Homework 03 - due Wednesday May 19th, 2021.

The first problem will help you understand the Singular Value Decomposition. The goal of the second problem is to practice, on a real system, with performance definition and robust stability (unstrucured uncertainty case) problems in the MIMO case. Do as much as you can.

Problem 1
Let the SISO plant be

$$G(s) = \frac{10}{(s+2)(s+10)}$$

Find a controller using typical SISO basic techniques (root locus, loop shaping, ...) such that:

- the steady state error w.r.t. a constant reference is zero
- there is good disturbance rejection for both sinusoidal disturbances d_1 and d_2 with frequencies up to 5 rad/s
- sinusoidal measurement noise in the range $[100, 1000]$ rad/s is sufficiently attenuated
- satisfactory transient behavior.

You decide what "good", "sufficiently" and "satisfactory" is (you choose the specs).

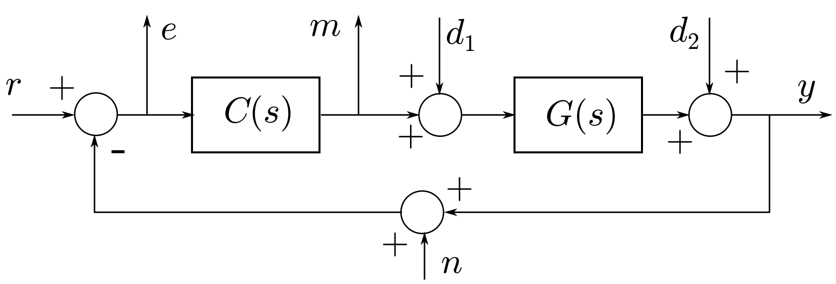

Figure 1: Control scheme

We now look at the control system in Fig. 1 as a MIMO system with 4 inputs r, d_1, d_2 and n while the 3 outputs are y, $e_r = r - y$ and the control input m: a non-square (3×4) MIMO system.

1. Write the closed-loop MIMO transfer function matrix $W(s)$ both symbolically (for generic $G(s)$ and $C(s)$) and numerically (for the given $G(s)$ and the controller $C(s)$ you designed).
2. For the given choices of $G(s)$ and $C(s)$, is the corresponding gain matrix (static gain) of $W(s)$ consistent with the closed-loop steady state specifications? How do you interpret the input and output directions corresponding to the different singular values?
3. Try to define and solve an almost equivalent (i.e. which solves similar specifications) mixed sensitivity control problem on the MIMO system.

Problem 2
Consider the 4 tanks experimental set up represented in Fig. 2 taken from [1]. For each tank we can describe the change of volume in time as

$$\frac{dV(t)}{dt} = (q_{\text{in}}(t) - q_{\text{out}}(t))$$

with $q_{\text{in}}(t)$ and $q_{\text{out}}(t)$ respectively the water inflow (volume/time) and outflow (volume/time). Denoting with c the cross section area of the outlet hole, Bernouilli's law gives

$$q_{\text{out}}(t) = c\sqrt{2g\,h(t)}$$

g being the gravity acceleration and h the water level in the tank. The pump generates a flow proportional to an applied voltage $u(t)$, i.e. $q(t) = k\,u(t)$, and the generic valve splits the flow into a flow to the lower tank $q_{\text{low}}(t)$ and one to the upper tank $q_{\text{up}}(t)$ according to a valve constant γ

$$q_{\text{low}}(t) = \gamma q(t) = \gamma k\,u(t), \qquad q_{\text{up}}(t) = (1-\gamma)k\,u(t), \qquad \gamma \in (0,1)$$

i.e. $q(t) = q_{\text{low}}(t) + q_{\text{up}}(t)$. A sensor provides an output voltage proportional (with constant k_t) to the water height h_i, therefore the two measured outputs (two sensors with same k_t) are

$$y_1 = k_t\,h_1, \qquad y_2 = k_t h_2$$

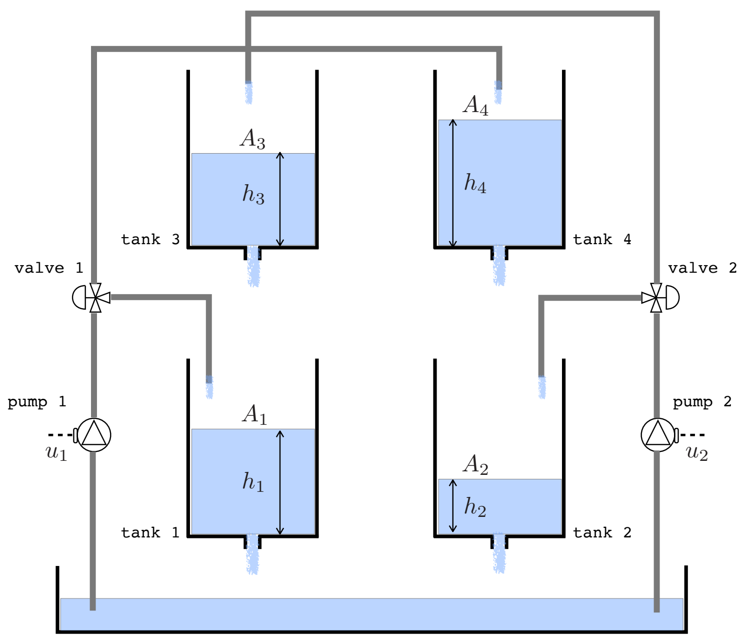

Figure 2: The 4-tanks experimental apparatus with their base areas A_i and water levels h_i

The model is given by (the subscript meaning should be evident)

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{c_1}{A_1}\sqrt{2gh_1} + \frac{c_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1}u_1\\
\frac{dh_2}{dt} &= -\frac{c_2}{A_2}\sqrt{2gh_2} + \frac{c_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2}u_2\\
\frac{dh_3}{dt} &= -\frac{c_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3}u_2\\
\frac{dh_4}{dt} &= -\frac{c_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4}u_1
\end{aligned}$$

Denote with u_i^0, h_i^0 and y_i^0 the corresponding values at an equilibrium. Define the variations

$$\Delta u_i = u_i - u_i^0, \qquad \Delta h_i = h_i - h_i^0, \qquad \Delta y_i = y_i - y_i^0$$

and

$$u = \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \qquad x = \begin{pmatrix} \Delta h_1 \\ \Delta h_2 \\ \Delta h_3 \\ \Delta h_4 \end{pmatrix}, \qquad y = \begin{pmatrix} \Delta y_1 \\ \Delta y_2 \end{pmatrix}$$

The corresponding linearized system is

$$\dot{x} = \begin{pmatrix} -\frac{1}{\tau_1} & 0 & \frac{A_3}{A_1\tau_3} & 0 \\ 0 & -\frac{1}{\tau_2} & 0 & \frac{A_4}{A_2\tau_4} \\ 0 & 0 & -\frac{1}{\tau_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{\tau_4} \end{pmatrix} x + \begin{pmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_4} & 0 \end{pmatrix}$$

$$y = \begin{pmatrix} k_t & 0 & 0 & 0 \\ 0 & k_t & 0 & 0 \end{pmatrix} x$$

with

$$\tau_i = \frac{A_i}{c_i}\sqrt{\frac{2h_i^0}{g}}$$

and the transfer function matrix is

$$G(s) = \begin{pmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 k_1 n_1}{1+\tau_1 s} & \frac{(1-\gamma_2)k_2 n_1}{(1+\tau_1 s)(1+\tau_3 s)} \\ \frac{(1-\gamma_1)k_1 n_2}{(1+\tau_2 s)(1+\tau_4 s)} & \frac{\gamma_2 k_2 n_2}{1+\tau_2 s} \end{pmatrix} \qquad \text{with} \quad n_i = \frac{k_t \tau_i}{A_i}$$

The corrected model can be found in [2].

1. For the tank system $G(s)$, define performance weights which give performance similar to the decentralized PI controller used in [1] but with the system and controller parameters of [3]. Solve the corresponding mixed sensitivity problem.
2. Use the uncertainty description of [3] and solve a robust stabilization problem (unstructured uncertainty case).
3. See if you can include the robust stability requirement in a mixed sensitivity problem with also the previous performance requirements.

References

[1] K. H. Johansson, "The quadruple-tank process: A multivariable laboratory process with an adjustable zero", *IEEE Transactions on control systems technology*, vol. 8, n. 3, pp. 456–465, 2000

[2] T. Roinila, M. Vilkko nad A.I Jaatinen, "Corrected Mathematical Model of Quadruple Tank Process", I*FAC Proceedings Volumes*, Vol. 41, Issue 2, 2008, pp. 11678-11683.

[3] R. Vadigepalli, E. P. Gatzke, and F. J. Doyle. "Robust control of a multivariable experimental four-tank system." *Industrial & engineering chemistry research*, vol. 40.8, 2001, pp. 1916–1927.