UNIVERSITY OF PADOVA

MASTER DEGREE IN PHYSICS

PATH INTEGRAL IN COSMOLOGY:THE RENORMALIZATION GROUP APPROACH

A small thesis for Cosmology exam

Author:
Luca Teodori
Student 1179540

Lecturer:
Prof. Matarrese Sabino

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1 Introduction

The path integral approach and renormalization group equations are mostly known in the quantum field theory setting. Here we will try to show how all the apparatus of perturbation theory and even Feynman diagrams can be used to solve problems of cosmological relevance. In particular, we will show how one can reorganize cosmological perturbation theory in order to compute quantities like the propagator (that is the two point correlator between density or velocity field fluctuations at different times) and the power spectrum beyond linear theory with a non perturbative approach, by exploiting renormalization group techniques. This if of great relevance since for example we can use it to probe the BAO (Barionic Acoustic Oscillations) region, which involves density fluctuations that are highly non-linear, as showed in [5].

In section 2 we will briefly review the derivation of the main equations we want to solve (continuity and Euler equations) obtained by the Vlasov equation in the single stream approximation; in section 3 we will reorganize these two formulas in a way such that standard instruments of perturbation theory and (classical) path integral (see [6]) can be applied to our case ([1] and [5]); in section 4 we indeed develop the path integral approach (where we will also implement gaussian distributed initial conditions), following mainly [5]; in section 5 we will appreciate how the new formalism we have built can be used to cast classical perturbation theory in terms of Feynman diagrams (we will follow [1]); in section 6, 7 we will instead see how to apply renormalization group techniques to our problem at hand and in section 8 we will resume the main results one can achieve with this approach [5]; finally in section 9 we will briefly see how to extend our results in the case of different universes with respect to De Sitter ones and how to implement initial conditions different from the gaussian ones ([4], [7]).

Regarding the notation we will use, since the formulas involved are quite lengthy, we have introduced numerous short cuts, the most prominent one being the integration over various momenta often being understood. This of course could lead to confusions, but we tried to clarify the various formulas when confusion may arise by writing explicitly the various dependencies of the quantities of interest (something that often we will omit when they can safely be understood from the context). All this in order not to get lost in a notation nightmare and focus on the important concepts one wants to convey.

2 Vlasov equation

Here we want to find the Vlasov equation corresponding to a system of self-gravitating collisionless particles in a Robertson-Walker background (in non linear Newtonian approximation).

To derive it, we start with the lagrangian for a single particle of mass m

$$\mathcal{L} = \frac{1}{2} m\dot{r}^2 - m\varphi(\mathbf{x}, t) \ ,$$

where $\mathbf{x} = \mathbf{r}/a$ is the comoving radius (with $a \equiv a(t)$ as the Robertson-Walker scale factor), $\dot{} \equiv d/dt$ and $\varphi(\mathbf{x}, t)$ is the Newtonian gravitational potential satisfying

$$\nabla^2_{\mathbf{x}}\varphi(\mathbf{x}, t) = 4\pi G a^2 \rho(\mathbf{x}, t) \ , \tag{2.1}$$

where $\rho(\mathbf{x}, t) = \rho_b(t)(1 + \delta(\mathbf{x}, t))$ is the energy density, which accounts for the background density $\rho_b(t)$ and the fluctuations $\delta(\mathbf{x}, t)$. Since if we change the lagrangian adding a total time derivative we won't change the equations of motion, we pass to

$$\mathcal{L}' = \mathcal{L} - \frac{\mathrm{d}\psi}{\mathrm{d}t} \ , \ \psi = \frac{1}{2} m a \dot{a} x^2 \ ,$$

and the new lagrangian becomes (we drop the prime)

$$\mathcal{L} = \frac{1}{2} m a^2 \dot{x}^2 - \frac{m}{2} a\ddot{a} x^2 - m\varphi = \frac{1}{2} m a^2 \dot{x}^2 - m\phi \ ,$$

where in the last step, with $\phi \equiv \varphi - \varphi_b$, we exploited the second Friedmann equation (without the pressure term) and the expression for the background gravitational potential (solution of (2.1) with $\rho(\mathbf{x}, t) = \rho_b(t)$), respectively

$$\ddot{a} = -\frac{4\pi G}{3} a\rho_b(t) \ ; \ \varphi_b = \frac{2}{3}\pi G \rho_b(t) a^2 x^2 \ .$$

We are interested in the collisionless Boltzmann equation, so we need the hamiltonian

$$\mathcal{H}_a = \mathbf{p} \cdot \mathbf{x} - \mathcal{L} = \frac{p^2}{2ma^2} + m\phi \ ,$$

where we used

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = m a^2 \dot{\mathbf{x}} \ .$$

The collisionless Boltzmann equation for the particle distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is

$$\mathbb{C}[f] = 0 = \frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} f + \dot{\mathbf{p}} \cdot \frac{\mathrm{d}f}{\mathrm{d}\mathbf{p}} \ ,$$

and using the Hamilton equations

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{H}_a}{\partial \mathbf{p}} = \frac{\mathbf{p}}{ma^2} \ ,$$
$$\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}_a}{\partial \mathbf{x}} = -m\boldsymbol{\nabla}\phi \ ,$$

we find the Vlasov equation

$$\boxed{\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{ma^2}\cdot\boldsymbol{\nabla}_{\mathbf{x}}f-m\boldsymbol{\nabla}_{\mathbf{x}}\phi\cdot\frac{\mathrm{d}f}{\mathrm{d}\mathbf{p}}=0} \tag{2.2}$$

Solving the non linear and non local Vlasov equation is not easy, so we consider its first two moments:

$$\rho(\mathbf{x},t)=\frac{m}{a^3}\int \mathrm{d}^3p\, f(\mathbf{x},\mathbf{p},t) \qquad \text{(zero moment)}$$
$$\mathbf{v}(\mathbf{x},t)=\frac{N}{ma}\int \mathrm{d}^3p\,\mathbf{p}f(\mathbf{x},\mathbf{p},t)=\frac{1}{a^4\rho}\int \mathrm{d}^3p\,\mathbf{p}f(\mathbf{x},\mathbf{p},t) \qquad \text{(first moment)}$$

where in the first we divided by a^3 since we are interested in the physical volume and not the comoving one, and on the last $N=(\int \mathrm{d}^3p\, f)^{-1}=m/a^3\rho$ is a normalization factor.

Now taking Vlasov equation, multiplying it by m and integrating over $\mathbf{p}$ we find, with $H=\dot{a}/a$ as the Hubble constant (in the following we won't write explicitly all the dependencies)

$$\boxed{\dot{\rho}+3H\rho+\frac{1}{a}\boldsymbol{\nabla}_{\mathbf{x}}\cdot(\rho\mathbf{v})=0} \tag{2.3}$$

that is the continuity equation, whereas if again we multiply Vlasov by p^i and integrate over $\mathbf{p}$ we find (after some work)

$$\frac{\partial v^i}{\partial t}+Hv^i+\frac{1}{a}v^j\frac{\partial v^i}{\partial x^j}=-\frac{1}{a}\frac{\partial\phi}{\partial x^i}-\frac{1}{a}\frac{\partial}{\partial x^j}\left(\Pi^{ij}\rho\right)\ ,$$

where at the end we have a second moment related quantity (that represents the dispersion of velocity)

$$\Pi^{ij}=\frac{\langle p^ip^j\rangle}{m^2a^2}-v^iv^j\ .$$

We see that an equation for the n-th moment contains also the $n+1$-th moment. To truncate this set of equations, we can assume for example that the Π^{ij} term is zero (i.e. negligible velocity dispersion, this is the so called single stream approximation), thus obtaining

$$\boxed{\frac{\partial\mathbf{v}}{\partial t}+H\mathbf{v}+\frac{1}{a}(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\frac{1}{a}\boldsymbol{\nabla}\phi\ ,} \tag{2.4}$$

that is the Euler equation for a non static background.

3 Reorganizing Euler theory

We are interested in equations (2.3) and (2.4), which we rewrite as[1]

$$\frac{\partial\delta}{\partial\tau}+\boldsymbol{\nabla}\cdot((1+\delta)\mathbf{v})=0\ , \qquad \text{(Continuity eq.)}$$
$$\frac{\partial\mathbf{v}}{\partial\tau}+\mathcal{H}\mathbf{v}+(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}=-\boldsymbol{\nabla}\phi\ , \qquad \text{(Euler eq.)}$$

[1] In the following all space derivatives are taken with respect to the comoving coordinate $\mathbf{x}$.

where we used the conformal time τ (defined through $dt = a(\tau)d\tau$) and the conformal expansion rate $\mathcal{H} = \mathrm{d}\log a/\mathrm{d}\tau = aH$ (note that we can write $H = \mathrm{d}\log a/\mathrm{d}t$); for the first, we also used the third Friedmann equation (with no pressure term since we're discussing cold dark matter[2]) $\dot{\rho}_b = -3H\rho_b$, in this way we deal only with the fluctuations δ. In the following we will restrict to an Einstein-de Sitter model for which $\Omega_m = 1$ and $\Omega_\Lambda = 0$; we will see what happens when one renounces to these and other assumptions in section 9. Note that exploiting this restriction and our definitions we can rewrite the Poisson equation as

$$\nabla^2\phi = \frac{3}{2}\mathcal{H}^2\delta \ . \tag{3.1}$$

This holds since we have

$$\nabla^2\phi = \nabla^2(\varphi - \varphi_b) = 4\pi G a^2(\rho - \rho_b) = 4\pi G a^2\rho_b\delta \ ,$$

and we can exploit the first Friedmann equation in the form

$$\mathcal{H}^2 = \frac{8\pi G}{3}a^2\rho_b \ ,$$

yielding indeed (3.1).

Since we can assume the velocity field to be irrotational[3], we can express the previous in terms of the divergence velocity $\theta(\mathbf{x},\tau) = \boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{x},\tau)$; so passing also in Fourier 3D space (we cannot extend the Fourier transform to time since we are not in Minkowski) we rewrite the continuity equation as

$$\frac{\partial\delta(\mathbf{k},\tau)}{\partial\tau} + \theta(\mathbf{k},\tau) + \int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\alpha(\mathbf{q},\mathbf{p})\theta(\mathbf{q},\tau)\delta(\mathbf{p},\tau) = 0 \ , \tag{3.2}$$

where we denoted with δ_D the Dirac delta function (in order not to make confusion with density fluctuations), whereas from Euler equation, taking first its divergence, we find

$$\frac{\partial\theta(\mathbf{k},\tau)}{\partial\tau} + \mathcal{H}\theta(\mathbf{k},\tau) + \frac{3}{2}\mathcal{H}^2\delta(\mathbf{k},\tau) + \int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\beta(\mathbf{q},\mathbf{p})\theta(\mathbf{q},\tau)\theta(\mathbf{p},\tau) = 0 \ . \tag{3.3}$$

The non linearity and non locality of Vlasov equation is encoded in the functions[4]

$$\alpha(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})\cdot\mathbf{p}}{p^2} \ , \ \beta(\mathbf{q},\mathbf{p}) = \frac{(\mathbf{p}+\mathbf{q})^2\mathbf{q}\cdot\mathbf{p}}{2q^2p^2} \ ;$$

[2]Cold dark matter is by definition non-relativistic (at least from the time of its decoupling) so we can safely put the pressure $P = 0$.

[3]A simple argument to justify this assumption is the following: using Helmoltz theorem, we can split $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$, with $\boldsymbol{\nabla}\times\mathbf{v}_\parallel = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{v}_\perp = 0$; insert $\mathbf{v}_\perp$ in the linearized Euler equation (i.e. without the advective term) without the source (since $\boldsymbol{\nabla}\phi$, being irrotational, cannot contribute to the divergenceless part of $\mathbf{v}$) and you find $\partial\mathbf{v}_\perp/\partial\tau + \mathcal{H}\mathbf{v}_\perp = 0 \implies \mathbf{v}_\perp \propto a^{-1}$, so the divergenceless part of the velocity dies out with the expansion of the universe.

[4]For the details on these derivations, see appendix B.1.

in fact, setting to zero both α and β, we recover the results of standard cosmological perturbations in the linearized Newtonian limit (see appendix B.2 for details).

We now reorganize (3.2) and (3.3); define first the doublet

$$\begin{pmatrix}\varphi_1(\mathbf{k},\eta)\\ \varphi_2(\mathbf{k},\eta)\end{pmatrix} \equiv e^{-\eta}\begin{pmatrix}\delta(\mathbf{k},\eta)\\ -\theta(\mathbf{k},\eta)/\mathcal{H}\end{pmatrix} , \tag{3.4}$$

where we used a new time variable $\eta = \log a/a_{in}$, with a_{in} the scale factor taken at a conveniently early time, when all the relevant scales are inside the linear regime (so little values of η correspond to a situation where linear theory can safely be applied). Define also the vertex functions (called in this way since they will represent the interaction in the path integral point of view)

$$\begin{aligned}
&\gamma_{121}(\mathbf{k},\mathbf{q},\mathbf{p}) = \frac{1}{2}\delta_D(\mathbf{k}+\mathbf{q}+\mathbf{p})\alpha(\mathbf{q},\mathbf{p}) \ ,\\
&\gamma_{222}(\mathbf{k},\mathbf{q},\mathbf{p}) = \delta_D(\mathbf{k}+\mathbf{q}+\mathbf{p})\beta(\mathbf{q},\mathbf{p}) \ ,\\
&\gamma_{abc} = 0 \text{ otherwise} \ , \ a,b,c = 1,2 \ ;
\end{aligned} \tag{3.5}$$

so the equations can be rewritten in the form

$$\boxed{(\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b(\mathbf{k},\eta) = e^{\eta}\int \mathrm{d}^3p\,\mathrm{d}^3q\,\gamma_{abc}(\mathbf{k},-\mathbf{q},-\mathbf{p})\varphi_b(\mathbf{p},\eta)\varphi_c(\mathbf{q},\eta)} \tag{3.6}$$

with

$$\Omega = \begin{pmatrix}1 & -1\\ -3/2 & 3/2\end{pmatrix}$$

(details on how to arrive at this are in appendix B.3).

We can immediately see the analogy with a field theory equation, where the term $\delta_{ab}\partial_\eta + \Omega_{ab}$ can be seen in momentum space as the inverse of the free propagator, whereas the right hand side corresponds to an interaction term.

4 The Path Integral

We now want to define an action giving the equation of motion (3.6); since a term like $\varphi_a\partial_\eta\varphi_a$ would vanish upon integration by parts, to define the action we need an auxiliary field $\chi_a(\mathbf{k},\eta)$, $a = 1,2$, whose physical meaning is associated to the initial conditions as it will become clear later. So[5]

$$\boxed{\begin{aligned}S[\boldsymbol{\varphi},\boldsymbol{\chi}] = \int \mathrm{d}\eta\,(&\chi_a(-\mathbf{k},\eta)(\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b(\mathbf{k},\eta)\\ &- e^{\eta}\gamma_{abc}(\mathbf{k},-\mathbf{q},-\mathbf{p})\chi_a(-\mathbf{k},\eta)\varphi_b(\mathbf{p},\eta)\varphi_c(\mathbf{q},\eta))\end{aligned}} \tag{4.1}$$

The variation of this action with respect to χ_a will straightforwardly yield the equation of motion (3.6). The system is classical, so the path that solves the equation of motion

[5]Here is implied an integration over internal momenta when needed. We will use this shortcut notation also in the following; this should not bring confusion as long as we make explicit the actual dependencies on the left hand side. We will also often forget about writing the explicit dependencies on momenta, with the exception on those cases where confusion may arise.

is unique; we call it $\boldsymbol{\varphi}_{cl}$. But still we want a path integral representation of this classical system, so we can use the functional Dirac delta and express the probability that the configuration $\boldsymbol{\varphi}$ is in $\boldsymbol{\varphi}(\eta_f)$ at a time η_f given that at $\eta = 0$ was at $\boldsymbol{\varphi}(0)$ as

$$P[\boldsymbol{\varphi}(\eta_f), \boldsymbol{\varphi}(0)] = \delta_D[\boldsymbol{\varphi}(\eta_f) - \boldsymbol{\varphi}_{cl}[\eta_f, \boldsymbol{\varphi}(0)]] \;, \tag{4.2}$$

where we remarked the fact that $\boldsymbol{\varphi}_{cl}$ is completely specified by the initial conditions. The corresponding path integral representation is (see appendix B.4)

$$P[\boldsymbol{\varphi}(\eta_f); \boldsymbol{\varphi}(0)] = \mathcal{N} \int \mathcal{D}''\boldsymbol{\varphi}\mathcal{D}\boldsymbol{\chi} e^{iS} \;, \tag{4.3}$$

where $\mathcal{D}''\boldsymbol{\varphi}$ means that upon integration I keep $\boldsymbol{\varphi}$ fixed at the two extrema $\eta = 0$ and $\eta = \eta_f$. We finally define the generating functional by summing over the final configurations $\boldsymbol{\varphi}(\eta_f)$ and, since we want to implement now statistical initial conditions (and not determined ones), averaging over the initial states as

$$\begin{aligned} Z[\mathbf{J}, \mathbf{K}; P^0] = \int & \mathcal{D}\boldsymbol{\varphi}(0)\mathcal{D}\boldsymbol{\varphi}(\eta_f)\mathcal{D}''\boldsymbol{\varphi}\mathcal{D}\boldsymbol{\chi}\mathcal{W}[\boldsymbol{\varphi}(0), P^0] \exp\Bigg(i \int_0^{\eta_f} d\eta \\ & \times (\chi_a(\delta_{ab}\partial_\eta + \Omega_{ab})\varphi_b - e^\eta \gamma_{abc}\chi_a\varphi_b\varphi_c + J_a(\mathbf{k}, \eta)\varphi_a(\mathbf{k}, \eta) + K_a(\mathbf{k}, \eta)\chi_a(\mathbf{k}, \eta))\Bigg) \end{aligned} \tag{4.4}$$

where we have introduced the sources $\mathbf{J}(\mathbf{k}, \eta), \mathbf{K}(\mathbf{k}, \eta)$ and the gaussian weight for the initial configuration

$$\mathcal{W}[\boldsymbol{\varphi}(0), P^0] = \exp\left(-\frac{1}{2}\varphi_a(\mathbf{k}, 0)\left[((P^0)^{-1})_{ab}(k)\right]\varphi_b(-\mathbf{k}, 0)\right) \;, \tag{4.5}$$

with P^0 the power spectrum of the initial configuration (dependent only on the module of $\mathbf{k}$). This expression for $\mathcal{W}$ obviously holds only for gaussian initial conditions, and we will use this assumption (for non gaussian initial conditions, the (4.5) assumes a more general expression, we will briefly discuss this in section 9). We note that in the partition function defined in (4.4) is encoded all the dynamical and statistical content of the continuity, Euler and Poisson equation we started with (equations (2.3), (2.4) and (3.1) respectively) *supplemented* with the initial power spectrum. The quantities of interest we obtain from (4.4) are not then the exact configurations of the field $\boldsymbol{\varphi}$ at a certain time η but rather its statistical properties (since we're assuming gaussian initial conditions).

The linear theory limit, i.e. the $e^\eta \gamma_{abc} \to 0$ limit, corresponds to the tree level of perturbation theory. In this limit the integrals can be computed and we obtain (see B.4)

$$\boxed{\begin{aligned} Z_0[\mathbf{J}, \mathbf{K}; P^0] = \exp\Bigg(- \int d\eta_1 \, \mathrm{d}\eta_2 \Big(& \frac{1}{2}(J_a(\mathbf{k}, \eta_1)P^L_{ab}(k, \eta_1, \eta_2)J_b(-\mathbf{k}, \eta_2)) \\ & + iJ_a(\mathbf{k}, \eta_1)g_{ab}(\eta_1, \eta_2)K_b(-\mathbf{k}, \eta_2)\Big)\Bigg) \;, \end{aligned}} \tag{4.6}$$

where $g_{ab}(\eta_1, \eta_2)$ is the Green function of the operator $\delta_{ab}\partial_\eta + \Omega_{ab}$ and P^L_{ab} is the power spectrum evolved at linear level, i.e.

$$P^L_{ab}(k, \eta_1, \eta_2) = g_{ac}(\eta_1, 0) g_{bd}(\eta_2, 0) P^0_{cd}(k) \; . \tag{4.7}$$

Now that we have an explicit expression for Z_0, we can compute all the interesting quantities of linear theory by making appropriate functional derivatives; for example the power spectrum, defined as[6]

$$\langle \varphi_a(\mathbf{k}, \eta_1) \varphi_b(\mathbf{k}', \eta_2) \rangle \equiv \delta_D(\mathbf{k} + \mathbf{k}') P_{ab}(k, \eta_1, \eta_2) \; , \tag{4.8}$$

is (at this linear level)

$$\delta_D(\mathbf{k} + \mathbf{k}') P^L_{ab}(k, \eta_1, \eta_2) = \frac{(-i)^2}{Z_0} \left. \frac{\delta^2 Z_0[\mathbf{J}, \mathbf{K}, P^0]}{\delta J_a(\mathbf{k}, \eta_1) \delta J_b(\mathbf{k}', \eta_2)} \right|_{\mathbf{J},\mathbf{K}=0} \; .$$

In the following we will assume that the initial perturbations $\delta(\mathbf{k}, 0)$ and $\theta(\mathbf{k}, 0)$ are proportional random fields, so that we can write[7]

$$\varphi_a(\mathbf{k}, 0) = u_a \delta_0(\mathbf{k}) \; , \tag{4.9}$$

for a certain two component vector $\mathbf{u}$, where $\delta_0(\mathbf{k})$ is the initial proportional random field. Note that with this assumption we can write

$$\delta_D(\mathbf{k} + \mathbf{k}') P_{ab}(k) \equiv \langle \varphi_a(\mathbf{k}, 0) \varphi_b(\mathbf{k}', 0) \rangle = \delta_D(\mathbf{k} + \mathbf{k}') u_a u_b P_0(k) \; ,$$

with $\delta_D(\mathbf{k} + \mathbf{k}') P_0(k) = \langle \delta_0(\mathbf{k}) \delta_0(\mathbf{k}') \rangle$.

Instead the propagator is

$$\delta_D(\mathbf{k} + \mathbf{k}') g_{ab}(\eta_1, \eta_2) = \frac{i}{Z_0} \left. \frac{\delta^2 Z_0[\mathbf{J}, \mathbf{K}, P^0]}{\delta J_a(\mathbf{k}, \eta_1) \delta K_b(\mathbf{k}', \eta_2)} \right|_{\mathbf{J},\mathbf{K}=0} \; .$$

Now the complete partition function with also the interaction term can be written as[8]

$$Z[\mathbf{J}, \mathbf{K}; P^0] = \exp\left(-i \int d\eta \, e^\eta \gamma_{abc} \left(\frac{-i\delta}{\delta K_a} \frac{-i\delta}{\delta J_b} \frac{-i\delta}{\delta J_c} \right) \right) Z_0[\mathbf{J}, \mathbf{K}; P^0] \; , \tag{4.10}$$

[6] The seemingly strange minus sign in (4.1) on the dependencies of the χ field on the momenta is mainly due to the fact that we want to write these relations involving the power spectrum in this standard form, i.e. with $\delta(\mathbf{k} + \mathbf{k}')$.

[7] At linear level this is certainly true, see appendix B.2.

[8] We are using the following property (we introduce $\langle \cdots \rangle$ meaning integration over momenta only here for clarity)

$$\frac{-i\delta}{\delta K_a} \frac{-i\delta}{\delta J_b} \frac{-i\delta}{\delta J_c} e^{i\langle K_i \chi_i + J_l \varphi_l \rangle} = \chi_a \varphi_b \varphi_c e^{i\langle K_i \chi_i + J_l \varphi_l \rangle} \; ,$$

so that $\exp\left(\left\langle V\left(\frac{-i\delta}{\delta K_a} \frac{-i\delta}{\delta J_b} \frac{-i\delta}{\delta J_c} \right) \right\rangle \right) \exp(i \langle K_i \chi_i + J_l \varphi_l \rangle) = \exp(\langle V(K_a, J_b, J_c) \rangle) \exp(\langle K_i \chi_i + J_l \varphi_l \rangle)$; this can be straightforwardly applied to (4.10).

and to this can be associated the following Feynman rules (by looking also at (4.6) for the propagator and the power spectrum rule):

Propagator: $a, 1 \;\text{———-----}\; b, 2 \;= -i g_{ab}(\eta_1, \eta_2)\;;$ (4.11)

Power spectrum: $a, 1 \;\text{—∘—}\; b, 2 \;= P^L_{ab}(\eta_1, \eta_2, k)\;;$ (4.12)

Interaction vertex: $a \; c \; b \;= -i e^{\eta} \gamma_{abc}(k_a, k_b, k_c)\;,$ (4.13)

where as usual we indicated field indices with a, b and time indices with 1, 2.

These Feynman rules can be used to compute the statistical configuration of the field $\boldsymbol{\varphi}(\eta)$ as we will see in the next section.

5 Feynman rules

Here we will briefly see how to cast perturbation theory in terms of Feynman diagrams without too many details since our main focus is the use of renormalization group techniques. Nevertheless we think that it is useful to put a small review about this topic here for a better understanding of the following.

Equations (3.2) and (3.3) can be solved writing the following perturbative expansions:

$$\delta(\mathbf{k}, \tau) = \sum_{n=1}^{\infty} \delta_n(\mathbf{k}) a^n(\tau)\;,$$

$$\theta(\mathbf{k}, \tau) = -\mathcal{H}(\tau) \sum_{n=1}^{\infty} \theta_n(\mathbf{k}) a^n(\tau)\;.$$

Plugging these in (3.2) and (3.3), one obtains $\delta_n(\mathbf{k})$ and $\theta_n(\mathbf{k})$ in terms of the linear fluctuations

$$\delta_n(\mathbf{k}) = \int \left(\prod_{i=1}^{n} \mathrm{d}^3\mathbf{q}_i \, \delta_0(\mathbf{q}_i) \right) \delta_D \left(\mathbf{k} - \sum_{i=1}^{n} \mathbf{q}_i \right) F_n(\mathbf{q}_1, \ldots, \mathbf{q}_n)\;,$$

$$\theta_n(\mathbf{k}) = \int \left(\prod_{i=1}^{n} \mathrm{d}^3\mathbf{q}_i \, \delta_0(\mathbf{q}_i) \right) \delta_D \left(\mathbf{k} - \sum_{i=1}^{n} \mathbf{q}_i \right) G_n(\mathbf{q}_1, \ldots, \mathbf{q}_n)\;,$$

where δ_0 is the initial random field and F_n and G_n have rather involved expressions. It is here that our Feynman rules greatly help in finding the expressions for these perturbative terms. Rewrite the previous in the formalism we have built

$$\varphi_a(\mathbf{k}, \eta) = \sum_{n=1}^{\infty} \varphi_a^{(n)}(\mathbf{k}, \eta)\;, \tag{5.1}$$

where

$$\varphi_a^{(n)}(\mathbf{k},\eta) = \int \left(\prod_{i=1}^{n} \mathrm{d}^3\mathbf{q}_i\, \delta_0(\mathbf{q}_i)\right) \delta_D\left(\mathbf{k} - \sum_{i=1}^{n} \mathbf{q}_i\right) \mathcal{F}_a^{(n)}(\mathbf{q}_1, \dots, \mathbf{q}_n) \ ,$$

with $\mathcal{F}_a^{(n)}$ as the analogous of F_n and G_n in our field formalism. Instead of computing the $\varphi_a^{(n)}$ in terms of their analytical expression (though certainly possible, see [1], but rather cumbersome), here we see how we can associate a set of diagrams to each $\varphi_a^{(n)}$. In the end we will obtain the same terms one would have obtained by doing all the computation with the explicit expressions for $\mathcal{F}_a^{(n)}$.

At first, let's start with a situation in which the initial conditions are fixed, so that we can in principle compute the exact configuration $\boldsymbol{\varphi}$ at a certain time η. To obtain the set of diagrams corresponding to the n-th order term in equation (5.1), one has to draw all topologically different tree diagrams (i.e. without loops) with $n-1$ vertices and n initial conditions (depicted as dashed lines in our diagrammatic representation)[9]. These diagrams are built as follows: start from the final time variable, then go from the left to the right, and when you encounter a vertex, bifurcate this line and continue this process for every line (or propagators) until one of their ends reaches an initial condition. Figure 5.1 certainly explains this better than words. The rightmost part of the diagrams correspond to the initial fields (time $\eta = 0$), each with momentum $\mathbf{k}_i$; every vertex correspond to an interaction happening at a time $0 \leq \eta_i \leq \eta_f$ (where η_f is the final time), coupling the incoming $\mathbf{k}_j$ and $\mathbf{k}_l$ to an outgoing $\mathbf{k}_k$; every propagator represents the linear evolution of a given mode $\mathbf{k}_i$ from a time η_j to a time η_k. So to construct the corresponding integral coming out of these diagrams, associate to all these terms the corresponding Feynman rule (4.11)-(4.13), to all free dashed lines a corresponding $\varphi_a(\mathbf{k}, 0)$ field and finally integrate over all intermediate time variables and momenta. As an example, write

$$\begin{aligned}\varphi_a^{(2)}(\mathbf{k},\eta_f) = &\int \mathrm{d}^3k_1\, \mathrm{d}^3k_2 \int_0^{\eta_f} \mathrm{d}\eta_s\, g_{ab}(\eta_f,\eta_s) e^{\eta_s} \gamma_{bcd}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) \\ &\times g_{ce}(\eta_s,0)\varphi_e(\mathbf{k}_1,0) g_{df}(\eta_s,0)\varphi_f(\mathbf{k}_2,0) \ ,\end{aligned} \tag{5.2}$$

$$\begin{aligned}\varphi_a^{(3)}(\mathbf{k},\eta_f) = &2\int \mathrm{d}^3k_1\, \mathrm{d}^3k_2 \int_0^{\eta_f} \mathrm{d}\eta_s\, g_{ab}(\eta_f,\eta_s) e^{\eta_s} \gamma_{bcd}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) \\ &\times g_{ce}(\eta_s,0)\varphi_e(\mathbf{k}_1,0) \varphi_d^{(2)}(\mathbf{k}_2,\eta_s) \ ,\end{aligned}$$

where on $\varphi_a^{(3)}$ the factor 2 comes from the fact that its two diagrams are topologically equivalent (and they give the same contribution) and we recognized the presence of the lower order diagram corresponding to $\varphi_d^{(2)}$.

Now we want to implement also the gaussian randomly distributed initial conditions (and not fixed ones); in this case we're not interested in the exact configuration $\boldsymbol{\varphi}(\mathbf{k},\eta_f)$ (which obviously is meaningless if initial conditions are not fixed), but rather in its statistical properties via

$$\langle \varphi_a(\mathbf{k}',\eta)\varphi_b(\mathbf{k},\eta)\rangle = \delta_D(\mathbf{k}+\mathbf{k}')P_{ab}(\mathbf{k},\eta) \ . \tag{5.3}$$

[9]Dashed lines correspond to the $\boldsymbol{\chi}$ field that, as we have already remarked and as one can see in equation (6.1), contains informations about the initial conditions.

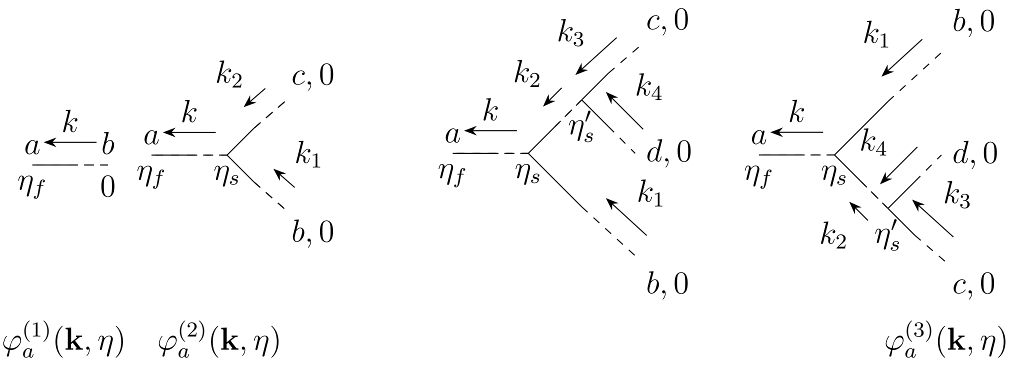


Figure 5.1: Diagrammatic representation of the first terms in (5.1). At each diagram can be associated an integral by means of our Feynman rules (4.11)-(4.13) as follows: attach to every free dashed line a corresponding $\varphi_a(\mathbf{k}, 0)$ field, use Feynman rules and integrate over all intermediate momenta and time steps. The last two diagrams belong to $\varphi^{(3)}(\mathbf{k}, \eta)$.

Use a perturbative expansion for the new quantity of interest, i.e. the power spectrum

$$P_{ab}(\mathbf{k}, \eta) = \sum_{l=0}^{\infty} P_{ab}^{(l)}(\mathbf{k}, \eta) \ ,$$

and putting (5.1) in (5.3), one finds that, insisting in considering the l index in $P_{ab}^{(l)}$ as the number of loops in a diagram,

$$\delta_D(\mathbf{k} + \mathbf{k}')P_{ab}^{(l)}(\mathbf{k}, \eta) = \sum_{m=1}^{2l+1} \left\langle \varphi_a^{(2l+2-m)}(\mathbf{k}', \eta)\varphi_b^{(m)}(\mathbf{k}, \eta) \right\rangle \ . \tag{5.4}$$

In order to understand the previous, recall that in one diagram, calling V the number of vertices, l the number of loops and E and I respectively the number of external and internal lines, one has $l = I - (V - 1)$ (a simple argument for this is the following: the number of loops is the number of independent internal momenta, I is the total number of internal momenta, each satisfying V conditions due to conservation of momentum at each vertex, minus one due to the overall conservation of momentum); then since each vertex comes with 3 lines crossing at the vertex, we have $3V$ lines to start with, out of which E are chosen to be external. The remaining $(3N - V)$ ones form $I = (3V - E)/2$ internal lines. The factor of one-half is due to the fact that each internal line is shared between two vertices. So considering that in our diagrams we will deal with only $E = 2$ (corresponding to the final configuration at η_f of the two fields) one has $V = 2l$. Doing the sum in the exponent in the RHS of (5.4), and considering that the number of vertices corresponding to a $\varphi_a^{(n)}$ term is $n - 1$, one has (change the index l in (5.4) to k to avoid confusions) $2k + 2 - m + m - 2 = 2k = l$, yielding the right number of loops.

To see how to build the Feynman rules for computing $P_{ab}(\mathbf{k}, \eta)$, consider the simple

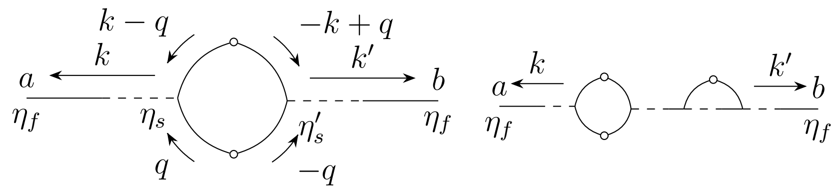


Figure 5.2: Diagrammatic representation of the power spectrum terms of $P_{ab}^{(l)}$. The first is one of the three 1 loop diagrams contributing to $P_{ab}^{(1)}$, the other is one of the 29 loop diagrams contributing to $P_{ab}^{(2)}$ (where we have suppressed the intermediate times and momenta for clarity).

case of $P_{ab}^{(0)}$; keeping in mind that the average acts only on the terms $\varphi_a(\mathbf{k}, 0)$,

$$\begin{aligned}\delta_D(\mathbf{k}+\mathbf{k}')P_{ab}^{(0)}(\mathbf{k},\eta) &= \left\langle \varphi_a^{(1)}(\mathbf{k}',\eta)\varphi_b^{(1)}(\mathbf{k},\eta)\right\rangle \\ &= g_{ac}(\eta,0)g_{bd}(\eta,0)\underbrace{\langle\varphi_a(\mathbf{k},0)\varphi_b(\mathbf{k}',0)\rangle}_{=\delta_D(\mathbf{k}-\mathbf{k}')P_{ab}(k)} = \delta_D(\mathbf{k}-\mathbf{k}')P_{ab}^L(k,\eta,0)\ ,\end{aligned}$$

where in the second step we used the Feynman rules for fixed initial conditions and on the last one we recognized the linear power spectrum defined in (4.7). One can see that also for the power spectrum we can give related Feynman rules: to draw each diagram that contributes to $P_{ab}^{(l)}$, just put the tree diagrams for $\varphi_a^{(m)}$ against one for $\varphi_b^{(2l+2-m)}$ with their initial fields (i.e. dashed lines) facing each other; pair the initial fields in all possible pairs, glue the pairs, then convert all glued initial fields in the corresponding linearized power spectrum; for example, for $P_{ab}^{(1)}$,

$$a,1 \text{ ——‐‐‐‐‐ } c,0 \quad \text{and} \quad d,0 \text{ ‐‐‐‐‐—— } b,1 \ \rightarrow\ a,1 \text{ —∘— } b,1\ ,$$

where the index 1 stands for final time and 0 for the initial one. We see that indeed we recover the same result obtained with the calculations if we use the Feynman rule for the power spectrum. Other examples are showed in figure 5.2, of which we write the expression for the first diagram we denote by $P'^{(1)}_{ab}$

$$\begin{aligned}P'^{(1)}_{ab}(\mathbf{k},\eta_f) =&2\int_0^{\eta_f}\mathrm{d}\eta_s\int_0^{\eta_f}\mathrm{d}\eta'_s\, g_{ac}(\eta_f,\eta_s)e^{\eta_s}\gamma_{cde}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})P_{df}^L(q,\eta_s,\eta'_s)\\ &\times P_{eg}^L(|\mathbf{k}-\mathbf{q}|,\eta_s,\eta'_s)e^{\eta'_s}\gamma_{fgh}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})g_{hb}(\eta_f,\eta'_s)\ ,\end{aligned}$$

where the factor 2 is there since you can construct an identical diagram corresponding to the interchange of the two fields.

We discovered that we can apply standard perturbation theory tools completely analogous to the ones pertaining quantum field theories, and we can (and should) go beyond this point by developing the necessary tools to apply also renormalization group techniques to cosmological problems.

6 The Renormalization Group approach

Note that we can rewrite equation (4.10) in the following way (details in appendix B.4)

$$Z[\mathbf{J},\mathbf{K};P^0] = \int \mathcal{D}\boldsymbol{\varphi}\,\mathcal{D}\boldsymbol{\chi}\,\exp\Bigg(\int d\eta_1\,\mathrm{d}\eta_2\,\Big(-\frac{1}{2}\chi_a(\eta_1)P^0_{ab}(\eta_1,\eta_2)\delta_D(\eta_1)\delta_D(\eta_2)\chi_b(\eta_2)$$

$$+\,i\chi_a(\eta_1)g^{-1}_{ab}(\eta_1,\eta_2)\varphi_b(\eta_2)\Big) + i\int \mathrm{d}\eta\,(-e^{\eta}\gamma_{abc}\chi_a\varphi_b\varphi_c + J_a\varphi_a + K_a\chi_a)\Bigg) \tag{6.1}$$

(we have put also the time dependence of the various quantities only where the proper dependencies could have been ambiguous and as always integration over all momenta is implied), where one can see how the power spectrum is coupled only to the field $\boldsymbol{\chi}$, showing the role of this field in encoding the informations on the statistic of the initial conditions. We will use this alternative form of the path integral in the following. The starting point for our formulation of the renormalization group approach is the introduction of a low pass filter $\Theta(\lambda,k)$ to the power spectrum,

$$P^0(k) \rightarrow P^0_\lambda(k) \equiv P^0(k)\Theta(\lambda,k)\ ; \tag{6.2}$$

in this way we introduce a fictitious universe described by $Z_\lambda[\mathbf{J},\mathbf{K},P^0] \equiv Z[\mathbf{J},\mathbf{K},P^0_\lambda]$ where all the fluctuations with momenta greater than λ are damped. Increasing the cut-off from $\lambda = 0$ to $\lambda = \infty$, linear and non linear effects of higher and higher fluctuations are gradually taken into account.

These RG methods are particularly suited in physical situations in which the scale you have to probe in measurements is well separated by the scale where one should control the "fundamental theory". Starting from the fundamental scale (here corresponding to low values of k), the RG flow describes the gradual inclusion of fluctuations at scales closer and closer to the ones relevant for measurements (here represented by region of high k).

In the following for simplicity we will just take the Heaviside function as the filter, $\Theta(\lambda,k) = \theta(\lambda - k)$. Now we can take the derivative with respect to λ (that is a scale parameter, analogous to those one introduces in renormalization in QFT) of the quantities of interest to obtain their RG equations (i.e. equations expressing the variations with respect to the scale parameter), for example for Z_λ we obtain

$$\begin{aligned}\partial_\lambda Z_\lambda &= \int \mathcal{D}\boldsymbol{\varphi}\,\mathcal{D}\boldsymbol{\chi}\,\overbrace{\exp(\cdots)}^{\text{see (6.1)}}\left(-\frac{1}{2}\right)\int d\eta_1\,\mathrm{d}\eta_2\,\chi_a(\eta_1)P^0_{ab}(\eta_1,\eta_2)\delta_D(\lambda-q)\delta_D(\eta_1)\delta_D(\eta_2)\chi_b(\eta_2)\\ &= \frac{1}{2}\int d\eta_1\,\mathrm{d}\eta_2\,P^0_{ab}(\eta_1,\eta_2)\delta_D(\lambda-q)\delta_D(\eta_1)\delta_D(\eta_2)\frac{\delta^2 Z_\lambda}{\delta K_a\delta K_b}\ ,\end{aligned}$$

where on the first step we used the fact that the derivative of Heaviside is the Dirac delta and on the last one we wrote all the part dependent on the fields in

$$\frac{\delta^2 Z_\lambda[\mathbf{J},\mathbf{K}]}{\delta K_a(\mathbf{k},\eta_1)\delta K_b(\mathbf{k},\eta_2)} = -\int \mathcal{D}\boldsymbol{\varphi}\,\mathcal{D}\boldsymbol{\chi}\,\chi_a(\mathbf{k},\eta_1)\chi_b(\mathbf{k},\eta_2)\exp(\cdots)\ .$$

Our main aim is to derive expressions for the full propagator and power spectrum (i.e. corresponding to the full interacting theory) by using renormalization group equations, and for this we need to introduce other quantities (which are familiar in QFT): one is

$$W[\mathbf{J},\mathbf{K}] = -i\log(Z[\mathbf{J},\mathbf{K}])\ , \tag{6.3}$$

known as the generating functional of connected diagrams, from which one can define the expectation values of the fields in the presence of a source as

$$\langle\varphi_a(\mathbf{k},\eta)\rangle \equiv \varphi_a^{(\mathrm{cl})}[\mathbf{J},\mathbf{K}] = \frac{\delta W[\mathbf{J},\mathbf{K}]}{\delta J_a(\mathbf{k},\eta)}\ ,\quad \langle\chi_b(\mathbf{k},\eta)\rangle \equiv \chi_b^{(\mathrm{cl})}[\mathbf{J},\mathbf{K}] = \frac{\delta W[\mathbf{J},\mathbf{K}]}{\delta K_b(\mathbf{k},\eta)} \tag{6.4}$$

(the name classical is due to the fact that they solve a classical equation of motion like (B.5)); another quantity is the Legendre transform of W

$$\Gamma[\boldsymbol{\varphi}^{(\mathrm{cl})},\boldsymbol{\chi}^{(\mathrm{cl})}] = W[\mathbf{J},\mathbf{K}] - \int \mathrm{d}\eta\,\mathrm{d}^3k\,(J_a\varphi_a^{(\mathrm{cl})} + K_b\chi_b^{(\mathrm{cl})})\ , \tag{6.5}$$

known as the effective action (or also as the generator of 1PI diagrams)[10]. Note that

$$\left.\frac{\delta^n\Gamma[\boldsymbol{\varphi}^{(\mathrm{cl})},\boldsymbol{\chi}^{(\mathrm{cl})}]}{\delta\varphi_{a_1}^{(\mathrm{cl})}\cdots\delta\varphi_{a_n}^{(\mathrm{cl})}}\right|_{\boldsymbol{\varphi}^{(\mathrm{cl})},\boldsymbol{\chi}^{(\mathrm{cl})}=0} = 0 \quad \forall n\geq 0\ ; \tag{6.6}$$

this represents 1PI diagrams with n $\boldsymbol{\varphi}$ field external legs[11]. To see that this is indeed zero, using the notation of section 5, we see that, with $E = n$,

$$3V = n + 2I\ ,\quad l = I-(V-1) \implies V = n + 2(l-1)\ ;$$

all vertices must have at most one external leg attached (otherwise the connection with another vertex must be with a single internal propagator or power spectrum, thus yielding a non 1PI diagram), so n of these V vertices have only an external leg and the other $2(l-1)$ have none; now since we cannot have χ field external legs, every χ field must be contracted with a φ field belonging to another vertex. One then see that a contribution to (6.6) must contain at least one loop, and at least one of them must be a φ loop, that vanishes (in the propagator there is an Heaviside function like $\theta(\eta_1-\eta_2)$, see (B.9), and in a loop at least one of the Heaviside vanishes since I have to return back to the initial time).

The quantities we're interested in are encoded in the second functional derivatives of W; at linear level, since using (4.6) we can write

$$W_0[\mathbf{J},\mathbf{K}] = \int d\eta_1\,\mathrm{d}\eta_2\left(\frac{i}{2}J_aP^L_{ab}J_b - J_a g_{ab}K_b\right)\ , \tag{6.7}$$

[10] 1PI = one particle irreducible, which are diagrams that "cannot be split by cutting only one line" and without external propagators.

[11] Actually the functional derivatives of the effective action (called also proper vertices functions) don't contain the expression for the external legs; what we mean here is that we're considering connected diagrams with n external legs which are also 1PI once its external legs are removed.

we have

$$\left.\frac{\delta^2 W_0[\mathbf{J},\mathbf{K}]}{\delta J_a(\mathbf{k},\eta_1)\delta J_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} = i\delta_D(\mathbf{k}+\mathbf{k}')P^L_{ab}(k,\eta_1,\eta_2) \ ,$$
$$\left.\frac{\delta^2 W_0[\mathbf{J},\mathbf{K}]}{\delta J_a(\mathbf{k},\eta_1)\delta K_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} = -\delta_D(\mathbf{k}+\mathbf{k}')g_{ab}(\eta_1,\eta_2) \ ,$$

so we can define the full propagator G_{ab} and the full power spectrum P_{ab} as

$$\begin{aligned}\left.\frac{\delta^2 W[\mathbf{J},\mathbf{K}]}{\delta J_a(\mathbf{k},\eta_1)\delta J_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} &\equiv i\delta_D(\mathbf{k}+\mathbf{k}')P_{ab}(k,\eta_1,\eta_2) \ ,\\ \left.\frac{\delta^2 W[\mathbf{J},\mathbf{K}]}{\delta J_a(\mathbf{k},\eta_1)\delta K_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} &\equiv -\delta_D(\mathbf{k}+\mathbf{k}')G_{ab}(\eta_1,\eta_2) \ .\end{aligned} \tag{6.8}$$

We can do an analogous procedure for the second derivatives of the effective action: writing the linear part plus the correction due to interaction[12], we have

$$\begin{aligned}\Gamma^{(2)}_{\varphi_a\chi_b} &\equiv g^{-1}_{ab} - \Sigma_{\varphi_a\chi_b} \ ,\\ \Gamma^{(2)}_{\chi_a\chi_b} &\equiv iP^0_{ab}\delta_D(\eta_1)\delta_D(\eta_2) + i\Phi_{ab} \ ,\end{aligned} \tag{6.9}$$

where we have defined for example

$$\delta_D(\mathbf{k}+\mathbf{k}')\Gamma^{(2)}_{\varphi_a\chi_b}(\mathbf{k},\mathbf{k}',\eta_1,\eta_2) \equiv \left.\frac{\delta^2\Gamma[\boldsymbol{\varphi}^{(\mathrm{cl})},\boldsymbol{\chi}^{(\mathrm{cl})}]}{\delta\varphi^{(\mathrm{cl})}_a(\mathbf{k},\eta_1)\delta\chi^{(\mathrm{cl})}_b(\mathbf{k}',\eta_2)}\right|_{\boldsymbol{\varphi}^{(\mathrm{cl})},\boldsymbol{\chi}^{(\mathrm{cl})}=0} . \tag{6.10}$$

One can show[13] that we can relate (6.8) and (6.9) and write

$$P_{ab} = P^{\mathrm{I}}_{ab} + P^{\mathrm{II}}_{ab} \ , \tag{6.11}$$

where

$$P^{\mathrm{I}}_{ab}(k,\eta_1,\eta_2) = G_{ac}(k,\eta_1,0)G_{bd}(k,\eta_2,0)P^0_{cd}(k) \ , \tag{6.12}$$

$$P^{\mathrm{II}}_{ab}(k,\eta_1,\eta_2) = \int_0^{\eta_1}\mathrm{d}\eta'\int_0^{\eta_2}\mathrm{d}\eta''\, G_{ac}(k,\eta_1,\eta')G_{bd}(k,\eta_2,\eta'')\Phi_{cd}(k,\eta',\eta'') \ , \tag{6.13}$$

and

$$G_{ab}(k,\eta_1,\eta_2) = (g_{ab} - \Sigma_{\varphi_a\chi_b})^{-1}(k,\eta_1,\eta_2) \ . \tag{6.14}$$

We are then ready to write the RG equation for W_λ (in the following a quantity labelled with λ is a quantity where the substitution (6.2) has been made)

$$\partial_\lambda W_\lambda = \frac{1}{2}\int \mathrm{d}\eta_3\,\mathrm{d}\eta_4\, P^0_{ab}(\eta_3,\eta_4)\delta_D(\lambda-q)\delta_D(\eta_3)\delta_D(\eta_4)\left(i\chi^{(\mathrm{cl})}_a\chi^{(\mathrm{cl})}_b + \frac{\delta^2 W_\lambda}{\delta K_a\delta K_b}\right) \ , \tag{6.15}$$

[12]The quantities defined here can be seen as an analogous of the self-energies one encounter in renormalization in QFT.

[13]Details regarding derivations of this and the following equations are in appendix B.5.

and thus for the propagator

$$
\begin{aligned}
-\delta_D(\mathbf{k}+\mathbf{k}')\partial_\lambda G_{ab,\lambda} &= \partial_\lambda \left.\frac{\delta^2 W_\lambda}{\delta J_a(\mathbf{k},\eta_2)\delta K_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} \\
&= \frac{1}{2}\int \mathrm{d}\eta_3\,\mathrm{d}\eta_4\, P^0_{ab}(\eta_3,\eta_4)\delta_D(\lambda-q)\delta_D(\eta_3)\delta_D(\eta_4) \left.\frac{\delta^4 W_\lambda}{\delta J_a \delta K_b \delta K_c \delta K_d}\right|_{\mathbf{J},\mathbf{K}=0} .
\end{aligned}
\tag{6.16}
$$

Defining

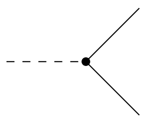
$$
\delta_D(\mathbf{k}_a+\mathbf{k}_b+\mathbf{k}_c+\mathbf{k}_d) W^{(4)}_{J_a K_b K_c K_d} \equiv \left.\frac{\delta^4 W_\lambda}{\delta J_a \delta K_b \delta K_c \delta K_d}\right|_{\mathbf{J},\mathbf{K}=0} , \tag{6.17}
$$

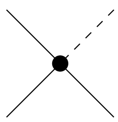
we can write it using multiple times the (B.12)

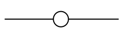
$$
\begin{aligned}
& W^{(4)}_{J_a K_b K_c K_d,\lambda}(\mathbf{k},\eta_1,-\mathbf{k},\eta_2,\mathbf{k}',\eta_3,-\mathbf{k}',\eta_4) \\
&= \int \mathrm{d}s_1 \ldots \mathrm{d}s_4\, G_{ae,\lambda}(k,\eta_1,s_1) G_{fb,\lambda}(k,\eta_2,s_2) G_{gc,\lambda}(k',\eta_3,s_3) G_{hd,\lambda}(k',\eta_4,s_4) \\
&\times \Bigg(-2\int \mathrm{d}s_5\,\mathrm{d}s_6\, G_{li,\lambda}(k-k',s_6,s_6) \Gamma^{(3)}_{\chi_e\varphi_h\varphi_l,\lambda}(\mathbf{k},s_1,-\mathbf{k}',s_4,-\mathbf{k}+\mathbf{k}',s_5) \\
&\times \Gamma^{(3)}_{\chi_i\varphi_g\varphi_f,\lambda}(\mathbf{k}-\mathbf{k}',s_6,\mathbf{k}',s_3,-\mathbf{k},s_2) + \Gamma^{(4)}_{\chi_e\varphi_f\varphi_g\varphi_h,\lambda}(\mathbf{k},s_1,-\mathbf{k},s_2,\mathbf{k}',s_3,-\mathbf{k}',s_4) \Bigg) .
\end{aligned}
\tag{6.18}
$$

Inserting the previous into (6.16) we obtain the RG equation for the propagator; since it is too long we won't write it explicitly, but still we can represent it diagrammatically as in figure 6.1, where this time we use

$a, 1$ $b, 2$ $\quad = -iG_{ab}(k,\eta_1,\eta_2)$; (Full propagator)

 $\quad = i\Gamma^{(3)}_{\chi\varphi\varphi,\lambda}$; (3-point vertex function)

 $\quad = i\Gamma^{(4)}_{\chi\varphi\varphi\varphi,\lambda}$; (4-point vertex function)

a  b $\quad = K_{ab,\lambda}$. (RG kernel)

The RG kernel is defined as

$$
K_{ab,\lambda}(k,\eta,\eta') = G_{ac}(k,\eta,0) G_{bd}(k,\eta',0) P^0_{cd}(k)\delta_D(\lambda-k) . \tag{6.19}
$$

Regarding the RG equation for the power spectrum, we have that P^{I}_{ab} is completely determined from the behavior of the propagator (since basically it depends only on it), whereas for $\partial_\lambda P^{\mathrm{II}}_{ab}(k,\eta_1,\eta_2)$, simply apply Leibniz to (6.13) and notice that we need the

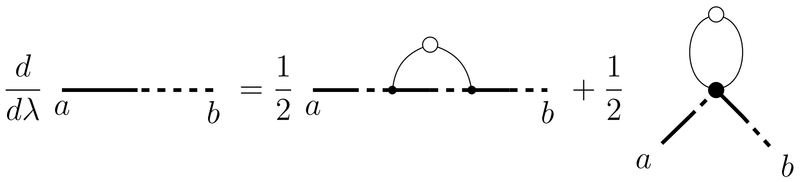


Figure 6.1: Diagrammatic representation of the RG equation for the propagator.

RG equation for Φ_{ab}. To find it, again with the same steps employed for the propagator, we find

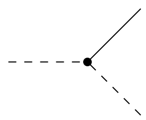
$$
\begin{aligned}
i\delta_D(\mathbf{k}+\mathbf{k}')\partial_\lambda P_{ab,\lambda} = \partial_\lambda \left.\frac{\delta^2 W_\lambda}{\delta J_a(\mathbf{k},\eta_2)\delta J_b(\mathbf{k}',\eta_2)}\right|_{\mathbf{J},\mathbf{K}=0} &= \frac{1}{2}\int \mathrm{d}\eta_3\,\mathrm{d}\eta_4\, P^0_{ab}(\eta_3,\eta_4) \\
\times\, \delta_D(\lambda-q)\delta_D(\eta_3)\delta_D(\eta_4)\left(i \left.\frac{\delta^2\chi_c^{(\mathrm{cl})}\chi_d^{(\mathrm{cl})}}{\delta J_a \delta J_b}\right|_{\mathbf{J},\mathbf{K}=0} + \left.\frac{\delta^4 W_\lambda}{\delta J_a\delta J_b\delta K_c\delta K_d}\right|_{\mathbf{J},\mathbf{K}=0}\right)&\,,
\end{aligned}
\tag{6.20}
$$

where this time the $\chi\chi$ term similar to the one in (6.15) does not vanish after the evaluation in $\mathbf{J},\mathbf{K}=0$ as in (6.16), but it yields

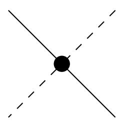
$$
\left.\frac{\delta^2\chi_c^{(\mathrm{cl})}\chi_d^{(\mathrm{cl})}}{\delta J_a\delta J_b}\right|_{\mathbf{J},\mathbf{K}=0} = 2G_{ca}G_{db} + 2\,\underbrace{\chi_c \left.\frac{\delta G_{ad}}{\delta J_b}\right|_{\mathbf{J},\mathbf{K}=0}}_{=0}\;;
$$

it is not difficult then to recognize that this previous term contributes to the P^{I}_{ab} RG equation by looking at (6.11), and also notice from (6.11) that the RG equation for Φ only (again using the same procedure we have done with the propagator) can be represented with the diagrammatic representation of figure 6.2 (notice the removed external propagators due to the expression of P^{II}_{ab}), with the new Feynman rules

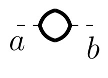
$a,1$ —×— $b,2$ $\quad = P_{ab}(k,\eta_1,\eta_2)\,,$ (Full power spectrum)



$= i\Gamma^{(3)}_{\chi\chi\varphi,\lambda}\;;$ (3-point vertex function)



$= i\Gamma^{(4)}_{\chi\chi\varphi\varphi,\lambda}\;;$ (4-point vertex function)



$= \Phi_{ab}\,,$

where also other two new types of vertex function appear (note that these vanish at tree level).

7 Solving Renormalization Group equations

We have all the RG equations we need, now in this section we will briefly sketch some methods to try to solve them.

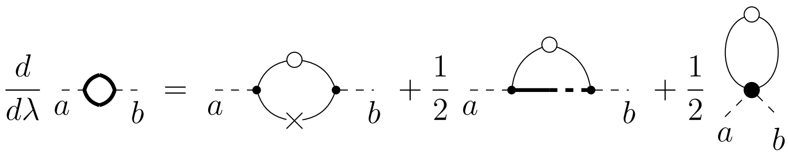


Figure 6.2: Diagrammatic representation of the RG equation for $\Phi_{ab,\lambda}$.

We have seen that the expression for the full propagator and the full power spectrum depends on the three and four-point vertex functions, which are themselves λ-dependent quantities. As a consequence of that, one can take their RG equations and find that they involve higher vertex functions. Similar to the Vlasov equation, one has then an infinite tower of coupled differential equations, so one can truncate them at a certain order as an approximation.

Here we will approximate the full RG flow by keeping a running power spectrum and propagator but we will keep the tree level vertex expression. This will imply that all diagrams involving other than the tree level vertex won't be considered, and we will approximate the full $\Gamma^{(3)}$ expression with the tree level one (obtained doing the proper functional derivatives on the vertex term of the action)

$$\Gamma^{(3)}_{\chi_a\varphi_b\varphi_c,\lambda}(\mathbf{k}, s_1, -\mathbf{q}, s_2, -\mathbf{k}+\mathbf{q}, s_3) \approx -2\left(\prod_{i=1}^{3}\delta_D(s-s_i)\right)e^s\gamma_{abc}(\mathbf{k}, -\mathbf{q}, -\mathbf{k}+\mathbf{q}) \ . \tag{7.1}$$

For the propagator, in this approximation the diagram that contributes on the RHS of figure 6.1 is the first (since the second contains a four-point vertex function). We can now compute the one-loop correction to the propagator as follows: just use the tree level (order $l=0$) quantities to express $W^{(4)}$ given in (6.18) (that is use (7.1) as the vertex term and g_{ab} as the propagator):

$$\begin{aligned} W^{(4),l=0}_{J_aK_bK_cK_d,\lambda} = -8\int \mathrm{d}s_1\,\mathrm{d}s_2\, e^{s_1+s_2} g_{ae}(s_1,\eta_1)g_{fb}(s_2,\eta_2)g_{gc}(\eta_3,s_1)g_{hd}(s_1,s_2) \\ \times\, \gamma_{ehl}\gamma_{igf}\, g_{li}(s_2,\eta_4)\ , \end{aligned} \tag{7.2}$$

then putting this in the RG equation (6.16), one can straightforwardly integrate with respect to λ (since the only dependence on λ of the RHS is in a delta term) and find, with the obvious initial condition

$$G_{ab,\lambda=0}(k,\eta_1,\eta_2) = g_{ab}(\eta_1,\eta_2)\ , \tag{7.3}$$

the following result for the propagator with the correction at one loop

$$\begin{aligned} G_{ab,\lambda=\infty} =& G_{ab}(k,\eta_1,\eta_2) = g_{ab}(\eta_1,\eta_2) + 4P^{(0)}(k)\gamma_{ehl}(\mathbf{k},\mathbf{k},-2\mathbf{k})\gamma_{igf}(2\mathbf{k},-\mathbf{k},\mathbf{k}) \\ &\times \int \mathrm{d}s_1\,\mathrm{d}s_2\, e^{s_1+s_2} g_{hd}(s_1,0)g_{gc}(s_2,0)g_{ae}(s_1,\eta_1)g_{li}(s_1,s_2)g_{fb}(s_2,\eta_2)\ . \end{aligned}$$

This could have also been obtained by means of Feynman diagrams, but the power of RG flow is to probe, in certain approximations, the non linear regime in a non perturbative way; to show this, we will see now how it allows us to do a resummation of an infinite amount of diagrams by doing only a loop integration.

Consider the $k \gg \lambda$ limit. We can write in this limit the kernel (6.19) with its linear expression as (looking also at (B.9))

$$K_{gh,\lambda}(q, s_3, s_4) \simeq u_g u_h \theta(s_3)\theta(s_4) P^{(0)}(q)\delta_D(\lambda - q) \ ,$$

where we approximate the vertex (3.5) with

$$u_f \gamma_{efg}(-\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q}) \simeq \delta_{eg} \frac{k}{2q} \cos \frac{\mathbf{k}\cdot\mathbf{q}}{kq} \ .$$

Plugging the (7.2) within these approximations in the RG equation for the propagator, we obtain (after trivial integrations over time steps)

$$\partial_\lambda G_{ab,\lambda}(k, \eta_1, \eta_2) = -g_{ab}(\eta_1, \eta_2) \frac{k^2}{3} \frac{(e_1^\eta - e_2^\eta)^2}{2} \int \mathrm{d}^3 q \, \delta_D(\lambda - q) \frac{P^0}{q^2} \ ,$$

and integrating it and taking the $\lambda \to \infty$ limit, with the initial condition (7.3), we end up with

$$\boxed{G_{ab,\lambda}(k, \eta_1, \eta_2) = g_{ab}(\eta_1, \eta_2) \exp\left(-k^2 \sigma_v^2 \frac{(e_1^\eta - e_2^\eta)^2}{2} \right)} \ , \tag{7.4}$$

where we defined the velocity dispersion

$$\sigma_v^2 \equiv \frac{1}{3} \int \mathrm{d}^3 q \, \frac{P^0}{q^2} \ .$$

This last result (the only one regarding the RG approach we see in detail here) tells us that an intrinsic UV cut off appears: fluctuations with large momenta are exponentially damped, so that the RG evolution freezes out for $\lambda \gg e^{-\eta}/\sigma_v$. This result cannot be obtained in perturbation theory (that predicts a divergent behavior in the UV region), but can be obtain by means of resummation of an infinite amount of diagrams as showed in [1]; here instead we obtained it with basically only one loop integration. This, as anticipated, shows the power of this approach.

For the power spectrum, one can follow an analogous procedure. Again allowing for a running propagator and power spectrum but keeping only the vertices at tree level, one has that the equation represented in 6.2 reduces to

$$\begin{aligned}\partial_\lambda \Phi_{ab,\lambda}(k, \eta_1, \eta_2) =& 4e^{\eta_1+\eta_2} \int \mathrm{d}^3 q \, \delta_D(\lambda - q) P^{\mathrm{I}}_{dc;\lambda}(q, \eta_1, \eta_2) P_{fe;\lambda}(|\mathbf{q}-\mathbf{k}|, \eta_1, \eta_2) \\ & \times \gamma_{adf}(\mathbf{k}, -\mathbf{q}, -\mathbf{k}+\mathbf{q}) \gamma_{bce}(-\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q})\end{aligned}$$

that corresponds to considering only the first diagram on the RHS (due to the other vertices that vanish at tree level). Then one can proceed to compute in this and other approximations the running full power spectrum (see [5]).

8 Main Results

Thanks to the RG approach, one can try to solve the continuity and Euler equations non-perturbatively. In particular, differently from the perturbative approach, one can see that the intrinsic cut-off (7.4) for large wavelenghts appear, whereas in perturbation theory the n-th order correction diverges as k^{2n}.

The comparison between the N-body simulations and the results obtained with these RG techniques is quite remarkable [5], and we remember that we took the approximation of considering only the tree-level vertex.

In conclusion, with this RG approaches one can probe analytically also the non-linear range of the power spectrum in the BAO region, which is of great interest since with that one can constraint various models of dark energy. In particular, results with this approach outclass the ones given by linear and one-loop perturbation theory, which for regions down to $z = 0$ redshift badly fail.

An improvement to what we showed here can come from the extension to other cosmologies and from the analysis of the possible effects of initial non-Gaussianity on non-linear scales.

9 Possible extensions

Here we will briefly see how to extend our previous discussion to include other cosmologies than the Einstein-De Sitter ones and how to include non gaussian initial conditions.

Including other cosmologies. In cosmologies other than the De-Sitter ones, we can write the Poisson equation as

$$\nabla^2\phi = 4\pi G a^2 \rho_b \delta = \frac{3}{2}\Omega_m \mathcal{H}^2 \delta \ , \tag{9.1}$$

where Ω_m is the usual cosmological density parameter

$$\Omega_m \equiv \frac{\rho_b}{\rho_c} \ , \quad \rho_c \equiv \frac{3H^2}{8\pi G} \ ,$$

that is one for an Einstein-De Sitter universe. Then equation (3.3) becomes

$$\frac{\partial\theta}{\partial\tau} + \mathcal{H}\theta + \frac{3}{2}\Omega_m\mathcal{H}^2\delta + \int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\beta(\mathbf{q},\mathbf{p})\theta(\mathbf{q})\theta(\mathbf{p}) = 0 \ .$$

One can obtain again the (3.6) by redefining η as

$$\eta = \ln\frac{D^+}{D_i^+} \ , \tag{9.2}$$

where D^+ is the linear growing decay mode (notice that, since for Einstein-De Sitter, $D^+ \propto a$ (see appendix B.2), we recover the usual definition of η); also redefine[14]

$$\begin{pmatrix}\varphi_1(\mathbf{k},\eta)\\ \varphi_2(\mathbf{k},\eta)\end{pmatrix} \equiv e^{-\eta}\begin{pmatrix}\delta(\mathbf{k},\eta)\\ -\theta(\mathbf{k},\eta)/\mathcal{H}f\end{pmatrix} \ , \tag{9.3}$$

[14]Look at appendix B.2 for implications of this extended approach to linear theory.

where $f \equiv d\ln D^+/d\ln a$. With this new redefinitions, equation (3.6) remains the same if we change

$$\Omega = \begin{pmatrix} 1 & -1 \\ -3\Omega_m/2f^2 & 3\Omega_m/2f^2 \end{pmatrix} .$$

Now since the ratio Ω_m/f^2 is close to unity over the entire lifetime of the universe for ΛCDM cosmologies, all the work we have done here can be applied also to them[15].

Extension to non-gaussian case. Taking into account non-gaussian initial conditions means that the power spectrum is not anymore all we need to know, since quantities like the bispectrum, trispectrum etc. are non vanishing. In particular the $\mathcal{W}$ function we introduced in (4.4) assumes a general form of the type

$$\begin{aligned}\mathcal{W}[\boldsymbol{\varphi}(0),\{C\}] = \exp\big(&-\varphi_a(\mathbf{k},0)C_a(\mathbf{k}) - \varphi_a(\mathbf{k}_1,0)C_{ab}(\mathbf{k}_1,\mathbf{k}_2)\varphi_b(\mathbf{k}_2,0) \\ &+ \varphi_a(\mathbf{k}_1,0)\varphi_b(\mathbf{k}_2,0)\varphi_c(\mathbf{k}_3,0)C_{abc}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) + \ldots\big) ,\end{aligned}$$

where the C functions are related to the initial n-points correlation functions. These extra terms will produce other terms in the partition function Z that will give rise to new Feynman rules that will affect the running of the propagator and power spectrum we discussed in section 6. An example of this approach where only the power spectrum and the bispectrum are retained is given in [4], where there are showed the new Feynman rules involving the new fundamental quantity of interest, that is the bispectrum, and how this affects the discussion we have made here.

Another approach to treat non-gaussianity (always truncating at the bispectrum) is given in [7]. The idea is to use iteratively the equation (3.6) as follows:

$$\begin{aligned}\partial_\eta \langle\varphi_a\varphi_b\rangle &= -\Omega_{ac}\langle\varphi_c\varphi_b\rangle + e^\eta\gamma_{acd}\langle\varphi_c\varphi_d\varphi_b\rangle - \Omega_{bc}\langle\varphi_a\varphi_c\rangle + e^\eta\gamma_{bcd}\langle\varphi_a\varphi_c\varphi_d\rangle , \\ \partial_\eta \langle\varphi_a\varphi_b\varphi_c\rangle &= -\Omega_{ad}\langle\varphi_d\varphi_b\varphi_c\rangle + e^\eta\gamma_{ade}\langle\varphi_d\varphi_e\varphi_b\varphi_c\rangle - \Omega_{bd}\langle\varphi_a\varphi_d\varphi_c\rangle \\ &\quad + e^\eta\gamma_{bde}\langle\varphi_a\varphi_d\varphi_e\varphi_c\rangle - \Omega_{cd}\langle\varphi_a\varphi_b\varphi_d\rangle + e^\eta\gamma_{cde}\langle\varphi_a\varphi_b\varphi_d\varphi_e\rangle , \\ \partial_\eta \langle\varphi_a\varphi_b\varphi_c\varphi_d\rangle &= \ldots .\end{aligned}$$

One immediately sees that the equation for the n-point correlation function contains the $n+1$-point one, so we have to truncate the infinite tower of equations by considering for example a vanishing trispectrum; recall that the definitions of bispectrum and trispectrum are respectively B_{abc} and Q_{abcd}, with

$$\langle\varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{q},\eta)\varphi_c(\mathbf{p},\eta)\rangle \equiv \delta_D(\mathbf{k}+\mathbf{q}+\mathbf{p})B_{abc}(\mathbf{k},\mathbf{p},\mathbf{q},\eta) ,$$

$$\begin{aligned}\langle\varphi_a(\mathbf{k},\eta)\varphi_b(\mathbf{p},\eta)\varphi_c(\mathbf{q},\eta)\varphi_d(\mathbf{r},\eta)\rangle &\equiv \delta_D(\mathbf{k}+\mathbf{q})\delta_D(\mathbf{p}+\mathbf{r})P_{ab}(\mathbf{k},\eta)P_{cd}(\mathbf{p},\eta) \\ &+\delta_D(\mathbf{k}+\mathbf{p})\delta_D(\mathbf{q}+\mathbf{r})P_{ac}(\mathbf{k},\eta)P_{bd}(\mathbf{q},\eta) + \delta_D(\mathbf{k}+\mathbf{r})\delta_D(\mathbf{q}+\mathbf{p})P_{ad}(\mathbf{q},\eta)P_{bc}(\mathbf{q},\eta) \\ &+\delta_D(\mathbf{k}+\mathbf{p}+\mathbf{q}+\mathbf{r})Q_{abcd}(\mathbf{k},\mathbf{q},\mathbf{p},\mathbf{r},\eta) .\end{aligned}$$

[15] This approximation does not work when we have that Ω_m is mode-dependent, since then the redefinition of η is ill-defined. This could be relevant for cosmologies accounting for massive neutrinos [2].

The equations we are interested to solve, imposing $Q_{abcd} = 0$, are then

$$\begin{aligned}\partial_\eta P_{ab}(\mathbf{k}) = -\Omega_{ac}(\mathbf{k})P_{cd}(\mathbf{k}) - \Omega_{bc}(\mathbf{k})P_{ac}(\mathbf{k}) + e^\eta \int \mathrm{d}^3q\, (\gamma_{acd}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) \\ \times B_{bcd}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) + (\gamma_{bcd}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k})B_{acd}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}))\end{aligned}$$

$$\begin{aligned}\partial_\eta B_{abc}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) &= -\Omega_{ad}(\mathbf{k})B_{dcb}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) - \Omega_{bd}(-\mathbf{q})B_{adc}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) \\ &-\Omega_{cd}(\mathbf{q}-\mathbf{k})B_{adb}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) + 2e^\eta \big(\gamma_{ade}(\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k})P_{db}(\mathbf{q})P_{ec}(\mathbf{k}-\mathbf{q}) \\ &+\gamma_{bde}(-\mathbf{q}, \mathbf{q}-\mathbf{k},\mathbf{k})P_{dc}(\mathbf{k}-\mathbf{q})P_{ea}(\mathbf{k}) + \gamma_{cde}(\mathbf{q}-\mathbf{k}, \mathbf{k}, -\mathbf{q})P_{da}(\mathbf{k})P_{ec}(\mathbf{q})\big)\end{aligned}$$

Note that the above equations are similar to RG equations where the flow parameter is time η instead of the filter λ, so that this approach can be legitimately called time renormalization group approach. Then one can continue by studying the corrections that this approach has on the non linear full propagator and power spectrum and bispectrum [7].

As a final comment, we must remark that the gaussian initial condition approximation is a very good one for our universe, but still non-gaussianity can come into play in the high-mass end of the power spectrum, i.e. on the scale of galaxy clusters, and also it can alter the clustering of dark matter halos inducing a scale-dependent bias on large scales (see [3]); so including non-gaussianity in RG approach is something of interest.

A Gaussian integrals

We show here how to solve what is basically the only functional integral we can exactly solve, the gaussian integral. Let's start with an example in a finite number of dimensions, define

$$Z[\mathbf{b}] = \mathcal{N} \int \left(\prod_{i=1}^{n} \mathrm{d}x_i \right) \exp\left(-\frac{1}{2} \sum_{ij} k_{ij} x_i x_j + i \sum_j b_j x_j \right) ,$$

where $i \sum_j b_j x_j$ is the source term and $\mathcal{N}$ is a normalization factor; then we make the change of variables

$$\mathbf{y} = ik^{-1}\mathbf{b},\ \mathbf{x} = \mathbf{y} + \mathbf{z} \ ;$$

with this the exponent becomes[16]

$$-\frac{1}{2}(x, k, x) + i(b, x) = -\frac{1}{2}(z, k, z) - \frac{1}{2}(b, k^{-1}, b) \ ,$$

so

$$Z[b] = \mathcal{N} \exp\left(-\frac{1}{2}(b, k^{-1}, b) \right) \int \left(\prod_{i=1}^{n} \mathrm{d}z_i \right) \exp\left(-\frac{1}{2}(z, k, z) \right) ,$$

now diagonalize k as $\sum k_{ij} z_i z_j = \sum \lambda_i \xi_i^2$, where λ_i are the eigenvalues of k. We thus end up with a standard gaussian integral, the final result is

$$Z[b] = \mathcal{N} (2\pi)^{n/2} (\det k)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(b, k^{-1}, b) \right) \ ;$$

imposing $Z[0] = 1$, we have

$$Z[b] = \exp\left(-\frac{1}{2}(b, k^{-1}, b) \right) \ .$$

All this procedure can be extended to the functional case

$$Z[J] = \mathcal{N} \int \mathcal{D}q \exp\left(-\frac{1}{2}(q, k, q) + i(q, J) \right) ,$$

yielding (with the proper normalization)

$$Z[J] = \exp\left(-\frac{1}{2}(J, k^{-1}, J) \right) \ . \tag{A.1}$$

[16]We are using here the following notation: $(x, k, x) \equiv \sum k_{ij} x_i x_j$.

B Some derivation details

B.1 More on derivation of (3.2) and (3.3)

Here we will discuss some details regarding the derivation of (3.2) and (3.3).

For equation (3.2), we should see how to arrange the Fourier transform of $\boldsymbol{\nabla}\cdot(\delta\mathbf{v})$; writing the Fourier transform with the notation $\mathcal{F}(f)$, we have (in the following we will omit the time dependence of the various quantities):

$$\mathcal{F}(\boldsymbol{\nabla}\cdot(\delta(\mathbf{k})\mathbf{v}(\mathbf{k}))) = i\mathbf{k}\cdot\mathcal{F}(\delta(\mathbf{k})\mathbf{v}(\mathbf{k})) = i\mathbf{k}\cdot(\mathcal{F}(\delta(\mathbf{k}))*\mathcal{F}(\mathbf{v}(\mathbf{k})))\ ;$$

doing explicitly the convolution and setting $\mathbf{k}=\mathbf{q}+\mathbf{p}$, we can rewrite the previous as

$$\mathcal{F}(\boldsymbol{\nabla}\cdot(\delta(\mathbf{k})\mathbf{v}(\mathbf{k}))) = i\int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\,(\mathbf{p}+\mathbf{q})\cdot\mathbf{v}(\mathbf{q})\delta(\mathbf{p})\ ;$$

our goal is to express this in terms of $\theta(\mathbf{k}) = i\mathbf{k}\cdot\mathbf{v}(\mathbf{k}) = ikv(k)$, where the last step comes from the hypothesis that $\mathbf{v}$ is irrotational and thus $\mathbf{k}\parallel\mathbf{v}(\mathbf{k})$. We need to rewrite only $\mathbf{q}\cdot\mathbf{v}(\mathbf{p})$ as

$$\mathbf{q}\cdot\mathbf{v}(\mathbf{p}) = qv\cos\theta = \frac{pq\cos\theta}{p^2}pv(\mathbf{p}) = -i\frac{\mathbf{p}\cdot\mathbf{q}}{p^2}\theta(\mathbf{p})\ , \tag{B.1}$$

where on the last step we exploited $\mathbf{p}\parallel\mathbf{v}$. Then the equation (3.2) easily follows.

Regarding (3.3), the third term comes from (remember that we must take before the divergence of Euler equation in order to express all with respect to $\theta(\mathbf{k})$)

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\phi = \nabla^2\phi = 4\pi G a^2\rho_b\delta = \frac{3}{2}\mathcal{H}^2\delta\ ,$$

where on the last step we exploited the first Friedmann equation for an Einstein-De Sitter universe with no cosmological constant, $a^2H^2 = \mathcal{H}^2 = 8\pi G a^2\rho_b/3$. Instead for the Fourier transform of the term $\boldsymbol{\nabla}\cdot((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v})$, with analogous procedure as before we arrive at

$$\mathcal{F}(\boldsymbol{\nabla}\cdot((\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v})) = -\int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})((\mathbf{p}+\mathbf{q})\cdot\mathbf{v}(\mathbf{q}))(\mathbf{q}\cdot\mathbf{v}(\mathbf{p}))\ ,$$

then with (B.1) we find

$$\int \mathrm{d}^3\mathbf{q}\,\mathrm{d}^3\mathbf{p}\,\delta_D(\mathbf{k}-\mathbf{q}-\mathbf{p})\,\frac{\mathbf{p}+\mathbf{q}}{p^2}\cdot\mathbf{q}\ \theta(\mathbf{q})\frac{\mathbf{p}\cdot\mathbf{q}}{p^2}\theta(\mathbf{p})\ ,$$

and then one easily arrives at (3.3)

B.2 Recovering linearized Cosmological perturbation results in Newtonian limit

Setting $\alpha\ ,\beta=0$ in (3.2) and (3.3) we arrive at

$$\frac{\partial\delta(\mathbf{k},\tau)}{\partial\tau} + \theta(\mathbf{k},\tau) = 0\ ,$$
$$\frac{\partial\theta(\mathbf{k},\tau)}{\partial\tau} + \mathcal{H}\theta(\mathbf{k},\tau) + \frac{3}{2}\mathcal{H}^2\delta(\mathbf{k},\tau) = 0\ .$$

Differentiate again the first with respect to τ and we find (we denote with $\dot{} \equiv \mathrm{d}/\mathrm{d}\tau$)

$$\ddot{\delta} + \dot{\theta} = \ddot{\delta} - \mathcal{H}\theta - \frac{3}{2}\mathcal{H}^2\delta = \ddot{\delta} + \mathcal{H}\dot{\delta} - \frac{3}{2}\mathcal{H}^2\delta = 0 \; . \tag{B.2}$$

In a matter dominated universe we have $a(t) = kt^{2/3}$, with k a proportionality constant, and thus $H = 2/3t$; so, since the relation between conformal time and cosmic time is

$$\tau(t) = \int_0^t \frac{dt'}{a(t')} = \frac{3}{k} t^{\frac{1}{3}} \; ,$$

we have

$$\mathcal{H} = aH = kt^{\frac{2}{3}} \frac{2}{3t} = \frac{2}{\tau} \; , \tag{B.3}$$

so (B.2) becomes

$$\ddot{\delta} + \frac{2}{\tau}\dot{\delta} - \frac{6}{\tau^2}\delta = 0 \; ;$$

assuming $\delta \propto \tau^\alpha$, substituting in the previous and then solving the resulting equation $\alpha^2 + \alpha - 6 = 0$, one finds $\alpha = 2 \, , -3$ and since $a(\tau) \propto \tau^2$, one recovers the known result

$$\delta(\mathbf{k}, \tau) = \delta(\mathbf{k}, \tau_i) \left(\frac{a(\tau)}{a(\tau_i)} \right)^m \; ,$$

with τ_i a reference time, $m = 1$ the growing mode and $m = -3/2$ the decaying mode. The equation for θ is then straightforward:

$$\theta(\mathbf{k}, \tau) = -\frac{\partial \delta(\mathbf{k}, \tau)}{\partial \tau} = -m\mathcal{H}\delta(\mathbf{k}, \tau) \; .$$

One then can see that the doublet $\boldsymbol{\varphi}$ from (3.4) becomes

$$\begin{pmatrix} \varphi_1(\mathbf{k}, \eta) \\ \varphi_2(\mathbf{k}, \eta) \end{pmatrix} = e^{-\eta} \delta(\mathbf{k}, \tau_i) \left(\frac{a(\tau)}{a(\tau_i)} \right)^m \begin{pmatrix} 1 \\ m \end{pmatrix} \; ,$$

where one sees that, exploiting the definition of η, the growing mode corresponds to $\boldsymbol{\varphi} = \text{const}$, and in general one can select the growing or the decay mode by selecting initial fields proportional to

$$\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \; , \quad \mathbf{v} = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix} \; ,$$

respectively. Note also that the previous implies that indeed the two components of the initial field are proportional to the same random field, as claimed in (4.9).

In the case of the extension to other cosmologies (section 9), the above equations simply become

$$\delta(\mathbf{k}, \tau) = \delta(\mathbf{k}, \tau_i) \frac{D^\pm(\tau)}{D^\pm(\tau_i)} \; , \quad \theta(\mathbf{k}, \tau) = -f(\tau)\mathcal{H}(\tau)\delta(\mathbf{k}, \tau_i) \frac{D^\pm(\tau)}{D^\pm(\tau_i)} \; .$$

where $D^\pm$ is the linear growth function (growing or decaying mode respectively); so one sees that with the redefinitions of η and the fields described in section 9, one has that the linear growing mode corresponds again to $\boldsymbol{\varphi} = \text{const}$.

B.3 Deriving (3.6)

In order to relate (3.6) with (3.3) and (3.2), we have to mainly focus on the left hand side of (3.6) (since then understanding how to match the right hand side is straightforward), so as a first thing we can see that the following holds:

$$\frac{\partial}{\partial\eta} = \frac{\partial\tau}{\partial a}\frac{\partial a}{\partial\eta}\frac{\partial}{\partial\tau} = \frac{1}{2\tau k}k\tau^2\frac{\partial}{\partial\tau} = \frac{\tau}{2}\frac{\partial}{\partial\tau} \ ,$$

where we used $a(\tau) = k\tau^2$ with k a proportionality constant. Then using also (B.3), the left hand side of (3.6) with the index $a = 1$ becomes

$$\mathrm{LHS} = e^{-\eta}\Big(-\delta + \frac{\tau}{2}\dot{\delta} + \delta + \frac{\tau}{2}\theta\Big) \ ,$$

that is indeed what we were looking for if we multiply it by $2e^{\eta}/\tau$. The case $a = 2$ is completely analogous.

B.4 Deriving some path integral formulas

Here we will see some details about the derivations of (4.3), (4.6), and (6.1). For the first, let's rewrite (4.2) as a sum over all the paths from $\boldsymbol{\varphi}(0)$ to $\boldsymbol{\varphi}(\eta_f)$:

$$P[\boldsymbol{\varphi}(\eta_f), \boldsymbol{\varphi}(0)] = \int \mathcal{D}''\boldsymbol{\varphi}\,\delta_D[\boldsymbol{\varphi}(\eta_f) - \boldsymbol{\varphi}_{cl}[\eta_f, \boldsymbol{\varphi}(0)]] \ ,$$

that is the expression for the path integral in classical mechanics (see also [6]).

Now to simplify the notation we express equation (3.6) schematically as $\hat{O}\varphi - \lambda\varphi = 0$, with λ representing the interaction term. We can then use the following property of the delta function: given the function $f(\varphi)$, if f has a zero in φ_{cl}, then we can formally write

$$\delta_D(\varphi - \varphi_{cl}) = \delta_D(f(\varphi))|f'(\varphi_{cl})| \ ; \tag{B.4}$$

using $f(\varphi) = \hat{O}\varphi - \lambda\varphi$ and collecting $|f'(\varphi_{cl})|$ in an overall constant factor, we have (using the integral representation of the delta)

$$\delta_D(\varphi - \varphi_{cl}) = \mathcal{N}\int \mathrm{d}\chi\, e^{i\int \mathrm{d}\eta\chi f(\varphi)} \ ,$$

with χ as the auxiliary field. It is not difficult to recognize the action (4.1) in the argument of the exponential, and this allows us to arrive at (without schematic notation)

$$P[\boldsymbol{\varphi}(\eta_f), \boldsymbol{\varphi}(0)] = \mathcal{N}\int \mathcal{D}''\boldsymbol{\varphi}\,\mathcal{D}\boldsymbol{\chi}\,e^{iS} \ .$$

Now regarding (4.6), we can reverse the previous reasoning: the piece affected by the integration over $\boldsymbol{\chi}$ becomes

$$\int \mathcal{D}\boldsymbol{\chi} e^{i\chi_a(f_a(\boldsymbol{\varphi})+K_a)} = \delta_D(f(\boldsymbol{\varphi}) + \mathbf{K}) \ ,$$

so that the part affected by $\int \mathcal{D}\boldsymbol{\varphi}(\eta_f)\, \mathcal{D}''\boldsymbol{\varphi}$ becomes

$$\int \mathcal{D}\boldsymbol{\varphi}(\eta_f)\, \mathcal{D}''\boldsymbol{\varphi}\, \delta_D(f(\boldsymbol{\varphi}) + \mathbf{K}) \exp\left(i \int \mathrm{d}\eta\, J_a \varphi_a\right) = \exp\left(i \int \mathrm{d}\eta\, J_a \tilde{\varphi_a}\right) ,$$

where $\tilde{\varphi_a}$, that comes from the Dirac delta integration, is the solution of the classical equation of motion with source K_a:

$$(\delta_{ab}\partial_\eta + \Omega_{ab})\tilde{\varphi_b}(\eta) = -K_a(\eta) , \tag{B.5}$$

which is straightforwardly given by

$$\tilde{\varphi_a}(\eta_1) = \varphi_a^0(\eta_1) - \int \mathrm{d}\eta_2\, g_{ab}(\eta_1, \eta_2) K_b(\eta_2) ,$$

with $g_{ab}(\eta_1, \eta_2)$ the Green function of the operator $\delta_{ab}\partial_\eta + \Omega_{ab}$ and $\varphi_a^0(\eta_1)$ the homogeneous solution of (B.5), i.e. the sourceless zeroth order solution, that is correlated to the initial configuration through[17]

$$\varphi_a(\eta) = g_{ab}(\eta, 0)\varphi_b(0) . \tag{B.6}$$

To see explicitly why the previous holds, use the Laplace transform on equation (3.6) with respect to the variable η and obtain (dropping the interaction term γ_{abc} that is not important now)[18]

$$\omega\delta_{ab} + \Omega_{ab}\varphi_b(\mathbf{k}, \omega) = \varphi_a(\mathbf{k}, \eta = 0) ;$$

now multiply the previous by the inverse of $\omega\delta_{ab} + \Omega_{ab} \equiv \sigma_{ab}^{-1}$, that is

$$\begin{pmatrix} 1+\omega & -1 \\ -3/2 & 3/2+\omega \end{pmatrix}^{-1} = \frac{1}{\omega(2\omega+5)} \begin{pmatrix} 2\omega+3 & 2 \\ 3 & 2\omega+2 \end{pmatrix} , \tag{B.7}$$

and obtain, doing the inverse Laplace transform, indeed the (B.6), with (taking $c > 0$ to take the standard retarded propagator)

$$g_{ab}(\eta, 0) = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}\omega_1}{2\pi i} \sigma_{ab}(\omega) e^{\omega\eta} .$$

[17]We drop the index zero since all our discussion here is at linear level.

[18]The Laplace transform is defined as $\mathcal{L}[f(\eta)](\omega) = \int_0^\infty e^{-\omega\eta} f(\eta)\, \mathrm{d}\eta$, where ω is a complex quantity. Regarding the properties we need, we see that for a derivative,

$$\mathcal{L}[f'(\eta)](\omega) = e^{-\omega\eta} f(\eta)\Big|_0^\infty + \omega \int_0^\infty \mathrm{d}\eta\, e^{-\omega\eta} f(\eta) = -f(0) + \omega\mathcal{L}[f(\eta)] .$$

Instead its inverse is given by the Mellin's inverse formula, that is

$$\mathcal{L}^{-1}[f(\eta)] = \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{\mathrm{d}\omega}{2\pi i} \mathcal{L}[f](\omega_1) e^{\omega\eta} ,$$

where c is a real number so that the contour path of integration is in the region of convergence of $\mathcal{L}[f]$.

Using residue theorem results, we can compute explicitly the previous; for $\eta < 0$, we have to close the circuit where we integrate on the complex plane on the right (since that, in order to have a negative exponential we must go to values with $\mathrm{Re}\,\omega > 0$) and since the only poles of the integrand correspond to $\omega = 0, -5/2$ we have that g_{ab} is zero, thus respecting causality; for $\eta > 0$, the circuit must be closed on the left, so it contains the two poles and thus the complete g_{ab} is (calling A_{ab} the matrix (B.7) without the factors at the denominator)

$$g_{ab}(\eta, 0) = \left(\operatorname*{Res}_{\omega=0} e^{\omega\eta} \frac{A_{ab}(\omega)}{2\omega+5} + \operatorname*{Res}_{\omega=-5/2} e^{\omega\eta} \frac{A_{ab}(\omega)}{\omega} \right) \theta(\eta) = \left(B_{ab} + C_{ab} e^{-\frac{5}{2}\eta} \right) \theta(\eta) \ , \quad \text{(B.8)}$$

with

$$B = \frac{1}{5} \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \ , \quad C = \frac{1}{5} \begin{pmatrix} 2 & -2 \\ -3 & 3 \end{pmatrix} \ .$$

The (B.8) can be easily extended to a generic initial time, yielding

$$g_{ab}(\eta_1, \eta_2) = \left(B_{ab} + C_{ab} e^{-\frac{5}{2}(\eta_1 - \eta_2)} \right) \theta(\eta_1 - \eta_2) \ . \tag{B.9}$$

g_{ab} is also the Green function of the operator $\delta_{ab}\partial_\eta + \Omega_{ab}$, i.e. the solution to

$$(\delta_{ab}\partial_{\eta_1} + \Omega_{ab}) g_{bc}(\eta_1, \eta_2) = \delta_{ac}\delta_D(\eta_1 - \eta_2) \ , \tag{B.10}$$

in fact

$$(\delta_{ab}\partial_{\eta_1} + \Omega_{ab}) \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}\omega_1}{2\pi i} \sigma_{bd}(\omega) e^{\omega(\eta_1-\eta_2)} = \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}\omega_1}{2\pi i} e^{\omega(\eta_1-\eta_2)} \overbrace{(\omega\delta_{ab} + \Omega_{ab})(\omega\delta_{bd} + \Omega_{bd})^{-1}}^{\delta_{ad}} \ ,$$

and then one can easily recognize the integral representation of the Dirac delta by a change of variable in the complex plane like $\omega = i\omega'$.

So we arrive at

$$\begin{aligned} Z_0[\mathbf{J}, \mathbf{K}; P^0] = & \exp\left(-i \int \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, J_a(\mathbf{k}, \eta_1) g_{ab}(\eta_1, \eta_2) K_b(-\mathbf{k}, \eta_2) \right) \\ & \times \int \mathcal{D}\boldsymbol{\varphi}(0) \mathcal{W}[\boldsymbol{\varphi}(0), P^0] \exp\left(i \int \mathrm{d}\eta\, J_a(\mathbf{k}, \eta) g_{ab}(\eta, 0) \varphi_b(\mathbf{k}, 0) \right) \ , \end{aligned}$$

and the integral in the last line is a simple gaussian integral, that can be solved with (A.1) using as the source term $J_a(\mathbf{k}, \eta) g_{ab}(\eta, 0)$, yielding indeed (4.6).

For (6.1), we have only to show that Z_0 from (4.6) can be written as

$$\begin{aligned} Z_0[\mathbf{J}, \mathbf{K}; P^0] = \int \mathcal{D}\boldsymbol{\varphi}\, \mathcal{D}\boldsymbol{\chi}\, \exp\Bigg(& -\frac{1}{2}\chi_a(0) P^0_{ab} \chi_b(0) + i \int d\eta_1\, \mathrm{d}\eta_2\, \chi_a g^{-1}_{ab} \varphi_b \\ & + i \int \mathrm{d}\eta\, (J_a\varphi_a + K_a\chi_a) \Bigg) \ , \end{aligned}$$

since then the interaction term can be included in the same way as in (4.10). φ enters only linearly in the exponent, so again use the delta integral representation to write

$$\int \mathcal{D}\boldsymbol{\varphi} \exp\left(i \int \mathrm{d}\eta \left(\int \mathrm{d}\eta'\, \chi_a g^{-1}_{ab} + J_b \right) \varphi_b \right) = \delta_D \left(\int \mathrm{d}\eta\, \chi_a g^{-1}_{ab} + J_b \right) \ ,$$

and from the integration over $\boldsymbol{\chi}$, exploiting (B.4) to write $\delta_D(\int \mathrm{d}\eta\, \chi_a g^{-1}_{ab} + J_b) \propto \delta_D(\boldsymbol{\chi} - \boldsymbol{\chi}_0)$ with

$$(\chi_0)_a(\eta_2) = -\int \mathrm{d}\eta_1\, J_a(\eta_1) g_{ab}(\eta_1, \eta_2)\ , \tag{B.11}$$

and reabsorbing the proportionality term in an unimportant overall constant, we indeed obtain (4.6).

B.5 Section 6 relations

Here we will try to explain the details of the derivations of some of the equations of section 6.

An important relation we will frequently use is the functional derivative chain rule, that is

$$\frac{\delta}{\delta J(z)} = \int \mathrm{d}y\, \frac{\delta\phi(y)}{\delta J(z)} \frac{\delta}{\delta\phi(y)}\ ; \tag{B.12}$$

with this we can see that (here we will use the shortcut notation ϕ_a to denote generically a $\varphi^{(\mathrm{cl})}_b$, $\chi^{(\mathrm{cl})}_c$ field, so a goes from 1 to 4, and same for the sources regrouped in J_a)

$$\begin{aligned}\frac{\delta\Gamma[\boldsymbol{\phi}]}{\delta\phi_a(\eta, \mathbf{k})} &= \int \mathrm{d}\eta'\, \mathrm{d}^3k'\, \frac{\delta J_a(\eta', \mathbf{k}')}{\delta\phi_a(\eta, \mathbf{k})} \overbrace{\frac{\delta W[\mathbf{J}]}{\delta J_a(\eta', \mathbf{k}')}}^{\phi_a(\eta', \mathbf{k}')} - \int \mathrm{d}\eta'\, \mathrm{d}^3k'\, \frac{\delta J_a(\eta', \mathbf{k}')}{\delta\phi_a(\eta, \mathbf{k})} \phi_a(\eta', \mathbf{k}') \\ &- \int \mathrm{d}\eta'\, \mathrm{d}^3k'\, J_a(\eta', \mathbf{k}') \delta_D(\eta - \eta') \delta_D(\mathbf{k} - \mathbf{k}') = -J_a(\eta, \mathbf{k})\ ,\end{aligned}$$

where the sum over a is not implied and we used (6.4) (note that from that you can see $J_a = J_a[\boldsymbol{\phi}]$). Now we can take the second derivative of Γ as

$$\left(\frac{\delta^2\Gamma[\boldsymbol{\phi}]}{\delta\phi(\eta, \mathbf{k})\delta\phi(\eta', \mathbf{k}')}\right)_{ab} = -\frac{\delta J_a(\eta, \mathbf{k})}{\delta\phi_b(\eta', \mathbf{k}')} = -\left(\frac{\delta\phi_b(\eta', \mathbf{k}')}{\delta J_a(\eta, \mathbf{k})}\right)^{-1} = -\left(\frac{\delta^2 W[\mathbf{J}]}{\delta J(\eta, \mathbf{k})\delta J(\eta', \mathbf{k}')}\right)^{-1}_{ab}, \tag{B.13}$$

so one sees that the 4×4 matrices $\Gamma^{(2)}$ and $W^{(2)}$ (whose definitions can be guessed from (6.10) and (6.17) respectively) are basically one the inverse of the other. At the linear level (using (4.6), (6.3) and (6.5) and restoring our usual notation),

$$\Gamma_0 = \int \mathrm{d}\eta\, \mathrm{d}\eta'\, \mathrm{d}^3k \left(\frac{i}{2} J_a P^L_{ab} J_b - J_a g_{ab} K_b\right) - \int \mathrm{d}\eta\, \mathrm{d}^3k \left(J_a \varphi^{(\mathrm{cl})}_a + K_a \chi^{(\mathrm{cl})}_a\right)\ ;$$

since[19]

$$\begin{aligned}\chi^{(\mathrm{cl})}_a(\eta, \mathbf{k}) &= \frac{\delta W}{\delta K_a(\eta, \mathbf{k})} = -\int \mathrm{d}\eta'\, J_b(\eta', -\mathbf{k}) g_{ba}(\eta', \eta)\ , \\ \varphi^{(\mathrm{cl})}_a(\eta, \mathbf{k}) &= \frac{\delta W}{\delta J_a(\eta, \mathbf{k})} = \int \mathrm{d}\eta' \left(i J_b(-\mathbf{k}, \eta') P^L_{ab}(k, \eta, \eta') - g_{ab}(\eta, \eta') K_b(-\mathbf{k}, \eta')\right)\ ,\end{aligned} \tag{B.14}$$

[19]Notice that we obtain the same result as in (B.11) for the $\boldsymbol{\chi}$ field.

so

$$\int \mathrm{d}\eta\, \mathrm{d}^3k \left(J_a\varphi_a^{(\mathrm{cl})} + K_a\chi_a^{(\mathrm{cl})}\right) = \int \mathrm{d}\eta\, \mathrm{d}\eta'\, \mathrm{d}^3k \left(iJ_aP^L_{ab}J_b - 2J_ag_{ab}K_b\right) \implies \Gamma_0 = -W_0 \ ,$$

and expressing all with respect to fields,

$$\Gamma_0 = \int \mathrm{d}\eta_1\, \mathrm{d}\eta_2\, \mathrm{d}^3k \left(\frac{i}{2}\chi_a^{(\mathrm{cl})}P^0_{ab}\chi_b^{(\mathrm{cl})}\delta_D(\eta_1)\delta_D(\eta_2) + \chi_a^{(\mathrm{cl})}g^{-1}_{ab}\varphi_b^{(\mathrm{cl})}\right) \ .$$

With this last equations one can now understand the split between linear and non linear part made in (6.9).

To derive (6.11)-(6.14), it suffices to compare

$$\begin{pmatrix} \mathbf{0} & g_{ab} - \Sigma_{\varphi_a\chi_b} \\ g_{ab} - \Sigma_{\varphi_a\chi_b} & P^0_{ab} - i\Phi_{ab} \end{pmatrix}^{-1} = -\begin{pmatrix} P_{ab} & G_{ab} \\ G_{ab} & \mathbf{0} \end{pmatrix} \ ,$$

where the zero block diagonal matrices are there due to (6.6).

For (6.15), we used $\partial_\lambda W_\lambda = -i(\partial_\lambda Z)/Z$ and rewrote it exploiting

$$\frac{\delta W}{\delta K_a} = -\frac{i}{Z}\frac{\delta Z}{\delta K_a} \implies \frac{\delta^2 W}{\delta K_a \delta K_b} = \underbrace{\frac{1}{Z^2}\frac{\delta Z}{\delta K_a}\frac{\delta Z}{\delta K_b}}_{-i\chi_a^{(\mathrm{cl})}\chi_b^{(\mathrm{cl})}} + \frac{i}{Z}\left(\int \mathcal{D}\boldsymbol{\varphi}\mathcal{D}\boldsymbol{\chi}\, \chi_a\chi_b \exp(\cdots)\right) \ .$$

Regarding the (6.18) using (B.12), we can see that for example (with a "reduced notation" just for simplicity)

$$\begin{aligned} \frac{\delta^3 W}{\delta J(\eta_1)\delta K(\eta_2)\delta K(\eta_3)} &= -\int \mathrm{d}s_1\, \frac{\delta\varphi^{(\mathrm{cl})}(s_1)}{\delta K(\eta_2)}\frac{\delta}{\delta\varphi^{(\mathrm{cl})}(s_1)}\frac{\delta^2 W}{\delta J(\eta_1)\delta K(\eta_2)} \\ = \int \mathrm{d}s_1\, G(s_1,\eta_2)\frac{\delta}{\delta\varphi^{(\mathrm{cl})}(s_1)}\left(\frac{\delta^2\Gamma}{\delta\varphi^{(\mathrm{cl})}(\eta_1)\delta\chi^{(\mathrm{cl})}(\eta_3)}\right)^{-1} &= \int \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, G(s_1,\eta_1)G(s_2,\eta_2) \\ \times\, G(s_3,\eta_3)\frac{\delta^3\Gamma}{\delta\varphi^{(\mathrm{cl})}(\eta_1)\delta\varphi^{(\mathrm{cl})}(\eta_2)\delta\chi^{(\mathrm{cl})}(\eta_3)} \ , \end{aligned}$$

where we used $\delta\varphi^{(\mathrm{cl})}/\delta K = \delta^2 W/\delta J\delta K$ (using (B.13)) that is the propagator (see (6.8)). An analogous reasoning is made for the full $W^{(4)}$ (keeping in mind that at the end the functional derivative is computed at $\mathbf{J},\mathbf{K} = 0$), yielding indeed (6.18).

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