Laboratorio 2 - Identification of parameters

Luca Schenato *

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1 Activity goals

The activity consists of estimating unknown parameters from data. The first part is based on synthetic data of abstract static systems, while the second part would require the identification of damping parameters for a DC motor.

2 Estimation of a scalar parameter

Consider the following measurement model:

$$y_i = \theta + v_i, \quad i = 1, \dots, N$$

where v_i is i.i.d. white Gaussin noise with unit variance, i.e. $v_i \sim \mathcal{N}(0, \sigma^2)$ where $\sigma = 1$.

1. *Recursive least squares (known noise variance)*: Let start with the assumption that σ is known, while $\theta = 1$ is the unknown parameter. Generate $N = 1000$ random measurement samples based on the model above using the MATLAB command `v=randn`. Implement the recursive least square, i.e. compute $\widehat{\theta}_n, n = 1, \dots, N$. Plot also the so-called $3 - \sigma$ (99%) confidence bound, i.e. $\sigma_n^\theta = \frac{\sigma}{\sqrt{n}}$ around $\widehat{\theta}_n$ and verify whether the true θ belongs to this interval.

2. *Recursive least squares (unknown noise variance)*: Repeat the previous problem by also estimating the variance of the noise $\widehat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (y_i - \widehat{y}_i^n)^2 = \frac{1}{n}\sum_{i=1}^n (y_i - \widehat{\theta}_n)^2$ and the variance of the estimate as $\widehat{\sigma}_n^\theta = \frac{\widehat{\sigma}_n}{\sqrt{n}}$.

*Department of Engineering (DEI), University of Padova.
Phone: +39-049-8277925 email: `schenato@dei.unipd.it`

3. *Recursive least squares with forgetting factor*: Let us now consider that the unknown parameter θ is not constant but goes through a large change, i.e.

$$y_i = \theta_i + v_i, \quad i = 1, \ldots, N$$

where

$$\theta_i = \begin{cases} 1 & \text{for} \quad i = 1, \ldots N \\ 2 & \text{for} \quad i = N+1, \ldots, 2N \end{cases}$$

Implement the recursive least squares with forgetting factor $\lambda = \{1, 0.99, 0.95, 0.9, 0.8\}$ and compare the estimate with the true value θ_i.

3 Model selection with cross validation

Let us consider the following true data generating model:

$$y = \sin 4x + \frac{1}{4x+1}$$

which is unknown. The data are generated according to the model

$$y_i = \sin 4x_i + \frac{1}{4x_i+1} + v_i, \quad i = 1, \ldots, N$$

where $N = 100$, $x_i = \frac{i-1}{99}$ and $v_i \sim \mathcal{N}(0, \sigma^2), \sigma = 0.2$ is i.i.d. Gaussian Noise whose variance is unknown. We will consider the following polynomial estimator:

$$\widehat{y} = \theta_0 + \theta_1 x + \cdots + \theta_m x^m = \phi(x)^T \theta$$

where $\theta = (\theta_0, \ldots, \theta_m)$ and $\phi(x) = (1, x, x^2, \ldots, x^m)$ and m represents the model complexity. Divide the N data points $\{(x_i, y_i)\}_{i=1}^N$ into two sets: the training set $\mathcal{D}_{tr}$ containing 80% of the data points and the validation $\mathcal{D}_{vl}$ containing the other 20%. Compute the least square estimator for $m = 0, \ldots, M$ with $M = 20$ using the training set where $\widehat{y}_i = \phi^T(x_i)\theta$, i.e.

$$\widehat{\theta}(m) = \arg\min_{\theta} V(\theta; \mathcal{D}_{tr}, m) = \arg\min_{\theta} \frac{1}{|\mathcal{D}_{tr}|} \sum_{i \in \mathcal{D}_{tr}} (y_i - \widehat{y}_i)^2$$

Compute also the average prediction error on the training set $\widehat{\sigma}_y^{tr}(m)$ and the average prediction error on the validation set $\widehat{\sigma}_y^{vl}(m)$ as a function of m, i.e.

$$\widehat{\sigma}_y^{tr}(m) := \sqrt{V(\widehat{\theta}(m); \mathcal{D}_{tr}, m)}, \quad \widehat{\sigma}_y^{vl}(m) := \sqrt{V(\widehat{\theta}(m); \mathcal{D}_{vl}, m)}$$

Verify that $\widehat{\sigma}_y^{tr}(m)$ is monotonically decreasing with m while $\widehat{\sigma}_y^{vl}(m)$ presents a minimum and in general $\widehat{\sigma}_y^{tr}(m) \leq \widehat{\sigma}_y^{vl}(m)$.

Note that in MATLAB the least square solution $\widehat{\theta} = (\Phi^T\Phi)^{-1}\Phi^T y$ can be computed using the command `theta_hat = Phi\y`. Also `y_hat = Phi*theta_hat` and `sigma_hat = norm(y-y_hat)/sqrt(T)` where T is the size of the vector y.

For the optimal m based on the validation set, plot $\widehat{y}(x)$ and the $3-\sigma$ (99%) confidence bound based on $\widehat{\sigma}_y^{vl}(m)$ and the true function $y(x)$.

4 Identification of discrete time linear systems

Consider the current-controller model of a DC motor given by

$$Y(s) = \frac{K_i K_t}{s(Js+b)} U(s) = P(s)U(s)$$

where K_i, K_t, J are known but the damping coefficient is unknown. If the previous system is discretized with sampling period T, the *exact* discretization given

$$Y(z) = \frac{b_1 z + b_0}{(z-1)(z+a_0)} U(z)$$

where

$$\tau = \frac{J}{b}, \quad b_0 = \frac{K_i K_t}{b}(\tau - (T+\tau)e^{-\frac{T}{\tau}}, \quad b_1 = \frac{K_i K_t}{b}((T-\tau) + \tau e^{-\frac{T}{\tau}}, \quad a_0 = -e^{-\frac{T}{\tau}}$$

Consider the following filtered input-output signals:

$$U_f(z) = \frac{1}{z} U(z), \quad Y_f(z) = \frac{z-1}{z} Y(z)$$

and note that

$$Y_f(z) = \frac{b_1 z + b_0}{z + a_0} U_f(z) = \frac{b_1 + b_0 z^{-1}}{1 + a_0 z^{-1}} U_f(z)$$

which can be equivalently be written in the time-domain as

$$y_f(t) = -a_0 y_f(t-1) + b_1 u_f(t) + b_0 u_f(t-1) = \underbrace{[-y_f(t-1) \; u_f(t) \; u_f(t-1)]}_{\phi(t)^T} \underbrace{[a_0 \; b_1 \; b_0]^T}_{\theta}$$

1. *Parameter identification:* Assuming to be able to collect time series $\{u_f(t), y_f(t)\}_{t=1}^N$ provide a recursive least square (RLS) estimate of the transfer function coefficients $(\widehat{a}_0(t), \widehat{b}_1(t), \widehat{b}_0(t))$ as function of t and compare them with the exact ones. Use the

SIMULINK code provided to generate data and write a MATLAB script off-line to estimate the RLS estimate. Noting that $a_0 = -e^{\frac{Tb}{J}}$ estimate the daming coefficient as

$$\widehat{b}(t) = \frac{J}{T}\ln(-\widehat{a}_0(t))$$

Recalling that

$$Var(\widehat{\theta}(t)) \approx P(t)\sigma_y^2 = (\sum_{k=1}^{t}\phi(k)\phi^T(k))^{-1}\sigma_y^2$$

and using as an estimate of the measurement noise variance σ_y^2 the following formula

$$\widehat{\sigma}_y^2(t) = \frac{1}{t}\sum_{k=1}^{t}(y_f(k) - \widehat{y}_f(k;t))^2 = \frac{1}{t}\sum_{k=1}^{t}(y_f(k) - \phi^T(k)\widehat{\theta}(t))^2$$

estimate the confidence bound on the estimated parameter a_0 using the 1-1 element of the matrix $P(t)\widehat{\sigma}_y^2(t)$, i.e.

$$\widehat{\sigma}_{\widehat{a}_0}(t) = \sqrt{P_{11}(t)\widehat{\sigma}_y^2(t)}$$

plot the $3-\sigma$ confidence bounds for b:

$$\widehat{b}_{bounds}(t) = \frac{J}{T}\ln(-\widehat{a}_0(t) \pm 3\widehat{\sigma}_{\widehat{a}_0}(t))$$

2. *Identification-based Tuning for Control Design:*

   Design a PD position controller

$$C(s) = K_p + K_d\frac{s}{1+\tau_c s}$$

   for the continuous time transfer function $P(s)$ with maximum overshoot of 10% and settling time $t_s = 0.5$ $[s]$. Consider a scenario where b is not known and a scenario after the identification of the damping coefficient. For the first scenario, since b is not known, the standard procedure is to be conservative and assuming $\widehat{b} = 0$, i.e. the transfer function to be used to design the PI controller is:

$$P_{b=0}(s) = \frac{K_i K_t}{Js^2}$$

   while in the second scenario you can use the transfer function with the estimated parameter:

$$P_{b=\widehat{b}(N)}(s) = \frac{K_i K_t}{Js^2 + \widehat{b}(N)s}$$

Compare the performance of the two controllers on the true DC motor model which uses the true b, i.e.

$$P_{b=b}(s) = \frac{K_i K_t}{Js^2 + bs}$$