Laboratorio 1 - Stabilizzazione di un Pendolo di Furuta

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## 1 Activity goals

The activity consists of designing an infinite–horizon LQR for stabilizing the Furuta's pendulum around its unstable equilibrium (i.e. a configuration with an upward–directed pendulum). The goal of the activity is to analyze how different choices of the weighting matrices in the quadratic cost yield different system responses. Also it is requested a comparison between the LQR approach with its associated Symmetric Root Locus and the standard procedure of Pole Placement. As an optional task, it requested to verify the possibility to stabilize the system using only one output via a Dynamic Observer.).

## 2 LQR design and experimental testing

Design a infinite–horizon LQR capable of stabilizing the Furuta's pendulum around its unstable equilibrium. Analyze how different choices of the weights $\mathbf{Q} = \mathbf{Q}^T \succeq 0$ and $r > 0$ in the quadratic cost

$$J_\infty = \frac{1}{2}\int_0^{+\infty} \mathbf{x}^T(t)\,\mathbf{Q}\,\mathbf{x}(t) + r\,u^2(t)\;dt$$

with $\mathbf{x} = \left[\alpha,\,\beta,\,\dot{\alpha}\;\dot{\beta}\right]^T$ yield different (optimal) closed–loop responses.

For any choice, analyze how the system reacts to an impulsive torque disturbance, which is generated by applying an impulsive voltage command to the D.C. motor (impulse height $= 3\,V$, impulse width $0.1\,s$, impulse initial time: after that the pendulum has been stabilized to the upward position).

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As starting points for the LQR design, consider the following tentative choices for the weights $\mathbf{Q}$ and $r$:

1. *expensive vs. cheap control cases*:

$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 1\}, \qquad r = 1000 \tag{1}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 1\}, \qquad r = 100 \tag{2}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 1\}, \qquad r = 10 \tag{3}$$

2. *different weights for* $\alpha$:

$$\mathbf{Q} = \mathrm{diag}\{10, 1, 1, 1\}, \qquad r = 1000 \tag{4}$$
$$\mathbf{Q} = \mathrm{diag}\{100, 1, 1, 1\}, \qquad r = 1000 \tag{5}$$
$$\mathbf{Q} = \mathrm{diag}\{1000, 1, 1, 1\}, \qquad r = 1000 \tag{6}$$

3. *different weights for* $\beta$:

$$\mathbf{Q} = \mathrm{diag}\{1, 10, 1, 1\}, \qquad r = 1000 \tag{7}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 100, 1, 1\}, \qquad r = 1000 \tag{8}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 1000, 1, 1\}, \qquad r = 1000 \tag{9}$$

4. *different weights for* $\dot{\alpha}$:

$$\mathbf{Q} = \mathrm{diag}\{1, 1, 10, 1\}, \qquad r = 1000 \tag{10}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 1, 100, 1\}, \qquad r = 1000 \tag{11}$$

5. *different weights for* $\dot{\beta}$:

$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 10\}, \qquad r = 1000 \tag{12}$$
$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 100\}, \qquad r = 1000 \tag{13}$$

6. *only relative values are important when defining the weights*:

$$\mathbf{Q} = \mathrm{diag}\{1, 1, 1, 1\}, \qquad r = 1000 \tag{14}$$
$$\mathbf{Q} = \mathrm{diag}\{0.001, 0.001, 0.001, 0.001\}, \qquad r = 1 \tag{15}$$

Verify that weights with different absolute values, but with identical relative values (i.e. with the same ratios between the values), yield the same optimal control input and optimal closed–loop response.

7. *no weighting for angular velocities* $\dot{\alpha}$ *and* $\dot{\beta}$:

$$\mathbf{Q} = \mathrm{diag}\{1,\,1,\,0,\,0\}, \qquad r = 1000 \tag{16}$$
$$\mathbf{Q} = \mathrm{diag}\{10,\,10,\,0,\,0\}, \qquad r = 1000 \tag{17}$$
$$\mathbf{Q} = \mathrm{diag}\{100,\,100,\,0,\,0\}, \qquad r = 1000 \tag{18}$$

Compare the results with the choices:

$$\mathbf{Q} = \mathrm{diag}\{1,\,1,\,1,\,1\}, \qquad r = 1000 \tag{19}$$
$$\mathbf{Q} = \mathrm{diag}\{10,\,10,\,10,\,10\}, \qquad r = 1000 \tag{20}$$

8. *correct/wrong choice of the weighting matrix* $\mathbf{Q}$:

$$\mathbf{Q} = \mathrm{diag}\{1,\,0,\,0,\,0\}, \qquad r = 1000 \tag{21}$$
$$\mathbf{Q} = \mathrm{diag}\{0,\,1,\,1,\,1\}, \qquad r = 1000 \tag{22}$$

Explain why the choice (21) yields a stabilizing LQR, while the choice (22) does not produce any result.

9. *Bryson's rule* (see also lecture notes):

$$\mathbf{Q} = \mathrm{diag}\left\{\frac{1}{\alpha_{max}^2},\,\frac{1}{\beta_{max}^2},\,\frac{1}{\dot{\alpha}_{max}^2},\,\frac{1}{\dot{\beta}_{max}^2},\right\}, \qquad r = \frac{1}{u_{max}^2} \tag{23}$$

As for $\alpha_{max}$, $\beta_{max}$, $\dot{\alpha}_{max}$, $\dot{\beta}_{max}$ and $u_{max}$, consider the following tentative values:

$$\alpha_{max} = 30^\circ \cdot \frac{\pi}{180^\circ} \tag{24}$$
$$\beta_{max} = 10^\circ \cdot \frac{\pi}{180^\circ} \tag{25}$$
$$\dot{\alpha}_{max} = \alpha_{max} \cdot (2\pi\,10) \tag{26}$$
$$\dot{\beta}_{max} = \beta_{max} \cdot (2\pi\,10) \tag{27}$$
$$u_{max} = 0.5\,K_t\,K_i \tag{28}$$

where $K_t = 0.071\,Nm/A$ is the D.C. motor torque constant and $K_i = 2\,A/V$ is the current driver transconductance gain.

## 3 LQR Symmetric Root Locus vs Pole Placement

Consider the SISO system where only the first output is used, i.e. $C_1 = [1\ 0\ 0\ 0]$, and consider $Q = C_1^T C_1$.

1. Compute explicitly the transfer function in terms of the systems parameters (moment of inertia, etc..). i.e. $P(s) = C_1(sI - A)^{-1}B$.

2. Compute open loop poles and zeros of such transfer function

3. Compute the Symmetric Root Locus for such transfer function.

4. Compute the location of the closed loop poles of the LQR solution for $r \to +\infty$ (*expensive control*).

5. Compute the location of the closed loop poles of the LQR solution for $r \to 0$ (*cheap control*). If there some goes to "infity", indicate the approximate expression in terms of $r$ of their location in the complex plane.

6. Evaluate the performance of the systems for $r = \{1, 10, 100, 1000\}$ and pick the solution which provides better performance from a simulation point of view.

7. Compute the alternative solution based on pole placement by placing the closed loop poles in the location

$$p_{12} = \omega_n \left( -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} \right), \quad p_{34} = \omega_n \left( -\frac{\sqrt{3}}{2} \pm j\frac{1}{2} \right)$$

   by choosing different values of $\omega_n$.

8. Compare the performance between the LQR and the Pole Placement approach by using the best performing values for $r$ and $\omega_n$.

## 4 LQR with Dynamic Observer (optional)

Verify that the system is observable from the first output only.

1. Design a Dynamic Observer based on Pole placement using as pole location of the observer
$$p_{12} = \omega_n \left( -\frac{\sqrt{2}}{2} \pm j\frac{\sqrt{2}}{2} \right), \quad p_{34} = \omega_n \left( -\frac{\sqrt{3}}{2} \pm j\frac{1}{2} \right)$$
where $\omega_n$ has to be selected to provide best performance.

2. Design a Dynamic Observer by using LQR approach. In this case we exploit duality theory and we consider the following control system
$$F \leftarrow A^T, \quad G \leftarrow C_1^T, \quad H \leftarrow B^T$$

3. Compute explicitly the transfer function in terms of the systems parameters (moment of inertia, etc..). i.e. $P(s) = H(sI - F)^{-1}G$.

4. Let $R = r$ and $Q = H^T H$. Compute the LQR gain for this system for different values of $r$, i.e. $\Gamma_{LQR} = \mathrm{lqr}(F, G, Q, R)$, and using duality theory use as gain for the observer
$$L_{LQR} = \Gamma_{LQR}^T$$
Choose the best value for $r$ in term of simulative performance.

5. Compare the Pole Placement design with the LQR design for the Dynamic Observer.