# Second Homework, Cart Pendulum Model Analysis and Control

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## Abstract

The aim of this paper is to analyze the dynamics of the cart pendulum system, study the behavior in the frequency and time domains, understanding the importance of the hidden dynamics and the limitations that arise when we deal with NMP unstable systems.

**Keywords**: Cart Pendulum, Mixsyn, Hinfsyn, MIMO, LQR, Limitations, Control System, Non-Minimum Phase, Stabilization, Optimal Control, Full State Feedback, Output Feedback

## 1. Cart Pendulum Model

The linearized dynamics of the cart and inverted pendulum model is a key example in the understanding of the non-minimum phase systems, since, as we can see from the figure, the control input of the system acts behind the center of gravity of the system. Other real examples of the non-minimum phase dynamics could be the airplane linearized dynamics, since in most common aircrafts the inputs acts below the center of gravity of the plane; another example could be a dinghy with an outboard motor. In this case seems logical to think that when we rotate the helm to go to the left, before we start going in that direction, the dinghy goes a bit to the right, causing the equivalent of an undershoot. This dynamic raise because we are try to changing the orientation of the dinghy's speed without using a torque, instead we use a single force behind the center of mass. From the control system point of view, the presence of Right Half Plane (R.H.P) zeroes and the presence of unstable poles, gives birth to bandwidth limitations for the senisitivity and complementary sensitivity functions.

In our approach we will first consider the nonlinear dynamic of the system, then we will linearize that around an unstable equilibrium point, considering the inverted pendulum in a vertical position. After that we will apply two approximations to simplify the problem, the first is about the neglection of the frictions, the other one is considering small angle approximation, and ignoring quadratic terms in the angular speed and position. Doing this we obtain a linear time invariant state space model, and we will build controller on the basis of this linaear model.

Another key point and big limatation in this analysis is that we will try to construct a controller that in reality should work on the real nonlinear plant, but we are considering the linearized dynamic, and this is a big limitation; we could maybe obtain better results by considering a nonlinear control strategy.

In the following figure the system is described as long as its parameters.

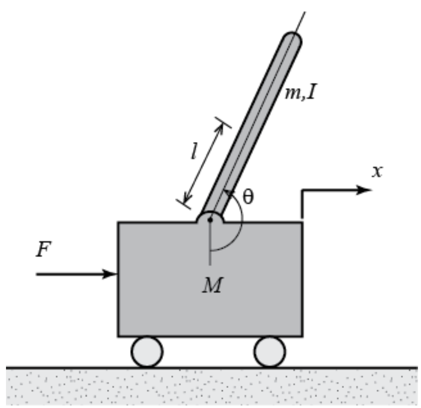


*Fig. 1 Inverted Pendulum on a Cart*

The state space model obtained is the following and it is dependent on the parameters chosen for the model:

$$\begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m^2 l^2 g}{D} & 0 & 0 \\ 0 & \frac{M_t m g l}{D} & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{D} \\ -\frac{ml}{D} \end{bmatrix} u \qquad (1)$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} x \qquad (2)$$

Where $x$ is the state variables vector, $y$ is the output vector that define a M.I.M.O. system. The disturbance matrix is not considered in this case of study.

For this system we can extract the input output transfer functions by doing a Laplace Transform and condisdering things from input output point of view. This result a vector of transfer functions.

In the Fig.2 the pole zero maps of the transfer functions are shown.

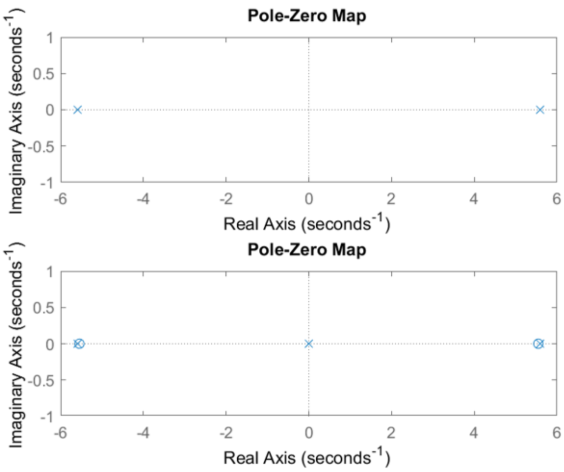


*Fig.2 Pole Zero map of the two Transfer Functions*

## 1. Limitations

Having a SISO plant, we can define the sensitivity and complementary sensitivity functions and the relation between them. Their sum at any frequency must be equal to one, and this imply that their magnitude can differ at most by 1 at any frequency.

Ideally, we would like to obtain S and T magnitudes both small to reject disturbances and noise, benefitting from the feedback action.

Moreover, we have found the **interpolations constraints** results, shown below:

If p is a RHP pole of the plant P(s) then:

$$\boldsymbol{T(p) = 1}$$

$$\boldsymbol{S(p) = 0}$$

If z is RHP zero of the plant P(s) then:

$$\boldsymbol{T(z) = 0}$$

$$\boldsymbol{S(z) = 1}$$

These constraints clearly impose a shape on the S and T to guarantee internal stability.

In these situations anyway there are other limitations due to the sensitivity integrals and related waterbed effects, that may cause the sensitivity function magnitude to have a peak gain bigger then one. This would cause the feedback action to amplify the error, instead of reducing it, in a specific range of frequencies.

This peak is unavoidable in practice, and this will cause that if we push the sensitivity down at some frequencies, it will increase in others, and will exist a frequency range in which the magnitude of S is bigger then one.

This will cause the distance from the critical point in the Nyquist plot of the L(s) to be less then one.

This result is summarized in the **First Waterbed Effect**:

Suppose that open loop transfer function L(s) is rational and has at least relative degree equal 2, and has Np RHP poles, then:

$$\int_0^{\infty} \ln|S(j\omega)|\, d\omega = \pi * \sum_{n=1}^{Np} Re(pi)$$

Where Np are the number of RHP poles and Re(pi) their real parts.

For a stable plant the right side of the equation is zero and the area in which sensitivity reduction is equal to the area of the sensitivity increase.

The right hand side of the equation, in presence of unstable poles, specify that the area in which the magnitude of S is bigger then one exceed the area of sensitivity reduction by an amount proportional to the sum of the distances of RHP poles from LHP, and this is the price we have to pay to guarantee stability and our sensitivity peak will increase.

In the case of RHP zeroes, the sensitivity function must satisfy another integral relationship.
First of all we must say that the presence of an unstable zero implies that the loop function enter in the unit circle around the point (-1,0) in Nyquist plot, and so the sensitivity magnitude will have a peak value bigger then 1.

**Second Waterbed Effect:**

Suppose that L(s) has a RHP zero z or a complex conjugate pair of zeros, and has Np RHP poles.
Then for closed loop stability the sensitivity function must satisfy:

$$\int_0^\infty \ln|S(j\omega)| * w(z,\omega)\, d\omega = \pi * \prod \frac{p_i + z}{\overline{p_\iota} - z}$$

Where the productory is on Np and $\overline{p_\iota}$ is the complex conjugate of an RHP pole.
The factor $w(z,\omega)$ differ depending on the real or complex nature of the zero, and cut the contribution to the integral for the frequencies bigger then z.

We note that when we have a RHP pole near an RHP zero, the term in the productory goes to infinity and the plant is in general impossible to be stabilized.

This second waterbed effect impose also a frequency limitation on the trade off between positive and negative area of S. In particular, they impose a constraint on the crossover frequency of the S. (The balance between the area must be satisfied in the frequencies before the RHP zero).

All this considerations are done considering frequencies in a linear scale, not a logaritmic one.

Let's now consider the cart pendulum problem.

## 2. Force – Angle Control

In this case we have relative degree equal to 2 and a single RHP pole at p = 5.5841.

To stabilize the system, we must require an input usage and we will also have a lower bound on the required bandwidth, that constraint us to act speedily in order to contrast the presence of the unstable pole.

We will need T(p) = 1.

We must satisfy:

$$|\mathbf{WT}(\mathbf{p})| < \mathbf{1}$$

In particular, after having set HF gains equal to 40 decibel and a very low value for the DC gain in the weight for complementary sensitivity, we obtain a bandwidth lower bound, equal approximately to twice the value of the unstable pole (around 11 rad/s).

This limitation implies that there will be a positive area in the sensitivity, but since there are no RHP zeroes, we could spread this area in all the frequencies above the lower bound for the bandwidth.

Below are shown the sensitivity functions when we move the desired bandwidth for T from 15 rad/s (in blue) to 1 rad/s (in red).

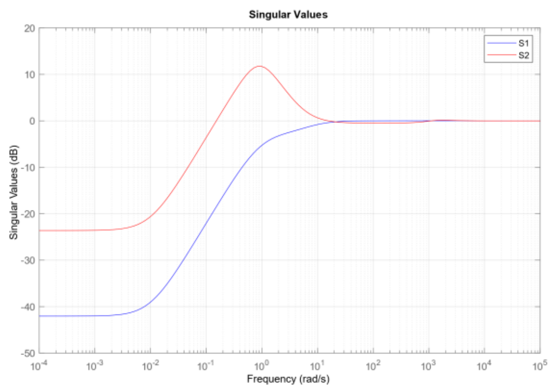


*Fig. 3 Sensitivity Analysis*

## 3. Force – Position Control

This situation is more critical since we have a RHP zero and a RHP pole, and they are also near each other. We can deduce from the second waterbed effect that the trade off between positive and negative area of the S must happen in a limited frequency range.

In this case, the term in the right hand side of the second waterbed effect will go to infinity since the denominator will go to zero (cause the pole and the zero are very near each other).
This can be tought as the fact that we will need an infinite positive area for the sensitivity function, and so the plant is in practice impossible to stabilize (Skogestad page 171).

So we could find a controller using hinfsyn or mixsyn, but it will not stabilize our closed loop plant, in particular we will have a stabilized response for the position, but the internal state of the system will diverge, as shown in the following plots.

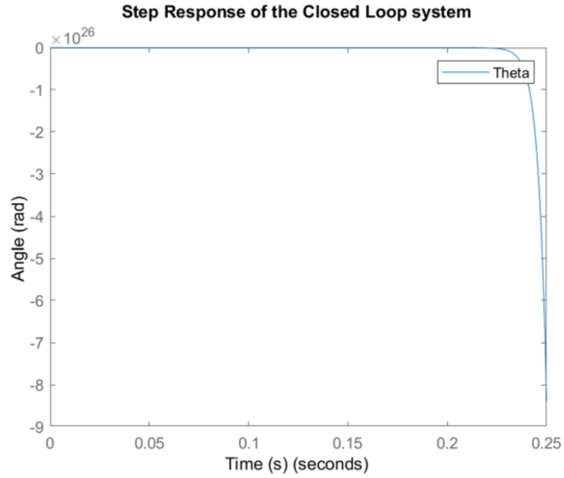


*Fig. 4 Cart Angle Step Response Behaviour*

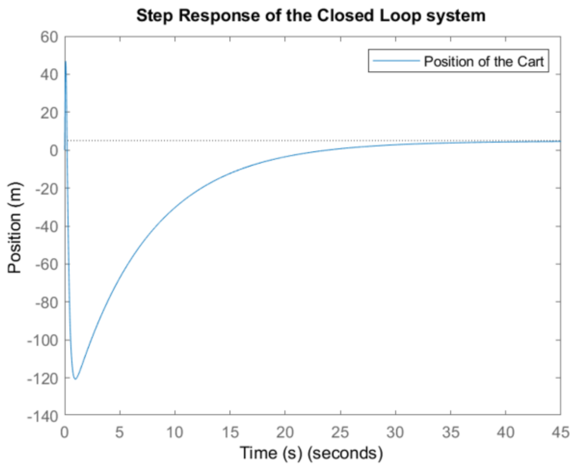


*Fig. 5 Cart Position Step Response Behaviour*

## References

[1] Surname A and Surname B 2009 *Journal Name* **23** 544