UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Non-linear evolution of cosmological power-spectra using Kinetic Field Theory approach

Thesis supervisor

Prof. Sabino Matarrese

Thesis co-supervisor

Prof. Matthias Bartelmann

Candidate

Luca Teodori

Academic Year 2018/2019

UNIVERSITY OF PADOVA

Abstract

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Non-linear evolution of cosmological power-spectra using Kinetic Field Theory approach

by Luca TEODORI

We use the kinetic field theory approach developed in [1–4] to study the evolution of simple primordial power spectra. Kinetic field theory is a microscopic, non equilibrium statistical field theory that can be accustomed to treat large scale structure formation. Due to its features, it overcomes some difficulties arising in other perturbative approaches, the most famous being shell crossing [5]. In particular, we will use the Born approximation applied in kinetic field theory setting, that was proven to be particularly trustworthy compared to N-body simulations [6], to probe the non linear evolution of some simplified primordial power spectra; one will be related to dark matter primordial power spectrum but with a changing small scale slope; we will also analyze Gaussian shape primordial power spectra; such spectra are cosmologically irrelevant, but nevertheless it is interesting to see what kinetic field theory predicts also for such cases. We found that all these evolved power spectra have a small scale fall-off that goes as k^{-3}; this behavior seems to be valid for a wide class of primordial power spectra. Another goal of this work is to analyze higher order correlators (like bispectrum) evolution within this framework.

Contents

List of Figures

List of Tables

List of Abbreviations, constants and parameters

Abbreviations	
KFT	**K**inetic **F**ield **T**eory
QFT	**Q**uantum **F**ield **T**eory
CMB	**C**osmic **M**icrowave **B**ackground
LSS	**L**arge **S**cale **S**tructure (formation)
ΛCDM	"Lambda" (dark energy) **C**old **D**ark **M**atter (models)
GSL	**G**nu **S**cientific **L**ibrary

Constants		
Speed of light	c	$2.997\,924\,58 \times 10^{8}\,\mathrm{m\,s^{-1}}$ (exact)
Hubble constant (today)	H_0	$70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$
Gravitational constant	G	$6.674\,08(31) \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$

Parameters		
Baryon density parameter (today)	$\Omega_{\mathrm{mb},0}$	0.04
Dark matter density parameter (today)	$\Omega_{\mathrm{md},0}$	0.26
Dark energy density parameter (today)	$\Omega_{\Lambda,0}$	0.70
Hubble parameter	h	0.7
Galactic variance per $8\,\mathrm{Mpc}\,h^{-1}$	σ_8	0.8

List of Symbols

Symbol	Description/Name	Find in text
κ	Curvature parameter	
$\vec{x}_j(t) := (\vec{q}_j(t), \vec{p}_j(t))^\top$	6 dimensional phase space vector for particle j	
$\mathcal{J} := \begin{pmatrix} 0 & \mathbb{1}_3 \\ -\mathbb{1}_3 & 0 \end{pmatrix}$	Symplectic matrix	(2.2)
$\boldsymbol{x}(t) := \vec{x}_j(t) \otimes \vec{e}_j$	Boldface symbols are Kronecker products of this kind	
$\vec{s}, \vec{k}, \vec{l}$	Fourier conjugate wavevectors for $\vec{x}, \vec{q}, \vec{p}$ respectively	
$\int_r := \int \frac{\mathrm{d}^3 k_r}{(2\pi)^3} \int \frac{\mathrm{d}^3 l_r}{(2\pi)^3} \int \mathrm{d}t$	Shortcuts for integrals	
$\vec{\chi}(t) := \begin{pmatrix} \vec{\chi}_{q_j}(t) \\ \vec{\chi}_{p_j}(t) \end{pmatrix}$	Auxiliary 6-dimensional microscopic field	(2.10)
$\mathcal{H}(\boldsymbol{x}, \boldsymbol{p}, t)$	Hamiltonian	
$\boldsymbol{E}(\boldsymbol{x})$	Equation of motion (defined by $\boldsymbol{E}(\boldsymbol{x}) = 0$)	(2.6)
$\mathcal{V}(\vec{q}, t)$	Interaction potential	
$v(\vec{q})$	Single particle interaction potential	(2.20)
$S[\boldsymbol{x}, \boldsymbol{\chi}]$	Functional action	(2.12)
$P_0(\boldsymbol{x}^{(\mathrm{i})})$	Initial phase-space distribution probability function	(2.47)
$\mathrm{d}\boldsymbol{\Gamma}_\mathrm{i} := \mathrm{d}\boldsymbol{x}^{(\mathrm{i})}\, P_0(\boldsymbol{x}^{(\mathrm{i})})$	Initial phase space integration measure	(2.13)
$\boldsymbol{J}(t) := \begin{pmatrix} \vec{J}_{q_j}(t) \\ \vec{J}_{p_j}(t) \end{pmatrix} \otimes \vec{e}_j$	Source field conjugate of $\boldsymbol{x}$	(2.14)
$\boldsymbol{K}(t) := \begin{pmatrix} \vec{K}_{q_j}(t) \\ \vec{K}_{p_j}(t) \end{pmatrix} \otimes \vec{e}_j$	Source field conjugate of $\boldsymbol{\chi}$	(2.14)
$Z_\mathrm{C}[\boldsymbol{J}, \boldsymbol{K}]$	Canonical generating functional	(2.15)

Continued on next page

Symbol	Description/Name	Find in text
$\bar{\boldsymbol{x}}(t)$	Classical solution of the free equation of motion	(2.17)
$\mathcal{G}(t,t') := \begin{pmatrix} g_{qq}\mathbb{1}_3 & g_{qp}\mathbb{1}_3 \\ g_{pq}\mathbb{1}_3 & g_{pp}\mathbb{1}_3 \end{pmatrix}(t,t')$	Free propagator	(2.18)
$\boldsymbol{\mathcal{G}}(t,t') := \mathcal{G}(t,t') \otimes \mathbb{1}_N$	Kronecker product for the propagator	(2.18)
$\rho(\vec{q},t)$	Density field	(2.19)
$B(\vec{q},t)$	Response field	(2.22)
$\Phi(\vec{s},t) := \begin{pmatrix} \Phi_f(\vec{s},t) \\ \Phi_B(\vec{s},t) \end{pmatrix}$	Klimontovich fields	(2.27)
$\sigma(\vec{s},\vec{t},\vec{s}',\vec{t}')$	Interaction potential in path integral	(2.29)
$H(\vec{s},t) := \begin{pmatrix} H_f(\vec{s},t) \\ H_B(\vec{s},t) \end{pmatrix}$	Source field conjugate of Φ	(2.30)
$\hat{\Phi}(\vec{s},t)$	Operator version of Φ	(2.33)
$\hat{\Phi}^{(1)}_\alpha(\vec{s},t)$	One particle Φ_α operator	(2.33)
$\hat{H}(\vec{s},t)$	Operator version of H	(2.35)
$\hat{b}_j(\vec{s},t)$	Response field operator	(2.34)
$\hat{S}_1$	Operator version of the interaction part of the action	(2.36)
$\delta(\vec{q},t)$	Density contrast	(2.38)
$Z_{\mathrm{C}}[\boldsymbol{J},\boldsymbol{K},H]$	Canonical full generating functional	(2.32)
$\vec{u}^{(\mathrm{i})}(\vec{q}^{(\mathrm{i})})$	Initial velocity field	(2.40)
$\psi(\vec{q}^{(\mathrm{i})})$	Initial irrotational velocity potential	(2.40)
$P_\delta(k,t)$	Density contrast power spectrum	(2.39)
$\vec{P}(\vec{q}^{(0)})$	Initial momentum dispersion	(2.43)
$\hat{\mathcal{C}}\left(\frac{\partial}{\mathrm{i}\partial \boldsymbol{p}^{(\mathrm{i})}}\right)$	Particle correlation operator	(2.50)
$G_{\alpha_1(1)\cdots\alpha_n(n)}(1,\ldots,n)$	n-point particle correlator	(2.52)
$G_{\rho(1)\rho(2)}(1,2)$	Density 2-point correlator	(2.51)
$C_{jk}(\vec{q}_{jk}) := \begin{pmatrix} C_{\delta\delta}\mathbb{1}_3 & C_{\delta p}\mathbb{1}_3 \\ C_{p\delta}\mathbb{1}_3 & \boldsymbol{C}_{pp}\mathbb{1}_3 \end{pmatrix}(\vec{q}_{jk})$	Covariance matrix	(2.48), (2.49)
$\boldsymbol{C}(\vec{q}_{jk}) := C_{jk}(\vec{q}_{jk}) \otimes \vec{e}_j \otimes \vec{e}_k$	Kronecker product of correlation matrix	(2.48)
$(\boldsymbol{L},\boldsymbol{L}_q,\boldsymbol{L}_p)(\vec{s}_1,t_1,\ldots,\vec{s}_m,t_m)$	Shift vectors of order m	(2.54), (2.56)

Continued on next page

Symbol	Description/Name	Find in text
σ_n^2	Generalized momentum variance	(2.58)
$(Q, Q_0, Q_D)(\vec{s}_1, t_1, \ldots, \vec{s}_m, t_m)$	Exponential free streaming damping factors of order m	(2.60)
$\vec{k}_{jk}$	Generalized wave vectors	(2.68)
$\mathcal{P}(k,t)$	Power spectrum from factorization of generating functional	(2.67)
$\xi_\psi(q_{jk})$	ψ Correlation function	(2.72)
$\pi_\parallel(\hat{k}_{jk}), \pi_\perp(\hat{k}_{jk})$	Projectors	(2.70)
$\lambda_\parallel, \lambda_\perp, a_\parallel(q), a_\perp(q), a_1(q), a_2(q)$	Factors arising in factorization of generating functional	(2.69),(2.71), (2.76)
$j_n(x)$	Spherical Bessel functions	(2.75)
$\mathcal{L}(t)$	Lagrangian	
$D_+(t)$	Linear growth factor	
η, τ	Time coordinate expressed using the linear growth factor	(3.11)(3.1)
$f(\eta), g(\eta)$	Factors arising in propagator of trajectories in expanding spacetime	(3.3), (3.4)
$\vec{f}(\tau)$	Effective force	(3.19)
$\langle S_I \rangle (\vec{k}, \tau)$	Interacting action averaged following Born approximation	(3.24)
$\bar{\mathcal{P}}(\vec{k}, \tau)$	Non linearly evolved power spectrum in Born approximation	(3.25)
$\left\langle \delta(\vec{k}_1) \cdots \delta(\vec{k}_n) \right\rangle_c$	Connected n-point density correlator	(5.8)
$B_\delta(k_1, k_2, \hat{k}_1 \cdot \hat{k}_2)$	Bispectrum	(5.10)
$T_\delta(k_1, k_2, k_3, \mu_{21}, \mu_{31}, \mu_{32})$	Trispectrum	(5.11)
$G^{(c)}_{\alpha_1(1)\cdots\alpha_n(n)}$	Generic n-th order cumulant	(D.1)
$G^{(c,0,l)}_{f(1)\cdots f(n)}$	Generic n-th order free cumulant for l particles	(D.8)
$Q(I_1, \ldots, I_l)$	Generalized free streaming damping	(D.9)
$\tilde{\Sigma}^{(l)}_C(I_1, \ldots, I_l)$	Sum of connected diagrams in the computation of free cumulants	(D.10)

Continued on next page

Symbol	Description/Name	Find in text
$\left(\mathcal{P}_{\delta_i\delta_j}, \mathcal{P}_{p_i\delta_j}, \mathcal{P}_{p_ip_j}, \mathcal{P}_{(\delta p)^2_{ij}}\right)(\vec{k}_{ij})$	Diagram lines factors	(D.11)

Chapter 1

Introduction

Large scale structures formation (LSS) studies necessitate some techniques to probe the non linear regime of structure formation. Right now, N-body simulations applied to this problem have reached an high degree of accuracy, but nevertheless, even with the computational power increase we experienced in recent years, they are time consuming and they cannot span a large range of parameters. Also they cannot shed light on the connection of physical properties of LSS with the fundamental physics. It is thus of interest to try to find out analytical means to probe the non linear regime of LSS formation. Tools that use standard perturbation theory [5], approaches using renormalization group techniques and related [7–9] and others [10, 11] showed that this is not easy to accomplish. The recently developed "kinetic field theory for cosmic structure formation" [1–4, 12] seems a very promising tool to probe semi-analytically the non linear regime. It is based on a microscopic statistical field theory, and enables to avoid notorious problems arising in classical approaches like shell-crossing.

In this thesis, we will use this Kinetic Field Theory (KFT) approach to analyze the non linear evolution of some simplified primordial power spectra. We will take into account the fluctuation density field of collisionless dark matter (we will thus ignore all the other species like baryons; nevertheless one can include the effects of baryons in KFT, see [13]) in a standard ΛCDM cosmological setting. In KFT framework, one can attempt to include gravitational interactions using a perturbative approach [1, 4] or using some non perturbative scheme [6]. We will focus on the latter approach for our analysis.

In this chapter we will show the notations and physical conventions we will use throughout the thesis; also in section 1.3 we will summarize the cosmological setting relevant for our work. In chapter 2 we review the KFT, in particular in section 2.4 we will show how the central object of KFT, the generating functional, can be fully factorized [4], something useful for what we want to do; in chapter 3 we derive the propagator, accounting for the free evolution and a part of the interactions, to use in our cosmological setting, [14, 15] and describe our non-perturbative approach to take into account gravitational interactions [6].

The original part of this work starts from chapter 4, where we describe the analysis of the primordial power spectra; in chapter 5 we show how the KFT formalism can be accustomed for the computations of higher order correlators like the bispectrum and the trispectrum and we will analyze the bispectrum behavior; finally in chapter 6 one can find the conclusions and possible future developments of this work.

On my github page[1], one can find the code I used to obtain the results shown in this thesis.

1.1 Notations

Here we resume the main notations we are using throughout the thesis.

1.1.1 Mathematical objects

In the following, we tried to make the notation as simple as possible but searching also to specify the kind of mathematical object we are dealing with.

We will use the vector symbol (like in $\vec{k}$) to indicate three dimensional vectors or generic n dimensional vectors (like $\vec{e}_j$, see (2.3)); a vector quantity will be written without the vector symbol when we want to consider its module (for example $k = |\vec{k}|$); quantities that are Kronecker products (defined in appendix A.1) are indicated with boldface symbols (like in $\boldsymbol{k}$); we didn't find necessary to use a particular way to denote matrices.

The functional integration in a function $x(t)$ is denoted as $\int \mathcal{D}x$, whereas the ordinary integration over a variable x is simply $\int \mathrm{d}x$ as usual.

1.1.2 Dependencies and integrals

To make the formulas not too lengthy, we will often use shortcuts for dependencies or integrals. Since we will work with many quantities with phase space coordinates dependencies, we will find it useful to put the whole dependencies in a single, integer-valued, label we will often generically indicate with the letter r. In general, $\pm r := \{\pm\vec{k}_r, \pm\vec{l}_r, t_r\}$, where $\vec{k}_r$, $\vec{l}_r$ are the Fourier conjugate of the real phase space coordinates $\vec{q}_r$, $\vec{p}_r$ respectively. The corresponding shortcut for the integrals will be

$$\int_r := \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \frac{\mathrm{d}^3l}{(2\pi)^3} \int_{t^{(\mathrm{i})}}^{t_\mathrm{f}} \mathrm{d}t \ ; \tag{1.1}$$

when we will specify the dependence in this way for a function that depends only on some of the phase space plus time quantities, the previous definitions will be restricted to only the relevant quantities (for example for a quantity A that depends only on $\vec{k}$ and t, $A(1) = A(\vec{k}_1, t_1)$ only).

Since we will also often not specify the dependencies at all when they are clear from the context, we decided nevertheless to clearly specify them in the list of symbols, that you can find in page xi, for quick reference.

[1] Link to the web page:https://github.com/lucateo/KFT_thesis_program .

1.1.3 Inner products

A generic inner product is denoted with the $\cdot$ symbol; its definition is slightly different (but still very intuitive) based on the quantities it acts on:

$$\begin{aligned}
&\text{Between ordinary vectors : } \vec{a}\cdot\vec{b} = a_j b_j \,; \\
&\text{Between Kronecker products : } \boldsymbol{a}\cdot\boldsymbol{b} = (\vec{a}_j\otimes\vec{e}_j)(\vec{b}_j\otimes\vec{e}_j) = \vec{a}_j\cdot\vec{b}_j \,; \\
&\text{Between tuples of fields : } H\cdot\Phi = (H_f \;\; H_B)(\Phi_f \;\; \Phi_B)^\top = H_{\alpha_j}\cdot\Phi_{\alpha_j} \,.
\end{aligned}$$

We can also extend the meaning of the inner product, meaning also integration over the variables the quantities that are multiplied depend on, indicating it with $\bullet$, so that

$$A(r)\bullet B(r) := \int_r A(-r)\cdot B(r) \,.$$

We finally remark that we will always use, unless it is clearly specified otherwise, the Einstein summation convention, i.e. summation over repeated indices is understood.

1.1.4 Fourier transform

We will use the following convention for the Fourier transform

$$f(\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} f(\vec{k}) \,,$$

where we indicate the Fourier transform of f simply by specifying its dependence on $\vec{k}$.

Conversely, the inverse Fourier transform will be

$$f(\vec{k}) = \int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} f(\vec{x}) \,,$$

and thus the integral representation of the Dirac delta will be

$$\delta_{\mathrm{D}}(\vec{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \,.$$

1.2 Physical conventions

The metric signature we will use is the mostly positive $(-+++)$, often used in general relativity. We will use natural units, with $c = \hbar = 1$. We will convert to standard units (the one pertaining the international system of units or usual astronomical units like Mpc) when we will do quantitative analysis.

Quantities evaluated at the initial time will be denoted with the superscript (i), whereas quantities evaluated at the present time will be denoted with the subscript 0; we will also often indicate quantities evaluated in free theory (that is setting the interaction term to zero) with the subscript 0, but we think confusion will not arise since the difference between the two will always be clear.

As initial time, we will take one early in the matter dominated era (so that the initial matter density parameter $\Omega_{\rm m}^{(\rm i)} \simeq 1$), and we will normalize the Robertson-Walker scale factor a to be unity today, $a_0 := a(t_0) = 1$.

1.3 Cosmological setting summary

Here we will briefly summarize the cosmological model we will assume for our discussion.

1.3.1 Standard ΛCDM models

All cosmological models start from the following two assumptions, regrouped in what is called cosmological principle:

- *Isotropy*. The universe, on large scales, is substantially isotropic;
- *Homogeneity*. The universe has no preferred place, i.e. it is homogeneous.

Of course this is true as a first approximation (but still a very good one, expecially for early times and/or enough large scales, as the CMB spectrum and large scale surveys show), otherwise we won't see any structure at all and we won't exist.

Upon an isotropic and homogeneous universe, the most general line element compatible with these two symmetries is the Robertson-Walker one, that in spherical coordinates reads[2]

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right) ,$$

where $a(t)$ is the scale factor, encoding the expansion behavior of the universe, and where we are using comoving coordinates[3]. The curvature parameter κ can assume the values $0, +1, -1$, representing a flat, closed and open universe respectively. This line element is invariant under the redefinitions $r \to \lambda r$, $a \to \lambda^{-1}a$, $\kappa \to \lambda^{-2}\kappa$, so that we can fix the normalization of a. In this work, we will choose $a(t_0) = 1$.

Putting this line element in Einstein equations for general relativity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} ,$$

together with the stress-energy tensor for a perfect fluid (the appropriate choice within the symmetries we are assuming)

$$T_{\mu\nu} = (\bar{P}(t) + \bar{\rho}(t))u_\mu u_\nu + \bar{P}(t)g_{\mu\nu} ,$$

[2]Recall that in the following we will use natural units, so in particular we set the speed of light $c = 1$.

[3]The Robertson-Walker line element has this simple form only in comoving coordinates, i.e. with coordinates comoving with the "cosmic fluid" (that can be represented by the CMB, so that comoving coordinates are the ones which belong to an observer that does not see the dipole moment of the CMB.).

we end up with the first two Friedmann equations[4]

$$\left(\frac{\dot{a}}{a}\right)^2 =: H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{\kappa}{a^2} + \frac{\Lambda}{3} ,$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{P}) + \frac{\Lambda}{3} ;$$

the third Friedmann equation, also called continuity equation, can be derived from the previous two or also from the zero component of the continuity equation $T^{\mu\nu}{}_{;\nu} = 0$, and reads

$$\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P}) = 0 .$$

Friedmann equations are two independent equations for three unknowns $a, \bar{\rho}, \bar{P}$, so one usually assumes an equation of state like

$$\bar{P} = w\bar{\rho} .$$

Exploiting the previous, using the third Friedmann equation, we have

$$\bar{\rho}(t) = \bar{\rho}(t_0)\left(\frac{a(t)}{a(t_0)}\right)^{-3(1+w)} ,$$

and since $a(t_0) = 1$, we can ignore it in the following. Exploiting also the first Friedmann equation, we can derive from the previous

$$a \propto t^{2/3(1+w)} \implies H = \frac{2}{3(1+w)} .$$

Different components of the universe are characterized by a different equation of state, in particular $w = 0$ for matter and $w = 1/3$ for radiation; using these behaviors, we can express the first Friedmann in terms of the various density parameters today

$$\Omega_{j,0} := \frac{8\pi G}{3H^2}\bar{\rho}_{j,0} ,$$

where we used 0 as a subscript for quantities evaluated at the present time, and the index j labels the various components (r = radiation, m=matter, md= (cold) dark matter, mb = baryons, Λ = dark energy). So the first Friedmann equation becomes

$$H^2 = H_0^2\left(\Omega_{\mathrm{r},0}a^{-4} + \Omega_{\mathrm{m},0}a^{-3} + \Omega_{\kappa,0}a^{-2} + \Omega_{\Lambda,0}\right) . \tag{1.2}$$

The parameter $\Omega_{\kappa,0}$, that refers to the contribution of the curvature, is very close to zero [16] so that in the following we will consider the curvature κ to be negligible (this is also coherent with inflationary models); the models in which $\kappa = 0$ are also called Einstein-De Sitter models. Instead $\Omega_{\mathrm{r},0}$ refers to radiation, and it is of order $\simeq 10^{-5}$, so it can be neglected during the matter and dark energy dominated epochs (the ones we will consider).

[4]The notation we are using is pretty standard, but just to be clear: $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant (it is meant to represent dark energy here), $\bar{P}(t)$ and $\bar{\rho}(t)$ are the (background) pressure and energy density of the cosmic fluid, dependent only on cosmic time (otherwise they would break homogeneity and isotropy),u_μ is the (covariant) four velocity and $\dot{} := \mathrm{d}/\mathrm{d}t$.

In this thesis, as a background cosmological model, we will use a standard ΛCDM model, i.e. a universe whose expansion and background values of quantities like energy density and pressure are described by Friedmann equations, with parameters [16]

$$H_0 = 100\,h\frac{\mathrm{km}}{\mathrm{s\,Mpc}}\ ,\ h = 0.7\ ,\ \Omega_\mathrm{m} = 0.3\begin{cases}\Omega_{\mathrm{mb},0} = 0.04\\ \Omega_{\mathrm{md},0} = 0.26\end{cases}\ ,\ \Omega_{\Lambda,0} = 0.7\ ,\ \Omega_{\mathrm{r},0} \simeq 0\ .$$

1.3.2 Fluctuations and power spectra

In order to explain the growth of structures in the universe, we must deal with fluctuations around the background values, in particular we must deal with

$$\delta = \frac{\rho - \bar\rho}{\rho}\ ,$$

where ρ is the full density, comprehensive of the background mean density $\bar\rho$ (obeying Friedmann equations) and the fluctuation $\bar\rho\delta$.

We can view the density field as a stochastic field[5]. Inflation predicts primordial fluctuations to be gaussian, and a gaussian random field is completely determined once we know the first two correlators, i.e. the mean and the two-point correlation function. Since for δ the mean is zero by definition, we need only

$$\left\langle \delta(\vec r, t)\delta(\vec r\,', t)\right\rangle =: \xi_\delta(\left|\vec r - \vec r\,'\right|)\ ,$$

where, due to homogeneity and isotropy, ξ can only depend on the distance between the two points considered. The average is done over all the possible stochastic realizations of different universes. Usually in cosmology one focuses on the power spectrum, that is defined, in Fourier space, as

$$\left\langle \delta(\vec k, t)\delta(\vec k', t)\right\rangle =: (2\pi)^3\delta_\mathrm{D}(\vec k + \vec k')P_\delta(k,t)\ ;$$

using this definition, it is easy to see that the power spectrum is the Fourier transform of ξ_δ (this fact os also known as the Wiener-Khintchine theorem).

Inflationary models predict the following primordial power spectrum

$$P_\delta^{(\mathrm{p})} \propto k^{n_\mathrm{s}}\ , \tag{1.3}$$

where the superscript (p) here means primordial (i.e. immediately after inflation) and n_s is the scalar spectral index; from CMB measurements, $n_\mathrm{s} \simeq 0.96$ [16]; it is therefore very near unity (that is, the primordial power spectrum after inflation is very near to a perfect Harrison-Zel'dovich spectrum, which has $n_\mathrm{s} = 1$).

In a full relativistic treatment, one should take into account relativistic effects arising for scales larger than the comoving Hubble radius $r_\mathrm{H} := 1/aH$; for example, modes that are outside the Hubble radius (that is modes whose typical comoving length is greater than r_H) cannot be in causal contact.

[5] We are forced to model our universe as a stochastic realization of a statistical ensemble of possibilities, since we do not have direct observational access to the primordial perturbations seeds (which will

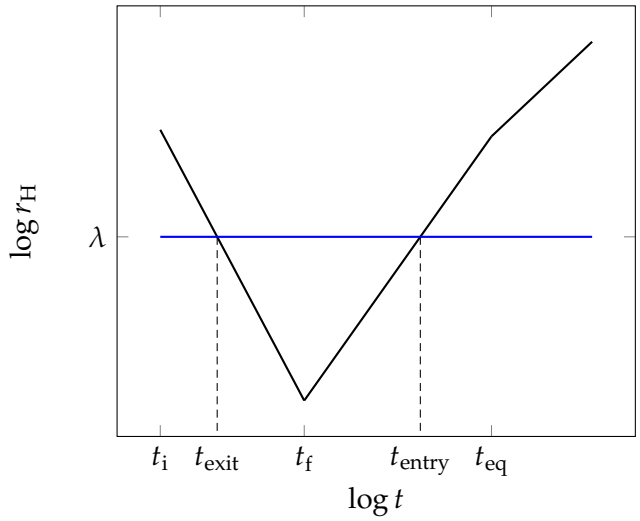


FIGURE 1.1: Behavior of the comoving Hubble radius with respect to time in log-log axes; t_i is the time when inflation begins, t_f when inflation ends, t_{eq} the equivalence between matter and radiation; here is represented a comoving scale λ, that exits the horizon at t_{exit} during inflation (as most of the modes relevant for structure formation do) and re-enters it at t_{entry} during radiation domination.

Of course r_H is not constant in time, in particular during accelerated expansion (as the one that hypothetically happened in the early universe during inflation, and the one is happening now, in dark energy dominated era) it shrinks, so that modes that were previously inside r_H can go out of it and they could be no more in causal contact. A picture of this behavior can be found in figure 1.1.

For scales smaller than the Hubble radius, the equation of motions reduce to the ones pertaining Newtonian gravity; the effects of the expansion of the universe are encoded only in the redefinition of the coordinates used (comoving coordinates are the proper ones) and in the redefinition of the gravitational potential.

1.3.3 Fluctuations evolution

To find the equations for the fluctuations, the proper setting is of course again general relativity; however, a full relativistic treatment must deal with subtleties like gauge problems and such (see [17]); for our purposes, since we will deal with cold dark matter (which is non relativistic) the Newtonian approach we mentioned in the previous section is more than enough.

Here we will briefly go through the Eulerian perturbation theory by exploiting the so called Vlasov equation for collisionless dark matter particles. Some of the results we will obtain in this section will be used in the main body of the thesis.

Lagrangian of collisionless dark matter. We want here to derive an expression for the lagrangian of collisionless dark matter particles in an expanding space-time. We will need this in order to obtain the Vlasov equation.

yield deterministic initial conditions) and also it is hopeless to study the evolution of a single, definite large scale structure due to the time scales involved. See [5] for more on this.

Consider at first the following lagrangian for a single particle of mass m

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 - m\varphi(\vec{q},t)\,,$$

where here we will consider $\vec{q} = \vec{r}/a$ as a comoving quantity, $\dot{} := \mathrm{d}/\mathrm{d}t$ and $\varphi(\vec{q},t)$ as the Newtonian gravitational potential satisfying

$$\nabla^2_{\vec{q}}\varphi(\vec{q},t) = 4\pi G a^2 \rho(\vec{q},t)\,, \tag{1.4}$$

where we recall $\rho(\vec{q},t) = \bar{\rho}(t)(1+\delta(\vec{q},t))$. Since if we change the lagrangian adding a total time derivative we won't change the equations of motion, we pass to

$$\mathcal{L}' = \mathcal{L} - \frac{\mathrm{d}\psi}{\mathrm{d}t}\,,\ \psi = \frac{1}{2}ma\dot{a}q^2\,,$$

and the new lagrangian becomes (we drop the prime)

$$\mathcal{L} = \frac{1}{2}ma^2\dot{q}^2 - \frac{m}{2}a\ddot{a}q^2 - m\varphi = \frac{1}{2}ma^2\dot{q}^2 - m\phi\,, \tag{1.5}$$

where in the last step, with $\phi := \varphi - \varphi_\mathrm{b}$, we exploited the second Friedmann equation (without the pressure term) and the expression for the background gravitational potential (solution of (1.4) with $\rho(\vec{q},t) = \bar{\rho}(t)$), respectively

$$\ddot{a} = -\frac{4\pi G}{3}a\bar{\rho}(t)\,;\ \varphi_\mathrm{b} = \frac{2}{3}\bar{\rho}(t)\pi G a^2 q^2\,.$$

Vlasov equation. We are interested in the collisionless Boltzmann equation, so we need the hamiltonian

$$\mathcal{H} = \vec{p}\cdot\vec{q} - \mathcal{L} = \frac{p^2}{2ma^2} + m\phi\,,$$

where we used

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{q}}} = ma^2\dot{\vec{q}}\,.$$

The collisionless Boltzmann equation for the particle distribution function $f(\vec{q},\vec{p},t)$ is

$$\mathbb{C}[f] = 0 = \frac{\partial f}{\partial t} + \dot{\vec{q}}\cdot\boldsymbol{\nabla}_{\vec{q}}f + \dot{\vec{p}}\cdot\frac{\mathrm{d}f}{\mathrm{d}\vec{p}}\,,$$

and using the Hamilton equations

$$\dot{\vec{q}} = \frac{\partial\mathcal{H}}{\partial\vec{p}} = \frac{\vec{p}}{ma^2}\,,$$
$$\dot{\vec{p}} = -\frac{\partial\mathcal{H}}{\partial\vec{q}} = -m\boldsymbol{\nabla}\phi\,,$$

we find the Vlasov equation

$$\boxed{\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2}\cdot\boldsymbol{\nabla}_{\vec{q}}f - m\boldsymbol{\nabla}_{\vec{q}}\phi\cdot\frac{\mathrm{d}f}{\mathrm{d}\vec{p}} = 0} \tag{1.6}$$

Continuity and Euler equations. Solving the non linear and non local Vlasov equation is not easy, so we consider its first two moments:

$$\rho(\vec{q},t) = \frac{m}{a^3}\int \mathrm{d}^3p\, f(\vec{q},\vec{p},t) \qquad \text{(zero moment)}$$

$$\vec{v}(\vec{q},t) = \frac{N}{ma}\int \mathrm{d}^3p\, \vec{p} f(\vec{q},\vec{p},t) = \frac{1}{a^4\rho}\int \mathrm{d}^3p\, \vec{p} f(\vec{q},\vec{p},t) \qquad \text{(first moment)}$$

where in the first we divided by a^3 since we are interested in the physical volume and not the comoving one, and on the last $N = (\int \mathrm{d}^3p\, f)^{-1} = m/a^3\rho$ is a normalization factor.

Now taking Vlasov equation, multiplying it by m and integrating over $\vec{p}$ we find

$$\dot{\rho} + 3H\rho + \frac{1}{a}\boldsymbol{\nabla}_{\vec{q}}\cdot(\rho\vec{v}) = 0\,, \tag{1.7}$$

that is the continuity equation, whereas if again we multiply Vlasov by p^i and integrate over $\vec{p}$ we find (after some work)

$$\frac{\partial v^i}{\partial t} + Hv^i + \frac{1}{a}v^j\frac{\partial v^i}{\partial x^j} = -\frac{1}{a}\frac{\partial\phi}{\partial x^i} - \frac{1}{a}\frac{\partial}{\partial x^j}\left(\Pi^{ij}\rho\right)\,,$$

where at the end we have a second moment related quantity (that represents the dispersion of velocity)

$$\Pi^{ij} = \frac{\langle p^i p^j\rangle}{m^2a^2} - v^iv^j\,,$$

where here the angle brackets means average over phase space coordinates. We see that an equation for the n-th moment contains also the $n+1$-th moment. To truncate this set of equations, we can assume for example that the Π^{ij} term is zero (i.e. negligible velocity dispersion, this is the so called single stream approximation), thus obtaining

$$\frac{\partial\vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\boldsymbol{\nabla})\vec{v} = -\frac{1}{a}\boldsymbol{\nabla}\phi\,, \tag{1.8}$$

that is the Euler equation for a non static background.

Fluctuation equations. Expressing (1.4) with respect to fluctuation δ, we obtain (exploiting the third Friedmann equation $\dot{\bar{\rho}} = -3H\bar{\rho}$ to get rid of the background density)

$$\dot{\delta} + \frac{1}{a}\boldsymbol{\nabla}\cdot((1+\delta)\vec{v}) = 0\,;$$

take its time derivative and exploit (1.8) and (1.4) to obtain, at linear level on the fluctuations δ, v, ϕ

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0\,. \tag{1.9}$$

This is the equation for fluctuations at linear level, valid as long as the fluctuations remain small. Using the first Friedmann equation, we can rewrite it as (assuming the relevant Ω to be unity)

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}H^2\delta = 0\,; \tag{1.10}$$

the analogous treatment for radiation (accounting properly for the pressure [18]) brings one to the equation

$$\ddot{\delta} + 2H\dot{\delta} - 4H^2\delta = 0\,.$$

In Einstein-De Sitter models, $H = 2/3(1+w)t$; with the ansatz $\delta \propto t^n$, we find

$$n = \begin{cases} 2/3\,,\ -1 & \text{matter dominated,} \\ \pm 1\,, & \text{radiation dominated.} \end{cases}$$

We clearly see that we have a growing mode and a decaying mode, whose definition can be respectively read in

$$\delta(t) = \delta(t^{(\mathrm{i})})\frac{D_\pm(t)}{D_\pm(t^{(\mathrm{i})})}\,.$$

The linearly evolved power spectrum, neglecting the decaying mode, will thus be $P_\delta^{\mathrm{linear}} \propto D_+^2 P_\delta^{(\mathrm{i})}$, where D_+ here is the growing mode for pressureless matter.

We can insert D_+ back in (1.10) and obtain (restoring Ω_m)

$$\ddot{D}_+ + 2H\dot{D}_+ - \frac{3}{2}H^2\Omega_\mathrm{m}D_+ = 0\,;$$

a good fit formula for the solution of the previous for the standard ΛCDM model is found to be (see [19–21])

$$D_+(a) = \frac{5a}{2}\Omega_\mathrm{m}\left(\Omega_\mathrm{m}^{4/7} - \Omega_\Lambda + \left(1+\frac{\Omega_\mathrm{m}}{2}\right)\left(1+\frac{\Omega_\Lambda}{70}\right)\right)^{-1}\,. \tag{1.11}$$

Throughout this thesis, whenever we have to compute cosmological quantities, we will use (1.2) and (1.11) respectively for Hubble parameter and for growth factor related quantities. In chapter 4 we will mostly evaluate quantities using as a time coordinate the scale factor a; to convert back to cosmic time, one can use simple redshift-time formulas like

$$\begin{aligned} t =& \frac{1}{H_0\Omega_{\Lambda,0}^{1/2}}\int_0^a \frac{\mathrm{d}a'}{a'}\left(\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}}a'^{-3}+1\right)^{-1/2} \\ =& \frac{1}{3H_0\Omega_{\Lambda,0}^{1/2}}\ln\frac{\epsilon+1}{\epsilon-1}\,,\ \ \epsilon := \left(\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}}a^{-3}+1\right)^{1/2}\,, \end{aligned} \tag{1.12}$$

obtained integrating (1.2) for a flat ΛCDM model. It will be useful sometimes to use the growing factor D_+, instead of the cosmic time t as a time variable; this is something we will also do, as one can see in section 3.1.

1.3.4 Power spectrum after radiation domination

According to the standard hot big bang model (see for example [18]), the Universe at very early stages underwent an inflationary expansion, and the quantum fluctuations of the hypothetical scalar field that causes it (the inflaton) are believed to be the seeds of density fluctuations that, thanks to Jean's instability mechanisms and the likes, will form the structures we see all around us. Theories of inflation predicts

the power spectrum to be very close (but not equal) to the Harrison-Zel'dovich one (1.3); then the universe underwent the radiation dominated era.

During radiation domination, we derived $D_+ \propto t \propto a^2$; but fluctuation modes that were inside the Hubble radius were suppressed by radiation pressure, whereas the ones that were outside of it keep increasing like a^2, before entering again. This translates in a suppression factor proportional to k^{-2}. The power spectrum at small scales will thus have a k^{-4} suppression during this period, until the matter dominated era.

Taking as initial time one early in the matter dominated era, approximating $n_\mathrm{s} \approx 1$, we have thus

$$P_\delta^{(\mathrm{i})}(k) \sim \begin{cases} k & k \ll k_\mathrm{eq} \\ k^{-3} & k \gg k_\mathrm{eq} \end{cases} , \tag{1.13}$$

where k_eq is the wavenumber corresponding to the Hubble radius at matter-radiation equivalence, $k_\mathrm{eq} \simeq 0.025\, h\,\mathrm{Mpc}^{-1}$. This is the typical behavior of power spectrum for dark matter at the beginning of matter dominated era, where the large scale behavior is still set by inflation whereas the small scale behavior feel the radiation damping. We remark that, at the time of matter-radiation equivalence, linear theory is still expected to hold.

We have a starting shape for the power spectrum; its overall amplitude can instead be determined by the galactic variance today in a scale $R = 8\ \mathrm{Mpc}\, h^{-1}$, defined by

$$\sigma_8 = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P_\delta(k, t_0) W_R^2(k) , \tag{1.14}$$

where $W_R(k)$ is the window function filtering a scale R. We will take a value of $\sigma_8 = 0.8$ throughout this thesis.

Eulerian theory we briefly sketched here and other perturbative approaches have their problems, and it is difficult to make them work also in the non linear regime. This is why one is interested in other possible approaches, attempting to at least tackle the weakly non-linear regime of LSS formation. Kinetic field theory is one of them, and we will introduce it to the reader on next chapter.

Chapter 2

Kinetic Field Theory approach

Kinetic Field Theory (KFT) is basically a non equilibrium statistical field theory for classical fields, in particular we will consider as fields the phase space coordinates $\vec{x}_j = (\vec{q}_j, \vec{p}_j)^\top$, where the index j, $1 \leq j \leq N$, labels the particle we are considering and N is the number of particles. These particles are meant to represent macroscopic objects like galaxies and such (the usual starting point in cosmology); notice that this is also the starting assumption of N-body simulations (see for example [22]).

For these particle phase space coordinates, the Hamilton equations

$$\partial_t \vec{x}_j = \mathcal{J} \partial_j \mathcal{H} \,, \tag{2.1}$$

hold, where $\mathcal{H}$ is the hamiltonian and $\mathcal{J}$ is the symplectic matrix

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbb{1}_3 \\ -\mathbb{1}_3 & 0 \end{pmatrix} \,, \tag{2.2}$$

and $\mathbb{1}_n$ is the n dimensional identity matrix.

It is important to remark at this point that, since the particles we are considering follow the hamiltonian flow, and since trajectories in phase space do not cross, we will avoid the notorious problem of shell crossing [5].

We want to consider systems of N particles, so we introduce the following Kronecker products:

$$\boldsymbol{q} = \vec{q}_j \otimes \vec{e}_j \,, \; \boldsymbol{p} = \vec{p}_j \otimes \vec{e}_j \,,$$

and also

$$\boldsymbol{x} = \vec{x}_j \otimes \vec{e}_j \,, \tag{2.3}$$

where $\vec{e}_j$ is the N dimensional vector whose the only non zero entry is the j-th one and it is equal to one, $(\vec{e}_j)_k = \delta_{jk}$, and the summation over j is implied. We will use boldface symbols for quantities that are Kronecker products, all defined in the same way as in (2.3), in the following. We introduce also the scalar product between two Kronecker products as

$$\boldsymbol{a} \cdot \boldsymbol{b} := \vec{a}_j \cdot \vec{b}_j \,.$$

With these definitions we can concisely express for example the Hamilton equations (2.1) for all N particles, as

$$\partial_t \boldsymbol{x} = \boldsymbol{F} \mathcal{H} \,, \tag{2.4}$$

with

$$\boldsymbol{F} := (\mathcal{J} \otimes \mathbb{1}_N) \cdot (\partial_j \otimes \vec{e}_j) \tag{2.5}$$

(more insights on Kronecker products are given in appendix A.1). The advantage of this notation with Kronecker products is thus clear, we can concisely express quantities that involve all the particles we are considering without getting mad with too many indices.

2.1 Generating functional formalism

We now develop, as it is common in statistical field theories, the generating functional for our system of N classical particles.

At first we rewrite (2.4) in the form

$$\boldsymbol{E}(\boldsymbol{x}) = \partial_t \boldsymbol{x} + \boldsymbol{F}\mathcal{H}_0 + \partial_{(-\mathcal{J}\boldsymbol{x}_q)}\mathcal{V} = 0\,, \tag{2.6}$$

where we split the free part of the equation of motion (here represented by $\mathcal{H}_0$) and the interaction part (with the interaction potential $\mathcal{V}(\vec{q}, t)$). We also used the notation

$$\partial_{(-\mathcal{J}\boldsymbol{x}_q)} := -\frac{\partial}{\partial \mathcal{J}\vec{x}_{q_j}} \otimes \vec{e}_j\,, \quad \vec{x}_{q_j} := \begin{pmatrix} \vec{q}_j \\ 0 \end{pmatrix}\,. \tag{2.7}$$

We want a path integral representation of our system. Since the system is classical, the path that solves the equation of motion is unique, call it $\boldsymbol{x}_{\mathrm{cl}}$; we can use the functional Dirac delta and express the probability that the configuration $\boldsymbol{x}$ is in $\boldsymbol{x}(t_{\mathrm{f}})$ at a time t_{f} given that at $t = t^{(\mathrm{i})}$ was at $\boldsymbol{x}(t^{(\mathrm{i})})$ as

$$P[\boldsymbol{x}(t_{\mathrm{f}}), \boldsymbol{x}(t^{(\mathrm{i})})] = \int \mathcal{D}''\boldsymbol{x}\, \delta_{\mathrm{D}}[\boldsymbol{x}(t_{\mathrm{f}}) - \boldsymbol{x}_{\mathrm{cl}}[t_{\mathrm{f}}, \boldsymbol{x}(t^{(\mathrm{i})})]]\,, \tag{2.8}$$

where we remarked the fact that $\boldsymbol{x}_{\mathrm{cl}}$ is completely specified by the initial conditions, and $\mathcal{D}''\boldsymbol{x}$ means that upon integration I keep $\boldsymbol{x}$ fixed at the two extrema $t = t^{(\mathrm{i})}$ and $t = t_{\mathrm{f}}$. This is the expression for classical path integral (see also [23]), and its meaning is simple: we are just summing over all paths $\boldsymbol{x}(t)$ at fixed extrema, and the delta ensures that the only contribution is given by the solution of the equation of motion (2.4).

We can exploit the integral representation of the delta to write

$$\delta_{\mathrm{D}}[\boldsymbol{f}(\boldsymbol{x})] = \int \mathcal{D}\boldsymbol{\chi} \exp\left(\mathrm{i}\int \mathrm{d}t\, \boldsymbol{\chi}\cdot\boldsymbol{f}(\boldsymbol{x})\right)\,, \tag{2.9}$$

with $\boldsymbol{f}$ a generic 6-N dimensional vector, where we introduced the auxiliary field

$$\boldsymbol{\chi}(t) := \begin{pmatrix} \vec{\chi}_{q_j}(t) \\ \vec{\chi}_{p_j}(t) \end{pmatrix} \otimes \vec{e}_j\,, \tag{2.10}$$

that we needed to perform the integration. We thus find (see appendix B.1 for details)

$$P[\boldsymbol{x}(t_{\mathrm{f}}), \boldsymbol{x}(t^{(\mathrm{i})})] = \int \mathcal{D}''\boldsymbol{x}\, \mathcal{D}\boldsymbol{\chi}\, \mathrm{e}^{\mathrm{i}S}\,, \tag{2.11}$$

where S is the action

$$S[\boldsymbol{x}, \boldsymbol{\chi}] = \int_{t^{(\mathrm{i})}}^{t_{\mathrm{f}}} \mathrm{d}t\, \boldsymbol{\chi}\cdot\boldsymbol{E}[\boldsymbol{x}]\,. \tag{2.12}$$

Introducing random initial conditions, we should average the (2.11) with

$$\int \mathrm{d}\boldsymbol{x}^{(\mathrm{i})}\, P_0(\boldsymbol{x}^{(\mathrm{i})}) =: \int \mathrm{d}\boldsymbol{\Gamma}_{\mathbf{i}} \ , \tag{2.13}$$

where $\boldsymbol{x}^{(\mathrm{i})} := \boldsymbol{x}(t = t^{(\mathrm{i})})$ and $P_0(\boldsymbol{x}^{(\mathrm{i})})$ is the probability assigned to the initial configuration $\boldsymbol{x}^{(\mathrm{i})}$. Notice that here we are not dealing with an integration over functions, hence this is an ordinary integration over $6N$ dimensions.

In the end the generating functional is obtained by introducing the source fields

$$\boldsymbol{J}(t) := \begin{pmatrix} \vec{J}_{q_j}(t) \\ \vec{J}_{p_j}(t) \end{pmatrix} \otimes \vec{e}_j \ , \quad \boldsymbol{K}(t) := \begin{pmatrix} \vec{K}_{q_j}(t) \\ \vec{K}_{p_j}(t) \end{pmatrix} \otimes \vec{e}_j \ , \tag{2.14}$$

yielding the canonical generating functional[1]

$$Z_{\mathrm{C}}[\boldsymbol{J}, \boldsymbol{K}] = \int \mathrm{d}\boldsymbol{\Gamma}_{\mathbf{i}} \int \mathcal{D}'\boldsymbol{x}\, \mathcal{D}\boldsymbol{\chi} \exp\left(\mathrm{i} \int_{t^{(\mathrm{i})}}^{t_{\mathrm{f}}} \mathrm{d}t \, (\boldsymbol{\chi} \cdot \boldsymbol{E}[\boldsymbol{x}] + \boldsymbol{J} \cdot \boldsymbol{x} + \boldsymbol{K} \cdot \boldsymbol{\chi}) \right) , \tag{2.15}$$

where we have integrated also over the final configurations, here represented by $\mathcal{D}'\boldsymbol{x}$ where the single prime means that the integration over the paths is made by keeping fixed only the initial conditions.

Looking at the free generating functional $Z_{\mathrm{C},0}$ (that is the previous with the substitution $\boldsymbol{E} \to \boldsymbol{E}_0 := \partial_t \boldsymbol{x} + \boldsymbol{F}\mathcal{H}_0$, this last step coming from (2.6)), we can perform some integrations to arrive at (see appendix B.1)

$$Z_{\mathrm{C},0}[\boldsymbol{J}, \boldsymbol{K}] = \int \mathrm{d}\boldsymbol{\Gamma}_{\mathbf{i}} \exp\left(\mathrm{i} \int_{t^{(\mathrm{i})}}^{t_{\mathrm{f}}} \mathrm{d}t \, \boldsymbol{J} \cdot \bar{\boldsymbol{x}} \right) , \tag{2.16}$$

with $\bar{\boldsymbol{x}}(t)$ as the solution of the free equation of motion coupled with the source $\boldsymbol{K}$, i.e.

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{G}}(t, t^{(\mathrm{i})})\boldsymbol{x}^{(\mathrm{i})} - \int_{t^{(\mathrm{i})}}^{t_{\mathrm{f}}} \mathrm{d}t' \, \boldsymbol{\mathcal{G}}(t, t')\boldsymbol{K}(t') \ , \tag{2.17}$$

and with the propagator

$$\boldsymbol{\mathcal{G}}(t, t') = \mathcal{G}(t, t') \otimes \mathbb{1}_N \ , \quad \mathcal{G}(t, t') = \begin{pmatrix} g_{qq}(t, t')\mathbb{1}_3 & g_{qp}(t, t')\mathbb{1}_3 \\ g_{pq}(t, t')\mathbb{1}_3 & g_{pp}(t, t')\mathbb{1}_3 \end{pmatrix} \ . \tag{2.18}$$

2.2 Collective fields

We want to use KFT in the cosmic structure formation setting, so we should now connect the microscopic phase space coordinates fields with the macroscopic ones, representing collective properties of the particle ensemble. Since the primary concern of cosmic structure formation regards density fluctuations, we shall certainly connect the collective density field with the particle ensemble properties. For N point particles with equal mass (here supposed to be unity only for simplicity), the

[1] We are dealing with a canonical ensemble since we are considering the number of particles N as fixed.

collective density field is given by a sum of delta contributions,

$$\rho(\vec{q},t) = \sum_{j=1}^{N} \delta_{\mathrm{D}}(\vec{q} - \vec{q}_j(t)) \,. \tag{2.19}$$

Now assume that we can write the potential as a sum over all point-particle potentials v as

$$\mathcal{V}(\vec{q},t) = \sum_j v(\vec{q} - \vec{q}_j(t)) \,; \tag{2.20}$$

with this (not severe) assumption we can recast in a useful form the interaction term $\boldsymbol{E}_1$ coming from (2.6),

$$\boldsymbol{E}_1(\boldsymbol{q}) := \begin{pmatrix} 0 \\ \partial_{\vec{q}_j} \end{pmatrix} \mathcal{V} \otimes \vec{e}_j \,;$$

we can express it in terms of the density field as

$$\mathcal{V} = \int \mathrm{d}^3 y \, v(\vec{q} - \vec{y}) \sum_{j=1}^{N} \delta_{\mathrm{D}}(\vec{y} - \vec{q}_j(t)) = \int \mathrm{d}^3 y \, v(\vec{q} - \vec{y}) \rho(\vec{y},t) \,. \tag{2.21}$$

Exploit the Dirac delta to write

$$\partial_{\vec{q}_j} \mathcal{V} = \int \mathrm{d}^3 q \, \delta_{\mathrm{D}}(\vec{q} - \vec{q}_j) \partial_{\vec{q}} \mathcal{V} = - \int \mathrm{d}^3 q \, (\partial_{\vec{q}} \delta_{\mathrm{D}}(\vec{q} - \vec{q}_j)) \mathcal{V} \,;$$

defining the response field

$$B(\vec{q},t) := \sum_j \vec{\chi}_{p_j} \cdot \partial_{\vec{q}} \delta_{\mathrm{D}}(\vec{q} - \vec{q}_j(t)) \,, \tag{2.22}$$

we can write the interaction part of the action, exploiting also (2.21), as

$$S_1 := \int \mathrm{d}t \, \boldsymbol{\chi} \cdot \boldsymbol{E}_1(\boldsymbol{x}) = - \int \mathrm{d}t \int \mathrm{d}^3 q \int \mathrm{d}^3 y \, B(\vec{q},t) v(\vec{q} - \vec{y}) \rho(\vec{y},t) \,. \tag{2.23}$$

From now on, we will use the shortcut notations for dependencies and integrals as stated in section 1.1.2, i.e. we indicate only the label r (generally an integer number) for the whole dependencies.

So expressing everything with respect to Fourier transform quantities the (2.23), one can easily arrive at

$$S_1 = -(2\pi)^3 \int_r B(-r) v(r) \rho(r) \delta_{\mathrm{D}}(\vec{l}_r) \,, \tag{2.24}$$

where $\delta_{\mathrm{D}}(\vec{l}_r)$ is there since the potential does not depend on momenta.

So we need the expression of ρ, B in Fourier space. Notice that, looking at (2.19) and (2.22), they involve a sum over all the particles, so that we can consider their so called one particle contribution. For the field density

$$\rho_j(\vec{q},t) := \delta_{\mathrm{D}}(\vec{q} - \vec{q}_j(t)) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot(\vec{q} - \vec{q}_j(t))} =: \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}} \rho_j(\vec{k},t) \,,$$

so the one particle contribution to density field in Fourier space is

$$\rho_j(r) = \mathrm{e}^{-\mathrm{i}\vec{k}_r \cdot \vec{q}_j(t_r)} \,; \tag{2.25}$$

analogously, the one particle contribution for the response field B in Fourier space will be

$$B_j(r) = \mathrm{i}\vec{k}_r \cdot \vec{\chi}_{p_j}(t_r)\rho_j(r) \,. \tag{2.26}$$

We thus introduce the so called Klimontovich doublet of collective fields

$$\Phi(r) = \begin{pmatrix} \Phi_f(r) \\ \Phi_B(r) \end{pmatrix} := \sum_{j=1}^{N} \begin{pmatrix} \rho_j(\vec{k}_r, t_r)\mathrm{e}^{-\mathrm{i}\vec{l}_r\cdot\vec{p}_j(t_r)} \\ B_j(\vec{k}_r, t_r)\mathrm{e}^{-\mathrm{i}\vec{l}_r\cdot\vec{p}_j(t_r)} \end{pmatrix} \,, \tag{2.27}$$

that reduces to ρ, B if they are evaluated at $\vec{l} = 0$. In the end we can write (explicit calculations are made in appendix B.3)

$$S_1 = \int_r \int_{r'} \Phi^\top(-r)\sigma(r,-r')\Phi(r) = \Phi \bullet \sigma \bullet \Phi \,, \tag{2.28}$$

with

$$\sigma(r,r') = -\frac{1}{2}(2\pi)^9 \delta_{\mathrm{D}}(t_r - t_r')\delta_{\mathrm{D}}(\vec{k}_r + \vec{k}_r')\delta_{\mathrm{D}}(\vec{l}_r)\delta_{\mathrm{D}}(\vec{l}_r')v(r) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,. \tag{2.29}$$

Since we have introduced these new fields, we should also introduce accordingly other source fields, which will couple to the collective fields in the generating functional, so we will introduce the term

$$\exp\left(\mathrm{i}\int_r H(-r)\cdot\Phi(r)\right) = \mathrm{e}^{\mathrm{i}H\bullet\Phi} \,,\; H(r) = \begin{pmatrix} H_f(r) \\ H_B(r) \end{pmatrix} \,, \tag{2.30}$$

inside $Z_{\mathrm{C},0}$[2].

To finally write the full generating functional, we only need the operator form of the new collective and source fields, so that, as it is common in QFT, we can write the interaction part as an exponential of operators acting on the free functional $Z_{\mathrm{C},0}$.

The operator form of (2.26) and (2.25) is given by the substitution

$$\boldsymbol{x}(t) \to -\mathrm{i}\frac{\delta}{\delta \boldsymbol{J}(t)} \,,\; \boldsymbol{\chi}(t) \to -\mathrm{i}\frac{\delta}{\delta \boldsymbol{K}(t)} \,, \tag{2.31}$$

so that we can write[3]

$$Z_{\mathrm{C},0}[H, \boldsymbol{J}, \boldsymbol{K}] = \exp\left(\mathrm{i}\int_r H(r)\cdot\hat{\Phi}(r)\right) Z_{\mathrm{C},0}[\boldsymbol{J}, \boldsymbol{K}] \,, \tag{2.32}$$

[2]Notice that the source field H depends also on wavenumbers $\vec{k}$ and $\vec{l}$ differently from J, K that depend only on time. This is coherent, since collective fields are macroscopic fields.

[3]Some insights on this approach in including terms on the generating functional by means of operators is given in appendix B.2.

where $\hat{\Phi}$ is the operator version of (2.27), whose explicit expression is, with $\vec{s} = (\vec{k} \;\; \vec{I})^\top$

$$\begin{aligned}\hat{\Phi}(r) &= \sum_j \begin{pmatrix} \exp\left(-\vec{s}_r \cdot \delta \Big/ \delta \vec{J}_j(t_r)\right) \\ \left(\vec{k}_r \cdot \delta \Big/ \delta \vec{K}_{p_j}(t_r)\right) \exp\left(-\vec{s}_r \cdot \delta \Big/ \delta \vec{J}_j(t_r)\right) \end{pmatrix} \\ &=: \sum_j \hat{\Phi}_j^{(1)}(r) =: \sum_j \begin{pmatrix} \hat{\Phi}_{f_j}^{(1)}(r) \\ \hat{\Phi}_{B_j}^{(1)}(r) \end{pmatrix}\end{aligned} \tag{2.33}$$

where one can see from the structure of $\hat{\Phi}_{B_j}$ that we can write

$$\hat{\Phi}_{B_j}^{(1)} = \hat{b}_j(r)\hat{\Phi}_{f_j}^{(1)}(r) \;, \quad \hat{b}_j(r) := \vec{k}_r \cdot \frac{\delta}{\delta \vec{K}_{p_j}(t_r)} \;. \tag{2.34}$$

We can now include the interaction term (2.28) again using its operator form, obtained with the substitution

$$\Phi(r) \to \hat{H}(r) := -\mathrm{i}\frac{\delta}{\delta H(-r)} \;, \tag{2.35}$$

so that

$$\hat{S}_1 = \int_r \int_{r'} \hat{H}^\top(-r)\sigma(r,-r')\hat{H}(r') = \hat{H} \bullet \sigma \bullet \hat{H} \;. \tag{2.36}$$

At last, the full generating functional can be written as

$$Z_{\mathrm{C}}[H, \boldsymbol{J}, \boldsymbol{K}] = \mathrm{e}^{\mathrm{i}\hat{S}_1}\mathrm{e}^{\mathrm{i}H\bullet\hat{\Phi}} Z_{\mathrm{C},0}[\boldsymbol{J}, \boldsymbol{K}] \;. \tag{2.37}$$

2.3 Initial phase space probability distribution

We need now to specify the initial probability distribution. Since our aim is to apply the machinery we developed to cosmic structure formation, the microscopic degrees of freedom need to be correlated in such a way that the corresponding density, a macroscopic (or collective, since it resumes a collective property of the particle ensemble we are considering) field, together with the comoving peculiar momenta $\vec{P}^{(\mathrm{i})}(\vec{q})$, become homogeneous and isotropic gaussian fields[4]. Calling the density contrast

$$\delta(\vec{q}, t) := \frac{\rho(\vec{q}, t) - \bar{\rho}}{\bar{\rho}} \;, \tag{2.38}$$

where $\bar{\rho}$ is the mean density, we recall that we can define the power spectrum as

$$\left\langle \delta(\vec{k}, t)\delta(\vec{k}', t) \right\rangle =: (2\pi)^3 \delta_{\mathrm{D}}(\vec{k} + \vec{k}') P_\delta(k, t) \;. \tag{2.39}$$

Since we are discussing random gaussian fields, the power spectrum (together with the mean) is all we need to know; what we should do now is to see how our assumptions, made on the macroscopic density, affects $P(\boldsymbol{x}^{(\mathrm{i})})$ that involves particles. The

[4]Here we will not deal with primordial non gaussianities, that to the best of our current knowledge (see [24]) are not relevant.

derivation is not straightforward, so we will do it in next subsection and summarize the main formulas and results in subsection 2.3.2.

2.3.1 Derivation of $P(\boldsymbol{x}^{(\mathrm{i})})$

In order to derive a $P(\boldsymbol{x}^{(\mathrm{i})})$ relevant for cosmic structure formation, the standard assumption is that the initial velocity field is irrotational[5], i.e. we can define a velocity potential ψ such that

$$\vec{u}^{(\mathrm{i})} = \boldsymbol{\nabla}\psi \,, \tag{2.40}$$

where in this whole section all the derivatives are done with respect to initial configuration variables $\vec{q}^{(\mathrm{i})}$. Writing

$$\vec{q}(t) = \vec{q}^{(\mathrm{i})} + b(t)\vec{u}^{(\mathrm{i})} \,, \tag{2.41}$$

for a certain monotonic function b with normalization $b(t^{(\mathrm{i})}) = 1$, we have, exploiting continuity

$$\begin{aligned} \rho(t)\,\mathrm{d}^3q(t) =& \rho(t^{(\mathrm{i})})\,\mathrm{d}^3q^{(\mathrm{i})} = \bar{\rho}\left|\det\left(\frac{\partial\vec{q}^{(\mathrm{i})}}{\partial\vec{q}(t)}\right)\right|\mathrm{d}^3q(t) \\ \implies \rho(t) =& \bar{\rho}\left|\det\left(\delta_{ij} + b(t)\frac{\partial^2\psi}{\partial q_i^{(\mathrm{i})}\partial q_j^{(\mathrm{i})}}\right)\right|^{-1} \end{aligned}$$

where $\bar{\rho} = \rho(t^{(\mathrm{i})})$ is the mean density, and we used (2.41) to simplify the expression of the jacobian; exploiting the generic approximation valid at linear level

$$\det(\mathbb{1} + A) = \mathrm{e}^{\ln\det(\mathbb{1}+A)} = \mathrm{e}^{\mathrm{Tr}\ln(\mathbb{1}+A)} \simeq \mathrm{e}^{\mathrm{Tr}\,A} \simeq 1 + \mathrm{Tr}\,A \,,$$

that holds for a generic $n \times n$ matrix A (considered as a perturbation), we can write (at linear level in the fluctuations)

$$\begin{aligned} \rho(t) \simeq& \bar{\rho}(1 + b(t)\nabla^2\psi)^{-1} \simeq \bar{\rho}(1 - b(t)\nabla^2\psi) =: \bar{\rho}(1 + \delta(t)) \\ \implies& \delta(t^{(\mathrm{i})}) =: \delta^{(\mathrm{i})} = -\nabla^2\psi \,. \end{aligned} \tag{2.42}$$

If we consider particles with unitary mass, we write for the initial momentum simply

$$\vec{P}^{(\mathrm{i})} = \boldsymbol{\nabla}\psi \,, \tag{2.43}$$

so that both the initial density contrast and the initial momentum are related by the velocity potential. Since the remaining discussion in this subsection involves quantities evaluated at initial time, we can drop the superscript (i) in the following.

We can write the probability for finding a particle labeled with j at position $\vec{q}_j$ with momentum $\vec{p}_j$ as

$$P(\vec{q}_j, \vec{p}_j) = \int \mathrm{d}\delta(\vec{q}_j) \int \mathrm{d}\vec{P}(\vec{q}_j)\, P(\vec{q}_j|\delta(\vec{q}_j))P(\vec{p}_j|\vec{P}(\vec{q}_j))P(\delta(\vec{q}_j), \vec{P}(\vec{q}_j)) \,; \tag{2.44}$$

[5]This assumption relies on the fact that, at linear level, vector perturbations usually decay (see [17])

now, recalling the definition of density given in (2.19), the probability that the particle we are considering is in $\vec{q}_j$, given the density value at the same point, will be proportional to this value[6], thus

$$P(\vec{q}_j|\rho(\vec{q}_j)) = \frac{\rho(\vec{q}_j)}{N} ,$$

where the factor N, the number of particles, comes from normalization condition $\int \rho(\vec{q})\,\mathrm{d}V = N$. This, using the definition of the density contrast (2.38), implies

$$P(\vec{q}_j|\delta(\vec{q}_j)) = \frac{\bar{\rho}}{N}(1+\delta(\vec{q}_j)) ;$$

using also

$$P(\vec{p}_j|\vec{P}(\vec{q}_j)) = \delta_{\mathrm{D}}(\vec{p}_j - \vec{P}(\vec{q}_j)) ,$$

that means that $\vec{p}_j$ and $\vec{P}(\vec{q}_j)$ have basically the same meaning in this context (remember we are evaluating everything at the initial time), we arrive at

$$P(\vec{q}_j, \vec{p}_j) = \frac{\bar{\rho}}{N}\int \mathrm{d}\delta(\vec{q}_j)\,(1+\delta(\vec{q}_j))P(\delta(\vec{q}_j), \vec{p}_j) .$$

The generalization to the full Kronecker product is straightforward,

$$P(\boldsymbol{q}, \boldsymbol{p}) = V^{-N}\prod_{j=1}^{N}\left(\int \mathrm{d}\delta(\vec{q}_j)\,(1+\delta(\vec{q}_j))P(\delta(\vec{q}_j), \vec{p}_j)\right) , \tag{2.45}$$

where $V = N\bar{\rho}^{-1}$.

We need now to find $P(\delta(\vec{q}_j), \vec{p}_j)$. To evaluate it, we consider its characteristic function; since we are discussing gaussian random fields with zero mean, the corresponding characteristic function, regrouping for brevity the random fields in the doublet

$$\boldsymbol{d} := \begin{pmatrix}\delta(\vec{q}_j)\\ \vec{P}(\vec{q}_j)\end{pmatrix}\otimes\vec{e}_j = \begin{pmatrix}\delta(\vec{q}_j)\\ \vec{p}_j\end{pmatrix}\otimes\vec{e}_j =: \begin{pmatrix}\boldsymbol{d}_\delta\\ \boldsymbol{d}_p\end{pmatrix} ,$$

will be

$$\Phi_{\boldsymbol{d}}(\boldsymbol{t}) = \exp\left(-\frac{1}{2}\boldsymbol{t}^\top\cdot\boldsymbol{C}\cdot\boldsymbol{t}\right) , \tag{2.46}$$

with $\boldsymbol{t}$ as the Fourier conjugate of $\boldsymbol{d}$ and $\boldsymbol{C}$ as the N-point covariance matrix

$$\boldsymbol{C} := C_{jk}\otimes(\vec{e}_j\otimes\vec{e}_k) =: \begin{pmatrix}C_{\delta_j\delta_k} & \vec{C}^\top_{\delta_j p_k}\\ \vec{C}_{p_j\delta_k} & \boldsymbol{C}_{p_j p_k}\end{pmatrix}\otimes(\vec{e}_j\otimes\vec{e}_k) =: \begin{pmatrix}\boldsymbol{C}_{\delta\delta} & \boldsymbol{C}_{\delta p}\\ \boldsymbol{C}_{p\delta} & \boldsymbol{C}_{pp}\end{pmatrix} ,$$

with

$$C_{jk} := \begin{pmatrix}\langle\delta(\vec{q}_j)\delta(\vec{q}_k)\rangle & \left\langle\delta(\vec{q}_j)\vec{P}(\vec{q}_k)\right\rangle^\top\\ \left\langle\vec{P}(\vec{q}_j)\delta(\vec{q}_k)\right\rangle & \left\langle\vec{P}(\vec{q}_j)\otimes\vec{P}^\top(\vec{q}_k)\right\rangle\end{pmatrix} .$$

[6]Here the label j means that we are considering the particle j, but the corresponding $\vec{q}_j$ can range over all the possible values the coordinate can take; what we are saying here is that, since the contribution to the density is a sum of Dirac delta contributions, the more "deltas" I have on the point $\vec{q}$, the more probable a particle can be found there, and if I have no deltas in the point $\vec{q}$, then of course I must have a zero probability to find any particle there.

We can connect every element of the covariance matrix with the power spectrum; in fact, from the definition (2.39) it immediately follows

$$C_{\delta_j\delta_k} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} P_\delta(k) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j-\vec{q}_k)} \; ;$$

for the others, it suffices to exploit (2.43) and the last equation in (2.42) to write, in Fourier space

$$\vec{P}(\vec{k}) = \mathrm{i}\frac{\vec{k}}{k^2}\delta(\vec{k}) \; ,$$

so that

$$\begin{aligned}
\left\langle \vec{P}(\vec{q}_j)\delta(\vec{q}_k) \right\rangle =& \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3} k^{-2}\vec{k} P_\delta(k) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j-\vec{q}_k)} \; , \\
\left\langle \vec{P}(\vec{q}_j) \otimes \vec{P}^\top(\vec{q}_k) \right\rangle =& -\int \frac{\mathrm{d}^3k}{(2\pi)^3} \vec{k}\otimes\vec{k}\, k^{-4} P_\delta(k) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j-\vec{q}_k)} \; .
\end{aligned}$$

The probability function we are searching will be the inverse Fourier of (2.46)

$$\begin{aligned}
P(\boldsymbol{d}) = & \int \frac{\mathrm{d}\boldsymbol{t}_p}{(2\pi)^{3N}} \exp\left(-\frac{1}{2}\boldsymbol{t}_p^\top \cdot \boldsymbol{C}_{pp} \cdot \boldsymbol{t}_p + \mathrm{i}\boldsymbol{t}_p \cdot \boldsymbol{p} \right) \\
& \times \int \frac{\mathrm{d}\boldsymbol{t}_\delta}{(2\pi)^N} \exp\left(-\frac{1}{2}\boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta\delta} \cdot \boldsymbol{t}_\delta - \boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta p} \cdot \boldsymbol{t}_p + \mathrm{i}\boldsymbol{t}_\delta \cdot \boldsymbol{\delta} \right) \; ;
\end{aligned}$$

now put the previous in (2.45), and define the part affected by the δ integration

$$I_1(\boldsymbol{t}_\delta) := \prod_{j=1}^N \left(\int \mathrm{d}\delta(\vec{q}_j)\, (1+\delta(\vec{q}_j)) \right) \exp(\mathrm{i}\boldsymbol{t}_\delta \cdot \boldsymbol{\delta}) \; ,$$

use the substitution $z_j := 1 + \delta(\vec{q}_j)$ to obtain

$$I_1(\boldsymbol{t}_\delta) = \exp\left(-\mathrm{i}\sum_{j=1}^N t_{\delta_j} \right) \prod_{j=1}^N \left(\int \mathrm{d}z_j\, z_j \mathrm{e}^{\mathrm{i}t_{\delta_j} z_j} \right) = \exp\left(-\mathrm{i}\sum_{j=1}^N t_{\delta_j} \right) \prod_{j=1}^N \left(-2\pi\mathrm{i}\frac{\partial}{\partial t_{\delta_j}} \delta_\mathrm{D}(t_{\delta_j}) \right) \; ,$$

where we exploited the integral representation of the delta on the last step. Then for the $\boldsymbol{t}_\delta$ integration,

$$\mathcal{C}(\boldsymbol{t}_p) := \int \frac{\mathrm{d}\boldsymbol{t}_\delta}{(2\pi)^N} \exp\left(-\frac{1}{2}\boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta\delta} \cdot \boldsymbol{t}_\delta - \boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta p} \cdot \boldsymbol{t}_p \right) I_1(\boldsymbol{t}_\delta) \; ,$$

exploit the derivatives of the Dirac deltas in I_1 to obtain

$$\mathcal{C}(\boldsymbol{t}_p) = \mathrm{i}^N \prod_{j=1}^N \left(\frac{\partial}{\partial t_{\delta_j}} \right) \exp\left(-\frac{1}{2}\boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta\delta} \cdot \boldsymbol{t}_\delta - \boldsymbol{t}_\delta^\top \cdot \boldsymbol{C}_{\delta p} \cdot \boldsymbol{t}_p - \mathrm{i}\sum_{j=1}^N t_{\delta_j} \right) \Bigg|_{\boldsymbol{t}_\delta = 0} \; ;$$

the explicit expression of the previous, playing with derivatives, can be seen to be

$$\mathcal{C}(\boldsymbol{t}_p) = \prod_{j=1}^{N}\left(1 - \mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\vec{t}_{p_m}\right) + \sum_{i<j} C_{\delta_i\delta_j}\prod_{\{n\}'}\left(1 - \mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\vec{t}_{p_m}\right) + \sum_{\{i<j,k<l\}'_{i<k}} C_{\delta_i\delta_j}C_{\delta_k\delta_l}\prod_{\{n\}'}\left(1 - \mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\vec{t}_{p_m}\right) + \dots ,$$

where the primed set $\{n\}'$ excludes the previous indices picked up on the previous sum, whereas the prime in $\{i<j, k<l\}'_{i<k}$ indicates that no label may be shared between pairs in any individual term of the sum.

Finally, for the $\boldsymbol{t}_p$ integration, we can write

$$\int \frac{\mathrm{d}\boldsymbol{t}_p}{(2\pi)^{3N}}\mathcal{C}(\boldsymbol{t}_p)\exp\left(-\frac{1}{2}\boldsymbol{t}_p^{\top}\cdot\boldsymbol{C}_{pp}\cdot\boldsymbol{t}_p + \mathrm{i}\boldsymbol{t}_p\cdot\boldsymbol{p}\right) = \hat{\mathcal{C}}\left(\frac{\partial}{\mathrm{i}\partial\boldsymbol{p}}\right)\int\frac{\mathrm{d}\boldsymbol{t}_p}{(2\pi)^{3N}}\exp\Big(-\frac{1}{2}\boldsymbol{t}_p^{\top}\cdot\boldsymbol{C}_{pp}\cdot\boldsymbol{t}_p + \mathrm{i}\boldsymbol{t}_p\cdot\boldsymbol{p}\Big) = \frac{1}{\sqrt{(2\pi)^{3N}\det(\boldsymbol{C}_{pp})}}\hat{\mathcal{C}}\left(\frac{\partial}{\mathrm{i}\partial\boldsymbol{p}}\right)\exp\left(-\frac{1}{2}\boldsymbol{p}^{\top}\cdot\boldsymbol{C}_{pp}^{-1}\cdot\boldsymbol{p}\right),$$

where on the first step we used the trick explained in appendix B.2 to put $\mathcal{C}$ outside the integral and on the last step we exploited the standard result for gaussian integrals (A.1).

2.3.2 Summary of initial phase space computations

The final expression for the initial phase space probability is

$$P_0(\boldsymbol{x}^{(\mathrm{i})}) = \frac{V^{-N}}{\sqrt{(2\pi)^{3N}\det(\boldsymbol{C}_{pp})}}\hat{\mathcal{C}}\left(\frac{\partial}{\mathrm{i}\partial\boldsymbol{p}^{(\mathrm{i})}}\right)\exp\left(-\frac{1}{2}\boldsymbol{p}^{(\mathrm{i})^{\top}}\cdot\boldsymbol{C}_{pp}^{-1}\cdot\boldsymbol{p}^{(\mathrm{i})}\right), \tag{2.47}$$

where $\boldsymbol{C}$ is the N-point covariance matrix, $\boldsymbol{C} := C_{jk}\otimes(\vec{e}_j\otimes\vec{e}_k)$, with

$$\begin{aligned} C_{jk} &:= \begin{pmatrix} C_{\delta_j\delta_k} & \vec{C}_{\delta_j p_k}^{\top} \\ \vec{C}_{p_j\delta_k} & \boldsymbol{C}_{p_j p_k}\end{pmatrix} \\ &:= \begin{pmatrix} \left\langle\delta^{(\mathrm{i})}(\vec{q}_j^{(\mathrm{i})})\delta^{(\mathrm{i})}(\vec{q}_k^{(\mathrm{i})})\right\rangle & \left\langle\delta^{(\mathrm{i})}(\vec{q}_j^{(\mathrm{i})})\vec{P}^{(\mathrm{i})}(\vec{q}_k^{(\mathrm{i})})\right\rangle^{\top} \\ \left\langle\vec{P}^{(\mathrm{i})}(\vec{q}_j^{(\mathrm{i})})\delta^{(\mathrm{i})}(\vec{q}_k^{(\mathrm{i})})\right\rangle & \left\langle\vec{P}^{(\mathrm{i})}(\vec{q}_j^{(\mathrm{i})})\otimes\vec{P}^{(i)^{\top}}(\vec{q}_k^{(\mathrm{i})})\right\rangle\end{pmatrix}; \end{aligned} \tag{2.48}$$

since in cosmic structure formation the initial momentum field is irrotational and related to density contrast via continuity equation, that is, with $\vec{u}^{(\mathrm{i})}$ as the initial velocity field and ψ the velocity potential,

$$\vec{u}^{(\mathrm{i})} = \boldsymbol{\nabla}\psi\,,\ \delta^{(\mathrm{i})} = -\nabla^2\psi\,,$$

we can express all the covariance matrix terms with respect to the density fluctuation power spectrum:

$$\begin{aligned}\left\langle \delta^{(\mathrm{i})}(\vec{q}_j^{\,(\mathrm{i})})\delta^{(\mathrm{i})}(\vec{q}_k^{\,(\mathrm{i})})\right\rangle =& \int \frac{\mathrm{d}^3k}{(2\pi)^3}P_\delta(k)\mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j^{\,(\mathrm{i})}-\vec{q}_k^{\,(\mathrm{i})})}\,,\\ \left\langle \vec{P}^{(\mathrm{i})}(\vec{q}_j^{\,(\mathrm{i})})\delta^{(\mathrm{i})}(\vec{q}_k^{\,(\mathrm{i})})\right\rangle =& \mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3}k^{-2}\vec{k}P_\delta(k)\mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j^{\,(\mathrm{i})}-\vec{q}_k^{\,(\mathrm{i})})}\,,\\ \left\langle \vec{P}^{(\mathrm{i})}(\vec{q}_j^{\,(\mathrm{i})})\otimes\vec{P}^{(i)^\top}(\vec{q}_k^{\,(\mathrm{i})})\right\rangle =& -\int \frac{\mathrm{d}^3k}{(2\pi)^3}\vec{k}\otimes\vec{k}\,k^{-4}P_\delta(k)\mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_j^{\,(\mathrm{i})}-\vec{q}_k^{\,(\mathrm{i})})}\,,\end{aligned} \tag{2.49}$$

and at last $\mathcal{C}$ is the operator

$$\begin{aligned}\hat{\mathcal{C}}\left(\frac{\partial}{\mathrm{i}\partial\boldsymbol{p}^{(\mathrm{i})}}\right) =& \prod_{j=1}^{N}\left(1-\mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\frac{\partial}{\mathrm{i}\partial\vec{p}_m^{\,(\mathrm{i})}}\right)+\sum_{i<j}C_{\delta_i\delta_j}\prod_{\{n\}'}\left(1-\mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\frac{\partial}{\mathrm{i}\partial\vec{p}_m^{\,(\mathrm{i})}}\right)\\ &+\sum_{\{i<j,k<l\}'_{i<k}}C_{\delta_i\delta_j}C_{\delta_k\delta_l}\prod_{\{n\}'}\left(1-\mathrm{i}\sum_{m=1}^{N}\vec{C}_{\delta_n p_m}\cdot\frac{\partial}{\mathrm{i}\partial\vec{p}_m^{\,(\mathrm{i})}}\right)+\dots\,,\end{aligned} \tag{2.50}$$

where the primed set $\{n\}'$ excludes the previous indices picked up on the previous sum, whereas the prime in $\{i<j,k<l\}'_{i<k}$ indicates that no label may be shared between pairs in any individual term of the sum.

2.4 Factorizing the free generating functional

Our primary interest regards the computation of the power spectrum, defined in (2.39), that is related to the density correlator. In our formalism, this can be seen to be given by

$$(2\pi)^3\delta_{\mathrm{D}}(\vec{k}_1+\vec{k}_2)P_\delta(k,t)=\hat{H}_{f_1}(1)\hat{H}_{f_2}(2)\;Z_{\mathrm{C}}[H]\Big|_{H,\vec{l}_1,\vec{l}_2=0}=:G_{\rho(1)\rho(2)}\,; \tag{2.51}$$

in general, we will thus need to evaluate density correlators of this kind. What we want to show now is that, in the free case, density correlators completely factorize.

A general correlator has the form

$$G_{\alpha_1(1)\cdots\alpha_n(n)}:=\langle\Phi_{\alpha_1}(1)\cdots\Phi_{\alpha_n}(n)\rangle=\hat{H}_{\alpha_1}(1)\cdots\hat{H}_{\alpha_n}(n)\;Z_{\mathrm{C}}[H]\Big|_{H=0}\,, \tag{2.52}$$

where α_j can stand for f or B (density or respond field), or other macroscopic field labels one could define from these two. Then, restricting to the free case, one sees, from (2.32),

$$\begin{aligned}G^{(0)}_{\alpha_1(1)\cdots\alpha_n(n)}=&\mathrm{e}^{\mathrm{i}H\bullet\hat{\Phi}}\hat{\Phi}_{\alpha_1}(1)\cdots\hat{\Phi}_{\alpha_n}(n)\;Z_{\mathrm{C},0}[\boldsymbol{J},\boldsymbol{K}]\Big|_{H,\boldsymbol{J},\boldsymbol{K}=0}=\\ =&\left(\sum_j\hat{\Phi}^{(1)}_{\alpha_{1,j}}(1)\right)\cdots\left(\sum_j\hat{\Phi}^{(1)}_{\alpha_{n,j}}(n)\right)\;Z_{\mathrm{C},0}[\boldsymbol{J},\boldsymbol{K}]\Big|_{\boldsymbol{J},\boldsymbol{K}=0}\,;\end{aligned} \tag{2.53}$$

it is clear then that the previous is a sum of one-particle contribution terms like

$$\hat{\Phi}^{(1)}_{\alpha_{1,1'}}(1)\cdots\hat{\Phi}^{(1)}_{\alpha_{n,n'}}(n)\; Z_{\mathrm{C},0}[\boldsymbol{J},\boldsymbol{K}]\Big|_{\boldsymbol{J},\boldsymbol{K}=0} ,$$

where prime indices label the particular particle. Since we are considering an ensemble of indistinguishable particles, the full correlator (2.53) will be a sum over all possible ways of combining N indices in subgroups of n elements. Every term of this sum will give an equal contribution, so what we need to consider is only one one-particle contribution with arbitrary assignment of indices. Then the real result, i.e. the sum, will contain only an overall, irrelevant constant factor.

Considering thus only n one-particle density operators, we can show that

$$\hat{\Phi}^{(1)}_{f_{j_1}}(1)\cdots\hat{\Phi}^{(1)}_{f_{j_n}}(n)\; Z_{\mathrm{C},0}[\boldsymbol{J},\boldsymbol{K}]\Big|_{\boldsymbol{J},\boldsymbol{K}=0} = Z_{\mathrm{C},0}[\boldsymbol{L},0] ,$$

with

$$\boldsymbol{L} = -\sum_{m=1}^{n}\vec{s}_m\delta_{\mathrm{D}}(t-t_m)\otimes\vec{e}_{j_m} = -\sum_{m=1}^{n}\begin{pmatrix}\vec{k}_m\\ \vec{l}_m\end{pmatrix}\delta_{\mathrm{D}}(t-t_m)\otimes\vec{e}_{j_m} , \tag{2.54}$$

where j_m labels the particle that corresponds to the dependence label m. In fact, using (2.16) and (2.33), we see that

$$\hat{\Phi}^{(1)}_{f_{jm}}(m)Z_{\mathrm{C},0}[\boldsymbol{J},\boldsymbol{K}] = \int \mathrm{d}\boldsymbol{\Gamma}_{\mathbf{i}}\exp\left(\mathrm{i}\int_{t^{(\mathrm{i})}}^{t_{\mathrm{f}}}\mathrm{d}t\,\boldsymbol{J}\cdot\bar{\boldsymbol{x}}\right)\mathrm{e}^{-\vec{s}_m\cdot\bar{x}_{jm}(t_m)} = Z_{\mathrm{C},0}[\boldsymbol{J}+\boldsymbol{L},\boldsymbol{K}] ,$$

where $\boldsymbol{L}$ is easily seen to be

$$\boldsymbol{L} = -\vec{s}_m\delta_{\mathrm{D}}(t-t_m)\otimes\vec{e}_{j_m} ,$$

and the (2.54) easily follows in the general case.

The action of $\hat{\Phi}_B$ operators in this case is then straightforward; from (2.34), we see that the only new thing is the action of $\hat{b}_j(r)$, that can be considered after the evaluation of all the $\hat{\Phi}^{(1)}_{f_j}(j)$ operators and after setting $\boldsymbol{J}=0$, so consider

$$\begin{aligned}\hat{b}_{j_r}(r)\; Z_{\mathrm{C},0}[\boldsymbol{L},\boldsymbol{K}]\Big|_{\boldsymbol{K}=0} =& \vec{k}_r\cdot\frac{-\mathrm{i}\delta}{\delta\vec{K}_{p_{j_r}}(t_r)}\Bigg\{\int\mathrm{d}t\sum_{i=1}^{n}\Bigg(\delta_{\mathrm{D}}(t-t_i)\vec{s}_i\cdot\Big[\mathcal{G}(t,t^{(\mathrm{i})})\vec{x}^{(\mathrm{i})}_{j_i}\\ &-\int_{t^{(\mathrm{i})}}^{t}\mathrm{d}t'\,\mathcal{G}(t,t')\vec{K}_{j_i}(t')\Big]\Bigg)\Bigg\}\; Z_{\mathrm{C},0}[\boldsymbol{L},\boldsymbol{K}]\Big|_{\boldsymbol{K}=0}\\ =&\mathrm{i}\sum_{i=1}^{n}\Big(\vec{k}_r\cdot\vec{k}_i g_{qp}(t_r,t_i)+\vec{k}_r\cdot\vec{l}_i g_{pp}(t_r,t_i)\Big)\delta_{j_r j_i}Z_{\mathrm{C},0}[\boldsymbol{L},0] ,\end{aligned}$$

where the last Kronecker delta refers to particle indices (the delta does not remove the sum, since the sum is made with respect to dependence labels, and different dependence labels can be shared by the same particle). Since response factors will appear with the interaction potential σ that contains $\delta(\vec{l})$ factors, usually the term $\vec{k}_r\cdot\vec{l}_i g_{pp}(t_r,t_i)$ on the last line can be safely put to zero.

We can give a more explicit expression if we consider only momentum-momentum correlations in (2.47), that is if we put $\mathcal{C}=1$; This approximation is justified as

long as we treat systems with unbound g_{qp} component of the propagator and for sufficiently late times[7]. Notice at first that we can write, using (2.16)

$$Z_{\mathrm{C},0}[\boldsymbol{L},0] = \int \mathrm{d}\boldsymbol{\Gamma}_{\mathrm{i}} \exp\Bigg(-\mathrm{i}\int \mathrm{d}t \sum_m \delta_{\mathrm{D}}(t-t_m)\Big(\vec{k}_m\cdot(g_{qq}(t_m,t^{(\mathrm{i})})\vec{q}^{(\mathrm{i})}_{j_m} + g_{qp}(t_m,t^{(\mathrm{i})})\vec{p}^{(\mathrm{i})}_{j_m}) \\ + \vec{l}_m\cdot(g_{pq}(t_m,t^{(\mathrm{i})})\vec{q}^{(\mathrm{i})}_{j_m} + g_{pp}(t_m,t^{(\mathrm{i})})\vec{p}^{(\mathrm{i})}_{j_m})\Big)\Bigg) =: \int \mathrm{d}\boldsymbol{\Gamma}_{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{L}_q\cdot\boldsymbol{q}^{(\mathrm{i})}+\mathrm{i}\boldsymbol{L}_p\cdot\boldsymbol{p}^{(\mathrm{i})}}\,, \tag{2.55}$$

where we defined

$$\begin{aligned}\boldsymbol{L}_q &= -\sum_m(\vec{k}_m g_{qq}(t_m,t^{(\mathrm{i})}) + \vec{l}_m g_{pq}(t_m,t^{(\mathrm{i})}))\otimes\vec{e}_{j_m} =: \sum_m \vec{L}_{q_{j_m}}\otimes\vec{e}_{j_m}\,,\\ \boldsymbol{L}_p &= -\sum_m(\vec{k}_m g_{qp}(t_m,t^{(\mathrm{i})}) + \vec{l}_m g_{pp}(t_m,t^{(\mathrm{i})}))\otimes\vec{e}_{j_m} =: \sum_m \vec{L}_{p_{j_m}}\otimes\vec{e}_{j_m}\,.\end{aligned} \tag{2.56}$$

Inserting (2.47) with $\mathcal{C}=1$,

$$Z_{\mathrm{C},0}[\boldsymbol{L},0] = \frac{V^{-N}}{\sqrt{(2\pi)^{3N}\det(\boldsymbol{C}_{pp})}}\int \mathrm{d}\boldsymbol{q}^{(\mathrm{i})}\,\mathrm{d}\boldsymbol{p}^{(\mathrm{i})}\exp\Bigg(-\frac{1}{2}\boldsymbol{p}^{(\mathrm{i})^\top}\cdot\boldsymbol{C}_{pp}^{-1}\cdot\boldsymbol{p}^{(\mathrm{i})} \\ +\mathrm{i}\boldsymbol{L}_p\cdot\boldsymbol{p}^{(\mathrm{i})}+\mathrm{i}\boldsymbol{L}_q\cdot\boldsymbol{q}^{(\mathrm{i})}\Bigg) = V^{-N}\int \mathrm{d}\boldsymbol{q}^{(\mathrm{i})}\exp\Big(-\frac{1}{2}\boldsymbol{L}_p^\top\cdot\boldsymbol{C}_{pp}\cdot\boldsymbol{L}_p+\mathrm{i}\boldsymbol{L}_q\cdot\boldsymbol{q}^{(\mathrm{i})}\Big)\,, \tag{2.57}$$

where on the last step we used the standard result (A.1).

It will be useful to consider

$$\boldsymbol{C}_{pp} = \frac{\sigma_1^2}{3}\mathbb{1}_3\otimes\mathbb{1}_N + \sum_{j\neq k} C_{p_jp_k}\otimes\vec{e}_j\otimes\vec{e}_k\,,$$

where

$$\sigma_n^2 := \int\frac{\mathrm{d}^3k}{(2\pi)^3}k^{2n-4}P_\delta(k)\,; \tag{2.58}$$

it is then clear that we split the term that involve the same particle correlator and different particle correlators. This reflects in

$$\begin{aligned}Q := \boldsymbol{L}_p^\top\cdot\boldsymbol{C}_{pp}\cdot\boldsymbol{L}_p &= \frac{\sigma_1^2}{3}\sum_j\vec{L}_{p_j}^2 + \sum_{j\neq k}\vec{L}_{p_j}^\top C_{p_jp_k}\vec{L}_{p_k}\\ &=: Q_0 - Q_{\mathrm{D}} + \sum_{j\neq k}\vec{L}_{p_j}^\top C_{p_jp_k}\vec{L}_{p_k}\,,\end{aligned} \tag{2.59}$$

with the definitions

$$Q_0 := \frac{\sigma_1^2}{3}\Bigg(\sum_j\vec{L}_{p_j}\Bigg)^2\,,\quad Q_{\mathrm{D}} := \frac{\sigma_1^2}{3}\sum_{j\neq k}\vec{L}_{p_j}\cdot\vec{L}_{p_k}\,; \tag{2.60}$$

these are meaningful, in fact we will see that Q_0 will vanish in the important case related to the power spectrum, and Q_{D} will be connected to the free streaming damping.

[7] It is still possible to exactly account also for the other correlations, see appendix D, but accounting for those gives only a very small correction at the price of a way heavier computational effort.

2.4.1 Deriving the factorization

Now we are ready to demonstrate the factorization of the generating functional. The derivation is quite technical, so we will summarize its most important results and formulas in section 2.4.2.

Start from

$$Z_{\mathrm{C},0}[\boldsymbol{L},0] = V^{-N} \int \mathrm{d}\boldsymbol{q}^{(\mathrm{i})} \exp\left(-\frac{1}{2} \boldsymbol{L}_p^\top \cdot \boldsymbol{C}_{pp} \cdot \boldsymbol{L}_p + \mathrm{i} \boldsymbol{L}_q \cdot \boldsymbol{q}^{(\mathrm{i})} \right) ; \tag{2.61}$$

the exponential actually depends only on l different particle labels due to the expression of $\boldsymbol{L}_p$ (see (2.56)), where l is the number of density operators acting on the generating functional. So, $N - l$ integrations yield only a volume factor V. For the other integrations, since $\boldsymbol{C}_{pp}$ depends only on the coordinate differences $\vec{q}_{ij} := \vec{q}_j^{(\mathrm{i})} - \vec{q}_k^{(\mathrm{i})}$, it is useful to rewrite the term in the exponential as

$$\boldsymbol{L}_q \cdot \boldsymbol{q} = \vec{L}_{q_1} \cdot \vec{q}_1^{(\mathrm{i})} + \vec{L}_{q_2} \cdot (\vec{q}_{21} + \vec{q}_1^{(\mathrm{i})}) + \ldots = \left(\sum_{j=1}^{l} \vec{L}_{q_j} \right) \cdot \vec{q}_1^{(\mathrm{i})} + \sum_{j=2}^{l} \vec{L}_{q_j} \cdot \vec{q}_{j1} ;$$

now the integration over the single $\vec{q}_1^{(\mathrm{i})}$ can be done without any effect on the $\boldsymbol{L}_p^\top \cdot \boldsymbol{C}_{pp} \cdot \boldsymbol{L}_p$ part (that depends only on coordinate differences), thus yielding a delta. Changing also the integration variable, we have then

$$Z_{\mathrm{C},0}[\boldsymbol{L},0] = \frac{(2\pi)^3}{V^l} \delta_{\mathrm{D}} \left(\sum_{j=1}^{l} \vec{L}_{q_j} \right) \prod_{j=2}^{l} \int \mathrm{d}^3 q_{j1} \exp\left(-\frac{1}{2} \boldsymbol{L}_p^\top \cdot \boldsymbol{C}_{pp} \cdot \boldsymbol{L}_p + \mathrm{i} \sum_{j=2}^{l} \vec{L}_{q_j} \cdot \vec{q}_{j1} \right) ;$$

now $\boldsymbol{C}_{pp}$ depends also on the other coordinate differences

$$\vec{q}_{ab} := \vec{q}_{a1} - \vec{q}_{b1} \, , \; b = 2, \ldots, (l-1) \, , \; a = (b+1), \ldots, l \, , \tag{2.62}$$

so the idea now is to integrate over all coordinate differences, by introducing appropriate deltas enforcing the constraint (2.62). So we write

$$\begin{aligned} Z_{\mathrm{C},0}[\boldsymbol{L},0] =& \frac{(2\pi)^3}{V^l} \delta_{\mathrm{D}} \left(\sum_{j=1}^{l} \vec{L}_{q_j} \right) \prod_{j>k}^{l} \int \mathrm{d}^3 q_{jk} \left(\prod_{a>b} \delta_{\mathrm{D}}(\vec{q}_{ab} - \vec{q}_{a1} + \vec{q}_{b1}) \right) \\ & \times \exp\left(-\frac{1}{2} \boldsymbol{L}_p^\top \cdot \boldsymbol{C}_{pp} \cdot \boldsymbol{L}_p + \mathrm{i} \sum_{j=2}^{l} \vec{L}_{q_j} \cdot \vec{q}_{j1} \right) . \end{aligned} \tag{2.63}$$

Write the deltas using their integral representation

$$\delta_{\mathrm{D}}(\vec{q}_{ab} - \vec{q}_{a1} + \vec{q}_{b1}) = \int \frac{\mathrm{d}^3 k_{ab}}{(2\pi)^3} \mathrm{e}^{\mathrm{i} \vec{k}'_{ab} \cdot (\vec{q}_{ab} - \vec{q}_{a1} + \vec{q}_{b1})} ;$$

we can then regroup all the phase factors appearing in (2.63) as

$$\begin{aligned} \sum_{j=2}^{l} \vec{L}_{q_j} \cdot \vec{q}_{j1} + \sum_{a>b} \vec{k}'_{ab} \cdot (\vec{q}_{ab} - \vec{q}_{a1} + \vec{q}_{b1}) &= \sum_{j=2}^{l} \left(\vec{L}_{q_j} - \sum_{c=2}^{j-1} \vec{k}'_{jc} + \sum_{d=j+1}^{l} \vec{k}'_{dj} \right) \cdot \vec{q}_{j1} \\ + \sum_{a>b} \vec{k}'_{ab} \cdot \vec{q}_{ab} &=: \sum_{j>k}^{l} \vec{k}_{jk} \cdot \vec{q}_{jk} \, , \end{aligned}$$

where a and b span over the indices subjected to the constraint in (2.62), and with

$$\vec{k}_{jk} := \begin{cases} \vec{L}_{q_j} - \sum_{c=2}^{j-1} \vec{k}'_{jc} + \sum_{d=j+1}^{l} \vec{k}'_{dj} & \text{for } k = 1, j \in \{2, \dots, l\} \\ \vec{k}'_{jk} & \text{otherwise} . \end{cases}$$

So (2.63) becomes (using also the splitting (2.59), and recalling that Q_0, Q_D do not depend on coordinate differences)

$$\begin{aligned} Z_{\mathrm{C},0}[\boldsymbol{L}, 0] = & \frac{(2\pi)^3}{V^l} \delta_\mathrm{D} \left(\sum_{j=1}^{l} \vec{L}_{q_j} \right) \mathrm{e}^{-(Q_0 - Q_\mathrm{D})/2} \prod_{a>b} \int \frac{\mathrm{d}^3 k_{ab}}{(2\pi)^3} \prod_{j>k} \\ & \times \int \mathrm{d}^3 q \, \mathrm{e}^{-\vec{L}_{p_j}^\top C_{p_j p_k} \vec{L}_{p_k} + \mathrm{i} \vec{k}_{jk} \cdot \vec{q}_{jk}} ; \end{aligned} \tag{2.64}$$

we have now only the last line of the previous to simplify.

Notice that we can correlate $C_{p_j p_k}$ with the velocity potential correlation function

$$\xi_\psi(q_{jk}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} P_\psi(k) \mathrm{e}^{\mathrm{i} \vec{k} \cdot \vec{q}_{jk}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} k^{-4} P_\delta(k) \mathrm{e}^{\mathrm{i} \vec{k} \cdot \vec{q}_{jk}} ,$$

(where on the last step we exploited (2.42) to write $P_\psi = k^{-4} P_\delta$). In fact, looking at the last of (2.50), we have

$$C_{p_j p_k} = -(\boldsymbol{\nabla} \otimes \boldsymbol{\nabla}) \xi_\psi(q_{jk}) .$$

Due to isotropy, ξ_ψ can only depend on the modulus of the coordinate vector difference, thus it is useful to express the derivatives with respect to it; writing only q as the argument of ξ_ψ for simplicity, we have

$$\boldsymbol{\nabla} \otimes \boldsymbol{\nabla} = (\hat{q} \otimes \hat{q}) \frac{d^2}{dq^2} + (\mathbb{1}_3 - \hat{q} \otimes \hat{q}) \frac{1}{q} \frac{\mathrm{d}}{\mathrm{d}q} ,$$

with $\hat{q}$ as the unit vector pointing in the direction of $\vec{q}$. In the end, using the previous two equations,

$$C_{p_j p_k} = -(\hat{q} \otimes \hat{q}) \xi''_\psi(q) - (\mathbb{1}_3 - \hat{q} \otimes \hat{q}) \frac{\xi'_\psi(q)}{q} .$$

We want to express it with respect to the projectors

$$\pi^{\parallel}_{jk} := \hat{k}_{jk} \otimes \hat{k}_{jk} , \quad \pi^{\perp}_{jk} := \mathbb{1}_3 - \pi^{\parallel}_{jk} ,$$

so we impose

$$(\hat{q} \otimes \hat{q}) \xi''_\psi(q) + (\mathbb{1}_3 - \hat{q} \otimes \hat{q}) \frac{\xi'_\psi(q)}{q} \stackrel{!}{=} a_\parallel(q) \pi^{\parallel}_{jk} + a_\perp(q) \pi^{\perp}_{jk} .$$

In order to obtain the expressions for $a_\parallel, a_\perp$, multiply the previous by the projectors $\pi^\parallel_{jk}$ and by $\pi^\perp_{jk}$ and take their trace; exploiting, with $\mu := \hat{k}_{jk} \cdot \hat{q}$,

$$\begin{aligned}
\mathrm{Tr}\,\pi^\parallel_{jk} &= \sum_{i=x,y,z} (\hat{k}_{jk})_i^2 = 1 = \mathrm{Tr}(\hat{q} \otimes \hat{q})\,, \\
\mathrm{Tr}\Big(\pi^\parallel_{jk}\pi^\parallel_{jk}\Big) &= \sum_{i,l}((\hat{k}_{jk})_i(\hat{k}_{jk})_l)^2 = \left(\sum_{i=x,y,z}(\hat{k}_{jk})_i^2\right)^2 = 1\,, \\
\mathrm{Tr}\Big(\pi^\parallel_{jk}\pi^\perp_{jk}\Big) &= 0\,, \\
\mathrm{Tr}\Big(\pi^\parallel_{jk}\hat{q}\otimes\hat{q}\Big) &= \sum_{i,l}((\hat{k}_{jk})_i\hat{q}_l)^2 = \left(\sum_{i=x,y,z}(\hat{k}_{jk})_i\hat{q}_i\right)^2 = \mu^2\,, \\
\mathrm{Tr}\Big(\pi^\perp_{jk}\hat{q}\otimes\hat{q}\Big) &= \mathrm{Tr}\Big(\pi^\parallel_{jk}(\mathbb{1}_3 - \hat{q}\otimes\hat{q})\Big) = 1-\mu^2\,, \\
\mathrm{Tr}\Big(\pi^\perp_{jk}(\mathbb{1}_3 - \hat{q}\otimes\hat{q})\Big) &= 1+\mu^2\,,
\end{aligned}$$

we end up with

$$a_\parallel(q) = \mu^2\xi''_\psi(q) + (1-\mu^2)\frac{\xi'_\psi(q)}{q}\,,\quad a_\perp(q) = \frac{1-\mu^2}{2}\xi''_\psi(q) + \frac{1+\mu^2}{2q}\xi'_\psi(q)\,.$$

Defining for simplicity

$$\lambda^\parallel_{jk} := \frac{\vec{L}^\top_{p_j}\pi^\parallel_{jk}\vec{L}_{p_k}}{g^2_{qp}(t,0)k^2_{jk}}\,,\quad \lambda^\perp_{jk} = \frac{\vec{L}^\top_{p_j}\pi^\perp_{jk}\vec{L}_{p_k}}{g^2_{qp}(t,0)k^2_{jk}}\,,$$

we can rewrite the last line in (2.64) as

$$\int \mathrm{d}^3q\, \mathrm{e}^{g^2_{qp}(t,0)k^2_{jk}(a_\parallel\lambda^\parallel_{jk}+a_\perp\lambda^\perp_{jk})}\mathrm{e}^{\mathrm{i}\vec{k}_{jk}\cdot\vec{q}} =: (2\pi)^3\delta_\mathrm{D}(\vec{k}_{jk}) + \mathcal{P}_{jk}\,, \tag{2.65}$$

with

$$\mathcal{P}_{jk}(k_{jk},t) = \int \mathrm{d}^3q\left(\mathrm{e}^{g^2_{qp}(t,0)k^2_{jk}(a_\parallel\lambda^\parallel_{jk}+a_\perp\lambda^\perp_{jk})} - 1\right)\mathrm{e}^{\mathrm{i}\vec{k}_{jk}\cdot\vec{q}}\,.$$

The particular splitting we made in (2.65) can be understood once we consider the application to the power spectrum, something we will do in section 2.4.3.

2.4.2 Factorization final result

In the end the one-particle contribution of l density operators $\hat{\Phi}_f\big|_{\vec{l}=0}$ to the correlator is

$$\begin{aligned}
Z_{\mathrm{C},0}[\boldsymbol{L},0] =& V^{-l}(2\pi)^3\delta_\mathrm{D}\left(\sum_{j=1}^{l}\vec{L}_{q_j}\right)\mathrm{e}^{-(Q_0-Q_\mathrm{D})/2}\prod_{2\le b<a}^{l}\int\frac{\mathrm{d}^3k_{ab}}{(2\pi)^3} \\
&\times\prod_{1\le k<j}^{l}\left((2\pi)^3\delta_\mathrm{D}(\vec{k}_{jk}) + \mathcal{P}_{jk}\right)\,,
\end{aligned} \tag{2.66}$$

with[8]

$$\mathcal{P}_{jk}(k_{jk},t) = \int \mathrm{d}^3q \left(\mathrm{e}^{g^2_{qp}(t,0)k^2_{jk}(a_\parallel \lambda^\parallel_{jk} + a_\perp \lambda^\perp_{jk})} - 1 \right) \mathrm{e}^{\mathrm{i}\vec{k}_{jk}\cdot\vec{q}} \,, \tag{2.67}$$

as the non linearly evolved power spectrum, as we will see, and

$$\vec{k}_{jk} := \begin{cases} \vec{L}_{q_j} - \sum_{b=2}^{j-1} \vec{k}'_{jb} + \sum_{a=j+1}^{l} \vec{k}'_{aj} & \text{for } k = 1, j \in \{2,\dots,l\} \\ \vec{k}'_{jk} & \text{otherwise} \,, \end{cases} \tag{2.68}$$

where the primed wave vectors are the Fourier conjugates of the coordinate differences $\vec{q}_{ij} := \vec{q}_i - \vec{q}_j$; also

$$\lambda^\parallel_{jk} := \frac{\vec{L}^\top_{p_j} \pi^\parallel_{jk} \vec{L}_{p_k}}{g^2_{qp}(t,0)k^2_{jk}} \,, \;\; \lambda^\perp_{jk} = \frac{\vec{L}^\top_{p_j} \pi^\perp_{jk} \vec{L}_{p_k}}{g^2_{qp}(t,0)k^2_{jk}} \,, \tag{2.69}$$

with the projectors (indicating as $\hat{k}$ the unit vector pointing in the direction of $\vec{k}_{jk}$)

$$\pi^\parallel_{jk} = \hat{k} \otimes \hat{k} \,, \;\; \pi^\perp_{jk} = \mathbb{1}_3 - \pi^\parallel_{jk} \,, \tag{2.70}$$

and with

$$a_\parallel(q) = \mu^2 \xi''_\psi(q) + (1-\mu^2)\frac{\xi'_\psi(q)}{q} \,, \;\; a_\perp(q) = \frac{1-\mu^2}{2}\xi''_\psi(q) + \frac{1+\mu^2}{2q}\xi'_\psi(q) \,, \tag{2.71}$$

where μ is the angle between $\vec{q}$ and $\vec{k}_{21}$, $\mu := \hat{q}\cdot\hat{k}_{21}$, and ξ_ψ is the correlation function for the initial velocity potential

$$\xi_\psi(q_{jk}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} P_\psi(k) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}_{jk}} = \int \frac{\mathrm{d}^3k}{(2\pi)^3} k^{-4} P_\delta(k) \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}_{jk}} \,. \tag{2.72}$$

We can simplify further these last expressions, in fact we can integrate on the solid angle to find

$$\xi_\psi(q) = \frac{1}{2\pi^2 q}\int_0^\infty \mathrm{d}k\, k^{-3} P_\delta(k) \sin(kq) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, k^{-2} P_\delta(k) j_0(kq) \,, \tag{2.73}$$

so that the derivatives are

$$\begin{aligned} \xi'_\psi(q) &= -\frac{q}{2\pi^2}\int_0^\infty \mathrm{d}k\, P_\delta(k) \frac{j_1(kq)}{kq} \,, \\ \xi''_\psi(q) &= \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\, P_\delta(k) \left(j_2(kq) - \frac{j_1(kq)}{kq} \right) \,, \end{aligned} \tag{2.74}$$

[8] Here q refers to the initial momentum integration, but since in the end is a dummy variable (i.e. it is always integrated), we will not explicitly put the superscript (i) for simplicity.

where j_0, j_1, j_2 are the spherical Bessel functions

$$\begin{aligned} j_0(x) :=& \frac{\sin x}{x} \,, \\ j_1(x) :=& \frac{\sin x}{x^2} - \frac{\cos x}{x} \,, \\ j_2(x) :=& \left(\frac{3}{x^2} - 1\right)\frac{\sin x}{x} - \frac{3\cos x}{x^2} \,. \end{aligned} \tag{2.75}$$

We can recast (2.71) as

$$a_\parallel(q) = a_1(q) + \mu^2 a_2(q) \,, \;\; 2a_\perp(q) = 2a_1(q) + (1-\mu^2)a_2(q) \,,$$

with

$$a_1(q) := -\frac{1}{2\pi^2}\int \mathrm{d}k\, P_\delta(k)\frac{j_1(kq)}{kq} \,, \;\; a_2(q) := \frac{1}{2\pi^2}\int \mathrm{d}k\, P_\delta(k) j_2(kq) \,. \tag{2.76}$$

2.4.3 Power spectrum

Specializing the (2.66) for the action of two density operators, we obtain

$$Z_0[\boldsymbol{L}, 0] = \frac{(2\pi)^3}{V^2}\delta_{\mathrm{D}}(\vec{L}_{q_1} + \vec{L}_{q_2})\mathrm{e}^{-(Q_0 - Q_{\mathrm{D}})/2}((2\pi)^3\delta_{\mathrm{D}}(\vec{k}_{21}) + \mathcal{P}_{21}) \,; \tag{2.77}$$

since it is a two-point density correlator, from (2.51), if computed at $\vec{l} = 0$ and at equal time, it contains informations regarding the free power spectrum. So in the following we will assume $\vec{l}_1 = \vec{l}_2 = 0$ and $t_1 = t_2 \equiv t$. The delta in (2.77) ensures $\vec{L}_{q_1} = -\vec{L}_{q_2}$ so that $\vec{k}_1 = -\vec{k}_2$, implying $Q_0 = 0$ and

$$Q_{\mathrm{D}} = -2\frac{\sigma_1^2}{3}k_1^2 g_{qp}^2(t, t^{(\mathrm{i})}) \,;$$

the only wave vector we have to consider is $\vec{k}_{21} = \vec{L}_{q_2} = -g_{qq}(t, t^{(\mathrm{i})})\vec{k}_1$; this implies that the projector $\pi_{21}^{\parallel}$ points in the same direction of $\vec{k}_1$; hence the (2.69) becomes

$$\lambda_{21}^{\parallel} = -1 \,, \;\; \lambda_{21}^{\perp} = 0 \,.$$

Collecting everything, we have (setting $g_{qq} \equiv 1$ as it is often the case)

$$Z_0[\boldsymbol{L}, 0] = \frac{(2\pi^3)}{V^2}\delta_{\mathrm{D}}(\vec{k}_1 + \vec{k}_2)\mathrm{e}^{-\sigma_1^2 g_{qp}^2(t, t^{(\mathrm{i})})k_1^2/3}((2\pi)^3\delta_{\mathrm{D}}(\vec{k}_{21}) + \mathcal{P}_{21}) \tag{2.78}$$

with

$$\mathcal{P}_{21} =: \mathcal{P} = \int \mathrm{d}^3 q \left(\mathrm{e}^{-g_{qp}^2(t, t^{(\mathrm{i})})k_1^2 a_\parallel} - 1\right)\mathrm{e}^{\mathrm{i}\vec{k}_1\cdot\vec{q}} \,. \tag{2.79}$$

This last term, as can be seen from (2.78) (ignoring the $\delta_{\mathrm{D}}(\vec{k}_{21})$ contribution, that only sets $\vec{k}_{21} = 0$, thus representing an unimportant contribution to the mean), is indeed the non linear evolved power spectrum apart for the exponential damping factor involving Q_{D}; this damping is due to free streaming, that at late times damps the

power spectrum given by the initial correlations; if interaction is not there to counterbalance this damping, structures cannot form and the power spectrum becomes flat (see [3] for more on the interplay between diffusion and interaction within KFT).

We can give an approximate expression for (2.67) and hence for (2.79) valid at early times and/or large scales; in fact in this situation, the argument of the exponential is little, so that we can write

$$\mathcal{P}_{ij} \simeq -g^2_{qp}(t,t^{(\mathrm{i})})k^2_{ij}\int \mathrm{d}^3q\,(a_{\parallel}(q)\lambda^{\parallel}_{ij}+a_{\perp}(q)\lambda^{\perp}_{ij})\mathrm{e}^{\mathrm{i}\vec{k}_{ij}\cdot\vec{q}}\,; \tag{2.80}$$

to evaluate the last integral, we will need to evaluate quantities of the form

$$I^{\alpha}_n(k) := 2\pi\int_0^{\infty}\mathrm{d}q\,q^n\xi^{(n)}_{\psi}(q)\int_{-1}^{1}\mathrm{d}\mu\,\mu^{\alpha}\mathrm{e}^{i\mu kq}\,,$$

with $\xi^{(n)}_{\psi}$ as the n-th derivative of (2.72); using

$$\int_{-1}^{1}\mathrm{d}\mu\,\mu^{\alpha}\mathrm{e}^{\mathrm{i}\mu kq} = \frac{1}{(\mathrm{i}k)^{\alpha}}\partial^{\alpha}_q\int_{-1}^{1}\mathrm{d}\mu\,\mathrm{e}^{\mathrm{i}\mu kq} = \frac{2}{(\mathrm{i}k)^{\alpha}}\partial^{\alpha}_q\left(\frac{\sin kq}{kq}\right) = \frac{2}{(\mathrm{i}k)^{\alpha}}\partial^{\alpha}_q j_0(kq) = \frac{2}{\mathrm{i}^{\alpha}}\partial^{\alpha}_{kq}j_0(kq)\,;$$

exploiting the recurrence relation of the Bessel functions

$$j_n(x) = (-x)^n\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n j_0(x)\,,$$

we have, exploiting also the expressions (2.73) and (2.74)

$$\begin{aligned}
I^0_1(k) =& -\frac{2}{\pi}\int_0^{\infty}\mathrm{d}q\int_0^{\infty}\mathrm{d}k'\,P_{\delta}(k')\frac{qj_1(k'q)}{k'}j_0(kq)\,,\\
I^0_2(k) =& \frac{2}{\pi}\int_0^{\infty}\mathrm{d}q\int_0^{\infty}\mathrm{d}k'\,P_{\delta}(k')q^2\left(j_2(k'q)-\frac{j_1(k'q)}{k'q}\right)j_0(kq)\,,\\
I^2_1(k) =& \frac{2}{\pi}\int_0^{\infty}\mathrm{d}q\int_0^{\infty}\mathrm{d}k'\,P_{\delta}(k')\frac{qj_1(k'q)}{k'}\left(j_2(kq)-\frac{j_1(kq)}{kq}\right),\\
I^2_2(k) =& -\frac{2}{\pi}\int_0^{\infty}\mathrm{d}q\int_0^{\infty}\mathrm{d}k'\,P_{\delta}(k')q^2\left(j_2(k'q)-\frac{j_1(k'q)}{k'q}\right)\left(j_2(kq)-\frac{j_1(kq)}{kq}\right);
\end{aligned}$$

so we have (using the expressions (2.71) for for $a_{\perp}$ and $a_{\parallel}$)

$$\begin{aligned}
\int \mathrm{d}^3q\,a_{\parallel}(q)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}} =& 2\pi\int_0^{\infty}q^2\int_{-1}^{1}\mathrm{d}\mu\,a_{\parallel}(q)\mathrm{e}^{\mathrm{i}kq\mu} = I^2_2(k)+I^0_1(k)-I^2_1(k) = -k^{-2}P_{\delta}(k_1)\,,\\
\int \mathrm{d}^3q\,a_{\perp}(q)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}} =& I^0_2(k)+I^0_1(k)+I^2_1(k)-I^2_2(k) = 0\,,
\end{aligned}$$

where we exploited the relation

$$\int_0^{\infty}\mathrm{d}x\,x^2j_n(\alpha x)j_n(\beta x) = \frac{\pi}{2\alpha^2}\delta_{\mathrm{D}}(\alpha-\beta)\,.$$

In the end, (2.80) becomes

$$\mathcal{P}_{ij}(k_{ij},t) \simeq -g^2_{qp}(t,0)\lambda^{\parallel}_{ij}P_{\delta}(k_{ij})\,; \tag{2.81}$$

for the power spectrum, $\lambda_{ij}^{\parallel} = -1$ so we have

$$\mathcal{P}(k_1, t) \simeq g_{qp}^2(t, t^{(\mathrm{i})}) P_\delta(k_1) \,. \tag{2.82}$$

This is thus the expected linear growth of the power spectrum, and this justifies, as we anticipated, the splitting we made in (2.65).

Chapter 3

Propagators and interactions

We have seen in the previous chapter how to compute the free correlators with some approximations. In order to go further and give specific (and computable) expressions of the quantities of interest, we need an initial power spectrum, the Green function for the free evolution and the interaction term; these three ingredients are basically the only parameters in our theory.

To study the evolution of some given initial power spectrum, we must first establish the free propagator that is more suitable to describe cosmological situations and a way to implement gravitational interactions. As we will see, it is possible to compute exactly the Green function for point particles in cosmology, but nevertheless one could be interested in improved forms, that could already contain part of the interaction (think about the Zel'dovich approximation); but to fully account for the interactions of course we are forced to rely on some approximation scheme; in this chapter we will introduce a non perturbative scheme, based on the Born approximation.

3.1 Trajectories in an expanding space-time

To describe the trajectories of collisionless dark matter particles in an expanding space time, we must refer to what we did in section 1.3.3; in particular, we will need the Lagrangian (1.5) we derived there and the Poisson equation (1.4).

We want here to use as a time coordinate the growth factor $D_+(a(t))$, that is we will use

$$\tau := D_+(t) - D_+^{(\mathrm{i})} \,, \tag{3.1}$$

with $D_+^{(\mathrm{i})}$ as the growth factor at the initial time (so that $\tau^{(\mathrm{i})} = 0$). We can relate the new time coordinate with the old one,

$$\frac{\mathrm{d}}{\mathrm{d}t} = HfD_+ \frac{\mathrm{d}}{\mathrm{d}\tau} \,, \tag{3.2}$$

with

$$f = \frac{\mathrm{d}\ln D_+}{\mathrm{d}\ln a} \,,\; H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \,. \tag{3.3}$$

Under this change of time coordinate, the lagrangian (1.5) changes, using the invariance of the action, as

$$S = \int \mathrm{d}t\, \mathcal{L}(\vec{q}, \mathrm{d}\vec{q}/\mathrm{d}t\,, t) = \int \mathrm{d}\tau\, \mathcal{L}'(\vec{q}, \mathrm{d}\vec{q}/\mathrm{d}\tau\,, \tau) = \int \mathrm{d}t\, \frac{\mathrm{d}\tau}{\mathrm{d}t}\mathcal{L}' \implies \mathcal{L}' = \frac{\mathrm{d}\tau}{\mathrm{d}t}\mathcal{L}\,;$$

using

$$g(\tau) := a^2 D_+ f H H^{(\mathrm{i})^{-1}}\,, \tag{3.4}$$

with $H^{(\mathrm{i})}$ as the initial Hubble parameter, we can write the new lagrangian, factorizing the constant $mH^{(\mathrm{i})}$, dropping the prime and redefining $\dot{} := \mathrm{d}/\mathrm{d}\tau$, as

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}, \tau) = \frac{g(\tau)}{2}\dot{q}^2 - v(\vec{q}, \tau)\,,$$

with the effective gravitational potential

$$v(\vec{q}, \tau) = \frac{a^2\phi}{g(\tau)H^{(\mathrm{i})^2}}\,.$$

The Poisson equation for the fluctuation ϕ can be written, exploiting (1.4), in terms of comoving densities $\rho_\mathrm{c}\,,\ \bar{\rho}_\mathrm{c}$ as

$$\nabla^2_{\vec{q}}\phi = \frac{4\pi G}{a}(\rho_\mathrm{c} - \bar{\rho}_\mathrm{c})\,; \tag{3.5}$$

recall that the matter density parameter is defined, specialized at the initial time, as

$$\Omega_m^{(\mathrm{i})} = \frac{8\pi G}{3}H^{(\mathrm{i})^2}\bar{\rho}^{(\mathrm{i})}\,; \tag{3.6}$$

since the mean mass density scales as $\bar{\rho} \propto a^{-3}$, the comoving $\bar{\rho}_\mathrm{c} = a^3\bar{\rho}$ is constant, so that we can write the previous substituting $\bar{\rho}^{(\mathrm{i})}$ with $\bar{\rho}_\mathrm{c}$. We can thus rewrite the Poisson (3.5) as

$$\nabla^2_{\vec{q}}v(\vec{q}, \tau) = \frac{3a}{2g(\tau)}\Omega_m^{(\mathrm{i})}\delta\,. \tag{3.7}$$

The canonically conjugate momentum is

$$\vec{p} := \frac{\partial\mathcal{L}}{\partial\dot{\vec{q}}} = g(\tau)\dot{\vec{q}}\,,$$

so the hamiltonian is

$$\mathcal{H} = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L} = \frac{p^2}{2g(\tau)} + v(\vec{q}, \tau)\,,$$

and the Hamilton equations yield

$$\dot{\vec{q}} = g^{-1}(\tau)\vec{p}\,,\ \ \dot{\vec{p}} = -\boldsymbol{\nabla}_{\vec{q}}v \implies \ddot{\vec{q}} + \frac{\dot{g}}{g}\dot{\vec{q}} + \boldsymbol{\nabla}\left(\frac{v}{g}\right) = 0\,. \tag{3.8}$$

We can rewrite them as

$$\dot{\vec{x}} + \begin{pmatrix} 0 & -g^{-1}\mathbb{1}_3 \\ 0 & 0 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ -\boldsymbol{\nabla}_{\vec{q}}v \end{pmatrix}\,,$$

whose retarded solution is (see also appendix A.3)

$$\begin{aligned}\vec{x}(\tau) &= \int_0^\tau \mathrm{d}\tau' \begin{pmatrix} 0 \\ -\boldsymbol{\nabla}_{\vec{q}} v \end{pmatrix} \exp\left(- \int_{\tau'}^\tau \mathrm{d}\tau'' \begin{pmatrix} 0 & -g^{-1}(\tau'')\mathbb{1}_3 \\ 0 & 0 \end{pmatrix} \right) \\ &= \int_0^\tau \mathrm{d}\tau' \begin{pmatrix} 0 \\ -\boldsymbol{\nabla}_{\vec{q}} v \end{pmatrix} \begin{pmatrix} \mathbb{1}_3 & \int_{\tau'}^\tau \mathrm{d}\tau'' \, g^{-1}(\tau'')\mathbb{1}_3 \\ 0 & \mathbb{1}_3 \end{pmatrix}\end{aligned}$$

where on the last step we exploited, with

$$A := \begin{pmatrix} 0 & \int_{\tau'}^\tau \mathrm{d}\tau'' \, g^{-1}(\tau'')\mathbb{1}_3 \\ 0 & 0 \end{pmatrix} ,$$

the fact that $A^2 = 0$, so that $\exp(A) = \mathbb{1}_6 + A$; this implies that the retarded Green function is

$$\mathcal{G}(\tau, \tau') = \begin{pmatrix} \mathbb{1}_3 & g_{qp}(\tau, \tau')\mathbb{1}_3 \\ 0 & \mathbb{1}_3 \end{pmatrix} \Theta(\tau - \tau') , \tag{3.9}$$

with Θ as the Heaviside function, and

$$g_{qp}(\tau, \tau') := \int_{\tau'}^\tau \frac{\mathrm{d}\tau''}{g(\tau'')} .$$

Thus the free trajectories are given by

$$\vec{q}^{(0)}(\tau) = \vec{q}^{(\mathrm{i})} + g_{qp}(\tau, 0)\vec{p}^{(\mathrm{i})} . \tag{3.10}$$

3.1.1 Alternative formulation

We can express the results of previous section also using as time coordinate

$$\eta := \ln\left(\frac{D_+}{D_+^{(\mathrm{i})}} \right) ; \tag{3.11}$$

it is easy to see that the choice of η instead of τ as a time coordinate does not change much the discussion we made in section 3.1; provided that one redefines

$$g(\eta) := \frac{g(\tau)}{D_+} ,$$

you can trust all the formulas we used in section 3.1 from (3.4) to (3.10) with the substitution $\tau \to \eta$ and $g(\tau) \to g(\eta) = a^2 H f H^{(\mathrm{i})\,-1}$.

If we decide to express the equation of motion in terms of a rescaled momenta

$$\vec{p}' := g^{-1}(\eta)\vec{p} ,$$

we have to compute its time derivative (in this subsection, $\dot{} := \mathrm{d}/\mathrm{d}\eta$)

$$\dot{\vec{p}}' = -g^{-1}(\eta)\boldsymbol{\nabla}_{\vec{q}} v - g^{-1}(\eta)\vec{p}'\dot{g}(\eta) ,$$

exploiting $Hf = D_+^{-1}\,\mathrm{d}D_+/\mathrm{d}t$, $\dot{D}_+/D_+ = 1$ and

$$\frac{\mathrm{d}^2 D_+}{\mathrm{d}t^2} + 2H\frac{\mathrm{d}D_+}{\mathrm{d}t} = \frac{3}{2}\Omega_m H^2 D_+ ,$$

(see (1.9), with $\delta = D_+\delta^{(\mathrm{i})}$) we have

$$\frac{\dot{g}}{g} = \frac{1}{a^2 f H}\left(2a\dot{a}\frac{1}{D_+}\frac{dD_+}{dt} + \frac{a^2}{Hf}\left(\frac{1}{D_+}\frac{d^2D_+}{dt^2} - \left(\frac{1}{D_+}\frac{\mathrm{d}D_+}{\mathrm{d}t}\right)^2\right)\right) = -\left(1 - \frac{3}{2}\frac{\Omega_m}{f^2}\right),$$

so that, redefining $V := g^{-1}v$ and dropping primes from the new momentum, we have

$$\begin{aligned}\dot{\vec{q}} &= \vec{p}\,,\\ \dot{\vec{p}} &= \left(1 - \frac{3}{2}\frac{\Omega_m}{f^2}\right)\vec{p} - \boldsymbol{\nabla}_{\vec{q}}V \approx -\frac{1}{2}\vec{p} - \boldsymbol{\nabla}_{\vec{q}}V\,,\end{aligned}$$

where on the last step we used $\Omega_m/f^2 \approx 1$, approximation that is valid for most of ΛCDM cosmologies (see [25]). In this approximation, the Green function, analogously to what we did previously, is

$$\mathcal{G}(\eta,\eta') = \exp\left(-\int_{\eta'}^{\eta}\mathrm{d}\eta''\begin{pmatrix}0 & -\mathbb{1}_3\\ 0 & \mathbb{1}_3/2\end{pmatrix}\right)\Theta(\eta-\eta')\,;$$

to compute the exponential, notice that, using schematically

$$\begin{pmatrix}0 & \mathbb{1}_3(\eta-\eta')\\ 0 & -\mathbb{1}_3(\eta-\eta')/2\end{pmatrix} =: \begin{pmatrix}0 & 2a\\ 0 & -a\end{pmatrix} =: A\,,$$

we have

$$A^2 = \begin{pmatrix}0 & -2a^2\\ 0 & a^2\end{pmatrix},\ A^3 = \begin{pmatrix}0 & 2a^3\\ 0 & -a^3\end{pmatrix} \implies \mathrm{e}^A = \begin{pmatrix}1 & 2(a - \frac{1}{2}a^2 + \frac{1}{3!}a^3 + \ldots)\\ 0 & 1 - a + \frac{1}{2}a^2 - \frac{1}{3!}a^3 + \ldots\end{pmatrix},$$

and thus it is easy to see that the free propagator for dark matter particles in Newtonian dynamics in this alternative approach will be

$$\mathcal{G}(\eta,\eta') = \begin{pmatrix}\mathbb{1}_3 & 2(1-\exp\left(-\frac{1}{2}(\eta-\eta')\right))\mathbb{1}_3\\ 0 & \exp\left(-\frac{1}{2}(\eta-\eta')\right)\mathbb{1}_3\end{pmatrix}\Theta(\eta-\eta')\,. \tag{3.12}$$

The new V obeys, looking at (3.7), the Poisson

$$\nabla^2_{\vec{q}}V(\vec{q},\eta) = \frac{3a}{2g^2(\eta)}\Omega_m^{(\mathrm{i})}\delta = \frac{3}{2}\frac{\Omega_m^{(\mathrm{i})}H_i^2}{a^3H^2f^2}\delta = \frac{3}{2}\frac{\Omega_m}{f^2}\delta \approx \frac{3}{2}\delta\,,$$

where on the third step we used

$$\Omega_m = \frac{8\pi G}{3H^2}\bar{\rho} = \frac{8\pi G}{3a^3H^2}\bar{\rho}_{\mathrm{c}} = \frac{\Omega_m^{(\mathrm{i})}}{a^3H^2}H_i^2\,,$$

where on the last step we exploited (3.6).

So in this setting, the single-particle gravitational potential in Fourier space, from

$$V(\vec{q},\eta) = \sum_{j=1}^{N} v(|\vec{q} - \vec{q}_j(\eta)|) \,,$$

and using the single-particle density contrast $\delta = \bar{\rho}^{-1}\delta_{\mathrm{D}}(\vec{q}) - 1$, reads

$$v(k) = -\frac{3}{2}\frac{1}{\bar{\rho}k^2} \,,$$

where we neglected the contribution given by the Fourier transformed unity $\hat{1}$, since the zero mode of the potential cannot affect correlators.

3.2 Improved Zel'dovich propagator

We would like to search for propagators that already capture part of the interaction, like for example the one given by the Zel'dovich approximation [26].

Inspired by (A.3), we search for solutions of the form

$$\vec{q}(\tau) = \vec{q}_0 + g_{qp}(\tau,0)\dot{\vec{q}}_0 + \int_0^\tau \mathrm{d}\tau'\, g_{qp}(\tau,\tau')\vec{f}(\tau') \,, \tag{3.13}$$

where $\vec{f}(\tau)$ can be regarded as an effective force term, whereas g_{qp} is now the improved propagator we are searching for. In the Zel'dovich approximation, the trajectories are straight ones provided that one uses as time coordinate the linear growing factor τ; the propagator within this approximation is thus

$$g_{qp}(\tau,\tau') = \tau - \tau' \,; \tag{3.14}$$

Taking two times the derivative of (3.13), we have

$$\ddot{\vec{q}}(\tau) = \ddot{g}_{qp}(\tau,0)\dot{\vec{q}} + \dot{g}_{qp}(\tau,\tau)\vec{f}(\tau) + \int_0^\tau \mathrm{d}\tau'\, \ddot{g}_{qp}(\tau,\tau')\vec{f}(\tau') \,.$$

The matching of the last of (3.8), that is the equation of the trajectories in an expanding space-time, with the previous using the Zel'dovich propagator (3.14) implies

$$\vec{f}(\tau) = -\frac{\dot{g}}{g}\dot{\vec{q}} - \boldsymbol{\nabla} v \,.$$

This is the effective force corresponding to the Zel'dovich approximation. An improvement to this picture (see also [14]) can be given by the following effective force

$$\vec{f}(\tau) = \frac{\dot{g}}{g}h(\tau)\dot{\vec{q}} - \boldsymbol{\nabla} v \,, \quad h(\tau) := \frac{1}{g} - 1 \,; \tag{3.15}$$

for early times, we have $h(\tau) \to 0$ so that with this choice you don't have an initial velocity dependent contribution[1]. With this new $\vec{f}$, we can rewrite (3.8) as

$$\ddot{\vec{q}}(\tau) - \dot{h}(\tau)\dot{\vec{q}}(\tau) = \vec{f}(\tau) \,.$$

This equation has the same structure of (A.2), so it has the same solution (A.3), that is

$$\dot{\vec{q}}(\tau) = \mathrm{e}^{h(\tau)}\dot{\vec{q}}_0 + \mathrm{e}^{h(\tau)} \int_0^\tau \mathrm{d}\tau' \, \vec{f}(\tau')\mathrm{e}^{-h(\tau')} \,; \tag{3.16}$$

this implies that the improved propagator is

$$g_{qp}(\tau, \tau') = \int_{\tau'}^\tau \mathrm{d}\tau'' \, \mathrm{e}^{h(\tau'')-h(\tau')} \,. \tag{3.17}$$

Notice that, since for late times $h(\tau) \to -1$, the previous reduces to the Zel'dovich propagator (3.14) times e^{-1}.

Finally, we want to find an expression for the effective force in this improved Zel'dovich picture. Using (3.16) and (3.15), we obtain the following equation for $\vec{f}$,

$$\vec{f}(\tau) = \frac{\dot{g}}{g} h(\tau)\mathrm{e}^{h(\tau)} \left(\dot{\vec{q}}_0 + \int_0^\tau \mathrm{d}\tau' \, \vec{f}(\tau')\mathrm{e}^{-h(\tau')} \right) - \boldsymbol{\nabla} v \,; \tag{3.18}$$

we want to put it in the form of (A.2), so redefine for simplicity

$$\alpha^{-1}(\tau) := \frac{\dot{g}}{g} h(\tau)\mathrm{e}^{h(\tau)} \,,$$

multiply the (3.18) by $\alpha(\tau)$, derive with respect to τ and obtain

$$\dot{\vec{f}} + \frac{\dot{\alpha}}{\alpha}\vec{f} - \frac{\mathrm{e}^{-h}}{\alpha}\vec{f} = -\frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha \boldsymbol{\nabla} v \,;$$

this has indeed the same form of (A.2), and has the particular solution (exploiting (A.3) and integrating by parts)

$$\vec{f}(\tau) = -\boldsymbol{\nabla} v - \frac{\dot{g}h}{g^2} \int_0^\tau \mathrm{d}\tau' \, g \boldsymbol{\nabla} v \,. \tag{3.19}$$

This can be seen as the effective force particles in this improved Zel'dovich picture feel, and we will use this to account for interactions in next section.

3.3 The Born approximation

Return back at equations (2.16) and (2.17); one way we could have to include interactions is to consider the interaction term together with the source term $\boldsymbol{K}$ in the solution of the equation of motion, that is

$$\bar{\boldsymbol{x}}(t) = \boldsymbol{\mathcal{G}}(t, t^{(\mathrm{i})})\boldsymbol{x}^{(\mathrm{i})} - \int_{t^{(\mathrm{i})}}^{t_\mathrm{f}} \mathrm{d}t' \, \boldsymbol{\mathcal{G}}(t, t') \begin{pmatrix} \boldsymbol{K}_q(t') \\ \boldsymbol{K}_p(t') + \boldsymbol{\nabla}_{\boldsymbol{q}} \mathcal{V} \end{pmatrix} \,;$$

[1] With this "improved Zel'dovich approximation", you reduce the removal of structures that affects the Zel'dovich approximation (in Zel'dovich approximation the propagator is unbounded, and structures are driven apart by inertial motion), see [14].

then to account for the action of the density operators, consider the (2.16) with $\boldsymbol{J} = \boldsymbol{L}$ and $\boldsymbol{K} = 0$ with our new solution of the equation of motion; this changes the (2.55) as (putting also $\vec{l} = 0$)

$$\begin{aligned} Z_{\mathrm{C}}[\boldsymbol{L}, 0] &= \int \mathrm{d}\boldsymbol{\Gamma}_{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{L}_q \cdot \boldsymbol{q}^{(\mathrm{i})} + \mathrm{i}\boldsymbol{L}_p \cdot \boldsymbol{p}^{(\mathrm{i})}} \exp\left(\mathrm{i} \sum_m \vec{k}_m \cdot \int_{t^{(\mathrm{i})}}^{t_m} \mathrm{d}t'\, g_{qp}(t_m, t') \boldsymbol{\nabla}_{j_m} \mathcal{V} \right) \\ &:= \int \mathrm{d}\boldsymbol{\Gamma}_{\mathrm{i}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{L}_q \cdot \boldsymbol{q}^{(\mathrm{i})} + \mathrm{i}\boldsymbol{L}_p \cdot \boldsymbol{p}^{(\mathrm{i})} + \mathrm{i}S_{\mathrm{I}}} , \end{aligned} \tag{3.20}$$

where we have the new interaction term

$$S_{\mathrm{I}} = \sum_m \vec{k}_m \cdot \int_{t^{(\mathrm{i})}}^{t_m} \mathrm{d}t'\, g_{qp}(t_m, t') \boldsymbol{\nabla}_{j_m} \mathcal{V} ; \tag{3.21}$$

what we want to do now is to consider an effective force for $\boldsymbol{\nabla}_{j_m} \mathcal{V}$ like the one in (3.19), with

$$\vec{f}_j = -\boldsymbol{\nabla}_j \mathcal{V} ,$$

where j labels the particle over which the effective force acts. Having in mind the formula for the power spectrum (2.78), taking into account the Dirac delta that ensures $\vec{k}_1 = -\vec{k}_2$, we can write

$$S_{\mathrm{I}} = -\vec{k}_1 \cdot \int_{t^{(\mathrm{i})}}^{t} \mathrm{d}t'\, g_{qp}(t, t') (\vec{f}_1(t') - \vec{f}_2(t')) .$$

We can consider for example the force acting on particle 1 as the sum of all forces acting on it due to other particles; so we can recast the force terms as

$$\vec{f}_1 - \vec{f}_2 = \sum_{j \neq 1}^{N} \vec{f}_{1j} - \sum_{j \neq 2}^{N} \vec{f}_{2j} = 2 f_{12} + \sum_{j=3}^{N} (\vec{f}_{1j} - \vec{f}_{2j}) ,$$

where we exploited Newton's third law $\vec{f}_{12} = -\vec{f}_{21}$. The sum involving particles with $j \geq 3$ can be put to zero, since in an isotropic field we should expect that the average of the forces that acts on particle one and two (not including their mutual interaction) should vanish. So we end up with

$$S_{\mathrm{I}} = -\vec{k}_1 \cdot \int_{t^{(\mathrm{i})}}^{t} \mathrm{d}t'\, g_{qp}(t, t') \vec{f}_{12}(t') .$$

For the potential, we will use the one we derived in (3.8), v/g, that, exploiting (3.7), obeys the Poisson

$$\nabla^2_{\vec{q}} \left(\frac{v}{g} \right) (\vec{q}, \tau) = \frac{3a}{2g^2(\tau)} \Omega_m^{(\mathrm{i})} \delta ,$$

so the one particle equivalent in Fourier space will obey

$$\bar{v}(\vec{k}, \tau) = -\frac{3a}{2g^2(\tau) \bar{\rho} k^2} \Omega_m^{(\mathrm{i})} =: -\frac{A(\tau)}{\bar{\rho} k^2} , \quad A(\tau) := \frac{3a}{2g^2(\tau)} \Omega_m^{(\mathrm{i})} ;$$

so we can write, in real space, the gradient of potential of particle 2 at the position of the particle 1 (that will enter in $\vec{f}_{12}$), as

$$\boldsymbol{\nabla}_{\vec{q}_1(\tau)} \bar{v}_2(\vec{q}, \tau) =: \boldsymbol{\nabla}_1 \bar{v}_2 = -\mathrm{i} \frac{A(\tau)}{\bar{\rho}} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{\vec{k}}{k^2} \mathrm{e}^{\mathrm{i}\vec{k} \cdot (\vec{q}_1(\tau) - \vec{q}_2(\tau))} .$$

Now enters the approximation: the idea is to substitute the previous by a suitable average over the particles 1 and 2, so that one can factorize out of the generating functional the whole interaction term[2]. This average can be done using the correlation function; in fact we can express the probability to find particle 1 and 2 at a separation $q_{12} := |\vec{q}_1 - \vec{q}_2|$ as

$$P_{12} = \bar{\rho}(1 + \xi_{12}(q_{12})) \,,$$

where ξ_{12} is the correlation function, representing the excess probability over the mean, random $\bar{\rho}$ one to find particle 2 at a distance q_{12} from particle 1. We use this probability to make the average, yielding, in Fourier space

$$\langle \boldsymbol{\nabla}_1 \bar{v}_2 \rangle (\vec{k}) = \bar{\rho} \int \mathrm{d}^3 q_{12}\, \xi_{12}(q_{12}) \boldsymbol{\nabla}_1 \bar{v}_2(q_{12}, \tau) \mathrm{e}^{-\mathrm{i}\vec{k}\cdot(\vec{q}_1 - \vec{q}_2)} \,; \tag{3.22}$$

using the convolution theorem, we can express the previous as the convolution of the Fourier transform of ξ_{12}, that is related to the power spectrum, and $\boldsymbol{\nabla}_1 \bar{v}_2$; for the power spectrum, we can use the linearly evolved one damped on free streaming scale, that is use

$$D_+^2 \bar{P}_\delta(k) := D_+^2 \mathrm{e}^{-\sigma_1^2 g_{qp}^2 k^2/3} P_\delta^{(\mathrm{i})}(k) \,.$$

So we can write (3.22) as

$$\langle \boldsymbol{\nabla}_1 \bar{v}_2 \rangle (\vec{k}, \tau) = -\mathrm{i} A(\tau) D_+^2 \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\vec{k} - \vec{k}'}{|\vec{k} - \vec{k}'|^2} \bar{P}_\delta(k') \,;$$

we can simplify the previous by integrating over the angles; with $\mu := \hat{k} \cdot \hat{k}'$, where hatted quantities are the corresponding unit vectors, and $y := k'/k$, we have

$$\begin{aligned} \int \frac{\mathrm{d}^3 k'}{(2\pi)^3} \frac{\vec{k} - \vec{k}'}{|\vec{k} - \vec{k}'|^2} \bar{P}_\delta(k') &= \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{d}\mu \int_0^\infty \mathrm{d}k'\, k'^2 \frac{k(\hat{k} - y\hat{k}')}{k^2(1 + y^2 - 2y\mu)} \bar{P}_\delta(k') \\ &= \frac{k^2 \hat{k}}{(2\pi)^2} \int_0^\infty \mathrm{d}y\, y^2 J(y) \bar{P}_\delta(ky) \,, \end{aligned}$$

where

$$J(y) := \int_{-1}^{1} \mathrm{d}\mu\, \frac{1 - \mu y}{1 + y^2 - 2\mu y} = 1 + \frac{1 - y^2}{2y} \ln \frac{1 + y}{|1 - y|} \,.$$

In the end the gravitational potential gradient average reads (by truncating the integration at $y = 1$ in order to cut modes on scales smaller than the density fluctuation considered)

$$\langle \boldsymbol{\nabla}_1 \bar{v}_2 \rangle (\vec{k}, \tau) = -\mathrm{i} A(\tau) D_+^2 \frac{k^2 \hat{k}}{(2\pi)^2} \int_0^1 \mathrm{d}y\, y^2 J(y) \bar{P}_\delta(ky) \,; \tag{3.23}$$

insert this in (3.19), yielding what we denote with $\left\langle \vec{f}_{12} \right\rangle$, in order to obtain the averaged interacting action

$$\langle S_\mathrm{I} \rangle (\vec{k}, \tau) = -\vec{k} \cdot \int_{t^{(\mathrm{i})}}^{\tau} \mathrm{d}\tau'\, g_{qp}(\tau, \tau') \left\langle \vec{f}_{12} \right\rangle (\vec{k}, \tau') \,. \tag{3.24}$$

[2]One can call such an approximation "Born approximation" since it is similar in spirit to the one used in scattering theory.

This averaged action does not depend anymore on initial positions and momenta, so we can factorize it out of the integral from (3.20) and obtain the final result for the non linear evolved power spectrum of KFT within our assumptions and approximations

$$\boxed{\bar{\mathcal{P}}(\vec{k},\tau) = \mathrm{e}^{Q_D/2+\mathrm{i}\langle S_I\rangle}\mathcal{P}(\vec{k},\tau)} \tag{3.25}$$

Chapter 4

Power spectra analysis

After all the efforts we made in the previous chapters, we are finally ready to analyze the non linear evolution of primordial power spectra using KFT approach.

As initial power spectra, in this work we will study the evolution of two types: one is

$$P_\delta^{(\mathrm{i})}(k) = Ak\left(1 + \left(\frac{k}{k_{\mathrm{eq}}}\right)^2\right)^{-n/2}, \tag{4.1}$$

with k_{eq} as the wavevector corresponding to the Hubble horizon scale at matter-radiation equality; for $n > 0$, this power spectrum goes like k for $k \ll k_0$, whereas it goes like k^{1-n} for $k \gg k_0$, thus mimicking the dark matter behavior (1.13) for $n = 4$. Its interest comes from the study of its small scale behavior; we thus would like to see, by varying n, what is its asymptotic behavior, if it always falls like k^{-3} or it has a certain dependence on the choice of n. We will call this initial spectrum the "dark matter power spectrum".

Another type of power spectrum we will analyze is the one with Gaussian shape

$$P_\delta^{(\mathrm{i})}(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma^2}\right); \tag{4.2}$$

a power spectrum like that is very unusual in cosmology, but nevertheless it could be interesting to see what KFT predicts for its evolution. We will call this initial spectrum the "Gaussian power spectrum".

The constant A appearing in front of these initial power spectra is a normalization constant, to be determined using the value for today's variance σ_8, whereas k_0 is a reference wavevector.

The evolution equation we will use will be the (3.25) derived within the Born approximation. Regarding the parameters and setting of our discussion:

- we will take as the initial time one near the beginning of matter dominated epoch, so that $\Omega_{\mathrm{m}}^{(\mathrm{i})} = 1$;
- as background cosmological model, we will assume a standard ΛCDM universe with $\Omega_{\mathrm{md}} = 0.26$, $\Omega_\Lambda = 0.7$, $\Omega_{\mathrm{mb}} = 0.04$, $h = 0.7$, $\sigma_8 = 0.8$ at zero redshift (i.e. today);
- the free propagator will be the improved Zel'dovich one (3.17), that already accounts for part of the interaction;

- interactions are taken into account within the non-perturbative Born approximation, using the effective force term (3.19) and the averaging scheme depicted in section 3.3.

We summarize here all the equations implied by (3.25) we should try to solve (written explicitly): from also (2.78), (2.79) (neglecting the $\delta_{\mathrm{D}}(\vec{k})$ term, that only sets $\vec{k}=0$, thus representing an unimportant contribution to the overall mean)

$$\bar{\mathcal{P}}(\vec{k},\tau)=\exp\left(-\frac{1}{3}\left(\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{P_\delta^{(\mathrm{i})}(k')}{k'^2}\right)g_{qp}^2(\tau,0)k^2+\mathrm{i}\,\langle S_{\mathrm{I}}\rangle\right)\left(\int\mathrm{d}^3q\right.$$
$$\left.\times\left(\exp\left\{-g_{qp}^2(\tau,0)k^2\left[\frac{1}{2\pi^2}\int_0^\infty\mathrm{d}k'\,P_\delta^{(\mathrm{i})}(k')\left((\hat{k}\cdot\hat{q})^2j_2(k'q)-\frac{j_1(k'q)}{k'q}\right)\right]\right\}-1\right)\mathrm{e}^{\mathrm{i}\vec{q}\cdot\vec{k}}\right);$$

with $\tau=D_+(t)-D_+^{(\mathrm{i})}$ and from (3.17)

$$g_{qp}(\tau,\tau')=\int_{\tau'}^{\tau}\mathrm{d}\tau''\exp\left(\left(a^2D_+fHH^{(\mathrm{i})^{-1}}\right)^{-1}(\tau'')-\left(a^2D_+fHH^{(\mathrm{i})^{-1}}\right)^{-1}(\tau')\right),$$

with $f=\mathrm{d}\ln D_+/\mathrm{d}\ln a$; and also from (3.24), (3.19)

$$\langle S_{\mathrm{I}}\rangle(\vec{k},\tau)=-2\vec{k}\cdot\int_0^\tau\mathrm{d}\tau'\,g_{qp}(\tau,\tau')\left(-\langle\boldsymbol{\nabla}_1\bar{v}_2\rangle(\vec{k},\tau')-\left(2\frac{\dot{a}}{a}+\frac{\dot{f}}{f}+\frac{\dot{H}}{H}\right)\right.$$
$$\left.\times\frac{1-a^2D_+fHH^{(\mathrm{i})^{-1}}}{a^4D_+^2f^2H^2H^{(\mathrm{i})^{-2}}}\int_0^{\tau'}\mathrm{d}\tau''\,(a^2D_+fHH^{(\mathrm{i})^{-1}})(\tau'')\,\langle\boldsymbol{\nabla}_1\bar{v}_2\rangle(\vec{k},\tau')\right),$$

and finally from (3.23)

$$\begin{aligned}\langle\boldsymbol{\nabla}_1\bar{v}_2\rangle(\vec{k},\tau)=&-\mathrm{i}\frac{3H^{(\mathrm{i})^2}}{2a^3D_+f^2H^2}\frac{k^2\hat{k}}{(2\pi)^2}\int_0^1\mathrm{d}y\,y^2\left(1+\frac{1-y^2}{2y}\ln\frac{1+y}{|1-y|}\right)\\&\times\exp\left(-\frac{1}{3}\left(\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{P_\delta^{(\mathrm{i})}(k')}{k'^2}\right)g_{qp}^2(\tau,0)k^2y^2\right)P_\delta^{(\mathrm{i})}(ky)\,.\end{aligned}\tag{4.3}$$

Apart for the initial power spectrum, what we need are the cosmological parameters $a(t), D_+(t)$ (and hence $H, H^{(\mathrm{i})}, f$), to be determined by using our background cosmological model.

4.1 Solving the integrals

The expression we need to analyze is not easy to handle, so we cannot expect to find an analytical solution for $\bar{\mathcal{P}}$.

4.1.1 Dark matter spectrum

The first integrals we can try to solve are the ones that do not depend on cosmological quantities; one is

$$\sigma_1^2 = \frac{A}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{k}\left(1+\left(\frac{k}{k_{\mathrm{eq}}}\right)^2\right)^{-n/2} = \frac{A}{2\pi^2}\int_0^\infty \frac{k}{\left(1+\left(k/k_{\mathrm{eq}}\right)^2\right)^{n/2}}\,;$$

for $n > 2$, this integral converges and can be easily solved (putting for example $x = k/k_0$), yielding

$$\sigma_1^2 = \frac{A}{2\pi^2}\frac{k_{\mathrm{eq}}^2}{n-2}\,,\quad n>2\,.$$

The integrals that involve the Bessel functions can be solved numerically using methods suitable for fast oscillating integrals (see for example the Levin collocation scheme in [27, 28]); we have

$$\begin{aligned} a_1(q) &= -\frac{A}{2q\pi^2}\int_0^\infty \mathrm{d}k'\left(1+\left(\frac{k'}{k_{\mathrm{eq}}}\right)^2\right)^{-n/2} j_1(k'q)\,,\\ a_2(q) &= \frac{A}{2\pi^2}\int_0^\infty \mathrm{d}k'\,k'\left(1+\left(\frac{k'}{k_{\mathrm{eq}}}\right)^2\right)^{-n/2} j_2(k'q)\,, \end{aligned}$$

the integral we have to solve then becomes

$$\mathcal{P}(k,\tau) = 2\pi\int_0^\infty \mathrm{d}q\,q^2\int_{-1}^1 \mathrm{d}\mu\left(\mathrm{e}^{-g_{qp}^2(\tau,0)k^2(a_1(q)+\mu^2a_2(q))}-1\right)\mathrm{e}^{\mathrm{i}qk\mu}\,; \tag{4.4}$$

the integral in μ can be seen to be

$$(4.4)_{\mu\text{-part}} = \int_{-1}^1 \mathrm{d}\mu\left(\mathrm{e}^{-g_{qp}^2(\tau,0)k^2(a_1(q)+\mu^2a_2(q))}-1\right)\cos(qk\mu)\,,$$

this comes from the fact that the imaginary part of the integrand, being an odd function in μ, gives zero. This last integral admits a representation involving the error function erf, where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t\,,$$

but from a numerical point of view, it is better to directly solve (4.4) using 2-D Levin collocation scheme.

The other integrals involve cosmological quantities, so we shall use some fit formula for the cosmological parameters of interest. On their implementation, we used as time coordinate the scale factor normalized to unity today; on the time integral one must then consider also the jacobian $\mathrm{d}\tau/\mathrm{d}a$ to take into account this change of coordinate. We recall that one can find back the cosmological time associated using for example the time-redshift formula (1.12).

We used (1.2) to determine the behavior of $H(a)$ and (1.11) to determine $D_+(a)$ and hence $f(a)$. Then the integrals, again, must be solved numerically, and we used some

standard integration routine like the 15 point Gauss-Kronrod rule, implemented in the Gnu Scientific Library (GSL) [29].

4.1.2 Gaussian spectrum

For the Gaussian power spectrum case, the results for the integrals we know how to solve is

$$\sigma_1^2 = \frac{A}{(2\pi^2)} \int_0^\infty \mathrm{d}k \exp\left(-\frac{(k-k_0)^2}{2\sigma^2}\right),$$

put $x = (k-k_0)/\sqrt{2}\sigma$ and the result is

$$\sigma_1^2 = \sqrt{\frac{\pi}{2}}\sigma\left(1 - \mathrm{erf}\left(-\frac{k_0}{\sqrt{2}\sigma}\right)\right).$$

Then the steps for the evaluation of the integrals are the same we have already explained in section 4.1.1.

4.2 Dark matter spectrum results

In table 4.1 we summarize the analysis of the dark matter power spectrum (4.1) by varying the index n, non-linearly evolved until the present time $a = 1$. The first remarkable thing one can notice is that the fall-off on small scales always goes like k^{-3} regardless of the initial index n (we remark that the initial power spectrum fall-off is k^{1-n}).

In figure 4.2, 4.3 we put some plots of different cases.

One can notice that, in cases where n is higher than 10, there is a change in the steepness for high values of k. Actually this change is always there, and what causes it is the decrease the averaged interacting action $\langle S_\mathrm{I} \rangle$ experiences for high k[1]. This will lead the Born averaged power spectrum to asymptotically reach the free non-linearly evolved one, that is $\exp(Q_\mathrm{D}/2)\mathcal{P}$.An example of this behavior, common for every case, is shown in figure 4.4. Since we expect the Born approximation to break down for such small scales, on the fall-off analysis we did in table 4.1 we didn't take into account this very large k behavior of the full $\bar{\mathcal{P}}$.

The analysis of cases with $n < 4$ gave some problems of numerical convergence; in general initial power spectra that are too enhanced on small scales (like the ones with "gentle" slopes) are not suitable for our numerical methods of integration, and are thus are left out from our analysis. We leave improvements to our code and analysis on such initial power spectra for future works.

4.3 Gaussian spectrum results

In this section we summarize the analysis we did for the gaussian spectrum initial condition. Such primordial spectra are cosmologically artificial, and this could mean

[1] This comes from the exponential free streaming damping factor in the averaged gradient of the potential, see equation (4.3).

n	$\bar{\mathcal{P}}$ fall-off	$\exp(Q_D/2)\mathcal{P}$ fall-off	k of $\bar{\mathcal{P}}$ peak [h Mpc]
4	-3.203241 ± 0.001225	-3.206744 ± 0.001685	0.175426
5	-3.001858 ± 0.000100	-3.001866 ± 0.000102	0.093005
6	-3.000029 ± 0.000004	-3.000031 ± 0.000004	0.070859
7	-3.000005 ± 0.000002	-3.000006 ± 0.000002	0.053986
8	-3.000004 ± 0.000002	-3.000004 ± 0.000002	0.045034
9	-3.000003 ± 0.000002	-3.000003 ± 0.000002	0.041131
10	-3.000002 ± 0.000003	-3.000001 ± 0.000003	0.037567
17	-3.049102 ± 0.018668	-3.000002 ± 0.000002	0.023876
18	-3.010402 ± 0.005326	-3.000002 ± 0.000003	0.023876
22	-3.000004 ± 0.000003	-3.000001 ± 0.000004	0.019917
26	-3.006900 ± 0.003935	-3.000000 ± 0.000003	0.018190
30	-3.028725 ± 0.012968	-3.000000 ± 0.000003	0.016614
35	-3.008317 ± 0.004770	-3.000007 ± 0.000004	0.015174

TABLE 4.1: Summary of the dark matter power spectrum results. The first column is the n index of the initial power spectrum in (4.1); the second and third column represent the fall-off $k^{-n_{\text{fall}}}$ on small scales of the non-linearly evolved Born approximated and free power spectrum respectively, and the last column is the value of k corresponding to the peak in $\bar{\mathcal{P}}$.

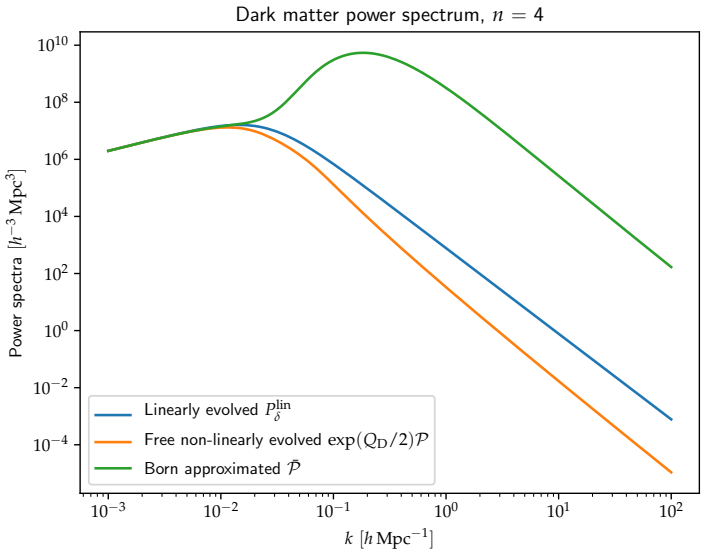


FIGURE 4.2: Non-linearly evolved dark matter power spectra plot for $n = 4$.

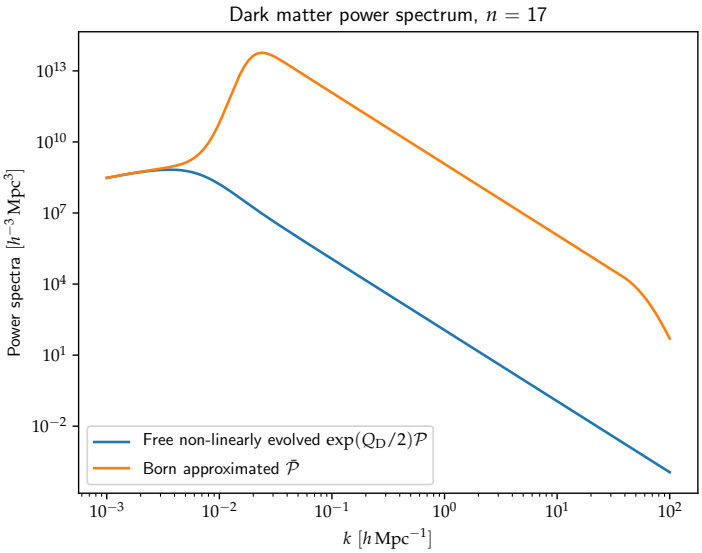


FIGURE 4.3: Non-linearly evolved dark matter power spectra plot for $n = 17$.

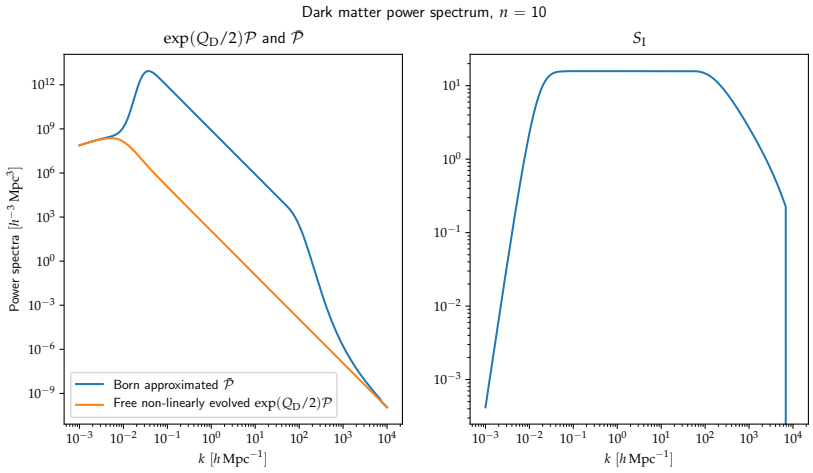


FIGURE 4.4: Non-linearly evolved dark matter power spectra plot for $n = 10$; here we show the behavior of very large scales, to be understood by noticing the sharp decrease of $\langle S_\mathrm{I} \rangle$ on very large scales, represented on the right.

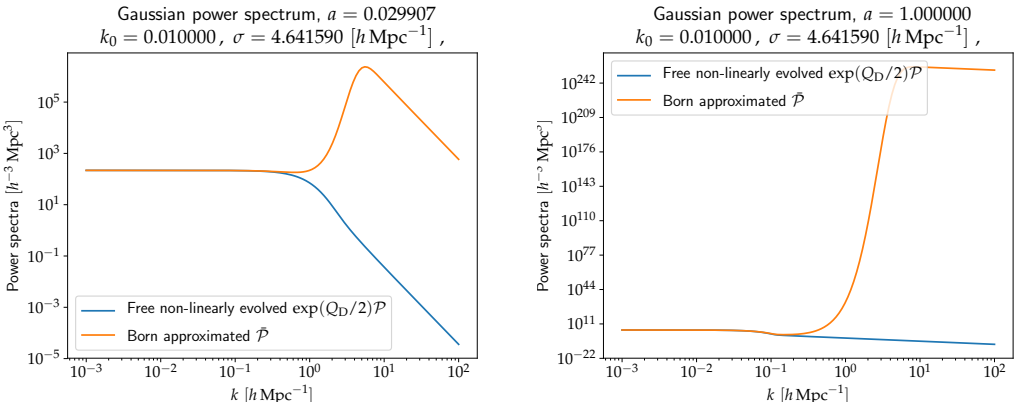


FIGURE 4.5: Comparison of non-linearly evolved gaussian spectra at different times. The one on the left is evolved until $a = 0.03$, the one on the right is evolved until $a = 1$ (today).

that the time scale when the Born approximation can fail could be much different. See for example figure 4.5 on the right; one easily see that for that choice of parameters, the Born approximated power spectrum just gets crazy if you let him evolve at $a = 1$. A similar behavior actually holds also for the dark matter power spectrum case, but you must let him evolve for much larger values of a (see figure 4.6). This because we expect the Born approximation not to hold for small scales (see [6]) and this suggests also its breakdown for very late times, as our analysis shows. This suggested to span also over earlier times for these initial Gaussian power spectra, where the Born approximation still gives a reasonable result (see 4.5 on the left).

In figure 4.7 we show the behavior of power spectra with the same parameters evolved at different times. That behavior is actually common for all the other cases, where the time scales seem to be dictated by high σ and/or high k_0 (that is, for high σ and/or k_0, the behavior one can see on figure 4.7 happens before and it is more pronounced, see for example figure 4.8). As a first guess, we can say that the main reason that causes this behavior is a too strong enhancing of power on middle/small scales. An excess of power on those scale is anyway expected to yield shorter time scales for power spectra evolution (an enhanced power on small scales means that collapse happens more easily). Such kind of power spectra could arise if dark matter is initially "sticky" so that power in small scales is more easily achieved; but this is strongly against what we observe today.

In tables 4.9-4.11 we show again also the slopes of the fall-off. Given the behavior of the Born approximated power spectrum case, to compute its slope we restricted to the part from the maximum value to the point in which it starts reaching asymptotically the free non-linearly evolved one, the same thing we did for the dark matter power spectrum case. We see also here that late-time evolution implies again a k^{-3} behavior.

Similarly to what happened on the dark matter spectrum case, we had problems of numerical convergence for initial power spectra with $k_0 \gtrsim 10\,h\,\mathrm{Mpc}^{-1}$, so they are not treated in our analysis.

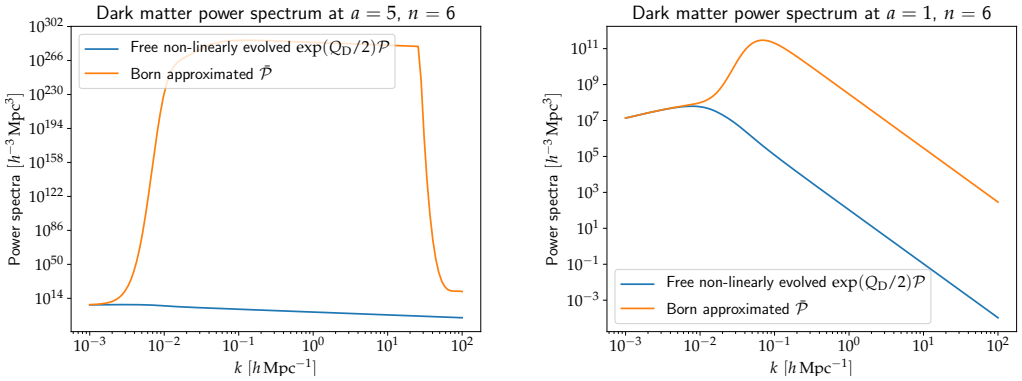


FIGURE 4.6: Non-linearly evolved power spectra for $n = 6$; on the left, it is evolved until $a = 5$ whereas on the right is evolved until today ($a = 1$). Notice how, in the case of very late times, the Born approximated spectrum gives unreasonable results, the same way the gaussian spectra do for earlier times.

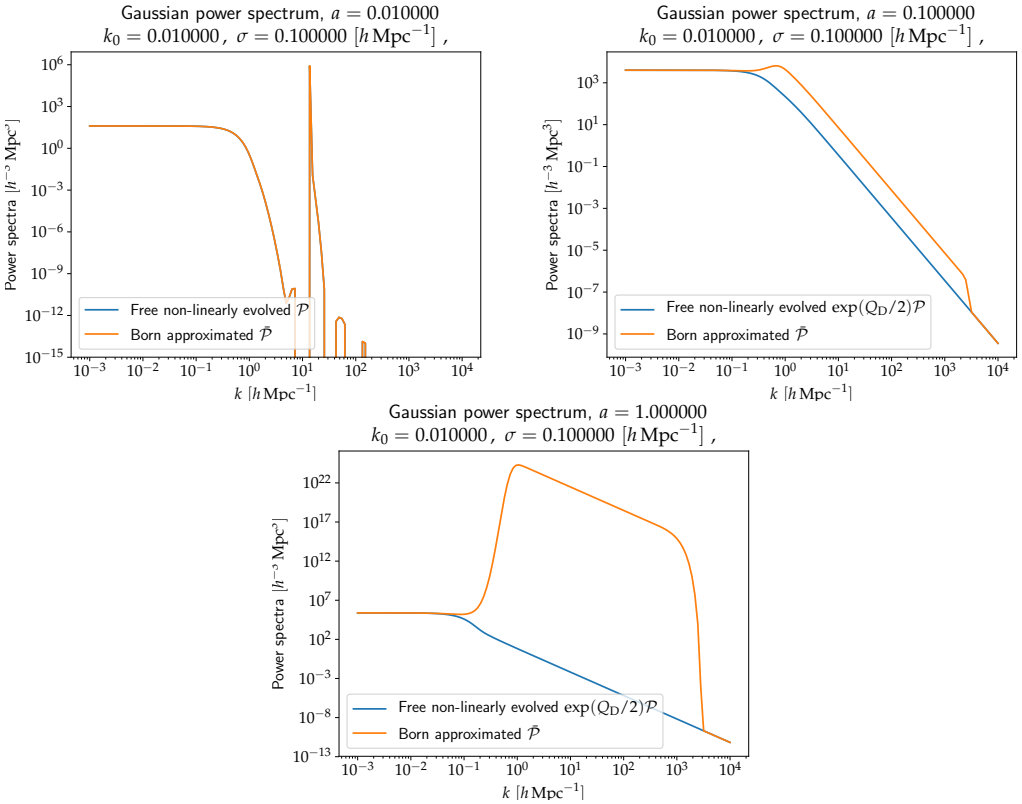


FIGURE 4.7: Gauss spectra plot corresponding to different times. This kind of evolution behavior is common to basically every case, with the difference that the time scales are different.

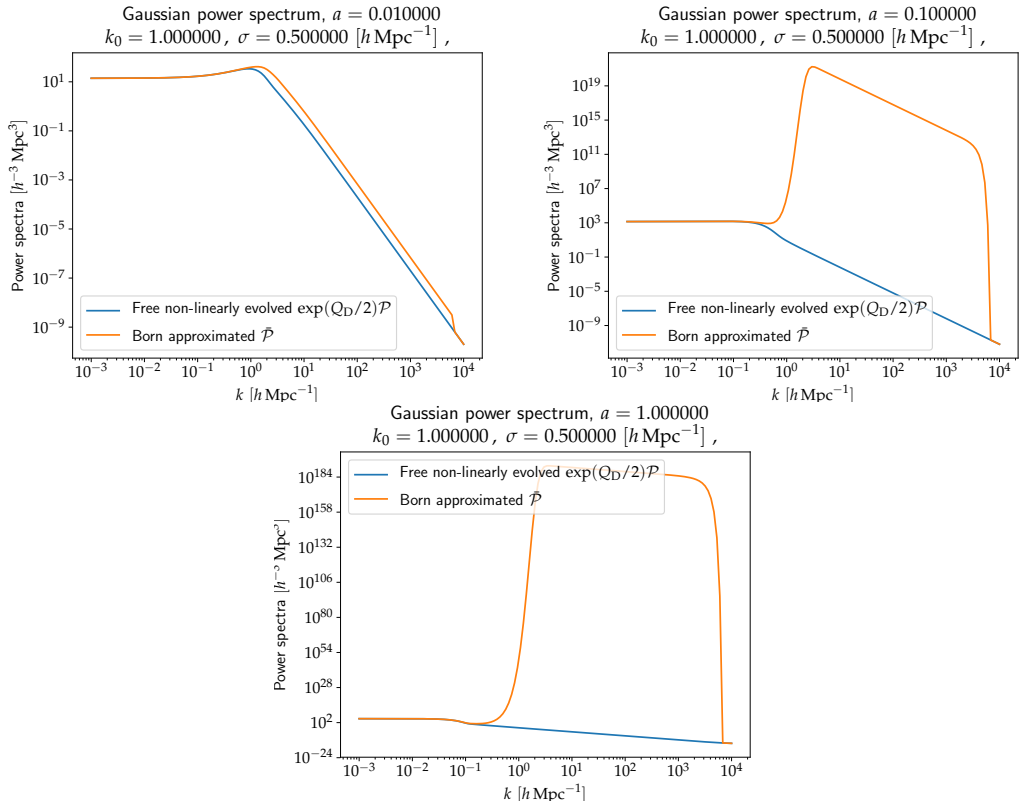


FIGURE 4.8: Gauss spectra plot for different times. Notice how, with respect to the case showed in figure 4.7, the time scale of evolution is shorter.

σ $[h\,\mathrm{Mpc}^{-1}]$	$\bar{\mathcal{P}}$ fall-off	$\exp(Q_D/2)\mathcal{P}$ fall-off	k of $\bar{\mathcal{P}}$ peak $[h\,\mathrm{Mpc}^{-1}]$
0.1000	-3.001808 ± 0.000233	-2.998890 ± 0.000150	1.075220
0.1668	-3.000999 ± 0.000129	-2.997981 ± 0.000272	1.385900
0.2154	-3.000729 ± 0.000094	-2.997233 ± 0.000372	1.573440
0.2783	-3.000527 ± 0.000068	-2.996183 ± 0.000511	1.786360
0.4642	-3.000204 ± 0.000027	-2.992615 ± 0.000979	2.614100
0.7743	-3.000098 ± 0.000014	-2.985564 ± 0.001886	3.369450
1.2915	-3.000030 ± 0.000011	-2.971784 ± 0.003608	4.343050
2.1544	-2.999981 ± 0.000009	-2.945460 ± 0.006761	5.597980
3.5938	-2.999861 ± 0.000021	-2.896717 ± 0.012287	7.215520
5.9948	-2.999486 ± 0.000068	-2.809006 ± 0.021759	9.300450

TABLE 4.9: Summary of the Gaussian power spectrum results for $k_0 = 0.01\,h\,\mathrm{Mpc}^{-1}$ evolved until $a = 1$.

σ [h Mpc^{-1}]	$\bar{\mathcal{P}}$ fall-off	$\exp(Q_D/2)\mathcal{P}$ fall-off	$\bar{\mathcal{P}}$ peak [h Mpc]
0.0100	-3.008955 $\pm$ 0.001162	-2.999745 $\pm$ 0.000035	0.389539
0.0215	-3.007050 $\pm$ 0.000913	-2.999626 $\pm$ 0.000051	0.502097
0.0464	-3.003693 $\pm$ 0.000476	-2.999330 $\pm$ 0.000091	0.734754
0.1000	-3.001655 $\pm$ 0.000214	-2.998582 $\pm$ 0.000192	1.075220
0.2154	-3.000512 $\pm$ 0.000066	-2.996627 $\pm$ 0.000453	1.786360
0.4642	-3.000187 $\pm$ 0.000025	-2.991422 $\pm$ 0.001134	2.614100
1.0000	-3.000060 $\pm$ 0.000009	-2.977516 $\pm$ 0.002899	3.825400
2.1544	-2.999955 $\pm$ 0.000038	-2.941308 $\pm$ 0.007245	5.597980
4.6416	-2.999713 $\pm$ 0.000038	-2.852180 $\pm$ 0.017127	8.191920

TABLE 4.10: Summary of the Gaussian power spectrum results for $k_0 = 0.1\,h\,\mathrm{Mpc}^{-1}$ evolved until $a = 1$.

σ [h Mpc^{-1}]	$\bar{\mathcal{P}}$ fall-off	$\exp(Q_D/2)\mathcal{P}$ fall-off	k of $\bar{\mathcal{P}}$ peak [h Mpc^{-1}]
0.5000	-3.000005 $\pm$ 0.000006	-2.946057 $\pm$ 0.006691	3.825400
0.6975	-2.999992 $\pm$ 0.000006	-2.938285 $\pm$ 0.007596	4.343050
0.9729	-2.999971 $\pm$ 0.000008	-2.926194 $\pm$ 0.008985	4.930750
1.3572	-2.999934 $\pm$ 0.000015	-2.908535 $\pm$ 0.010975	5.597980
1.8932	-2.999875 $\pm$ 0.000018	-2.883433 $\pm$ 0.013745	6.355500
2.6410	-2.999770 $\pm$ 0.000031	-2.848200 $\pm$ 0.017554	7.215520

TABLE 4.11: Summary of the Gaussian power spectrum results for $k_0 = 1\,h\,\mathrm{Mpc}^{-1}$ evolved until $a = 1$.

4.4 Physical meaning

The main result of our analysis is the small scale fall-off of non-linearly evolved power spectra. In general, KFT prediction can be helpful in the study of the evolution of such steep (large n for dark matter-like spectrum case) or localized (gaussian case) initial power spectra, since numerical simulations are unable to study such spectra due to inevitable shot noise. Thanks to KFT, we saw that for a wide class of initial power spectra, small scale fall-off follows a k^{-3} behavior. This asymptotic behavior is expected if one looks for asymptotic series expansion for $\mathcal{P}$ (as the recent work [30] is trying to show), and our analysis for these cases confirms this.

The large n case for dark-matter like initial power spectra is interesting also because axions (or axion-like particles) are indeed expected to yield initial power spectra with a similar steep shape [31, 32].

Chapter 5

Higher order correlators

We now want to analyze higher-order correlators like bispectrum and trispectrum in a similar fashion we did for the power spectrum in the previous chapters. Differently from the power spectrum case, a mean-field approach to include interactions is not known; one could then rely on perturbation theory (see [1],[12]).

Here we will focus our interest only on the behavior of free non-linearly evolved higher order correlators and we leave the analysis including interactions (within perturbation theory or within a new developed mean-field approach) to future work.

We have then to specialize (2.66) for the correlator we need. Let's say we want to compute a density correlator of order l; we will need to evaluate the products

$$\prod_{1\leq k<j}^{l}\left((2\pi)^3\delta_{\mathrm{D}}(\vec{k}_{jk})+\mathcal{P}_{jk}(\vec{k}_{jk})\right)=:\prod_{1\leq k<j}^{l}(\Delta+\mathcal{P})_{jk}\,,$$

where we have defined $\Delta_{jk}:=\Delta(\vec{k}_{jk}):=(2\pi)^3\delta_{\mathrm{D}}(\vec{k}_{jk})$ for convenience. There are $n=l(l-1)/2$ factors appearing in the previous product, and in terms of Δ and $\mathcal{P}$ factors, we can schematically write

$$\prod_{1\leq k<j}^{l}(\Delta+\mathcal{P})_{jk}\sim\sum_{m=0}^{n}\binom{n}{m}\Delta^{n-m}\mathcal{P}^{m}\,; \tag{5.1}$$

of course actually the Δ and $\mathcal{P}$ factors differ due to the particular $\vec{k}_{jk}$ they depend on, but the useful splitting we can make is between internal and external wavevectors, i.e. wavevectors which are integrated and which are not, respectively. Looking at (2.68), we see that there are $n-l+1$ internal wavevectors and thus $l-1$ external ones. We can then further decompose (5.1) as

$$\begin{aligned}\binom{n}{m}\Delta^{n-m}\mathcal{P}^{m}=&\sum_{\substack{e=0\,,\,l-1\geq e\,,\\ n-l+1\geq m-e}}^{m}\binom{l-1}{e}\binom{n-l+1}{m-e}\\&\times\mathcal{P}_{\mathrm{ext}}^{e}\mathcal{P}_{\mathrm{int}}^{m-e}\Delta_{\mathrm{ext}}^{l-e-1}\Delta_{\mathrm{int}}^{n-l+1-m+e}\,,\end{aligned} \tag{5.2}$$

where quantities with the subscript int refer to the ones that depend on internal wavevectors (that is $\vec{k}_{jk}$ with $k\neq 1$) and quantities with the subscript ext refer to the ones that depend on external wavevectors (that is $\vec{k}_{jk}$ with $k=1$). To properly understand how to get advantages from this splitting, it is better to make the explicit example for $l=3$; since we want to arrive at an explicit expression for the bispectrum, this will prove useful anyway.

So for the case $l = 3$, we have one internal wavevector $\vec{k}_{32} = \vec{k}'_{32}$ and two external (look at (2.68))

$$\vec{k}_{21} = \vec{L}_{q_2} + \vec{k}'_{32} \,, \;\; \vec{k}_{31} = \vec{L}_{q_3} - \vec{k}'_{32} \,.$$

The relevant terms, from (5.2) and (5.1), are

$$\Delta^3 = \Delta_{\rm int}\Delta^2_{\rm ext} \implies \int \frac{{\rm d}^3 k_{32}}{(2\pi^3)} \Delta_{32}\Delta_{31}\Delta_{21} = \prod_{j=2}^{3} \Delta(\vec{L}_{q_j}) \,, \tag{5.3}$$

$$\begin{aligned} 3\Delta^2\mathcal{P} =& 2\mathcal{P}_{\rm ext}\Delta_{\rm ext}\Delta_{\rm int} + \Delta^2_{\rm ext}\mathcal{P}_{\rm int} \underset{\int {\rm d}^3 k_{32}/(2\pi)^3}{\Longrightarrow} \Delta(\vec{L}_{q_2})\mathcal{P}_{31}(\vec{L}_{q_3}) \\ &+ \Delta(\vec{L}_{q_3})\mathcal{P}_{21}(\vec{L}_{q_2}) + \Delta(\vec{L}_{q_2} + \vec{L}_{q_3})\mathcal{P}_{32}(\vec{L}_{q_3}) \,, \end{aligned} \tag{5.4}$$

$$\begin{aligned} 3\Delta\mathcal{P}^2 =& 2\mathcal{P}_{\rm ext}\Delta_{\rm ext}\mathcal{P}_{\rm int} + \Delta_{\rm int}\mathcal{P}^2_{\rm ext} \underset{\int {\rm d}^3 k_{32}/(2\pi)^3}{\Longrightarrow} \mathcal{P}_{31}(\vec{L}_{q_2} + \vec{L}_{q_3})\mathcal{P}_{32}(-\vec{L}_{q_2}) \\ &+ \mathcal{P}_{21}(\vec{L}_{q_2} + \vec{L}_{q_3})\mathcal{P}_{32}(\vec{L}_{q_3}) + \mathcal{P}_{21}(\vec{L}_{q_2})\mathcal{P}_{31}(\vec{L}_{q_3}) \,, \end{aligned} \tag{5.5}$$

$$\mathcal{P}^3 = \mathcal{P}^2_{\rm ext}\mathcal{P}_{\rm int} \underset{\int {\rm d}^3 k_{32}/(2\pi)^3}{\Longrightarrow} \int \frac{{\rm d}^3 k_{32}}{(2\pi^3)} \mathcal{P}_{32}(\vec{k}_{32})\mathcal{P}_{21}(\vec{L}_{q_2} + \vec{k}_{32})\mathcal{P}_{31}(\vec{L}_{q_3} - \vec{k}_{32}) \,. \tag{5.6}$$

On the second and third line, we exploited the various Δ to perform the internal integration, and we can also exploit the overall Dirac delta in (2.66) to set $\vec{L}_{q_2} + \vec{L}_{q_3} = -\vec{L}_{q_1}$.

It is then clear what are the main steps one can do to simplify (5.2) for a generic l:

- use the $\Delta_{\rm int}$ factors, if any, to integrate and set to zero the corresponding internal wavevector;
- use the $\Delta_{\rm ext}$ factors, if any, to integrate over the remaining internal wavevectors, in such a way to remove as much internal wavevectors as possible from the $\mathcal{P}$ arguments.

5.1 The expression for the bispectrum

To recover the full expression for the bispectrum, write (2.66), with $l = 3$, as

$$\begin{aligned} Z_{{\rm C},0}[\boldsymbol{L}, 0] =& V^{-3}(2\pi)^3\delta_{\rm D}\left(\sum_{j=1}^{3}\vec{L}_{q_j}\right){\rm e}^{-(\mathrm{Q}_0 - \mathrm{Q}_{\rm D})/2}\int \frac{{\rm d}^3 k_{32}}{(2\pi)^3} \\ &\times \prod_{1\le k<j}^{3}\Big((2\pi)^3\delta_{\rm D}(\vec{k}_{jk}) + \mathcal{P}_{jk}\Big) \,; \end{aligned} \tag{5.7}$$

the overall Dirac delta, given the expressions of $\vec{L}_{q_j}$, sets $\vec{k}_1 = -\vec{k}_2 - \vec{k}_3$; since $\vec{L}_{p_j} = -g_{qp}(\tau,0)\vec{k}_j$, this also implies $\vec{L}_{p_1} = -\vec{L}_{p_2} - \vec{L}_{p_3}$, so that, from (2.59), we have

$$Q_0 = 0\,; \;\; Q_{\rm D} = -\frac{2}{3}\sigma_1^2 g_{qp}^2(k_2^2 + k_3^2 + \vec{k}_2\cdot\vec{k}_3)\,.$$

We need now to compute the last line of (5.7) by using the splitting (5.3)-(5.6); we basically need to determine, looking also at (2.67), the $\lambda^{\parallel}_{jk}$, $\lambda^{\perp}_{jk}$ factors in (2.69) for the relevant $\mathcal{P}_{jk}(\vec{k}_{jk})$.

As an example, let's determine $\lambda^{\parallel}_{31}$ for $\mathcal{P}_{31}(\vec{L}_{q_2} + \vec{L}_{q_3})$ on (5.5); there, $\vec{k}_{32}$ is set to $-\vec{L}_{q_2}$ due to the Δ_{21} factor, so

$$\vec{k}_{31} = \vec{L}_{q_2} + \vec{L}_{q_3} = -\vec{k}_3 - \vec{k}_2 = \vec{k}_1\,,$$

where the last step comes from the overall Dirac delta. So $\hat{k}_{31}$ points in the same direction of $\vec{k}_1$, and thus we have

$$\vec{L}^{\top}_{p_3}\pi^{\parallel}_{31}\vec{L}_{p_1} = g_{qp}^2(\tau,0)\vec{k}_3^{\top}\pi^{\parallel}_{31}\vec{k}_1 = g_{qp}^2(\tau,0)\vec{k}_3\cdot\vec{k}_1\,,$$

so

$$\lambda^{\parallel}_{31} = \frac{\vec{k}_3\cdot\vec{k}_1}{k_1^2} = -\frac{\vec{k}_2\cdot\vec{k}_1}{k_1^2} - 1\,, \;\; \lambda^{\perp}_{31} = 0\,.$$

Similar reasoning can be made for all the other terms.

Connected contributions. When one studies an n-th order correlator, one is interested to the connected contribution, that is the part of the correlator that involves the n-body interaction only[1]. In general, we can write

$$\left\langle \delta(\vec{k}_1)\cdots\delta(\vec{k}_n)\right\rangle_{\rm c} = \left\langle \delta(\vec{k}_1)\cdots\delta(\vec{k}_n)\right\rangle - \sum_{S\in\sigma(\{1,\ldots,n\})}\prod_{s\in S}\left\langle \delta(\vec{k}_{s(1)})\cdots\delta(\vec{k}_{s(\#s)})\right\rangle_{\rm c}\,, \tag{5.8}$$

where the subscript c means connected, S belongs to the proper partitions (that is partition not including the set itself) $\sigma(\{1,\ldots,n\})$ of n elements, and s is a subset of $\{1,\ldots,n\}$, belonging to a particular partition S with $\#s$ elements. For a connected 3-rd order correlator, we have

$$\begin{aligned}\left\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\delta(\vec{k}_3)\right\rangle_{\rm c} = \left\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\delta(\vec{k}_3)\right\rangle - \left\langle \delta(\vec{k}_1)\right\rangle\left\langle \delta(\vec{k}_2)\right\rangle\left\langle \delta(\vec{k}_3)\right\rangle \\ -\Big(\left\langle \delta(\vec{k}_1)\right\rangle\left\langle \delta(\vec{k}_2)\delta(\vec{k}_3)\right\rangle + \text{cyc.}\Big)\,,\end{aligned}$$

where cyc. means cycling over the indices $\{1,2,3\}$. It is then clear that (5.3) and (5.4) are non-connected contributions to the 3-rd order correlator. In general, as we will see in section 5.3, all the terms corresponding to $m < n - l + 1$ in (5.1) are non-connected contributions and can thus be neglected.

[1]Indeed a connected contribution is characterized by the fact that, by letting a coordinate difference among two points go to infinity, then the whole contribution vanishes (something that is not true for non-connected contribution). This can also be seen as a criterium to determine whether a contribution is connected or not.

Final result. In the end, using the notation

$$\mathcal{P}(\vec{k}\,,\lambda^{\parallel}\,,\lambda^{\perp}) = \int \mathrm{d}^3q \left(\mathrm{e}^{g_{qp}^2 k^2(a_{\parallel}\lambda^{\parallel}+a_{\perp}\lambda^{\perp})}-1\right)\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}}\,, \tag{5.9}$$

we obtain the following final expression for the correlator (not including (5.3) and (5.4))

$$\begin{aligned} Z_{\mathrm{C},0}[\boldsymbol{L},0] =& V^{-3}(2\pi)^3\delta_{\mathrm{D}}\Big(\vec{k}_1+\vec{k}_2+\vec{k}_3\Big)\exp\left(-\frac{\sigma_1^2}{3}g_{qp}^2(\tau,0)(k_1^2+k_2^2+\vec{k}_1\cdot\vec{k}_2)\right)\\ &\times\mathcal{P}\left(k_1\,,-\frac{\vec{k}_2\cdot\vec{k}_1}{k_1^2}-1,0\right)\mathcal{P}\left(k_2\,,-\frac{\vec{k}_2\cdot\vec{k}_1}{k_2^2}-1\,,0\right)+\mathcal{P}\left(k_1\,,\frac{\vec{k}_2\cdot\vec{k}_1}{k_1^2}\,,0\right)\\ &\times\mathcal{P}\left(|\vec{k}_1+\vec{k}_2|\,,-\frac{\vec{k}_2\cdot\vec{k}_1+k_2^2}{k_1^2+k_2^2+2\vec{k}_1\cdot\vec{k}_2}\,,0\right)+\mathcal{P}\left(k_2\,,\frac{\vec{k}_2\cdot\vec{k}_1}{k_2^2}\,,0\right)\\ &\times\mathcal{P}\left(|\vec{k}_1+\vec{k}_2|\,,-\frac{\vec{k}_2\cdot\vec{k}_1+k_1^2}{k_1^2+k_2^2+2\vec{k}_1\cdot\vec{k}_2}\,,0\right)+\mathcal{PPP}\text{ term}\Bigg\}\\ =& V^{-3}(2\pi)^3\delta_{\mathrm{D}}\Big(\vec{k}_1+\vec{k}_2+\vec{k}_3\Big)B_\delta(k_1,k_2,\hat{k}_1\cdot\hat{k}_2)\,, \end{aligned} \tag{5.10}$$

where on the last line B_δ is the bispectrum and as $\mathcal{PPP}$ term we mean the integral in (5.6). The last equality holds since we are excluding from the generating functional expression the non-connected contributions (5.3) and (5.4).

We see then that, from a computational point of view, we reduce to compute factors of the type (5.9), something we have already done for the power spectrum. Notice that everything depends only on the three scalar quantities $k_1\,,k_2$ and $\hat{k}_1\cdot\hat{k}_2$, as expected for a bispectrum in an homogeneous and isotropic background.

Unfortunately, we do not have a reliable and efficient way to compute integrals involving multiple $\mathcal{P}$; as long as they are not evolved to much later times, such terms, since the initial amplitude of $\mathcal{P}$ is low, are expected to be subdominant with respect to the other ones, so we will simply neglect them in the following.

5.1.1 Early time bispectrum

We can recast the expression for the free bispectrum, ignoring the $\mathcal{PPP}$ term, as

$$\begin{aligned} B_\delta(k_1,k_2,\mu) =& \exp\left(-\frac{\sigma_1^2}{3}g_{qp}^2(\tau,0)(k_1^2+k_2^2+k_1k_2\mu)\right)\\ &\times\left(\mathcal{P}\left(k_1\,,-\frac{\vec{k}_2\cdot\vec{k}_1}{k_1^2}-1,0\right)\mathcal{P}\left(k_2\,,-\frac{\vec{k}_2\cdot\vec{k}_1}{k_2^2}-1\,,0\right)+\text{cyc.}\right)\,, \end{aligned}$$

with $\mu=\hat{k}_1\cdot\hat{k}_2$ and cyc means cycling over the indices $\{1,2,3\}$ (recall that then $\vec{k}_3$ can be expressed in terms of the other two wavevectors).

In the limit of early times or large scales, we can recast the previous, using (2.81) and neglecting the exponential damping factor, as

$$\begin{aligned} B_\delta &\simeq g_{qp}^4 \left(\left(\frac{\vec{k}_2 \cdot \vec{k}_1}{k_1^2} + 1 \right) \left(\frac{\vec{k}_2 \cdot \vec{k}_1}{k_2^2} + 1 \right) P_\delta(k_1) P_\delta(k_2) + \text{cyc.} \right) \\ &=: g_{qp}^4 \Big(F(\vec{k}_1, \vec{k}_2) P_\delta(k_1) P_\delta(k_2) + \text{cyc.} \Big) \,, \end{aligned}$$

with $F(\vec{k}_1, \vec{k}_2)$ resembling very closely the $2F_2$ kernel one finds in Eulerian perturbation theory [5] once one recasts it in the following form:

$$F(\vec{k}_1, \vec{k}_2) = 1 + \frac{\vec{k}_1 \cdot \vec{k}_2}{k_1 k_2} \left(\frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{(\vec{k}_1 \cdot \vec{k}_2)^2}{k_1^2 k_2^2} \,.$$

We remark that this is the same result one finds in [1], obtained within KFT with a different method.

5.2 Bispectrum analysis

5.3 Other higher order correlators

The cases of the other higher order correlators are quite similar to what we did for the bispectrum, only more involved as one could easily expect.

For a generic l correlator, we have (exploiting the overall delta to set $\vec{k}_l = -\sum_i^{l-1} \vec{k}_i$)

$$Q_0 = 0 \,, \quad Q_\text{D} = -\frac{2}{3} \sigma_1^2 g_{qp}^2(\tau, 0) \left(\sum_{i<j}^{l-1} (\vec{k}_i \cdot \vec{k}_j) + \sum_{i=1}^{l-1} k_i^2 \right) ;$$

from the previous one can read that a generic l-th order correlator will only depend on $l-1$ modules and $(l-1)(l-2)/2$ angles, or $l(l-1)/2$ independent parameters among the possible combinations of moduli and angles (this is not surprising, it simply comes from homogeneity and isotropy assumptions). Then one can use (5.2) and (5.1) for the case of interest, but you would end up with many uninteresting non-connected contributions.

In general, to obtain directly the connected contributions out of a correlator, one can use cumulants obtained from the logarithm of the generating functional[2], as detailed in appendix D. The generic expression for a cumulant is in (D.7), but here we are interested to an l-th order density cumulant; neglecting all the shot noise terms (that is we consider only the term with $l = n_f$ on the sum in (D.7)), equation (D.8) becomes

$$G^{(\text{c},0,l)}_{\rho(1)\cdots\rho(l)} = \bar{\rho}^l (2\pi)^3 \delta_\text{D} \left(\sum_{i=1}^{l} \vec{k}_i \right) \text{e}^{-Q(I_1,\ldots,I_l)} \tilde{\Sigma}^{(l)}_\text{C}(I_1, \ldots, I_l) \,,$$

[2]The fact that the logarithm of the generating functional generates only connected contributions is a well-known result of statistical mechanics and QFT, see any standard textbook on QFT.

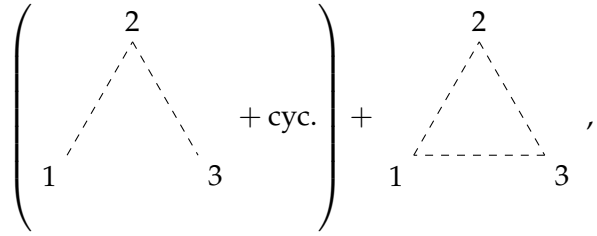
where here the sum over the field labels $\{I_1, \ldots, I_l\}$ reduces to only one term, since there is only one way to assign l labels to l non-empty sets. We also recall that a ρ-cumulant can be obtained by a f-cumulant by setting all $\vec{l}_i = 0$. In this setting, we have

$$\mathrm{e}^{-Q(I_1,\ldots,I_l)} = \mathrm{e}^{Q_\mathrm{D}/2} = \exp\left(-\frac{1}{3}\sigma_1^2 g_{qp}^2(\tau,0)\left(\sum_{i<j}^{l-1}(\vec{k}_i\cdot\vec{k}_j) + \sum_{i=1}^{l-1}k_i^2\right)\right),$$

so that everything is analogous to the results obtained with the correlators. The key difference lies on the evaluation of $\tilde{\Sigma}_\mathrm{C}^{(l)}$; by neglecting all the initial density-density and density-momentum correlations, the only type of diagrams we need for its evaluation are the ones involving only $\mathcal{P}_{p_i p_j} = \mathcal{P}_{ij}$, on the third line in (D.11). Then the Feynman rules detailed in appendix D reduce to the following:

1. draw l numbered vertices and write all possible connected diagrams;
2. any pair of particles can be connected by at most one line, and subdiagrams of the form  are forbidden;
3. to a line connecting points i and j (considering always $i > j$) it is associated the generalized vector $\vec{k}_{ij}$; for every closed loop, choose a generalized momentum $\vec{k}_{ij}$ corresponding to one of the lines forming the loop, and integrate over it;
4. to assign a generalized momenta to all the remaining line, use (2.68), assigning $\vec{k}_{ij} = 0$ to the lines that do not appear in the diagram;
5. every line assigned in this way will correspond to a $\mathcal{P}_{ij}(\vec{k}_{ij})$ term.

For the bispectrum, our Feynman rules generate the following diagrams:

$\tilde{\Sigma}_\mathrm{C}^{(3)}(1,2,3) =$ 

,

corresponding indeed to the terms we have considered in (5.10).

5.3.1 The trispectrum case

As an example, and as a possible comparison to what we have seen previously in this chapter using directly correlators, let's see the trispectrum case ($l = 4$).

We can split all the connected diagrams contributing to $\tilde{\Sigma}_\mathrm{C}^{(4)}$ using as a criterium the number of lines a diagram has. In general, for an l-vertices diagram we need at least $l-1$ lines to have connected contribution, the maximum number of lines will be $l(l-1)/2$ and any diagram with at least $l(l-1)/2 - l + 2$ will be a connected one for sure (since there is no way to leave a point without any connection with the remaining diagram with that many lines). There are $n!$ ways, with $n = l(l-1)/2$, to draw generic l-vertices diagrams with arbitrary number of lines $m \in \{0, \ldots, n\}$, and the number of diagrams with m lines will be $\binom{n}{m}$. All these $n!$ diagrams are

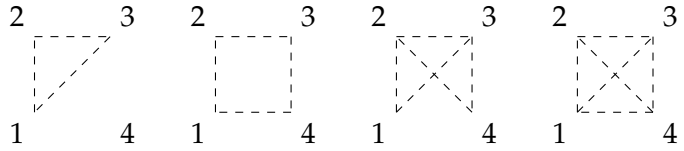
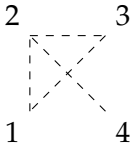


FIGURE 5.1: Examples of possible diagrams for the evaluation of 4-th order correlator. The first diagram on the left is one of the 4 non-connected diagrams with 3 lines; the other 3 are obtained simply cycling over its indices.

on one-to-one correspondence with the factors in (5.1), with the number of lines m corresponding to the number of factors $\mathcal{P}$, and the number of lines $n - m$ that are not in the diagram corresponding to Δ factors. We immediately see then that all the terms in (5.1) with $m < l - 1$ are not connected whereas terms with $m \geq n - l + 2$ are connected for sure.

For $l = 4$, we have terms with $3, 4, 5, 6$ lines; the ones with $m \geq 4$ lines are connected for sure, whereas for $m = 3$, we have 4 contributions which are non connected, see figure 5.1. So we have a total of 38 connected diagrams to evaluate. As an example, take the following diagram:

$F :=$  ;

in this diagram there is a closed loop, so we can keep for example $\vec{k}_{32}$ as wavevector over which we integrate. $\vec{k}_{43}$ and $\vec{k}_{41}$ correspond to lines which are not in the diagram we are considering, so we set them to zero. Then the other generalized momenta are determined by

$$\begin{cases} \vec{k}_{21} = -\vec{k}_2 + \vec{k}_{32} + \vec{k}_{42} \\ \vec{k}_{31} = -\vec{k}_3 - \vec{k}_{32} + \vec{k}_{43} \\ \vec{k}_{41} = -\vec{k}_4 - \vec{k}_{42} - \vec{k}_{43} = 0 \end{cases} \Longrightarrow \begin{cases} \vec{k}_{21} = -\vec{k}_2 + \vec{k}_{32} - \vec{k}_4 \\ \vec{k}_{31} = -\vec{k}_3 - \vec{k}_{32} \\ \vec{k}_{42} = -\vec{k}_4 \end{cases} ,$$

so this diagram will correspond to the term

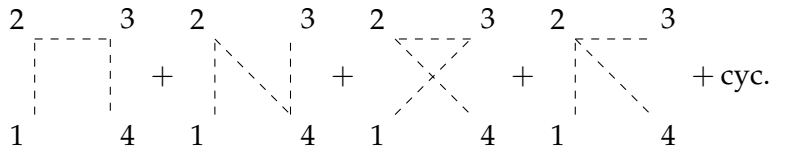
$$F = \int \frac{\mathrm{d}^3 k_{32}}{(2\pi)^3} \mathcal{P}_{31}(-\vec{k}_3 - \vec{k}_{32}) \mathcal{P}_{21}(-\vec{k}_2 + \vec{k}_{32} - \vec{k}_4) \mathcal{P}_{32}(\vec{k}_{32}) \mathcal{P}_{42}(-\vec{k}_4) \, .$$

Comparing with the result one would have obtained using (5.1) and (5.2), we see that the previous diagram correspond to a $\Delta_{\mathrm{int}} \Delta_{\mathrm{ext}} \mathcal{P}_{\mathrm{int}}^2 \mathcal{P}_{\mathrm{ext}}^2$ contribution (we recall that lines correspond to $\mathcal{P}$ factors, whereas the absence of a line correspond to a Δ factor). In particular

$$F = \int \frac{\mathrm{d}^3 k_{32} \, \mathrm{d}^3 k_{42} \, \mathrm{d}^3 k_{43}}{(2\pi)^3 (2\pi)^3 (2\pi)^3} \Delta(\vec{k}_{43}) \Delta(\vec{k}_{41}) \mathcal{P}_{42}(\vec{k}_{42}) \mathcal{P}_{31}(\vec{k}_{31}) \mathcal{P}_{21}(\vec{k}_{21}) \mathcal{P}_{32}(\vec{k}_{32}) \, .$$

If we want to neglect terms with multiple $\mathcal{P}$ integrations, we should consider only connected diagrams with $l - 1$ lines (since all the $n - l + 1$ integrations are removed

by the $n - l + 1$ Δ factors, see (5.2)), so for the trispectrum we should consider



where cyc. means cycling over the indices $\{1, 2, 3, 4\}$ separately for all 4 diagrams. Actually we can compute the first diagram and then map the indices accordingly to obtain the second and the third (for example to obtain the second diagram from the first, map $1 \to 1$, $2 \to 2$, $3 \to 4$ and $4 \to 3$). The fourth is topologically different so we must compute it explicitly. This yields (calling $\mu_{ij} := \hat{k}_i \cdot \hat{k}_j$)

$$
\begin{aligned}
T_\delta(k_1, k_2, k_3, \mu_{21}, \mu_{31}, \mu_{32}) = \exp\left(-\frac{\sigma_1^2}{3} g_{qp}^2 \left(k_1 k_2 \mu_{21} + k_2 k_3 \mu_{32} + k_1 k_3 \mu_{31} + k_1^2 + k_2^2 + k_3^2\right)\right) \\
\left\{ \mathcal{P}\left(k_1, \frac{k_2 \mu_{21}}{k_1}, 0\right) \mathcal{P}\left(|\vec{k}_1 + \vec{k}_2|, \frac{k_3 k_2 (\mu_{31}\mu_{21} + \mu_{32})}{|\vec{k}_1 + \vec{k}_2|^2}, \frac{k_3 k_2 \mu_{31} \mu_{21}}{|\vec{k}_1 + \vec{k}_2|^2}\right) \mathcal{P}\left(k_4, \frac{k_3 \mu_{43}}{k_4}, 0\right) \right. \\
+ \mathcal{P}\mathcal{P}\mathcal{P}[1 \to 1, 2 \to 2, 3 \to 4, 4 \to 3] + \mathcal{P}\mathcal{P}\mathcal{P}[1 \to 1, 2 \to 3, 3 \to 2, 4 \to 4] \quad (5.11) \\
+ \mathcal{P}\mathcal{P}\mathcal{P}[1 \to 1, 2 \to 2, 3 \to 4, 4 \to 3] + \mathcal{P}\left(k_1, \frac{k_2 \mu_{21}}{k_1}, 0\right) \mathcal{P}\left(k_3, \frac{k_2 \mu_{32}}{k_3}, 0\right) \\
\left. \times \mathcal{P}\left(k_4, \frac{k_2 \mu_{42}}{k_4}, 0\right) + \text{cyc.} \right\},
\end{aligned}
$$

where $\mathcal{P}[\ldots]$ is a shorthand to indicate the same term in the second line of the previous but with different indices. We recall that, due to the constraint $\vec{k}_4 = -\vec{k}_1 - \vec{k}_2 - \vec{k}_3$, both k_4 and μ_{4j}, $j = 1, 2, 3$ are not free parameters. Again, we can perform the same steps we did for the bispectrum in section 5.1.1 to compute the trispectrum at early times, and the structure one obtains is analogous to the one found in Eulerian perturbation theory [33, 34].

Chapter 6

Conclusions

In this thesis, we used the KFT approach for the study of LSS, applied to the analysis of non-linear evolution of some simplified primordial power spectra.

KFT is a statistical field theory involving microscopic degrees of freedom that can be accustomed to describe LSS. As most of statistical field theories and QFT, its central object is a generating functional (2.37), and correlators of macroscopic fields (like density-contrast field, of primary importance in cosmology) are obtained through appropriate functional derivatives. We summarized the basics of KFT, together with the cosmological framework we used, in the first three chapters of this thesis.

On the analysis of the two classes of power spectra (4.1), (4.2) in chapter 4, we found that the small scale fall-off of the corresponding evolved power spectra goes as k^{-3} for a lot of different cases. This is coherent with the findings in the recent work [30], that employs asymptotic series to (2.79), showing indeed what we confirmed for some case studies.

With more appropriate functional derivatives, KFT can treat also the evolution of higher-order correlators. In chapter 5, we investigated the bispectrum behavior and gave the outline on how to compute other higher order cumulants, for example the trispectrum.

Bispectrum ...

KFT is in its early stages of development, and a lot has still to be done (see [12] for a recent review). In this work, regarding the higher order analysis, we investigated only the free case. Including interactions is not an easy task; perturbative approaches [1, 2] have been developed, but their implementation is numerically challenging. A possible future work could consist in finding a non-perturbative way to include them, maybe in the spirit of the Born approximation [6] used for the simple power spectrum.

Appendix A

Mathematical tools

A.1 Kronecker products

A Kronecker product between two generic matrices A, B, with A an $m \times n$ matrix and B a $p \times q$ one, is defined as

$$A \otimes B := \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \cdots & A_{mn}B \end{pmatrix} .$$

The Kronecker product has the following properties:

$$\begin{aligned} &A \otimes (B + C) = A \otimes B + A \otimes B \,, \; (A + B) \otimes C = A \otimes C + B \otimes C \,, \\ &(kA) \otimes B = k(A \otimes B) = A \otimes (kB) \,, \\ &A \otimes (B \otimes C) = (A \otimes B) \otimes C \,, \\ &(A \otimes B)^\top = A^\top \otimes B^\top \,, \\ &\operatorname{Tr}(A \otimes B) = \operatorname{Tr} A \operatorname{Tr} B \,; \end{aligned}$$

Also if A, C and B, D can be separately multiplied, we have

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \,.$$

The idea behind our usage of the Kronecker product is to encode the many particle structure of the quantities we use on the second factor of the product. As an example, we see the explicit expression behind (2.4):

$$\partial_t \boldsymbol{x} = \partial_t \begin{pmatrix} q_j^x \vec{e}_j \\ q_j^y \vec{e}_j \\ \vdots \\ p_j^z \vec{e}_j \end{pmatrix} = \partial_t \begin{pmatrix} q_1^x \\ q_2^x \\ \vdots \\ q_N^x \\ q_1^y \\ \vdots \\ p_N^z \end{pmatrix} = \begin{pmatrix} (\mathcal{J}\partial_j)^{(1)} \vec{e}_j \\ (\mathcal{J}\partial_j)^{(2)} \vec{e}_j \\ \vdots \\ (\mathcal{J}\partial_j)^{(6)} \vec{e}_j \end{pmatrix} \mathcal{H} = \begin{pmatrix} (\mathcal{J}\partial_1)^{(1)} \\ (\mathcal{J}\partial_2)^{(1)} \\ \vdots \\ (\mathcal{J}\partial_N)^{(1)} \\ (\mathcal{J}\partial_2)^{(1)} \\ \vdots \\ (\mathcal{J}\partial_N)^{(6)} \end{pmatrix} \mathcal{H} \,,$$

where $(\mathcal{J}\partial_j)^{(i)}$ is the i-th component of $\mathcal{J}(\partial_{\vec{q}_j} \;\; \partial_{\vec{p}_j})^\top$. These are indeed the $6N$ Hamilton equations of the system.

Our compact notation thus allows us to treat these $6N$ dimensional quantities with little effort.

A.2 Gaussian integrals

We show here how to solve what is basically the only functional integral we can exactly solve, the gaussian integral. Let's start with an example in a finite number of dimensions, define

$$Z[\vec{b}] = \mathcal{N} \int \left(\prod_{i=1}^{n} \mathrm{d}x_i \right) \exp\left(-\frac{1}{2} \sum_{ij} k_{ij} x_i x_j + \mathrm{i} \sum_j b_j x_j \right) ,$$

where $\mathrm{i} \sum_j b_j x_j$ is the source term and $\mathcal{N}$ is a normalization factor; then we make the change of variables

$$\vec{y} = \mathrm{i} k^{-1} \vec{b}, \;\; \vec{x} = \vec{y} + \vec{z} \, ;$$

with this the exponent becomes

$$-\frac{1}{2} x_i k_{ij} x_j + \mathrm{i} b_i x_i = -\frac{1}{2} z_i k_{ij} z_j - \frac{1}{2} b_i k_{ij}^{-1} b_j \, ,$$

so

$$Z[\vec{b}] = \mathcal{N} \exp\left(-\frac{1}{2} b_i k_{ij}^{-1} b_j \right) \int \left(\prod_{i=1}^{n} \mathrm{d}z_i \right) \exp\left(-\frac{1}{2} z_i k_{ij} z_j \right) ,$$

now diagonalize k as $\sum k_{ij} z_i z_j = \sum \lambda_i \xi_i^2$, where λ_i are the eigenvalues of k. We thus end up with a standard gaussian integral, the final result is

$$Z[\vec{b}] = \mathcal{N} (2\pi)^{n/2} (\det k)^{-1/2} \exp\left(-\frac{1}{2} b_i k_{ij}^{-1} b_j \right) ;$$

imposing $Z[0] = 1$, we have

$$Z[\vec{b}] = \exp\left(-\frac{1}{2} b_i k_{ij}^{-1} b_j \right) .$$

All this procedure can be extended to the functional case

$$Z[J] = \mathcal{N} \int \mathcal{D}q \exp\left(-\frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, q(x) k(x,y) q(y) + \mathrm{i} \int \mathrm{d}x \, q(x) J(x) \right) ,$$

yielding (with the proper normalization)

$$Z[J] = \exp\left(-\frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, J(x) k^{-1}(x,y) J(y) \right) . \tag{A.1}$$

A.3 Solving linear inhomogeneous equations

Consider a non homogeneous equation of the type

$$(\partial_t + a(t)) f(t) = b(t) \, ; \tag{A.2}$$

an homogeneous solution is simply given by

$$f_0(t) = f(0) \exp\left(- \int_0^t \mathrm{d}t' \, a(t') \right) .$$

To find a particular solution, let's apply the variation of constants method: use the trial solution

$$f(t) = C(t) \exp\left(- \int_0^t \mathrm{d}t' \, a(t') \right) ,$$

insert it in (A.2) to obtain

$$\exp\left(- \int_0^t \mathrm{d}t' \, a(t') \right) \partial_t C(t) = b(t) \implies C(t) = \int^t \mathrm{d}t' \, b(t') \exp\left(\int_0^{t'} \mathrm{d}t'' \, a(t'') \right) ,$$

so that we can write the particular solution as

$$\begin{aligned} f(t) &= \exp\left(- \int_0^t \mathrm{d}t' \, a(t') \right) \int^t \mathrm{d}t' \, b(t') \exp\left(- \int_{t'}^0 \mathrm{d}t'' \, a(t'') \right) \\ &= \int^t \mathrm{d}t' \, b(t') \exp\left(- \int_{t'}^t \mathrm{d}t'' \, a(t'') \right) . \end{aligned}$$

The full solution can be written as

$$f(t) = G(t,0) f(0) + \int_0^t \mathrm{d}t' \, G(t,t') b(t') , \tag{A.3}$$

with the Green function corresponding to (A.2)

$$G(t,t') = \exp\left(- \int_{t'}^t \mathrm{d}t'' \, a(t'') \right) \Theta(t - t') .$$

Appendix B

Some derivation details

B.1 Classical path integral

Here we will see some details about the derivations of (2.11), (2.16),

We can use the following property of the delta function: given the function $\boldsymbol{E}(\boldsymbol{x})$, if $\boldsymbol{E}$ has a zero in $\boldsymbol{x}_{\mathrm{cl}}$, then we can formally write

$$\delta_{\mathrm{D}}(\boldsymbol{x}-\boldsymbol{x}_{\mathrm{cl}}) = \delta_{\mathrm{D}}(\boldsymbol{E}(\boldsymbol{x}))\big|\boldsymbol{E}'(\boldsymbol{x}_{\mathrm{cl}})\big| \; ; \tag{B.1}$$

collecting $|\boldsymbol{E}'(\boldsymbol{x}_{\mathrm{cl}})|$ in an unimportant overall constant factor, and using the integral representation of the delta (2.9) we have, from (2.8), indeed the (2.11).

Now regarding (2.16), we can reverse the previous reasoning: the piece affected by the integration over $\boldsymbol{\chi}$ in the free part of (2.15) becomes

$$\int \mathcal{D}\boldsymbol{\chi}\, \mathrm{e}^{\mathrm{i}\boldsymbol{\chi}\cdot(\boldsymbol{E}_0(\boldsymbol{x})+\boldsymbol{K})} = \delta_{\mathrm{D}}(\boldsymbol{E}_0(\boldsymbol{x})+\boldsymbol{K}) \; ,$$

so that the part affected by $\int \mathcal{D}'\boldsymbol{x}$ becomes

$$\int \mathcal{D}'\boldsymbol{x}\, \delta_{\mathrm{D}}(\boldsymbol{E}_0(\boldsymbol{x})+\boldsymbol{K}) \exp\left(\mathrm{i}\int \mathrm{d}t\, \boldsymbol{J}\cdot\boldsymbol{x}\right) = \exp\left(\mathrm{i}\int \mathrm{d}t\, \boldsymbol{J}\cdot\bar{\boldsymbol{x}}\right) \; ,$$

where $\bar{\boldsymbol{x}}$, that comes from the Dirac delta integration, is the solution of the classical equation of motion with source $\boldsymbol{K}$ as written in (2.17).

B.2 Including interaction terms

Here we will see the logic behind the inclusion of interaction terms, common in QFT, as seen for example in (2.37).

Imagine we have a function of a quantity ϕ we call $f(\phi)$, and that we have the expression

$$A(J) := \int \mathrm{d}\phi\, f(\phi)\mathrm{e}^{\mathrm{i}J\phi} \; ,$$

if for example $f(\phi) = \exp(\phi)$, then we can think of it in terms of its Taylor expansion

$$\exp(\phi) = 1 + \phi + \frac{1}{2}\phi^2 + \dots \; ;$$

now a function of an operator is defined in terms of the function's Taylor expansion as well, like for example

$$\exp\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}J}\right) := 1 - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}J} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}J^2} + \dots ,$$

so

$$\exp\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}J}\right)\mathrm{e}^{\mathrm{i}J\phi} = \left(1 + \phi + \frac{1}{2}\phi^2 + \dots\right)\mathrm{e}^{\mathrm{i}J\phi} = \mathrm{e}^{\phi}\mathrm{e}^{\mathrm{i}J\phi} ;$$

in general this trick works for every sufficiently regular function, so that

$$f\left(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}J}\right)\int \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}J\phi} = \int \mathrm{d}\phi\, f(\phi)\mathrm{e}^{\mathrm{i}J\phi} .$$

With this technique, we can pull out of an integral for example the interaction terms by exploiting the source terms (like $\boldsymbol{J}, \boldsymbol{K}$ of the main body of this thesis) and this trick.

B.3 Deriving (2.28)

At first, we should explicitly show the validity of (2.24); from (2.23), by expressing everything in terms of Fourier transform,

$$\begin{aligned}
S_1 &= -\int \mathrm{d}t \int \mathrm{d}^3q \int \mathrm{d}^3y \int \frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{q}}B(t,\vec{k}) \int \frac{\mathrm{d}^3n}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{n}\cdot(\vec{q}-\vec{y})}v(\vec{n}) \int \frac{\mathrm{d}^3m}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{m}\cdot\vec{y}}\rho(t,\vec{m}) \\
&= -\int \mathrm{d}t \int \mathrm{d}^3n\, \mathrm{d}^3m\, \frac{\mathrm{d}^3k}{(2\pi)^3} \underbrace{\int \frac{\mathrm{d}^3q}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{q}\cdot(\vec{k}+\vec{n})}}_{\delta_{\mathrm{D}}(\vec{k}+\vec{n})} \underbrace{\int \frac{\mathrm{d}^3y}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{y}\cdot(\vec{m}-\vec{n})}}_{\delta_{\mathrm{D}}(\vec{m}-\vec{n})} B(t,\vec{k})v(\vec{n})\rho(t,\vec{m}) \\
&= -(2\pi)^3 \int \mathrm{d}t\, \frac{\mathrm{d}^3n}{(2\pi)^3}B(t,-\vec{n})v(\vec{n})\rho(t,\vec{n}) \int \frac{\mathrm{d}^3l}{(2\pi)^3}\delta_{\mathrm{D}}(\vec{l}) ,
\end{aligned}$$

that indeed it is the (2.24) once one uses our notation; now we write explicitly (2.28)

$$\begin{aligned}
S_1 = -\frac{1}{2}(2\pi)^9 \int \frac{\mathrm{d}^3k_r}{(2\pi)^3}\frac{\mathrm{d}^3l_r}{(2\pi)^3}\frac{\mathrm{d}^3l_{r'}}{(2\pi)^3}\frac{\mathrm{d}^3k_{r'}}{(2\pi)^3}\,\mathrm{d}t_r\,\mathrm{d}t_{r'}\,\delta_{\mathrm{D}}(\vec{k_r}-\vec{k_{r'}})\delta_{\mathrm{D}}(\vec{l_r})\delta_{\mathrm{D}}(\vec{l_{r'}})v(r) \\
\times\left(\Phi_f(r)\Phi_B(-r) + \Phi_f(-r)\Phi_B(+r)\right)
\end{aligned}$$

then, since $\Phi_f(r) = \rho(r)$, $\Phi_B(r) = B(r)$ when evaluated at $\vec{l_r} = 0$, we have indeed the equivalence of (2.24) and (2.28).

Appendix C

Numerical techniques

In order to numerically solve the integrals we met in this work, we used some C++ libraries implementing GSL integration routines[1] [29]. In particular, we dealt with fast-oscillating integrals, and an efficient implementation of their integrations is the Levin collocation scheme [27, 28], which we will briefly review here.

C.1 Levin collocation scheme

Consider integrals of the form

$$I := \int_a^b \mathrm{d}x\, f(x) S(rx) \,, \tag{C.1}$$

where S is an oscillatory function dependent on a parameter r. If r is large, we are dealing with a very fast oscillatory function, so its integration with standard techniques can become quite heavy computationally (that is we need a lot of function evaluations).

To tackle this problem, consider integrals of the form

$$J := \int_a^b \mathrm{d}x\, \vec{S}(x) \cdot \vec{f}(x) \,,$$

where $\vec{S}$ and $\vec{f}$ are now n dimensional vectors, the first of independent rapidly oscillating functions and the second of independent non rapidly oscillating functions (you then can return to the (C.1) case by considering $n = 1$ and $S(x) \to S(rx)$). Assume that $\vec{S}$ satisfy the set of ordinary differential equations

$$\vec{S}'(x) = A(x)\vec{S}(x) \,, \tag{C.2}$$

where the apex means derivation over x.

The idea is to find a vector $\vec{R}$ such that

$$(\vec{S} \cdot \vec{R})' = \vec{S} \cdot \vec{f} \,, \tag{C.3}$$

so that the solution of the integral is simply

$$J = \vec{S}(b) \cdot \vec{R}(b) - \vec{S}(a) \cdot \vec{R}(a) \,.$$

[1] The code used for this thesis can be found here.

Exploiting (C.2) and (C.3), we have that $\vec{R}$ should satisfy

$$\vec{R}' + A\vec{R} = \vec{f}\,; \tag{C.4}$$

we reconduced thus the problem of finding a solution for J to find an approximate solution for the differential equation (C.4).

Now we want to use a collocation scheme to solve this problem. An l-th order collocation approximation to (C.4) is defined as the function $\vec{R}_l$ that satisfies

$$(\vec{R}_l)_i(x) = \sum_{k=1}^{l} c_{ik} u_{ik}(x)\ ,\quad i = 1, \ldots, n\ ,$$

where u_{ik} are some basis functions and the coefficients c_{ik} are determined by the collocation condition

$$\vec{R}_l'(x_j) + A\vec{R}_l(x_j) = \vec{f}(x_j)\ ,$$

where x_j, $j = 1, \ldots, l$ are chosen points in the interval $[a, b]$.

We now illustrate the Levin collocation appoach. Divide the interval $[a, b]$ in m equal subintervals $[t_{j-1}, t_j]$, with $1 \leq j \leq m$, $t_j = a + hj$ with $h = (b-a)/m$ as the length of the interval. Then apply a l-th order collocation method on each subinterval using the basis functions $u_{ik} = x^{k-1}$, $k = 1, \ldots, l$. The resulting integrals will come from the sum of the approximated $R_{j,l}$ one obtains with such a procedure, where j labels the solution on the interval $[t_{j-1}, t_j]$. So we can write the approximated result for the integral as

$$J_{l,h} = \sum_{j=1}^{m} \vec{S}(t_j) \cdot \vec{R}_{j,l}(t_j) - \vec{S}(t_{j-1}) \cdot \vec{R}_{j-1,l}(t_{j-1})\ ,$$

where we made explicit that the approximation is done considering h as the length of the subinterval and l as the order of the collocation method.

The proof that this method yields indeed good results on determining integrals with fast oscillatory integrands can be found in [28]. The advantages of the Levin collocation scheme involve the efficiency in evaluating the integrals together with the small error associated.

Appendix D

Exact free cumulants

In this appendix we summarize how one could in principle take into account exactly also the density-density and density-momentum correlations, i.e. how to take fully into account the action of $\hat{\mathcal{C}}$ in (2.50). However, including all these new contributions yield very small corrections, that's why we neglected them in the main text of the thesis. Also, we focus on cumulants rather than correlators, so that we focus only on connected contributions to the various correlators. Here we will only see the main ideas, all the derivations of the results we show here can be found in [2].

D.1 Cumulant

In our formalism, a cumulant can be obtained from the generating functional by functional derivatives, in particular a generic n-th order cumulant is defined as

$$G^{(\mathrm{c})}_{\alpha_1(1)\cdots\alpha_n(n)} := \langle \Phi_{\alpha_1}(1)\cdots\Phi_{\alpha_n}(n)\rangle_c = \hat{H}_{\alpha_1}(1)\cdots\hat{H}_{\alpha_n}(n)\,\ln Z_{\mathrm{C}}[H]\Big|_{H=0}\,, \tag{D.1}$$

where α_j can stand for f or B (density or respond field), or other macroscopic field labels as we will see in the following. The superscript c here means connected. Notice that cumulants are obtained by the logarithm of the partition function, that is indeed the generator of connected correlators[1]. In particular the power spectrum will be

$$G^{(\mathrm{c})}_{\rho(1)\rho(2)} := G^{(\mathrm{c})}_{f(1)f(2)}\Big|_{\vec{l}_1=\vec{l}_2=0} = (2\pi)^3\delta_{\mathrm{D}}(\vec{k}_1+\vec{k}_2)P_\delta(k_1,t_1,t_2)\,. \tag{D.2}$$

One could legitimately ask why are we interested also in cumulants involving the response fields, ad the answer is that we need them if we want to compute perturbatively quantities like the full power spectrum. In fact what we can do is to compute cumulants with the interaction term set to zero (free cumulants), and then expand the interaction term in (2.37) to find the perturbative corrections in terms of free cumulants. As an example, we can try to derive the expression for the full power spectrum at first order in the interaction term σ.

Let's apply (D.1) to write

$$G^{(\mathrm{c})}_{f(1)f(2)} \simeq \hat{H}_f(1)\hat{H}_f(2)\,\ln\big((1+\mathrm{i}\hat{S}_1)Z_{\mathrm{C},0}[H]\big)\Big|_{H=0}\,,$$

[1] This is a standard result of QFT.

where we expanded a first order $\exp(\mathrm{i}\hat{S}_1)$; then notice that we can write the logarithm as

$$\ln\left(Z_{\mathrm{C},0}\left(1+\mathrm{i}\frac{\hat{S}_1 Z_{\mathrm{C},0}}{Z_{\mathrm{C},0}}\right)\right) = \ln Z_{\mathrm{C},0} + \ln\left(1+\mathrm{i}\frac{\hat{S}_1 Z_{\mathrm{C},0}}{Z_{\mathrm{C},0}}\right) \simeq \ln Z_{\mathrm{C},0} + \mathrm{i}\frac{\hat{S}_1 Z_{\mathrm{C},0}}{Z_{\mathrm{C},0}} \ ;$$

$\hat{S}_1$ contains both $\hat{H}_f$ and $\hat{H}_B$, it is clear then that, applying it to the non-connected generating functional $Z_{\mathrm{C},0}$, we will obtain quantities related to connected cumulants involving also response fields; writing schematically $\hat{S}_1 = 2\mathrm{i}\hat{H}_f \bullet \sigma_{fB} \bullet \hat{H}_B$ (notice that $\sigma_{fB} = \sigma_{Bf}$), exploiting

$$\begin{aligned}\frac{\hat{H}_f\hat{H}_B Z_{\mathrm{C},0}}{Z_{\mathrm{C},0}} &= \hat{H}_f\hat{H}_B \ln Z_{\mathrm{C},0} + \frac{1}{Z^2_{\mathrm{C},0}}\hat{H}_f Z_{\mathrm{C},0}\hat{H}_B Z_{\mathrm{C},0} \\ &= \hat{H}_f\hat{H}_B \ln Z_{\mathrm{C},0} + \hat{H}_f \ln Z_{\mathrm{C},0}\hat{H}_B \ln Z_{\mathrm{C},0} \ ,\end{aligned}$$

we arrive at

$$\begin{aligned}G^{(\mathrm{c})}_{f(1)f(2)} =& G^{(\mathrm{c},0)}_{f(1)f(2)} + 2\mathrm{i}\Big(G^{(\mathrm{c},0)}_{ff(2)} \bullet \sigma_{Bf} \bullet G^{(\mathrm{c},0)}_{Bf(1)} + G^{(\mathrm{c},0)}_{ff(1)} \bullet \sigma_{Bf} \bullet G^{(\mathrm{c},0)}_{Bf(2)}\Big) \\ &+ 2\mathrm{i}\Big(G^{(\mathrm{c},0)}_{fBf(1)f(2)} \bullet \sigma_{Bf}\Big) + \mathcal{O}(\sigma^2) \ .\end{aligned} \tag{D.3}$$

where $G^{(\mathrm{c},0)}$ are the free cumulants.

Since response fields will be always paired with an interaction term, it is useful to introduce the dressed response field

$$\Phi_{\mathcal{F}(r)} := \int_{r'} \sigma_{fB}(r,-r')\Phi_B(r') \ , \tag{D.4}$$

so that in the previous one can substitute all B cumulants with $\mathcal{F}$ ones.

D.2 Results

Here we resume all the important formulas for the exact computation of free cumulants. We recall that here we won't do much of the derivations, which can be found in [2].

A general free cumulant in KFT can be written as

$$\begin{aligned}G^{(\mathrm{c},0)}_{\alpha_1(1)\cdots\alpha_n(n)} &= \sum_{l=1}^{\infty} \hat{\Phi}^{(l)}_{\alpha_1}(1)\cdots\hat{\Phi}^{(l)}_{\alpha_n}(n)\ \frac{1}{l!}\hat{\Sigma}^{(l)}_C(1,\ldots,l)\,\mathrm{Tr}^{(l)}\Big|_{H,\mathbf{J},\mathbf{K}=0} \\ &=: \sum_{l=1}^{\infty} G^{(\mathrm{c},0,l)}_{\alpha_1(1)\cdots\alpha_n(n)} \ ,\end{aligned} \tag{D.5}$$

where l index indicates quantities referring to canonical ensemble with l particles[2], $\hat{\Phi}^{(l)}_{\alpha_j}(j)$ are the quantities corresponding to the canonical (2.33) with $N = l$, and

$$\mathrm{Tr}^{(l)} = \exp\left(\mathrm{i}\int_r H(-r)\cdot\hat{\Phi}^{(l)}(r)\right)\int \mathrm{d}\mathbf{x}^{(\mathrm{i})}\,\bar{\rho}^l P_{\sigma_1^2}(\mathbf{p}^{(\mathrm{i})})\mathrm{e}^{\mathrm{i}\int \mathrm{d}t\mathbf{J}\cdot\bar{\mathbf{x}}}\,,$$

with $\bar{\rho}$ the mean density, and with Kronecker products built for l particles, $\bar{\mathbf{x}}(t)$ is the l particle version of (2.17) and

$$P_{\sigma_1^2}(\mathbf{p}^{(\mathrm{i})}) = \frac{1}{(6\pi\sigma_1^2)^{3l/2}}\exp\left(-\frac{\mathbf{p}^{(\mathrm{i})}\cdot\mathbf{p}^{(\mathrm{i})}}{6\sigma_1^2}\right),$$

as the uncorrelated gaussian distribution defined by σ_1^2 whose expression in terms of the power spectrum is (2.58). Finally, $\hat{\Sigma}^{(l)}_{\mathrm{C}}(1,\dots,l)$ is the operator acting on $\mathrm{Tr}^{(l)}$ representing the sum of all connected l particle diagrams (with representative set of particles $\{1,\dots,l\}$) obtained with the following line types:

$$\begin{aligned}
&\overset{i}{\bullet}\!\!-\!\!-\!\!\bullet\overset{j}{} = C_{\delta_i\delta_j}\exp\Big(-\hat{\bar{\chi}}^{\top}_{p_i}(t^{(\mathrm{i})})C_{p_ip_j}\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})})\Big) =: \hat{C}_{\delta_i\delta_j}(\hat{\bar{\chi}}_{p_i}(t^{(\mathrm{i})}),\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))\,;\\
&\overset{i}{\bullet}\!\!-\!\!-\!\!\bullet\overset{j}{} = (-\mathrm{i}\vec{C}_{\delta_ip_j}\cdot\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))\exp\Big(-\hat{\bar{\chi}}^{\top}_{p_i}(t^{(\mathrm{i})})C_{p_ip_j}\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})})\Big)\\
&\qquad =: \hat{C}_{\delta_ip_j}(\hat{\bar{\chi}}_{p_i}(t^{(\mathrm{i})}),\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))\,;\\
&\overset{i}{\bullet}\!\!-\!\!-\!\!\bullet\overset{j}{} = \exp\Big(-\hat{\bar{\chi}}^{\top}_{p_i}(t^{(\mathrm{i})})C_{p_ip_j}\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})})\Big) - 1 =: \hat{C}_{p_ip_j}(\hat{\bar{\chi}}_{p_i}(t^{(\mathrm{i})}),\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))\,;\\
&\overset{i}{\bullet}\!\!-\!\!\circ\!\!-\!\!\bullet\overset{j}{} = (-\mathrm{i}\vec{C}_{\delta_ip_j}\cdot\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))(-\mathrm{i}\vec{C}_{\delta_jp_i}\cdot\hat{\bar{\chi}}_{p_i}(t^{(\mathrm{i})}))\\
&\qquad\times\exp\Big(-\hat{\bar{\chi}}^{\top}_{p_i}(t^{(\mathrm{i})})C_{p_ip_j}\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})})\Big) =: \hat{C}_{(\delta p)^2_{ij}}(\hat{\bar{\chi}}_{p_i}(t^{(\mathrm{i})}),\hat{\bar{\chi}}_{p_j}(t^{(\mathrm{i})}))\,;
\end{aligned}\tag{D.6}$$

where $\vec{C}_{\delta_ip_j}, C_{p_ip_j}, C_{\delta_i\delta_j}$ can be read in (2.49); the possible diagrams are subjected to the following three rules:

- **No self-correlation rule:** subdiagrams of the form $\circlearrowleft\bullet$ for any line are forbidden;
- **No two particle loop rule:** any pair of particles can be connected directly by at most one line;
- **δ-line rule:** no particle can have more than one solid δ line attached to it.

On the computation of cumulants, recall that, from (2.33), we can write

$$\hat{\Phi}^{(l)}_{\alpha_j} = \sum_{k=1}^{l}\hat{\Phi}^{(1)}_{\alpha_j,k}\,;$$

we say that, in an expression, a particle k carries a field label $\alpha_j(r)$ if an operator $\hat{\Phi}^{(1)}_{\alpha_j(r),k}$ is present, and also that a field label grouping $\{I_1,\dots,I_l\}$ is characterized by l sets, where a generic I_k set contains all the labels r (that is the momenta and times)

[2]On the computation of free cumulants it is better to use the gran canonical ensemble and exploit the Mayer cluster expansion, see [2] for details.

and α_j (that is the type of the field) corresponding to all the $\hat{\Phi}^{(1)}_{\alpha_j,k}(r)$ for the given particle labelled with k, acting on the expression we are considering.

We can apply the following restriction to truncate the sum in (D.5):

- **Homogeneity restriction:** if a particle k does not carry a field label in one of the terms given by the sum in

$$\hat{\Phi}^{(l)}_{\alpha_1}(1)\cdots\hat{\Phi}^{(l)}_{\alpha_n}(n) = \sum_{k=1}^{l}\hat{\Phi}^{(1)}_{\alpha_1,k}(1)\cdots\sum_{k=1}^{l}\hat{\Phi}^{(1)}_{\alpha_n,k}(n)\,,$$

  then the associated term in $G^{(0,l)}_{\alpha_1(1)\cdots\alpha_n(n)}$ vanishes;

- **Causality restriction:** non vanishing contributions are given by those field label groupings $\{I_1,\ldots,I_l\}$ that have the following property: for every I_k, there exists at least one label r corresponding to a Φ_f field such that $t_r \geq t'_r$ for all Φ_B labels $r' \in I_k$;

these restrictions together imply the truncation scheme

$$G^{(\mathrm{c},0)}_{f(1)\cdots f(n_f)B(1')\cdots B(n'_B)} = \sum_{l=1}^{n_f} G^{(c,0,l)}_{f(1)\cdots f(n_f)B(1')\cdots B(n'_B)}\,. \tag{D.7}$$

This last result is very important, since it tells us that we need to evaluate only a finite number of terms to compute *exactly* any free cumulant.

In the case of density cumulants, we can rewrite (D.5) as

$$\boxed{G^{(\mathrm{c},0,l)}_{f(1)\cdots f(n)} = \bar{\rho}^l(2\pi)^3\delta_{\mathrm{D}}\left(\sum_{r=1}^{n}\vec{L}_{q,r}(t^{(\mathrm{i})})\right)\sum_{\{I_1,\ldots,I_l\}}\mathrm{e}^{-Q(I_1,\ldots,I_l)}\tilde{\Sigma}^{(l)}_{\mathrm{C}}(I_1,\ldots,I_l)\,,} \tag{D.8}$$

where the sum is over all the possible label groupings, and

$$\mathrm{e}^{-Q(I_1,\ldots,I_l)} := \exp\left(-\frac{\sigma_1^2}{6}\sum_{j=1}^{l}\vec{L}^2_{p,I_j}(t^{(\mathrm{i})})\right)\,, \tag{D.9}$$

is the generalization of Q_{D} in (2.59), whereas the generalization of (2.56) is

$$\vec{L}_{q,I_j}(t) := \sum_{r\in I_j}\vec{L}_{q,r}(t) := -\sum_{r\in I_j}(\vec{k}_r g_{qq}(t_r,t) + \vec{l}_r g_{pq}(t_r,t))\,,$$
$$\vec{L}_{p,I_j}(t) := \sum_{r\in I_j}\vec{L}_{p,r}(t) := -\sum_{r\in I_j}(\vec{k}_r g_{qp}(t_r,t) + \vec{l}_r g_{pp}(t_r,t))\,;$$

instead

$$\tilde{\Sigma}^{(l)}_{\mathrm{C}}(I_1,\ldots,I_l) := \prod_{j=1}^{l-1}\int \mathrm{d}^3 q^{(\mathrm{i})}_{jl}\,\mathrm{e}^{\mathrm{i}\vec{L}_{q,I_j}(t^{(\mathrm{i})})\cdot\vec{q}^{(\mathrm{i})}_{jl}}\Sigma^{(l)}_{\mathrm{C}}(I_1,\ldots,I_l)\,, \tag{D.10}$$

where

$$\vec{q}^{(\mathrm{i})}_{il} := \vec{q}^{(\mathrm{i})}_{i} - \vec{q}^{(\mathrm{i})}_{l}\,,$$

are the relative coordinates to the coordinate $\vec{q}^{(\mathrm{i})}_{l}$, $\Sigma^{(l)}_{C}(I_1,\ldots,I_l)$ is $\hat{\Sigma}^{(l)}_{C}$ with the substitutions $\hat{\vec{\chi}}_{p_j} \to \vec{L}_{p,I_j}$ (so it will be not an operator anymore); we can rewrite it as an

integral over all possible relative coordinates[3] $\vec{q}_{ij} := \vec{q}^{\,(\mathrm{i})}_{il} - \vec{q}^{\,(\mathrm{i})}_{jl}$, with $i < j$, using what we call the generalized momenta (2.68) and the new redefinitions of the lines in (D.6):

$$
\begin{aligned}
\overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\bullet} &= \mathcal{P}_{\delta_i\delta_j}(\vec{k}^{\,(\mathrm{i})}_{ij}) \qquad := \int \mathrm{d}^3 q^{(\mathrm{i})}_{ij}\, \mathrm{e}^{\mathrm{i}\vec{k}^{\,(\mathrm{i})}_{ij}\cdot\vec{q}^{\,(\mathrm{i})}_{ij}}\, \hat{C}_{\delta_i\delta_j}(\vec{L}_{p,I_i}, \vec{L}_{p,I_j})\,; \\
\overset{i}{\bullet}\!\!-\!\!\cdot\!\!\cdot\!\!\overset{j}{\bullet} &= \mathcal{P}_{\delta_i p_j}(\vec{k}^{\,(\mathrm{i})}_{ij}) \qquad := \int \mathrm{d}^3 q^{(\mathrm{i})}_{ij}\, \mathrm{e}^{\mathrm{i}\vec{k}^{\,(\mathrm{i})}_{ij}\cdot\vec{q}^{\,(\mathrm{i})}_{ij}}\, \hat{C}_{\delta_i p_j}(\vec{L}_{p,I_i}, \vec{L}_{p,I_j})\,; \\
\overset{i}{\bullet}\!\!-\!\!\cdot\!\!-\!\!\cdot\!\!\overset{j}{\bullet} &= \mathcal{P}_{p_i p_j}(\vec{k}^{\,(\mathrm{i})}_{ij}) \qquad := \int \mathrm{d}^3 q^{(\mathrm{i})}_{ij}\, \mathrm{e}^{\mathrm{i}\vec{k}^{\,(\mathrm{i})}_{ij}\cdot\vec{q}^{\,(\mathrm{i})}_{ij}}\, \hat{C}_{p_i p_j}(\vec{L}_{p,I_i}, \vec{L}_{p,I_j})\,; \\
\overset{i}{\bullet}\!\!-\!\!\circ\!\!-\!\!\overset{j}{\bullet} &= \mathcal{P}_{(\delta p)^2_{ij}}(\vec{k}^{\,(\mathrm{i})}_{ij}) \quad := \int \mathrm{d}^3 q^{(\mathrm{i})}_{ij}\, \mathrm{e}^{\mathrm{i}\vec{k}^{\,(\mathrm{i})}_{ij}\cdot\vec{q}^{\,(\mathrm{i})}_{ij}}\, \hat{C}_{(\delta_p)^2_{ij}}(\vec{L}_{p,I_i}, \vec{L}_{p,I_j})\,.
\end{aligned}
\tag{D.11}
$$

Notice that $\mathcal{P}_{p_i p_j} = \mathcal{P}_{ij}$ defined in (2.67).

The rules to properly compute (D.10) are then the following:

1. Choose an arrangement of l dots representing particles carrying the labels I_j, $j = 1, \ldots, l$;
2. Draw all possible connected diagrams, subjected to the three rules stated above, with the lines (D.11);
3. For any closed loop, assign to one of the lines forming the loop a momentum $\vec{k}^{\,(\mathrm{i})}_{ij}$ and integrate over it
4. We need to assign to every line a generalized momenta $\vec{k}^{\,(\mathrm{i})}_{ij}$, so pick a vertex j that has only one line with undetermined momentum and use

$$
\vec{L}_{q,I_j} - \sum_{i=1}^{j-1} \vec{k}^{\,(\mathrm{i})}_{ij} + \sum_{i=j+1}^{l} \vec{k}^{\,(\mathrm{i})}_{ji} = 0\,, \tag{D.12}
$$

   to fix its momentum. Incoming momenta from vertices with smaller labels are counted positive while outgoing momenta to vertices with larger labels are counted negative. Momenta associated to vertices not connected to j are counted as zero.
5. Repeat the same steps for all vertices to fix all momenta of the lines.

Power spectrum case. Let's apply these rules to the computation of $G^{(\mathrm{c},0,2)}_{f(1)\cdots f(n)}$. It is a 2-particle quantity, so we must consider all possible diagrams with 2 vertices, which are

$$
\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \;+\; \overset{1}{\bullet}\!\!-\!\!\cdot\!\!-\!\!\overset{2}{\bullet} \;+\; \overset{1}{\bullet}\!\!-\!\!\cdot\!\!-\!\!\overset{2}{\bullet} \;+\; \overset{1}{\bullet}\!\!-\!\!\cdot\!\!-\!\!\overset{2}{\bullet} \;+\; \overset{1}{\bullet}\!\!-\!\!\circ\!\!-\!\!\overset{2}{\bullet}\,.
$$

We have to fix the generalized momenta; picking vertex $j = 1$ we have that, for every diagram, by applying the (D.12),

$$
\vec{k}^{\,(\mathrm{i})}_{12} = \vec{L}_{q,I_1}\,.
$$

[3] All these steps are a generalization to what we did in 2.4.1, see [2].

So

$$\tilde{\Sigma}^{(2)}_{\mathrm{C}}(I_1, I_2) = \mathcal{P}_{\delta_1\delta_2}(\vec{L}_{q,I_1}) + \mathcal{P}_{p_1\delta_2}(\vec{L}_{q,I_1}) + \mathcal{P}_{\delta_1 p_2}(\vec{L}_{q,I_1}) + \mathcal{P}_{p_1p_2}(\vec{L}_{q,I_1}) + \mathcal{P}_{(\delta p)^2_{12}}(\vec{L}_{q,I_1}) \,,$$

and using (D.8) we have

$$\begin{aligned} G^{(\mathrm{c},0,2)}_{f(1)\cdots f(n)} &= \bar{\rho}^2(2\pi)^3\delta_{\mathrm{D}}\left(\sum_{r=1}^{n}\vec{L}_{q,r}(t^{(\mathrm{i})})\right) \\ &\quad\times \sum_{\{I_1,I_2\}}\exp\left(-\frac{\sigma_1^2}{6}\left(\vec{L}^2_{p,I_1}(t^{(\mathrm{i})}) + \vec{L}^2_{p,I_2}(t^{(\mathrm{i})})\right)\right)\tilde{\Sigma}^{(2)}_{\mathrm{C}}(I_1, I_2) \,, \end{aligned} \tag{D.13}$$

where the sum is over all the possible ways to put n particle labels into the two non-empty sets I_1, I_2. The case of the power spectrum can be obtained from the previous with $n = 2$ and replacing f-cumulant with ρ-cumulant (that is evaluate it at $\vec{l} = 0$); by neglecting $C_{\delta p}, C_{\delta\delta}$ factors, $\tilde{\Sigma}^{(2)}_{\mathrm{C}}$ reduces to $\tilde{\Sigma}^{(2)}_{\mathrm{C}}(I_1, I_2) = \mathcal{P}_{p_1p_2}(\vec{L}_{q,I_1})$; since there is only one way to assign 2 labels to the two non-empty sets $\{I_1, I_2\}$, we end up with

$$\begin{aligned} G^{(\mathrm{c},0,2)}_{\rho(1)\rho(2)} &= \bar{\rho}^2(2\pi)^3\delta_{\mathrm{D}}\left(\vec{k}_1 + \vec{k}_2\right)\exp\left(-\frac{\sigma_1^2}{3}g^2_{qp}(t, t^{(\mathrm{i})})k_1^2\right)\mathcal{P}_{p_1p_2}(k_1) \\ &= \bar{\rho}^2(2\pi)^3\delta_{\mathrm{D}}\left(\vec{k}_1 + \vec{k}_2\right)\mathrm{e}^{\mathrm{Q}_{\mathrm{D}}/2}\mathcal{P}_{21} \,. \end{aligned}$$

We thus obtain, as expected, the same result we obtained in section 2.4.3.

Including response fields. The generic cumulant including also response field is not difficult to derive from (D.8), in fact applying $n = n_f + n'_B$ field operators, with obvious notation, looking at (2.34) one can notice that we can reconduce to the n density operator case, since $\hat{\Phi}^{(1)}_{f_j}$ and $\hat{b}_j$ operators commute so that we can push to the right all density operators, including the ones arising from $\hat{\Phi}^{(l)}_B$. We have then to be careful to associate to all the terms of the resulting sums the corresponding $\hat{b}_j$ factors. To understand this, consider an explicit example with $l = 2$, $n_f = 2$ (the minimum amount we can consider due to restriction relations) and $n'_B = 3$, so we have to consider

$$\sum_{j=1}^{2}\hat{b}_j(1')\hat{\Phi}^{(1)}_{f_j}(1')\sum_{j=1}^{2}\hat{b}_j(2')\hat{\Phi}^{(1)}_{f_j}(2')\sum_{j=1}^{2}\hat{b}_j(3')\hat{\Phi}^{(1)}_{f_j}(3')\sum_{j=1}^{2}\hat{\Phi}^{(1)}_{f_j}(1)\sum_{j=1}^{2}\hat{\Phi}^{(1)}_{f_j}(2) \,,$$

so every term will have a number n'_B of $\hat{b}_j(r')$ operators and n density operators. Each term of the sum can be characterized by a label set I_1, I_2, for example the term corresponding to the choice $I_1 = \{1', 3', 2\}$ and $I_2 = \{2', 1\}$ (recall that, due to causality condition, for this term to be different from zero it should be $t_1 > t_{2'}$ and $t_2 > t_{1'}, t_{3'}$), is

$$\hat{b}_1(1')\hat{b}_1(3')\hat{b}_2(2')\hat{\Phi}^{(1)}_{f_1}(1')\hat{\Phi}^{(1)}_{f_1}(3')\hat{\Phi}^{(1)}_{f_1}(2)\hat{\Phi}^{(1)}_{f_2}(2')\hat{\Phi}^{(1)}_{f_2}(1) \,,$$

so we can still reconduce to a sum over the possible choices of $\{I_1, \ldots, I_l\}$ but we have to keep track of the $\hat{b}_j(r')$ factors, and we will do this introducing the notation

$\hat{b}_{I(r)} := \hat{b}_j(r)$, that is, given a label grouping $\{I_1, \ldots, I_l\}$, $I(r)$ associates the particle index j corresponding to the label r in the choice of the label grouping. In the previous example, $I(1')$ associates the index 1 since the $1'$ label is in I_1.

In the end, since the action of $\hat{b}_j(r')$ corresponds to insert the term

$$b_j(r) = \mathrm{i}\vec{L}_{p,I_j}(t_r) \cdot \vec{k}_r \,,$$

we can write

$$\boxed{\begin{aligned} G^{(\mathrm{c},0,l)}_{f(1)\cdots f(n_f)B(1')\cdots B(n'_B)} &= \bar{\rho}^l (2\pi)^3 \delta_{\mathrm{D}} \left(\sum_{r=1}^{n} \vec{L}_{q,r}(t^{(\mathrm{i})}) \right) \\ &\times \sum_{\{I_1,\ldots,I_l\}} \mathrm{e}^{-Q(I_1,\ldots,I_l)} b_{I(1')} \cdots b_{I(n'_B)} \tilde{\Sigma}_{\mathrm{C}}(I_1, \ldots, I_l) \,, \end{aligned}} \tag{D.14}$$

that is the general formula for computing exactly any free cumulant.

The generalization to $\Phi_{\mathcal{F}}$ cumulants is straightforward: looking at (D.4), the only change is

$$G^{(\mathrm{c},0,l)}_{f(1)\cdots f(n_f)\mathcal{F}(1')\cdots\mathcal{F}(n'_B)} = \left(\prod_{r=1}^{n_B} \int_{r''} \sigma_{fB}(r', -r'') \right) G^{(0,l)}_{f(1)\cdots f(n_f)B(1')\cdots B(n'_B)} \,.$$

Acknowledgements

Bibliography

[1] Matthias Bartelmann et al. "A microscopic, non-equilibrium, statistical field theory for cosmic structure formation". In: *New Journal of Physics* 18.4 (2016), p. 043020. DOI: 10.1088/1367-2630/18/4/043020. URL: https://doi.org/10.1088%2F1367-2630%2F18%2F4%2F043020.

[2] Felix Fabis et al. "Kinetic field theory: exact free evolution of Gaussian phase-space correlations". In: *Journal of Statistical Mechanics: Theory and Experiment* 2018.4 (2018), p. 043214. DOI: 10.1088/1742-5468/aab850. URL: https://doi.org/10.1088%2F1742-5468%2Faab850.

[3] Johannes Dombrowski, Felix Fabis, and Matthias Bartelmann. "Kinetic field theory: cosmic structure formation and fluctuation-dissipation relations". In: *Journal of Physics Communications* 2.2 (2018), p. 025020. DOI: 10.1088/2399-6528/aaaa8f. URL: https://doi.org/10.1088%2F2399-6528%2Faaaa8f.

[4] Matthias Bartelmann et al. "Kinetic field theory: effects of momentum correlations on the cosmic density-fluctuation power spectrum". In: *New Journal of Physics* 19.8 (2017), p. 083001. DOI: 10.1088/1367-2630/aa7e6f. URL: https://doi.org/10.1088%2F1367-2630%2Faa7e6f.

[5] Francis Bernardeau et al. "Large-scale structure of the Universe and cosmological perturbation theory". In: *Physics Reports* 367.1 (2002), pp. 1 –248. ISSN: 0370-1573. DOI: https://doi.org/10.1016/S0370-1573(02)00135-7. URL: http://www.sciencedirect.com/science/article/pii/S0370157302001357.

[6] Matthias Bartelmann et al. "Analytic calculation of the non-linear cosmic density-fluctuation power spectrum in the Born approximation". In: (2017). arXiv: 1710.07522 [astro-ph.CO].

[7] Martín Crocce and Román Scoccimarro. "Renormalized cosmological perturbation theory". In: *Phys. Rev. D* 73 (6 2006), p. 063519. DOI: 10.1103/PhysRevD.73.063519. URL: https://link.aps.org/doi/10.1103/PhysRevD.73.063519.

[8] Sabino Matarrese and Massimo Pietroni. "Resumming cosmic perturbations". In: *Journal of Cosmology and Astroparticle Physics* 2007.06 (2007), pp. 026–026. DOI: 10.1088/1475-7516/2007/06/026. URL: https://doi.org/10.1088%2F1475-7516%2F2007%2F06%2F026.

[9] Massimo Pietroni. "Flowing with time: a new approach to non-linear cosmological perturbations". In: *Journal of Cosmology and Astroparticle Physics* 2008.10 (2008), p. 036. DOI: 10.1088/1475-7516/2008/10/036. URL: https://doi.org/10.1088%2F1475-7516%2F2008%2F10%2F036.

[10] Mark P. Hertzberg. "Effective field theory of dark matter and structure formation: Semianalytical results". In: *Phys. Rev. D* 89 (4 2014), p. 043521. DOI: 10.1103/PhysRevD.89.043521. URL: https://link.aps.org/doi/10.1103/PhysRevD.89.043521.

[11] Takahiko Matsubara. "Resumming cosmological perturbations via the Lagrangian picture: One-loop results in real space and in redshift space". In: *Phys. Rev. D* 77 (6 2008), p. 063530. DOI: 10.1103/PhysRevD.77.063530. URL: https://link.aps.org/doi/10.1103/PhysRevD.77.063530.

[12] Matthias Bartelmann et al. "Cosmic Structure Formation with Kinetic Field Theory". In: (2019). arXiv: 1905.01179 [astro-ph.CO].

[13] Daniel Geiss et al. "Resummed Kinetic Field Theory: Using Mesoscopic Particle Hydrodynamics to Describe Baryonic Matter in a Cosmological Framework". In: (2018). arXiv: 1811.07741 [astro-ph.CO].

[14] Matthias Bartelmann. "Trajectories of point particles in cosmology and the Zel'dovich approximation". In: *Phys. Rev. D* 91 (8 2015), p. 083524. DOI: 10.1103/PhysRevD.91.083524. URL: https://link.aps.org/doi/10.1103/PhysRevD.91.083524.

[15] Robert Lilow et al. "Resummed Kinetic Field Theory: general formalism and linear structure growth from Newtonian particle dynamics". In: *Journal of Cosmology and Astroparticle Physics* 2019.04 (2019), pp. 001–001. DOI: 10.1088/1475-7516/2019/04/001. URL: https://doi.org/10.1088%2F1475-7516%2F2019%2F04%2F001.

[16] N. Aghanim et al. "Planck 2018 results. VI. Cosmological parameters". In: (2018). arXiv: 1807.06209 [astro-ph.CO].

[17] Karim A. Malik and David Wands. "Cosmological perturbations". In: *Physics Reports* 475.1 (2009), pp. 1 –51. ISSN: 0370-1573. DOI: https://doi.org/10.1016/j.physrep.2009.03.001. URL: http://www.sciencedirect.com/science/article/pii/S0370157309000635.

[18] P.P. Coles and F. Lucchin. *Cosmology: The Origin and Evolution of Cosmic Structure*. Wiley, 2002. ISBN: 9780471489092. URL: https://books.google.it/books?id=uUFVb-DHtCwC.

[19] Ofer Lahav et al. "Dynamical effects of the cosmological constant". In: *Monthly Notices of the Royal Astronomical Society* 251.1 (July 1991), pp. 128–136. ISSN: 0035-8711. DOI: 10.1093/mnras/251.1.128. eprint: http://oup.prod.sis.lan/mnras/article-pdf/251/1/128/18523334/mnras251-0128.pdf. URL: https://doi.org/10.1093/mnras/251.1.128.

[20] Sean M. Carroll, William H. Press, and Edwin L. Turner. "The Cosmological Constant". In: *Annual Review of Astronomy and Astrophysics* 30.1 (1992), pp. 499–542. DOI: 10.1146/annurev.aa.30.090192.002435. eprint: https://doi.org/10.1146/annurev.aa.30.090192.002435. URL: https://doi.org/10.1146/annurev.aa.30.090192.002435.

[21] Sean M. Carroll. "The Cosmological Constant". In: *Living Reviews in Relativity* 4.1 (2001), p. 1. ISSN: 1433-8351. DOI: 10.12942/lrr-2001-1. URL: https://doi.org/10.12942/lrr-2001-1.

[22] Yasushi Suto. "Exploring Nonlinear Gravitational Clustering Phenomena with Cosmological N-Body Simulations". In: *Progress of Theoretical Physics* 90.6 (Dec. 1993), pp. 1173–1228. ISSN: 0033-068X. DOI: 10.1143/ptp/90.6.1173. eprint: http://oup.prod.sis.lan/ptp/article-pdf/90/6/1173/5387770/90-6-1173.pdf. URL: https://doi.org/10.1143/ptp/90.6.1173.

[23] Riccardo Penco and Danilo Mauro. "Perturbation theory via Feynman diagrams in classical mechanics". In: *European Journal of Physics* 27.5 (2006), pp. 1241–1249. DOI: 10.1088/0143-0807/27/5/023. URL: https://doi.org/10.1088%2F0143-0807%2F27%2F5%2F023.

[24] Planck Collaboration et al. "Planck 2015 results - XVII. Constraints on primordial non-Gaussianity". In: *A&A* 594 (2016), A17. DOI: 10.1051/0004-6361/201525836. URL: https://doi.org/10.1051/0004-6361/201525836.

[25] Martín Crocce and Román Scoccimarro. "Memory of initial conditions in gravitational clustering". In: *Phys. Rev. D* 73 (6 2006), p. 063520. DOI: 10.1103/

PhysRevD.73.063520. URL: https://link.aps.org/doi/10.1103/PhysRevD.73.063520.

[26] Y.B. Zel'Dovich. "Gravitational instability: an approximate theory for large density perturbations." In: *aap* 500 (1970), pp. 13–18. URL: https://ui.adsabs.harvard.edu/abs/1970A&A.....5...84Z.

[27] David Levin. "Fast integration of rapidly oscillatory functions". In: *Journal of Computational and Applied Mathematics* 67.1 (1996), pp. 95 –101. ISSN: 0377-0427. DOI: https://doi.org/10.1016/0377-0427(94)00118-9. URL: http://www.sciencedirect.com/science/article/pii/0377042794001189.

[28] David Levin. "Analysis of a collocation method for integrating rapidly oscillatory functions". In: *Journal of Computational and Applied Mathematics* 78.1 (1997), pp. 131 –138. ISSN: 0377-0427. DOI: https://doi.org/10.1016/S0377-0427(96)00137-9. URL: http://www.sciencedirect.com/science/article/pii/S0377042796001379.

[29] M. Galassi et al. *GNU Scientific Library Reference Manual*. 2018. URL: https://www.gnu.org/software/gsl/.

[30] Sara Konrad and Matthias Bartelmann. "Power spectra for particles on Zel'dovich trajectories with initial momentum correlations". In: (2019).

[31] David J. E. Marsh. "Axion Cosmology". In: *Phys. Rept.* 643 (2016), pp. 1–79. DOI: 10.1016/j.physrep.2016.06.005. arXiv: 1510.07633 [astro-ph.CO].

[32] Carsten Littek. "Kinetic Field Theory: Momentum-Density Correlations and Fuzzy Dark Matter". PhD thesis. University of Heidelberg, 2018. DOI: 10.11588/heidok.00025108. URL: http://www.ub.uni-heidelberg.de/archiv/25108.

[33] David J. Baumgart and J. N. Fry. "Fourier Spectra of Three-dimensional Data". In: *apj* 375 (1991), p. 25. DOI: 10.1086/170166.

[34] Gero Jürgens and Matthias Bartelmann. "Perturbation theory trispectrum in the time renormalization approach". In: *Monthly Notices of the Royal Astronomical Society* 428.4 (Nov. 2012), pp. 3173–3182. ISSN: 0035-8711. DOI: 10.1093/mnras/sts260. eprint: http://oup.prod.sis.lan/mnras/article-pdf/428/4/3173/18461561/sts260.pdf. URL: https://doi.org/10.1093/mnras/sts260.