PCV

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Abstract

We show what we have done to reproduce the results from [1]. We used pseudo-spectral methods to solve an extension to 2D Navier-Stokes equations that uses a piecewise constant viscosity (PCV) to model the forcing due to active agents. This PCV model allows one to regulate more precisely the strenght and range of forced wavenumbers. This model shows a phase transition to the formation of vortices at the largest lenght scale. Also we tried a burst analysis and characterize the range of the parameters for which this can happen in our model

FIG. I.1: Remarkable demonstration of polar order in a sardine school

FIG. I.2: Bacterial "turbulence" in a sessile drop of Bacillus sub-tilis viewed from below through the bottom of a petri dish. Gravity is perpendicular to the plane of the picture, and the horizontal white line near the top is the air-water-plastic contact line. The central fuzziness is due to collective motion. The scale bar is 35 μm.

I. ACTIVE FLUIDS

Active matter: unifying characteristic is that they are composed of self-driven units, active particles, each capable of converting stored or ambient free energy into systematic movement. The interaction of active particles with each other, and with the medium they live in, gives rise to highly correlated collective motion and mechanical stress. A distinctive, indeed, defining feature of active systems compared to more familiar nonequilibrium systems is the fact that the energy input that drives the system out of equilibrium is local, for example, at the level of each particle, rather than at the system's boundaries as in a shear flow.

II. SOLVING NUMERICALLY NAVIER STOKES

We used ω- ψ formalism, so from the incompressible Navier Stokes,

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\boldsymbol{\nabla} P}{\rho} + \nu \nabla^2 \mathbf{u} \ , \tag{II.1}$$

apply the curl on both sides to reduce to an equation for the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$ (recalling that for incompressible flows $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{\nabla} u_i) \times \partial_i \mathbf{u} + u_i \partial_i \boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} \ . \tag{II.2}$$

In 2 dimensions, one has that the curl is defined as

$$\boldsymbol{\nabla} \times (u_x \hat{\mathbf{x}} + u_y \hat{\mathbf{y}}) \equiv (\partial_x u_y - \partial_y u_x) \hat{\mathbf{z}} \ . \tag{II.3}$$

We can consider the vorticity as a scalar function (I need only one scalar component to fully describe it) and also we have

$$(\boldsymbol{\nabla} u_i) \times \partial_i \mathbf{u} = (\boldsymbol{\nabla} \cdot \mathbf{u})(\boldsymbol{\nabla} \times \mathbf{u}) = 0 \tag{II.4}$$

where on the last step we used incompressibility. So the final equation is

$$\partial_t \omega + \partial_y \psi \partial_x \omega - \partial_x \psi \partial_y \omega = \nu \nabla^2 \omega \ , \tag{II.5}$$

where we have used the so called stream function

$$\frac{\partial \psi}{\partial y} = u_x \ , \quad \frac{\partial \psi}{\partial x} = -u_y \implies \omega = -\nabla^2 \psi \ . \tag{II.6}$$

In Fourier space this becomes

$$\partial_t \hat{\omega} = -i(k_x \widehat{u_x \omega} + k_y \widehat{u_y \omega}) - \nu k^2 \hat{\omega} \ , \quad \hat{\psi} = \hat{\omega}/k^2 \ . \tag{II.7}$$

To treat numerically the non linearity, we can use pseudo spectral methods. The idea is to solve the spatial part of the equation in Fourier space trying to avoid to directly Fourier transform the non-linear term (which would involve a convolution, a heavy process from the computational point of view) but rather perform the multiplication in real space. Then the time evolution is performed numerically. In formulas, calling $N \equiv -i(k_x \widehat{u_x \omega} + k_y \widehat{u_y \omega})$ the non linearity,

$$\hat{N} = -ik_x \mathcal{F}(\mathcal{F}^{-1}(\hat{u}_x) \cdot \mathcal{F}^{-1}(\hat{\omega})) - ik_y \mathcal{F}(\mathcal{F}^{-1}(\hat{u}_y) \cdot \mathcal{F}^{-1}(\hat{\omega})) \ . \tag{II.8}$$

Going back and forth from real space to the Fourier one is way better than doing the convolution since fast Fourier transform algorithms (FFT) can perform the Fourier transform in $\mathcal{O}(N \log N)$ steps rather than $\mathcal{O}(N^2)$.

Since the problem in Navier-Stokes is only the non linearity, we can implement the integrating factor method to exactly solve the linear part and applying numerical methods for the non linear part, so write

$$\partial_t (e^{\nu k^2 t} \hat{\omega}) = e^{\nu k^2 t} \hat{N} \ , \tag{II.9}$$

such that the exact linear solution is encoded on the exponential factor. Then apply explicit second order Runge-Kutta to evolve it in time,

$$\begin{aligned} \hat{\omega}^* =& e^{-\nu k^2 \delta t/2} \Big(\hat{\omega}(t) + 0.5 \delta t \hat{N}(\hat{\omega}(t)) \Big) \ , \\ \hat{\omega}(t+\delta t) =& e^{-\nu k^2 \delta t/2} \Big(e^{-\nu k^2 \delta t/2} \hat{\omega}(t) + \delta t \hat{N}(\hat{\omega^*}) \Big) \ . \end{aligned} \tag{II.10}$$

III. PCV MODEL

A typical extension to Navier-Stokes equations is

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \boldsymbol{\nabla} P = \boldsymbol{\nabla} \cdot \sigma \ , \tag{III.1}$$

with

$$\sigma_{ij} = (\Gamma_0 - \Gamma_2 \nabla^2 + \Gamma_4 (\nabla^2)^2)(\partial_i u_j + \partial_j u_i) \ . \tag{III.2}$$

To model more precisely the forcing of wave numbers due to bacteria, introduce a piecewise constant viscosity (PCV):

$$\hat{\nu}(k) = \begin{cases} \nu_0 > 0 \quad \text{for} \quad k < k_{\min} \ , \\ \nu_1 < 0 \quad \text{for} \quad k_{\min} \leq k \leq k_{\max} \ , \\ \nu_2 > 0 \quad \text{for} \quad k > k_{\max} \ ; \end{cases}$$

so that the equation to solve is

$$\partial_t \hat{\omega} - \hat{N} = -\hat{\nu}(k) k^2 \hat{\omega} \ .$$

To numerically solve it, we used the same method we described for Navier-Stokes equations.

A. Technical details

a. Stability. Since we used an explicit method, we had to use an adaptive time step using Courant-Friedrichs-Lewy (CFL) time condition, that is

$$\delta t = \frac{\Delta x}{u_{\max}} = \frac{L}{N u_{\max}} \ , \tag{III.3}$$

where L is the length of the grid, N the grid spacing and $u_{\max}$ the modulus of the maximum velocity.

b. Aliasing. As noted in [2], in order to prevent aliasing we put the modes with $k \geq 2k_{max}/3$ to zero always before a non-linear evaluation.

B. Scaling relations

We saw that scaling $\nu(k)$ does not change the pattern followed by the solution $\mathbf{u}$; in fact, non dimensionalizing our Navier Stokes equation extension using

$$u = Uu' \ , \quad x = Lx' \implies t = \frac{L}{U} t' \ , \quad k = \frac{1}{L} k' \ , \quad \omega = \frac{U}{L} \omega' \ , \tag{III.4}$$

we have

$$\partial_{t'} \hat{\omega'} - \hat{N'} = -\frac{\hat{\nu}(k)}{UL} k'^2 \hat{\omega'} \ , \tag{III.5}$$

N	ν_0	ν_1/ν_0	ν_2/ν_0	k_{min}	k_{max}
256	10^{-4}	-0.25-(-7.0)	10	33	40

TABLE III.1: Parameters used in our analysis.

where on the RHS we have the Reynolds number $R_e = UL/\hat{\nu}(k)$. Since the space grid is never changed in our simulations, we expect that the only variable upon scaling of ν is the typical velocity; increasing the viscosity means that the velocity is greater, and if indeed we have that

$$\nu \to \lambda\nu \implies u \to \lambda u \ , \tag{III.6}$$

we can conclude that the Reynolds number remains the same, meaning indeed that the pattern of the fluids with according scaled variables is the same if (III.6) holds.

In the end we thus fixed a scale for ν by fixing ν_0 and analyzed the range of parameters resumed in table III.1 (the only free parameter is then ν_1).

IV. BURST ANALYSIS

Pattern formation arising when forcing only one mode: patterns initially form but then, due to non linearities, the pattern changes. [3]

Appendix A: The code

Here we show an example of the python code of the main program we used for this work

[1] M. Linkmann, G. Boffetta, M. Marchetti, and B. Eckhardt, (2018).

[2] S. A. Orszag, Journal of the Atmospheric Science (1971).

[3] O. Mickelin and J. S. et al, Physical Review Letters (2018).