also show the return per trade, $r(a, m, c)$, and the drawdown measures for the optimal entry and exit levels. The table clearly shows the drawdown measures decreasing as the entry level becomes further from the process' mean. In the cases shown, the drawdown is an order of magnitude less than the return per trade.

It is widely accepted that financial processes exhibit non-Gaussian and non-stationary behaviour and this has been shown in several important studies [18–21]. Thus, price returns of spreads and other asset combinations cannot be characterised by a Gaussian distribution. In this example the drawdown measures calculated for the Ornstein–Uhlenbeck process should be expected to underestimate the true drawdown of the financial process being modelled. Characterising the statistical nature of such spreads remains the focus of further study.

5. Summary

In this paper we have presented a framework for the optimal trading of Itô diffusion processes with transaction costs. It was shown how the total length of a trade can be expressed as a first passage type problem and expressions for the distribution and density functions for the trade length were obtained. The density functions were used to derive the expected value and variance of the trading frequency. We constructed expressions for the expected rate of profit and the variance of the profit for the trading strategy. These expressions are formulated in terms of the return per trade and the trading frequency.

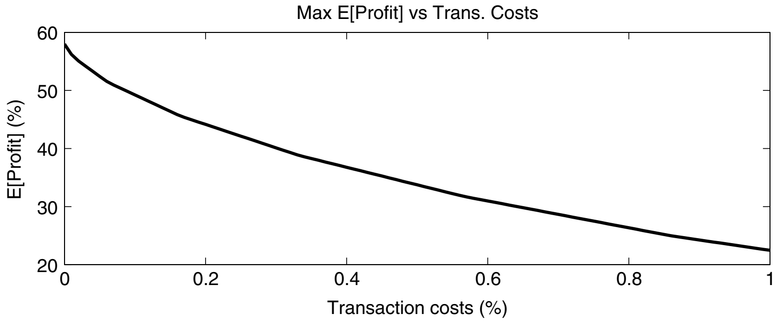


Fig. 6. Maximum expected profit vs transaction cost for the Ornstein–Uhlenbeck strategy.

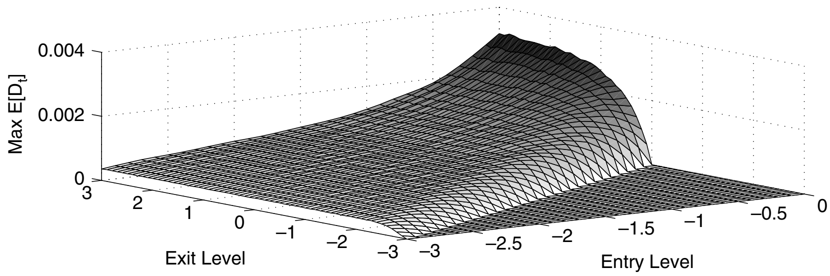


Fig. 7. The maximum expected drawdown for the Ornstein–Uhlenbeck process as a function of trade entry and exit levels.

Table 1
Results for the strategy optimisation applied to the Ornstein–Uhlenbeck process with $\alpha = 180.9670$, $\sigma = 0.1538$ and $0 < t < 1$. The entry and exit levels are expressed in units of the steady-state standard deviation, θ.

TC	$\mathbb{E}[T]$	μ_P	Entry	Exit	$r(a, m, c)$	$\max \mathbb{E}[D_t]$	Q_{95}
0.0010	0.0174	0.4950	−0.5881	0.5932	0.0086	0.0017	0.0091
0.0020	0.0231	0.4454	−0.7541	0.7558	0.0103	0.0015	0.0085
0.0030	0.0275	0.4057	−0.8675	0.8728	0.0112	0.0014	0.0081
0.0040	0.0317	0.3718	−0.9674	0.9684	0.0118	0.0013	0.0076
0.0050	0.0355	0.3420	−1.0519	1.0504	0.0121	0.0013	0.0072
0.0060	0.0391	0.3152	−1.1228	1.1223	0.0123	0.0012	0.0070
0.0070	0.0431	0.2909	−1.2000	1.1941	0.0125	0.0011	0.0068
0.0080	0.0471	0.2687	−1.2672	1.2625	0.0127	0.0011	0.0065
0.0090	0.0510	0.2482	−1.3232	1.3259	0.0126	0.0011	0.0063
0.0100	0.0552	0.2293	−1.3840	1.3869	0.0127	0.0010	0.0060
0.0125	0.0660	0.1878	−1.5201	1.5214	0.0124	0.0009	0.0057
0.0150	0.0787	0.1532	−1.6481	1.6544	0.0121	0.0009	0.0055
0.0175	0.0934	0.1240	−1.7738	1.7734	0.0116	0.0008	0.0051
0.0200	0.1115	0.0996	−1.8932	1.8958	0.0111	0.0008	0.0048
0.0250	0.1631	0.0622	−2.1362	2.1358	0.0101	0.0007	0.0043
0.0300	0.2441	0.0368	−2.3641	2.3667	0.0090	0.0006	0.0042
0.0350	0.3941	0.0204	−2.6096	2.6050	0.0081	0.0005	0.0037
0.0400	0.6620	0.0106	−2.8413	2.8395	0.0070	0.0004	0.0033
0.0450	1.2827	0.0050	−3.1160	3.0908	0.0065	0.0003	0.0025
0.0500	2.6178	0.0022	−3.3594	3.3549	0.0058	0.0002	0.0013

The drawdown is an important consideration when implementing a trading strategy and should be considered as a constraint when choosing optimal entry and exit points for a trade. We presented two measures for the drawdown in this paper. The maximum expected drawdown measures the average drawdown that we are likely to see over the course of a trade. The maximum Q_{95} drawdown is a measure of the largest drawdown that is likely to occur at the 95% confidence level.

The method was applied to the Ornstein–Uhlenbeck process. This process was chosen as it is commonly used as a model for spread trading. We illustrated the example by calibrating the model to a spread formed from a dual-listed security. The expected trade length and expected return rates were calculated as functions of trade entry and exit level. It was shown that for the unconstrained case the optimal trading levels are not only symmetric, but occur much closer to the mean of the process than is commonly accepted in practice. This result highlights the important effect that trading frequency has on the profitability of a strategy. The drawdown measures were calculated for the optimal strategies and it was shown how they varied with entry/exit level and transaction cost. In the studied example we found the expected drawdown to be an order of magnitude less than the return per trade.

While the strategy itself is simple, *i.e.* buying low and selling high, it is possible to apply the method to more complex strategies on different stochastic processes. It is well known that financial processes exhibit non-Gaussian behaviour such as power law distributions and long-memory. A model such as the Ornstein–Uhlenbeck process will underestimate the maximum expected drawdown and Q_{95} drawdown if applied to such financial processes. Extensions for future work include applying the method to processes that include non-stationary terms, using a simulation based approach to extend the method to non-Gaussian processes and using real-world data to examine the viability and even the existence of tradable market inefficiencies.

Acknowledgements

The author would like to thank Jim Clark of Lacima Group and Ivan Kaverzine of Deutsche Bank Australia, Equities and Futures Trading for several inspiring discussions in relation to the practical implementation of quantitative trading strategies.

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